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Buildings: Interactions with Algebra and Geometry

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ABSTRACT. The focus of the conference was on buildings and their applications. Buildings are combinatorial structures (metric cell complexes) which may be viewed as simultaneous generalizations of trees and projective spaces. There is a rich class of groups acting on buildings; the action can often be used to obtain structural results about the group itself. On the other hand, buildings and related metric spaces — such as Riemannian symmetric spaces, p -adic symmetric spaces, metric CAT(0)-complexes — form an interesting class of geometric structures with a high degree of symmetry. In the last years, there were several new developments in the applications of buildings in arithmetic geometry, Riemannian geometry, representation theory, and geometric group theory. The workshop brought together experts from these areas whose work is related to buildings.

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Introduction by the Organisers

The focus of the conference was on buildings and their applications. Buildings are combinatorial structures (metric cell complexes) which may be viewed as simultaneous generalizations of trees and projective spaces. There is a rich class of groups acting on buildings; the action can often be used to obtain structural results about the group itself. On the other hand, buildings and related metric spaces — such as Riemannian symmetric spaces, p -adic symmetric spaces, metric CAT(0)-complexes — form an interesting class of geometric structures with a high degree of symmetry. In the last years, there were several new developments in the applications of buildings in arithmetic geometry, Riemannian geometry, representation theory, and geometric group theory. The workshop brought together experts from these areas whose work is related to buildings. This mixture worked

out quite well and led to many discussions and interactions between people coming from different areas.

A somewhat similar workshop *Buildings and Curvature* took place in 2004 in Oberwolfach. Then, the focus was more on Riemannian and metric geometry. The present workshop *Buildings: Interactions with Algebra and Geometry* took a more algebraic turn. There were 50 participants and 25 lectures. We had a substantial group of young participants, some of which could present their work at the conference. Both the young and the senior speakers succeeded remarkably well in presenting their work in a way which was accessible to people working in neighboring, but different areas.

There were several talks reporting on essential progress in abstract building theory: R. Weiss presented his revision of the classification for locally finite affine buildings and A. Thuillier reported on a new approach to compactifications of Bruhat-Tits buildings and Berkovich spaces. For spherical buildings, C. Parker presented recent results on convex subsets, and K. Brown on displacement lengths of automorphisms in non-spherical buildings. However, most of the talks concerned applications in various areas. P. Littelmann gave a survey on applications of buildings in representation theory of algebraic groups. The talks by B. Rémy and G. Willis underlined the increasing importance of locally finite buildings in the new structure theory of locally compact totally disconnected groups. There were talks on constructions of lattices using buildings by A. Thomas and R. Gramlich. K.-U. Bux and K. Wortman presented strong and recent results on finiteness properties of arithmetic groups. U. Görtz reported on affine Deligne-Lusztig varieties and the reduction of Shimura varieties, which leads to interesting (and hard) combinatorial questions about folded galleries.

The organizers had the clear impression that the workshop worked very well, that it led to new scientific collaboration, to new ideas and that it brought new geometric insights. We would like to thank the institute and its friendly and helpful staff. The additional support by the NSF for young researches from the US is gratefully acknowledged.

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Abstracts

Levi decomposition and Moufang twin buildings

PETER ABRAMENKO

In their paper [3], Kac and Peterson generalize (among many other things) the classical Levi decomposition of parabolic subgroups of algebraic groups to parabolic subgroups of Kac-Moody groups. It was observed later by B. Rémy (see [5], Section I.6) that this Levi decomposition can be further generalized to groups with an RGD-system in the sense of [6], provided that Tits's more general commutator relations (RGD1) are replaced by Rémy's more special axiom $(DRJ1)_{lin}$. In joint work with B. Mühlherr (see [1]), necessary and sufficient criteria for the existence of a Levi decomposition of parabolic subgroups of general RGD-systems (not necessarily satisfying $(DRJ1)_{lin}$) are derived. An important tool is the Moufang twin building associated to an RGD-system.

We are now going to describe these results in detail. Let (W, S) be a Coxeter system, $\Sigma = \Sigma(W, S)$ the associated Coxeter complex and Φ the set of roots of Σ . Let further $(G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system in the sense of [6], and denote by H the intersection of the normalizers of all U_α 's. Note that this RGD-system gives rise to a Moufang twin building $\Delta = (\Delta_+, \Delta_-)$. We fix a "fundamental" twin apartment (Σ_+, Σ_-) in Δ and identify Σ_+ with Σ . Now given any residue R in Σ , we want to derive a Levi decomposition for the parabolic subgroup $G_R = \{g \in G \mid gR = R\}$. We denote by R° the residue in Σ_- opposite to R and set $\Phi_R = \{\alpha \in \Phi \mid R \text{ is on the boundary of } \alpha\}$, $\Psi_R = \{\alpha \in \Phi \mid R \text{ is in the interior of } \alpha\}$. Then the "Levi factor" of G_R is by definition the group $L = L(R, \Sigma) = H \langle U_\alpha \mid \alpha \in \Phi_R \rangle = G_R \cap G_{R^\circ}$. For any chamber c in R , we set $U_c = \langle U_\alpha \mid c \in \alpha \in \Phi \rangle$ and denote by $U_{R,c}$ the normal closure of $\langle U_\alpha \mid \alpha \in \Psi_R \rangle$ in U_c . The "unipotent radical of R " is now defined as the normal closure of $\langle U_\alpha \mid \alpha \in \Psi_R \rangle$ in G_R . It is easy to see that we always have $G_R = LU_R$. We say that G_R admits a *Levi decomposition* if $L \cap U_R = \{1\}$, that is if G_R is the semidirect product of L and U_R .

Theorem 1. *The following are equivalent:*

- (1) G_R admits a Levi decomposition
- (2) U_R acts freely on the set R^{op} of residues in Δ_- which are opposite to R
- (3) $U_{R,c}$ acts trivially on R
- (4) $U_{R,c} = U_R$, i.e. $U_{R,c}$ is normal in G_R
- (5) There exists a subgroup U' of U_c which acts trivially on R and transitively on R^{op} ; in that case necessarily $U' = U_R$
- (6) For any β in Ψ_R , U_β acts trivially on R
- (7) For any prenilpotent pair $\{\alpha, \beta\}$ with $\alpha \in \Phi_R$ and $\beta \in \Psi_R$, the commutator $[U_\alpha, U_\beta]$ acts trivially on R
- (8) For any prenilpotent pair $\{\alpha, \beta\}$ with $\alpha \in \Phi_R$ and $\beta \in \Psi_R$, $[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \gamma \in \Psi_R \cap (\alpha, \beta) \rangle$

So condition 8. in Theorem 1 gives the necessary and sufficient condition under which G_R admits a Levi decomposition. (The “open interval” (α, β) is defined as in [6], Section 3.3.) The natural question arises when this condition is satisfied. We say that a twin building satisfies (co) if for any chamber c of that twin building, the set of all chambers opposite to c is (gallery) connected. (Concerning the significance of (co), which is satisfied for “almost all” 2-spherical twin buildings, see [4].) However, we shall not require (co) for Δ but only for the residual twin building (R, R°) .

Theorem 2. *If (R, R°) satisfies (co), then all the equivalent conditions in Theorem 1 are satisfied.*

We finally remark that for general RGD-systems, G_R need not admit a Levi decomposition. But in order to prove this, one has to construct new RGD-systems which do not satisfy condition 8. above (and so in particular not $(DRJ1)_{lin}$). Equivalently, one has to construct new Moufang twin buildings, which is work in progress with the first step being provided by [2]. The easiest examples of parabolic subgroups of RGD-systems not admitting a Levi decomposition can be constructed for Coxeter systems of rank at least 3 where all off-diagonal entries of the Coxeter matrix are ∞ and all root groups U_α are of order 2.

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Cohomology of buildings and finiteness properties of S -arithmetic groups over function fields

HELMUT BEHR

1. Report on Borel-Serre's computation of (Alexander-Spanier-) cohomology of spherical and affine buildings for connected semi-simple algebraic groups, defined over a non-archimedean local field k (see [2]).

Proposition 1. *Let G be connected semi-simple of k -rank l , Y_t the Tits-building of $G(k)$ with analytic topology. X the Bruhat-Tits-building, R a ring and M a R -module. Then the cohomology with compact supports*

$$H_c^i(X; M) \simeq H^{i-1}(Y_t; M) \text{ for } i \geq 1,$$

(canonical $G(k)$ - R -module isomorphisms) in particular

$$\begin{aligned} H_c^i(X; M) &= 0 & \text{for } i \neq l & \quad \text{and} \\ H_c^l(X; M) &= M & \text{for } l = 0 \end{aligned}$$

Corollary. *If P is a minimal parabolic k -subgroup of G with unipotent radical U and $C_c^\infty(-; M)$ denotes locally constant functions with compact support, then*

$$H_c^l(X; M) \simeq C_c^\infty(U(k); M) \quad \text{for } l > 0,$$

especially it is free for $M = \mathbb{Z}$.

2. Report on results for finiteness properties (finite generation, finite presentation, types FP_n and F_n) for S -arithmetic groups over function fields between 1959 and 2007 (cf. the references in [3]).

Let F be a function field (c.e. $[F : \mathbb{F}_q(t)] < \infty$) with a finite set S of places, F_v the completion of F with respect to $v \in S$, G an absolutely almost simple algebraic group, defined over F with global rank $rk_F G = r$ and local ranks $rk_{F_v} G = r_v (v \in S)$, set $d := \sum_{v \in S} r_v$. $X = \prod_{v \in S} X_v$ is the product of Bruhat-

Tits-buildings X_v for $G(F_v)$ and a S -arithmetic subgroup Γ of $G(F)$ is discrete in $\prod_{v \in S} G(F_v)$. Call G anisotropic, if $rk_F G = 0$, then X/Γ is compact (Godement) and Γ is of type FP_∞ (see [2]), otherwise G is called F -isotropic.

Proposition 2 (Bux-Wortman 2007:[3]). *For an absolutely almost simple isotropic F -group G the S -arithmetic subgroup Γ cannot be of type FP_d (or F_d).*

All known results make plausible the following

Conjecture. *If G is isotropic, then Γ is of type FP_{d-1} .*

3. We shall sketch a proof for this conjecture, which uses the vanishing part of Borel-Serre's result and a variation of K. Brown's finiteness criterion with cohomology preserving direct limits (see [1]).

Let $C = C(X)$ be the chain-complex for the polysimplicial X . By induction on dimension k we construct complexes

$$C' = C'(k) \quad \text{for } 0 \leq k \leq d-1, \quad d = \sum_{v \in S} r_v = \sum_{v \in S} \dim X_v$$

with finitely generated projective (fgp) $\mathbb{Z}[\Gamma]$ -modules $C'(k)_n$ with

$$\begin{aligned} C'(k)_n &= 0 && \text{for } n > k, \\ \tilde{H}_n(C'(k)) &= 0 && \text{for } n < k \\ f_k : C'(k) &\rightarrow C && \text{and mapping cone } C''(k) \end{aligned}$$

(by definition: $C''_n = C_n \oplus C'_{n-1}$, $\delta''(c, c') = (\delta c + f c', -\delta' c')$)
and obtain a long exact sequence

$$(1) \quad \dots \rightarrow \tilde{H}_n(C) \rightarrow \tilde{H}_n(C'') \rightarrow \tilde{H}_{n-1}(C') \rightarrow \tilde{H}_{n-1}(C) \rightarrow \tilde{H}_{n-1}(C'') \rightarrow \dots$$

since C is acyclic we have $\tilde{H}_n(C) = 0$ for all n and therefore

$$(1a) \quad \tilde{H}_n(C'') \simeq \tilde{H}_{n-1}(C') \text{ for all } n \geq 1.$$

Start the induction for $k = 0$ with a vertex $x_0 \in X$, $X_0 := \Gamma x_0$, so $C(X_0)_0 \simeq \mathbb{Z}[\Gamma/\Gamma_0]$ ($\Gamma_0 = \text{stab}_\Gamma x_0$), in order to define

$$C'(0)_0 := \mathbb{Z}[\Gamma] \xrightarrow{f_0} C(X_0)_0 \subset C(X)_0.$$

Then $\tilde{H}(C'(0)) = \Gamma$ (augmentation ideal) is f.g. iff Γ is so. It is well known, that Γ is f.g. if $d \geq 2$, so is $\tilde{H}_0(C'_0) \simeq H_1(C''(0))$. The idea is to attach the "1-cells of $C''(0)$ to $C'(0)$ " to kill $\tilde{H}_0(C'(0))$ - more precisely: Choose a fgp P , which projects on $H_1(C''(0))$ and define $C'(1)_1 := P$, $C'(1)_0 := C'(0)_0$.

For further progress we need cohomology with compact supports and have again a long exact sequence.

$$(2) \quad \dots \rightarrow \tilde{H}_c^{n-1}(C'') \rightarrow \tilde{H}_c^{n-1}(C) \rightarrow \tilde{H}_c^{n-1}(C') \rightarrow \tilde{H}_c^n(C'') \rightarrow \tilde{H}_c^n(C) \rightarrow \dots$$

and for $n \leq d - 1$ there is $\tilde{H}_c^n(C) = 0$ by Borel-Serre, so

$$(2a) \quad \tilde{H}_c^{n-1} \simeq \tilde{H}_c^n(C''), \quad n - 1 \leq d - 2.$$

Lemma 1. C' chain-complex of fgp with $\tilde{H}_n(C') = 0$ for $n < k$, M arbitrary \mathbb{Z} -module, then

$$\tilde{H}_c^k(C' : M) \simeq \tilde{H}^k(C'; M \otimes \mathbb{Z}[\Gamma]) \simeq \text{Hom}_{\mathbb{Z}[\Gamma]}(\tilde{H}_k(C'); M \otimes \mathbb{Z}[\Gamma]).$$

Corollary. $\tilde{H}_c^{k+1}(C''; M) \simeq \text{Hom}_{\mathbb{Z}[\Gamma]}(\tilde{H}_{k+1}(C''); M \otimes \mathbb{Z}[\Gamma])$
by (1a) + (2a).

Lemma 2. If for any direct system $\{N_i\}$ with $\varinjlim N_i = 0$ also $\varinjlim \text{Hom}_R(N; N_i) = 0$ (R a ring, N R -module), then N is f.g. (over R).

Lemma 3. $C'(k)$ is a complex of fgp, so $\tilde{H}^n(C'_k)$ preserves direct limits, so also $\tilde{H}_c^n(C'_k)$.

Furthermore $\tilde{H}_c^n(C)$ preserves direct limits for $n < k$, since it vanishes by [2] for all \mathbb{Z} -modules M - together with Lemma 3 this is also true for $\tilde{H}_c^n(C'')$ by sequence (2). Thus the 3 lemmata imply.

Proposition 3. *If $C'(k)$ exists (by induction), then for $k+1 \leq d-1$ $\tilde{H}_{k+1}(C''(k))$ is finitely generated.*

Induction step $k \rightarrow k+1$: Choose a fgp. P , projecting on $Z_{k+1}(C''(k))$:

$$\begin{array}{ccc} P & \xrightarrow{\varphi_2} & Z_k(C'(k)) \\ \varphi_1 \downarrow & & \downarrow f_k \\ C_{k+1} & \xrightarrow{\delta_k} & C_k \end{array}$$

because $Z_{k+1}(C''(k))$ is a fiber-product.

Define:

$$C'(k+1)_n = \begin{cases} C'(k)_n & \text{for } n < k+1 \\ P & \text{for } n = k+1 \\ 0 & \text{for } n > k+1 \end{cases}$$

and extend the maps f_k to f_{k+1} by $f_{k+1}(P) = \varphi_1(P)$ and δ'_k to δ_{k+1} by $\delta_{k+1}(P) = \varphi_2(P)$. The homology sequence for the pair $(C''(k), C''(k+1))$ shows, that also $\tilde{H}_{k+1}(C''(k+1)) = 0$ and by (1) $H_k(C') = H_k(C) = 0$.

Finally $C' := C'(d-1)$ is a complex of fgp. $\mathbb{Z}[\Gamma]$ -modules with $\tilde{H}_n(C') = 0$ for $n \leq d-2$, so we have a projective resolution

$$(*) \quad C'_{d-1} \rightarrow C'_{d-2} \rightarrow \dots \rightarrow C'_0 \rightarrow \mathbb{Z} \rightarrow 0$$

which proves

Theorem. Γ is of the type FP_{d-1} .

Question. *Is it possible to obtain under the assumption that Γ is even of type FP_d - using (*) and by Borel-Serre's result, that for isotropic G the top cohomology $\tilde{H}_c^d(C; M)$ is not finitely generated - a contradiction, which would be a new proof for proposition 2.*

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Automorphisms of non-spherical buildings have unbounded displacement

KENNETH S. BROWN

(joint work with Peter Abramenko)

A well-known folklore result says that a nontrivial automorphism ϕ of a thick Euclidean building X has unbounded displacement. Here we are thinking of X as a metric space, and the assertion is that there is no bound on the distance that ϕ moves a point. [For the proof, consider the action of ϕ on the boundary X_∞ at infinity. If ϕ had bounded displacement, then ϕ would act as the identity on X_∞ , and one would easily conclude that $\phi = \text{id}$.] We generalize this result to buildings that are not necessarily Euclidean. We work with buildings Δ as combinatorial objects, whose set \mathcal{C} of chambers has a discrete metric (“gallery distance”). We say that Δ is of *purely infinite type* if every irreducible factor of its Weyl group is infinite.

Theorem 1. *Let ϕ be a nontrivial automorphism of a thick building Δ of purely infinite type. Then ϕ , viewed as an isometry of the set \mathcal{C} of chambers, has unbounded displacement.*

It is possible to prove this by using the Davis CAT(0) realization of Δ and arguing as in the Euclidean case. (But more work is required in the non-Euclidean case.) We instead give an elementary proof based on Weyl distance and the following lemma about Coxeter groups:

Lemma 2. *Let (W, S) be a purely infinite Coxeter system. If $w \neq 1$ in W , then there exists $s \in S$ such that:*

- (a) $w^{-1}sw \notin S$.
- (b) $l(sw) > l(w)$.

This can be restated geometrically in terms of the Coxeter complex $\Sigma = \Sigma(W, S)$:

Lemma 3. *Let Σ be a purely infinite Coxeter complex. If C and D are distinct chambers in Σ , then C has a wall H with the following two properties:*

- (a) H is not a wall of D .
- (b) H does not separate C from D .

Our proof of Lemma 3 is based on the interpretation of $\Sigma(W, S)$ as a “slice” of the Tits cone X . The lemma turns out to follow from the fact that X does not contain any pair $\pm x$ of nonzero antipodal points; this is due to Vinberg. See [1] for further details.

The theorem has the following group-theoretic corollary, which was our original motivation in proving it:

Corollary 4. *Let Δ and \mathcal{C} be as in the theorem, and let G be a group of automorphisms of Δ . If there is a bounded set of representatives for the G -orbits in \mathcal{C} , then G has trivial center.*

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Connectivity of horospheres and finiteness properties of S -arithmetic subgroups of groups of global rank 1

KAI-UWE BUX

(joint work with Kevin Wortman)

Let \mathbf{G} be a non-commutative, absolutely almost simple group scheme defined over the global function field K . Let S be a non-empty set of places and let \mathcal{O}_S be the ring of S -integers in K . Let $d := \sum_{v \in S} \text{rank}_{K_v}(\mathbf{G})$ be the sum of the local ranks of \mathbf{G} at the places in S . We are interested in finiteness properties of the S -arithmetic group $\mathbf{G}(\mathcal{O}_S)$.

Recall that a group Γ is of type F_m if Γ admits an Eilenberg-Mac Lane space of dimension m . The group is of type FP_m if there is a projective resolution of the integers \mathbf{Z} , regarded as a $\mathbf{Z}\Gamma$ -module with trivial Γ -action, that is finitely generated in dimensions $\leq m$.

In previous work [1], we have shown the following:

Theorem 1. *If the global rank of \mathbf{G} is non-zero, then $\mathbf{G}(\mathcal{O}_S)$ is not of type FP_d .*

Our main result complements the above negative statement by the positive result for groups of global rank 1:

Theorem 2. *If \mathbf{G} has K -rank 1, then $\mathbf{G}(\mathcal{O}_S)$ is of type F_{d-1} .*

This can be viewed as evidence for the conjecture that, if the global rank of \mathbf{G} is non-zero, $\mathbf{G}(\mathcal{O}_S)$ is of type FP_{d-1} but not of type FP_d .

In the proof, we study the action of $\mathbf{G}(\mathcal{O}_S)$ on the product $X := \prod_{v \in S} X_v$ of Euclidean buildings associated to $\mathbf{G}(K_v)$. Using reduction theory, we find a $\mathbf{G}(\mathcal{O}_S)$ -invariant family of pairwise disjoint open horoballs in X so that the complement is cocompact. The theorem then follows from standard topological criteria for finiteness properties and the following result about connectivity of horospheres in Euclidean buildings:

Theorem 3. *Let $X = X_1 \times \cdots \times X_n$ be a product of irreducible Euclidean buildings. Then, the spherical building at infinity decomposes as a join of spherical buildings $X^\infty = X_1^\infty \times \cdots \times X_n^\infty$. Let $e \in X^\infty$ be a point at infinity and suppose that e is not contained in any sub-join of the X_i^∞ , i.e., the Busemann function β corresponding to e is not orthogonal to any of the factors X_i . Then, any level set $\beta^{-1}(t)$, i.e., any horosphere centered at e is $(\dim X - 2)$ -connected.*

The proof of this theorem relies on combinatorial Morse theory.

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On the isometry group of a locally compact CAT(0) space

PIERRE-EMMANUEL CAPRACE

(joint work with Nicolas Monod)

A CAT(0) space is a metric space which is non-positively curved in the sense of A. D. Alexandrov. In other words, it is a geodesic metric space none of whose geodesic triangle is thicker than a congruent geodesic triangle in the Euclidean plane. In the case of complete spaces, this definition has been characterized by F. Bruhat and J. Tits as follows. A complete metric space X is CAT(0) if and only if for all $x, y \in X$, there exists $m \in X$ satisfying the following condition:

$$\text{For all } z \in X, d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2.$$

We refer to [BH99] for a detailed treatment of the basic theory. The following well known fact yields essential examples of CAT(0) spaces: A simply connected Riemannian manifold is CAT(0) if and only if it has non-positive sectional curvature. Other examples of CAT(0) spaces are provided by geometric realizations of buildings of arbitrary type. In particular, locally compact CAT(0) spaces provide a natural category which involves Riemannian symmetric spaces of non compact type, Bruhat-Tits buildings and products of those.

As mentioned in the introduction of [BH99], the CAT(0) property has been mainly used so far to study *discrete* groups acting properly by isometries. This note reports on a joint work with Nicolas Monod [CM], whose purpose is to collect basic facts on the *full* isometry group of a locally compact CAT(0) space. This group is endowed with a canonical topology – namely the topology of uniform convergence on compacta – which turns it into a locally compact second countable topological group; it is generally non-discrete.

Theorem 1. *Let X be a proper CAT(0) space with finite-dimensional Tits boundary. Assume that $\text{Is}(X)$ has no global fixed point in ∂X .*

Then there is a canonical closed convex $\text{Is}(X)$ -stable $\text{Is}(X)$ -minimal subset $X' \subset X$ such that $G = \text{Is}(X')$ admits a canonical decomposition

$$G \cong (S_1 \times \cdots \times S_p \times (\mathbf{R}^n \rtimes \mathbf{O}(n)) \times D_1 \times \cdots \times D_q) \rtimes F \quad (p, q, n \geq 0)$$

where S_i is a virtually connected simple Lie group with trivial centre, D_j is a totally disconnected irreducible group with trivial amenable radical and F is a finite permutation group of possibly isomorphic factors. Moreover, all non-trivial normal, subnormal or ascending subgroups of D_j are still irreducible with trivial amenable radical and trivial centraliser in D_j .

Given a CAT(0) space X and a group G acting on X by isometries, a G -stable subset Y is called G -minimal if it does not contain any proper closed convex G -stable subset. If G acts without fixed point at infinity, then X contains a G -minimal subspace, by a simple application of Zorn's lemma. In order to avoid uninteresting degeneracies, it is natural to restrict to CAT(0) spaces whose isometry group acts minimally; adding this assumption for X in Theorem 1 would allow one to avoid the distinction between X and the subspace X' .

We remark that in some naturally arising situations, the hypotheses of the theorem are automatically satisfied:

- If $X/\text{Is}(X)$ is compact, then ∂X is finite-dimensional [Kle99, Th. C].
- If X has no Euclidean factor and if a discrete group Γ acts minimally, properly and cocompactly on X , then Γ (and a fortiori $\text{Is}(X)$) has no fixed point at infinity [AB98, Cor. 2.7].

The first step in the proof of the theorem is to obtain a canonical “de Rham decomposition” of the underlying space:

Theorem 2. *Let X be a proper CAT(0) space with finite-dimensional Tits boundary, such that $\text{Is}(X)$ acts minimally. Then X admits a canonical isometric splitting*

$$X \cong \mathbf{R}^n \times Z_1 \times \cdots \times Z_m \quad (n, m \geq 0)$$

with each Z_i irreducible and $\neq \mathbf{R}^0, \mathbf{R}^1$. Every isometry of X preserves this decomposition upon permuting possibly isometric factors Z_i .

Note that such a statement fails to hold for *any* proper CAT(0) space, due notably to the possible presence of infinite-dimensional compact factors. A general “de Rham decomposition” theorem has been obtained for locally compact geodesic metric spaces in [FL06], under a local assumption of finite-dimensionality. It is however not clear *a priori* whether this assumption holds in the setting of Theorem 2.

The next step is to analyze the structure of the isometry group of irreducible CAT(0) spaces:

Theorem 3. *Let X be a proper irreducible CAT(0) space with finite-dimensional Tits boundary. If $G < \text{Is}(X)$ acts minimally without fixed point at infinity, then any nontrivial normal subgroup N of G still has these properties.*

This statement may be viewed as a weak geometric form of simplicity for the group G . It has the following consequence on the algebraic structure of normal subgroups:

Corollary 4. *If $N < G$ is nontrivial and normal, then:*

- (i) N is not amenable.
- (ii) N does not split nontrivially as a direct product.
- (iii) N has trivial centraliser in $\text{Is}(X)$.

Theorem 1 follows from Theorem 2 and Corollary 4, combined with the solution to Hilbert's fifth problem.

In the classical situation of symmetric spaces, the conditions arising in Theorem 3 have the following algebraic interpretation:

Let G be a connected noncompact simple Lie group of rank at least 2 and X be the associated symmetric space. A subgroup $\Gamma < G$ is Zariski dense if and only if Γ acts minimally on X without fixed point at infinity.

The ‘only if’ part is due to B. Kleiner and B. Leeb [KL05]; the other implication is easier and may be deduced from Theorem 3. This remark suggests that the following result could be viewed as a geometric version of Borel’s density theorem:

Theorem 5. *Let G be a locally compact group and $\Gamma < G$ be a lattice. If G acts continuously, minimally, without fixed point at infinity on a proper CAT(0) space without nontrivial Euclidean factor, then the induced Γ -action still has these properties.*

Combining Theorem 5 with (the easy direction of) the preceding remark, one obtains in particular a new proof of Borel’s density theorem.

The results presented here may be used to obtain information on discrete group actions; here is an illustration:

Theorem 6. *Let $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ with $n \geq 3$ and X be a proper CAT(0) space whose isometry group acts cocompactly without fixed point at infinity. Given any Γ -action on X by isometries, there exists a closed convex Γ -stable Γ -minimal subset $Y \subset X$ on which the given Γ -action extends to a continuous $\mathrm{SL}_n(\mathbf{R})$ -action by isometries.*

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Compactly supported cohomology of buildings

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(joint work with Jan Dymara, Tadeusz Januszkiewicz, Boris Okun)

Suppose (W, S) is a Coxeter system. A subset $T \subset S$ is *spherical* if it generates a finite subgroup of W . \mathcal{S} denotes the poset of spherical subsets of S . Let Φ be a building in the sense of [6] (i.e., Φ is a set of chambers, equipped with a family of adjacency relations indexed by S and a W -valued distance function, $\Phi \times \Phi \rightarrow W$).

Let A denote the set of finitely supported \mathbf{Z} -valued functions on Φ (i.e., A is the free abelian group on Φ). For each $T \in \mathcal{S}$, A^T denotes the set of $f \in A$ which are constant on all residues of type T . If $U \supset T$, then $A^U \subset A^T$. We show that the quotient, $D^T := A^T / \sum_{s \in S-T} A^{T \cup \{s\}}$, is free abelian. Let \hat{A}^T be a summand of A^T which projects isomorphically onto D^T .

Decomposition Theorem. For each $T \in \mathcal{S}$,

$$A^T = \bigoplus_{U \supset T} \hat{A}^U.$$

Suppose X is a CW complex and that $\{X_s\}_{s \in S}$ is a mirror structure over S on X (defined in [2, p.63]). For a cell c of X or point $x \in X$, put

$$S(c) := \{s \in S \mid c \subset X_s\}, \quad S(x) := \{s \in S \mid x \in X_s\}.$$

The X -realization of Φ is the quotient space $\mathcal{U}(\Phi, X) := (\Phi \times X) / \sim$, where \sim is the equivalence relation defined by $(\phi, x) \sim (\phi', x')$ if and only if $x = x'$ and ϕ, ϕ' belong to the same $S(x)$ -residue. (Here Φ has the discrete topology.) For each $T \subset S$, put

$$X_T := \bigcap_{s \in T} X_s \quad \text{and} \quad X^T := \bigcup_{s \in T} X_s.$$

We are primarily interested in the case where $X = K$, the geometric realization of the poset \mathcal{S} and where K_s is the geometric realization of $\mathcal{S}_{\geq \{s\}}$ (cf. [2, Chap.7]).

For any subgroup B of A and $T \subset S$, put $B^T := B \cap A^T$. We have a ‘‘coefficient system’’ $\mathcal{I}(B)$ on X , giving a cochain complex

$$\mathcal{C}^i(X; \mathcal{I}(B)) := \bigoplus_{c \in X^{(i)}} B^{S(c)},$$

where $X^{(i)}$ denotes the set of i -cells in X . Let $\mathcal{H}^*(X; \mathcal{I}(B))$ denote the cohomology groups of this cochain complex. The Decomposition Theorem gives us a direct sum decomposition of coefficient systems

$$\mathcal{I}(A) = \bigoplus_{T \in \mathcal{S}} \mathcal{I}(\hat{A}^T),$$

leading to the following.

Theorem.

$$\mathcal{H}^*(X; \mathcal{I}(A)) = \bigoplus_{T \in \mathcal{S}} \mathcal{H}^*(X; \mathcal{I}(\hat{A}^T)) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \hat{A}^T.$$

If X is compact and if, for each cell $c \subset X$, $S(c)$ is spherical, then $\mathcal{H}^*(X; \mathcal{I}(A)) = H_c^*(\mathcal{U}(\Phi, X))$. This gives our main result, the following corollary.

Corollary. (cf. [3, 4]).

$$H_c^*(\mathcal{U}(\Phi, K)) = \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \hat{A}^T.$$

When Φ is an irreducible affine building, this corollary is the classical computation of Borel-Serre [1]. When $\Phi = W$ (the thin building) or when Φ is right-angled, proofs can be found in [3]. A version of the general result is claimed in [5]; however, there is a mistake in the proof.

Our proof of the Decomposition Theorem is modeled on a homological argument from [4] for a similar result. The key to the proof is a calculation for the standard realization of Φ where X is the simplex Δ of dimension $\text{Card}(S) - 1$ with its codimension-one faces indexed by S . The Decomposition Theorem follows from the next result (and some similar statements).

Theorem. $\mathcal{H}^*(\Delta; \mathcal{I}(A))$ is concentrated in the top degree ($= \text{Card}(S) - 1$) and is a free abelian group.

We also need versions of this which assert the concentration in the top degree of $\mathcal{H}^*(\sigma, \sigma^U; \mathcal{I}(A))$, where σ ranges over the spherical faces of Δ and U over the subsets of S .

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Buildings have finite asymptotic dimension

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(joint work with Thomas Schick)

A metric space X has asymptotic dimension $\leq n$ if for every $D > 0$ there exists a cover $U = U_0 \cup U_1 \cup \dots \cup U_n$ of X , such that each U_i is D -disjoint and uniformly bounded. D -disjointness means that any two elements of U_i are at least D apart; in other words, their $D/2$ -neighbourhoods are disjoint. Uniform boundedness means that there exists a common upper bound on the diameters of all sets in U_i . A good introduction to asymptotic dimension is [1].

Dranishnikov and Januszkiewicz proved that Coxeter groups have finite asymptotic dimension (cf. [2]). A Coxeter group has several geometric realisations: the set W of group elements with the word-length metric; Cayley graph; Davis complex. These are, however, quasi-isometric, and asymptotic dimension is an invariant of quasi-isometries. Similarly, a building has the following (quasi-isometric)

metric realisations: the set of chambers with the gallery distance; the Davis realisation with its $CAT(0)$ metric.

Theorem. *Let X be a building with Weyl group W . Then the asymptotic dimension of X is equal to that of W .*

The proof goes as follows. We choose a chamber B in X and consider the B -based folding (aka retraction) map $\pi: X \rightarrow W$. Then we take a cover $U = U_0 \cup U_1 \cup \dots \cup U_n$ of W as in the definition of asymptotic dimension and pull it back to X via π . D -disjointness is preserved, but uniform boundedness fails: the farther is $V \subset W$ from $\pi(B)$, the larger is $\pi^{-1}(V)$. To rectify this, we subdivide each $\pi^{-1}(V)$ into its connected components. We prove that uniform boundedness is restored; however, this time D -disjointness fails: distinct components of $\pi^{-1}(V)$ can be close to each other. The golden mean is to subdivide each $\pi^{-1}(V)$ into its intersections with the connected components of π^{-1} of the $D/2$ -neighbourhood of V .

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Wagoner complexes - from Tits' Lemma to group homology

JAN ESSERT

Let G be a group with a spherical root datum. Let Σ be the standard apartment in the associated spherical building. To every simplex $s \in \Sigma$, we associate the unipotent radical U_s of the corresponding standard parabolic subgroup P_s . Note that the group generated by all U_s is the little projective group G^\dagger of G .

Definition 1. *The associated Wagoner complex $\mathcal{W}(G)$ is the flag complex over all cosets of such unipotent radicals:*

$$\mathcal{W}(G) := \text{Flag}\{gU_s : g \in G, s \in \Sigma\}.$$

This definition is a generalisation of the complexes J. Wagoner originally defined in [4] for buildings of type A_n . He used these complexes for an alternative definition of Quillen K-Theory.

Wagoner complexes admit a lot of structure — they are covered by two different kinds of subcomplexes isomorphic to barycentric subdivisions of the apartments of the building. There is also a canonical projection onto the corresponding building inducing an epimorphism on singular homology. The aforementioned subcomplexes are mapped bijectively onto (barycentric subdivisions of) apartments of the building. Here, we only focus on $\pi_0(\mathcal{W})$ and $\pi_1(\mathcal{W})$.

Definition 2 (Tits). *The Steinberg group \hat{G} is the direct limit of the direct system consisting of all root groups U_α and all groups $U_{[\alpha,\beta]} := \langle U_\gamma : (\alpha \cap \beta) \subseteq \gamma \rangle$.*

Lemma 3. *We have $\hat{G} \cong \varinjlim U_s$.*

The following theorem is due to P.-E. Caprace, see [2]. “In most cases” means basically that the statement is true, if it is true for the rank 2 case. The rank 2 case is examined by Tom de Medts and Katrin Tent in [3].

Theorem 4 (Caprace). *In most cases, the Steinberg group \hat{G} is the universal central extension of the little projective group G^\dagger .*

We invoke a theorem by Abels and Holz, see [1, Corollary 2.5], to get the following statement, which is a generalisation of Wagoner’s corresponding result in [4].

Theorem 5.

$$\begin{aligned}\pi_0(\mathcal{W}(G)) &\cong G/G^\dagger \\ \pi_1(\mathcal{W}(G)) &\cong \ker(\hat{G} \twoheadrightarrow G^\dagger)\end{aligned}$$

If the conditions of Theorem 4 are satisfied, we have

$$\pi_1(\mathcal{W}(G)) \cong H_2(G^\dagger).$$

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Affine Deligne Lusztig varieties

ULRICH GÖRTZ

(joint work with Thomas Haines, Robert Kottwitz, Daniel Reuman)

Let \mathbb{F}_q be a finite field, $k \supset \mathbb{F}_q$ an algebraic closure, $L = k((\epsilon))$ the field of Laurent series, and $\mathfrak{o} = k[[\epsilon]]$ the ring of power series. Let G be a split reductive group over \mathbb{F}_q . To simplify the exposition below, we assume that G is semisimple and simply connected. We denote by σ the Frobenius on k , L (acting by $\sum_i a_i \epsilon^i \mapsto \sum_i a_i^q \epsilon^i$), $G(k)$ and $G(L)$. We fix a split maximal torus and a Borel subgroup $T \subset B \subset G$, and denote by $I \subset G(\mathfrak{o})$ the standard Iwahori subgroup. We have the Bruhat-Iwahori decomposition

$$G(L) = \coprod_{w \in W_a} IwI,$$

the union ranging over the affine Weyl group W_a of G .

Definition 1. Let $b \in G(L)$, and $x \in W_a$. The affine Deligne-Lusztig variety associated to b and x is

$$X_x(b) = \{g \in G(L)/I; g^{-1}b\sigma(g) \in IxI\}.$$

If we identify $G(L)/I$ with the set of alcoves in the Bruhat-Tits building of the group G over L , then we can express the definition as follows: $X_x(b)$ is the set of all alcoves C such that the Weyl distance between C and its “Frobenius translate” $b\sigma C$ (for the twisted Frobenius automorphism $b\sigma$) is the fixed element w . We can view the quotient $G(L)/I$ as the (set of k -valued points of) the affine flag variety for G , an ind-scheme over k . Then $X_x(b)$ is a locally closed subset and is a k -scheme (with the reduced scheme structure) which is locally of finite type over k . In general, $X_x(b)$ will have infinitely many irreducible components.

The definition of affine Deligne-Lusztig varieties is obviously analogous to the definition of usual Deligne-Lusztig varieties [1], which live in the finite-dimensional flag variety G/B . There are however a number of important differences. In the finite-dimensional case, the parameter b is irrelevant, and one therefore usually sets $b = 1$. This is a consequence of Lang’s theorem. Moreover, the Deligne-Lusztig variety associated to an element w of the finite Weyl group W is smooth and equi-dimensional of dimension $\ell(w)$, the length of w . On the other hand, for many pairs (b, x) the affine Deligne-Lusztig variety $X_x(b)$ is in fact empty; it is a difficult problem to decide which affine Deligne-Lusztig varieties are non-empty and to determine their dimensions. This is the question we focus on in the sequel.

We mention that there is an obvious variant where I is replaced by the “maximal compact” group $G(\mathfrak{o})$. This case is in fact much better understood than the Iwahori case; for instance there is a relatively simple criterion determining which affine Deligne-Lusztig varieties are non-empty, and a formula for their dimension. This formula was conjectured by Rapoport [3]; in [2] the conjecture was reduced to the so-called superbasic case, and this case was treated by Viehmann [5].

Apart from being interesting in their own right, affine Deligne-Lusztig varieties are important for studying the reduction of Shimura varieties in positive characteristic. For details, see [2], in particular section 5.10.

As an example, let us discuss the case of $G = SL_2$, $b = 1$. Let C be an alcove in the Bruhat-Tits tree. The Frobenius automorphism preserves the distance of the alcove to the rational building (i. e. the building over \mathbb{F}_q , the fixed point set of σ). If C lies in the rational building, then $\sigma C = C$, so the Weyl distance we get is the identity element in W_a . If the distance of C to the rational building is $d > 0$, then the distance of σC to the rational building is d , as well, and it is easy to see that as a consequence (because alcoves outside the rational building must be moved by σ) the distance from C to σC is $2d - 1$. This implies that their Weyl distance has odd length. There are two elements of any given odd length in W_a . However it is not hard to see that both these elements can be obtained as the Weyl distance of C and σC for some C . So the result in this case is that $X_x(1)$ is non-empty if and only if $x = \text{id}$ or $\ell(x)$ is odd. In the general case, it is much harder to describe the result in such a succinct way.

Nevertheless, using results about σ -conjugacy classes in unipotent groups, one can prove the following criterion which reduces the question of non-emptiness to a question about orbit intersections in the group, or the affine flag variety. Denote by $U \subset B$ the unipotent radical. For an element w of the finite Weyl group W , write ${}^wU := wUw^{-1}$.

Proposition 2. *Let $b = \epsilon^\nu$ be a translation element. Then $X_x(b) \neq \emptyset$ if and only if there exists $w \in W$ such that*

$${}^wU(L)\epsilon^{w\nu}I \cap IxI \neq \emptyset.$$

This criterion can be reformulated as a purely combinatorial method to determine non-emptiness of affine Deligne-Lusztig varieties by analyzing the images of end points of galleries in the building of a fixed type, under retractions “from infinity” with respect to finite Weyl chambers. With some more effort, one can also determine the dimensions of non-empty affine Deligne-Lusztig varieties in this way. See [2]. Furthermore, the criterion can be generalized to the case of general b ; the group U then has to be replaced by a group of the form $(I \cap M)N$, where P is a parabolic subgroup with Levi decomposition $P = MN$. This generalization will be discussed in a forthcoming paper. The criterion can be used to produce examples (with a computer program); see [2].

We have the following conjecture, which is an extended version of the conjecture stated in [4]. We consider two maps from W_a to W . The map η_1 is just the projection from W_a to W . To describe the second map, we identify W with the set of Weyl chambers. We define $\eta_2(x) = w$, where w is the unique element in W such that $w^{-1}x\mathbf{a}$ is contained in the dominant chamber, where \mathbf{a} is the base alcove corresponding to I . We say that $x \in W_a$ lies in the shrunken Weyl chambers, if $U_\alpha \cap xI \neq U_\alpha \cap I$ for all finite roots α (where U_α denotes the corresponding root subgroup). Furthermore, let S denote the set of (finite) simple reflections.

Conjecture 3. *a) Let $x \in W_a$ be an element of the shrunken Weyl chambers. Then $X_x(1) \neq \emptyset$ if and only if*

$$\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in W \setminus \bigcup_{T \not\subseteq S} W_T,$$

and in this case

$$\dim X_x(1) = \frac{1}{2} (\ell(x) + \ell(\eta_2(x)^{-1}\eta_1(x)\eta_2(x)))$$

b) If $[b]$ is an arbitrary σ -conjugacy class, then there exists $n_0 \in \mathbb{Z}_{\geq 0}$, such that for all $x \in W_a$ of length $\ell(x) \geq n_0$, we have

$$X_x(b) \neq \emptyset \iff X_x(1) \neq \emptyset.$$

There is a large amount of evidence for the conjecture, provided by a computer program which relies on the combinatorial criterion stated above. For groups of type A_2 , C_2 part a) of the conjecture was proved by Reuman [4]. We can prove one direction of part a) of the conjecture in general. In fact, for this direction

the assumption that x lies in the shrunken Weyl chambers is not necessary. The converse direction is not true for all x , however.

Theorem 4. *Let $x \in W_a$, and write $x = \epsilon^\lambda w$, $w \in W$. Assume that $x \notin W$ and that $\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in \bigcup_{T \subsetneq S} W_T$. Then $X_x(1) = \emptyset$.*

Note that if x is contained in the finite Weyl group W , then clearly $X_x(1) \neq \emptyset$ (it contains the corresponding finite Deligne-Lusztig variety).

In our proof of the theorem, the key point is that we exhibit a criterion which implies that for certain $x \in W_a$ the map

$$I \times I_M x I_M \longrightarrow IxI, \quad (i, m) \mapsto i^{-1}m\sigma(i),$$

is surjective. Here M is a Levi subgroup of G which contains x , and $I_M = I \cap M$. For $x = \text{id}$, $M = A$, this just says that I is a single σ -conjugacy class, which is well known. It is clear that if x has this property, then there are strong restrictions on $X_x(b)$ being non-empty, and a careful analysis of the situation yields the theorem. Details will be published in a forthcoming paper.

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Lattices from involutions of Kac-Moody groups

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(joint work with Bernhard Mühlherr)

Let (W, S) be a Coxeter system with root system Φ , infinite W , and finite S . Let Λ be a centre-free group with a locally finite root group datum $\{U_\alpha\}_{\alpha \in \Phi}$, i.e., the root subgroups U_α are finite. The group Λ contains a twin BN -pair. The completion $\overline{\Lambda}_+$ of Λ with respect to the topology of uniform convergence on compact sets of the building G/B_+ is a totally disconnected locally compact group $\overline{\Lambda}_+$.

A *Phan involution* θ of a group Λ with a twin BN -pair B_+ , B_- , N is an automorphism of Λ such that (i) $\theta^2 = \text{id}$, (ii) $B_+^\theta = B_-$, and (iii) θ centralises the Weyl group N/T . It induces an involution, also called *Phan involution* and also denoted by θ , of the corresponding twin building $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ which interchanges both halves isometrically and maps a chamber to an opposite one. For a Phan involution θ and a chamber $c \in \mathcal{C}_+$ define $\delta^\theta(c) := \delta_*(c, c^\theta)$.

Note that $(\delta^\theta(c))^2 = 1$. A Phan involution is called *distance-transitive* if for each element $r \in \text{Inv}^\theta(W) := \{\omega \in W \mid \text{there exists a chamber } c \text{ such that } \delta^\theta(c) = \omega\}$ the centraliser Γ of θ in Λ acts transitively on the set $\mathcal{C}_r^\theta := \{c \in \mathcal{C}_+ \mid \delta^\theta(c) = r\}$.

Theorem 1. *Let Λ be a group with a non-spherical locally finite root group datum $\{U_\alpha\}_{\alpha \in \Phi}$, let $q = \min_{\alpha \in \Phi} \{|U_\alpha|\}$, and let θ be a distance-transitive Phan involution of Λ . Then the centraliser Γ of θ in Λ is a lattice in the group $\overline{\Lambda}_+$ if the series $\sum_{w \in W} \frac{1}{q^{l(w)}}$ converges.*

Proof. The group $\overline{\Lambda}_+$ is locally compact. The centraliser Γ of θ is a discrete subgroup of $\overline{\Lambda}_+$, because $\Gamma \cap U_+ = \{1\}$. It remains to establish the finiteness of the volume of $\Gamma \backslash \overline{\Lambda}_+$. By distance-transitivity the orbits of Γ on \mathcal{B}_+ are parametrised by the elements of the set $\text{Inv}^\theta(W)$. For each $r \in \text{Inv}^\theta(W)$ fix a chamber $c_r \in \mathcal{C}_r^\theta$. By Serre's criterion it suffices to prove that the series $\sum_{r \in \text{Inv}^\theta(W)} \frac{1}{|\text{Stab}_\Gamma(c_r)|}$ converges. For this we use the following result.

Lemma 2. *Let r be an involution and $w \in W$ such that $l(wrw^{-1}) = l(r) - 2l(w)$.*

- (1) *Let $c \in \mathcal{C}_r^\theta$ and let d be a chamber at distance w from c . Then $\delta^\theta(d) = wrw^{-1}$.*
- (2) *Let $d \in \mathcal{C}_{wrw^{-1}}^\theta$. Then there exists a unique chamber c with distance w^{-1} from d such that $\delta^\theta(c) = r$.*

To continue with the proof of Theorem 1 fix $r \in \text{Inv}^\theta(W)$. Then there exists a spherical subset $J \subseteq S$ and an element $w_1 \in W$ such that $r = w_1 w_J w_1^{-1}$ with $l(w) = 2l(w_1) + l(w_J)$ where w_J denotes the longest element in W_J . There exist at least $q^{l(w_1)}$ chambers at distance w_1 from c . By distance-transitivity and Lemma 2 the stabiliser $\text{Stab}_\Gamma(c_r)$ acts transitively on the set of these chambers. Therefore $|\text{Stab}_\Gamma(c_r)| \geq q^{l(w_1)}$. Hence the series $\sum_{r \in \text{Inv}^\theta(W)} \frac{1}{|\text{Stab}_\Gamma(c_r)|}$ is dominated by the series

$$\sum_{J \subseteq S} \sum_{\text{spherical } w \in W} \frac{1}{q^{l(w)}}.$$

Since S is finite, there are only finitely many spherical subsets J of S , so that the latter series converges if and only if $\sum_{w \in W} \frac{1}{q^{l(w)}}$ converges. \square

Remark 3. Let G be Kac-Moody group defined over \mathbb{F}_{q^2} and let θ be the Chevalley involution twisted by the field automorphism of order two. Since θ is semi-linear, for each panel P of the twin building \mathcal{B} of G we have $\{x \in P \mid \text{proj}_P \theta(x) = x\} \neq P$, where proj_P denotes the coprojection in \mathcal{B} onto the spherical residue P , so that distance-transitivity of θ follows from the transitivity of the action of $\text{Stab}_G(\theta)$ on the ‘‘big cell’’ $\mathcal{C}_{1_W}^\theta$. The latter is implied by the fact that over a finite field any two semilinear Phan involutions are conjugate.

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Lattices in locally algebraic representations of $\mathrm{GL}_{d+1}(F)$

ELMAR GROSSE-KLÖNNE

Let F be a local non-Archimedean field with ring of integers \mathcal{O}_F , uniformizer $\pi \in \mathcal{O}_F$ and residue field k_F . Fix a positive integer $d \geq 1$ and let X denote the Bruhat-Tits building of the group $\mathrm{PGL}_{d+1}(\mathcal{O}_F)$. Given a j -simplex σ of X , a set M of vertices of X is called *f-stable with respect to σ* if there is a lattice chain

$$\pi_F \mathcal{L}_j \subset L_M \subset L_0 \subset L_1 \subset \dots \subset L_j$$

(each of these inclusions being strict) of full rank \mathcal{O}_F -lattices in F^{d+1} such that the vertices of σ are represented by the collection $\{F^\times L_0, F^\times L_1, \dots, F^\times L_j\}$ and such that the elements of M are represented by all $F^\times L$ with $\pi_F \mathcal{L}_j \subsetneq L \subset L_M$.

Theorem 1 (Local acyclicity criterion). *Let \mathcal{V} be a (homological) coefficient system on X , let $1 \leq j \leq d$ and suppose that for each $j-1$ -simplex η and each f-stable (w.r.t. η) set of vertices M the sequence*

$$\bigoplus_{y, y' \in M} \mathcal{V}(\eta \cup \{y, y'\}) \longrightarrow \bigoplus_{y \in M} \mathcal{V}(\eta \cup \{y\}) \longrightarrow \mathcal{V}(\eta)$$

(where in the first term the direct sum is taken over all pairs $\{y, y'\}$ of elements of M such that $\eta \cup \{y, y'\}$ is a $(j+1)$ -simplex) is exact. Then $H_j(X, \mathcal{V}) = 0$.

Now assume $\mathrm{char}(F) = 0$. Let V_1 be a finite length smooth admissible representation of $G = \mathrm{GL}_{d+1}(F)$ on an E -vector space, for a finite field extension E of F , and suppose V_1 is generated by its invariants under a congruence subgroup in G . Let V_2 be an F -rational (algebraic) representation of G . We are asking for G -stable \mathcal{O}_E -free lattices in $V_1 \otimes_F V_2$ (by definition, $V_1 \otimes_F V_2$ is a locally algebraic representation of G).

Let \mathcal{V} denote the G -equivariant coefficient system on X which to a vertex σ assigns $V^{U_\sigma} \otimes_F V_2$, where $U_\sigma \subset G$ is a certain pro- p -subgroup fixing σ . By a theorem of Schneider and Stuhler, $H_0(X, \mathcal{V}) \cong V_1 \otimes V_2$. Let $\mathcal{L} \subset \mathcal{V}$ denote a G -equivariant coefficient system such that each $\mathcal{L}(\sigma)$ is an \mathcal{O}_E -lattice in $\mathcal{V}(\sigma)$. Applying theorem 1 to \mathcal{V}/\mathcal{L} (and $j = 0$) we get a "local" (on X) criterion for $H_0(X, \mathcal{L})$ being a lattice (as searched for) in $H_0(X, \mathcal{V}) \cong V_1 \otimes V_2$.

Cases in which this has been verified are (the cases (a) and (b) as well as the general strategy are due to M.F.Vigneras):

- (a) $d = 1$, V_1 the twisted Steinberg representation, V_2 arbitrary.
- (b) $d = 1$, V_1 a tamely ramified principal series representation (satisfying a certain condition on its exponents), V_2 trivial.
- (c) $d = 2$, V_1 the twisted Steinberg representation, V_2 with weights small compared to the cardinality of k .

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Kostant convexity for affine buildings

PETRA HITZELBERGER

In [3] Kostant proves a convexity theorem for semisimple Lie groups. We prove an analog of his result for arbitrary thick simplicial affine buildings and give an application for groups with affine BN -pairs.

We first introduce some notation. Let X be a simplicial affine building, A an apartment in X . Denote by $B = \{\alpha_i, i \in I\}$ the base of the associated irreducible root system R , by W^a the affine and by W the spherical Weyl-group. Weyl-chambers (sometimes called sectors) S in A are translates of the metric closure of a connected component of $A \setminus \{H_{i,0}, i \in I\}$ under W^a . The parallel class S^∞ of a Weyl-chamber S is a chamber in the spherical building $\Delta = \partial X$ at infinity.

We fix an apartment A in X and a special vertex 0 in A .

Retractions. *There are two natural retractions from X onto A :*

- (1) *for each chamber $c \in X$ the canonical retraction $r_{A,c}$ centered at $c \in X$.*
- (2) *the retraction $\rho_{A,S}$ centered at $S^\infty \in \Delta$ for each Weyl-chamber S in A . This is the unique retraction ρ such that for each apartment B containing a sub-Weyl-chamber of S the restriction $\rho|_B$ is an isomorphism fixing $A \cap B$ pointwise.*

We abbreviate $r := r_{A,c_f}$ and $\rho := \rho_{A,-C_f}$, where we denote by c_f the fundamental chamber in A and by $-C_f$ the opposite of the fundamental Weyl-chamber.

Denote by $\bar{\omega}_i$ the fundamental coweights. The sets $\bar{H}_{i,k} := \{x \in A \mid \langle x^\vee, \bar{\omega}_i \rangle = k\}$ are called *dual hyperplanes*. A *dual convex set* is a finite intersection of dual half-apartments $(\bar{H}_{i,k})^\pm$. The *dual convex hull* $\text{conv}^*(Y)$ of a set Y is the smallest dual convex set containing Y . Our result then reads as follows

Theorem. *Let X be a thick affine building, A an apartment in X . Let r and ρ be defined as above and let y be a special vertex in A . Then*

$$\rho(r^{-1}(W.y)) = \text{conv}^*(W.y) \cap \{ \text{vertices } x \in A \text{ of type } \text{type}(y) \}.$$

Extending the maps ρ and r to galleries in X that start at 0 , the proof of the theorem can be reduced to the following two facts:

Proposition 1. *All vertices in $\text{conv}^*(W.y)$ of type $\text{type}(y)$ are endpoints of positively folded galleries of fixed type t , where t is the type of a minimal gallery connecting the origin 0 with y . And all such endpoints are contained in this set.*

Proposition 2. *Every positively folded gallery of Proposition 1 has a preimage under ρ which is a minimal gallery starting at 0 .*

In [2] Gaussent and Littelmann study irreducible highest weight representations of semisimple algebraic or compact Lie groups. They establish a character formula in terms of positively folded galleries. Using their result one can prove Proposition 1. Note that Proposition 1 is just a statement about affine Coxeter complexes. One therefore is completely independent of the building in which the apartment originally "lived in".

AN APPLICATION

Let G be a group acting on an affine building X by automorphisms. Assume that G has an affine BN -pair and assume that B splits as $B = UT$. Let K be the stabilizer of 0 in G . Then G has an Iwasawa decomposition

$$G = BK = UTK$$

where U is transitive on the set of all apartments containing subsectors of $-\mathcal{C}_f$ and T is the group of translations in A . Special vertices in A can therefore be identified with cosets of K in G . The origin 0 then corresponds to K . Note that an arbitrary point in X can be described as utK , $u \in U, t \in T$. One can prove that the retraction ρ is, in this context, given by $utK \mapsto tK$. The set $r^{-1}(W.y)$ is the same as the K -orbit of y for all $y = tK \in A$. The following is a direct consequence of the theorem stated above.

Corollary.

$$\rho(KtK) = \text{conv}^*(tK)$$

or, equivalently,

$$Ut'K \cap KtK \Leftrightarrow t'K \in \text{conv}^*(tK),$$

since $\rho^{-1}(t'K) = Ut'K$.

The proof of " \Rightarrow " is well known and can be found in [1]. For partial results of " \Leftarrow " and related questions see for example [2, 4, 5, 6] and [7].

OPEN QUESTIONS

We are interested in the following questions.

- Is there another, purely geometric, proof of Proposition 1, that does not rely on representation theory?
- If yes, is there a generalization of the theorem to nondiscrete (or even Λ -) affine buildings?
- Let c be a chamber in A . How can one describe the set $\rho(r^{-1}(W.c))$?

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MV-cycles, galleries and spherical Hecke algebras

PETER LITTELMANN

(joint work with Stéphane Gaussent, Christoph Schwer)

The original aim of the work was to give a geometrization of the path model a representation [13] of a complex semisimple algebraic group G . The idea was to connect the combinatorics of crystals [12] via the path model with the work of Mirković and Vilonen [15] on the intersection cohomology of Schubert varieties in the affine Grassmannian \mathcal{G} of its Langlands dual group \check{G} . The work of Mirković and Vilonen can be viewed as geometrization of the classical Satake isomorphism.

It turned out that, at least in the classical case, i.e. semisimple algebraic groups, that the language of galleries in an affine building is the appropriate tool to achieve the goal. It is one of the surprising facts that, at about the same time, Kapovich, Leeb and Millson [8, 9, 10] investigated geodesic triangles in symmetric spaces and buildings and recovered the (part of) the path model of a representation. These two very different approaches meet in applications determining structure constants for the spherical Hecke algebra [8, 18].

Recall that the affine Grassmannian \mathcal{G} (over the complex numbers) is the quotient $\mathcal{G} = \check{G}(\mathbb{C}((t)))/\check{G}(\mathbb{C}[[t]])$. As a $\check{G}(\mathbb{C}[[t]])$ -variety, \mathcal{G} decomposes [3] into the disjoint union of orbits $\mathcal{G}_\lambda = \check{G}(\mathbb{C}[[t]]) \cdot \lambda$, where λ runs over all dominant characters of G (= co-characters of \check{G}).

The closure $X_\lambda = \overline{\mathcal{G}_\lambda}$ of such an orbit is a finite dimensional projective variety (in terms of Kac–Moody groups, it is a Schubert variety). The intersection cohomology of this variety is closely connected to the irreducible representation $V(\lambda)$ of G of highest weight λ . Lusztig [14] has shown that the Poincaré series of the stalks of the intersection cohomology sheaf in a point $x \in \mathcal{G}_\mu$, $\mu \preceq \lambda$, coincides with a q -version of the weight multiplicity of μ in $V(\lambda)$. Mirković and Vilonen construct in [15] a canonical basis of $\mathbb{H}^\bullet(X_\lambda)$, represented by certain cycles called MV-cycles in the following. These MV-cycles can be viewed as an explicit basis of the irreducible G -representation $V(\lambda)$ of highest weight λ .

In our combinatorial setting, the language of paths is replaced by the language of galleries in an apartment, and LS-paths are replaced by LS-galleries. The translation between the two settings is rather straightforward.

Consider a Demazure–Hansen–Bott–Samelson desingularization $\hat{\Sigma}(\lambda)$ of X_λ , for simplicity assume that λ is regular. Fixing such a desingularization is equivalent to fix a minimal gallery γ_λ joining the origin and λ . The set of all galleries of

this type in the affine Tits building (associated to \check{G}) has a natural structure of a projective algebraic variety: by [2], the points of $\hat{\Sigma}(\lambda)$ can be identified with the galleries of type γ_λ in this affine Tits building, and the T -fixed points in $\hat{\Sigma}(\lambda)$ correspond in this language exactly to galleries of type γ_λ in the apartment fixed by the choice of T .

The homology of $\hat{\Sigma}(\lambda)$ has a basis given by Białyński–Birula cells, which are indexed by the T -fixed points in $\hat{\Sigma}(\lambda)$. The connection with galleries is obtained as follows: We show that the retraction from $-\infty$ of the building onto the apartment (fixed by the choice of T) induces on the level of galleries a map from $\hat{\Sigma}(\lambda)$ onto the set of galleries of type γ_λ , such that the fibres are precisely the Białyński–Birula cells.

We determine those galleries of type γ_λ in the fixed apartment γ such that the associated cell has a non-empty intersection S_γ with \mathcal{G}_λ (identified with an open subset of $\hat{\Sigma}(\lambda)$), these are exactly the positively folded galleries. In this context the folding operators e_α, f_α associated to a simple root play an important rôle.

We show that the closure $\overline{S_\gamma} \subset X_\lambda$ is an MV-cycle if and only if γ is a LS-gallery. The galleries can also be used to derive lot of information about the cycles (dimension, affine open subsets of the form $\mathbb{C}^a \times (\mathbb{C}^*)^b, \dots$). Summarizing: We identify the Demazure–Hansen–Bott–Samelson desingularization $\hat{\Sigma}(\lambda)$ with the set of all galleries of type γ_λ in the affine building, starting in 0. We denote by $\Gamma(\gamma_\lambda)$ the set of all galleries in the fixed apartment which start in 0 and which are of type γ_λ . Let $\Gamma^+(\gamma_\lambda) \subset \Gamma(\gamma_\lambda)$ the subset of positively folded galleries and let $\Gamma_{LS}(\gamma_\lambda)$ be the subset of LS-galleries. We refer to the galleries in the fixed apartment also as *combinatorial galleries*, the *dimension* of a positively folded combinatorial gallery is the number of load-bearing walls for the gallery, i.e., roughly speaking, the number walls which are either folding hyperplanes for the gallery or which are crossed by the gallery in positive direction.

Theorem. [4] a) *The set of LS-galleries $\Gamma_{LS}(\gamma_\lambda)$ in $\Gamma(\gamma_\lambda)$ is stable under the folding operators. Let $B(\gamma_\lambda)$ be the directed colored graph having as vertices the set of LS-galleries $\Gamma_{LS}(\gamma_\lambda)$, and put an arrow $\delta \xrightarrow{\alpha} \delta'$ with color α between two galleries if $f_\alpha(\delta) = \delta'$. Then this graph is connected, and it is isomorphic to the crystal graph of the irreducible representation $V(\lambda)$ of G^\vee of highest weight λ . Let $e(\delta)$ be the endpoint of a gallery, then*

$$\text{Char } V(\lambda) = \sum_{\delta \in \Gamma_{LS}(\gamma_\lambda)} \exp(e(\delta)).$$

b) *Let $\hat{\mathcal{G}}_\lambda \subset \hat{\Sigma}(\lambda)$ be the pre-image of the orbit $\mathcal{G}_\lambda \subset X_\lambda$ with respect to the desingularization map $\hat{\Sigma}(\lambda) \rightarrow X_\lambda$, in the language of galleries this are exactly the minimal galleries of type γ_λ . The retraction of the affine building induces a map $r_{\gamma_\lambda} : \hat{\mathcal{G}}_\lambda \rightarrow \Gamma^+(\gamma_\lambda)$ onto the set of all positively folded galleries of type γ_λ in the fixed apartment. For such a gallery δ , the fibre $r_{\gamma_\lambda}^{-1}(\delta)$ is naturally equipped with the structure of an irreducible quasi-affine variety, it is the intersection of a Białyński–Birula cell of $\hat{\Sigma}(\gamma_\lambda)$ with $\hat{\mathcal{G}}_\lambda$. The dimension of the fibre is equal to the*

combinatorially defined dimension $\dim \delta$, and $r_{\gamma_\lambda}^{-1}(\delta)$ admits a finite decomposition into a union of subvarieties, each being a product of \mathbb{C} 's and \mathbb{C}^* 's. In particular, the fibre admits a canonical open and dense subvariety isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$, where a and b can be determined from combinatorial properties of δ .

If ν is a Weyl group conjugate of λ , then it follows that the fibre is an affine space, this has also been shown by Ngo and Polo (using a different approach) in [16]. To recover the MV-cycles, let p be the bijection $p : \hat{\mathcal{G}}_\lambda \rightarrow \mathcal{G}_\lambda$, and for $\delta \in \Gamma^+(\gamma_\lambda)$ write $Z(\delta)$ for the closure of $p(r_{\gamma_\lambda}^{-1}(\delta))$ in X_λ . Denote by X_λ^μ the union of the $Z(\delta)$ for all $\delta \in \Gamma^+(\gamma_\lambda)$ having μ as target ($\forall \mu \prec \lambda$):

$$Z(\delta) = \overline{p(r_{\gamma_\lambda}^{-1}(\delta))} \subset X_\lambda, \quad X_\lambda^\mu = \bigcup_{\delta \in \Gamma^+(\gamma_\lambda), e(\delta)=\mu} Z(\delta).$$

In group theoretic terms, we have $X_\lambda^\mu = \overline{U^-(\mathcal{K})\mu \cap \mathcal{G}_\lambda}$, where $U^- \subset B^-$ is the unipotent radical. The special rôle played by the LS-galleries is that the corresponding $Z(\delta)$ are precisely the irreducible components of X_λ^μ .

Theorem.[4] *The irreducible components of X_λ^μ are given by the $Z(\delta)$ for δ a LS-gallery, i.e. $X_\lambda^\mu = \bigcup_\delta Z(\delta)$, where δ runs only over all LS-galleries in $\Gamma^+(\gamma_\lambda)$ having as target $e(\delta) = \mu$. These irreducible components are precisely the MV-cycles.*

Most of the construction also makes sense over the field $\mathcal{K} = \mathbb{F}_q((t))$ instead of $\mathcal{K} = \mathbb{C}((t))$. One can naturally associate [18] to each positively folded gallery $\delta \in \Gamma^+(\gamma_\lambda)$ a polynomial $L_\delta(q)$, roughly speaking the polynomial is a product of powers of q (for a positive crossings of a walls) and $(q-1)$'s (for positive foldings). For a weight μ let $L_{\lambda,\mu}(q)$ be the sum of the polynomials $L_\delta(q)$ for all galleries $\delta \in \Gamma^+(\gamma_\lambda)$ ending in μ . Geometrically the polynomial $L_{\lambda,\mu}(q)$ describes the number of points in $U^-(\mathcal{K})\mu \cap \mathcal{G}_\lambda$ over the finite field \mathbb{F}_q [1, 7, 18]. In the following we consider q as a variable.

Let Λ be the weight lattice of G and let W be its Weyl group, we use the Satake isomorphism to identify $\mathbb{Z}[q^{\pm \frac{1}{2}}][\Lambda]^W$ with the spherical Hecke algebra with equal parameters associated to the root system of G . Under this isomorphism, Hall-Littlewood polynomials correspond (up to some factor) to the Macdonald basis and the monomial symmetric functions correspond to the monomial basis of the spherical Hecke algebra. A third basis is given by the Schur polynomials, the characters of the complex irreducible representations of the complex algebraic group G . Using this language, the character formula in the theorem can be reformulated as follows: the coefficients appearing in the expansion of Schur polynomials in terms of monomial symmetric functions can be calculated using the gallery model of the representation. The q -version of this statement was given by Schwer:

Theorem.[18] *The coefficients appearing in the expansion of Hall-Littlewood polynomials in terms of monomial symmetric functions can be calculated using the gallery model of the representation. More precisely, the coefficient of the monomial*

basis element m_μ , μ a dominant weight, in the extension of the Hall-Littlewood polynomial $p_\lambda(q^{-1})$, is the polynomial $L_{\lambda,\mu}$, up to a scaling factor.

In particular we see: $L_{\lambda,\mu} \neq 0$ if and only if $\mu \leq \nu$. Specializing q at some prime power we get $L_{\lambda,\mu} > 0$ for all $\mu \leq \lambda$. This was shown by Rapoport [17] for the case of spherical Hecke algebras of a reductive group over a local field.

To decompose a product of Schur polynomials into the sum of Schur polynomials amounts to describe how the tensor product of two irreducible complex representations of the complex semisimple group G decomposes into irreducible components. The fact that the set of LS-galleries of a fixed type has the structure of a crystal can be reformulated as follows [4]:

The structure constants appearing in the decomposition of the product of two Schur polynomials into the sum of Schur polynomials can be calculated using the gallery model of the representations. More precisely, let s_λ, s_μ be the Schur polynomials associated to the dominant weights λ, μ , then the multiplicity, with which the Schur polynomial s_ν occurs in the expansion of the product $s_\lambda \cdot s_\mu$ equals the number of LS-galleries δ of type γ_λ such that δ ends in $\nu - \mu$, and the gallery δ , shifted by the weight μ , is completely contained in the dominant Weyl chamber. Such a gallery is called a μ -dominant gallery.

Schwer gives in [18] a q -analogue of this formula, he associates to a μ -dominant gallery δ a polynomial $C_\delta(q)$, which is similarly defined as the polynomial $L_\delta(q)$.

Theorem.[18] *For dominant weights λ, μ, ν let*

$$p_\lambda(q^{-1})p_\mu(q^{-1}) = \sum_{\nu} q^{-\langle \rho, \mu - \lambda + \nu \rangle} C_{\lambda,\mu}^{\nu} p_{\nu}(q^{-1})$$

be an expansion of a product of two Hall-Littlewood polynomials. Then the coefficients $C_{\lambda,\mu}^{\nu}$ can be calculated using the gallery model of the representation. Up to scaling factors, the coefficient $C_{\lambda,\mu}^{\nu}$ is the sum of the $C_\delta(q)$, where the sum runs over all μ -dominant galleries ending in $\nu - \mu$.

As a consequence one can show that $C_{\lambda,\mu}^{\nu} \neq 0$ if the corresponding coefficient for the tensor product decomposition is different from zero. Specializing q to a prime power one gets that $C_{\lambda,\mu}^{\nu} > 0$ if the tensor product decomposition coefficient is > 0 . For equal parameters this is proven in [11] by Kapovich and Millson, and also by Haines [6] by geometric arguments using the affine Grassmanian.

For further applications, for example commutation formulas in the extended affine Hecke algebra, see [18].

The path model mentioned at the beginning exists for irreducible highest weight representations of arbitrary Kac-Moody algebras, together with corresponding combinatorial character and tensor product decomposition formulas [13]. A first step in generalizing the point of view of the above to this much more general setting has been done in [5]. They give the definition of a kind of building I for a symmetrizable Kac-Moody group over a field K endowed with a discrete valuation and with a residue field containing \mathbb{C} . Due to some bad properties, they call this I a hovel, and not a building. Nevertheless, I has some good properties, for example

the existence of retractions with center a sector-germ. This enables them to generalize many results proved in the semi-simple case. In particular, if $K = \mathbb{C}((t))$, the geodesic segments in I , with a given special vertex as end point and a good image under some retraction, are parametrized by a Zariski open subset of \mathbb{C}^N . This dimension N is maximal when the retraction is a LS-path, and should correspond to some generalization of a MV-cycle.

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Divergence in lattices in semisimple Lie groups and graphs of groups

SHAHAR MOZES

(joint work with Cornelia Druţu, Mark Sapir)

Divergence functions of a metric space estimate the length of a path connecting two points A, B at distance $\leq n$ avoiding a large enough ball around a third point C . We characterize groups with non-linear divergence functions as groups having cut-points in their asymptotic cones. That property is weaker than the property of having Morse (rank 1) quasi-geodesics. Using our characterization of Morse quasi-geodesics, we give a new proof of the theorem of Farb-Kaimanovich-Masur that states that mapping class groups cannot contain copies of irreducible lattices in semi-simple Lie groups of higher ranks. It also gives a generalization of the result of Birman-Lubotzky-McCarthy about solvable subgroups of mapping class groups not covered by the Tits alternative of Ivanov and McCarthy.

We show that any group acting acylindrically on a simplicial tree or a locally compact hyperbolic graph always has “many” periodic Morse quasi-geodesics (i.e. Morse elements), so its divergence functions are never linear. We also show that the same result holds in many cases when the hyperbolic graph satisfies Bowditch’s properties that are weaker than local compactness. This gives a new proof of Behrstock’s result that every pseudo-Anosov element in a mapping class group is Morse.

On the other hand, we conjecture that lattices in semi-simple Lie groups of higher rank always have linear divergence. We prove it in the case when the \mathbb{Q} -rank is 1 and when the lattice is $\mathrm{SL}_n(\mathcal{O}_S)$ where $n \geq 3$, S is a finite set of valuations of a number field K including all infinite valuations, and \mathcal{O}_S is the corresponding ring of S -integers.

The fundamental group of period domains over finite fields

SASCHA ORLIK

Period domains over finite fields are open subvarieties of flag varieties defined by a semi-stability condition. They were introduced and discussed by M. Rapoport in [R]. In this talk we discussed the determination of their fundamental groups which answers a question raised in loc.cit.

Let G be a reductive group over a finite field k . We fix an algebraic closure \bar{k} of k and denote by $\Gamma = \Gamma_k$ the corresponding absolute Galois group of k . Let \mathcal{N} be a conjugacy class of \mathbb{Q} -1-PS of $G_{\bar{k}}$. We denote by $E = E(G, \mathcal{N})$ the reflex field of the pair (G, \mathcal{N}) . This is a finite extension of k which is characterized by its Galois group $\Gamma_E = \{\sigma \in \Gamma \mid \nu \in \mathcal{N} \implies \nu^\sigma \in \mathcal{N}\}$. Every \mathbb{Q} -1-PS ν induces via Tannaka formalism a \mathbb{Q} -filtration \mathcal{F}_ν over \bar{k} of the forgetful fibre functor $\omega^G : \mathrm{Rep}_k(G) \rightarrow \mathrm{Vec}_k$ from the category of algebraic G -representations over k into the category of k -vector spaces. Two \mathbb{Q} -1-PS are called par-equivalent if they define the same \mathbb{Q} -filtration. There exists a smooth projective variety $\mathcal{F}(G, \mathcal{N})$ over E

with

$$\mathcal{F}(G, \mathcal{N})(\bar{k}) = \{ \nu \in \mathcal{N} \text{ modulo par-equivalence} \}.$$

The variety is a generalized flag variety for G_E . More precisely, by a lemma of Kottwitz [K], there is a \mathbb{Q} -1-PS $\nu \in \mathcal{N}$ which is defined over $E = E(G, \mathcal{N})$. Thus we may write $\mathcal{F}(G, \mathcal{N}) = G_E/P$, where $P = P(\nu)$ is the parabolic subgroup of G_E attached to ν . Further, after fixing a maximal torus and a Borel subgroup in G , we may suppose that ν is contained in the closure $\bar{C}_{\mathbb{Q}}$ of the corresponding rational Weyl chamber $C_{\mathbb{Q}}$.

A point $x \in \mathcal{F}(G, \mathcal{N})(\bar{k})$ is called semi-stable if the induced filtration $\mathcal{F}_x(\text{Lie}(G)_{\bar{k}})$ on the adjoint representation $\text{Lie}(G)_{\bar{k}} = \text{Lie}(G) \otimes_k \bar{k}$ of G is semi-stable. The latter means that for all k -subspaces U of $\text{Lie}(G)$, the following inequality is satisfied

$$\frac{1}{\dim U} \left(\sum_y y \cdot \dim \text{gr}_{\mathcal{F}|U_{\bar{k}}}^y(U_{\bar{k}}) \right) \leq \frac{1}{\dim \text{Lie}(G)} \left(\sum_y y \cdot \dim \text{gr}_{\mathcal{F}}^y(\text{Lie}(G)_{\bar{k}}) \right).$$

In [DOR] it is shown that there is an open subvariety $\mathcal{F}(G, \mathcal{N})^{ss}$ of $\mathcal{F}(G, \mathcal{N})$ parametrizing all semi-stable points, i.e. $\mathcal{F}(G, \mathcal{N})(\bar{k})^{ss} = \mathcal{F}(G, \mathcal{N})^{ss}(\bar{k})$. This open subvariety $\mathcal{F}(G, \mathcal{N})^{ss}$ is called the *period domain* to (G, \mathcal{N}) .

The most prominent example of a period domain is the Drinfeld upper half plane $\Omega_k^{(\ell+1)} = \mathbb{P}_k^\ell \setminus \cup \mathbb{P}(H)$ where H runs through all k -rational hyperplanes of $k^{\ell+1}$. This space corresponds to the pair (G, \mathcal{N}) where $G = \text{PGL}_{\ell+1, k}$ and $\nu = (x_1, x_2, \dots, x_\ell) \in \bar{C}_{\mathbb{Q}}$ with $x_1 > x_2$ and $x_1 + \ell \cdot x_2 = 0$. Here we identify $\bar{C}_{\mathbb{Q}}$ as usual with $(\mathbb{Q}^{\ell+1})_+^0 = \{(x_1, \dots, x_{\ell+1}) \in \mathbb{Q}^{\ell+1} \mid \sum_i x_i = 0, x_1 \geq x_2 \geq \dots \geq x_{\ell+1}\}$. The period domain $\Omega_k^{(\ell+1)}$ is isomorphic to a Deligne-Lusztig variety and admits therefore interesting étale coverings, cf. [DL].

Period domains only depend on their adjoint data, cf. [DOR]. More precisely, let G_{ad} be the adjoint group of G , and let \mathcal{N}_{ad} be the induced conjugacy class of \mathbb{Q} -1-PS of G_{ad} . Then

$$\mathcal{F}(G, \mathcal{N})(\bar{k})^{ss} \cong \mathcal{F}(G_{\text{ad}}, \mathcal{N}_{\text{ad}})(\bar{k})^{ss}.$$

Also if G splits into a product $G = \prod_i G_i$, the corresponding period domain splits into products, as well. Thus for formulating our main result, we may assume that G is k -simple adjoint. Hence there is an absolutely simple adjoint group G' over a finite extension k' of k with $G = \text{Res}_{k'/k} G'$. In this case $\mathcal{N} = (\mathcal{N}_1, \dots, \mathcal{N}_t)$ is given by a tuple of conjugacy classes \mathcal{N}_j of \mathbb{Q} -1-PS of G'_k , where $t = |k' : k|$. Thus ν is given by a tuple of \mathbb{Q} -1-PS $\nu = (\nu_1, \dots, \nu_t)$.

Our main result is the following. Let ℓ be the (absolute) rank of G' . We denote by π_1 the functor which associates to a variety its geometric fundamental group.

Theorem 1. *Let G be absolutely simple adjoint over k . Then $\pi_1(\mathcal{F}(G, \mathcal{N})^{ss}) = \{1\}$ unless $G = \text{PGL}_{\ell+1, k}$ and $\nu = (x_1 \geq x_2 \geq \dots \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})_+^0$ with $x_2 < 0$ or $x_\ell > 0$. In the latter case we have $\pi_1(\mathcal{F}(G, \mathcal{N})^{ss}) = \pi_1(\Omega_k^{(\ell+1)})$.*

More generally, let $G = \text{Res}_{k'/k} G'$ be k -simple adjoint. Then $\pi_1(\mathcal{F}(G, \mathcal{N})^{ss}) = \{1\}$ unless $G' = \text{PGL}_{\ell+1, k'}$ and such that the following two conditions are satisfied.

Write $\nu_i = (x_1^{[i]} \geq x_2^{[i]} \geq \dots \geq x_{\ell+1}^{[i]}) \in (\mathbb{Q}^{\ell+1})_+^0$, $i = 1, \dots, t$. Then there is a unique $1 \leq j \leq t$, such that

(i) $x_2^{[j]} < 0$ or $x_\ell^{[j]} > 0$.

(ii) $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$ if $x_2^{[j]} < 0$ resp. $\sum_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}$ if $x_\ell^{[j]} > 0$.

In the latter case we have $\pi_1(\mathcal{F}(G, \mathcal{N})^{ss}) = \pi_1(\Omega_{k'}^{(\ell+1)})$.

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Convexity in Buildings

CHRISTOPHER PARKER

(joint work with Katrin Tent)

Assume that Δ is a spherical building and that Ω is a convex subcomplex of Δ . Then we have the following conjecture due to Tits’

Conjecture 1 (Tits’ Centre Conjecture). *Suppose that Δ is a spherical building and Ω is a convex subcomplex of Δ . Then either*

- 1 *For each residue R in Ω there exists a residue R' in Ω with R' opposite R in Δ .*
- 2 *The stabiliser of Ω in $\text{Aut}(\Delta)$ fixes some residue in Ω (the centre).*

In the case when the building Δ is of classical type we have the following theorem.

Theorem 2 (Mühlherr and Tits [2]). *The Centre Conjecture holds for all buildings with no factors of type F_4 , E_6 , E_7 or E_8 .*

The proof of this theorem uses the point line realizations of the classical buildings and relies on a theorem due to Serre which we now state.

Theorem 3 (Serre [3]). *The following are equivalent.*

- 1 *Each residue R in Ω has an opposite in Ω . (So option (1) of the conjecture holds.)*
- 2 *Each residue R of maximal type (each vertex) in Ω has an opposite in Ω .*

Extending the results of Mühlherr and Tits to the exceptional buildings is complicated by the fact that the point line realizations of the buildings are more complicated. We are however able to prove the following theorem.

Theorem 4. *The Centre Conjecture holds for all buildings of type F_4 .*

The proof of Theorem 4 involves detailed calculations in metasymplectic spaces. We heavily use the point line description of such spaces provided by Cohen [1]. Since the Coxeter diagram of type F_4 has so many symmetries, the most detailed calculations are only needed when the convex set contains points. When attempting to generalize our work to buildings of type E_6 , E_7 and E_8 , the complexity of the problem grows substantially. For this reason we have proved a generalization of the above theorem of Serre. It reads as follows as is proved using the properties of projections.

Theorem 5. *Assume that Δ is a spherical irreducible building and that Ω is a convex chamber subcomplex of Δ . If every maximal residue of some fixed type in Ω has an opposite in Ω , then every residue in Ω has an opposite in Ω .*

By regarding the buildings of type E_6 , E_7 and E_8 as parapolar spaces, we expect that we will be able to use Theorem 5 to prove the Centre Conjecture for convex chamber subcomplexes. We are currently working towards this goal.

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Automorphism groups of buildings: flat rank and contraction groups

BERTRAND RÉMY

(joint work with Udo Baumgartner, Jacqui Ramagge, George Willis)

We illustrated the fact that building theory provides non-classical examples for the general theory of totally disconnected locally compact groups (TDLC groups for short). The latter theory was initiated about fifteen years ago by G. Willis [8]. One theoretic motivation is the study of the group of connected components of an arbitrary locally compact group. The theory of TDLC groups was recently used to study almost normal subgroups of arithmetic groups via suitable completions (Y. Shalom and G. Willis). For general notions on TDLC groups, we refer to the extended summary written by G. Willis in the same proceedings.

More specifically in this talk, we explained that when the TDLC group under consideration is a closed subgroup of the full automorphism group $\text{Aut}(X)$ of a locally finite thick building X , some topological group invariants are computable and can be related to the geometry of the involved building. (Here, by building we mean the non-positively curved metric realization of the abstract combinatorial structure.) Of course one has to assume transitivity conditions to be able to say something.

Recall now that Kac-Moody groups were defined by J. Tits [7] by means of a presentation; we restrict ourselves to the case where the ground field is finite. Their nice combinatorics implies that such a group, say Λ , acts on the product $X_- \times X_+$ of two locally finite buildings. The Λ -action on $X = X_{\pm}$ is no longer discrete and we can produce interesting TDLC groups by considering the closure of the latter action in $\text{Aut}(X)$, say G , endowed with the compact open topology. The class of these complete Kac-Moody groups seems to be of interest since it is at good distance from the classical examples given by semisimple algebraic groups over local fields like $\text{SL}_n(\mathbf{Q}_p)$. Indeed, such a group G is a locally pro- p group which is topologically simple when X has irreducible Weyl group, but it is not linear whenever the Weyl group has a non-affine non-spherical factor [6] – it contains in fact finitely generated simple groups.

We focused on two topics: the flat rank and the contraction groups of TDLC groups obtained as highly transitive automorphism groups of buildings.

Flat rank of automorphism groups of buildings.— Roughly, the flat rank of a TDLC group G is an integral invariant merely relying on the topological group structure of G : the space $\mathcal{B}(G)$ of all compact open subgroups and its natural metric are the basic tools to define it. A flat subgroup H of G is a closed subgroup for which there exists a single compact open subgroup minimizing all the displacement functions of the conjugations attached to H . Then such a group has a natural quotient that is free abelian. The *flat rank* of G , denoted $\text{flat-rk}(G)$, is the supremum of the \mathbf{Z} -ranks of the so-obtained free abelian groups.

Here is a summary of the main result of [2], which applies in fact to a more general combinatorial class of automorphism groups of buildings.

Theorem 1. *Let Λ be a Kac-Moody group over a finite field, with Weyl group W . We denote by G the geometric completion of Λ , i.e. the closure of the Λ -action in the full automorphism group of the positive building of Λ . Then we have:*

$$\text{alg-rk}(W) \leq \text{flat-rk}(G) \leq \text{geom-rk}(W),$$

where $\text{alg-rk}(W)$ is the maximal \mathbf{Z} -rank of abelian subgroups of W and $\text{geom-rk}(W)$ is the maximal dimension of the embedded Euclidean spaces in the non-positively curved geometric realization of the Coxeter complex of W .

This result has to be combined with a more recent result of P.-E. Caprace and F. Haglund showing that the equality $\text{alg-rk}(W) = \text{geom-rk}(W)$ actually holds for any Coxeter group [4]. The consequence is that in the above conditions, we have:

$$\text{alg-rk}(W) = \text{flat-rk}(G) = \text{geom-rk}(W).$$

Moreover for any semisimple algebraic group \mathbf{G} over a local field F , we have the equality: $\text{flat-rk}(\mathbf{G}(F)) = \text{rk}_F(\mathbf{G})$, where $\text{rk}_F(\mathbf{G})$ is the dimension of a maximal F -split torus in \mathbf{G} . An easy trick with (Kac-Moody-)Dynkin diagrams then allows to produce a sequence of topologically simple non-linear TDLC groups of arbitrary

large flat rank (while $\{\mathrm{SL}_n(\mathbf{Q}_p)\}_{n \geq 2}$ is a sequence of abstractly simple linear groups of arbitrary large flat rank).

Contraction groups.— The second topic considered in the talk is related to dynamics in topological groups. Let h be a non topologically periodic element in a topological group G . The *contraction group* attached to h is by definition the group U_h defined by:

$$U_h = \{g \in G : \lim_{n \rightarrow \infty} h^n g h^{-n} = 1\}.$$

If X is a locally compact CAT(0)-space, G equals $\mathrm{Isom}(X)$ and h is a hyperbolic isometry with repelling point $-\xi$, then the bigger group

$$P_h = \{g \in G : \text{the sequence } \{h^n g h^{-n}\}_{n \geq 0} \text{ is relatively compact}\}$$

contains U_h as a normal subgroup and is always closed: it is equal to the stabilizer $\mathrm{Stab}_G(-\xi)$. We refer to [5, Chap. III] for a quite general presentation of contraction subgroups of locally compact groups and their usefulness, and to [3] for twelve equivalent characterisations of closedness of contraction groups.

Here is a summary of the main result of [1] (as before, it applies to a more general combinatorial class of automorphism groups of buildings).

Theorem 2. *Let Λ be a Kac-Moody group over a finite field, with infinite irreducible Weyl group W . We denote by G the geometric completion of Λ and we pick a hyperbolic translation $h \in G$. Then the contraction group U_h is not closed whenever W is non-affine, that is whenever the associated buildings are not Euclidean.*

This is another illustration of the fact that sometimes the group-theoretic behaviours of automorphism groups of Euclidean and of non-Euclidean buildings are in sharp contrast. In the classical case of Bruhat-Tits buildings, i.e., when X is the Bruhat-Tits building of a semisimple algebraic group \mathbf{G} over a local field F and when $G = \mathbf{G}(F)$, then the group P_h is the rational points of a parabolic F -subgroup and U_h is the rational points of its unipotent radical; this can be used to prove strong approximation in positive characteristics (G. Prasad).

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Hyperbolic rank of affine buildings

VIKTOR SCHROEDER

In this lecture we discussed certain asymptotic invariants of metric spaces, in particular the hyperbolic rank, compare [BS1], [BS2]. Let X be a geodesic metric space. The hyperbolic rank of X , denoted by $\text{rank}_h X$, is defined as the maximal topological dimension of $\partial_\infty Y$ among all Gromov hyperbolic spaces Y which can be quasi-isometrically embedded into X (the notion introduced by M. Gromov). The hyperbolic rank is a quasi-isometry invariant of X .

It is not difficult to show that for a hyperbolic space X we have $\text{rank}_h = \dim \partial_\infty X$. In particular $\text{rank}_h \mathbb{H}^n = n - 1$ for the hyperbolic space \mathbb{H}^n and $\text{rank}_h T = 0$ for a tree T .

Let T be now the binary tree. In [DS2] we showed the surprising fact that $\text{rank}_h(T \times T) = 1$ by constructing a quasiisometric embedding of the hyperbolic plane \mathbb{H}^2 into $T \times T$. A farreaching generalization was proved in [BDS]. This result shows in particular that one cannot (in general) expect a logarithmic law of the form $\text{rank}_h(X_1 \times X_2) = \text{rank}_h X_1 + \text{rank}_h X_2$ for product spaces.

The case of products of trees leads naturally to the open question of determine the hyperbolic rank of other affine buildings. The result for product of trees suggest that an affine building of rank n has hyperbolic rank $n - 1$.

As a model problem for this question one should consider the following: Let X be a given affine building of type \tilde{A}_2 . Is it possible to construct a quasi-isometric embedding $\mathbb{H}^2 \rightarrow X$?

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Some free constructions of polygons, twin trees and CAT(1)-spaces

KATRIN TENT

Spherical buildings of rank 2 are generalized polygons and are most easily defined as follows: A generalized n -gon Γ is a bipartite graph of diameter n , girth $2n$ and valencies at least 3.

We start with the following theorem:

Theorem 1. ([5]) *Let Γ be a generalized n -gon where $n \leq 6$. Assume that $G \leq \text{Aut}(\Gamma)$ acts so that for each path $\gamma = (x_0, x_1, x_2)$ the stabilizer of all neighbours of x_1 acts transitively on the set of all $2n$ -cycles containing γ . Then G is an algebraic or classical group of relative rank 2.*

In contrast to this result, by free amalgamation one obtains generalized n -gons Γ for all $n \geq 3$ such that for each path $\gamma = (x_0, x_1, x_2)$ and any finite set $A \subseteq \Gamma_1(x_1)$ the stabilizer of $\gamma \cup A$ acts transitively on the set of all $2n$ -cycles containing γ .

Now fix a non-principal ultrafilter μ and for each $n \geq 3$ a generalized n -gon Γ_n as above. Put $G_n = \text{Aut}(\Gamma_n)$. Then the ultraproduct $(\Gamma^*, G^*) = \Pi_\mu(\Gamma_n, G_n)$ is a forest, i.e. a collection of trees with a natural distance and codistance function inherited from the distance function on the Γ_n . This distance function takes values in the ultrapower $\Pi_\mu \mathbb{N}$. The connected components of a set of opposite vertices are twin or multiple trees in the sense of Ronan and Tits [3]. The subgroup of G^* leaving each of the trees invariant has a twin (or multiple) BN-pair. In contrast to the rigid examples constructed by Ronan and Tits these groups acts highly transitively on the neighbours of any vertex.

After scaling each metric space Γ_n , which has diameter n , by the factor $2\pi/n$, each Γ_n with the induced metric has diameter 2π , and the ultraproduct $\bar{\Gamma}^*$ of these scaled polygons is a 1-round space; i.e. any two points of $\bar{\Gamma}^*$ are contained in an isometrically embedded 1-sphere. The group G^* acts faithfully on $\bar{\Gamma}^*$ and one can show that it acts transitively on the set of pointed 1-spheres contained in $\bar{\Gamma}^*$.

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Existence and covolumes of lattices for Davis complexes

ANNE THOMAS

Let G be a locally compact topological group with Haar measure μ . A *lattice* in G is a discrete subgroup $\Gamma < G$ such that the covolume $\mu(\Gamma \backslash G)$ is finite. A lattice Γ is *uniform* if $\Gamma \backslash G$ is compact. Given G , some basic questions are:

- (1) Does G admit a (uniform or nonuniform) lattice? How may we construct lattices in G ?
- (2) What is the set of covolumes of lattices in G ? In particular, is this set discrete?

In the classical setting, where G is a Lie group, the answers to these questions are well-understood. For example, let G be a noncompact simple real Lie group, such as $PSL(n, \mathbb{R})$. Borel [B] proved that G admits uniform and nonuniform lattices, and Margulis' famous arithmeticity theorem [M] states that in higher rank, all lattices in G are arithmetic. For covolumes, Wang [W] proved that (for G not equal to $PSL(2, \mathbb{R})$ or $PSL(2, \mathbb{C})$) the set of covolumes of lattices in G is discrete. If G is an algebraic group over a nonarchimedean local field, such as $PSL(n, \mathbb{Q}_p)$, the strong finiteness result of Borel–Prasad [BP] implies that the set of covolumes of lattices in G is also discrete.

We consider lattices in $G = \text{Aut}(X)$, where X is a locally finite polyhedral complex (such as a locally finite building). A normalisation of Haar measure due to Serre [S] implies that $\Gamma < G$ discrete is a lattice exactly when the series $\sum |\Gamma_x|^{-1}$ converges, where the sum is over vertices x in a fundamental domain for Γ .

Our main result is:

Theorem 1. *Let X be the Davis complex for a Coxeter group W . Let $G = \text{Aut}(X)$. Suppose that*

- G is nondiscrete, and
- the finite nerve L of W is “symmetric enough” (see below).

Then G admits a nonuniform lattice Γ , and an infinite family of uniform lattices (Γ_n) such that $\mu(\Gamma_n \backslash G) \rightarrow \mu(\Gamma \backslash G)$. Hence the set of covolumes of lattices in G is nondiscrete.

A simple combinatorial condition due to Haglund–Paulin [HP] characterises those Davis complexes X such that $G = \text{Aut}(X)$ is nondiscrete. The condition that L be “symmetric enough” requires that certain vertices of L be in the same orbit of the group of label-preserving automorphism of L , and that finite spherical subgroups of W involving these vertices of L have index 2 subgroups which are also finite Coxeter groups. If $\dim(X) = 2$, it is enough for L to be a graph such that $\text{Aut}(L)$ is transitive on vertices, and that all finite $m_{ij} = m$ with m even. This class of complexes includes (subdivisions of) certain two-dimensional buildings.

The proof of Theorem 1 uses Haefliger’s theory of complexes of groups (see [BH]) to construct the lattices Γ and Γ_n as fundamental groups of complexes of groups. Covering theory for complexes of groups is used to show that Γ and each of the Γ_n are contained in $\text{Aut}(X)$.

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Bruhat-Tits theory from Berkovich’s point of view: realizations and compactifications of buildings

AMAURY THUILLIER

(joint work with Bertrand Rémy, Annette Werner)

Let G be a semisimple linear algebraic group over a non-Archimedean local field k and let $\mathcal{B}(G, k)$ denote the affine building of $G(k)$. Several compactifications of this locally compact topological space have been defined: beside the well-known construction of Borel-Serre ([3]), one should mention Landvogt’s ”polyhedral compactification” ([5]) and its generalizations recently investigated by Werner ([7], [9], see also [8]).

While working out new foundations for analytic geometry over a non-Archimedean field ([1]), Vladimir Berkovich sketched a way to think about the building $\mathcal{B}(G, k)$ in relationship with the analytic spaces associated to G and its flag varieties. Our aim is to give a detailed exposition of Berkovich’s ideas and in particular to show that they provide a nice geometric framework to handle Landvogt’s and Werner’s compactifications. Roughly speaking, the core of this approach is to rely heavily on the functoriality of the building with respect to field extensions and to work with *arbitrary* non-Archimedean fields extending k in order to force the group to act transitively on the building.

Our starting point is the following result, which holds for any connected semisimple linear group G over a field k equipped with a non-trivial and complete non-Archimedean absolute value. Since no hypothesis is made on k , the existence of $\mathcal{B}(G, k)$ is assumed here.

THEOREM 1 — *For any point x in $\mathcal{B}(G, k)$, there exists a unique affinoid subgroup G_x of G^{an} satisfying the following condition:*

$$G_x(K) = \text{Fix}_{G(K)}(x)$$

for any non-Archimedean extension K of k .

In this statement, G^{an} denotes the analytic space associated to G . The underlying set consists of all multiplicative seminorms on the coordinate algebra $k[G]$ of $G = \text{Spec}(k[G])$ extending the absolute value of k , and G^{an} is a group object in the category of analytic spaces (just like group schemes are group objects in the category of schemes). Affinoid groups are analogue of affine group schemes in this context. In particular, G_x is a compact analytic subgroup of G^{an} , and thus this theorem should be seen as an appropriate generalization of the correspondence between vertices of $\mathcal{B}(G, k)$ and maximal bounded subgroups of $G(k)$ (from which a group theoretic description of Landvogt's compactification restricted to vertices has been worked out in [4]).

If G is split and if x is a special vertex, Bruhat-Tits theory attaches to x a Chevalley group scheme over the integers of k and G_x is then the corresponding affinoid group. The general case is reduced to the previous one by faithfully flat descent.

It turns out that each G_x has a unique maximal point $\vartheta(x)$ determining it completely. Hence we obtain a map $\vartheta : \mathcal{B}(G, k) \rightarrow G^{an}$, which is continuous, injective and equivariant with respect to the $G(k)$ -action by conjugation on the target. In particular, it follows from Berkovich's approach that the building of $G(k)$ can be realized as a set of multiplicative seminorms on the algebra $k[G]$.

The scheme $\text{Par}(G)$ of parabolic subgroups of G is a projective variety over k whose points in an extension K of k are the parabolic subgroups of $G \otimes_k K$. Notice that $\text{Par}(G, k)$ is never empty since it contains the trivial parabolic subgroup G and let $\Pi(k) = \text{Par}(G, k)/G(k)$ denote the set of $G(k)$ -conjugacy classes ("types") of parabolic subgroups of G . For a given type $t \in \Pi(k)$, we let $\text{Par}_t(G)$ denote the corresponding connected component of $\text{Par}(G)$. Moreover, $\mathcal{B}(G, k)$ is the product of the buildings of all almost simple factors of G , and we will let $\mathcal{B}_t(G, k)$ denote the quotient of $\mathcal{B}(G, k)$ obtained by removing each almost simple factor of G on which t is trivial.

Proceeding as above, one defines for each type $t \in \Pi(k)$ a canonical continuous and $G(k)$ -equivariant map $\vartheta_t : \mathcal{B}(G, k) \rightarrow \text{Par}_t(G)^{an}$ which factors through an injection $\mathcal{B}_t(G, k) \hookrightarrow \text{Par}_t(G)^{an}$. The target is a compact topological space and, when k is a local non-Archimedean field, one gets a compactification $\overline{\mathcal{B}}_t(G, k)$ of $\mathcal{B}_t(G, k)$ by taking its closure in $\text{Par}_t(G)^{an}$. In order to describe this compactification, one has to notice that each parabolic subgroup $P \in \text{Par}(G, k)$ defines a closed subvariety $X_t(P)$ of $\text{Par}_t(G)$, namely the image of the canonical morphism $\text{Par}_t(P) \rightarrow \text{Par}_t(G)$. We call P *t-relevant* if P is maximal among all $Q \in \text{Par}(G, k)$ defining the same subvariety, and it is readily seen that each parabolic subgroup is contained in a unique *t-relevant* one. For example: if $G = PGL(V)$ and if δ is the type of flags (L, V) where L is a line, then δ -relevant parabolics are those corresponding to flags (W, V) , where W is a strict linear subspace of V , and $\overline{\mathcal{B}}_\delta(PGL(V), k)$ is the compactification described in [7].

THEOREM 2 — 1. We have the following stratification

$$\overline{\mathcal{B}}_t(G, k) = \bigcup_{P \text{ } t\text{-relevant}} \mathcal{B}_t(P/\text{Rad}(P), k).$$

2. The stabilizer in $G(k)$ of a point x lying in the stratum $\mathcal{B}_t(P/\text{Rad}(P), k)$ is the semi-direct product $G_x(k) = R(k) \rtimes (P/R)_x(k)$, where R is the biggest subgroup of P acting trivially on $X_t(P)$ and $(P/R)_x(k)$ is the parahoric subgroup of $(P/R)(k)$ fixing x in $\mathcal{B}_t(P/\text{Rad}(P), k) = \mathcal{B}(P/R, k)$.

3. (Mixed Bruhat decomposition) Any two points x, y in $\overline{\mathcal{B}}_t(G, k)$ lie in a common compactified apartment $\overline{A}_t(S, k)$ and

$$G(k) = G_x(k)N(k)G_y(k),$$

where N is the normalizer of the maximal split torus S .

If we let t vary in $\Pi(k)$, the natural morphisms between the flag varieties $\text{Par}_t(G)$ induce maps between these compactifications which merely collapse some factors of the buildings appearing in the decomposition above. Moreover, the compactification of a given apartment is easily described in root theoretic terms.

We finally describe the representation theoretic viewpoint on these compactifications. Let \emptyset denote the type of Borel subgroups of G . According to the highest weight theory, an absolutely irreducible representation $\rho : G \rightarrow PGL(V)$ induces a morphism $\underline{\rho} : \text{Par}_{\emptyset}(G) \rightarrow \mathbb{P}(V)$ which factors through a closed immersion $\text{Par}_{t(\rho)}(G) \hookrightarrow \mathbb{P}(V)$ for some $t(\rho) \in \Pi(k)$. Set $X(V, k) = \mathcal{B}(PGL(V), k)$ and $\overline{X}(V, k) = \overline{\mathcal{B}}_{\delta}(PGL(V), k)$; the injection $\overline{X}(V) \rightarrow \mathbb{P}(V)^{an}$ has a canonical section τ (see [2] and [7]).

THEOREM 3 — Let $\rho : G \rightarrow PGL(V)$ be an absolutely irreducible representation of G .

1. The compactification of $\mathcal{B}(G, k)$ defined in [9] is $\overline{\mathcal{B}}_{t(\rho)}(G, k)$.
2. The map $\tau \circ \underline{\rho} \circ \vartheta_{\emptyset} : \mathcal{B}(G, k) \rightarrow \overline{X}(V, k)$ induces a homeomorphism between $\overline{\mathcal{B}}_{t(\rho)}(G, k)$ and the closure of its image in $\overline{X}(V, k)$.
3. Any Landvogt map $\mathcal{B}(G, k) \rightarrow \mathcal{B}(PGL(V), k)$ (see [6]) induces a homeomorphism between $\overline{\mathcal{B}}_{t(\rho)}(G, k)$ and the closure of its image in $\overline{X}(V, k)$.

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2-Dimensional affine apartment systems

HENDRIK VAN MALDEGHEM

(joint work with Koen Struyve)

In a series of rather long papers [1, 4, 5, 6, 8], I (jointly with Guy Hanssens in the first quoted paper) investigate in detail two classes of rank 3 affine buildings (namely, those of type \tilde{A}_2 and those of type \tilde{C}_2) and characterizes the corresponding spherical buildings at infinity (which are projective planes and generalized quadrangles, respectively). This leads to many new examples of such affine buildings, explicitly defined and with knowledge of the automorphism groups. Originally, the characterization made use of the notion of a *discrete valuation* on the algebraic structures that coordinatize projective planes and generalized quadrangles, but in later papers [7, 9], the valuation was defined directly on the geometry. The hope was that with such a direct definition, the case of type \tilde{G}_2 , which was the only remaining case in rank 3, would become treatable with much less effort. One of the reasons why it did *not* is that, although the paper [7] provides the exact condition for a generalized hexagon with valuation, the lack of symmetry in the formulae prevents us from deducing a *general* formulae independent of the type, and hence from (1) further generalization to non-discrete valuations, and (2) composing a type-free proof.

In the present research, we start such a type-free approach, which must eventually lead to a characterization of all irreducible 2-dimensional affine apartment systems. In fact, we present half of such a characterization here. More in particular, any irreducible 2-dimensional affine apartment system gives rise to a generalized polygon with a specific valuation, by which we mean, with the terminology of [7], an explicitly defined weight sequence. One of the crucial observations to achieve this is to slightly modify, or re-scale, the valuation as defined from a rank 3 affine building as defined in [9]. Roughly speaking, the valuation between two elements as defined in [9] counted the graph theoretic distance between two vertices in the simplicial complex related to the affine building. The purpose was to end up with a natural number. But taking the Euclidean distance instead puts much more symmetry into the picture, and at the same time provides a closed formula for the weight sequences. Also the non-discrete case can clearly be included in a natural way.

Our first main result is the following theorem.

Theorem 1. *Let Λ be an irreducible 2-dimensional symmetric affine apartment system (as defined in [2] or [3]). Let Λ^∞ be the irreducible rank 2 spherical building at infinity, and assume that it is the building of a generalized n -gon, $n \geq 3$. Let v be an arbitrary vertex of Λ , let C_1^∞, C_2^∞ be two adjacent chambers of Λ^∞ and let C_1, C_2 be the two respective corresponding sectors emerging from v . Then $C_1 \cap C_2$ contains a common ray of C_1 and C_2 ; let r_1, r_2 be the respective other rays of C_1, C_2 . Then we denote by $v(C_1^\infty, C_2^\infty)$ the length of $r_1 \cap r_2$ (is a real number or infinity). Then the following conditions are satisfied.*

- (V1) *In each panel of Λ^∞ there exists a pair of chambers C_1^∞ and C_2^∞ such that $v(C_1^\infty, C_2^\infty) = 0$.*
- (V2) *$v(C_1^\infty, C_2^\infty) = \infty$ if and only if $C_1^\infty = C_2^\infty$.*
- (V3) *$v(C_1^\infty, C_2^\infty) < v(C_2^\infty, C_3^\infty)$ implies $v(C_1^\infty, C_2^\infty) = v(C_1^\infty, C_3^\infty)$, for three pairwise adjacent chambers $C_1^\infty, C_2^\infty, C_3^\infty$ of Λ^∞ .*
- (V4) *For every closed non-stammering gallery $(C_0^\infty, C_1^\infty, \dots, C_{2n-1}^\infty, C_{2n}^\infty = C_0^\infty)$ of length $2n$, one has*

$$\sum_{i=1}^{n-1} \sin(i\pi/n)v(C_{i-1}^\infty, C_{i+1}^\infty) + \sum_{i=n+1}^{2n-1} \sin(i\pi/n)v(C_{i-1}^\infty, C_{i+1}^\infty) = 0.$$

A function v on the pairs of adjacent chambers of the spherical rank 2 building of a generalized n -gon satisfying (V1), (V2), (V3) and (V4) is called a *valuation*. So Λ^∞ is endowed with uncountably many different valuations (one for each vertex v of the affine building Λ). It is not so difficult to see that the family of all these valuations uniquely define Λ (after all, each of them “is” a vertex of the building, but still, one has to put a structure on this family so that we can recognize the apartments). The conjecture, however, is that only one such valuation suffices, not only to reconstruct, but also to define Λ . This conjecture has been proved for values of n up to 6 already.

Theorem 2. *Let Λ^* be the spherical building of a generalized n -gon, with $3 \leq n \leq 6$. Then Λ^* admits a valuation if and only if Λ^* is isomorphic to the building at infinity of a 2-dimensional symmetric affine system of apartments.*

In particular, this proves the long-standing conjecture stated in [7] for affine buildings of type \tilde{G}_2 . In the discrete case, we can now state the following result.

Theorem 3. *Let Λ^* be the spherical building of a generalized n -gon, with $n \geq 3$. Then Λ^* is isomorphic to the building at infinity of an irreducible affine rank 3 building if and only if there is a map u defined on the pairs of adjacent or equal chambers of Λ^* satisfying the following properties.*

- (U1) *In each panel of Λ^* there exists a pair of chambers C_1^* and C_2^* such that $u(C_1^*, C_2^*) = 0$.*
- (U2) *$u(C_1^*, C_2^*) = \infty$ if and only if $C_1^* = C_2^*$.*
- (U3) *$u(C_1^*, C_2^*) < u(C_2^*, C_3^*)$ implies $u(C_1^*, C_2^*) = u(C_1^*, C_3^*)$, for three pairwise adjacent chambers C_1^*, C_2^*, C_3^* of Λ^* .*

(U4) *There exists a sequence $(a_1, a_2, \dots, a_{n-1}; b_{n+1}, b_{n+2}, \dots, b_{2n-1})$ of natural numbers such that for every closed non-stammering gallery $(C_0^*, C_1^*, \dots, C_{2n-1}^*, C_{2n}^* = C_0^*)$ of length $2n$, with $C_0 \cap C_1$ a panel of prescribed and fixed type, one has*

$$\sum_{i=1}^{n-1} a_i u(C_{i-1}^*, C_{i+1}^*) - \sum_{i=n+1}^{2n-1} b_i u(C_{i-1}^*, C_{i+1}^*) = 0.$$

Note that in the discrete case of affine buildings of rank 3, only the values 3, 4, 6 for n turn up. They give rise to the following mutually dual sequences (the duality is caused by the fact that the type of $C_0 \cap C_1$ is prescribed in (U4); taking the other type changes the sequence in the other):

$$n = 3 \quad (a_1, a_2; b_4, b_5) = (1, 1; 1, 1).$$

$$n = 4 \quad (a_1, a_2, a_3; b_5, b_6, b_7) \in \{(1, 1, 1; 1, 1, 1), (1, 2, 1; 1, 2, 1)\}.$$

$$n = 6 \quad (a_1, a_2, a_3, a_4, a_5; b_7, b_8, b_9, b_{10}, b_{11}) \in \\ \{(1, 1, 2, 1, 1; 1, 1, 2, 1, 1), (1, 3, 2, 3, 1; 1, 3, 2, 3, 1)\}.$$

Note that Theorem 3 is slightly stronger than Theorem 3 restricted to $n = 3, 4, 6$. Indeed, we do not require that $n \in \{3, 4, 6\}$; this just follows from the fact that the numbers a_i and b_j are natural numbers; also the exact values of these numbers follow automatically, see [7].

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Locally finite affine buildings

RICHARD M. WEISS

In [3] (which relies on many of the results in [1]), Tits classified (irreducible) affine buildings of dimension at least three in terms of the building at infinity and a valuation of its root datum. Using [4], this result can be extended to affine buildings of dimension two under the assumption that the building at infinity

satisfies the Moufang property. (This “Moufang assumption” holds automatically when the dimension of the affine building is at least three.)

For *locally finite* affine buildings, the algebraic structure which determines the building at infinity is defined over a local field. By invoking various classical (and less classical) results about skew-fields, involutions, anisotropic quadratic forms, etc. over local fields, the various possibilities for the building at infinity can be determined much more precisely than in the general case.

There are three families of locally finite affine buildings of dimension at least two satisfying the Moufang assumption which are defined over purely inseparable extensions of local fields, one of each type \tilde{B}_2 , \tilde{F}_4 and \tilde{G}_2 . It turns out that the remaining locally finite affine buildings of dimension at least two satisfying the Moufang assumption fall naturally into thirty-two families. The buildings in these thirty-two families are precisely the affine buildings associated to absolutely simple algebraic groups defined over a local field. In particular, they correspond to the thirty-two “local indices” of spherical rank at least two in Tits’s famous Corvallis lecture notes [2]. For details, see the forthcoming book [5].

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Totally Disconnected Groups and Buildings

GEORGE WILLIS

(joint work with Udo Baumgartner)

The natural action of the totally disconnected locally compact group $G = SL_n(\mathbb{Q}_p)$ on its affine Tits building, X , relates algebraic features of G to geometric features of X . This talk discusses ideas for extending this relationship between algebraic and geometric features to general totally disconnected locally compact groups.

A starting point for both the algebraic study of totally disconnected groups and of their geometric representations is a theorem of van Dantzig, [5, Theorem II.7.7], that each such group has a base of neighbourhoods of 1 consisting of compact open subgroups. The set of all compact open subgroups of G will be denoted by $\mathcal{B}(G)$. Any two subgroups $V, W \in \mathcal{B}(G)$ are commensurable because $V \cap W$ is an open subgroup of V and W and hence has finite index in both. The set $\mathcal{B}(G)$ is a metric space with the metric d defined by

$$(1) \quad d(V, W) = \log([V : V \cap W][W : V \cap W]).$$

1. ALGEBRAIC FEATURES OF GROUPS

In the case of $SL_n(\mathbb{Q}_p)$, each Cartan subgroup of the group is contained in the stabilizer of exactly one apartment and this establishes a one-to-one correspondence between Cartan subgroups and apartments. Similarly, the Iwahori subgroups of the groups correspond to the fixators of alcoves.

These special subgroups have counterparts in a general totally disconnected group G , which are defined as follows. Let α be an automorphism of G .

- The *scale of α* is the positive integer

$$s(\alpha) := \min \{[\alpha(V) : \alpha(V) \cap V] \mid V \in \mathcal{B}(G)\}.$$

- Any subgroup $V \in \mathcal{B}(G)$ at which the minimum is attained is said to be *minimising for α* .

- A group \mathcal{H} of automorphisms of G is said to be *flat* if there is $V \in \mathcal{B}(G)$ that is minimizing for every $\alpha \in \mathcal{H}$.

Then flat groups correspond to Cartan subgroups of $SL_n(\mathbb{Q}_p)$ and the minimizing subgroups to Iwahoris.

To see how this correspondence works out in more detail, define the *uniscalar subgroup* of the flat group \mathcal{H} to be

$$\mathcal{H}_1 := \{\alpha \in \mathcal{H} \mid s(\alpha) = 1 = s(\alpha^{-1})\} = \{\alpha \in \mathcal{H} \mid \alpha(V) = V\}.$$

There are analogues for \mathcal{H} of root homomorphisms and root groups as follows.

Theorem *Let \mathcal{H} be a finitely generated flat group. Then there is $r \in \mathbb{N}$ such that*

$$\mathcal{H}/\mathcal{H}_1 \cong \mathbb{Z}^r.$$

- (1) *For any V minimizing for \mathcal{H} , there is $q \in \mathbb{N}$ such that*

$$V = V_0 V_1 \cdots V_q,$$

where V_j are closed subgroups with, for every $\alpha \in \mathcal{H}$:

$\alpha(V_0) = V_0$; and either $\alpha(V_j) \leq V_j$ or $\alpha(V_j) \geq V_j$ for every $j \in \{1, \dots, q\}$.

- (2) *Then $\tilde{V}_j := \bigcup_{\alpha \in \mathcal{H}} \alpha(V_j)$ is a closed subgroup of G for every $j \in \{1, \dots, q\}$, and*

- (3) *there are positive integers s_j and homomorphisms $\rho_j : \mathcal{H} \rightarrow \mathbb{Z}$ such that*

$$\Delta_j(\alpha|_{\tilde{V}_j}) = s_j^{\rho_j(\alpha)}, \text{ for all } \alpha \in \mathcal{H} \text{ and } j \in \{1, \dots, q\},$$

where Δ_j denotes the modular function on $\text{Aut}(\tilde{V}_j)$.

The numbers r , q and s_j and the homomorphisms ρ_j depend only on \mathcal{H} .

Geometric ideas are present in the proof of the theorem because V factors, for each $\alpha \in \mathcal{H}$, as a product $V = V_{\alpha+} V_{\alpha-}$, where α expands $V_{\alpha+}$ and shrinks $V_{\alpha-}$. Each such factoring separates \mathcal{H} into ‘half-spaces’ consisting of those automorphisms which expand and shrink $V_{\alpha+}$ and the argument proceeds by successively using elements of \mathcal{H} to refine the factoring of V .

The exponent r in the theorem is defined to be the *rank* of the flat group of automorphisms \mathcal{H} . Here are some flatness criteria.

Theorem Let \mathcal{H} be a group of automorphisms of the totally disconnected group G

- (1) If \mathcal{H} is finitely generated and nilpotent, then it is flat.
- (2) If \mathcal{H} is polycyclic, then it is virtually flat.

Another important structural feature is the *contraction group* of the automorphism α , which is defined to be $U_\alpha := \{x \in G \mid \alpha^n(x) \rightarrow 1 \text{ as } n \rightarrow \infty\}$. It is clear that $U_\alpha \leq V_{\alpha--} := \bigcup_{k \geq 0} \alpha^{-k}(V_{\alpha-})$. That U_α is non-trivial whenever $s(\alpha^{-1}) > 1$ is shown in [1] and the important question of when U_α is closed is also examined in this paper. A metrizable hypothesis required in [1] is removed in [7] and the detailed structure of contraction groups when they are closed described in [6].

2. GEOMETRIC REPRESENTATIONS

It is desirable to have a representation of G on a geometric structure that reflects the relationships between minimizing subgroups and flat groups of inner automorphisms. Although these algebraic notions have geometric interpretations, and although representations may be defined that are canonical up to quasi-isometry, there is no such geometric representation for general groups as yet.

2.1. Geometric Interpretation of Minimizing Subgroups and Flatness.

(1) The compact open subgroup V is minimizing if and only if $\{\alpha^n(V)\}_{n \in \mathbb{Z}}$ is a geodesic in $(\mathcal{B}(G), d)$, i.e., $d(V, \alpha^n(V)) = |n|d(V, \alpha(V))$ for all n .

(2) If $\mathcal{H} \leq \text{Aut}(G)$ is flat and has rank n , then for each $V \in \mathcal{B}(G)$ the orbit $\mathcal{H}.V$ is quasi-isometric to \mathbb{Z}^n . In the reverse direction, if $\mathcal{H}.V$ is bounded, i.e. quasi-isometric to \mathbb{Z}^0 , then there is $V' \in \mathcal{B}(G)$ that is stable under \mathcal{H} , [9, 4], so that \mathcal{H} has rank 0. This is extended in [3] to show that, under the additional hypothesis that balls in $(\mathcal{B}(G), d)$ are finite, if $\mathcal{H}.V$ is quasi-isometric to \mathbb{Z}^n , then \mathcal{H} is virtually flat and of rank n .

2.2. Geometries up to Quasi-Isometry.

The Rough Cayley Graph [8]: For G compactly generated and $V \in \mathcal{B}(G)$ the coset space G/V can be equipped with a locally finite graph structure so that the natural action of G is by graph automorphisms. This graph depends on the choice of V and on the generating set but all rough Cayley graphs are quasi-isometric.

Conjugacy orbits in $\mathcal{B}(G)$: For $V \in \mathcal{B}(G)$, the conjugation action of G on the metric space $(\{xVx^{-1} \mid x \in G\}, d)$, where d is the metric defined in (1), is transitive and by isometries. Different choices of V give spaces that are quasi-isometric.

If $N_G(V) = V$, which is the case when $G = SL_n(\mathbb{Q}_p)$ and V a maximal element of $\mathcal{B}(G)$, these constructions give the same set. In general, $N_G(V)$ need not be a compact subgroup of G and $\mathcal{B}(G)$ might not have maximal elements.

2.3. Directions of Automorphisms.

A canonical geometric space on which G acts is described in [2]. However this space does not generalize the affine Tits building of $SL_n(\mathbb{Q}_p)$, but rather its spherical Bruhat-Tits building at infinity.

The space is defined in terms of the notion of *direction* of those automorphisms of G having scale bigger than 1 which is defined in [2] – an equivalence relation is

defined on sequences $\{\beta^n(V)\}_{n \geq 0} \subset \mathcal{B}(G)$, for $V \in \mathcal{B}(G)$ and $\beta \in \text{Aut}(G)$, and the direction of α is its equivalence class. A pseudo-metric is defined on the set of directions of inner automorphisms and the corresponding completed metric space is $(\overline{\partial G}, \delta)$, the *space of directions for G* .

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 $\text{SL}_n(\mathbb{Z}[t])$ is not FP_{n-1}

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(joint work with Kai-Uwe Bux, Amir Mohammadi)

Little is known about the finiteness properties of $\text{SL}_n(\mathbb{Z}[t])$ for arbitrary n .

In 1959 Nagao proved that if k is a field then $\text{SL}_2(k[t])$ is a free product with amalgamation [Na]. It follows from his description that $\text{SL}_2(\mathbb{Z}[t])$ and its abelianization are not finitely generated.

In 1977 Suslin proved that when $n \geq 3$, $\text{SL}_n(\mathbb{Z}[t])$ is finitely generated by elementary matrices [Su]. It follows that $H_1(\text{SL}_n(\mathbb{Z}[t]), \mathbb{Z})$ is trivial when $n \geq 3$.

More recent, Krstić-McCool proved that $\text{SL}_3(\mathbb{Z}[t])$ is not finitely presented [Kr-Mc].

It's also worth pointing out that since $\text{SL}_n(\mathbb{Z}[t])$ surjects onto $\text{SL}_n(\mathbb{Z})$, that $\text{SL}_n(\mathbb{Z}[t])$ has finite index torsion-free subgroups.

In this talk we provide a generalization of the results of Nagao and Krstić-McCool mentioned above for the groups $\mathbf{SL}_n(\mathbb{Z}[t])$.

Theorem A. *If $n \geq 2$, then $\mathbf{SL}_n(\mathbb{Z}[t])$ is not of type FP_{n-1} .*

Recall that a group Γ is of type FP_m if there exists a projective resolution of \mathbb{Z} as the trivial $\mathbb{Z}\Gamma$ module

$$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where each P_i is a finitely generated $\mathbb{Z}\Gamma$ module.

In particular, the Theorem A implies that there is no $K(\mathbf{SL}_n(\mathbb{Z}[t]), 1)$ with finite $(n-1)$ -skeleton, where $K(G, 1)$ is the Eilenberg-Mac Lane space for G .

Outline of proof. The general outline of the proof is modeled on the proofs in [Bu-Wo 1] and [Bu-Wo 2], though some important modifications have to be made to carry out the proof in this setting.

As in [Bu-Wo 1] and [Bu-Wo 2], our approach is to apply Brown's filtration criterion [Br]. Here we will examine the action of $\mathbf{SL}_n(\mathbb{Z}[t])$ on the locally infinite Euclidean building for $\mathbf{SL}_n(\mathbb{Q}((t^{-1})))$. The infinite groups that arise as cell stabilizers for this action are of type FP_m for all m , which is a technical condition that is needed for our application of Brown's criterion.

We will demonstrate the existence of a family of diagonal matrices that will imply the existence of a "nice" isometrically embedded codimension 1 Euclidean space in the building for $\mathbf{SL}_n(\mathbb{Q}((t^{-1})))$. In [Bu-Wo 1] analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because $\mathbb{Q}((t^{-1}))$ is not locally compact, our treatment is quite a bit more hands on.

We use translates of portions of the codimension 1 Euclidean subspace found above to construct spheres in the Euclidean building for $\mathbf{SL}_n(\mathbb{Q}((t^{-1})))$ (also of codimension 1). These spheres will lie "near" an orbit of $\mathbf{SL}_n(\mathbb{Z}[t])$, but will be nonzero in the homology of cells "not as near" the same $\mathbf{SL}_n(\mathbb{Z}[t])$ orbit. Theorem A then follows from Brown's criterion.

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