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Optimal Control of Coupled Systems of PDE

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Abstract. Current research in the control of PDEs is focused on highly nonlinear coupled systems of partial differential equations that arise from diverse applications in engineering and science. Dealing with associated control problems calls for a careful analysis of such systems, efficient numerical methods for differential equations and powerful techniques of numerical optimization. The program of the conference contained a blend of associated talks. Systems modelling quantum effects, dynamic fluid-structure interaction, the coupling of heat transport or fluid flow with electromagnetic fields and compressible flows were subject of the talks. Main aspects of control theory were state-constrained optimal control, mesh-adaptivity and a posteriori error estimation, feedback control, free material and shape optimization, controllability and observability. The conference tightened the links between applications, numerics, and analysis with some emphasis on the analytic aspects.

Mathematics Subject Classification (2000): 49J20,35Qxx,65K05.

Introduction by the Organisers

The international conference *Optimal Control of Coupled Systems of PDE*, was held March 2nd–March 8th, 2008, organized by K. Kunisch (Karl-Franzens-University Graz), G. Leugering (University of Erlangen-Nürnberg), J. Sprekels (Weierstrass Institute of Applied Analysis and Stochastics Berlin) and Fredi Tröltzsch (Technische Universität Berlin). 44 participants attended this meeting and followed 33 talks on optimal control and related topics.

Mathematically, the control of partial differential equations (PDEs) is concerned with the following type of problems: The solution of a PDE (the state of the system) should be influenced in a desired way by the choice of certain control functions or control parameters (the controls), which may occur in different terms of the differential equation. If the controls are to minimize a certain functional related to the state, then an *optimal control problem* is posed. If the domain underlying the PDE is subject of the control, then a *shape optimization problem* is given. For evolution equations, it can be required to move the solution from a given initial state exactly to a desired final state. This is the question of exact *controllability*.

Optimization and control of partial differential equations continues to be a very active field of research. Scientists working in different fields came together to report on their contributions to the numerical analysis of control problems. It is remarkable that optimal control is a challenge for researchers with backgrounds in related fields such as the theory of systems of nonlinear PDEs, numerical methods for solving them, large scale nonlinear optimization, or the numerical simulation and optimization of complex processes in engineering or medical science.

This diversity was reflected by the conference program. Talks were focused on

- applications of optimal control to the thermistor problem, crystal growth, quantum mechanics or aviation
- state-constrained optimal control problems
- controllability and observability of the Navier-Stokes equations and of systems for fluid-structure interaction; feedback control
- Hamilton-Bellman-Jacobi equations
- models for the interaction of electromagnetic fields, heat transport and fluid flow
- mesh adaptivity, a-posteriori and a-priori error estimates for the solutions of optimal control problems
- the application of numerical techniques such as semismooth Newton methods, multilevel techniques or domain decomposition
- first- and second-order optimality conditions for the optimal controls of nonlinear systems of PDEs arising from different applications
- modal control
- the optimal shape design of electromagnetic systems or thin shells and on free material optimization.

All these issues are currently subject of active research. In extensive and lively discussions, the participants of the workshop produced new mathematical ideas and tightened connections of joint cooperation.

Workshop: Optimal Control of Coupled Systems of PDE

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Abstracts

Recent Advances in the Analysis of Pointwise State-Constrained Elliptic Optimal Control Problems

Eduardo Casas
(joint work with Fredi Tröltzsch)

We consider state-constrained elliptic control problems of the type

$$(\mathrm{P}) \begin{cases} \min J(u) = \displaystyle\int_\Omega (y_u(x) - y_d(x))^2\,dx + \frac{N}{2}\int_\Omega u(x)^2\,dx \\ \text{subject to } \ (y_u, u) \in (C(\bar\Omega) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ \alpha \le u(x) \le \beta \quad \text{for a.e. } \ x \in \Omega, \\ a \le y_u(x) \le b \ \ \forall x \in K, \end{cases},$$

where y_u is the solution of the elliptic boundary value problem

$$\begin{aligned} -\Delta\, y + d(y) &= u + e \quad \text{in } \Omega, \\ y &= 0 \qquad\ \ \text{on } \Gamma \end{aligned}$$

associated with u.

Here, $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ is a bounded Lipschitz domain, d is a monotone increasing C^2-function with locally Lipschitz second-order derivative, $\alpha < \beta$, $a < b$, and $N > 0$ are fixed real numbers, y_d, $e \in L^2(\Omega)$ are given functions, and K is a non-empty compact subset of $\bar\Omega$.

We address several questions that are important for a numerical analysis of this class of problems. In particular, these are the following issues:

- First-order necessary optimality conditions of KKT type
- Second-order sufficient optimality conditions
- Regularity of optimal controls
- Uniqueness of Lagrange multipliers.

The first-order necessary optimality conditions for a locally optimal control $\bar u$ are well known. They include an equation for the adjoint state $\bar\varphi$, regular Borel measures as Lagrange multipliers for the pointwise state constraints, the classical complementarity conditions and the projection formula

$$\bar u(x) = \mathrm{Proj}_{[\alpha,\beta]}\left(-\frac{1}{N}\bar\varphi(x)\right).$$

Second-order sufficient conditions for this class of problems were studied first in [3] with a critical cone that, considering sets of first-order sufficiency as in [4], covered active constraints in a rather implicit way. In the recent paper [1], a more natural form of the critical cone is found that seems to be sharp.

The adjoint state $\bar\varphi$ belongs to the space $W^{1,s}(\Omega)$ for all $s < n/(n-1)$, hence the projection formula above suggests that $\bar u$ belongs to the same space. However, we are able to prove more: It holds that $\bar u \in H^1_0(\Omega)$.

If the optimal state is active only in a finite number of points, then the associated Lagrange multipliers are Dirac measures concentrated in the active points so that the adjoint state becomes singular in these points. At first glance, this indicates that $\bar{u}$ should be singular there as well. However, the contrary holds true: By the projection formula, the control bounds α and β cut off the singularities of $\bar{u}$ so that $\bar{u}$ is Lipschitz in this case; this fact was proven in [2].

Does this property hold also for the problem (P)? Unfortunately, the answer is negative. We have found a counterexample of the type (P) with linear elliptic equation, where the optimal control is not Lipschitz. In this example, the optimal state is active in a sequence of points converging to an active point.

For the convergence of numerical algorithms, the uniqueness of Lagrange multipliers is important. We have the following sufficient condition for uniqueness:

Define, for $\varepsilon > 0$,

$$\Omega_\varepsilon = \{x \in \Omega : \alpha + \varepsilon < \bar{u}(x) < \beta - \varepsilon\}$$

and the active set

$$K_0 = \{x \in K : \ \bar{y}(x) = a \text{ or } \bar{y}(x) = b\}.$$

Assume the existence of some $\varepsilon > 0$ such that

$$T : L^2(\Omega_\varepsilon) \to C(K_0), \text{ defined by } Tv = z_v, \text{ has a dense range},$$

where $z_v \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution to

$$\left\{ \begin{array}{rcll} Az_v + d'(\bar{y})z_v &=& v & \text{in } \Omega \\ z_v &=& 0 & \text{on } \Gamma, \end{array} \right.$$

and v is extended by zero to the whole domain Ω. Then there exists a unique Lagrange multiplier $\mu \in M(K)$ associated with the state constraints.

It can be shown that this regularity property is sufficient for the standard linearized Slater condition.

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Intrinsic Methods in the Theory of Thin and Asymptotic Shells

Michel C. Delfour

Many hypersurfaces ω in $\mathbf{R}^N$ can be viewed as the boundary or a subset of the boundary Γ of an open subset Ω of $\mathbf{R}\,R^N$. In such cases the associated *oriented distance function* b_Ω to the underlying set Ω completely describes the surface ω: its (outward) normal is the gradient ∇b_Ω, its first, second, third, ..., and N-th fundamental forms are $\nabla b_\Omega \otimes \nabla b_\Omega$, its Hessian $D^2 b_\Omega$, $(D^2 b_\Omega)^2$, ... and $(D^2 b_\Omega)^{N-1}$ restricted to the boundary Γ ([10], [15, Chapter 8, § 5]). In addition, a fairly complete intrinsic theory of Sobolev spaces on $C^{1,1}$-surfaces is available in [7].

In the theory of thin shells, the asymptotic model, when it exists, only depends on the choice of the *constitutive law*, the *midsurface*, and the subspace of the space of solutions that properly handles the loading applied to the shell. A central issue is how rough this midsurface can be to still make sense of asymptotic *membrane shell* and *bending equations* without ad hoc mechanical or mathematical assumptions. It is possible for a general $C^{1,1}$-midsurface with or without boundary such as a sphere, a donut, or a closed reservoir. Moreover, it can be done without local maps, local bases, and Christoffel symbols via the purely intrinsic methods developed by Delfour and Zolésio starting in 1992 with [11] and in a number of subsequent papers [12, 13, 14, 4, 5, 6, 8, 9, 2]. In the classical theory of shells (cf. for instance [3]), the *midsurface* ω is defined as the image of a flat smooth bounded connected domain U in $\mathbf{R}^2$ via a $C^{(2)}$-immersion $\varphi : U \to \mathbf{R}^3$. When U is sufficiently smooth and the thickness sufficiently small, the associated *tubular neighborhood* $\mathbb{S}_h(\omega)$ *of thickness* $2h$ is a Lipschizian domain that is identified with a *thin shell of thickness* $2h$ *around* ω. Anicic, LeDret, and Raoult [1] relaxed the classical assumptions by introducing in 2004 a family of surfaces ω that are the image of a connected bounded open Lipschitzian domain U in $\mathbf{R}^2$ by a bi-Lipschitzian mapping φ with the assumption that the normal field defined almost everywhere is globally Lipschizian. Such surfaces are called *K-regular patches* by LeDret [16]. ¿From this, they construct a *tubular neighborhood* $\mathbb{S}_h(\omega)$ of thickness $2h$ around the surface and show that for sufficiently small h the *tubular neighborhood mapping* is bi-Lipschitzian.

We first prove that the surfaces of [1] (or K-regular patches) are $C^{1,1}$-surfaces with a bounded measurable second fundamental form. It was already known that $C^{1,1}$-surfaces have a globally Lipschitzian normal field, but it was not, a priori, clear whether midsurfaces generated in the parametrized set-up of [1] would be strictly rougher than $C^{1,1}$ or not. Moreover, since a K-regular patch does not see the singularities of the underlying bi-Lipschitzian parametrization, the G_1-junction of K-regular patches along a join developed in [16] generates a new K-regular patch that is a $C^{1,1}$ surface and the join is in fact $C^{1,1}$. Proofs are given for an hypersurface in $\mathbf{R}^N$, $N \geq 2$, since they are independent of the dimension. Secondly, we show that such tubular neighborhoods can be completely specified by the *algebraic distance* to ω and that they are generally not Lipschitzian domains in $\mathbf{R}^3$ since their tangential smoothness is not effectively controlled by the assumptions of [1] as illustrated by our Example of a bi-Lipschitzian parametrization of the plane that does not transform a Lipschizian domain into a Lipschizian domain. This means that classical results from three-dimensional linear elasticity over Lipschitzian domains cannot be directily applied to the class of thin shells considered in [1]. Therefore, $C^{1,1}$ is still the currently available minimum smoothness to make sense of asymptotic *membrane shell* and *bending equations*.

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Weak Solutions to a Model for Global Heat Transfer Arising in Crystal Growth from the Melt with Applied Magnetic Fields

Pierre-Étienne Druet

In crystal growth from the melt, the possibility of influencing the motion of the melt and the global temperature distribution in the furnace by means of applied magnetic fields nowadays receives increasing interest. Realistic geometrical situations are described in the paper [6]. We propose a model for the interaction between the melt flow, the applied magnetic field and the heat transfer phenomena. We then present results on the weak solvability of the underlying coupled system of PDE.

The model. We assume that the melt flow is governed by the Boussinesq approximation of the Navier-Stokes equations for a viscous, incompressible, electrically conducting and heat-conducting fluid

$$\rho_1 \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla p + \mathrm{div} v(\eta(\theta)\, D\, v) + f(\theta) + j \times B\,, \quad \text{in }]0, T[\times \Omega_1\,,$$

together with the incompressibility constraint $\mathrm{div} vv = 0$. Here, v and p respectively denote the velocity and the pressure of the fluid, Ω_1 is the set occupied by the melt (the crucible), $D\,v$ represents the tensor of deformations, f is the force responsible for buoyancy, and $j \times B$ is the Lorentz force. ρ_1 denotes a reference density of the melt, and η its viscosity.

The temperature distribution is searched in a domain $\Omega \supset \Omega_1$ (typically the entire furnace). We can model the global heat transfer with the equations

$$\rho\, c_V \left(\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta \right) = \mathrm{div} v(\kappa(\theta)\, \nabla \theta) + \frac{|j|^2}{\mathfrak{s}(\theta)} \quad \text{in }]0, T[\times \Omega\,,$$

where θ denotes the absolute temperature, $\mathfrak{s}$ is the electrical conductivity, and κ the heat conductivity. Note that $v \neq 0$ only in $]0, T[\times \Omega_1$, and that the Joule heating $|j|^2/\mathfrak{s}$ is neglected in the fluid according to the Boussinesq approximation.

The domain Ω is assumed to enclose a connected transparent cavity Ω_0. We account for nonlocal radiation effects for the heat flux at the boundary of this cavity by using the jump condition

$$\left[-\kappa(\theta)\, \frac{\partial \theta}{\partial \vec{n}} \right] = G(\sigma\, \theta^4) \quad \text{on }]0, T[\times \partial\Omega_0\,,$$

with a linear operator G, and the Stefan-Boltzmann constant σ.

We describe the electromagnetic effects by means of the low-frequency approximation of Maxwell's equations with applied current. Of course, the region of extension of the electromagnetic fields $\tilde{\Omega}$ is larger than Ω. Denoting by E the electric field strength, by B the vector of the magnetic induction, by H the magnetic field strength, and by j the current density, we have

$$\mathrm{curl} E + \frac{\partial B}{\partial t} = 0\,, \quad \mathrm{curl} H = j, \quad \text{in }]0, T[\times \tilde{\Omega}\,,$$

supplemented by Ohm's law in the conductors. The magnetic induction B has to be divergence free in $]0,T[\times\tilde{\Omega}$. We assume linear constitutive relations, that is $B = \mu\, H$, and $D = \mathfrak{e}\, E$ in $]0,T[\times\tilde{\Omega}$. In order to model the current source, we assume that the current j_g originating from an applied voltage is given in a part of the electrical conductors $]0,T[\times\tilde{\Omega}_{c_0}$. Thus, Ohm's law and Ampère's law can be rewritten as

$$\mathrm{curl} H = \mathfrak{s}(\theta)\left(-\frac{\partial A}{\partial t} + v \times B\right) + j_g, \quad \text{in }]0,T[\times\tilde{\Omega}_c\,,$$

with $j_g \neq 0$ only in $]0,T[\times\tilde{\Omega}_{c_0}$. The magnetic potential A is related to B by $\mathrm{curl} A = B$. Natural interface condition for the electromagnetic fields are assumed in the interior of the domain at the boundaries between different materials.

Results. We briefly mention the results attained in [4], [3] on the solvability of the heat equation with nonlocal radiation terms and right-hand side L^1, as well as the results of [2] on the higher integrability of the Lorentz force. This motivates the following main results discussed in the talk:

(1) The existence of stationary weak solutions $(v,\, H,\, \theta)$ in the class $W^{1,2}(\Omega_1)\times L^2_{\mathrm{curl}}(\tilde{\Omega}) \times W^{1,p}(\Omega)$, $(p < 3/2)$ can be proved under reasonable assumptions on the data. Uniqueness is obtained, as expected, only for small data (see [1]).

(2) In the case of evolution problems, we prove the existence of global weak solutions $(v,\, H,\, \theta)$ in the class $W^{1,2}(Q_1) \times L^2_{\mathrm{curl}}(\tilde{Q}) \times W^{1,0}_p(Q)$, $(p < 5/4)$ with a positive defect measure $\nu \in \mathcal{M}(Q)$ concentrated in the boundary of the heaters. Uniqueness is obtained, by constant coefficients, only for strong solutions (see [5]).

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Numerical Solution of Hamilton-Jacobi Equations in High-Dimensions

Maurizio Falcone
(joint work with E. Carlini, E. Cristiani and R. Ferretti)

The computation of optimal controls is one of the primary goals in control theory. The classical approach based on Pontryagin's Maximum Principle leads to a two point boundary value problem for a system of ordinary differential equations describing the dynamics and the co-state and provide necessary conditions for optimality. Although several numerical methods have been proposed for that problem [14, 4], it is well known that it this approach is limited by the following difficulties: a) the initialization of the co-state equations to start the numerical procedure can be very difficult and often requires weeks to be solved, b) the optimal control which is obtained is open-loop and c) the optimal state-costate couple (y^*, u^*) just satisfies necessary conditions so there is no guarantee that it corresponds to a global minimum for the cost functional on the space of admissible controls [14]. Despite those limitations, the approach based on the Pontryagin's Maximum Principle has been up to now the only approach which has produced feasible solutions for real industrial problems.
The second classical approach to the solution of optimal control problems is Dynamic Programming. In this approach, a central role is played by the value function v of the problem, defined as

$$v(x) = \inf_{u(\cdot)\in\mathcal{U}} J(x, u(\cdot)) \tag{1}$$

and the reconstruction of control is done starting from the Bellman equation associated to the problem once the value function has been found (here x will represent the initial position of the system, $\mathcal{U}$ will be the set of admissible controls and J is the cost functional). The development of the theory of viscosity solutions in the last twenty years has finally given a general framework for the characterization of the value function in all the classical problems of deterministic and stochastic control theory (finite horizon, infinite horizon, minimum time, optimal switching, impulsive control, pursuit-evasion games see e.g. [2, 15, 12] and the references therein). In fact, under very general assumptions, one is able to prove that the unique viscosity solution of the Bellman (resp. Isaacs) equation associated to the control (resp. game) problem is the value function. The advantage of this approach is twofold. It gives a precise characterization of the global optimum for the problem and it allows to obtain optimal control in feed-back form. This has motivated a large research activity on the development of numerical methods to solve the Bellman (Isaacs) equation as well as on the numerical sythesis of feedback controls based on the knowledge of an approximate value function (see e.g. [10, 11, 3, 13] and references therein).
The main drawback of this approach is due to the well known "course of dimensionality" of Dynamic Programming. In fact, the numerical solution of a nonlinear partial differential equation in high-dimension is a difficult task and requires new ideas and algorithms. Let us mention some of the techniques which have been

developed to overcame this huge computational task opening the way to the solution of real problems at least when the model can be described by a dozen of state variables (problems with hundreds of variables are still out of reach).

Domain decomposition
This a numerical technique which allows to divide the computation on a domain Ω into a series of problems on sub-domains Ω_i, where $\Omega = \cup_i \Omega_i$. This means that the global number of variables N is splitted into subsets of size N_i. Every subproblem is assigned to a processor and the global solution is obtained via a parallel algorithm which collects the informations from every processor. The main difficulty in this technique is define proper boundary conditions on the ̈virtual interfaces ̈separating the sub-domains, since there is no a-priori knowledge on these interfaces. Although this technique has been mainly developed for linear PDEs, a result for Bellman equations can be found in [5].

Interpolation in high dimension
The approximatin schemes based on Dynamic Programming always require the computation of the approximate value function $v(x_i + \Delta t \Phi(x_i, \Delta t, a))$ to obtain an approximation of $v(x_i)$ (where x_i represents a node in our grid and Φ is the Henrici function corresponding to a one-step method for the dynamics). Then we need an interpolation method to recover the value at $x_i + \Delta t \Phi(x, a)$ by the knowledge of the value functions on the nodes of the grid. In high dimension even linear interpolation can be a rather complicated task. In [6] is proposed an interpolation technique based on a tree structure which is accurate and efficient in any dimension.

Linearization and max-plus algebras
Max-plus methods have been explored for the solution of first-order, nonlinear Hamilton-Jacobi-Bellman partial differential equations and corresponding nonlinear control problems, e.g. in [16, 1, 7]. These methods exploit the max-plus linearity of the associated semigroups. In fact, although the problems are nonlinear, the semigroups are linear in the max-plus sense. The interesting point is that these methods have been used successfully to compute solutions for deterministic optimal control problems, although they have been shown to provide a consistent speed-up in the computation of the value function only for particular Hamiltonians.

Fast Marching Methods
The idea which is behind the development of Fast Marching Methods is rather simple: to concentrate the computational effort of the iterative method on a subset of the grid (called the Narrow Band) and proceed little by little in the computation of the value function saving in CPU time and memory allocations. This approach has been shown to be particularly effective for the eikonal equation corresponding

to the minimum time problem, where the information "flows" from the target to the exterior domain. For that problem, it is natural to dynamically up-date the Narrow Band starting from the first neighbors of the target, then proceeding to the second neighbors and so on. The speed-up with respect to the classical fixed point iteration on the whole grid is a factor 10 (see e.g. [8, 9]). The extension of this technique to more general Hamiltonians is under study.

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Semi-Automatic Transition from Simulation to Optimization in MDO Context

Nicolas R. Gauger
(joint work with Andreas Griewank, Adel Hamdi, Emre Özkaya)

The talk concerns the development of mathematical methods, algorithmic techniques and software tools for the transition from simulation to optimization. We focus in particular on applications in aerodynamics to optimize wing shapes in a multi-disciplinary design optimization (MDO) context. The methodology is applicable to all areas of scientific computing, where large scale governing equations involving discretized PDEs are treated by custom made fixed point solvers. To exploit the domain specific experience and expertise invested in these simulation tools we propose to extend them in a semi-automated fashion. First they are augmented with an adjoint solver to obtain (reduced) derivatives and then this sensitivity information is immediately used to determine optimization corrections. In other words, rather than applying an outer optimization loop we prefer the 'one-shot' strategy of pursuing optimality simultaneously with the goals of primal and adjoint feasibility.

For a given objective $f(y,u)$ we require to fulfill the state equation $c(y,u) = 0$, which is numerically solved by the fixed point iteration $y_{k+1} = G(y_k, u)$. Here $u \in U$ is a design vector, which may be kept fixed as $c(y,u) = 0$ is solved for the corresponding state vector $y = y_*(u) \in Y$. We assume a uniform contraction rate $\|G_y\| \leq \rho < 1$ and define the shiftet Lagrangian function

$$N(y, \bar{y}, u) = f(y,u) + G(y,u)^\top \bar{y} \ .$$

Rather than first fully converging the primal state using

$$y_{k+1} = G(y_k, u) \rightarrow \text{primal feasibility at } y_*$$

and then fully converging the dual state applying

$$\bar{y}_{k+1} = N_y(y, \bar{y}_k, u) \rightarrow \text{dual feasibility at } \bar{y}_*$$

before finally performing an "outer" optimization loop

$$u_{k+1} = u_k - B_k^{-1} N_u(y, \bar{y}, u_k) \rightarrow \text{optimality at } u_* \ ,$$

we suggest an extended single-step one-shot iteration of the form

$$\begin{bmatrix} y_{k+1} \\ \bar{y}_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} G(y_k, u_k) \\ N_y(y_k, \bar{y}_k, u_k)^\top \\ u_k - B_k^{-1} N_u(y_k, \bar{y}_k, u_k)^\top \end{bmatrix} .$$

For computing the optimization correction $u_{k+1} - u_k$ one has to choose as a preconditioner the symmetric positive definite matrix $B_k \succ 0$.

Deriving (sufficient) conditions on B to ensure contractivity of the extended single-step one-shot iteration has proven difficult. Instead, we look for descent on the augmented Lagrangian

$$L^a(y, \bar{y}, u) = \frac{\alpha}{2}\|G(y,u) - y\|^2 + \frac{\beta}{2}\|N_y(y, \bar{y}, u)^\top - \bar{y}\|^2 + N - \bar{y}^\top y \ ,$$

where $\alpha > 0$ and $\beta > 0$.
It turns out that descent is guaranteed for all large positive B with $\beta = \frac{2}{c}$, $\alpha = \frac{2c}{(1-\rho)^2}$, while $c = \|N_{yy}\|$. A suitable B is given by

$$B = \alpha G_u^\top G_u + \beta N_{yu}^\top N_{yu} + N_{uu} \simeq \nabla_u^2 L^a \ .$$

We present first numerical results for the drag reduction of a RAE2822 airfoil at transonic flight conditions by the use of the derived single-step one-shot approach. The underlying PDEs are the compressible Euler equations. The adjoint flow solver as well as all needed derivatives are generated by Automatic Differentiation (AD) tools.
Finally, we present a methodology for aero-structural wing designs and discuss how to extend it for single-step one-shot.

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A Semismooth Newton Method for Solving Elliptic Equations with Gradient Constraints

Roland Griesse
(joint work with Karl Kunisch)

We investigate iterative methods for the numerical solution of an elliptic partial differential equation (PDE) with gradient constraints,

$$\text{(1)} \qquad \max\{-\Delta u - f, \quad |\nabla u| - g\} = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary Γ. This problem was originally studied in [1], where sufficient conditions for existence, uniqueness and regularity results of the type $u \in W^{1,\infty}(\Omega)$ and $u \in W^{2,p}_{loc}(\Omega)$ were obtained. Besides its own inherent interest, the investigation of (1) is motivated by portfolio optimization problems, which are more involved, due to the appearance of singular coeffiencts and possibly unbounded domains.

In the present work [2] we aim for the efficient numerical treatment of (1). We analyze semi-smooth Newton methods for an appropriately defined family of approximating problems. It is verified that this approximation is consistent in the sense that the solutions to this family of approximating problems converge to the solution of (1) and that the semi-smooth Newton method converges super-linearly for each member of the family. Differently from previous applications of semi-smooth Newton methods, (1) does not directly arise from a variational setting.

For the sake of brevity, we discuss here only the multi-dimensional case and refer to [2] for details and proofs.

Assumption. *Suppose that* $\Omega \subset \mathbb{R}^d$, $d \geq 2$ *is a bounded domain with a smooth boundary* Γ. *We assume that* $f \in C^1(\overline{\Omega})$, $f > 0$ *and* $g \in C^2(\overline{\Omega})$, $g \geq 0$.

Instead of treating (1) directly, we consider the regularized formulation

$$-\Delta u + \gamma \max\{0, |\nabla u|^2 - g^2\} = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma \tag{2}$$

for an increasing sequence of parameters $\gamma \geq 0$.

Proposition. *Eq.* (2) *has a unique solution* $u \in W^{3,q}(\Omega) \cap H_0^1(\Omega)$, $1 \leq q < \infty$.

Theorem (Convergence as $\gamma \to \infty$)**.** *The unique solutions* u_γ *of* (2) *converge to the unique solution* $u \in W^{1,\infty}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ *of* (1) *in the following sense:*

$$\begin{aligned} u_\gamma &\to u \quad && \text{in } C(\overline{\Omega}) \\ \nabla u_\gamma &\to \nabla u && \text{in } W^{1,p}(\Omega') \text{ for all } 1 \leq p < \infty, \text{ and in } C(\Omega') \\ \Delta u_\gamma &\rightharpoonup \Delta u && \text{in } L^p(\Omega') \text{ for all } 1 \leq p < \infty, \end{aligned}$$

for every $\Omega' \subset\subset \Omega$.

We state the semismooth Newton algorithm for the solution of (2), combined with an outer loop for increasing the regularization parameter γ, as Algorithm 1. Sufficient conditions for the local superlinear convergence of the inner **while** loop are given in [2, Theorem 3.6, Lemma 3.8 and Theorem 3.10].

Algorithm 1 Semi-smooth Newton method in the multi-dimensional case

1: Choose initial u and $\gamma \geq 0$ and set $n = 0$
2: **while** not converged **do**
3:   **while** not converged **do**
4:     Set
$$A_n = \{x \in \Omega : |\nabla u_n| > g\}$$
5:     Solve for $\delta u \in H^2(\Omega) \cap H_0^1(\Omega)$
$$-\Delta \delta u + 2\,\gamma\,\chi_{A_n} \nabla u_n \cdot \nabla \delta u = \Delta u_n - \gamma \max\{0, |\nabla u_n|^2 - g^2\} + f \quad \text{in } \Omega$$
6:     Update $u_{n+1} = u_n + \delta u$ and increase n
7:   **end while**
8:   Increase γ
9: **end while**

As an example, we consider the following problem on the unit disk Ω in $\mathbb{R}^2$:

$$f(x) = 0.9 + x_1, \qquad g(x) = \begin{cases} 0.1 & \text{for } \|x\| \leq 0.3 \\ 0.4 & \text{for } \|x\| > 0.3. \end{cases}$$

For an increasing sequence of parameters from $\gamma = 1$ to $\gamma = 3.16 \cdot 10^4$, the method converged with a total of 43 semismooth Newton steps to the solution depicted in Figure 1. Each Newton step was stabilized using a streamline upwind Petrov-Galerkin (SUPG) scheme.

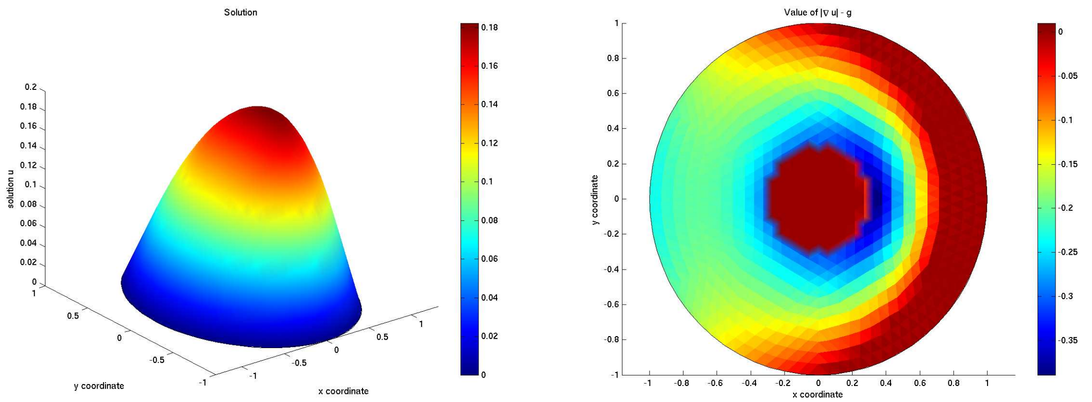

Figure 1. The figure shows the final iterate u_γ (left) and the value of $|\nabla u_\gamma| - g$ (right).

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Optimal Switching Boundary Control

Martin Gugat

Switching (that is 0-1) decisions are essential in many engineering control problems. As an illustration for this type of problems, consider a system governed by the wave equation on a finite interval with Dirichlet boundary control on both sides. The problem is to steer the state to rest in the given finite time T. We have a complementarity constraint: At each moment, only one nonzero control value is allowed. Thus we switch between two modes: Control at the end 0 of the string only or control at the end 1 of the string only. The objective function is the L^2 norm of the controls. This yields the nonconvex optimal control problem $\mathcal{S}$ defined below: Let $y_0 \in H^1(0,1)$, $y_1 \in L^2(0,1)$, $T \geq 2$ be given.

$$(1) \quad \mathcal{S} \begin{cases} \text{minimize} & \int_0^1 [u_{-1}(t)]^2 + [u_1(t)]^2 \, dt \text{ subject to} \\ \text{Initial Cond.} & y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ x \in (0,1) \\ \text{Boundary Cond.} & y(0,t) = u_{-1}(t),\ y(1,t) = u_1(t),\ t \in (0,T) \\ \text{Complementarity} & \\ \text{Constr.} & u_{-1}\, u_1 = 0 \in L^1(0,T) \\ \text{Wave Eqn.} & y_{tt}(x,t) = y_{xx}(x,t),\ (x,t) \in (0,1) \times (0,T) \\ \text{End Cond.} & y(x,T) = 0,\ y_t(x,T) = 0,\ x \in (0,1). \end{cases}$$

For $t \in (0,1)$ and a real number r define the Riemann invariants

$$(2) \qquad r_1(t,r) = (1/2)\, [-r + y_0(t) + \int_0^t y_1(s)\, ds],$$

$$(3) \qquad r_2(t,r) = (1/2)[r + y_0(1-t) - \int_0^{1-t} y_1(s)\, ds].$$

We give a sufficient condition for the existence of a solution of $\mathcal{S}$:
Theorem[Existence] Assume that $T \geq 2$ and that the following *genericity condition* holds: For all real r, the sets

$$\{t \in (0,1) : r_1(t,r) = 0\},\ \{t \in (0,1) : r_2(t,r) = 0\}$$

have measure zero.

Then $\mathcal{S}$ has a solution.
Proof: Define the natural number $k = \lfloor T \rfloor = \max\{j \in N : j \leq T\}$ and the real number $\Delta = T - \lfloor T \rfloor \geq 0$. Let

$$d(t) = \begin{cases} k & \text{if} \quad t \in (0,\Delta), \\ k-1 & \text{if} \quad t \in (\Delta,1). \end{cases}$$

For $t \in (0,1)$ and real numbers x and r, define the function

$$h(x,t,r) = \begin{cases} \frac{r_1(t,r)^2}{d(t)+1-x} + \frac{r_2(t,r)^2}{x} & \text{if} \quad x \neq 0 \text{ and } x \neq d(t)+1, \\ \frac{r_1(t,r)^2}{d(t)+1} & \text{if} \quad x = 0, \\ \frac{r_2(t,r)^2}{d(t)+1} & \text{if} \quad x = d(t)+1. \end{cases}$$

For a real number r, define the real-valued function H as

$$H(r) = \int_0^1 \min_{\kappa \in \{1,\ldots,d(t)\}} h(\kappa,t,r)\, dt.$$

Remark 1: *In [2], the function H is represented in the form $H(r) = \int_0^1 h(\kappa_f(t,r),t,r)\, dt$ with the corresponding definition of the natural number $\kappa_f(t,r)$ for the general case without genericity assumption.*

In [2] it is shown that a solution of $\mathcal{S}$ exists if and only if there exists a real number r_* that minimizes H, that is $H(r_*) = \inf_r H(r)$.

Here we show that the genericity assumption implies that the function H is lower semicontinuous.

Let $(r_k)_k$ denote a sequence of real numbers converging to r_∞. Then by Fatou's Lemma (see [1]) we have

$$\begin{aligned}
\liminf_{k\to\infty} H(r_k) &= \liminf_{k\to\infty} \int_0^1 \min_{\kappa\in\{1,\dots,d(t)\}} h(\kappa,t,r_k)\,dt \\
&\geq \int_0^1 \liminf_{k\to\infty} \min_{\kappa\in\{1,\dots,d(t)\}} h(\kappa,t,r_k)\,dt \\
&\geq \int_0^1 \min_{\kappa\in\{1,\dots,d(t)\}} h(\kappa,t,r_\infty)\,dt = H(r_\infty).
\end{aligned}$$

Since $\lim_{r\to\infty} H(r) = \infty = \lim_{r\to-\infty} H(r)$, the lower semicontinuity of H implies that H attains its infimum, which in turn implies the existence of a solution of $\mathcal{S}$.

Remark 2: *Note that the optimal value of problem $\mathcal{S}$ is equal to* $\inf_r H(r)$. *For a minimizing sequence for $\mathcal{S}$, every point of the sequence must satisfy the complementarity constraint. However, this does* not *imply that the complementarity constraint also holds for each weak limit point of the sequence.*

An explicit description of the optimal controls is given in [2]. *It is derived using d'Alembert's solution. In* [3], *the problem without complementarity is analysed using Fourier series and moment problems.*

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Shape Optimization in External Bernoulli Free Boundary Problems

Jaroslav Haslinger

(joint work with Raino A. E. Mäkinen, Jukka I. Toivanen)

This contribution deals with optimization of the shape of the free boundary in external Bernoulli free boundary problems and, in particular, with numerical realization of such problems. For details we refer to [1]. The state problem reads as follows:

given a bounded set $\omega \subset \mathbb{R}^2$ and $\gamma < 0$,
Find $\Omega \supset \overline{\omega}$ and a function $u : \Omega \setminus \overline{\omega} \to \mathbb{R}$ such that

$$(\mathcal{P}(\omega)) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{\omega} \\ u = 1 & \text{on } \partial\omega \\ u = 0,\ \dfrac{\partial u}{\partial n} = \gamma & \text{on } \partial\Omega. \end{cases}$$

Next we shall consider ω to be a control variable by means of which the shape of Ω will be controlled. Denote $\mathcal{O}$ a family of all admissible ω:s. $\mathcal{O}$ will be chosen in such a way that $(\mathcal{P}(\omega))$ has a unique solution $(\Omega(\omega), u(\omega))$ for every $\omega \in \mathcal{O}$.

Finally let $J : (\Omega(\omega), u(\omega)) \to \mathbb{R}$ be a cost functional. We consider the following optimization problem:

$$(\mathbb{P}) \qquad \begin{cases} \text{Find } \omega^* \in \mathcal{O} \quad \text{such that} \\ J(\Omega(\omega^*), u(\omega^*)) \leq J(\Omega(\omega), u(\omega)) \quad \forall \omega \in \mathcal{O}. \end{cases}$$

In our computations we used the following types of cost functionals:

$$J_1(\Omega(\omega), u(\omega)) = \rho(\partial\Omega(\omega), \Gamma_t),$$

$$J_2(\Omega(\omega), u(\omega)) = \frac{1}{2}\|u(\omega) - z_d\|^2_{0,\Omega(\omega)\setminus\overline{\omega}}, \quad z_d \in L^2_{loc}(\mathbb{R}^2) \text{ given},$$

where ρ is a distance between the free boundary $\partial\Omega(\omega)$ and a target Γ_t. The admissible family $\mathcal{O}$ consists of all bounded, star-like domains with respect to all points in $B_\delta(0)$, $\delta > 0$ given. For such $\mathcal{O}$, problem $(\mathcal{P}(\omega))$ has a unique solution $(\Omega(\omega), u(\omega))$ for every $\omega \in \mathcal{O}$, $\partial\Omega(\omega)$ is of class C^∞ and $\Omega(\omega)$ is star-like with respect to $B_\delta(0)$, as well. In addition, the following stability result holds:

$$\Delta(\partial\omega_n, \partial\omega) \to 0 \implies \Delta(\partial\Omega(\omega_n), \partial\Omega(\omega)) \to 0, \quad n \to \infty,$$

where

$$\Delta(\partial\omega_1, \partial\omega_2) := \sup\{\,|\ln\lambda| \;:\; \lambda\partial\omega_1 \cap \partial\omega_2 \neq \emptyset\},$$
$$\lambda\partial\omega_1 = \{\lambda x \;:\; x \in \partial\omega_1\}$$

(for the proof see [2]). On the basis of these results one can show the existence of at least one solution to $(\mathbb{P})$.

Using that $\Omega(\omega)$, $\omega \in \mathcal{O}$, is star-like as well, the cost functional J_1 can be specified as follows:

$$J_1(\Omega(\omega), u(\omega)) = \int_0^{2\pi} (g_\omega(\theta) - \hat{g}(\theta))^2\, d\theta,$$

where $g_\omega, \hat{g}$ are 2π-periodic functions describing $\partial\Omega(\omega)$ and Γ_t, respectively.

Since the problem is very ill-conditioned, one has to use a robust and reliable method enabling us to solve $(\mathcal{P}(\omega))$ with a high accuracy. It turned out that the so-called pseudo-solid formulation of Bernoulli free boundary problem fulfills all these requirements. Numerical results of several model examples were presented.

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Domain Decomposition and Model Reduction for the Optimal Control of Coupled Systems with Local Nonlinearities

Matthias Heinkenschloss

(joint work with D. C. Sorensen, K. Sun)

This work is concerned with the efficient simulation or efficient optimization of coupled parabolic systems with spatially localized nonlinearities. In the latter case, localized nonlinearities many be due to optimization variables (parameters to be identified, domain shapes, or controls) acting only in a small spatial subdomain.

We decompose the problem into subproblems such that the nonlinear part is concentrated in one subdomain. The linear subproblems in the other subdomains are reduced using balanced truncation model reduction. A crucial observation is that the interface conditions lead to auxiliary inputs and outputs that need to be included in the model reduction along with the original inputs and outputs. The reduced subdomain problems are then combined with the nonlinear subdomain problem to form the reduced coupled problem.

Our numerical tests have shown that the error of the input-to-output map of the original problem and that of the reduced problem are of the order of the error introduced by the balanced truncation model reduction on the linear subdomain problems. This indicates that the guaranteed bounds for the error between the input-to-output map of the original problem and that of the reduced problem that so far only existed for linear time invariant systems can be extended to systems with spatially localized nonlinearities.

As a model problem we consider the heat equation with a local nonlinearity. Let $\Omega \in \mathbb{R}^2$ be decomposed into three subdomains $\Omega_1 = (-21,-5)\times(-6,6)$, $\Omega_2 = (-5,5)\times(-1,1)$ and $\Omega_3 = (5,21)\times(-6,6)$ with interface Γ_{12} between Ω_1, Ω_2 and interface Γ_{23} between Ω_2, Ω_3. Given functions $c_k\colon \Omega_k \to \mathbb{R}$, $k = 1, 2$, and $c_2\colon \Omega_2 \times \mathbb{R} \to \mathbb{R}$ we consider the differential equations

$$\frac{\partial y_k}{\partial t}(x,t) - \nabla\cdot(c_k(x)\nabla y_k(x,t)) = 0, \quad (x,t) \in \Omega_k \times (0,T),\ k = 1,3,$$

$$\frac{\partial y_2}{\partial t}(x,t) - \nabla\cdot(c_2(x, y_2(x,t))\nabla y_2(x,t)) = 0, \quad (x,t) \in \Omega_2 \times (0,T),$$

with interface conditions

$$y_1(x,t) = y_2(x,t),\ (c_1(x)\nabla y_1(x,t))\cdot n + (c_2(x,y_2(x,t))\nabla y_2(x,t))\cdot n = 0,\ x \in \Gamma_{12},$$

$$y_2(x,t) = y_3(x,t),\ (c_2(x,y_2(x,t))\nabla y_2(x,t))\cdot n + (c_3(x)\nabla y_3(x,t))\cdot n = 0,\ x \in \Gamma_{23},$$

boundary conditions $y_1(x,t) = g_1(x,t), x \in \{-21\}\times(-6,6)$, $y_3(x,t) = g_3(x,t), x \in \{21\}\times(-6,6)$, $n\cdot(c_k\nabla y_k) = 0$, $x \in \partial\Omega_N = \partial\Omega\setminus(\{-21\}\times(-6,6)\cup\{21\}\times(-6,6))$, and initial conditions. In our example $T = 8\pi$, $c_1 = 40$, $c_2 = 25y_2 + 25$, $c_3 = 40$.

The boundary data g_1, g_2 are viewed as the inputs into the system and we are interested in the outputs

$$z_k(t) = \int_{\partial\Omega_k\cap\partial\Omega_N} y_k(x,t)ds,\ k = 1,3, \quad z_2(t) = \int_{\partial\Omega_2\cap\partial\Omega_N} y_2(x,t)ds.$$

The model problem is semi-discretized in space using piecewise linear finite elements. The subdomain structure is introduced in a standard way [5, 3, 4]. We apply balanced truncation model reduction [1, 2], to the linear systems corresponding to subdomains Ω_1, Ω_3. The interface conditions on Γ_{12} and Γ_{23} lead to auxiliary inputs and outputs for the subdomain model problems. See [3, 4] for details.

The table in Figure 1 shows the reduction in problem size. The reduction is limited by the size of the spatial discretization of the interfaces Γ_{12}, Γ_{23}, and by the size of the spatial discretization of the subdomain Ω_2 in which the nonlinearity is located. The plot in Figure 1 shows that outputs of the full and the reduced order system are in excellent agreement. This is due to the guaranteed error bounds between full and reduced order model generated by balanced truncation, and by the proper inclusion of the interface condition into the model reduction.

r	full model size	reduced model size
1	467	91
2	2117	221
3	4649	401
4	8230	604
5	12326	896

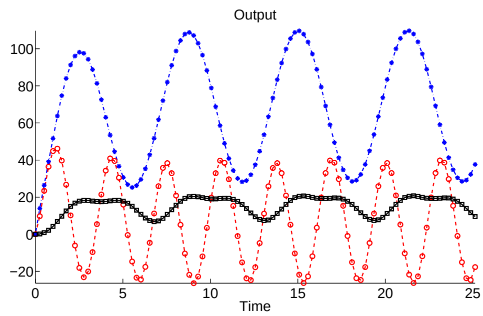

Figure 1. Left: Sizes of the full and the reduced order models for various spatial meshes. Right: Outputs 1, 2, 3 of the full order system, plotted using dotted, dashed and solid lines, respectively, and of the reduced order system, plotted using $*$, $\circ$ and $\Box$, respecitively, for inputs $g_1(x,t) = 2 + 3\sin(t)$, $g_3(x,t) = 4\sin(2t)$.

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Some Recent Advances in State Constrained Optimal Control of Partial Differential Equations

Michael Hintermüller
(joint work with M. Hinze, K. Kunisch)

Throughout we focus on the following model problem of an optimal control problem for an elliptic partial differential equation with pointwise constraints on the state variable:

$$\text{(1)}\qquad \begin{aligned} &\text{minimize } J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u-u_d\|^2_{L^2(\Omega)} =: J_\gamma(y,u)\\ &\text{over } (y,u) \in H^2(\Omega)\cap H^1_0(\Omega)\times L^2(\Omega) =: X\\ &\text{subject to } Ay = u \text{ in } \Omega, \quad y \le \psi \text{ a.e. in } \Omega, \end{aligned}$$

where $y_d, u_d \in L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, sufficiently regular (or convex, polygonal), A is a second order linear elliptic partial differential operator, $\psi \in H^2(\Omega)$ with $\psi|_{\partial\Omega} > 0$, and $\alpha > 0$. It is well-known that the Lagrange multiplier associated to the pointwise almost everywhere (a.e) constraint is measure-valued. This poses problems for numerical algorithms. In order to circumvent this difficulty we propose to consider the regularized problem

$$\text{(2)}\qquad \begin{aligned} &\text{minimize } J(y,u) + \frac{1}{2\gamma}\|(\bar\lambda + \gamma(y-\psi))_+\|^2_{L^2(\Omega)}\\ &\text{subject to } Ay = u \text{ in } \Omega. \end{aligned}$$

Here $\bar\lambda \in L^{2+\epsilon}(\Omega)$, $\bar\lambda \ge 0$ and $\epsilon > 0$, is a shift parameter mimicking a smooth original Lagrange multiplier and $\gamma > 0$ is the regularization parameter. Moreover, $(\cdot)_+$ represents the $\max(0,\cdot)$ pointwise a.e. Let $x_\gamma := (y_\gamma, u_\gamma)$ denote the unique solution of (2) with associated adjoint state p_γ and $\lambda_\gamma := (\bar\lambda + \gamma(y_\gamma - \psi))_+$. The first order necessary and sufficient conditions for (2) are

$$\begin{aligned} &A^* p_\gamma + \lambda_\gamma + y_\gamma - y_d = 0,\\ &\alpha(u_\gamma - u_d) - p_\gamma = 0,\\ &Ay_\gamma - u_\gamma = 0, \end{aligned}$$

where A^* is the adjoint of A. Observe that for $\gamma \to \infty$ we have that x_γ converges strongly in X to $x^* = (y^*, u^*)$, the optimal solution of (1). Considering the primal-dual path induced by γ and defined by $\mathcal{P} := \{(x_\gamma, p_\gamma, \lambda_\gamma) : \gamma > 0\}$ we have that the path is Lipschitz-continuous. Under the assumption that $\text{meas}(\{\bar\lambda + \gamma(y_\gamma - \psi) = 0\}) = 0$ the path is even differentiable (strongly in the primal variables x_γ and weakly in the dual variable p_γ). Moreover, the optimal value functional $V(\gamma) = J_\gamma(y_\gamma, u_\gamma)$ is continuously differentiable. Based on these theoretical observations one may design a primal-dual path-following algorithm. In fact, for fixed γ the first order conditions of (2) can be solved by a semismooth Newton method which converges locally superlinearly in function space. Then, due to the properties of $V(\gamma)$ a safeguarded update strategy for γ may be developed. The resulting algorithm together with its convergence analysis and numerical tests can be found in [1].

From a numerical point of view it is important to intertwine the update of γ and the mesh size of discretization h. Indeed, given h the errors due to discretization and due to regularization need to be balanced. Increasing γ beyond the associated threshold value yields no further improvement in accuracy with respect to the original solution (y^*, u^*) of (1). In the corresponding analysis the overall error (here written for the control only) is split according to

$$\|u_\gamma^h - u^*\|_{L^2(\Omega)} \leq \|u_\gamma^h - u_\gamma\|_{L^2(\Omega)} + \|u_\gamma - u^*\|_{L^2(\Omega)}. \tag{3}$$

A piecewise linear continuous finite element discretization for the state is used which induced an associate discretization for the control space. Then the second error term on the right hand side can be estimated by

$$\|u_\gamma - u^*\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\alpha}} \left(h^{1-\frac{d}{p}} + \gamma^{-\frac{1}{2}} h^{-\frac{d}{2}}\right)^{\frac{1}{2}} + \frac{C}{\sqrt{\alpha\gamma}} \|\bar{\lambda}\|^2_{L^2(\Omega)} \tag{4}$$

for some $0 < h \leq 1$ and $p \geq 1$, and a uniform constant $C > 0$. Here the parameter h occurs due to the fact that the objective functional allows us to handle the L^2-norm of the constraints violation, while first order optimality requires an L^∞-estimate. We closed this gap by utilizing estimates for the L^2-projector of the continuous state space onto the discrete one. The second error, i.e., the first term in the right hand side in (3) is estimates by

$$\|u_\gamma^h - u_\gamma\|_{L^2(\Omega)} \leq C h^{1-\frac{d}{4}}. \tag{5}$$

Bounds of the type (4) and (5) hold as well for the state in the $H^1(\Omega)$-norm.

A uniform (in γ) quadratic order in h for the error in u impossible to achieve as this would require a uniform bound of y_γ, respectively p_γ, in $H^2(\Omega)$. In this context only

$$\|u_\gamma^h - u_\gamma\|_{L^2(\Omega)} \leq \frac{C}{\alpha} \gamma h^2 \tag{6}$$

is available. All proof details, a guideline for adjusting γ and h and numerical test (also under weaker requirements compared to those needed by the theory) can be found in [3].

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A Priori and a Posteriori Error Control for Elliptic Control Problems with Pointwise Constraints

Michael Hinze
(joint work with Klaus Deckelnick and Andreas Günther)

In this talk we discuss a priori and a posteriori finite element discretization concepts for elliptic control problems with pointwise constraints. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded domain with a smooth boundary and outward pointing unit normal ν, and let $A := -\Delta + Id$. We denote by $\mathcal{M}(\bar{\Omega})$ the space of Radon measures which is defined as the dual space of $C^0(\bar{\Omega})$ and endowed with the norm

$$\|\mu\|_{\mathcal{M}(\bar{\Omega})} = \sup_{f \in C^0(\bar{\Omega}), |f| \leq 1} \int_{\bar{\Omega}} f d\mu.$$

State constraints. For a given function $u \in L^2(\Omega)$ we denote by $y = \mathcal{G}(u)$ the solution of the Neumann problem

$$Ay = u \text{ in } \Omega, \ \partial_\nu y = 0 \text{ on } \partial\Omega. \tag{1}$$

It is well known that $y \in H^2(\Omega)$ and

$$\|y\|_{H^2} \leq C\|u\|, \tag{2}$$

where $\|u\|_{L^2}$ denotes the L^2-norm. We now consider the following control problem

$$\min_{u \in L^2(\Omega)} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u - u_0|^2 \quad \text{subject to } y = \mathcal{G}(u) \text{ and } y(x) \leq b(x) \text{ in } \Omega. \tag{3}$$

Here, $\alpha > 0$ and $y_0, u_0 \in H^1(\Omega)$ as well as $b \in W^{2,\infty}(\Omega)$ are given functions.

The analysis of (3) is well understood. From [2, 3] we deduce the existence of a unique solution $u \in L^2(\Omega)$ to problem (3). Moreover,

Theorem 1. *A function $u \in L^2(\Omega)$ is a solution of (3) if and only if there exist $\mu \in \mathcal{M}(\bar{\Omega})$ and $p \in L^2(\Omega)$ such that with $y = \mathcal{G}(u)$ there holds*

$$\int_\Omega pAv = \int_\Omega (y - y_0)v + \int_{\bar{\Omega}} v d\mu \qquad \forall v \in H^2(\Omega) \text{ with } \partial_\nu v = 0 \text{ on } \partial\Omega \tag{4}$$

$$p + \alpha(u - u_0) = 0 \qquad \text{a.e. in } \Omega \tag{5}$$

$$\mu \geq 0, \ y(x) \leq b(x) \text{ in } \Omega, \text{ and } \qquad \int_{\bar{\Omega}} (b - y) d\mu = 0. \tag{6}$$

A finite element approximation of problem (3) is developed in [7, 8]. It uses variational discretization of controls [11] and reads

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{2}\|u - u_{0,h}\|^2 \quad \text{subject to } y_h = \mathcal{G}_h(u) \text{ and } y_h(x_j) \leq b(x_j) \text{ for } j = 1, \ldots, m, \tag{7}$$

where $\mathcal{G}_h$ denotes the finite element solution operator associated to piecewise linear, continuous Ansatz functions on a triangulation with nodes $x_i, i = 1, \ldots, m$.

We note that (7) still is an infinite-dimensional problem. It admits a unique solution. Moreover, Theorem 1 holds accordingly for discrete associates y_h, u_h, p_h, and μ_h, where the discrete measure has the representation $\mu_h = \sum_{i=1}^m \mu_i \delta_{x_i}$, with δ_{x_i} denoting the Dirac measure concentrated at x_i. From [7, 8] we have $\|\mu_h\|_{\mathcal{M}(\bar\Omega)} \le C$ uniformly in h, and also deduce

$$\alpha\|u-u_h\|^2 + \|y-y_h\|^2 \le C(\|u\|,\|u_h\|)\left\{\|y-y_h(u)\| + \|y^h(u_h)-y_h\|\right\} +$$
$$+ C(\|\mu\|_{\mathcal{M}(\bar\Omega)}, \|\mu_h\|_{\mathcal{M}(\bar\Omega)})\left\{\|y-y_h(u)\|_\infty + \|y^h(u_h)-y_h\|_\infty\right\} + \alpha\|u_0-u_{0,h}\|^2,$$

where $y^h(u_h) := \mathcal{G}(u_h)$, $y_h(u) := \mathcal{G}_h(u)$. Now let us assume $u, u_h \in L^\infty(\Omega)$ uniformly in h (see [9] for an example). Then we deduce from the previous estimate

$$\|u-u_h\| + \|y-y_h\|_{H^1} \le Ch|\log h|, \tag{8}$$

where we have assumed $\|u_0-u_{0,h}\| \le Ch$, and used the fact that $\|\mathcal{G}(v)-\mathcal{G}_h(v)\|_\infty \le Ch^2|\log h|^2$ for $v \in L^\infty(\Omega)$, with $C > 0$ denoting a generic constant. Estimate (8) is optimal and with a slightly lower order also proved in [12, Th.7.1].

Constraints on the gradient. Let $r > d$ be given. We consider the control problem

$$\min_{u\in L^r(\Omega)} J(u) = \frac{1}{2}\int_\Omega |y-y_0|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \text{ s. t. } y = \mathcal{G}(u) \text{ and } |\nabla y(x)| \le \delta \text{ in } \Omega, \tag{9}$$

where we now consider Dirichlet boundary conditions in (1). The analysis for (9) is provided in [4]. A finite element approximation of (9) following the lines of (7) is discussed in [6]. Let (u, y) be the unique solution of (9) and let (u_h, y_h) denote the unique solution to the variational discrete counterpart of (9), where the states are discretized with piecewise linear, continuous finite elements. Then there exists $h_1 > 0$ such that

$$\|y-y_h\| \le Ch^{\frac{1}{2}(1-\frac{d}{r})}, \text{ and } \|u-u_h\|_{L^r} \le Ch^{\frac{1}{r}(1-\frac{d}{r})} \text{ for all } 0 < h \le h_1.$$

A finite element approximation of (9) for $r = 2$ and additional L^∞-bounds on the controls using mixed finite elements is provided in [5]. Approximation of states and gradients in (9) with the lowest order Raviart Thomas element allows to prove the error estimate

$$\|u-u_h\| + \|y-y_h\| \le Ch^{\frac{1}{2}}|\log h|^{\frac{1}{2}}.$$

Both finite element approaches require to prove uniform bounds on the discrete multipliers associated to the discretized gradient constraints of (9). Uniform error estimates of finite element approximations to elliptic equations then deliver the respective results.

Goal oriented adaptivity. In [10] we propose an extension of the DWR method discussed in [1] to optimal control problems with state (and control) constraints based on variational discretization [11]. For problem (3) and its associated finite element approximation it delivers the representation

$$J(y,u) - J(y_h,u_h) = \frac{1}{2}\left(\rho^y(p - i_h p) + \rho^p(y - i_h y)\right) + \\ + \frac{1}{2}\left(\langle \mu + \mu_h, y_h - y\rangle + (\lambda + \lambda_h, u_h - u)\right),$$

where λ, λ_h denote multipliers associated to control constraints (if present) and the residual functionals ρ^y, ρ^p are given by

$$\rho^y(\cdot) := -a(y_h, \cdot) + (u_h, \cdot), \text{ and } \rho^p(\cdot) := J_y(y_h, u_h)(\cdot) - a(\cdot, p_h) + \langle \mu_h, \cdot\rangle.$$

Here, $a(\cdot,\cdot)$ denotes the bilinear form associated to A. Let us note that no control residuals appear in this representation since the variational discretization avoids the discretization of the controls. Continuous state and adjoint appearing in this representation are substituted by appropriate discrete counterparts. Furthermore, two methods are proposed to cope with the numerical evaluation of the multiplier μ appearing in this expression. We emphasize, that our concepts also extends to problem (9) where only the expression $\langle \mu + \mu_h, y_h - y\rangle$ has to be replaced by $\langle -\mathrm{div}(\vec{\mu} + \vec{\mu_h}), y_h - y\rangle$ with $\vec{\mu} = 1/\delta \nabla y \mu$,and $\vec{\mu_h} = 1/\delta \nabla y_h \mu$, with μ, μ_h denoting the multipliers associated to the gradient constraints.

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Goal Oriented Mesh Adaptivity for Control and State Constrained Elliptic Optimal Control Problems

Ronald H.W. Hoppe
(joint work with Michael Hintermüller)

We are concerned with the application of the goal oriented weighted dual approach to finite element discretized control and state constrained elliptic optimal control problems. In particular, we want to derive a posteriori error estimators for a quantity of interest related to the discretization error whose local contributions serve as indicators for a local refinement of the triangulations.
Given a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, functions $y^d, \psi \in L^2(\Omega)$, and a regularization parameter $\alpha > 0$, we consider the control constrained distributed optimal control problem

$$\begin{aligned}
&\text{Minimize} \quad && J(y,u) \ := \ \frac{1}{2}\ \|y - y^d\|_{0,\Omega}^2 \ + \ \frac{\alpha}{2}\ \|u\|_{0,\Omega}^2 \ , \\
&\text{over} && (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \ , \\
&\text{subject to} && a(y,v) \ = \ (u,v)_{0,\Omega} \ , \ v \in H_0^1(\Omega) \ , \\
& && u \in K \ := \ \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} \ ,
\end{aligned}$$

where $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$. The optimality conditions give rise to an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\sigma \in L^2(\Omega)$ such that the quadruple (y,p,u,σ) satisfies

$$\begin{aligned}
&a(y,v) = (u,v)_{0,\Omega} \quad , \quad a(p,v) = (y^d - y, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega), \\
&\alpha\ u = p - \sigma \ , \ \sigma \ \ge \ 0 \ , \ u \le \psi \ , \ (\sigma, u - \psi)_{0,\Omega} = 0.
\end{aligned}$$

We denote by V_ℓ the finite element space of continuous, piecewise linear functions, by W_ℓ the linear space of elementwise constants with respect to a simplicial triangulation $\mathcal{T}_\ell(\Omega)$, and we approximate the state in V_ℓ and the control in W_ℓ. Referring to $y_\ell^d \in V_\ell$ and $\psi_\ell \in W_\ell$ as a desired discrete state and a discrete control constraint, the optimality conditions for an optimal pair (y_ℓ, u_ℓ) of the finite element discretized control problem invoke a discrete adjoint state $p_\ell \in V_\ell$ and a discrete adjoint control $\sigma_\ell \in W_\ell$ such that the quadruple $(y_\ell, u_\ell, p_\ell, \sigma_\ell)$ satisfies

$$\begin{aligned}
&a(y_\ell, v_\ell) = (u_\ell, v_\ell)_{0,\Omega} \ , \ a(p_\ell, v_\ell) = (y_\ell^d - y, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell, \\
&\alpha\ u_\ell = M_\ell p_\ell - \sigma_\ell \ , \ \sigma_\ell \ge 0 \ , \ u_\ell \le \psi_\ell \ , \ (\sigma_\ell, u_\ell - \psi_\ell)_{0,\Omega} = 0,
\end{aligned}$$

where $M_\ell v_\ell|_T := |T|^{-1} \int_T v_\ell \ dx, T \in \mathcal{T}_\ell(\Omega)$.
We choose the objective functional J as the quantity of interest and introduce an associated Lagrangian $\mathcal{L} : H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by means of

$$\mathcal{L}(y,u,p,\sigma) := J(y,u) + (\nabla y, \nabla p)_{0,\Omega} - (u,p)_{0,\Omega} + (\sigma, u - \psi)_{0,\Omega}.$$

We refer to $\mathcal{L}_\ell : V_\ell \times W_\ell \times V_\ell \times W_\ell \to \mathbb{R}$ as its discrete counterpart. Setting $(x,\sigma) \in X \times L^2(\Omega), x := (y,u,p) \in X := H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$, and assuming that $(x_\ell, \sigma_\ell) \in X_\ell \times W_\ell, x_\ell := (y_\ell, u_\ell, p_\ell) \in X_\ell := V_\ell \times W_\ell \times V_\ell$, are the solutions of the continuous and discrete problem, we have the following error representation

$$J(y,u) - J_\ell(y_\ell, u_\ell) = -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) + (\sigma, u_\ell - \psi)_{0,\Omega} + osc_\ell(x_\ell),$$

where $osc_\ell(x_\ell)$ is an oscillation term consisting of data oscillations. We note that this error representation reduces to the one derived in [1] in the unconstrained case. Evaluating the second derivative of the Lagrangian further, we arrive at the error estimate

$$\begin{aligned} &|J(y,u) - J_\ell(y_\ell, u_\ell)| \ \le \\ &\le \ C \sum_{T \in \mathcal{T}_\ell(\Omega)} \Big(\omega_T^y \rho_T^y + \omega_T^p \rho_T^{p,1} + \omega_T^u \rho_T^{p,2}\Big) + \mu_\ell(x,\sigma) \ + \ osc_\ell(x, x_\ell) \ + \ osc_\ell. \end{aligned}$$

Here, ρ_T^y and $\rho_T^{p,\nu}, 1 \le \nu \le 2$, are L^2-norms of the residuals associated with the state and the adjoint state with suitable dual weights ω_T^y, ω_T^p and ω_T^u. Moreover, $\mu_\ell(x,\sigma)$ represents a primal-dual mismatch in complementarity and $osc_\ell(x, x_\ell)$ stands for a further oscillation term which can be made fully a posteriori (see [2] for details).

In the state constrained case, we assume that for any control $u \in L^2(\Omega)$ the state y belongs to $W^{1,r}(\Omega)$ for some $r > 2$ and satisfies the constraints $y \in K := \{v \in C(\overline{\Omega}) \mid v(x) \le \psi(x), x \in \overline{\Omega}\}$. In this case, the adjoint state p is in $W^{1,s}(\Omega)$ with s being conjugate to r and the multiplier σ for the state constraints is a measure living in the dual space $\mathcal{M}(\Omega)$ of $C(\overline{\Omega})$. We approximate y, p, u by continuous, piecewise linear finite elements with respect to a simplicial triangulation $\mathcal{T}_\ell(\Omega)$ and σ by a linear combination of Dirac delta functionals in the nodal points of the triangulation. As in the control constrained case, we can derive an error representation in terms of the associated Lagrangian which leads to the estimate

$$\begin{aligned} &|J(y,u) - J_\ell(y_\ell, u_\ell)| \ \le \\ &\le \ C \sum_{T \in \mathcal{T}_\ell(\Omega)} \Big(\omega_T^y \rho_T^y + \omega_T^p \rho_T^p + \omega_T^\sigma \rho_T^\sigma\Big) + \mu_\ell(x,\sigma) \ + \ osc_\ell(x, x_\ell) + osc_\ell. \end{aligned}$$

Here, $\rho_T^y, \rho_T^p, \rho_T^\sigma$ are residuals in the state, adjoint state, and the multiplier with appropriate dual weights taking into account the specific function space setting, $\mu_\ell(x,\sigma)$ refers to a primal-dual mismatch in complementarity, and $osc_\ell(x, x_\ell), osc_\ell$ represent data oscillation terms (see [3] for details).

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An Optimal Feedback Solution to Quantum Control Problems

Kazufumi Ito
(joint work with Karl Kunisch)

The talk discusses control of quantum systems described by the Schrödinger equation are considered. Feedback control laws are developed for orbit tracking via controlled Hamiltonians and their asymptotic properties are analyzed. Numerical integrations via time-splitting is also investigated and used to demonstrate the feasibility of the proposed feedback laws.

Consider a quantum system with internal Hamiltonian $\mathcal{H}_0$ prepared in the initial state $\Psi_0(x)$, where x denotes the relevant spatial coordinate. The state $\Psi(x,t)$ satisfies the time-dependent Schrödinger equation (we set $h = 1$). In the presence of an external interaction taken as an electric field modeled by a coupling operator with amplitude $\epsilon(t) \in R$ and a time independent dipole moment operator μ, the new Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \epsilon(t)\mu$ gives rise to the following dynamical system to be controlled,

$$i\,\frac{\partial}{\partial t}\Psi(x,t) = (\mathcal{H}_0 + \epsilon(t)\mu)\Psi(x,t), \quad \Psi(x,0) = \Psi_0(x). \tag{1}$$

where $\mathcal{H}_0$ is a positive, closed, self-adjoint operator in the Hilbert space H, and $\mu \in \mathcal{L}(\mathcal{H})$ is self-adjoint. Let X be the complexified Hilbert space corresponding to H. We normalize the initial state by $|\Psi_0|_X = 1$.

We consider the control problem of driving the state $\Psi(t)$ of (1) to an orbit $\mathcal{O}$ of the uncontrolled dynamics

$$i\,\frac{d}{dt}\mathcal{O}(t) = \mathcal{H}_0\mathcal{O}(t), \tag{2}$$

specifically to the one that corresponds to an eigen-state or the manifold spanned by finite many eigen-states. An element $\psi \in dom\,(\mathcal{H}_0)$ is an eigen-state of $\mathcal{H}_0$ if $\mathcal{H}_0\psi = \lambda\,\psi$ for $\lambda > 0$. Then, the corresponding orbit is given by

$$\mathcal{O}(t) = e^{-i(\lambda t - \theta)}\psi, \tag{3}$$

where $\theta \in [0, 2\pi)$ is the phase factor. We have $|\mathcal{O}(t)|_X = 1$ if ψ is normalized as $|\psi|_H = 1$. We assume that the family of eigenfunctions $\{\psi_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $\mathcal{H}_0$ and that the associated eigenvalues λ_k are arranged in increasing order.

We employ a variational approach based on either of the two Lyapunov functionals

$$\begin{aligned} V_1(\Psi(t), \mathcal{O}(t)) &= \tfrac{1}{2}|\Psi(t) - \mathcal{O}(t)|_X^2 \\ V_2(\Psi(t), \mathcal{O}(t)) &= \tfrac{1}{2}\,(1 - |(\mathcal{O}(t), \Psi(t))_X|^2). \end{aligned} \tag{4}$$

These variational procedures were previously discussed in [1, 2], In general we consider the tracking to

(5) $$\mathcal{O}(t) = \sum_{k=1}^{N} A_k e^{-i(\lambda_k t - \theta_k)} \psi_k,$$

where $\{(\lambda_k, \psi_k)\}_{k=1}^N$ are the first N eigen-pairs of $\mathcal{H}_0$ and $\sum_{k=1}^N A_k^2 = 1$.

Since $|\Psi(t)|_X = 1$ for all $t \geq 0$ V_1 can equivalently be expressed as

(6) $$V_1(\Psi(t), \mathcal{O}(t)) = 1 - Re\,(\mathcal{O}(t), \Psi(t))_X.$$

The second functional is motivated by the fact that $V_2(\Psi, \mathcal{O}) = 0$ if and only if $\Psi = e^{i\theta}\mathcal{O}$ where the phase $\theta \in [0, 2\pi)$ is arbitrary. As a consequence we choose time-independent and set $\mathcal{O}(t) = \mathcal{O}$ for the functional V_2. It is shown that

(7) $$\frac{d}{dt} V_1(\Psi(t), \mathcal{O}(t)) = \epsilon(t)\, Im\,(\mathcal{O}(t), \mu\Psi(t))_X.$$

Thus, if we set

(8) $$\epsilon(t) = -\frac{1}{\beta} Im\,(\mathcal{O}(t), \mu\Psi(t))_X = F_1(\Psi(t), \mathcal{O}(t)),$$

then

(9) $$\frac{d}{dt} V_1(\Psi(t), \mathcal{O}(t)) = -\beta\,|\epsilon(t)|^2.$$

Similarly, we have

(10) $$\frac{d}{dt} V_2(\Psi(t), \mathcal{O}) = \epsilon(t)\, Im\left(\overline{(\mathcal{O}, \Psi(t))_X}\ (\mathcal{O}, \mu\Psi(t))_X\right).$$

If we let

(11) $$\epsilon(t) = -\frac{1}{\beta} Im\left(\overline{(\mathcal{O}, \Psi(t))_X}\,(\mathcal{O}, \mu\Psi(t))_X\right) = F_2(\Psi(t), \mathcal{O}),$$

then similarly as above

(12) $$\frac{d}{dt} V_2(\Psi(t), \mathcal{O}) = -\beta\,|\epsilon(t)|^2.$$

The main objective of the talk is is to analyze the asymptotic tracking properties of these two feedback laws . Sufficient conditions will be obtained which guarantee orbit tracking for functional V_1 and manifold tracking for V_2. In order to obtain improved tracking capability we shall also analyze multiple control potentials of the form

(13) $$\tilde{\mu}(t) = \sum_{j=1}^{m} \epsilon_j(t)\, \mu_j.$$

The feedback law F_1 is optimal in the sense that $\epsilon(t) = F_1(\Psi(t), \mathcal{O}(t))$ minimizes

$$\int_0^T \frac{\beta}{2}(|\epsilon|^2 + |F_1(\Psi(t), \mathcal{O}(t)|^2)\, dt \;+\; V_1(\Psi(T), \mathcal{O}(T)).$$

An operator splitting method for solving (1) is discussed, The feasibility of the proposed feedback law is demonstrated by integrating the closed loop dynamics using the operator splitting method for a test example.

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Free Material Optimization: Towards the Stress Constraints

Michal Kocvara
(joint work with Michael Stingl)

The goal of the presentation is to find a formulation of stress constraint in the free material optimization (FMO) problem that would be computationally tractable and would lead to reasonable and expected results. The underlying model was introduced in [2] and later developed in [7] and [1]. The design variable is the full elastic stiffness tensor that can vary from point to point; it should be physically available but is otherwise not restricted. This problem gives the best physically attainable material and can be considered the "ultimate" generalization of the structural optimization problem.

The standard FMO problem deals with compliance and weight. However, in practise, it is usually the local stress that should be controlled. An often causes of structural failure is high stress, so it is desirable to keep it within given limits during the optimization process.

To control the stress in material optimization is, however, not an easy task; see, e.g., [3]. The first problem to be faced is how to measure stress, i.e., what kind of failure criteria should be used. This question is even more complicated in the FMO case when we design the material itself. In this presentation we opted for a (local) integral measure of the norm of the stress tensor. The second problem is technical. The optimization problem with stress constraints is a difficult mathematical program that is almost impossible to solve by available optimization software. In addition, the variables are matrices (the discretized elastic stiffness tensor) and vectors (displacements) that appear in the constraints in a nonlinear way. Hence we face a nonlinear semidefinite programming problem.

We consider the standard problem of linear elasticity, discretized by the finite element method. After discretization, the equilibrium equation becomes $SA(E)u = f$ where $A(E)$ is the global stiffness matrix depending linearly on the elastic stiffness tensor E, and u and f are the vectors of displacements and external forces, respectively. In FMO the design variable is the elastic stiffness tensor E which is a function of the space variable x (see [2]). The only constraints on E are that it is physically reasonable, i.e., that E is symmetric and positive semidefinite. As a “cost” of E we use the trace of E. After the discretization, *minimum weight single-load FMO problem* becomes

$$
\begin{aligned}
(1)\qquad & \min_{u,E_1,\ldots,E_m} \sum_{i=1}^{m} \mathrm{Tr}(E_i) \\
& \text{subject to } E_i \succeq 0, \quad i=1,\ldots,m \\
& \qquad \underline{\rho} \le \mathrm{Tr}(E_i) \le \overline{\rho} \quad i=1,\ldots,m \\
& \qquad f^T u \le \gamma \\
& \qquad A(E)u = f\,.
\end{aligned}
$$

Problem (1) is a mathematical programming problem with linear matrix inequality constraints and standard nonlinear constraints, the nonlinear semidefinite programming problem. Recently, there is not much software available for these problems. We solve the problem by a modified version of our software package PENNON that can be used to the solution of problems of type (1); see [4, 6].

In engineering practise, it is not (only) the compliance but some measure of local strain that should be controlled. In the continuous formulation, we would work with pointwise stresses, i.e., we would restrict the norm $\|\sigma(x)\|$ for all $x \in \Omega$. However, in the finite element approximation we use the primal formula (working with displacements) and it is a well-known fact that, generally, evaluation of stresses (from displacements) at points may be rather inexact. Hence we will consider the following integral form of stress and strain constraints $\int_{\Omega_i} \|\sigma\|^2 \le s_\sigma |\Omega_i|$; here Ω_i is the i^{th} finite element and $|\Omega_i|$ its volume. The integrals will be further approximated by the Gaussian intergation formulas, as in the finite element interpolation.

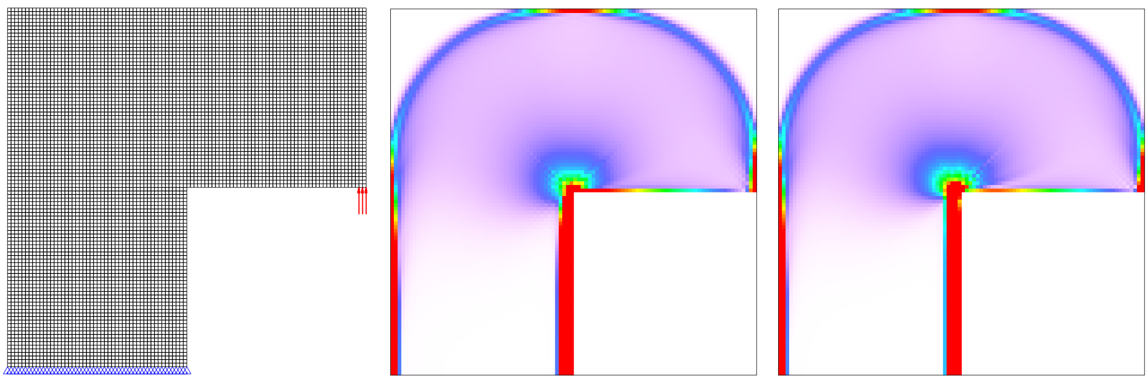

FIGURE 1. L-shape example: mesh and boundary conditions; FMO solution without stress constraints; FMO solution with stress constraints.

Figure 1 presents results of a standard test example, the L-shaped domain. We first solved the FMO problem without stress constraints; the optimal distribution of the trace of E is presented in the middle picture. The maximal stress in this result was 0.201 (located at the re-entrant corner). Then we added a stress constraint with upper bound 0.03125. The most-right picture in Figure 1 shows the corresponding optimal solution. We can see that the re-entrant corner is replaced by an approximate arc, with the goal to smoothen the corner and remove the stress singularity. For more details, see [5].

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Boundary Feedback Control in Dynamic Fluid-Structure Interactions

Irena Lasiecka

Introduction. We consider a boundary control system for a Fluid Structure Interaction Model. This system describes the motion of an elastic structure inside a viscous fluid with interaction taking place at the boundary of the structure, and with the possibility of controlling the dynamics from this boundary. Our aim is to construct a real time feedback control based on solution to the Riccati Equation. The difficulty of the problem under study is due to the unboundedness of control act ion, which is typical in boundary control problems. It is known that Riccati feedback (unbounded) controls may develop strong singularities which destroy the well-posedness of Riccati e quations. This makes computational implementations problematic, to say the lea st. However, as shown recently, this pathology does not happen for certain classes of unbounded control systems, The latter are referred as Singular Estimate Control Systems (SECS) [2, 3]. For such systems there is a full Riccati theory in place, which leads to the wellposedness of feedback dynamics generated by an appropriate trace restriction of the Riccati operator [2, 3]. (SECS) systems are defined as follows. Let $\mathcal{A}$ be a generator of a C_0 semigroup $e^{\mathcal{A}t}$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}$ - unbounded control operator- be such that $\mathcal{B} \in \mathcal{L}(U \to [D(A^*)]')$, where U a suitable control space. Consider the dynamics: $y_t = \mathcal{A}y + \mathcal{B}g \in [\mathcal{D}(\mathcal{A}^\star)]'$.

With this dynamics we associate observation operators $G \in \mathcal{L}(\mathcal{H}, W)$ where W - Hilbert space. and we wish to minimize $J(u,g) = \int_0^T |u|_U^2 dt + |G(y(T)|_W^2$.

Defintion 1. *We say that the system generated by the quadruple $(\mathcal{A}, \mathcal{B}, G)$ is SECS system iff the following singular estimate holds with some $0 \leq \gamma < 1$.*

$$|Ge^{\mathcal{A}t}\mathcal{B}g|_{\mathcal{H}} \leq \frac{C}{t^\gamma}|g|_U, \ 0 < t \leq 1. \tag{1}$$

Remark 1. *Note that when $\mathcal{B}$ is bounded from $U \to \mathcal{H}$, or e^{At} is analytic and $\mathcal{B}$ is relatively bounded with respect to $\mathcal{A}$, singular estimate in (1) is automatically satisfied. Thus, SECS systems are proper extensions of both control systems with bounded controls and analytic systems with relatively bounded control operators (e.g. boundary controls).*

Our objective is to show that boundary control problem arising in fluid structure interaction falls in the class of Singular Estimate Control Systems (SECS). This is due to heyperbolic-parabolic coupling within the structure. SECS estimate allows for application of the theory in [3], which then leads to wellposedness of Riccati equations and of Riccati feedback synthesis. In addition to the theoretical results, an explicit formulation of the Differential Riccati equation associated with this control system provides a basis for an effective computational treatment of the system under consideration.

Boundary Control Problem for a Fluid-Structure Interaction The mathematical model under consideration is the following. Let $\Omega \subset R^n$ be a bounded domain with an interior region Ω_s and an exterior region Ω_f. The boundary Γ_f is the outer boundary of the domain Ω while Γ_s is the boundary of the region Ω_s which also borders the exterior region Ω_f and where the interaction of the two systems take place. Let u be a function defined on Ω_f representing the velocity of the fluid while the scalar function p represents the pressure. w, w_t is the displacement and velocity functions of the solid Ω_s. ν is the unit outward normal vector with respect to the domain Ω_s. The boundary-interface control represented by $g \in L_2([0.T]; L_2(\Gamma_s))$ is active on the boundary Γ_s. We introduce the Cauchy Polya tensor given by $\mathcal{T}(u,p) \equiv \epsilon(u) - pI$, where $\epsilon(u) \equiv \nabla u + \nabla^T u$. defined by elastic strain tensor is giwen by $\sigma(w) \equiv 2\nu\epsilon(w) + \lambda Tr\epsilon(u)\delta_{ij}$. Given any $g \in L_2([0,T]; L_2(\Gamma_s))$, and initial conditions $y(0) = (u(0), w(0), w_t(0)) \in \mathcal{H}$, where $\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$ with $H = \{u \in L_2(\Omega)f), div\ u = 0, u = 0,\ on\ \Gamma_f\}$ we are seeking a quadruple $(u, w, w_t, p)(t) \in \mathcal{H} \times L_2(\Omega_f)$ that satisfy the following system:

$$\begin{aligned} u_t - div\mathcal{T}(u,p) = 0,\ div\ u = 0,\ in\ \Omega_f \times [0,T] \\ w_{tt} - div\sigma(w) = 0\ \Omega_s \times [0,T] \\ w_t = u, on\ \Gamma_s \times [0,T], u = 0, on\ \Gamma_f \times [0,T] \\ \sigma(w) \cdot \nu = \mathcal{T}(u,p) \cdot \nu + g,\ on\ \Gamma_s \times [0,T]. \end{aligned} \tag{2}$$

We are interested in the following Bolza problem : minimize with respect to all $g \in L_2([0,T]; \Gamma_s)$ the following functional:

$J(u,g) = \int_0^T |g(s)|^2_{L_2(\Gamma_s)} ds + |u(T,.) - u_T|^2_{L_2(\Omega_f)}$.
A distinctive feature of the control problem under consideration is the fact that control functions g actuate on the interface between the two media. This leads to very singular kernels in integral representation of the gain operator . The latter is the main technical difficulty of the problem under study and leads to controlled blow up of the gain operator at the terminal time.
Main Results With $\mathcal{A}$ denoting generator associated with $(u, w, w_t) \in \mathcal{H}$ in equation (3), $\mathcal{B}$ the corresponding boundary operator and G projection on the fluid component , we have the following:

- **Singular estimate**. $|Ge^{\mathcal{A}t}\mathcal{B}g|_{\mathcal{H}} \leq \frac{C}{t^{1/4+\epsilon}} |g|_{L_2(\Gamma_s)}$
- **Singular feedback synthesis** There exists a positive selfadjoint $P(t) \in \mathcal{L}(\mathcal{H})$ such that $J(g^0, y^0) = (P(0)y_0, y_0)_{\mathcal{H}}$, where $y = (u, w, w_t)$. The optimal control $g^0(t,.) = -\mathcal{B}^\star P(t) y^0(t)$ and the following singular estimate holds $|\mathcal{B}^* P(t) y^0(t)|_{L_2(\Gamma_s)} \leq C \frac{|y(0)|_{\mathcal{H}}}{|T-t|^{1/4+\epsilon}}$
- **Riccati equation** $P(t)$ is a unique solution to a corresponding Riccati Equation.

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Optimal Control of the Thermistor problem

Christian Meyer
(joint work with Dietmar Hömberg, Joachim Rehberg, Wolfgang Ring)

The talk deals with the optimal control of the thermistor problem that models the conductive heat transfer in a conductor produced by an electric current. This leads to the following quasi-linear system of partial differential equations (PDEs):

$$\partial_t\theta - \text{div}(\kappa\nabla\theta) = (\sigma(\theta)\nabla\varphi)\cdot\nabla\varphi \quad \text{in Q} \tag{1}$$

$$\nu\cdot\kappa\nabla\theta + \alpha\theta = \alpha\theta_l \quad \text{on}\Sigma \tag{2}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega \tag{3}$$

$$-\text{div}(\sigma(\theta)\nabla\varphi) = 0 \quad \text{in } Q \tag{4}$$

$$\nu\cdot\sigma(\theta)\nabla\varphi = u \quad \text{on } \Sigma_0 \tag{5}$$

$$\varphi = 0 \quad \text{on } \Sigma\backslash\Sigma_0, \tag{6}$$

with a Lipschitz domain $\Omega \subset \mathbb{R}^2$, $Q = \Omega\times]0,T[$, $\Sigma = \partial\Omega\times]0,T[$, and $\Sigma_0 = \Gamma_0\times]0,T[$, where Γ_0 denotes a fixed part of $\partial\Omega$. Moreover, θ represents the temperature, while φ is the electric potential. Furthermore, θ_l and θ_0 are given functions, and u is the control that can be interpreted as a current induced on Γ_0. A possible application for this coupled system of PDEs is the hardening of steel workpieces via the Joule effect.

Our aim is to adjust the control u such that

$$J(\theta,u) := \frac{1}{2}\,\|\theta(T)-\theta_d\|^2_{L^2(\Omega_m)} + \frac{\beta}{2}\,\|u\|^2_{L^2(\Sigma_0)} \tag{7}$$

is minimized subject to (1)–(6) and the following inequality constraints

$$u_a \le u(t,x) \le u_b \quad \text{a.e. on } \Sigma_0 \tag{8}$$

$$\theta_a(t,x) \le \theta(t,x) \le \theta_b(t,x) \quad \text{a.e. in } Q. \tag{9}$$

Here, (8) reflects the maximum available electrical power, whereas (9) prevents melting of the material which is crucial in view of hardening applications.

Notice that (9) represents a pointwise state constraint that is known to be numerically and theoretically challenging to handle. To be more precise, the generalized Karush-Kuhn-Tucker (KKT) theory requires to consider the state constraints in the space of continuous functions. The continuity of solutions to (1)–(6) is shown by using maximum parabolic regularity results in the spirit of [3, 4]. To be more precise it is proven that

$$(\varphi,\theta) \in L^\infty(]0,T[;W^{1,q}(\Omega)) \times W^{1,r}(]0,T[;W^{1,q'}(\Omega)^*) \cap L^r(]0,T[;W^{1,q}(\Omega))$$

provided that $u \in L^\infty(]0,T[;L^2(\Gamma_0))$ and that θ_0 and θ_l are sufficiently smooth. Here, q is a fixed number in $]2,4[$ and r satisfies $r > 2q/(q-2)$ such that

$$W^{1,r}(]0,T[;W^{1,q'}(\Omega)^*) \cap L^r(]0,T[;W^{1,q}(\Omega)) \hookrightarrow C([0,T];C(\bar{\Omega})).$$

Afterwards the linearized state system is discussed by similar arguments leading to the continuous Fréchet differentiability of the control-to-state operator by means of the implicit function theorem. In presence of pointwise state constraints as in (9) the generalized KKT theory implies existence of Lagrange multipliers in $C(\bar{Q})^*$ which can be identified with the space of regular Borel measures (cf. for instance [2]). The arising adjoint equation involving measures as inhomogeneity is analyzed by using a duality argument in the spirit of Amann [1]. This gives existence and uniqueness of the adjoint state in $L^{r'}(]0,T[;W^{1,q'}(\Omega))^2$, i.e., the adjoint state is not weakly differentiable w.r.t. time. The statement of first-order necessary optimality conditions under a Slater-type assumption then follows standard arguments.

The optimal control problem is solved numerically by means of a Moreau-Yosida type regularizaton of the state constraints (see for instance [5]). The feasibility of this approach is afterwards demonstrated by the example of hardening a gear rack used in the automotive industry. Figure 1 shows a cut-out of the optimal temperature distribution corresponding to the free optimization of a gear rack with simplified geometry. The aim is to achieve a uniform temperature of 800 K in the teeth at end time $T = 1.0\,s$.

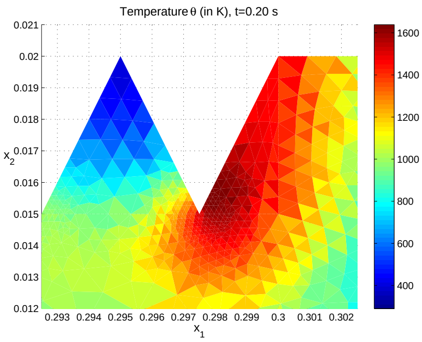

Figure 1. Temperature distribution in case of free optimization at $t = 0.2\,s$.

One observes that at time $t = 0.2\,s$ the temperature in the depicted region exceeds the melting temperature which has to be prevented. By means of the regularized state constraints it is possible to force the temperature not to surpass the melting temperature of approximately 1000 K. However, the optimal temperature distribution at end time in the teeth differs significantly from the desired 800 K as shown in Figure 2.

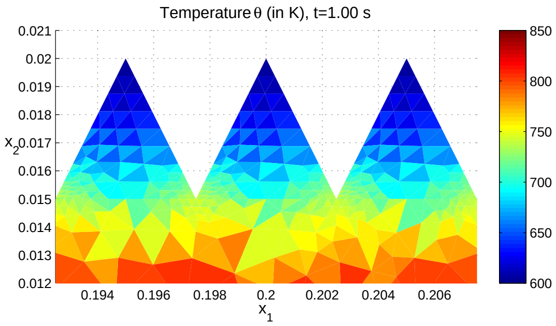

Figure 2. Temperature distribution in case of constrained optimization at end time.

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Optimal Feedback Control of Constrained Parabolic Systems in Uncertainty Conditions

Boris S. Mordukhovich

This paper is devoted to developing an efficient procedure to design a *suboptimal feedback control* regulator acting in the *Dirichlet boundary conditions* of a multidimensional linear *parabolic system* with *hard/pointwise constraints* on the state and control variables under distributed *uncertain perturbations*. Problems of this type are among the most challenging and difficult in control theory while being among the most important for various applications. The original motivation for our development came from practical design problems of automatic control of the soil groundwater regime in irrigation engineering networks functioning under uncertain weather and environmental conditions; see [8] for technological descriptions and modeling.

The system dynamics in the problem under consideration is given by the multidimensional *linear parabolic equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \quad \text{a.e. in } \ Q := [0,T]\times\Omega, \\ y(0,x) = 0, \quad x\in\Omega, \\ y(t,x) = u(t), \quad (t,x)\in\Sigma := [0,T]\times\partial\Omega \end{cases} \tag{1}$$

with *controls* $u(\cdot)$ acting in the Dirichlet boundary conditions and distributed *perturbations* $w(\cdot)$ on the right-hand side of the parabolic equation. In (1), A is a *self-adjoint* and *uniformly strongly elliptic operator* on $L^2(\Omega)$ defined by

$$Ay := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) - cy, \tag{2}$$

where $\Omega\subset I\!R^n$ is an open bounded domain with the closure $\mathrm{cl}\,\Omega$ and the boundary $\partial\Omega$ that is supposed to be a sufficiently smooth $(n-1)$-dimensional manifold, and where $T>0$ is a fixed time bound.

The sets of *admissible controls* U and *admissible perturbations* W are given, respectively, by the relationships

$$U := \Big\{u\in L^\infty[0,T]\Big| -\alpha\le u(t)\le\alpha \ \text{ a.e. } \ t\in[0,T]\Big\}, \tag{3}$$

$$W := \Big\{w\in L^\infty[0,T]\Big| -\beta\le w(t)\le\beta \ \text{ a.e. } t\in[0,T]\Big\} \tag{4}$$

with some fixed bounds $\alpha,\beta>0$. Note that control and perturbation functions look similarly via the *pointwise* constraints in (3) and (4)—except they are situated in the different parts of the parabolic system (1)—while their roles in the feedback control problem formulated below are completely opposite.

It has been well recognized that the *Dirichlet boundary conditions* as in (1) offer the *least regularity* properties of the parabolic dynamics and occur to be the *most challenging* in control theory; see, e.g., [3, 6, 11, 14, 17] with various results, discussions, and references therein. In particular, a lower regularity of feasible controls in (3) is *not sufficient* for the existence of classical solutions to the initial-boundary value problem in (1), while for any feasible pair $(u, w) \in U \times W$ there is a *unique generalized solution* $y \in L^2(Q)$ to the parabolic system (1); see, e.g., [7]. Having this in mind, fix a point $x_0 \in \Omega$ from the space domain and suppose that we are able to *collect information* about the system motion/performance $y(t, x_0)$ at this point. Since the domain Ω is *open* and $u, w \in L^\infty[0, T]$, we can *pointwisely* evaluate $y(t, x_0)$ for any $x_0 \in \Omega$; see, e.g., [1, Theorem 3.9].

A crucial requirement on the system performance (originally motivated by the groundwater control problem in [8]) is to keep the motion $y(t, x_0)$ within the given distance $\eta > 0$ from the initial equilibrium state $y(x, 0) \equiv 0$ for the whole dynamic process. This means imposing the *pointwise state constraints* on the motion under observation

$$-\eta \le y(t, x_0) \le \eta \quad \text{a.e. } \ t \in [0, T]. \tag{5}$$

As mentioned, perturbations $w(\cdot)$ in (1) are *uncertain*, i.e., they are not known a priori; the only information available on perturbations is the *bound* β of their admissible variations. The main goal of boundary controls $u(\cdot)$ in (1) is to keep the motion $y(t, x_0)$ within the state constraints (5) for *all admissible perturbations* $w(\cdot)$ from (4). Clearly, it *cannot* be done in any (prescribed) *open loop* $u = u(t)$, and so control actions in the boundary conditions of (1) should be formed depending on the *current position* $y(t, x_0)$ under observation. This means that we have to design a *feedback control regulator* in the boundary conditions as a function of the state position $\xi \in I\!\!R^n$, where ξ is generated by the dynamic system (1) via the moving point of observation $y(t, x_0)$ for each $t \in [0, T]$.

To formalize this procedure, we consider a function $f\colon I\!\!R \to I\!\!R$ satisfying the composite *summability condition*

$$\big|f\big(\gamma(t)\big)\big| \in L^1[0, T] \quad \text{whenever} \quad \gamma(t) \in L^2[0, T] \tag{6}$$

and construct boundary controls in (1) via the *feedback law*

$$u(t) := f\big(y(t, x_0)\big), \quad t \in [0, T]. \tag{7}$$

Thus boundary controls $u(t)$ in (1) are fully determined via (7) by the choice of a *feedback function/regulator* $f = f(\xi)$. We say that such a function f defines a *feasible regulator* if it satisfies the summability condition (6), generates controls $u(t)$ by (7) belonging to the admissible set U from (3), and keeps the corresponding motions $y(t, x_0)$ of the parabolic system (1) within the prescribed constraint area (5) for *every* admissible perturbation $w \in W$ from (4). The set of all feasible regulators is labeled as $\mathcal{F}$.

To estimate the *quality* of feasible regulators $f = f(\xi)$, we consider the *cost functional*

$$J(f) := \max_{w \in W} \Big\{ \int_0^T \big|f\big(y(t, x_0)\big)\big| \, dt \Big\}, \tag{8}$$

which is an *energy-type* functional with respect to controls (7) in the boundary conditions of (1) subject to the symmetric constraints (3). The *maximum* operation in (8) reflects the required *control energy* needed to neutralize the adverse effect of the *worst perturbations* from (4) and to keep the state performance within the prescribed area (5).

The *minimax feedback control problem* (P) studied in this paper is as follows:

$$\text{minimize} \;\; J(f) \;\; \text{over} \;\; f \in \mathcal{F}, \tag{9}$$

i.e., to find an *optimal feedback control* $\overline{f} = \overline{f}(\xi)$ that minimizes the energy-type cost functional (8) over the set $\mathcal{F}$ of all feasible regulators, provided of course that $\mathcal{F} \neq \emptyset$.

It has been well recognized in control theory and applications that *feedback* control problems are the most challenging and important for any type of dynamical systems, while PDE systems provide additional difficulties and much less investigated in comparison, e.g., with the ODE dynamics; see more discussions and references in [11]. Furthermore, significant complications come from *pointwise/hard constraints* on control and (much more) state functions; the latter are of high nontriviality even for open-look control problems, especially in the case of Dirichlet boundary control (see, in particular, the afore-mentioned publications [3, 14, 11, 17]). We are not familiar with any device applicable to the problem (P) under consideration among a variety of approaches and results available in the theories of differential games, H_∞-control, Riccati's feedback synthesis, etc.; see, e.g., [2, 4, 5, 6] and the references therein.

In this paper we develop and significantly extend the approach to solving the feedback control problem (P), which was initiated in [9] for the case of the one-dimensional heat equation in (1); see also [10, 11, 13] for partial results reported for Dirichlet boundary controls of multidimensional parabolic systems and [15] for the cases of controls in the Neumann and mixed (Robin) boundary conditions.

Our approach is essentially based on certain underlying features of the parabolic dynamics, particularly on the *monotonicity property* of transients, which is eventually related to the fundamental *Maximum Principle* for parabolic equations; see Section 2. Due to this property and the specific structures of the cost functional (8) and boundary controls in (1), we are able to select the *worst perturbations* in the area (4) for the class of *nonincreasing* and *odd feedbacks* (7) and then to study the corresponding *open-loop* optimal control problem with *pointwise state constraints* as a reaction of the parabolic system to the worst perturbations. Using the *spectral* Fourier type representation of solutions to the parabolic system (1) and assuming the *positivity* of the *first eigenvalue* of the elliptic operator A in (2)—which is often the case— we observe the *dominance* of the *first term* in the exponential series representation of solutions to (1) as $t \to \infty$. This allows us to justify an *efficient approximation* of the open-loop optimal control problem for the parabolic system under consideration by that for the corresponding *ODE system* with state constraints on a sufficiently *large* time interval. Moreover, the approximating ODE optimal control problem is solved *exactly*—under some requirements on the initial data of (P)— by constructing *yet another approximation* of state constraints, employing the *Pontryagin maximum principle* that provides *necessary and sufficient* optimality conditions for the unconstrained approximating problems with both *bang-bang* and *singular modes* of optimal controls, and then by passing to the limit while meeting the state constraints. It happens in this way (due to specific features of the ODE problems under consideration approximating the parabolic dynamics) that the *state constraints* surprisingly occur to be a *regularization factor*, which simplifies the structure of optimal controls, especially when the time interval becomes bigger and bigger ($T \to \infty$)—this reveals the fundamental *turnpike property* of such dynamic systems expanding to the *infinite horizon*.

Thus using the ODE approximation described above, we justify an easily implemented *suboptimal* (or *near-optimal*) *structures* of optimal controls in both *open-loop* and *closed-loop* modes and then *optimize their parameters* along the *parabolic dynamics*. This allows us arrive at a *three-positional feedback regulator* $f = f(\xi)$ in (7) acting via the Dirichlet boundary conditions of (1) that ensures the required state performance (5) under the fulfillments of all the constraints in (P) for *every feasible perturbation* from (4) providing a *near-optimal response* of the closed-loop control system in the case of *worst perturbations*.

The feedback control design constructed in this way leads us to the *highly nonlinear* closed-loop system (1) and (7), where $f(\xi)$ is a *discontinuous* three-positional regulator. The system may loose *robust stability* (in the large) and maintain the state performance (5) in a unacceptable *self-vibrating regime*. Developing a *variational approach* to robust stability that reduces the stability issue to a certain open-loop optimal control problem on the *infinite horizon*, we establish efficient conditions for robust stability of the closed-loop system whenever $t \geq 0$ in terms of the initial data of problem (P) and parameters of the three-positional feedback regulator.

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Optimal Control of Conductive-Radiative Temperature Fields Generated via Electromagnetic Heating

Peter Philip

The presented optimal control problem is motivated by the aim of aiding and optimizing crystal growth methods, such as vapor growth (e.g. SiC, AlN) and Czochralski growth (e.g Si or GaAs). In particular, the research is directed at modeling and controlling conductive-radiative heat transfer, where the heat sources are generated via electromagnetic heating. In each of the abovementioned crystal growth situations, the quality of the as-grown crystal as well as the crystal's growth rate are strongly influenced by the field of the temperature gradient in the vicinity of the growing crystal's surface. Here, in the gas phase between the SiC crystal and the SiC powder source, it is desirable to have the radial temperature gradient close to zero and to have the vertical temperature gradient sufficiently large to guarantee a viable growth rate. This leads to the following optimization problem:

$$\text{(1)} \qquad \text{Minimize} \quad J(y) := \frac{1}{2} \int_{\Omega_{\mathrm{g}}} |\nabla y - z|^2,$$

where Ω_{g} is the domain of the gas phase, y denotes absolute temperature, and z is the desired field of the temperature gradient. The temperature needs to satisfy heat equations with appropriate interface and boundary conditions. Thus, the minimization (1) is subject to the PDE constraints

$$(2) \qquad -\mathrm{div}(\kappa_{\mathrm{s}}\nabla y) = f(y) \quad \text{in } \Omega_{\mathrm{s}}, \quad \text{and} \quad -\mathrm{div}(\kappa_{\mathrm{g}}\nabla y) = 0 \quad \text{in } \Omega_{\mathrm{g}},$$

where Ω_{s} is the domain of the solid parts of the growth apparatus, κ_{s} and κ_{g} are the corresponding thermal conductivities, and f denotes the heat sources due to the electromagnetic heating (the gas phase is assumed to be electrically insulating, such that there are no heat sources in Ω_{g}). On the outer boundary $\partial(\Omega_{\mathrm{s}} \cup \Omega_{\mathrm{g}})$, emission according to the Stefan-Boltzmann law provides the outer boundary condition $\kappa_{\mathrm{s}}\nabla y \bullet \vec{n}_{\mathrm{s}} - \sigma\,\epsilon\,(y_{\mathrm{ext}}^4 - y^4) = 0$, where $\vec{n}_{\mathrm{s}}$ is the outer unit normal vector to the solid, $\sigma \in \mathbb{R}^+$ represents the Boltzmann radiation constant, $\epsilon \in [0,1]$ represents the emissivity of the solid surface, and $y_{\mathrm{ext}} \in \mathbb{R}^+$ represents the external temperature. In contrast to the outer boundary condition, due to radiative interaction, the interface condition on $\Sigma := \Omega_{\mathrm{s}} \cap \Omega_{\mathrm{g}}$ is nonlocal:

$$(3) \qquad (\kappa_{\mathrm{g}}\nabla y)\restriction_{\Omega_{\mathrm{g}}} \bullet \vec{n}_{\mathrm{s}} + \sigma\,\epsilon\left(K(G^{-1}(\epsilon\,y^4)) - y^4\right) = (\kappa_{\mathrm{s}}\nabla y)\restriction_{\Omega_{\mathrm{s}}} \bullet \vec{n}_{\mathrm{s}} \quad \text{on } \Sigma,$$

where $G(\rho) := \rho - (1-\epsilon)\,K(\rho)$, $K(\rho)(x) := \int_\Sigma \Lambda(x,\tilde{x})\,\omega(x,\tilde{x})\,\rho(\tilde{x})\,\mathrm{d}\tilde{x}$,

$$\Lambda(x,\tilde{x}) := \begin{cases} 0 & \Sigma \cap]x,\tilde{x}[\neq \emptyset, \\ 1 & \Sigma \cap]x,\tilde{x}[= \emptyset, \end{cases} \qquad \omega(x,\tilde{x}) := \frac{\big(\vec{n}_{\mathrm{s}}(\tilde{x}) \bullet (x-\tilde{x})\big)\big(\vec{n}_{\mathrm{s}}(x) \bullet (\tilde{x}-x)\big)}{\pi\big((\tilde{x}-x) \bullet (\tilde{x}-x)\big)^2}.$$

The heat sources $f(y)$ are determined from Maxwell's equations. Assuming that all domains are axisymmetric with a sinusoidal alternating voltage imposed in N disconnected rings leads to the following simplified model, based on ideas from [3]. The heat sources $f(y)$ can be computed from the current density j via $f(y) = |j(y)|^2/(2\,\sigma_{\mathrm{c}}(y))$, where σ_{c} denotes the electrical conductivity, and, using cylindrical coordinates (r,ϑ,z), $j = -i\omega\,\sigma_{\mathrm{c}}(y)\,\phi + \frac{\sigma_{\mathrm{c}}(y)\,u_k}{2\pi r}$ in the k-th coil ring, $j = -i\omega\,\sigma_{\mathrm{c}}(y)\,\phi$ in all other conducting materials, where i denotes the imaginary unit, ω is the angular frequency, u_k, $k = 1,\dots,N$, are the prescribed total voltages in the respective rings, and ϕ is a complex-valued magnetic scalar potential, satisfying

$$(4a) \qquad -\nu\mathrm{div}\frac{\nabla(r\phi)}{r^2} = 0 \quad \text{in the gas phase,}$$

$$(4b) \qquad -\nu\mathrm{div}\frac{\nabla(r\phi)}{r^2} + \frac{i\,\omega\sigma_{\mathrm{c}}(y)\phi}{r} = \frac{\sigma_{\mathrm{c}}(y)\,u_k}{2\pi r^2} \quad \text{in the } k\text{-th coil ring,}$$

$$(4c) \qquad -\nu\mathrm{div}\frac{\nabla(r\phi)}{r^2} + \frac{i\,\omega\sigma_{\mathrm{c}}(y)\phi}{r} = 0 \quad \text{in other conducting materials,}$$

where ν denotes the magnetic reluctivity, i.e. the reciprocal of the magnetic permeability. The system (4) is completed by the following interface and boundary conditions: $((\nu\restriction_{\mathrm{M}_1}/r^2)\nabla(r\phi)\restriction_{\mathrm{M}_1}) \bullet \vec{n}_{\mathrm{M}_1} = ((\nu\restriction_{\mathrm{M}_2}/r^2)\nabla(r\phi)\restriction_{\mathrm{M}_2}) \bullet \vec{n}_{\mathrm{M}_1}$ on interfaces between materials M_1 and M_2, where $\restriction$ denotes the restriction to the respective material, and $\vec{n}_{\mathrm{M}_1}$ denotes the outer unit normal vector to M_1. It is also assumed that ϕ is continuous throughout the whole domain and that $\phi = 0$ both on the symmetry axis $r = 0$ and sufficiently far from the growth apparatus.

As the quantities facilitating the control are actually the voltages u_k, letting $u := (u_1, \dots, u_k)$, (1) should be rewritten in the form

$$\text{(5)} \qquad \text{Minimize} \quad \tilde{J}(u) := J\big(y(u)\big) = \frac{1}{2}\int_{\Omega_g} |\nabla y(u) - z|^2.$$

So far, (5) constitutes a complicated, but finite-dimensional optimization problem. However, the special challenge now arises due to the following *pointwise state constraints* (6) that arise due to the fact that the apparatus would be destroyed for $y > y_{\max}$ and, to guarantee the growth of the desired crystal modification, one needs to impose a certain temperature range on the crystal's growth surface Γ_c.

$$\text{(6)} \qquad y \le y_{\max} \text{ on } \Omega_s, \quad y_{c,\min} \le y \le y_{c,\max} \text{ on } \Gamma_c.$$

Finally, one has the control constraints $0 \le u \le U_{\max}$.

In [2], the above problem was treated without the pointwise state constraints and under the assumption that the heat sources f could be controlled directly, completely disregarding the equations for ϕ, establishing first-order necessary optimality conditions. The full system, as described above, has been optimized numerically in [1]. The goal now lies in also establishing theoretical results (e.g. first-order necessary optimality conditions) for the full system.

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Unique Continuation Property Near a Corner and Its Fluid-Structure Controllability Consequences

Jean-Pierre Puel
(joint work with Axel Osses)

This is a work in collaboration with Axel Osses which is published in [3].
The problem we are going to study came from a question arising in approximate controllability for a linear model of fluid-structure interaction. Let us describe this problem first.

Let T be a positive number and Ω be a bounded open set of $\mathbb{R}^2$ with boundary Γ. This boundary is made of two parts : a rigid part Γ_R and an elastic part Γ_E. Inside Ω we have a viscous incompressible fluid whose motion is governed by a Stokes equation. On the rigid part of the boundary, the fluid obeys the usual no-slip boundary condition which is a homogeneous Dirichlet condition for the velocity. The elastic part of the boundary is an elastic membrane or a plate, which is coupled with the fluid movement of course and on which we can act with a control (distributed on the boundary). This model can be described by the following system.

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = 0 \quad \text{in } \Omega \times (0,T) \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T) \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_R \times (0,T) \tag{3}$$

$$u = \frac{\partial \eta}{\partial t} n \quad \text{on } \Gamma_E \times (0,T) \tag{4}$$

$$u(0) = u_0 \quad \text{in } \Omega \tag{5}$$

$$\frac{\partial^2 \eta}{\partial t^2} + \mathbf{B}\eta = -\sigma(u,p)n \cdot n + h \quad \text{on } \Gamma_E \times (0,T) \tag{6}$$

$$\eta(t) \in H_0^2(\Gamma_E) \cap L_0^2(\Gamma_E) \quad \text{a.e. in } (0,T) \tag{7}$$

$$\eta(0) = \eta_0,\ \frac{\partial \eta}{\partial t}(0) = \eta_1 \quad \text{on } \Gamma_E, \tag{8}$$

where $\mathbf{B}$ is a selfadjoint differential operator which is uniformly elliptic in $H_0^2(\Gamma_E)$, and where h is the control function.

Using standard arguments it is easy to show that approximate controllability for this system is equivalent to a unique continuation property for the adjoint system. The adjoint system has the following form

$$-\frac{\partial \mathbf{z}}{\partial t} - \nu\Delta \mathbf{z} + \nabla q = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{9}$$

$$\operatorname{div} \mathbf{z} = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{10}$$

$$\mathbf{z} = 0 \quad \text{on} \quad \Gamma_R \times (0,T) \tag{11}$$

$$\mathbf{z} \cdot \tau = 0 \quad \text{on } \Gamma_E \times (0,T) \tag{12}$$

$$\mathbf{z} \cdot \mathbf{n} = \frac{\partial \varphi}{\partial t} \quad \text{on} \quad \Gamma_E \times (0,T) \tag{13}$$

$$\mathbf{z}(T) = \mathbf{z}_T \quad \text{in} \quad \Omega \tag{14}$$

$$\frac{\partial^2 \varphi}{\partial t^2} + \mathbf{B}\varphi - \sigma(\mathbf{z},q)\mathbf{n} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_E \times (0,T) \tag{15}$$

$$\varphi(t) \in H_0^2(\Gamma_E) \quad \text{a.e. in} \quad (0,T) \tag{16}$$

$$\varphi(T) = \varphi_T,\ \frac{\partial \varphi}{\partial t}(T) = \psi_T \quad \text{on} \quad \Gamma_E. \tag{17}$$

If

$$\frac{\partial \varphi}{\partial t} = 0 \quad \text{on} \quad \Gamma_E \times (0,T)$$

which is the additional condition occuring in the unique continuation property, then

$$z \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_E,$$

and we have in fact

$$z = 0 \ \text{ on } \ \Gamma.$$

Then the system decouples and we obtain from the second equation

$$q = \text{Constant} \quad \text{on} \quad \Gamma_E.$$

The question is then : does this imply that

$$z = 0 \quad \text{and} \quad q = \text{Constant} \quad \text{in} \quad \Omega \times (0,T)?$$

Developping z on the orthonormal basis of eigenfunctions for the Stokes operator and using analyticity in time, it turns out that the question can be asked for eigenfunctions of the Stokes operator.
Let (u,p) be solution of

$$-\Delta u + \nabla p = \lambda u \quad \text{in } \Omega, \tag{18}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega, \tag{19}$$

$$u = 0, \quad \text{on } \Gamma, \tag{20}$$

such that

$$p = \text{constant} \quad \text{on } \Gamma_E. \tag{21}$$

Does this imply

$$u = 0 \text{ and } p = \text{constant} \quad \text{in} \quad \Omega?$$

As we are in dimension 2, the Stokes problem is equivalent to the following problem of order 4 by setting

$$u = \nabla^{\perp} w$$

$$\Delta^2 w = -\lambda \Delta w \quad \text{in } \Omega \tag{22}$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \tag{23}$$

$$\frac{\partial \Delta w}{\partial n} = 0 \quad \text{on } \Gamma_E \text{ (additional condition)} \tag{24}$$

The unique continuation property (UCP) is now : does this imply $w = 0$?

The case of a disc $B(0,1)$ gives a counterexample.
Take $\varphi \neq 0$ such that

$$-\Delta \varphi = \lambda \varphi \quad \text{in} \quad \Omega = B(0,1)$$
$$\varphi = 0 \quad \text{on } \Gamma$$
$$\varphi = \varphi(r).$$

Then if

$$u_1 = -\varphi(r) \sin\theta$$
$$u_2 = \varphi(r) \cos\theta$$

$u = (u_1, u_2)$ is solution of the Stokes eigenvalue problem with $p = 0$.

A conjecture is then : The disc the only domain for which we have a non zero eigenfunction. This is related to the Schiffer conjecture which is set for the Laplace operator.

When Γ is analytic, having the aditional condition on Γ_E or on the whole of Γ is the same. When Γ is not analytic, the situation is different and a subconjecture would be : if Γ is not analytic, then the UCP is true.

We show here that when Γ_R and Γ_E make an angle (different from $\frac{3\pi}{2}$) then UCP is valid.

1. Main result

Consider a circular sector of $\mathbb{R}^2$ centered at the origin described in polar coordinates

$$G = \{(r,\theta),\ 0 < r < r_0,\ 0 < \theta < \theta_0\}$$

Let Ω be a lipschitz bounded open subset of $\mathbb{R}^2$ with a corner at the origin such that

$$\Omega \cap B(0, r_0) = G.$$

We define

$$\Gamma_E = \{(r,0),\ 0 < r < r_0\}, \quad \Gamma_R = \{(r,\theta_0),\ 0 < r < r_0\}.$$

Our result for the biharmonic problem is the following (we have a similar result for the Laplace operator which is easier to prove).

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a lipschitz bounded subset with a corner of angle $0 < \theta_0 < 2\pi$ at the origin and assume that*

$$\theta_0 \neq \pi, \quad \theta_0 \neq \frac{3\pi}{2},$$

then any weak solution $w \in H^2(\Omega)$ of the problem

$$\Delta^2 w = -\lambda \Delta w \quad \text{in } \Omega \tag{25}$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \tag{26}$$

$$\frac{\partial \Delta w}{\partial n} = 0 \quad \text{on } \Gamma_E \tag{27}$$

vanishes in Ω.

Remark 2. *The result cannot be proved by a local argument as shown by the counterexample for the disc.*

The proof requires several steps.

In a first step, we show that any solution w of (25)-(27) is C^∞ near the origin. This is done by a careful study of the possible singularity (see [1]) which we prove to be zero when the additional condition is fulfilled.

In a second step we perform a power series expansion of the solution near the origin, and studying an infinite number of linear systems, we prove that the Taylor expansion of the solution at the origin is zero, so that the origin is a zero of infinite oder.

In a third step, using a result of Kondratiev, Koslov and Mazya ([2]) we show that the only possibility for w is to vanish everywhere, which is the announced result.

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Null Controllability of a Heat – Solid Structure Model

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(joint work with M. Vanninathan)

We are interested in the null controllability of a heat-solid structure model which is a simplified version of a more realistic fluid-solid structure coupled system introduced in [2] and [3]. Our simplified model is written down below.

Let $\mathcal{O}$ be a simply connected bounded domain in $\mathbb{R}^2$ with a regular boundary Γ_e, and let S be a simply connected domain in $\mathcal{O}$, of regular boundary Γ_i. We suppose that $\bar{S} \subset \mathcal{O}$, and we set $\Omega = \mathcal{O} \setminus \bar{S}$. Thus $\Gamma = \Gamma_e \cup \Gamma_i$ is the boundary of Ω and $\Gamma_e \cap \Gamma_i = \emptyset$. We denote by n the unit normal on Γ pointed outward to Ω. We consider a heat conducting medium occupying the open set Ω and a solid (which models an oscillator) occupying the closed set $\bar{S}$ described by the following coupled system:

$$
\begin{aligned}
&\phi' - \Delta\phi = f && \text{in } Q,\\
&\phi = 0 && \text{on } \Sigma_e,\\
&\phi = r' \cdot n && \text{on } \Sigma_i,\\
&\phi(0) = \phi^0 && \text{in } \Omega,\\
&r'' + r = -\int_{\Gamma_i} \partial_n \phi\, n && \text{in } (0,T),\\
&r(0) = r^0 \quad \text{and} \quad r'(0) = r^1 && \text{in } \mathbb{R}^2.
\end{aligned} \tag{1}
$$

In this setting $Q = \Omega \times (0,T)$, $T > 0$, $\Sigma_e = \Gamma_e \times (0,T)$, $\Sigma_i = \Gamma_i \times (0,T)$. The function ϕ represents the temperature in the medium Ω. The vector $r \in \mathbb{R}^2$ is the displacement of the solid part S which performs a simple harmonic motion with an external forcing term which depends on ϕ. Apart from this, we see that the boundary condition on Σ_i also couples ϕ with r. This model is an example of a coupling between a parabolic equation and (a finite dimensional) hyperbolic one. The forcing term f in the conducting medium is a function used to control the heat-solid structure system. Let ω be a nonempty open subset of Ω such that $\omega \subset\subset \Omega$. We will take the forcing term f in the form $f = u(x,t)\chi_{\omega\times(0,T)}(x,t)$ where u is a function defined in Q, and $\chi_{\omega\times(0,T)}$ is the characteristic function of $\omega \times (0,T)$. With a suitably chosen control of the above form, we establish null controllability of the above system.

In our model the domain occupied by the heat medium is fixed. This means that the displacement of the structure is an infinitesimal displacement. This kind of assumption is meaningful in some fluid-structure interaction problems (see e.g. [2, 3]). Moreover, controllability results for fluid-structure interaction problems sometimes rely on controllability results for a linearized model, and for the linearized model the domain occupied by the fluid is fixed (see [8]).

Comparing Carleman estimates for the heat equation or for the linearized Navier-Stokes equations with the ones obtained for fluid-solid coupled systems, we notice that there are additional difficulties due to the interface between the solid and the fluid. To overcome this kind of difficulty it is assumed in [8] that the solid is a disk, whereas in [1] some symmetry assumption is assumed on the solid [1, Assumption (1.9)]. In our case it is not necessary to make such an assumption. In [9] we prove that the solution to equation (1) (or similarly the solution to an adjoint system) obeys the following Carleman inequality

(2)

$$\begin{aligned} & s\,\lambda^2 \int_Q e^{-2s\beta}\,\frac{e^{\lambda\eta}}{t(T-t)}|\nabla\phi|^2 + s^3\,\lambda^4 \int_Q e^{-2s\beta}\,\frac{e^{3\lambda\eta}}{t^{3k}(T-t)^{3k}}|\phi|^2 \\ & + \int_0^T e^{-2s\beta}|_{\Gamma_i}\left(|r''|^2+|r|^2\right) + s^3\,\lambda^3 \int_0^T \frac{e^{3\lambda\eta}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}} e^{-2s\beta}|_{\Gamma_i}|r'|^2 \\ & \leq C\left\{\int_Q e^{-2s\beta}\,|f|^2 + s^3\lambda^4 \int_{\omega\times(0,T)} e^{-2s\beta}\frac{e^{3\lambda\eta}}{t^{3k}(T-t)^{3k}}|\phi|^2 + \int_0^T e^{-2s\beta}|_{\Gamma_i}|r|^2\right\}. \end{aligned}$$

Here $\beta(x,t) = \frac{e^{\lambda K_1} - e^{\lambda\eta(x)}}{t^k(T-t)^k}$ with $k \geq 2$, $K_1 > \max_{x\in\overline{\Omega}}|\eta(x)|$ and η is a positive function whose critical points are located in ω, which is constant on Γ and whose normal derivative at Γ is non positive and equal to -1 at Γ_i(see [9]). By comparing the terms corresponding to the same weights in $s^n\,\lambda^m$, we can notice that Carleman estimate (2) is very similar to the one established in [8, estimate (3.34)], except for $\int_0^T e^{-2s\beta}|_{\Gamma_i}(|r''|^2+|r|^2)$ in the LHS in (2) and for $\int_0^T e^{-2s\beta}|_{\Gamma_i}|r|^2$ in the RHS in (2), for which there is no analogue in [8, estimate (3.34)]. The terms $|r'|^2$ and $|r|^2$ appearing in integrals the LHS of (2) are used in an essential way to eliminate $\int_0^T e^{-2s\beta}|_{\Gamma_i}|r|^2$ in the RHS. The new inequality obtained for the solution to the adjoint system is next used to prove our null controllability result.

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Regularization and Discretization of State Constrained Optimal Control Problems

Arnd Rösch
(joint work with Klaus Krumbiegel, Christian Meyer)

Optimal control problems with pointwise state and control constraints are a field of very active research. However, there are lot of interesting directions: The direct discretization of such problems is analyzed in [2], [6].

The weak smoothness properties of optimal solutions and certain numerical arguments were the reason to study also regularization methods. We mention here Path-following methods [3], Lavrentiev regularization [7], source representation techniques [8] and the virtual control concept [5].

The discretization error of a regularized problem is estimated in [4]. A tuning of regluarization and discretization parameters is proposed in [1].

Here, we will study simultaneously the regularization error and the discretization error for the virtual control approach. Let us discuss the problem

$$(\mathcal{P})\qquad \begin{array}{llll} \min J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} & & & \\ \text{subject to} \quad Ay+y=0 & & & \text{in } \Omega \\ \qquad \partial_n y = u & & & \text{on } \Gamma \\ \qquad y \ge y_c & & \text{a.e.} & \text{in } \Omega' \\ \qquad a \le u \le b & & \text{a.e.} & \text{on } \Gamma. \end{array}$$

where A denotes a second order elliptic operator. We assume that the inner subdomain Ω' has a positive distance to the boundary Γ of the domain $\Omega \in R^d$, $d = 2, 3$.

Introducing a virtual control v, we obtain the regularized problem

$$(\mathcal{P}_\varepsilon)\qquad \begin{array}{llll} \min J^v_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} & & & \\ \text{subject to} \quad Ay+y=\phi(\varepsilon)v & & & \text{in } \Omega \\ \qquad \partial_n y = u & & & \text{on } \Gamma \\ \qquad y \ge y_c - \xi(\varepsilon)v & & \text{a.e.} & \text{in } \Omega' \\ \qquad a \le u \le b & & \text{a.e.} & \text{in } \Gamma. \end{array}$$

Theorem 1. *The following error estimate is satisfied*

$$\|\bar{u} - \bar{u}_\varepsilon\| \leq c \left(\frac{\xi(\varepsilon) + \phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{1}{d+1}}.$$

Theorem 1 delivers an error estimate for the regularization error. Next, we discuss a semidiscretization approach: The partial differential equation is discretized by conform finite elements. The control is not discretized. Due to the optimality condition, the optimal control is also a finite element function if the control constraints are completely inactive. Active control constraints leads to additonal kinks in the control.

The numerical analysis benefits essentially from the fact that the inner subdomain Ω' has a positive distance to the boundary Γ. Usually the regularity of the control is limited by the regularity of the adjoint state. The regularity of the adjoint state is determined by the measure part of the Lagrange multiplier associated to the state constraints. This is not the case for our problem. The regularity of the adjoint state near the boundary is not influenced by the measure part of the Lagrange multiplier. Moreover, we are able to show that the discretized adjoint states and the resulting controls are bounded in higher norms. This is used to show the final result:

Theorem 2. *The error estimate*

$$\|\bar{u} - \bar{u}^h_\varepsilon\| \leq c \left(\frac{\xi(\varepsilon) + \phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{1}{d+1}} + ch|\log h|^{1/2}.$$

is valid where h denotes the mesh size of the finite element discretization.

The derived error estimate decouples the discretization and the regularization error. The regularization via the virtual controls has the following advantages:

- uniqueness of adjoint variables,
- lower condition numbers of the involved matrices,
- easy implementation in a primal dual active set strategy.

In our numerical approach we choose the regularization parameters (functions) in such a way that the two errors are balanced.

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Modal Control for Partial Differential Equations

Thomas I. Seidman

We consider systems governed by differential equations whose evolution in time can switch, necessarily discontinuously, between distinct *modes.* [For a discussion of relevant modelling considerations, see, e.g., [3].] The evolving state of such a system is then the pair $[x, m]$ where $x = x(t)$ denotes the usual (PDE) state, now referred to as the *continuous component* and the modal index $\mu = \mu(t) \in \mathcal{M}$ is referred to as the *discrete component* of the state. For present purposes we think of the *switching function* $\mu(\cdot)$ as the control for this system. [For technical reasons it will actually be important to think of μ here as the sequence of modes $(\mu_0, \mu_1, \ldots)$ together with the lengths of the *interswitching intervals*, accepting the possibility that one or more of these lengths may degenerate to 0 while retaining this sequence — important, since applications may require that $\mathcal{M}$ be viewed as the nodes of a directed graph. We do, however, avoid *Zeno points*, requiring that there be only finitely many modal transitions in any finite time interval.]

For ODE-governed problems such *hybrid systems* have recently been the object of considerable investigation and we argue here that such questions are also significant for PDE-governed problems. While other considerations may also be of interest — e.g., stabilization to a (small?) global attractor — we here envision three canonical results:

Theorem 1: *Under appropriate hypotheses, treating $\mu(\cdot)$ as data, the system will be well-posed in some suitable sense.*

Introducing a suitable cost functional and treating $\mu(\cdot)$ as an open-loop control,
Theorem 2: *Under appropriate hypotheses there exists an optimal control μ, minimizing the cost.*

Introducing an appropriate notion of *feedback* based on suitable sensors
Theorem 3: *Under appropriate hypotheses the feedback controlled system will be well-posed in some suitable sense.*

As a specific example, we consider a class of problems involving transport and reaction on a graph, for which one motivating example might be a chemical reaction process with modal transitions consisting of the opening or closing of valves, so regulating the graph geometry, or of turning pumps on or off, or ... This example represents joint work with G. Leugering and F. Hante [2]. [Related examples, although not covered by the currently existing results, might include gas dynamics (a network of gas pipelines for which one turns compressors on and off at certain nodes) or traffic flow (a network of streets or highways with control by signal lights).]

The domain of our example is a graph $\mathfrak{G}$ consisting of nodes (vertices) $v_i \in \mathfrak{V}$ and edges $e_j \in \mathfrak{E}$. In each mode m we have, along each edge, a reaction transport equation of the form

$$u_t + (au)_s = f(\cdot, u) \tag{1}$$

for some concentration, noting that both the flow velocity a and the reaction rate f may depend on t and on the spatial variable s along the edge as well as on the particular edge and on the mode m. For any of a variety of spaces for u we may take our notion of solution to be given by the classical *method of characteristics*, cf., e.g., [1]. In addition, we have, at each node, an allocation of incoming fluxes to edges requiring flux data outgoing from that node; this allocation will in general be mode-dependent and may either be conservative, satisfying Kirchhoff's Law, or alternatively may involve storage in a buffer at the node. [If the latter, we may also have an ODE for reaction within the buffer.]

Of particular interest is the interplay between the requirements of our theorems and the choices of spaces for the state u. For Theorem 3 the modal *switching rules* defining the feedback must, at each moment of time, select an action — either continuing in the current mode or switching to a next mode — in a way depending on the current mode and the sensor values. We wish to take as sensors a finite set of evaluations of u at points interior to some edges so our spaces must allow for such point evaluations to be meaningful. We might like, then, to work with spaces of continuous functions, but the possibility of modal switching can be expected to introduce discontinuities even for smooth initial data and these discontinuities will then propagate so we work with *piecewise continuous functions* as solutions. [An alternative might be functions of bounded variation.] It is, then, necessary to consider an appropriate notion of convergence of such functions so as to ensure the avoidance of Zeno points while preserving some suitable interpretation of well-posedness. This requires us to formulate our notion of 'piecewise continuous functions' so as to permit degenerate 'intervals of continuity,' much as for μ above. Further, the relevant sense of well-posedness for Theorems 1,3 becomes an upper semicontinuity of the solution sets: the limit of solutions is a solution.

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Modeling and Shape Sensitivity Analysis for Compressible, Stationary Navier-Stokes Equations

Jan Sokolowski
(joint work with Pavel I. Plotnikov)

Modeling of stationary Fourier-Navier-Stokes equations is considered in [6]. It is shown, that the model is well-posed, there exist weak solutions of boundary value problems posed in bounded domains, subject to inhomogeneous boundary conditions. The first boundary value problem for elliptic-hyperbolic system of equations is analysed in [6]. The shape sensitivity analysis is performed in [5] for Navier-Stokes boundary value problems, in the case of small perturbations of the so-called *approximate solutions*. The approximate solutions are determined from the linear Stokes problem. The small perturbations are given by solutions to nonlinear boundary value problem [1]. The uniqueness of small solutions for the nonlinear problem is shown. The differentiability of solutions with respect to the coefficients of differential operators is obtained, which leads to the shape differentiability of the drag functional. The shape gradient of the drag functional is derived in the classical and useful for computations form, an appropriate adjoint state is introduced to this end. The shape derivatives of solutions to the Navier-Stokes equations are given by smooth functions, however the shape differentiability can be shown only in a weak norm. The proposed method of shape sensitivity analysis is general, and can be used to establish the well-posedness for distributed and boundary control problems as well as for inverse problems in the case of the state equations in the form of compressible Navier-Stokes equations. The differentiability of solutions to the Navier-Stokes equations with respect to the data leads to the first order necessary conditions for a broad class of optimization problems.

Shape optimization for compressible Navier-Stokes equations is important for applications and it is investigated from numerical point of view in the field of scientific computations, however the mathematical analysis of such problems is not available in the existing literature. One of the reasons is the lack of the existence results for inhomogeneous boundary value problems for such equations posed in bounded domains. The results established in the paper give in particular the first order necessary optimality conditions for a class of shape optimization problems for compressible Navier-Stokes equations.

Our results for the Fourier-Navier-Stokes and the Navier-Stokes boundary value problems can be described according to the following plan.

Mathematical modeling, well posedness of solutions to the boundary value problems. The most general setting for such analysis is introduced in [6] and covers the Fourier-Navier-Stokes boundary value problems in bounded domains with inhomogeneous boundary conditions. We point out, that in [2] the diatomic gases are considered and the existence of solutions for the mathematical models is shown. The shape differentiability of solutions is proved in [5] for the Navier-Stokes boundary value problems in bounded domains with inhomogeneous boundary conditions.

The drag functional is minimized, and the same approach can be used for more general problems of shape optimization including the lift maximization and the optimization of the density distribution at the outlet of the flow domain.
Framework for the shape sensitivity analysis. The new results are derived for small perturbations of the approximate solutions to compressible Navier-Stokes equations. In [5] the shape sensitivity analysis is performed with respect to the adjugate matrix defined for the Jacoby matrix of a given domain transformation mapping. Our approach allows for substantial simplification of the sensitivity analysis compared to the existing results obtained in the case of incompressible fluids by using the velocity or perturbation of identity methods of shape sensitivity analysis.
Material derivatives of solutions to compressible Navier-Stokes equations in the fixed domain setting are obtained in [5]. The shape differentiability of solutions for compressible Navier-Stokes boundary value problems is shown in weak norms i.e., in the norms of the negative, fractional Sobolev spaces for the hyperbolic component of the boundary problem i.e., the transport equation, however the obtained material derivatives are sufficiently regular in order to obtain the shape gradients given by some functions, and such a result is actually very useful for possible application of numerical methods of shape optimization of the level set type - since the shape gradients are the coefficients of the non linear hyperbolic equation.
Shape gradient of the drag functional is determined by means of the complicated adjoint state, and we observe that the expression obtained for the gradient is sufficiently smooth and given by a function, it implies that e.g., the level set method can be employed for numerical solution of the shape optimization for the drag minimization.
Many related results on the resolution of compressible Navier-Stokes equations can be found in [1]-[6].

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Towards Highly Parallel Mesh Adaptation for Large-Scale PDE Applications

Georg Stadler
(joint work with Carsten Burstedde, Omar Ghattas, Tiankai Tu, Lucas C. Wilcox)

Many of the recent supercomputers are built by connecting large numbers of standard processors with a fast network. These machines will allow us to solve systems of partial differential equations (PDEs) with the high accuracy that is often needed in applications. However, new scalable algorithms and implementations are required to make effective use of these systems. The main focus of this talk is to present our recent effort to develop an approach for parallel mesh adaptation that works efficiently on tens of thousands of processors.

Already over the last 20 years parallelization of PDE solvers has been an active field of research (see e.g., [1]). There are essentially two approaches: structured adaptive mesh refinement (SAMR) uses a hierarchy of logically rectangular grid patches. Historically, it has mainly been used for finite volume/finite difference simulations for hyperbolic equations. Unstructured adaptive mesh refinement (UAMR) allow for more general, usually conforming meshes and can handle PDEs in variational form. Our approach can be seen as in between SAMR and UAMR: We use hexahedral meshes with hanging nodes; the meshes are based on parallel octrees to store the mesh information, i.e., on a tree structure that is always distributed among all the processors. A major problem in parallel adaptive mesh coarsening and refinement is the redistribution of the mesh after an adaption step – this problem is known as the load balancing problem. We use a space filling curve ordering to load balance the tree that underlies the mesh. Our discretization is based on finite elements, where hanging nodes are eliminated by algebraic constraints to guarantee continuity.

Our driving applications for mesh adaption are geophysical systems governed by PDEs, in particular the simulation of mantle convection. Here, refinement and coarsening are essential to resolve the varying spatial scales. Convection in the mantle is the principal control on the thermal and geological evolution of the Earth. A simplified model for mantle convection is given by a time-dependent advection-diffusion equation, coupled with a stationary incompressible Stokes equation:

$$\text{(AD)} \qquad \frac{\partial T}{\partial t} + u \cdot \nabla T - \nabla^2 T - \gamma = 0,$$

$$\text{(S1)} \qquad \nabla \cdot \left[\eta(T) \left(\nabla u + \nabla^\top u\right)\right] - \nabla p + Ra\, T e_r = 0,$$

$$\text{(S2)} \qquad \nabla \cdot u = 0.$$

Here, T, $u = (u_1, u_2, u_3)$ and $\eta(T)$ denote temperature, velocity and viscosity, respectively. Moreover, $\gamma \geq 0$ is the internal heating production rate, Ra the Rayleigh number and e_r the radial direction. Typically, $\eta(T) = e^{-ET}$ with $E \sim 5 - 12$ leading to a highly varying viscosity in the Stokes equation (S1), (S2). While currently we are mainly focusing on solving the above forward problem, ultimately are interested in the inverse problem of recovering the temperate distribution millions of years ago from today's measurements.

Resolving mantle convection phenomena at faulted plate boundaries requires $\sim$ 1km resolution. On a uniform mesh of the Earth, this results in $\sim 10^{12}$ elements, well beyond the reach of even next generation supercomputers. Using adaptive mesh refinement we expect a 3 orders of magnitude reduction in the number of elements, making the solution of these problems feasible.

For our performance analysis we split the governing equations for mantle convection in the advection-diffusion equation (AD) with u assumed to be given and in the Stokes equation (S1), (S2) with given temperature T. We discretize the equations with trilinear finite elements and use streamline upwind / Petrov–Galerkin (SUPG) stabilization for (AD) and polynomial pressure stabilization for the Stokes equation [2]. For the solution of the Stokes equation we employ the preconditioned minimal residual method (MINRES), where as preconditioner we use one V-cycle of algebraic multigrid. Since in our application (AD) is advection-dominated, we use an explicit time stepping method for its solution. Based on an error indicator, the mesh is dynamically adjusted.

We present weak scaling results for our algorithms. Weak scaling refers to simultaneously increasing the number of processors and the problem size such that the number of finite elements per processor remains approximately constant. For the solution of the advection-diffusion equation (AD) on dynamically adjusting meshes, the time per time step increases only by 20% as we scale up from one to 16 000 processors. Moreover, only less than 5% of total runtime is needed for mesh coarsening and refinement. Solving the Stokes equation, we observe an increase in time by a factor of only 2.5 as we go from 1 to 4000 processors (and increase the problem size by the same factor).

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A New Method for the Solution of Multi-Disciplinary Free Material Optimization Problems

Michael Stingl
(joint work with M. Kocvara, G. Leugering)

Given a design body Ω with a Lipschitz boundary Γ, which is clamped on a part $\Gamma_0 \subset \Gamma$ and subjected to a set of external load functions $f_k \in L_2(\Gamma_1)^3,\ k \in \mathbb{K} = \{1, 2, \ldots, K\},\ \Gamma_1 \subset \Gamma$, the *worst-case multiple-load Free Material Optimization problem* can be stated as:

$$\text{(1)} \qquad \inf_{E \in \mathbb{E}} \max_{k \in \mathbb{K}} c_k(E) \quad \text{subject to} \quad v(E) \leq V\,.$$

Here the *set of admissible materials* is given as

$$\mathbb{E} := \left\{ E \in L^\infty(\Omega)^{6\times 6} \mid E = E^\top, E \succeq \underline{\rho} I, \mathrm{Tr}(E) \leq \overline{\rho} \text{ a. e. in } \Omega \right\},$$

$v(E)$ measures the total stiffness of the body which is limited by the bound V and the compliance functionals $c_k(E),\ k \in \mathbb{K}$, are defined by the formula

$$c_k(E) = \int_{\Gamma_1} f_k(x)^\top u_{E,k}(x)\ \mathrm{d}x,$$

where $u_{E,k}$ are the unique solutions of the boundary value problem of linear elasticity for all $k \in \mathbb{K}$. For more details about problem (1), such as existence of optimal solutions and convergent discretization schemes, the interested reader is referred to [1, 2] and the references therein. During the recent years, the authors tried to extend the basic problem statement by various constraints, such as eigenfrequency constraints or constraints on displacements or stresses; see [4, 3]. These additional constraints complicate the numerical treatment of the problem considerably. Discretization by the finite element method leads to a series of large-scale nonlinear semidefinite programming problems, which are beyond reach of all existing semidefinite programming solvers. Only recently the authors proposed a new first order method based on the concept of sequential convex programming (see [5]), which is able to solve the discrete counterpart of problem (1) with a realistic number of load cases and a sufficiently fine finite element mesh. The method is based on the approximation of convex functionals ϕ mapping from $\mathbb{S} = \mathbb{S}^{d_1} \times \mathbb{S}^{d_2} \times \ldots \times \mathbb{S}^{d_m}$ to $\mathbb{R}$, where $\mathbb{S}^{d_i}$ is the space of symmetric matrices of order d_i $(i = 1, 2, \ldots, m)$ by a sequence of strongly convex and separable functions of the form

$$
\begin{aligned}
\phi_{\bar{Y}}^{L,U,\tau}(Y) \quad := \quad & f(\bar{Y}) + \\
& \sum_{i=1}^{m} \left\langle \nabla_{+}^{i}\phi(\bar{Y}), (U_i - \bar{Y}_i)(U_i - Y_i)^{-1}(U_i - \bar{Y}_i) - (U_i - \bar{Y}_i) \right\rangle_{\mathbb{S}^{d_i}} - \\
& \sum_{i=1}^{m} \left\langle \nabla_{-}^{i}\phi(\bar{Y}), (\bar{Y}_i - L_i)(Y_i - L_i)^{-1}(\bar{Y}_i - L_i) - (\bar{Y}_i - L_i) \right\rangle_{\mathbb{S}^{d_i}} + \\
& \sum_{i=1}^{m} \tau_i \left\langle (Y_i - \bar{Y}_i)^2, (U_i - Y_i)^{-1} + (Y_i - L_i)^{-1} \right\rangle_{\mathbb{S}^{d_i}} , \qquad (2)
\end{aligned}
$$

which are first order approximations of ϕ in the neighborhood of the point $\bar{Y} \in \mathbb{S}$. Here $Y = (Y_1, Y_2, \ldots, Y_m)$, L_i and U_i are the so called lower and upper asymptotes guaranteeing $L_i \preceq Y_i \preceq U_i$, $\nabla_{+}^{i}\phi(\bar{Y})$ and $\nabla_{-}^{i}\phi(\bar{Y})$ are projections of the partial derivatives of f w.r.t. Y_i on the cone of positive and negative symmetric matrices, respectively, and τ_i are positive constants for all $i = 1, 2, \ldots, m$.

In this study the authors propose a generalization of the method presented in [5] – originally aiming at the solution of convex semidefinite programming problems – which allows for the inclusion of nonconvex constraints in the problem statement. A generalized algorithm is presented. The new algorithm is tested by means of Free Material Optimization problems including constraints on displacements. It is demonstrated that the generalized algorithm is able to solve these large-scale nonconvex semidefinite programming problems to local optimality with acceptable performance.

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Optimal Control Methods in Shape Optimization

Dan Tiba

Shape optimization or optimal design problems are now a well-known branch of the calculus of variations and there exists a rich literature devoted to various approaches for their study. We just quote the classical reference [4] and the recent monograph [2] for a comprehensive introduction to the subject and for relevant references. Here, we report on two new methods, based on optimal control theory, for the search of optimal geometries, [1, 3].

The following model problem is analyzed:

$Min_{\Omega} F(y, \Omega)$,

$\int_{\Omega} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 y v\right] dx = \int_{\Omega} f v dx, y \in H_0^1(\Omega),\ \forall v \in H_0^1(\Omega)$,

where $a_{ij}, a_0 \in L^{\infty}(\Omega)$, $a_0 \geq 0$, $f \in H^{-1}(D)$ and $(a_{ij})_{i,j=1,d}$ satisfies the usual ellipticity condition. Here $\Omega \subset D$ is a general open set not necessarily connected and D is a bounded Lipschitzian domain in the Euclidean space R^d. Various constraints may be imposed on y and Ω.

As examples of cost functionals, we mention

$$F(y, \Omega) = \int_{\Omega} j(x, y(x), \nabla y(x)) dx$$

or, if some given measurable subset $E \subset D$ exists such that $E \subset \Omega$ for any admissible domain :

$$F(y, \Omega) = \int_{E} j(x, y(x), \nabla y(x)) dx$$

where $j : D \times R \times R^d \to R$ is some Carathéodory mapping satisfying certain convexity and growth conditions. Notice that nonzero Dirichlet boundary conditions may be considered as well via a usual translation argument.

1. Approximation

Let $X(D)$ be a space of continuous functions in $\bar{D}$. Standard finite element spaces may play the role of $X(D)$. For any $g \in X(D)$, we introduce the admissible domain $\Omega = \Omega_g = int\{x \in D \mid g(x) \geq 0\}$. and we say that g is a parametrization of Ω. If the admissible class of open sets should contain E, we impose the constraint $g \geq 0$ in E.

If H is the Heaviside function, then $H(g)$ provides the characteristic function of $\overline{\Omega}_g$. We introduce the following penalized approximating extension of the boundary value problem with solution $y = y_g$ from $\Omega = \Omega_g$ to D, in the weak formulation

$\int_D \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial y_\epsilon}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 y_\epsilon v + \frac{1}{\epsilon}(1 - H^\epsilon(g)) y_\epsilon v\right] dx = \int_D f v dx,$

$y_\epsilon \in H_0^1(D), \forall v \in H_0^1(D)$.

Here H^ϵ is a smoothing of the Yosida approximation of the maximal monotone extension of the Heaviside function on the real line.

Theorem 1. *If $\Omega = \Omega_g$ is of class C, then $y_\epsilon \to y_g$ weakly in $H^1(\Omega_g)$ on a subsequence.*

Remark Due to the above theorem, we shall approximate the original shape optimization problem by an optimal control problem in D with the same cost functional and with the above approximating extension of the equation as state system. The mapping g plays the role of the control unknown. State constraints may be included as well and treated by standard methods from optimal control theory. Existence of an optimal pair, optimality conditions and gradient methods can be applied as well.

If the cost functional has the form

$$\int_E (y - y_d)^2 dx$$

where $y_d \in L^2(E)$ is given and the constraint $g \geq 0$ in E is valid, we have :

Theorem 2. *The directional derivative in point $g \in X(D)$ and in the direction $w \in X(D)$ is given by*

$$\frac{1}{\epsilon} \int_D (H^\epsilon)'(g) w y_\epsilon p dx$$

where y_ϵ is the solution of the penalized state equation and $p \in H_0^1(D)$ is the solution of the adjoint system :

$$\int_D \left[\sum_{i,j=1}^d a_{ji} \frac{\partial p}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 pv + \frac{1}{\epsilon}(1 - H^\epsilon(g)pv \right] dx = \int_E (y_\epsilon - y_d) v dx.$$

Remark Another approach based on geometric controllability properties may be extended to Neumann, Robin or other boundary conditions [1, 3]. If the elliptic operator is the Laplacian, the obtained approximated extension (for Dirichlet B.C.) is given by the system

$$\begin{array}{rcll} -\Delta y & = & f - \frac{1}{\epsilon}(1 - H(g))^2 p & \text{in } D, \\ -\Delta p & = & (1 - H(g))y & \text{in } D, \\ y & = & p = 0 & \text{in } \partial D. \end{array}$$

Some numerical examples are also indicated in [3].

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Parabolic-Hyperbolic Fluid-Structure Interaction: Semigroup Well-Posedness, Spectral Analysis, Strong and Uniform Stability, Backward Uniqueness

Roberto Triggiani

We consider a linear version of an established parabolic-hyperbolic coupled system of PDEs, which models fluid-structure interaction in dimensions $d = 2, 3$, with coupling taking place a the interface between the two media. The structure is modeled by the system of dynamic elasticity (canonically, the d-dimensional wave equation); which is hyperbolic. The structure is immersed in a fluid, which is modeled by the linear version of the Navier-Stokes equations; which is parabolic. At the interface of the two media the coupling involves two conditions: (i) matching the velocity of the structure and the velocity of the fluid; (ii) matching of the normal components of the stress tensor. The structure is fixed but oscillating. Henceforth, let $\{w, w_t, u\}$ denote the state of the system: here w is the d-dimensional displacement of the structure, w_t its velocity, while u is the d-dimensional fluid velocity. The system is defined on a bounded domain $\Omega = \Omega_s \cup \Omega_f$, consisting of the domain Ω_s of the structure which is immersed in (surrounded by) the domain Ω_f of the fluid. The boundary $\Gamma_s = \partial\Omega_s$ represents the interface between the two media, structure immersed in fluid. Γ_f represents the external boundary of the fluid. In its simplified canonical version (with the d-dimensional wave equation in place of the system of dynamic elasticity), the model is

$$\begin{cases} \begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } (0,T] \times \Omega_f = Q_f; \\ \text{div } u = 0 & \text{in } Q_f; \\ w_{tt} - \Delta w - w = 0 & \text{in } (0,T] \times \Omega_s = Q_s; \end{cases} \\ u|_{\Gamma_f} \equiv 0 & \text{on } (0,T] \times \Gamma_f \equiv \Sigma_f; \\ \text{B.C.} \begin{cases} u = w_t & \text{on } (0,T] \times \Gamma_s \equiv \Sigma_s; \\ \dfrac{\partial u}{\partial \nu} - \dfrac{\partial w}{\partial \nu} = p\nu & \text{on } \Sigma_s; \end{cases} \\ \text{I.C. } u(0,\cdot) = u_0;\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega. \end{cases}$$

The model is valid for small but rapid oscillations of the interface.

(1) The first result is a semigroup well-posedness theorem on the space $\mathcal{H}$ of finite energy $\{w, w_t, u\} \in H^1(\Omega_s) \times L_2(\Omega_s) \times L_2(\Omega_f)$, with $L_2(\Omega_f)$ being divergence free and having the boundary condition $f \cdot ot\nu = 0$ on Γ_f. Thus, the map

$$\{w_0, w_1, u_0\} \longrightarrow \{w(t), w_t(t), u(t)\}$$

generates a strongly continuous contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$.

(2) Next, the spectrum of the generator $\mathcal{A}$ on the imaginary axis is analyzed. First $\lambda = 0$ is always an eigenvalue with a one-dimensional eigenspace, which is explicitly characterized. Second, for most geometries of the structure, $\lambda = 0$ is

the only eigenvalue of $\mathcal{A}$ on the imaginary axis (e.g., when the boundary Γ_s of the structure is partially flat, or is partially spherical). However, in the case where Ω_s is a sphere, there are countably many additional eigenvalues on the imaginary axis, which may be explicitly identified.

(3) Thus, at any rate, the semigroup $e^{\mathcal{A}t}$ is *not* strongly stable on $\mathcal{H}$: at best, it is strongly stable on $\mathcal{H}$ factored out the one-dimensional eigenspace, when $\lambda = 0$ is the only eigenvalue on the imaginary axis. The remaining of the imaginary axis lies in the resolvent set of $\mathcal{A}$.

(4) The resolvent $R(\lambda, \mathcal{A})$ of the generator is not compact on $\mathcal{H}$ (in fact, on the displacement component space $H^1(\Omega)s)$).

(5) Next, to obtain the stability of the semigroup, not only strong stability in $\mathcal{H}$, but even uniform stability in $\mathcal{L}(\mathcal{H})$, dissipation needs to be inserted at the interface. Thus, the new B.C. will be

$$u = w_t + \frac{\partial w}{\partial \nu} \quad (\nu \text{ inward to } \Omega_s).$$

This uniform stabilization result of the new semigroup does no require any geometrical conditions on Ω_s.

(6) Finally, returning again to the original model $(*)$ written before, the backward uniqueness property holds true:

$$e^{\mathcal{A}T} x = 0, \quad x \in \mathcal{H}, \quad T > 0 \Rightarrow x = 0.$$

This property cannot be taken for granted as the semigroup couples a parabolic and a hyperbolic component.

From Exact Observability to Inverse Problems: a Functional Analytic Approach

Marius Tucsnak

(joint work with Carlos Alves, Ana Leonor Silvestre, Takï Takahashi)

1. Introduction

The connection of exact observability of infinite dimensional systems with inverse problems for PDE's has been remarked and used in several papers (see, for instance, [1]). Our aim is to give a general framework for this approach, by using a functional analytic approach and to give new applications, using the new exact observability results from [2].

Let X and Y be two Hilbert spaces, let $A : \mathcal{D}(A) \to X$ be the generator of a strongly continuous group $\mathbb{T}$ in X and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. We consider the system

$$\dot{z}(t) = Az(t) + \lambda(t) f, \quad z(0) = 0, \tag{1}$$

$$y(t) = Cz(t) \qquad (\tau \in [0, \tau]). \tag{2}$$

where $z_0 \in X$ and f is can be chosen in a space larger than X, denoted by Z'. More precisely Z' is the dual with respect to the pivot space X of the space Z defined by

$$Z = (\beta I - A)^{-1}(X + C^* Y).$$

The main result of this work is

Theorem 1. *Assume that the pair (A, C) is exactly observable in some time $\tau_0 > 0$ and that λ is a given C^1 function, with $\lambda(0) \neq 0$. Then for every $\tau > \tau_0$, the map $\mathbb{F}_\tau \in \mathcal{L}(Z', L^2([0,\tau], Y))$ defined by*

$$(\mathbb{F}_\tau f)(t) = y(t) \qquad (t \in (0,\tau)),$$

is one to one and

$$\|f\|_{Z'} \leq C_\tau \|y\|_{L^2([0,\tau],Y)}\,. \tag{3}$$

Sketch of the proof. For each $\tau > 0$ we define the operator

$$\Psi_\tau \in \mathcal{L}(X_1, L^2([0,\tau]; Y))$$

by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \quad \text{for} \quad t \in [0,\tau]. \tag{4}$$

It has been shown in Tucsnak and Weiss [3]. that, for each $\tau > \tau_0$, there exist two constants $M_\tau, m_\tau > 0$ such that, for every $f \in Z'$, we have

$$M_\tau \|f\|_{Z'} \geq \|\Psi_\tau f\|_{\left[H^1_R((0,\tau);Y)\right]'} \geq m_\tau \|f\|_{Z'}, \tag{5}$$

where $H^1_R((0,\tau);Y) = \left\{u \in H^1(0,\tau;Y) \mid u(\tau) = 0\right\}$. Let S be the integral operator

$$(Sg)(t) = \int_0^t \lambda(t-s) g(s) \mathrm{d}s \qquad (g \in L^2((0,\tau);Y)). \tag{6}$$

By adapting a classical result it can be shown that S admits a unique extension $\widetilde{S}$ which is an isomorphism from $\left[H^1_R((0,\tau);Y)\right]'$ onto $L^2((0,\tau);Y)$. With the above notation we have $y = (\widetilde{S} \circ \Psi_\tau)(f)$. Consequently,

$$\|y\|_{L^2([0,\tau],Y)} \geq C_S \|\Psi_\tau f\|_{\left[H^1_R((0,\tau);Y)\right]'} \geq C_S m_\tau \|f\|_{Z'} \qquad (f \in Z').$$

□

The above result shows that, for a given intensity, the location of sources of given intensity can be done in a stable way by using exactly observable outputs. We also give sufficient conditions for recovering both the location and the intensity of the source. The main application concerns identification of sources for the

Euler-Bernoulli plate equation. More precisely, consider the initial value problem, modeling the vibrations of a hinged Euler-Bernoulli plate,

$$\text{(7)} \qquad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \Delta^2 w = \lambda(t)\delta_\xi & \text{in } (0,\tau)\times\Omega, \\ w = \Delta w = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ w(0,x) = w_0(x), \quad \dfrac{\partial w}{\partial t}(0,x) = w_1(x) & x\in\Omega; \end{cases}$$

where $\xi \in \Omega$ and δ_ξ is the Dirac mass concentrated in ξ. Then the following result holds.

Theorem 2. *Let $\tau > 0$, let $\Omega \subset \mathbb{R}^2$ be a bounded set and let Γ be a nonempty open subset of $\partial\Omega$ such that either $\partial\Omega$ is smooth and Γ satisfies the geometric optics conditions, or Ω is a rectangle and Γ contains a non void vertical and a non void horizontal subset. Let $\varepsilon > 0$ and let $\xi^{(1)}, \xi^{(2)} \in \Omega$ be two points in Ω, each one at distance at least ε from $\partial\Omega$. Assume that $\lambda \in H^1(0,\tau)$ with $\lambda(0) \neq 0$, $w_0 \in H_0^1(\Omega)$, $w_1 \in H^{-1}(\Omega)$ and denote $y^{(j)} = \left.\frac{\partial w^{(j)}}{\partial \nu}\right|_\Gamma$, $j \in \{1,2\}$, where $w^{(j)}$ is the solution of (7) with $\xi = \xi^{(j)}$, $j \in \{1,2\}$.*

Then there exists $\delta > 0$, depending only on Ω, Γ, ε and τ such that

$$\|y^{(1)} - y^{(2)}\|_{L^2(0,\tau;L^2(\Gamma))} \geq \delta|\xi^{(1)} - \xi^{(2)}|.$$

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Multilevel Semismooth Newton Methods for 3D Elastic Contact Problems

Michael Ulbrich

(joint work with Stefan Ulbrich)

In this talk, the efficient application of semismooth Newton methods [2, 3] to 3D elastic contact problems is considered [4]. The contact problem is regularized in such a way that the complementarity condition allows for a semismooth reformulation. Error estimates for regularized solutions in terms of the regularization parameter are given and superlinear convergence of the semismooth Newton's method is established. A block elimination is performed to prepare the Newton system for the application of multigrid. Solving the Newton system then essentially reduces to the solution of an elliptic subsystem with special structure. Using abstract multilevel theory, in particular the results of [1, 5], a suitably designed multigrid V-cycle for the elliptic subsystem is developed and analyzed. The efficiency of the resulting multigrid preconditioned semismooth Newton method is documented by numerical tests for contact problems in 3D. The computations are performed for a finite element discretization of a human mandible. Uniform as well as adaptive mesh refinements are considered.

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A Posteriori Error Estimators for Control Constrained Optimization with PDEs Based on Interior Point or Semismooth Residuals

Stefan Ulbrich

In this talk we present a general concept to show that appropriate weigthed residuals of the optimality system, which are motivated by the analysis of interior-point methods in function space [6], can be used to construct reliable a posteriori error estimators for approximate solutions of control constrained optimization problems with PDEs. Moreover, we show that alternatively also semismooth residuals of the optimality system can be used. So far there exist only quite a few results on a posteriori error estimators for optimal control problems with control or state constraints. Residual based error estimators for elliptic problems are considered in [2, 3, 4, 5] and goal oriented error estimators in [1, 7].

We consider control constrained optimization problems of the form

$$\min_{y\in Y, u\in U} J(y,u) \text{ s. t. } c(y,u)=0,\ l\le u\le r, \tag{1}$$

where Y is a Banach space, $U=L^2(\Omega)$, $\Omega\subset\mathbb{R}^n$ open and bounded, is the control space and $l,r\in L^\infty(\Omega)$, $\operatorname{essinf}(r-l)>0$. We identify the Hilbert space U with its dual U^*. Our main assumptions are that with a Banach space Λ the mappings $J:Y\times U\to\mathbb{R}$, $c:Y\times U\to\Lambda$ are twice continuously differentiable with bounded derivatives on bounded subsets of $Y\times U$, for all $u\in U$ there exists a unique solution $y=y(u)\in Y$ of $c(y,u)=0$, and $c_y(y,u)\in\mathcal{L}(Y,\Lambda)$ has a uniformly bounded inverse on bounded subsets of $Y\times U$. This framework applies, e.g., to distributed and boundary control problems for elliptic and parabolic PDEs, the Navier-Stokes equations and to regularized contact or obstacle problems.

The Lagrangian of (1) is given by

$$\ell(y,u,\lambda,z_l,z_r)=J(y,u)+\langle\lambda,c(y,u)\rangle_{\Lambda^*,\Lambda}-(z_l,u-l)_2-(z_r,r-u)_2$$

with multipliers $(\lambda, z_l, z_r) \in \Lambda^* \times U^* \times U^*$. Under our assumptions for any local solution $(\bar{y}, \bar{u})$ of (1) there exist multipliers $(\bar{\lambda}, \bar{z}_l, \bar{z}_r) \in \Lambda^* \times U^* \times U^*$ such that

$$F(\bar{w}) := \big(\ell_y(\bar{y}, \bar{u}, \bar{\lambda}, \bar{z}_l, \bar{z}_r), \ell_u(\bar{y}, \bar{u}, \bar{\lambda}, \bar{z}_l, \bar{z}_r), c(\bar{y}, \bar{u}), \bar{z}_l(\bar{u} - l), \bar{z}_r(r - \bar{u})\big) = 0.$$

where $F : W_2 \to Y^* \times L^2 \times \Lambda \times L^1 \times L^1$ with $W_2 := Y \times L^2 \times \Lambda^* \times L^2 \times L^2$ is continuously differentiable. We assume that in a neighborhood of $\bar{w}$ the following second order sufficiency condition holds: The reduced Hessian

$$\hat{H}(w) := (\ell_{uu} + c_u^* c_y^{-*} \ell_{yy} c_y^{-1} c_u - c_u^* c_y^{-*} \ell_{yu} - \ell_{uy} c_y^{-1} c_u)(w)$$

satisfies with some constant $\alpha > 0$

$$(v, \hat{H}(w)v) \geq \alpha \|v\|_2^2 \quad \forall\, v \in L^2(\Omega). \tag{2}$$

Also refined second order conditions are possible, see below. We have the relation

$$DF(w, \bar{w})(w - \bar{w}) = F(w)$$

with an appropriate mean value derivative. We define the space $W_2' := Y^* \times L^2 \times \Lambda \times L^2 \times L^2$ and consider the scaled equation

$$S(w, \bar{w}) DF(w, \bar{w})(w - \bar{w}) = S(w, \bar{w}) F(w) \in W_2'$$

with the multiplicative scaling $S(w, \bar{w}) = \operatorname{diag}(I, I, I, 2(U_l + \bar{U}_l + Z_l + \bar{Z}_l)^{-1}, 2(U_r + \bar{U}_r + Z_r + \bar{Z}_r)^{-1})$, where $U_l = (u - l)I$, $U_r = (r - u)I$, $Z_l = z_l I$, $Z_r = z_r I$ etc. are multiplication operators.

Now let $w \in W_2^\circ := \{w = (y, u, \lambda, z_l, z_r) \in W_2 \ : \ l < u < r,\ z_l > 0,\ z_r > 0\}$ be arbitrary. Under the assumption that (2) holds for all points on the line segment $[w, \bar{w}]$ we show by estimating the norm of $(S(w, \bar{w}) DF(w, \bar{w}))^{-1}$ similar as in [6] that $\|S(w, \bar{w}) F(w)\|_{W_2'}$ is a reliable and efficient error bound for $\|w - \bar{w}\|_{W_2}$. More precisely, we show the following.

With $\frac{1}{\epsilon} = \max\left\{1, \|\hat{H}\|_{L^2, L^2}, \frac{1}{\sqrt{\alpha}}\right\}$ there exists a constant $C > 0$, which depends on $\|c_u^* c_y^{-*} \ell_{yy} c_y^{-1} - \ell_{uy} c_y^{-1}\|_{\Lambda, L^2}$, $\|c_u^* c_y^{-*}\|_{Y^*, L^2}$ and can be chosen uniformly on bounded sets, such that with $\rho := S(w, \bar{w}) F(w)$ the estimates hold

$$\|u - \bar{u}\|_{2, \Omega_1} \leq \tfrac{C}{\sqrt{\epsilon}} \|\rho\|_{W_2'}, \ \|u - \bar{u}\|_{2, \Omega_2} \leq \tfrac{C}{(1 + \sqrt{\alpha})\epsilon} \|\rho\|_{W_2'}, \ \|u - \bar{u}\|_{2, \Omega_3} \leq \tfrac{C}{\sqrt{\alpha\epsilon}} \|\rho\|_{W_2'},$$

where

$$\Omega_1 = \{\min(\hat{u}_l, \hat{u}_r) \leq \epsilon\},\ \Omega_2 = \{\epsilon < \min(\hat{u}_l, \hat{u}_r) \leq \tfrac{1}{2}\},\ \Omega_3 = \{\min(\hat{u}_l, \hat{u}_r) > \tfrac{1}{2}\}.$$

This implies the global bound $\|w - \bar{w}\|_{W_2} \leq \max\left(\frac{C}{\sqrt{\epsilon}}, \frac{C}{\sqrt{\alpha\epsilon}}\right) \|S(w, \bar{w}) F(w)\|_{W_2'}$.

Ω_1 can be interpreted as the set of likely active points, Ω_2 as the set of perhaps active points and Ω_3 as the set of likely inactive points. The worst estimate for $u - \bar{u}$ is obtained on Ω_3 with a leading constant of $O(1/\alpha)$. For the case of the Poisson equation a global estimate with leading constant of $O(1/\alpha)$ was derived in [2]. Our technique is universal and gives improved estimates on Ω_1, Ω_2. The assumption (2) can be weakened to hold only for $\operatorname{supp}(v) \subset \Omega_2 \cup \Omega_3$.

On the other hand, the error bound is efficient, i.e., it exists a constant $C' > 0$ with $\|w - \bar{w}\|_{W_2} \geq C' \|S(w, \bar{w}) F(w)\|_{W_2'}$.

To obtain an implementable error estimator for an estimate w resulting from a conformal discretization, we observe that

$$\|S(w,\bar{w})F(w)\|_{W_2'} \le \|\ell_y(w)\|_{Y^*} + \|\ell_u(w)\|_{L^2} + \|c(y,u)\|_\Lambda + \left\|\frac{u_l z_l}{u_l+z_l}\right\|_{L^2} + \left\|\frac{u_r z_r}{u_r+z_r}\right\|_{L^2}$$

and the right hand side is easy to estimate elementwise on a triangulation if we have reliable local error estimators for the residual $\|\ell_y(w)\|_{Y^*}$ of the adjoint equation and the residual $\|c(y,u)\|_\Lambda$ for the state equation at hand.

Finally we show that also the residual of a semismooth formulation of the optimality system yields a reliable and efficient error bound.

Based on the proposed error estimator we show numerical results for an adaptive semismooth Newton method applied to elliptic control problems (using the ZZ-estimator for the adjoint and state residual) and to a regularized elastic 3D contact problem (using an averaging estimator of Carstensen for the elasticity equation).

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Efficient Computation of the Tikhonov Regularization Parameter by Adaptive Finite Element Methods

Boris Vexler

(joint work with Anke Griesbaum, Barbara Kaltenbacher)

In this talk we consider parameter identification problems in partial differential equations and develop a multilevel inexact Newton method for determining an optimal regularization parameter in Tikhonov regularization, see [2] for more details.

The state variable u in a Hilbert space V is given as the solution of the (possible nonlinear) equation written in a weak form:

$$a(q,u)(v) = f(v) \quad \forall v \in V. \tag{1}$$

Here, the variable q denotes the unknown parameter from a parameter (Hilbert) space Q. The state equation is assumed to possess a unique solution for each

$q \in Q$, which allows for a definition of the solution operator $S\colon Q \to V$. Moreover, a measurement operator $C\colon V \to G$ maps the state variable u into the space of measurements G.

The corresponding parameter identification problem is formulated as follows: Given noisy measurements g^δ with

$$\|g - g^\delta\|_G \leq \delta$$

one has to estimate the unknown parameter q such that

$$C(S(q)) = g^\delta.$$

In a variety of situations this problem is ill-posed, i.e., the solution q of the above equation (if exists) does not depend continously on the measurements. Therefore, regularization methods are necessary for a stable numerical solution of the parameter identification problem. One of the well-known regularization techniques is Tikhonov regularization leading to the following optimal control problem which depends on the regularization parameter β:

$$\text{minimize } J(\beta, q, u) = \frac{1}{2}\|C(u) - g^\delta\|_G^2 + \frac{1}{2\beta}\|q\|_Q^2, \quad \text{subject to (1).} \tag{2}$$

The regularization parameter β should be chosen in such a way, that the solution of this optimal control problem denoted by (q^β, u^β) is close to the ideal solution $(q^\dagger, u^\dagger)$ for the problem without noise, see, e.g., [1] for precise definitions. A well-established strategy for choosing the Tikhonov parameter β is the discrepancy principle: The parameter β^* should be chosen as the solution of the following one-dimensional equation

$$i(\beta^*) = \tau^2\delta^2 \tag{3}$$

with some $\tau \geq 1$ and the function $i\colon \mathbb{R}_+ \to \mathbb{R}_+$ given as

$$i(\beta) = \|C(u^\beta) - g^\delta\|_G^2.$$

In [1] Newton's method for solving equation (3) is analyzed. The corresponding algorithm would require evaluation of the function $i(\beta)$ and its derivative $i'(\beta)$ in each step. However, $i(\beta)$ is not avaliable since it depends on the exact solution of the infinite dimensional optimal control problem (2). Therefore, one should replace the function $i(\beta)$ by its discrete analog $i_h(\beta)$ defined as

$$i_h(\beta) = \|C(u_h^\beta) - g^\delta\|_G^2,$$

where $(q_h^\beta, u_h^\beta) \in Q_h \times V_h$ is the solution of the discretized version of the optimal control problem (2). In [2] we describe and analyze an inexact multilevel Newton's method for solution of (3), where we replace $i(\beta)$ by $i_h(\beta)$ in each Newton step and control the choice of the discrete finite element spaces Q_h and V_h adaptively using appropriate a posteriori error estimates. The finite element spaces Q_h and V_h should be chosen on the one hand as coarse as possible to save numerical effort and on the other hand fine enough to preserve quadratic convergence of the method to the solution β^* of (3). This algorithm is sketched below:

(1) Choose initial guess $\beta^0 > 0$, initial discretization Q_{h_0}, V_{h_0}, set $k = 0$
(2) Solve discrete optimal control problem, compute $(q_h^{\beta^k}, u_h^{\beta^k})$
(3) Evaluate $i_h(\beta_k)$, $i'_h(\beta_k)$
(4) Evaluate error estimators

$$|i(\beta_k) - i_{h_k}(\beta_k)| \leq \eta^I, \qquad |i'(\beta_k) - i'_{h_k}(\beta_k)| \leq \eta^K$$

(5) If the accuracy requirements for η^I, η^K are fulfilled, set

$$\beta^{k+1} = \beta_k - \frac{i_{h_k}(\beta^k) - \tau^2\delta^2}{i'_{h_k}(\beta^k)}$$

(6) else: refine discretization $h_k \to h_{k+1}$ using local information from η^I, η^K
(7) if stopping criterion is fulfilled: break
(8) else: Set $k = k + 1$ and go to 2.

In [2] we present error estimators used in the step (4) of the algorithm, discuss an efficient strategy for evaluation of $i'_h(\beta)$, and provide accuracy requirements for the step (5) allowing for quadratic convergence of the method. Moreover, we discuss the stopping criterion for the step (7) and prove convergence of the solution q_h^β with computed $\beta = \beta_{k^*}$ to the ideal solution $q^\dagger$ as δ tends to 0.

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On the Numerical Verification of Optimality Conditions

Daniel Wachsmuth

(joint work with Arnd Rösch)

Let us consider the model problem

$$\text{(1)} \qquad \min_{u\in U} f(u).$$

Here, $D \subset \mathbb{R}^n$ is a domain, $U := L^2(D)$ the space of admissible controls, the function $f : U \to \mathbb{R}$ is supposed to be twice continuously Fréchet differentiable. We have in mind a general non-convex optimal control problem as motivation for the model problem, where the additional partial differential equation is already eliminated by means of a solution operator.

The standard first-order necessary optimality conditions for (1) are given by $f'(\bar{u}) = 0$. Second-order sufficient optimality conditions (SSC) are then formulated as: there is a $\alpha > 0$ such that it holds $f'(\bar{u}) = 0$ and $f''(\bar{u})[v, v] \geq \alpha\|v\|_U^2 \quad \forall v \in U$.

If $\bar{u}$ fulfills (SSC) then one has: $\bar{u}$ is locally optimal, $\bar{u}$ is stable with respect to small perturbations, local convergence of optimization methods (e.g. SQP) and approximation schemes (e.g. FEM). Hereby, the sufficient condition is an essential pre-requisite to show all these nice properties, see e.g. [2]. But how can we verify it? There are several possibilities:

- the original problem is convex, thus (SSC) is satisfied automatically,
- the solution $\bar{u}$ is known, one has to check (SSC) per hand.

However, in practice these conditions are not fulfilled. Instead, the typical situation is the following:

- the problem is not known to be convex,
- the solution $\bar{u}$ is unknown (sometimes even existence is not clear),
- only a numerical approximation $\bar{u}_h$ can be provided.

Hence, given a approximative solution $\bar{u}_h$ the following questions arise

- is $\bar{u}_h$ in the neighborhood of a stationary point?
- is $\bar{u}_h$ in the neighborhood of a local minimum?
- is $\bar{u}_h$ in the neighborhood of a local minimum that satisfies (SSC)?

Only if we can answer the third question positively we can apply all the results cited above that rely on (SSC).

We will now give answers to all this questions without requiring conditions on the unknown solution $\bar{u}$. Only information about $\bar{u}_h$ will be used.

Assumption 1. Let us assume that there are positive constants ϵ, α, M such that the following conditions are satisfied:

$$\|f'(\bar{u}_h)\|_U \le \epsilon,$$
$$f''(\bar{u}_h)[v,v] \ge \alpha \|v\|_U^2 \quad \forall v \in U,$$
$$|f''(u_1)(v_1,v_2) - f''(u_2)(v_1,v_2)| \le M\|u_1-u_2\|_U \|v_1\|_U \|v_2\|_U \quad \forall u, v_1, v_2, v_3 \in U.$$

Since $\bar{u}_h$ is unknown, there is a chance that we can check numerically whether these conditions are satisfied or not. If we are able check this conditions, then we can also compute the constants α, ϵ, M. How this can be done depends of course on the structure of the underlying problem. See for instance [1], where we applied these ideas to an optimal control problem subject to a semi-linear elliptic equation.

Remarks. The following things are essential in Assumption 1 as we will show in a forthcoming article.

1. *It is important to take the L^2-norms in Assumption 1 instead of $\mathbb{R}^n$-norms.*
2. *Furthermore, one cannot use U_h instead of U as the test space in*

$$f''(\bar{u}_h)[v_h, v_h] \ge \alpha \|v\|_U^2 \quad \forall v \in U_h,$$

   which would correspond to computing eigenvalues of the discrete Hessian.
3. *The Lipschitz condition on f'' can be replaced by a local bound in a neighborhood of $\bar{u}_h$.*

By Taylor expansion, we obtain the following estimate for the objective

$$f(u) - f(\bar{u}_h) \geq -\epsilon r + \frac{\alpha}{2} r^2 - \frac{M}{6} r^3 \quad \forall u \in U : \|u - \bar{u}_h\|_U = r. \tag{2}$$

Let us assume, that this polynomial admits positive values for some positive r.

Assumption 2. There is a positive number r_+ such that it holds

$$-\epsilon r_+ + \frac{\alpha}{2} r_+^2 - \frac{M}{6} r_+^3 > 0 \text{ and } \alpha - M r_+ > 0.$$

The coefficients of the polynomial are known, hence the check of this assumption reduces to compute roots of a polynomial.

Theorem. Let the Assumptions 1 and 2 be satisfied. Then there exists a local minimum $\bar{u}$ of the original problem in the neighborhood of $\bar{u}_h$ that satisfies (SSC), and it holds

$$\|\bar{u} - \bar{u}_h\|_U < r^+.$$

Let us emphasize that the Theorem is at first an existence result. Under some conditions on the approximation $\bar{u}_h$ there exists a solution of the original problem. This solution then also satisfies the sufficient optimality condition. Hence, we can answer the question at the beginning of this note. Furthermore, the Theorem provides us by an error bound, which is by its nature even computable.

We will report on the verification of Assumptions 1 and 2 for an optimal control problem subject to a semilinear elliptic equation in a forthcoming paper.

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Hidden Boundary Smoothness for the Solution of Maxwell Equation via Extractor Identity

Jean-Paul Zolesio

This talk deals with the regularity at the boundary for electromagnetic 3D time-depending Maxwell equations solution E, H. We show a hidden regularity result at the boundary for Electric field on a metallic obstacle. We consider a domain Ω with boundary Γ on which the boundary condition $E_\Gamma = 0$ applies . Assuming free divergence initial data $E_i \in H^i(\Omega, R^N)$, $i = 0, 1$ and free divergence curent $J \in L^2(0, \tau, L^2(\Omega, R^N))$; we derive that, at the boundary , the magnetic field verifies $H \in H^1(0, \tau, L^2(\Gamma, R^3))$ while $curl E \in L^2(0, \tau, L^2(\Gamma, R^3))$. The proof makes use of the *Extractor technique* introduced at ICIAM 1995 ([5]) and several papers([6]),[3][2],[1]); we first derive that $\frac{\partial}{\partial t} E$ and $(DE.n)_\Gamma$ are in $L^2(]0, \tau[\times \Gamma, R^3)$.

Assume the boundary Γ to be a C^2 manifold. Let (E_0, E_1, J) be divergence free vectors fields in

$L^2(I, H^1(\Omega, R^3)) \times L^2(I, H^1(\Omega, R^3)) \times H^1(I, L^2(\Omega, R^3))$.

with zero tangential componants: $(E_0)_\Gamma = 0$, $(E_0)_\Gamma = 0$.

The maxwell problem has a unique solution

$E \in C^0(\bar{I}, H^1(\Omega, R^3)) \cap C^1(\bar{I}, L^2(\Omega, R^3))$

Let (E_0^k, E_1^K, J^k) be divergence free vector fields in

$L^2(I, H^2(\Omega, R^3)) \times L^2(I, H^2(\Omega, R^3)) \times H^1(I, H^1(\Omega, R^3))$.

and converging when $k \to \infty$ to (E_0, E_1, J) strongly in $L^2(I, H^1(\Omega, R^3)) \times L^2(I, H^1(\Omega, R^3)) \times H^1(I, L^2(\Omega, R^3))$.

The associated solution $E^k \in C^0(\bar{I}, H^2(\Omega, R^3)) \cap C^1(\bar{I}, H^1(\Omega, R^3))$ strongly converges to E in

$C^0(\bar{I}, H^1(\Omega, R^3)) \cap C^1(\bar{I}, L^2(\Omega, R^3))$, as $k \to \infty$. There exists a constant $M > 0$ such that $\forall k$:

$$(1) \qquad |\; ||E_t^k||^2_{L^2(I\times\Gamma)^3} + ||(DE^k.n)_\Gamma||^2_{L^2(I\times\Gamma)^3} - ||\nabla_\Gamma(E^k.n)||^2_{L^2(I\times\Gamma)^3} \;| \le M$$

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