Mathematisches Forschungsinstitut Oberwolfach

Report No. 42/2008

Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

Organised by
Gerhard Knieper, Bochum
Leonid Polterovich, Tel-Aviv
Leonid Potyagailo, Lille

September 7th – September 13th, 2008

Abstract. The workshop "Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry" brought together experts in these different but close and interactive mathematical directions. The activities included talks by world renown experts and by graduate students on their recent achievements, as well as an open problem session. In addition to extended abstracts of the talks, the present report contains a list of open problems.

Mathematics Subject Classification (2000): 20Fxx, 22E40, 22E46, 22F10, 37D05, 37D40, 37K65, 53C22, 53Dxx.

Introduction by the Organisers

The workshop "Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry", organized by Gerhard Knieper, Leonid Polterovich and Leonid Potyagailo took place the second time in Oberwolfach. The general idea to organize a joint meeting of specialists belonging to three different but close and interacting fields turned out to be really fruitful. The aim of our workshop is to give a possibility to people working in the above fields to discuss their results and open problems from different points of view. We are sure that such conferences need to exist not only on the level of World or European Congresses but also in much less official and restrictive atmosphere inherent for Oberwolfach conferences.

Our last meeting brought together world renown senior experts in our fields as well as talented mathematicians in the beginning in their careers. Let us describe shortly the directions of the given talks. The presented talks were devoted to:

1. Hamiltonian diffeomorphisms and rigidity properties of symplectic manifolds (L. Butler, O. Cornea, M. Entov, S. Hohloch, V. Humiliere, G. Noetzel, F. Le Roux, E. Opshtein, F. Schlenk, F. Zapolsky)

2. Geometric group theory (D. Calegari, V. Gerasimov, G. Levitt, F. Dahmani, N. Peyerimhoff, R. Sharp)
3. Riemannian manifolds and their invariants (M. Burger, D. Burago, D. Kotschick).
4. Dynamical systems and ergodic theory (K. Fraczek, E. Glasmachers, A. Katok)
5. Subgroups of algebraic linear groups and their geometry (E. Breuillard, P.-E. Caprace, D. Witte Morris).
6. Probabilistic methods in group theory (A. Erschler, K. Fujiwara)
7. Spectral invariants and their asymptotics (A. Karlsson, I. Polterovich).

Historically, one of the motivating subjects being in the center of our joint interests was the theory of quasi-morphisms and bounded cohomology. In the present time there exist many other subjects which are in our common interests: study of groups of diffeomorphisms using methods of symplectic topology (e.g. the talk of F. Le Roux where he constructed a new candidate for a non-trivial normal subgroup of the group of area-preserving homeomorphisms); dynamical methods describing the geometry of the mapping class group and the group of outer automorphisms of free groups (G. Levitt); probabilistic methods in the geometric group theory (A. Erschler and K. Fujiwara); solving equations in free groups and diophantine geometry (F. Dahmani); strong Tits alternative for linear groups and its consequences: uniform exponential growth and uniform decay of return probability (E. Breuillard); spectral theory for combinatorial Laplacian of Cayley graphs of finite abelian groups (A. Karlsson) etc.

Another important issue of our meetings is that they attract many students working in these subjects. Several students (or recent students) gave 30 minutes talks during our informal evening sessions attended by all participants. The students have been actively participating in the discussions during the conference.

We continued our tradition by organizing the open problem session on Wednesday night successfully chaired by Anatole Katok. We enclose below the problems formulated during this session and hope that they will stimulate further research in our fields. We strongly believe that the tradition of such joint meetings representing several research directions should continue in Oberwolfach.

Workshop: Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

Table of Contents

Abstracts

Multiplicative asymptotics for the growth of periodic orbits in hyperbolic flows: beyond the Anosov case

Anatole Katok

The general program is to extend Margulis' classical result to broad classes of flows with non-uniformly hyperbolic behavior. The first substantial progress in the area was made by Gerhard Knieper in the late 1990's. He considered geodesic flows on manifolds of non-positive curvature of geometric rank one and obtained principal geometric and dynamical ingredients for proving multiplicative asymptotics in that case. Several of my Ph.D. students worked on the multiplicative asymptotic problem over a number of years. Recently one of them, Bryce Weaver, considered a particular case of surfaces where curvature changes sign and parabolic closed geodesics prevent uniform hyperbolicity. Such examples were considered by Donnay and Burns-Gerber form the ergodic theory point of view. They showed the geodesic flows to be Bernoulli with respect to the Liouville measure. Weaver follows the Knieper approach and constructs the measure of maximal entropy. The principal ingredient in his work is careful study of behavior of horocycles near singularities which allows him to carry out a modified version of the Margulis construction for the maximal entropy measure. While Weaver's Ph.D. thesis results on the asymptotic growth of closed geodesics fall somewhat short of desired multiplicative asymptotics, the remaining steps look technical and require application of well-understood ergodic theory tools.

I present an outliner of principal points in the Weaver work and will discuss further directions in this area, both in the low-dimensional and higher-dimensional cases.

Stable commutator length in free groups

Danny Calegari

Let G be a group, and let $[G,G]$ denote the commutator subgroup. For any $g \in [G,G]$, the *commutator length* of g, denoted $\mathrm{cl}(g)$, is the least number of commutators in G whose product is equal to g. The *stable commutator length* of g, denoted $\mathrm{scl}(g)$, is the limit

$$\mathrm{scl}(g) := \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$$

Stable commutator length extends in a natural way to a pseudo-norm on $B_1^H(G)$, the (real) vector space of (group) 1-boundaries, modulo the subspace H spanned by expressions of the form $g - hgh^{-1}$ and $g^n - ng$ for $g,h \in G$ and $n \in \mathbb{Z}$.

A function $\varphi : G \to \mathbb{R}$ is a *quasimorphism* if there is some least real number $D(\varphi)$ (called the *defect*) such that

$$|\varphi(g) + \varphi(h) - \varphi(gh)| \le D(\varphi)$$

for all $g, h \in G$. If additionally $\varphi(g^n) = n\varphi(g)$ for all $g \in G$ and $n \in \mathbb{Z}$, we say φ is *homogeneous*. The vector space of homogeneous quasimorphisms on G is denoted $Q(G)$, and contains the subspace $H^1(G)$ of (real valued) homomorphisms. The defect defines a norm on Q/H^1, making it into a Banach space. There is a duality between quasimorphisms and scl called *Bavard duality*, which says that for any chain $C = \sum t_i g_i$ in B_1, where $t_i \in \mathbb{R}$ and $g_i \in G$, there is an equality

$$\mathrm{scl}(C) = \sup_{\varphi \in Q/H^1} \frac{\sum t_i \varphi(g_i)}{2D(\varphi)}$$

The main theorems I discussed in my talk were the following:

Theorem 1 (Rationality). *Let F be a free group. Then the following are true.*

1. *The* scl *pseudo-norm is piecewise rational linear on $B_1^H(F)$*
2. *Every rational chain rationally bounds an extremal surface (one projectively realizing the minimum of $-\chi$)*
3. *There is a fast polynomial time algorithm to compute the norm in finite-dimensional subspaces*

The proof is geometric, reducing computation to a finite rational linear programming problem.

Theorem 2 (Rigidity). *Let F be a free group, and let S be a compact orientable surface with $\pi_1(S) = F$.*

1. *The chain ∂S in $B_1^H(F)$ is projectively contained in the interior of a top dimensional face of the* scl *norm ball*
2. *The unique homogeneous quasimorphism dual to this face (up to H^1 and scaling) is the rotation quasimorphism* rot_S *associated to the action of $\pi_1(S)$ on the circle at infinity coming from a hyperbolic structure on S.*

Corollaries of these theorems include:

1. If G is a graph of free groups amalgamated along cyclic subgroups, every nonzero class in $H_2(G; \mathbb{Q})$ is projectively realized by an injective closed surface subgroup
2. When F is a free group, the image $\mathrm{scl}(F)$ in $\mathbb{R}/\mathbb{Z}$ is exactly equal to $\mathbb{Q}/\mathbb{Z}$; this answers in the negative a well-known question of Bavard

Experiments suggest that for words w in F of fixed length, the frequency with which $\mathrm{scl}(w) = p/q$ as a reduced fraction satisfies a power law $\mathrm{freq}(p/q) \sim q^{-\delta}$. Ongoing research suggests this can be explained in terms of *surgery*; if $G = *_i A_i$ where each A_i is free abelian, and $w \in [G, G]$, there is a linear programming problem which computes $\mathrm{scl}_G(w)$. If $\rho_i : A_i \to \mathbb{Z}$ are a family of surjective homomorphisms, there is an induced surjection to a free group $\rho : G \to F$. It turns out that there is a linear programming problem which computes $\mathrm{scl}_F(\rho(w))$ which is related in a precise way (depending on the ρ_i) to the problem which computes $\mathrm{scl}_G(w)$. The distribution of values of $\mathrm{scl}_F(\rho(w))$ under families of maps ρ gives rise to power laws as above. A surprising corollary of the method of proof gives the following:

Corollary 3. *For any nonzero integers n, m the map $a \to a^n$, $b \to b^m$ induces an isometry of the* scl *norm on $B_1^H(F_2)$.*

The question of describing all isometries of the scl norm, even for a free group, seems difficult but interesting.

References

[1] D. Calegari, *Stable commutator length is rational in free groups*, preprint, arXiv:0802.1352
[2] D. Calegari, *Surface subgroups from homology*, Geom. Topol. **12** (2008), 1995–2007
[3] D. Calegari, *Faces of the scl norm ball*, preprint, arXiv:0807.0395
[4] D. Calegari, *scl*, Monograph, to be published by the MSJ; available from the author's webpage
[5] D. Calegari, *Surgery and scl in free product of Abelian groups*, preprint, in preparation

Ordering manifolds by maps of non-zero degree

Dieter Kotschick
(joint work with C. Löh)

The existence of a continuous map $M \longrightarrow N$ of non-zero degree defines an interesting transitive relation, denoted $M \geq N$, on the homotopy types of closed oriented manifolds [4, 2, 1]. In every dimension, homotopy spheres represent an absolutely minimal element for this relation. In dimension two, the relation coincides with the order given by the genus, and substantial results are now known in dimension three as well.

In general, if M dominates N, then M is at least as complicated as N. For example, $M \geq N$ implies that the Betti numbers of M are at least as large as those of N and that the fundamental group of M surjects onto a finite index subgroup of the fundamental group of N. However, these necessary conditions are in general very far from being sufficient, and the relation $\geq$ is poorly understood in higher dimensions. Nevertheless, interesting results about it have been obtained for two different kinds of targets N: either N is assumed to be highly connected, or N is assumed to be negatively curved, or at least to have a large universal covering in a suitable sense. In the highly connected case the methods of algebraic topology have been successfully applied to the study of the domination relation in the work of Duan and Wang. At the opposite end of the spectrum, for manifolds with large universal coverings, interesting results have been obtained via more geometric methods. These include Gromov's theory of bounded cohomology [2], most notably the concept of simplicial volume, and the theory of harmonic maps, as applied to the domination question by Siu, Sampson, Carlson–Toledo and others, cf. [1] and the literature quoted there.

In this talk I reported on joint work with C. Löh, see [3], in which we proved that certain manifolds cannot be dominated by any non-trivial product of closed manifolds. One of our motivations stems from Gromov's discussion of functorial semi-norms on homology. Gromov suggested that many interesting homology classes should not be representable by products (of surfaces) and singled out the fundamental classes of irreducible locally symmetric spaces as specific candidates. As a special case of our results, we confirmed Gromov's suggestion in the following general form: if P is any non-trivial product of closed manifolds and N is a closed irreducible locally symmetric space of non-compact type, then $P \not\geq N$. Another motivation for results of the type $P \not\geq N$ comes from the study of diffeomorphism groups, where the special case $(M \times S^1) \not\geq N$ occurs.

Our methods, while inspired by differential geometry and by Gromov's theory of the simplicial volume [2], are, for the most part, elementary. We combine basic homotopy theory with the discussion of certain purely algebraic properties of fundamental groups. More specifically, we translate domination by products on the level of manifolds into properties of the corresponding fundamental groups. As the images of the fundamental groups of the factors commute in the fundamental group of the target and generate a subgroup of finite index, domination by a product forces the fundamental group of the target to have a certain amount of commutativity. This alone is often enough to prove $P \not\geq N$.

For the formulation of our results we need to introduce some terminology. The first definition is due to Gromov [2].

Definition 1. *A closed, oriented, connected n-manifold N is called essential if $H_n(c_N)([N]) \neq 0 \in H_n(B\pi_1(N);\mathbb{Z})$, where $c_N\colon N \longrightarrow B\pi_1(N)$ classifies the universal covering of N. It is rationally essential if $H_n(c_N)([N]) \neq 0$ in $H_n(B\pi_1(N);\mathbb{Q})$.*

Sufficient conditions to ensure essentialness are: asphericity, non-zero simplicial volume, enlargeability, or the non-vanishing of certain asymptotic invariants, like the minimal volume entropy or the spherical volume. All these properties, except possibly the last one, actually ensure rational essentialness.

The crucial property of fundamental groups that we require is encapsulated in the second definition:

Definition 2. *An infinite group Γ is not presentable by a product if, for every homomorphism $\varphi\colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ onto a subgroup of finite index, one of the factors Γ_i has finite image $\varphi(\Gamma_i) \subset \Gamma$.*

Using these definitions, the naturality of classifying maps leads to the following result, proved in [3]:

Theorem 1. *If N is a closed, oriented, connected rationally essential manifold whose fundamental group is not presentable by a product, then $P \not\geq N$ for any non-trivial product P of closed, oriented, connected manifolds.*

This is complemented by an algebraic result, also proved in [3], providing examples of groups not presentable by products:

Theorem 2. *The following groups are not presentable by products:*

- (H) *hyperbolic groups that are not virtually cyclic,*
- (N-P) *fundamental groups of closed Riemannian manifolds of non-positive sectional curvature of rank one and of dimension ≥ 2,*
- (MCG) *mapping class groups of closed oriented surfaces of genus ≥ 1.*

The rank occurring in statement (N-P) can be taken to be either the geometric rank of the Riemannian metric, or the rank of the fundamental group. It is a result of Ballmann–Eberlein that the two agree.

Of course, as the groups in Theorem 2 are of geometric origin, the proof uses information gleaned from geometry. The case of fundamental groups of strictly negatively curved manifolds is contained as a special case in both (H) and (N-P). For these groups it is an elementary application of Preissmann's theorem to show that they are not presentable by products. Our proofs of the cases (H) and (MCG) follow the same line of argument, using the fact that most elements of those groups have small centralisers.

Theorem 2, particularly statement (H), shows that our methods are well suited to the study of targets that have some sort of negative curvature property. This is also true for the applications of the simplicial volume and of harmonic maps mentioned earlier. However, in contrast with those techniques, our methods also apply to manifolds and groups that are not non-positively curved at all. For instance, the mapping class groups of surfaces of genus ≥ 2 occurring in Theorem 2 are not hyperbolic because they contain Abelian subgroups of large rank. In fact, they do not even have any semi-simple actions by isometries on CAT(0)-spaces.

It turns out that the property not to be presentable by a product is quite common, and is also a consequence of many other weak negative curvature conditions.

In [3], Löh and I also studied some limiting cases of the above theorems. These concern situations in which certain manifolds may be dominated by products, but only under very restrictive circumstances. These restrictive circumstances then lead to a rigidity result. Here is an example of such a result from [3]:

Theorem 3. *Let N be a closed, oriented, connected Riemannian manifold with non-positive sectional curvature and Γ its fundamental group. If the dimension of N is at least two, then the following properties are equivalent:*

(1) *$P \geq N$ for some non-trivial product P of closed, oriented manifolds,*
(2) *the fundamental group Γ is presentable by a product,*
(3) *some finite index subgroup of Γ splits as a non-trivial direct product,*
(4) *there is a finite covering of N diffeomorphic to a non-trivial product $N_1 \times N_2$ of closed, oriented manifolds N_i.*

Thus, a non-positively curved manifold is dominated by a product if and only if it is virtually diffeomorphic to a product. The result about locally symmetric spaces of non-compact type mentioned earlier is in fact a special case of Theorem 3.

References

[1] J. A. Carlson and D. Toledo, *Harmonic mapping of Kähler manifolds to locally symmetric spaces*, Publ. Math. I.H.E.S. **69** (1989), 173–201.
[2] M. Gromov, *Volume and bounded cohomology*, Publ. Math. I.H.E.S. **56** (1982), 5–99.
[3] D. Kotschick and C. Löh, *Fundamental classes not representable by products*, Preprint arXiv:0806.4540 .
[4] J. W. Milnor and W. P. Thurston, *Characteristic numbers of 3-manifolds*, Enseign. Math. **23** (1977), 249–254; reprinted in J. W. Milnor, *Collected Papers*, Vol. **2**, Publish or Perish 1995.

Dynamical aspects of spectral asymptotics

Iosif Polterovich

The asymptotic behaviour of eigenvalues and eigenfunctions of the Laplacian on a Riemannian manifold is closely linked to the dynamical properties of the geodesic flow. This could be observed, for instance, by comparing the growth of the remainder in Weyl's law for the distribution of eigenvalues on a round sphere and on a flat torus. A related quantity that also has interesting dynamical features is the spectral function of the Laplacian:

$$N_{x,y}(\lambda) = \sum_{\lambda_i < \lambda} \phi_i(x)\phi_i(y).$$

Here x and y are points on a n-dimensional compact Riemannian manifold M, $0 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots$ are the eigenvalues, and ϕ_i form an orthonormal basis of eigenfunctions: $\Delta\phi_i = \lambda_i^2\phi_i$, $\int_M \phi_i\phi_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

A classical result due to Avakumovič and Levitan states that $N_{x,y}(\lambda) = O(\lambda^{n-1})$ for any $x \neq y$. This bound is sharp and attained when x and y are diametrically opposite points on a round sphere. In [JP] it was shown that

$$N_{x,y}(\lambda) \neq o\left(\lambda^{\frac{n-1}{2}}\right) \tag{1}$$

on any manifold if x and y are not conjugate along any shortest geodesic joining them. Most likely, the non-conjugacy condition is purely technical and (1) is true for all points x and y on any manifold. The estimate (1) is also sharp: indeed,

$$N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}}\right) \tag{2}$$

for any two *non*–diametrically opposite points x and y on a round sphere [LPS].

Such a difference between the growth of the spectral function at opposite and non-opposite points on a sphere has a simple dynamical explanation: if x and y are diametrically opposite, *any* geodesic emanating from x hits y. This produces a strong singularity in the wave kernel. The wave kernel and the spectral function are related via the Fourier transform, and therefore bigger singularity in the wave kernel yields faster growth of the spectral function.

It turns out that on an arbitrary compact n-dimensional Riemannian manifold M, the spectral function grows *on average* similarly to (2). In [LPS] we show that for any finite measure ν on $\mathbb{R}_+$ and any $x \in M$, there exists a subset $M_{x,\nu} \subset M$ of full measure, such that

$$\int_0^\infty \left| \frac{N_{x,y}(\lambda)}{1+\lambda^{\frac{n-1}{2}}} \right|^2 d\nu(\lambda) \ < \ \infty\,, \quad \forall y \in M_{x,\nu}\,. \tag{3}$$

Note that the set $M_{x,\nu}$ depends on the measure ν and, in particular, (3) does not imply that $N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}}\right)$ for all x and almost all y on any manifold. In fact, this is false: as was shown in [JP], for all $x \neq y$ on a negatively curved manifold,

$$N_{x,y}(\lambda) \neq o\left(\lambda^{\frac{n-1}{2}} (\log\lambda)^{C-\delta}\right)$$

for any $\delta > 0$. Here $C > 0$ is a positive constant depending on the dynamical characteristics of the geodesic flow, such as the topological entropy and the Sinai-Ruelle-Bowen potential.

Estimate (3) could be interpreted as follows. For fixed $x \neq y$, the spectral function is an oscillating function of λ. Estimate (3), together with the lower bound (1), indicates that the amplitudes of the spectral function are on average of order $\lambda^{\frac{n-1}{2}}$. In particular, average growth of the spectral function does not feel the dynamics and depends only on the dimension of the manifold.

In order to study the frequencies of oscillations of the spectral function, in [LPS] we use the theory of almost periodic functions (see [KMS, Bl] for an application of this theory in a related context). Recall that the space B^p, $p \geq 1$, of Besicovitch almost periodic functions [Be] is defined as the completion of the linear space of all finite trigonometric sums $\sum_{k=1}^N a_k e^{i\theta_k x}$ with $a_k \in \mathbb{C}$ and $\theta_k \in \mathbb{R}$ with respect to the seminorm

$$||f||_{B^p} = \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x)|^p dx \right)^{1/p}.$$

For each real-valued function $f \in B^p$, there exists a sequence of real numbers θ_k called the *frequencies* of f, such that

$$\lim_{N\to\infty} ||f - \sum_{k=1}^{N} a_k \sin(\theta_k x + \alpha_k)||_{B^p} = 0, \tag{4}$$

where the coefficients $a_k \in \mathbb{R}$ and the phase shifts $\alpha_k \in \mathbb{R}$ are some constants. If (4) holds, we write $f \sim \sum a_k \sin(\theta_k x + \alpha_k)$.

In [LPS] we formulate the following problem. Assume, for simplicity, that M is a surface (a similar question could be asked for manifolds of any dimension). Does there exist $p \geq 1$, such that for all $x \in M$ and almost all $y \in M$,

$$\frac{N_{x,y}(\lambda)}{1+\sqrt{\lambda}} \sim \frac{2}{(2\pi)^{\frac{3}{2}}} \sum_{\gamma\in\Gamma_{x,y}} \frac{\sin(\lambda\, l(\gamma) - \frac{\pi}{4} - m(\gamma)\frac{\pi}{2})}{l(\gamma)\sqrt{|J(l(\gamma)|}} \tag{5}$$

in the Besicovitch space B^p ? Here $\Gamma_{x,y}$ is the set of all geodesic segments between x and y, $l(\gamma)$ is the length of γ, $m(\gamma)$ is the Morse index of γ and $J(t)$ is the orthogonal Jacobi field along γ with the initial conditions $J(0) = 0$, $J'(0) = 1$. Note that the right–hand side of (5) is well-defined for all x and y that are not conjugate along any geodesic segment joining them, which is true for all x and almost all y on any surface.

Formula (5) suggests that the frequencies of the spectral function $N_{x,y}(\lambda)$ have a dynamical meaning: they are the lengths of geodesic segments joining x and y.

In [LPS] we prove (5) with $p = 2$ for round spheres and flat tori. It is likely that the same is also true for surfaces of revolution and Liouville tori, as well as for negatively curved surfaces. In fact, we are not aware of any counterexample to (5) with $p = 2$. Note that even in the weakest case $p = 1$, formula (5) would imply that the rescaled spectral function has a limit distribution, which is a property of independent interest.

References

[Be] A. Besicovitch. *Almost periodic functions.* Dover Publications, 1954.

[Bl] P. Bleher. *Distribution of energy levels of a quantum free particle on a surface of revolution.* Duke Math. J. 74 (1994), no. 1, 45–93.

[JP] D. Jakobson, I. Polterovich, *Estimates from below for the spectral function and for the remainder in local Weyl's law.* Geom. Funct. Anal. 17, no. 3 (2007), 806-838.

[KMS] D. Kosygin, A. Minasov, and Y. Sinai. *Statistical properties of the spectra of Laplace-Beltrami operators on Liouville surfaces.* Russian Math. Surveys 48 (1993), no. 4, 1–142.

[LPS] H. Lapointe, I. Polterovich, Yu. Safarov, *Average growth of the spectral function on a Riemannian manifold.* arXiv:0803.4171, 1-35. To appear in Comm. Partial Differential Equations.

The Maslov cocycle, smooth structures and real-analytic complete integrability

Leo T. Butler

1. Abstract

This talk proves two main results. First, it is shown that if Σ is a smooth manifold homeomorphic to the standard n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ and H is a real-analytically completely integrable convex hamiltonian on $T^*\Sigma$, then Σ is diffeomorphic to $\mathbf{T}^n$. Second, it is proven that for some topological 7-manifolds, the cotangent bundle of each smooth structure admits a real-analytically completely integrable riemannian metric hamiltonian.

2. Introduction

One of the most intriguing facts of differential topology is that a topological manifold may admit several distinct smooth structures. An important smooth invariant of a smooth manifold is the cotangent bundle, and a smooth dynamical system on the cotangent bundle can reflect the smooth structure. I show in this talk how the existence of a real-analytically integrable convex hamiltonian on the cotangent bundle is a non-trivial smooth invariant.

2.1. **Complete integrability.** The cotangent bundle of a smooth manifold Σ admits a canonical symplectic form $\omega = \sum \dot{y}_i \wedge \dot{x}^i$, where x^i are coordinates on Σ and y_i are the induced fibre coordinates. A symplectic form permits one to define a Poisson algebra structure on $C^\infty(T^*\Sigma)$ and consequently each smooth function $H : T^*\Sigma \to \mathbf{R}$ induces a hamiltonian vector field X_H defined by

$$X_H = \{H, \ \} \qquad \Longrightarrow X_H = \begin{cases} \dot{x}^i &= \frac{\partial H}{\partial y_i}, \\ \dot{y}_i &= -\frac{\partial H}{\partial x^i}. \end{cases} \tag{1}$$

A first integral of the hamiltonian vector field X_H is a smooth function F which Poisson commutes with H: $\{H, F\} = 0$. If X_H has $n = \dim \Sigma$ functionally independent first integrals $F_1, \ldots, F_n$, and the first integrals pairwise Poisson commute, then the compact regular level sets $\{F_1 = c_1, \ldots, \ F_n = c_n\}$ are n-dimensional lagrangian tori and the flow of X_H is translation-type. In this case, one says that X_H is *completely integrable*; if the first integrals are real-analytic, one says that X_H is real-analytically completely integrable.

2.2. **Geometric semisimplicity.** Let us abstract the notion of complete integrability. A smooth flow $\varphi : M \times \mathbf{R} \to M$ is *integrable* if there is an open, dense subset $R \subset M$ that is covered by angle-action charts which conjugate φ to a translation-type flow on the tori of $\mathbf{T}^k \times \mathbf{R}^l$. There is an open dense subset $L \subset R$ fibred by φ-invariant tori; let $f : L \to B$ be the induced smooth quotient map and let $\Gamma = M - L$ be the *singular set*. If Γ is a tamely-embedded polyhedron, then φ is said to be *k-semisimple* with respect to (f, L, B), or just semisimple [12]. Of most interest is when φ is a hamiltonian flow on a cotangent bundle or possibly a regular iso-energy surface.

Definition 1 (*c.f.* [32, 12])**.** *A hamiltonian flow is* geometrically semisimple *if it is semisimple with respect to* (f, L, B) *and* f *is a lagrangian fibration. It is* finitely geometrically semisimple *if, in addition, each component of* B *has a finite fundamental group.*

In this case, the lagrangian-ness of the fibres of f implies that φ is completely integrable, so geometric semisimplicity is a topologically-tame type of complete integrability. Taimanov [32] introduced a related notion of geometric simplicity, see sections 2.2-2.3 of [12] for further discussion. If φ is real-analytically completely integrable, then the triangulability of real-analytic sets implies that φ is finitely geometrically semisimple; in fact, in this case B may be taken to be a disjoint union of open balls. On the other hand, geometric semisimplicity is a weaker property than real-analytic complete integrability [12]. A basic question is:

Question 1. *What are the obstructions to the existence of a geometrically semisimple (resp. semisimple, completely integrable) flow?*

2.3. **Main Results.** Recall that a topological n-torus is a topological manifold that is homeomorphic to the standard n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. An exotic n-torus is a topological n-torus that is not diffeomorphic to $\mathbf{T}^n$. Exotic n-tori may be constructed by connect summing with exotic spheres, but not all arise this way.

Theorem 2. *If Σ is an exotic n-torus, then there are no* finitely geometrically semisimple *convex hamiltonians on $T^*\Sigma$. In particular, there are no real-analytically completely integrable convex hamiltonians on $T^*\Sigma$.*

The obstruction here is the *smooth* structure of the configuration space. This is the first result that shows that a smooth invariant may preclude real-analytic complete integrability; as such, it prompts several questions.

Question 3. *If Σ is an exotic torus, does there exist a completely integrable convex hamiltonian on $T^*\Sigma$?*

It is also possible to show that there are no completely integrable riemannian metrics on an exotic torus that are completely integrable via a geodesic equivalence. This suggests that the answer to Question 3 may be *no*.

It is important to note that the definitiveness of Theorem 2 is not general and may be atypical. The Gromoll-Meyer exotic 7-sphere is a biquotient $\mathrm{Sp}(2)//\mathrm{Sp}(1)$ and so it inherits a submersion metric from the bi-invariant metric on $\mathrm{Sp}(2)$ [20]. Paternain and Spatzier [29] proved the real-analytic complete integrability of the geodesic flow of this submersion metric. On the other hand, the remaining 12 unoriented diffeomorphism classes of the 7-sphere are not known to possess such geodesic flows.

Question 4. *Do all exotic* 7*-spheres admit a real-analytically completely integrable convex hamiltonian?*

And, more generally,

Question 5. *What are the smooth obstructions to the existence of a geometrically semisimple convex hamiltonian?*

This talk does not directly address Questions 3–5, but it is able to answer the question for some classes of topological 7-manifolds with more than one smooth structure. A *Witten-Kreck-Stolz space* $M_{k,l}$ is the smooth 7-manifold obtained by quotienting $S^5 \times S^3$ by the action of U_1 given by the representation $z \mapsto z^k \cdot I \oplus z^l \cdot I : \mathrm{U}_1 \to \mathrm{U}_3 \oplus \mathrm{U}_2$, where k and l are coprime integers. Kreck and Stolz showed that $M_{k,l}$ has a maximum of 28 smooth structures; and, with modest conditions on k and l, this maximum is attained and each smooth structure is represented by some $M_{k',l'}$ [21]. One can use the work of Mykytyuk and Panasyuk [27] to show that

Theorem 6. *There is a real-analytically completely integrable convex hamiltonian on the cotangent bundle of each Witten-Kreck-Stolz space. In particular, if $l = 0 \bmod 4$, $l = 0, 3, 4 \bmod 7$, $l \neq 0$ and $\gcd(k, l) = 1$, then each one of the* 28 *diffeomorphism classes of $M_{k,l}$ is the configuration space of a real-analytically completely integrable convex hamiltonian.*

The convex hamiltonian in all cases may be taken to be the hamiltonian induced by the round metrics on S^5 and S^3. For each Witten-Kreck-Stolz space, there is an S^1 fibre bundle $S^1 \hookrightarrow M_{k,l} \to \mathbf{C}P^2 \times \mathbf{C}P^1$. In [10] (resp. [11]) Bolsinov and Jovanović prove, *inter alia*, the real-analytic non-commutative (resp. complete) integrability of the geodesic flows of certain homogeneous metrics on $\mathbf{C}P^2 \times \mathbf{C}P^1$, see especially [11, Remark 3.4]. However, they do not directly consider the interesting class of Witten-kreck-Stolz spaces.

We also prove a similar result for the Eschenburg and Aloff-Wallach 7-manifolds. These manifolds are obtained through a quotient of SU_3 by a subgroup $V \cong \mathrm{U}_1$ of the maximal torus of $\mathrm{SU}_3 \times \mathrm{SU}_3$. The existence of real-analytically completely integrable geodesic flows on some special Eschenburg spaces was proven by Paternain & Spatzier and Bazaikin [29, 4]. The results of the present paper extend their work. Kruggel [22] has obtained a complete list of invariants that classify the smooth structures on most Eschenburg spaces. It is unknown if each topological

Eschenburg space admits the maximum 28 smooth structures and each smooth structure is itself an Eschenburg space. Numerical computations [15, 14] suggest this may be true for some families of Eschenburg spaces.

References

[1] S. Aloff and N. R. Wallach. An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures. *Bull. Amer. Math. Soc.*, 81:93–97, 1975.

[2] V.I. Arnold. On a characteristic class entering into conditions of quantization. *Funct. Anal. Appl.*, 1:1–13, 1967.

[3] V.I. Arnold. Sturm theorems and symplectic geometry. *Funct. Anal. Appl.*, 19(4):1–10, 1985.

[4] Ya. V. Bazaikin. Double quotients of Lie groups with integrable geodesic flows. *Siberian Math. J.*, 41(3):419–432, 2000.

[5] M. Bialy. On the number of caustics for invariant tori of Hamiltonian systems with two degrees of freedom. *Ergodic Theory Dynam. Systems* 11(2):273–278, 1991.

[6] M. L. Bialy and L. V. Polterovich. Geodesic flows on the two-dimensional torus and "commensurability-incommensurability" phase transitions. *Funktsional. Anal. i Prilozhen.*, 20(4):9–16, 96, 1986.

[7] M. L. Bialy and L. V. Polterovich. Lagrangian singularities of invariant tori of Hamiltonian systems with two degrees of freedom. *Invent. Math.*, 97(2):291–303, 1989.

[8] M. L. Bialy and L. V. Polterovich. Hamiltonian diffeomorphisms and Lagrangian distributions. *Geom. Funct. Anal.*, 2(2):173–210, 1992.

[9] M. L. Bialy and L. V. Polterovich. Hamiltonian systems, Lagrangian tori and Birkhoff's theorem. *Math. Ann.* 292(4):619–627, 1992.

[10] A.V. Bolsinov and B. Jovanović. Non-commutative integrability, moment map and geodesic flows. *Ann. Global Anal. Geom.*, 23(4):305–322, 2003.

[11] A.V. Bolsinov and B. Jovanović. Complete involutive algebras of functions on cotangent bundles of homogeneous spaces. *Math. Z.*, 246(1-2):213–236, 2004.

[12] L. T. Butler. Invariant fibrations of geodesic flows. *Topology*, 44(4):769–789, 2005.

[13] L. T. Butler. Geometry and real-analytic integrability. *Regul. Chaotic Dyn.*, 11(3):363–369, 2006.

[14] L. T. Butler. Smooth structures on Eschenburg spaces: numerical computations. *preprint.*

[15] T. Chinburg, C. Escher and W. Ziller. Topological properties of Eschenburg spaces and 3-Sasakian manifolds. *Math. Ann.* 339(1):3–20, 2007.

[16] G. Contreras, J.-M. Gambaudo, R. Iturriaga and G.P. Paternain. The asymptotic Maslov index and its applications. *Ergodic Theory Dynam. Systems* 23(5):1415–1443, 2003.

[17] J.J. Duistermaat. On the Morse index in variational calculus. *Advances in Math.* 21(2):173–195, 1976.

[18] H. R. Dullin, J. M. Robbins, H. Waalkens, S. C. Creagh, and G. Tanner. Maslov indices and monodromy. *J. Phys. A*, 38(24):L443–L447, 2005.

[19] J.-H. Eschenburg. New examples of manifolds with strictly positive curvature. *Invent. Math.*, 66(3):469–480, 1982.

[20] D. Gromoll and W. Meyer. An exotic sphere with nonnegative sectional curvature. *Ann. of Math. (2)*, 100:401–406, 1974.

[21] M. Kreck and S. Stolz. A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry. *Ann. of Math. (2)*, 127(2):373–388, 1988.

[22] B. Kruggel Homeomorphism and diffeomorphism classification of Eschenburg spaces. *Q. J. Math.*, 56(4):553–577, 2005.

[23] F. Lalonde and J.-C. Sikorav. Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents. *Comment. Math. Helv.*, 66(1):18–33, 1991.

[24] G. Luke and A.S. Mishchenko. *Vector bundles and their applications.* Mathematics and its Applications, 447. Kluwer Academic Publishers, Dordrecht, 1998.

[25] V. S. Matveev. Projectively equivalent metrics on the torus. *Differential Geom. Appl.*, 20(3):251–265, 2004.
[26] A. S. Miščenko and A. T. Fomenko. Euler equation on finite-dimensional Lie groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(2):396–415, 471, 1978.
[27] I. V. Mykytyuk and A. Panasyuk. Bi-Poisson structures and integrability of geodesic flow on homogeneous spaces. *Transform. Groups*, 9(3):289–308, 2004.
[28] G. P. Paternain. *Geodesic flows*, volume 180 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.
[29] G. P. Paternain and R. J. Spatzier. New examples of manifolds with completely integrable geodesic flows. *Adv. Math.*, 108(2):346–366, 1994.
[30] L. Polterovich Monotone Lagrange submanifolds of linear spaces and the Maslov class in cotangent bundles. *Math. Z.*, 207(2):217–222, 1991.
[31] M. Rudnev and V. Ten. Integrability versus topology of configuration manifolds and domains of possible motions. *Arch. Math. (Basel)*, 86(1):90–96, 2006.
[32] I. A. Taimanov. Topological obstructions to the integrability of geodesic flows on nonsimply connected manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(2):429–435, 448, 1987.
[33] P. Topalov and V. S. Matveev. Geodesic equivalence via integrability. *Geom. Dedicata*, 96:91–115, 2003.
[34] C. Viterbo. A new obstruction to embedding Lagrangian tori. *Invent. Math.*, 100(2):301–320, 1990.
[35] C. Viterbo. Exact Lagrange submanifolds, periodic orbits and the cohomology of free loop spaces. *J. Differential Geom.*, 47(3):420–468, 1997.
[36] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.
[37] M. Wang and W. Ziller. Einstein metrics with positive scalar curvature. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, volume 1201 of *Lecture Notes in Math.*, pages 319–336. Springer, Berlin, 1986.
[38] E. Witten. Search for a realistic Kaluza-Klein theory. *Nuclear Phys. B*, 186(3):412–428, 1981.

Characterization of Riemannian metrics on T^2 with and without positive topological entropy

Eva Glasmachers
(joint work with Gerhard Knieper)

Let (T^2, g) be a Riemannian two-dimensional torus. By the properties of single geodesics we classify the geodesic flow in the two cases of positive and vanishing topological entropy. In other words we formulate sufficient and necessary conditions for vanishing topological entropy of the geodesic flow depending on the behavior of the geodesics. The motivatation for these criteria comes from the properties of the class of minimal geodesics on the universal covering $\mathbb{R}^2$. These geodesics have the following properties:

(1) Two different minimal geodesics intersect at most once.
(2) Minimal geodesics correspond to Euclidean lines in the following sense: There exists a positive constant $D > 0$ such that
a) for a minimal geodesic c there exists a line l_c such that $d(c, l_c) \leq D$ and
b) for a line l there exists a minimal geodesic c_l such that $d(l, c_l) \leq D$.

According to the slope of the Euclidean lines we induce the notion of a rotation number $\alpha \in \mathbb{R} \cup \{\infty\}$ for each minimal geodesic.
Let $\mathcal{M}_\alpha = \{c : \mathbb{R} \to \mathbb{R}^2 \mid c \text{ is minimal with rotation number } \alpha\}$.

(3) For α irrational the set $\mathcal{M}_\alpha$ is ordered, i.e. two minimal geodesics with the same rotation number do not intersect. For α rational and $\alpha = \infty$, $\mathcal{M}_\alpha$ decomposes into disjoint sets $\mathcal{M}_\alpha^{per}$, $\mathcal{M}_\alpha^{\pm}$ and $\mathcal{M}_\alpha^{\mp}$. Each of these sets is ordered. An easy consequence of this ordering property is that no minimal geodesic intersects its $\mathbb{Z}^2$-translates on the universal covering.

Applying these properties proved by H. M. Morse [7], G. A. Hedlund [4] and V. Bangert [2] we could show the following result: When we consider only inital conditions of minimal geodesics denoted by $\tilde{S}T^2 = \{v \in ST^2 \mid c_v \text{ with } \dot{c}_v(0) = v$ is minimal$\}$ then for the restricted geodesic flow $\tilde{\phi}^t : \tilde{S}T^2 \to \tilde{S}T^2$ the topological entropy $h_{top}(\tilde{\phi}^t)$ vanishes.

Considering arbitrary geodesics on the universal covering we distinguish the following types of half-geodesics $c^+ : \mathbb{R}^{\geq 0} \to \mathbb{R}^2$.

- c^+ is *unbounded*, i.e. for all $K \subset \mathbb{R}^2$ compact there exists $t_0 \geq 0$ with $c^+(t) \notin K$ for all $t \geq t_0$.
- c^+ is *bounded*, i.e. there exists K compact with $c^+(t) \in K$ for all $t \geq 0$.
- c^+ is *oscillating*, i.e. there exist K compact and two sequences $s_n, t_n \to \infty$ such that $c^+(s_n) \in K$ and $||c^+(t_n)|| \to \infty$.

Analogously we distinguish unbounded, bounded and oscillating half-geodesics $c^- : \mathbb{R}^{\leq 0} \to \mathbb{R}^2$. For unbounded half-geodesics we generalize the notion of a rotation number introduced for minimal geodesics:

Definition: *Let c^+ be unbounded then, if the limit exists, let*

$$\delta(c^+) = \lim_{t \to \infty} \frac{c^+(t)}{||c^+(t)||} \in S^1$$

be the forward rotation direction and with $\delta(c^+) = (x, y)$ let

$$\rho(c^+) = \frac{y}{x} \in \mathbb{R} \cup \{\infty\}$$

be the forward rotation number of c^+.

Analogously we define the backward rotation direction and the backward rotation number of a unbounded half-geodesic $c^- : \mathbb{R}^{\leq 0} \to \mathbb{R}^2$.

Let us demand the intersection properties of minimal geodesics for arbitrary geodesics as well as the existence of rotation numbers. Then we get the following theorems for the topological entropy of the geodesic flow $\phi^t : ST^2 \to ST^2$:

Theorem I:
Let $h_{top}(\phi^t) = 0$. Then, for all lifts of geodesics on T^2 there exists the forward rotation number.

Theorem II:
If no lift of a periodic geodesic intersects its translates then $h_{top}(\phi^t) = 0$.
If lifts of all geodesics do not intersect their translates then this condition is even equivalent to the flatness of the torus.

Theorem I is motivated by an analogous result for orbits of twist maps by S. B. Angenent [1]. Several times in the proof of Theorem I we construct a separating set of initial conditions of closed geodesics on the unit tangent bundle such that its cardinality grows exponentially with the length of these geodesics. This positive growth rate is a lower bound for the topological entropy. A central tool for the construction of these geodesics is the so-called Curve-Shortening-flow, see e.g. [3]. By geodesic barriers we force the solutions under this flow to take a predetermined shape. In the proof of Theorem II we assume that the topological entropy is positive which implies by a well known result of A. Katok [6] the existence of a Smale-horseshoe on the unit tangent bundle. Studying the geodesics involved we conclude that under the given intersection assumption the periodic orbits in the horseshoe cannot grow exponentially with their length, which is a contradiction. For the second part of Theorem II we show that for each rational rotation number minimal geodesics foliate T^2. A result by N. Innami [5] then implies that in this case the torus is already flat.

References

[1] S. B. Angenent, *Monotone recurrence relations, their Birkhoff orbits and topological entropy*, Ergod. Th. and Dynam. Sys. **10** (1990), 15-41.
[2] V. Bangert, *Mather Sets for Twist Maps and Geodesics on Tori*, Dynamics Reported **1** (1988), 1-56.
[3] M. A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), no. 1, 71-111.
[4] G. A. Hedlund, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*, Ann. of Math. **33** (1932), 719-739.
[5] N. Innami, *Families of geodesics which distinguish flat tori*, Math. J. Okayama Univ. **28** (1986), 207-217.
[6] A. Katok, *Nonuniform Hyperbolicity and Structure of Smooth Dynamical Systems* (English), Proc. Int. Congr. Math. Warszawa 1983, **2** (1984), 1245-1253.
[7] H. M. Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. **26** (1924), 25-60.

Floer homology for homoclinic tangles

Sonja Hohloch

We point out a relation between two important topics of symplectic dynamical systems — homoclinic points and Lagrangian Floer homology. We construct a new symplectic invariant for homoclinic tangles called *primary homoclinic Floer homology.* This talk is mainly based on [7].

Let (M, ω) be a symplectic manifold and ϕ a symplectomorphism with hyperbolic fixed point x. For symplectomorphisms the (un)stable manifolds $W^u := W^u(x, \phi)$

and $W^s := W^s(x,\phi)$ are *Lagrangian submanifolds.* Thus the set of *homoclinic points* $\mathcal{H} := W^s \cap W^u$ can be seen as the intersection set associated to the *non-compact Lagrangian intersection problem* (W^s, W^u). This motivates the construction of Lagrangian Floer homology [3, 4, 5] for homoclinic tangles. Due to (up to now unsolved) compactness problems with certain moduli spaces we only work in dimension two, where we can replace the analysis by combinatorics [2, 6]. Thus from now on let (M,ω) be $(\mathbb{R}^2, dx \wedge dy)$ or a symplectic closed two-dimensional manifold with genus $g \geq 1$.

The main obstacle is the abundance of intersection points. If there is one (transverse) homoclinic point then Smale's horseshoe formalism assures the existence of infinitely many. Also, if we divide by the $\mathbb{Z}$-action of ϕ on $\mathcal{H}$ the quotient $\mathcal{H}/\mathbb{Z}$ still has infinitely many elements. This prevents the well-definedness of the usual Floer differential on $\mathcal{H}$ or $\mathcal{H}/\mathbb{Z}$. Classical methods, like for example the action filtration [1], fail to overcome this problem.

Nevertheless, there is a natural subset of $\mathcal{H}$ on which the Floer differential is well-defined: Denote by $[p,q]_s$ resp. $[p,q]_u$ the segment between p and q in W^s resp. W^u. We call p contractible if the loop $[p,x]_s \cup [p,x]_u$ is contractible in M. We denote by $\mathcal{H}_{[x]} \subset \mathcal{H}$ the set of contractible homoclinic points and call

$$\mathcal{H}_{pr} := \{p \in \mathcal{H}_{[x]} \backslash \{x\} \mid\,]p,x[_s \cap\,]p,x[_u \cap \mathcal{H}_{[x]} = \emptyset\}$$

the set of *primary* points. $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/\mathbb{Z}$ is finite and we denote the equivalence class of $p \in \mathcal{H}_{pr}$ by $\langle p \rangle$. The Maslov index μ induces a grading $\mu : \tilde{\mathcal{H}}_{pr} \to \mathbb{Z}$ and we define analogously to classical Lagrangian Floer homology

$$C_k := C_k(x,\phi) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p \rangle)=k}} \mathbb{Z}\langle p \rangle,$$

$$\partial \langle p \rangle := \sum_{\substack{\langle q \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q \rangle)=\mu(\langle p \rangle)-1}} m(\langle p \rangle, \langle q \rangle)\langle q \rangle,$$

$$H_* := H_*(x,\phi) := \frac{\ker \partial}{\operatorname{Im} \partial}.$$

The well-definedness of ∂ and the proof of $\partial \circ \partial = 0$ are tricky combinations of dynamical and combinatorial arguments.

H_* is invariant under so called *contractibly strongly intersecting (symplectic) isotopies.* The proof has to combine analytical *and* combinatorial arguments since a primary point $p \in \mathcal{H}_{pr}$ might vanish (analogously arise) in two ways:

- p vanishes as intersection point,
- p persists as intersection point, but is no longer primary.

The invariance implies an existence and bifurcation criterion for homoclinic points and the fixed point. We conjecture that Hamiltonian isotopies are naturally strongly intersecting.

In the two-dimensional situation, H_* also can be defined for nonsymplectic diffeomorphisms, but there is no natural invariance. Thus H_* is a symplectic invariant.

H_* is invariant under conjugacy of ϕ. Moreover, there is an estimate comparing the ranks of $H_*(x, \phi)$ and $H_*(x, \phi^n)$. Another version, called *chaotic primary homoclinic Floer homology*, also takes the chaos *near* a homoclinic tangle into account and gives rise to a symplectic zeta function.

H_* is the first invariant which takes the *algebraic* interaction of homoclinic points into account. Moreover, H_* is simultanously a *semi-global and semi-local* invariant: On the one hand, the branches and homoclinic points can lie anywhere on the manifold, but, on the other hand, we are (up to now) bound to contractible points. There is no direct way to relate $H_*(x, \phi)$ to the topology of M or W^s and W^u.

References

[1] A. Abbondandolo, M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), no. 2, 254 – 316

[2] S. de Silva, *Products in the symplectic Floer homology of Lagrangian intersections* Ph. D. thesis, Merton College, Oxford 1998

[3] A. Floer, *A relative Morse index for the symplectic action* Comm. Pure Appl. Math., Vol. **41** (1988), 393 – 407

[4] A. Floer, *The unregularized gradient flow of the symplectic action* Comm. Pure Appl. Math., Vol. **41** (1988), 775 – 813

[5] A. Floer, *Morse theory for Lagrangian intersections* J. Diff. Geom. **28** (1988), 513 – 547

[6] R. Gautschi, J. Robbin, D. Salamon, *Heegard splittings and Morse-Smale flows* Preprint ETH Zürich, October 2002

[7] S. Hohloch, *Floer homology for homoclinic tangles*, Ph.D. thesis, University of Leipzig 2008

Area spaces: First Steps.

Dmitri Burago

(joint work with S. Ivanov)

This research is motivated by the idea of studying spaces using areas instead of lengths. In particular, one can think of a space formed by closed curves equipped with a function that mimics the "minimal filling area". Hence the first question we asked ourselves was: "How much information does this space capture in case of a Riemannian manifold?".

For a Riemannian manifold M, let $\mathcal{S}_m(M)$ denote the abelian group of all m-dimensional Lipschitz chains with integer coefficients and $\mathcal{B}_m(M)$ the image of the boundary map $\partial : \mathcal{S}_{m+1}(M) \to \mathcal{S}_m(M)$.

The group $\mathcal{B}_1(M)$ is equipped with a (possibly non-homogeneous) semi-norm $|\gamma|_F = \inf\{area(s) : s \in \mathcal{S}_2(M), \partial s = \gamma\}$ In other words, $|\gamma|_F$ is the filling area of γ in M. This yields a semi-metric d_F on $\mathcal{B}_1(M)$: $d_F(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2|_F$.

Let $\mathcal{B}(M)$ denote the associated metric space $\mathcal{B}_1(M)/d_F$, that is, we identify γ_1 and γ_2 if $d_F(\gamma_1, \gamma_2) = 0$. We refer to $\mathcal{B}(M)$ as the *cycle space* of M.

This space is not complete. This is an Abelian group equipped with an invariant intrinsic metric induced by the filling norm. Let $\tilde{B}(M)$ denote the completion of $\mathcal{B}(M)$, which is thus also a geodesic space. Note that the norm $|\cdot|_F$ and the abelian group structure naturally extend to $\tilde{B}(M)$.

In dimension 2, the only information this "loop space" carries is the area of the surface. Even the genus cannot be recovered. However, it is easy to see that an area preserving diffeomorphism between higher dimensional Riemannian manifolds is an isometry. With a little bit more care one can show that if a diffeomorphism between M and M' induces an isomorphism between $\tilde{B}(M)$ and $\tilde{B}(M')$, then the diffeomorphism is an isometry. The main result discussed in this talk is a far-streched generalization of this observation. It says that the space $\tilde{B}(M)$ carries complete information about M if the dimension of M is at least 3:

Theorem. Let M, M' be compact Riemannian manifolds (possibly with boundaries) of dimension ≥ 3. Suppose that the metric spaces $\tilde{B}(M)$ and $\tilde{B}(M')$ are isometric via a homogeneous isometry Φ (that is, $\Phi(ks) = k\Phi(s)$ for all $k \in \mathbb{Z}$.) Then M and M' are isometric. Moreover, every homogeneous isometry $\Phi : \tilde{B}(M) \to \tilde{B}(M')$ is induced by a Riemannian isometry $\phi : M \to M'$ so that $\phi_* = \Phi$.

Note that we require that the isometry between $\tilde{B}(M)$ and $\tilde{B}(M')$ is homogeneous, that is the structure we deal with remembers which cycles are multiples of other cycles. It would be extremely surprising if this condition is essential, but so far we could not remove it.

The result immediately raises many questions. First of all, one would like to define "area spaces" similarly to "length spaces", with an ultimate goal of defining convergence and proving compactness theorems. This could be helpful in obtaining inequalities involving areas in situations without a good control over lengths. A possible convergence could be a certain type of convergence of "loop spaces", so in the limit one could get an abstract "loop space" without an underlying manifold. Two different surfaces Σ_1 and Σ_2 of the same area multiplied by a circle or length ε when $\varepsilon \to 0$ already give an interesting example (for the "loop spaces" for the surfaces are the same). The main difficulty is that $\tilde{B}(M)$ is full of strange "garbage". For instance, consider a circle of radius $\frac{1}{5}$ centered at every integer point of a huge 3-D cube of size n, and re-scale the cube down to size $n^{-\frac{1}{2}}$. One gets a "cycle" whose filling area is $\pi/25$ contained in a microscopic chunk of space. There are "cycles" of huge filling norm formed by "dust" of microscopic circles spread all over the manifold. These ugly cycles have nothing to do with objects we may be ultimately interested in, but they caused a lot of trouble when we were working on the proof of the main theorem, and they cause a lot of problems in our attempts to define convergence. Note that it is very unlikely that the recovery result is stable. Spaces of this type even tend to be universal (like in two dimensions) rather than unique for each manifold (and, probably, this would be the case if we considered cycles over reals rather than integer cycles, though this is not known). The theorem is based on certain rigidity. Perhaps this (plausible) lack of stability could even be good, for whatever convergence one defines for "area spaces", it should not imply metric convergence.

Furthermore, even though $\tilde{B}(M)$ uniquely determines M, it would be nice to recover basic properties of M directly from $\tilde{B}(M)$ It is not clear what one can conclude from the assumption that there is a short map from $\tilde{B}(M)$ to $\tilde{B}(M')$, and what one should assume to guarantee the existence of an area non-increasing map with certain topological properties (say, of degree one).

Finally, one can run the "recovery construction" used in the proof of the theorem for an abstract Abelian group with an invariant metric. Of course, this metric should be a length metric, so one does not want to look at examples like $\mathbb{Z}$. Furthermore, as we mentioned earlier, it is crucial that we work with currents with integer coefficients, so perhaps one should think of an Abelian group which sits "as a lattice" in the tensor product of this group and $\mathbb{R}$ equipped with an invariant intrinsic metric. This is somewhat similar to looking at analogs of topological properties in C^* algebras that are not algebras of functions on a space. We have not tried studying any examples yet.

Poisson-Furstenberg boundary for random walks on solvable groups

Anna Erschler

Let G be a finitely generated group and μ be a probability measure on G. Consider the random walk on G with transition probabilities $p(x|y) = \mu(x^{-1}y)$, starting at the identity.

The space of infinite trajectories G^∞ is equipped with the measure which is the image of the infinite product measure under the following map from G^∞ to G^∞:

$$(x_1, x_2, x_3, \dots) \to (x_1, x_1x_2, x_1x_2x_3, \dots).$$

Consider two infinite trajectories X and Y. We say that that they are equivalent if there exits N, C such that for any $i > N$ $X_i = Y_{i+C}$. Consider the measurable hull of this equivalence relation in the space of infinite trajectories. The quotient by the obtained equivalence relation is called *Poisson-Furstenberg boundary.*

Equivalently, the poisson boundary is the space of ergodic components of the time shift in the path space G^∞.

Poisson-Furstenberg boundary is often called also *Poisson boundary*, and its σ-field is also called *invariant σ-field.*

We recall that a function $F : G \to \mathbb{R}$ is called μ-harmonic, if for all $g \in G$ $F(g) = \sum_{h\in G} f(gh)\mu(h)$. It is known that the group G admits nonconstant positive harmonic functions with respect to some measure μ with the support generating G if and only if the Poisson-Furstenberg boundary of the random walk is non-trivial. The boundary can be defined in terms of bounded harmonic functions.

It is known that if the support of μ generates a non-amenable group, then the Poisson-Furstenberg boundary is non-trivial and that any amenable group G admits a symmetric measure with the support generating G such that the boundary of random walk is trivial (Kaimanovich, Vershik and Rosenblatt). First examples of symmetric random walks on amenable groups with nontrivial Poisson boundary were constructed by V.A.Kaimanovich and A.M.Vershik. They have shown that simple random walk on wreath products of $\mathbb{Z}^d$, $d \geq 3$ with a finite group (with at least two elements) has non-trivial Poisson boundary.

We recall that the *wreath product* of the groups A and B is a semidirect product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts: if $a \in A$, $f : A \to B, f \in \sum_A B$, then $f^a(x) = f(a^{-1}x), x \in A$. Let $A \wr B$ denote the wreath product.

By definition, any element of the wreath product is a pair (a, f), $a \in A$, $f : A \to B$ is such that for all but finite number of a we have $f(a) \neq e_B$, where e_B is a neutral element of B. We say that the support of f is a set of e such that $f(a) \neq e_B$.

In [1] V.A.Kaimanovich and A.M.Vershik have shown that for a simple random walk on $\mathbb{Z}^d$, $d \geq 3$ the value of configuration at any given point of the base $A = \mathbb{Z}^d$ (since the random walk on A is transient), and this implies non-triviality of the exit boundary. In [3] V.A.Kaimanovich has shown that a similar argument works also for measures with finite first moment. In that paper the case $A = \mathbb{Z}^d$, $B = \mathbb{Z}/2\mathbb{Z}$ has been considered, but the same argument works for arbitrary A and B: if the measure defining the random walk has finite first moment, than with probability 1 we can assign to each trajectory $X_1 = (a_1, f_1)$, $X_2 = (a_2, f_2)$, $\ldots$ on C the limit configuration $f : A \to B$, for all a_* $F(a_*) = \lim f_i(a_*)$.

Observe that the limit configuration is the same for any two infinite trajectories that coincide after some moment (X_i, X'_i are such that $X_i = X'_{i+K}$ for some constant K and any sufficiently large i). Note that the space of limit configurations carries a measure which is a projection of the measure on the space of infinite trajectories. Note also that C acts on the space of limit configuration by shifts, and that this action commutes with the action of C on infinite trajectories. A space with such property is called *a μ-boundary* of the random walk G, μ. It is known that any μ boundary is always a quotient of the Poisson boundary.

If we assume additionally that support of the measure μ generates C, then this lemma implies that the Poisson boundary of the random walk C, μ is not trivial. Indeed, assume the contrary. Then there is a configuration F such that with probability 1 the limit configuration in equal to F. Then for any $x \in C$ with probability 1 all trajectories starting from x have the same limit configuration F_x. Note, that for any $y \in supp\mu$ $F_x = yF_{xy}$. If support μ generates C, this can not happen.

It is known, that moreover, the boundary is non-trivial for any non-degenerate finite entropy measure μ as above such that the projection of the random walk on A is transient (see [5].

The following theorem states that for $C = A \wr B$, under some assumption on A and μ, the μ-boundary of limit configuration for the random walk C, μ is equal to the Poisson-Furstenberg boundary of C, μ.

Theorem Let μ be a non-degenerate measure on $C = A \wr B$ (the support of μ generates C as a group), $A = \mathbb{Z}^d$, $d \geq 5$, $\#B \geq 2$. Assume that the third moment of μ is finite. Then the Poisson boundary is equal to the space of limit configuration.

The theorem gives an answer for $d \geq 5$ for the question of Vershik and Kaimanovich about the boundary of simple random walks on $\mathbb{Z}^d \wr B$ ([1]). The question remains open in dimension 3 and 4. We discuss some partial results in these dimensions. Until now there were no known results about complete description of Poisson-Furstenberg boundary on wreath products $\mathbb{Z}^d \wr B$ for simple random walk (as it was considered in [1]) or for any other symmetric random walks. There were some results, however, about the non-reversible case. N.James and Yu.Peres has shown in [4] that the number of visits of points of the base provides a complete description of the Poisson-Furstenberg boundary of a certain measure on $\mathbb{Z}^d \wr \mathbb{Z}^+$.

One has known the complete description of the Poisson-Furstenberg boundary for the following finitely generated groups (under certain conditions on the decay of the probability measure defining the random walk): discrete subgroups in semi-simples Lie group (Ledrappier) for the case of discrete subgroups of $Sl(d, R)$, (Furstenberg) for a particular case of an infinitely supported measure, "Furstenberg approximation", Kaimanovich for the general case), free groups (Dynkin, Maliutov) for simple random walk on standard generators, Derriennic for measures with finite support), more generally for hyperbolic groups (Ancona for measures with finite support, Kaimanovich for measures of finite entropy and with finite logarithmic moments; groups with infinitely many ends (Kaimanovich), the mapping class group (Kaimanovich, Masur), braid groups (Farb, Masur), Coxeter groups (Karlsson, Margulis). Sometimes it is easier to identify the boundary for certain non-symmetric random walks, rather than for symmetric ones. It was done for random walks on wreath product $\mathbb{Z}^d \wr \mathbb{Z}/pZ$ with a non-zero drift of the projection on $\mathbb{Z}^d$ [3], for random walk solvable Baumslag-Solitar groups with a non-zero drift of the projection on $\mathbb{Z}$ [3], and, more generally, for such random walks on the group of rational affinities (Brofferio). Note that in the last two examples simple random walks have trivial boundary.

Our argument can be applied not only to wreath products, but for some other solvable groups and group extensions. In particular, we give the complete description of the Poisson boundary for free metabelian groups on d generators, $d \geq 5$, answering in dimension ≥ 5 a question of A.M.Vershik [2].

Finally we discuss measures with slow decay and provide examples of random walks on wreath products, where the Poisson-Furstenberg boundary is non-trivial, and there is no non-trivial partition of the boundary which can be defined in terms of finite configurations.

References

[1] V.A. Kaimanovich, A.M. Vershik, *Random walks on discrete groups: Boundary and entropy*, [J] Ann. Probab. **11**, (1983), 457-490.

[2] A.M. Vershik, *Dynamic theory of growth in groups: Entropy, boundaries, examples*, Russ. Math. Surv. **55**, No.4, (2000), 667-733; translation from Usp. Mat. Nauk **55**, No.4, (2000) 59-128.

[3] V.A.Kaimanovich, *Poisson boundaries of random walks on discrete solvable groups*, [A] Heyer, Herbert (ed.), Probability measures on groups X. Proceedings of the tenth Oberwolfach conference, held November 4-10, 1990 in Oberwolfach, Germany. New York, NY: Plenum Publishing Corporation.(1991), 205-238.
[4] N. James, Yu. Peres, *Cutpoints and exchangeable events for random walks*, [J] Theory Probab. Appl. **41**, No.4, (1996), 666-677 and Teor. Veroyatn. Primen. **41**, No.4, (1996), 854-868.
[5] A. Erschler, *Liouville property for groups and manifolds*, Inv. Mathematicae, **155**, 2004, 55-80.

Asymptotics of spectral invariants of discrete tori in the continuum limit

Anders Karlsson
(joint work with Gautam Chinta, Jay Jorgenson)

We study the spectral theory of the combinatorial Laplacian for sequences of Cayley graphs of finite abelian groups when the orders of the cyclic factors tend to infinity at comparable rates. We establish an asymptotic expansion of the determinant of the combinatorial Laplacian. This extends works of physicists and mathematicians, in particular a well-known asymptotic formula in dimension 2 ([3], [2]).

The zeta-regularized determinant of the Laplacian of the limiting real torus appears as a constant in this expansion. On the other hand, by a classical theorem the determinant of the combinatorial Laplacian of a finite graph divided by the number of vertices equals the number of spanning trees, called the complexity, of the graph. As a result, we establish a link between the complexity of the Cayley graphs of finite abelian groups and heights of real tori, cf. the discussion in [4].

It is also known that one can express spectral determinants on discrete tori using trigonometric functions and spectral determinants on real tori in terms of modular forms on general linear groups. Another interpretation of our analysis is thus to establish a relation between limiting values of certain products of trigonometric functions and modular forms.

The heat kernel analysis which is basic to our approach uses a careful study of I-Bessel functions $I_x(t)$. Our methods extend to prove the asymptotic behavior of other spectral invariants through degeneration, such as special values of spectral zeta functions and Epstein-Hurwitz zeta functions.

The main theorem in more detail: Let $N(u) = (n_1(u), n_2(u), ..., n_d(u))$ be a sequence of vectors in u such that $N(u)/u \to A = (\alpha_1, ..., \alpha_d)$ as $u \to \infty$. Set $V(N) = n_1 n_2 ... n_d$.

Theorem 1 ([1]). *Let* $\det' \Delta_{N(u)}$ *be the determinant of the combinatorial Laplacian on* $\mathbb{Z}^d/N(u)\mathbb{Z}^d$ *and let* $\det' \Delta_A$ *be the determinant of the Laplacian on* $\mathbb{R}^d/A\mathbb{R}^d$ *(defined through regularization). Then*

$$\log \det \Delta_{N(u)} = V(N(u))\mathcal{I}_d(0) + 2\log u + \log \det \Delta_A + o(1)$$

as $u \to \infty$ and where

$$\mathcal{I}_d(0) = \log 2d - \int_0^\infty (e^{-2dt}(I_0(2t)^d - 1)\frac{dt}{t}.$$

For general d, the main term has previously been studied by physicists and mathematicians ([5] obtained previously our expression for the main term) . For $d = 2$, our expansion specializes to a formula in [2]:

Theorem 2 (d=2)**.** *The following holds*

$$\log\det\Delta_{(n_1,n_2)} = n_1 n_2 \frac{4G}{\pi} + \log n_1 n_2 + \log(\left|\eta(i\alpha_2/\alpha_1)^4\right| \alpha_2/\alpha_1) + o(1),$$

where G is Catalan's constant and η is Dedekind's eta-function.

The constant $4G/\pi$ in the main term is a celebrated result from the early 1960s [3]. We show moreover for $d = 2$ how to improve the error term in this formula and obtain an explicit expansion up to arbitrary degree of polynomial decay.

For $d = 1$, the expansion reads in explicit form

$$2\log n = 2\log u + 4\pi\log(2\pi/\alpha)\zeta_{\mathbb{Q}}(0) - 4\zeta'_{\mathbb{Q}}(0) + o(1),$$

where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function.

References

[1] G. Chinta, J. Jorgenson, and A. Karlsson, *Zeta functions, heat kernels and spectral asymptotics on degenerating families of discrete tori*, Arxive preprint, 34 pp.

[2] B. Duplantier, F. David, *Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice*, Journal of Statistical Physics, **51** (1988), 327–434.

[3] P.W. Kasteleyn, *The statistics of dimers on a lattice, I The number of dimer arrangements on a quadratic lattice*, Physica **27** (1961), 1209-1225.

[4] P. Sarnak, *Determinants of Laplacians; heights and finiteness.* In: Analysis, et cetera, Academic Press, Boston, MA, 1990, 601–622.

[5] A. Sokal, A. Starinets, *Pathologies of the large-N limit for RP^{N-1}, CP^{N-1}, QP^{N-1} and mixed isovector/isotensor-models*, Nuclear Phys. B. **601** (2001), 425–502.

C^0-rigidity of the Poisson brackets

Michael Entov

(joint work with Leonid Polterovich)

A mainstream topic of modern symplectic topology is the study of rigidity properties of subsets and Hamiltonian diffeomorphisms of symplectic manifolds. A number of recent developments show that there is another manifestation of symplectic rigidity which takes place on function spaces associated to a symplectic manifold.

Namely, let (M, ω) be a connected symplectic manifold (open or closed). Denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M equipped with the Poisson bracket $\{F, G\}$. Write $\|\cdot\|$ for the standard *uniform norm* (also called the C^0*-norm*) on it: $\|F\| := \max_{x\in M} |F(x)|$.

We will concentrate on a number of problems concerning the behavior of certain (non-linear) functionals $\Phi : C_c^\infty(M) \times C_c^\infty(M) \to \mathbb{R}$, based on Poisson brackets, with respect to the uniform norm. Given such a Φ and $\epsilon > 0$, put

$$\overline{\Phi}_\epsilon(F,G) = \inf_{||F-F'||\leq\epsilon, ||G'-G||\leq\epsilon} \Phi(F',G') ,$$

$$\overline{\Phi}(F,G) := \liminf_{F',G' \xrightarrow{C^0} F,G} \Phi(F,G) = \liminf_{\epsilon\to 0} \overline{\Phi}_\epsilon(F,G). \tag{1}$$

We call Φ *lower semicontinuous* (with respect to the uniform norms) if

$$\Phi(F,G) = \overline{\Phi}(F,G), \tag{2}$$

and *weakly robust* if $\overline{\Phi}(F,G) > 0$ whenever $\Phi(F,G) > 0$. In case Φ is lower semicontinuous, we are interested in the convergence rate of $\overline{\Phi}_\epsilon(F,G)$ to $\overline{\Phi}(F,G)$ as $\epsilon \to 0$.

Now the definition of the Poisson bracket $\{F,G\}$ of two smooth functions $F, G \in C_c^\infty(M)$ involves first derivatives of the functions. Thus *a priori* there is no restriction on possible changes of $\{F,G\}$ when F and G are slightly perturbed in the uniform norm. Amazingly such restrictions do exist: F.Cardin and C.Viterbo [2] discovered that the functional $(F,G) \mapsto \|\{F,G\}\|$ is weakly robust. Moreover,

Theorem 1 ([3])**.** *The functional $(F,G) \mapsto \max\{F,G\}$ (and hence also the functional $(F,G) \mapsto \|\{F,G\}\|$) is lower semicontinuous.*

In the case $\dim M = 2$ the lower semicontinuity for this functional was first proved by F.Zapolsky [7] by methods of two-dimensional topology.

In the proof of Theorem 1 we use the following result from "hard" symplectic topology due to D.McDuff [5, 6]: Denote by $Ham^c(M)$ the group of Hamiltonian diffeomorphisms of M generated by Hamiltonian flows with compact support. Then sufficiently small segments of one-parameter subgroups of the group $Ham^c(M)$ of Hamiltonian diffeomorphisms of M minimize the "positive part of the Hofer length" among all paths on the group in their homotopy class with fixed end points. In fact, our method readily generalizes to any (in general, "infinite-dimensional") Lie group equipped with a bi-invariant (Finsler) semi-norm, provided sufficiently short segments of 1-parameter subgroups are minimal geodesics. It would be interesting to formalize this remark and to find new significant examples.

In a subsequent independent work [1] L.Buhovsky found a different proof of Theorem 1 along with a sharp estimate on the rate of convergence for the lower semicontinuous functional $(F,G) \mapsto \|\{F,G\}\|$. To state this result we put

$$\Psi(F,G) := ||\{\{\{F,G\},F\},F\} + \{\{\{F,G\},G\},G\}|| . \tag{3}$$

One can show that $\Psi(F,G) > 0$ provided $\{F,G\} \not\equiv 0$. With this notation, Buhovsky derived the following 2/3-law: There exists $\epsilon_0(F,G) > 0$ such that for any $0 < \epsilon < \epsilon_0(F,G)$

$$\Phi(F,G) - \overline{\Phi}_\epsilon(F,G) \leq C \cdot \Psi(F,G)^{\frac{1}{3}} \epsilon^{\frac{2}{3}} , \tag{4}$$

where $C > 0$ is a numerical constant. Buhovsky's proof of (4) is based on an ingenious application of the energy-capacity inequality. We have been able to reprove (4) by our methods as well.

Similar questions can be asked about rigidity of iterated Poisson brackets. The following results provide an evidence for the lack of C^0-rigidity for iterated Poisson brackets of three or more functions.

Theorem 2 ([3]). *Let M^{2n} be a symplectic manifold. Then*

(a) **For any** *$2n+1$ smooth functions $F_1, \dots, F_{2n+1} \in C_c^\infty(M)$ there exist functions $F'_1, \dots, F'_{2n+1} \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to $F_1, \dots, F_{2n+1}$ which satisfy the following relation:*

$$\{F'_1, \{F'_2, \dots \{F'_{2n}, F'_{2n+1}\}\} \dots\} \equiv 0.$$

(b) **There exist** *3 functions $F, G, H \in C_c^\infty(M)$ satisfying $\{F, \{G, H\}\} \not\equiv 0$ such that there exist smooth functions $F', G', H' \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to F, G, H and satisfying the condition*

$$\{F', \{G', H'\}\} \equiv 0.$$

Having seen that, in general, there is no rigidity for iterated of Poisson brackets of three or more functions we explore the case of iterated Poisson brackets of just two functions. Let us focus on non-negative functionals $\Phi^v(F, G)$ of the form

$$\Phi^v(F,G) = v_1 \cdot \max\{\{F,G\},F\} - v_2 \cdot \min\{\{F,G\},F\}$$
$$+ v_3 \cdot \max\{\{F,G\},G\} - v_4 \cdot \min\{\{F,G\},G\} ,$$

where $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ is a non-zero vector with non-negative entries.

Theorem 3 ([4]).

(i) *If either $v_3 = v_4 = 0$* **or** *$v_1 = v_2 = 0$, Φ^v is weakly robust but not lower semicontinuous.*

(ii) *If at least one of v_1, v_2 is positive* **and** *at least one of v_3, v_4 is positive, Φ^v is lower semicontinuous.*

The failure of lower semicontinuity in (i) is shown by means of a local two-dimensional example which can be implanted into a Darboux chart on a arbitrary symplectic manifold.

The weak robustness in (i) follows from the classical Landau-Hadamard inequality and the lower semicontinuity of the single Poisson bracket.

The proof of the lower semicontinuity in (ii) is based on the following estimate (we do not know whether the asymptotics here is sharp):

Theorem 4 (Convergence rate). *Let*

$$\Phi(F, G) := \max\{\{F, G\}, F\} + \max\{\{F, G\}, G\}.$$

Then for every $\epsilon > 0$

$$\Phi(F,G) - \overline{\Phi}_\epsilon(F,G) \le C(F,G)\cdot \epsilon^{\frac{1}{3}} \,, \tag{5}$$

where $C(F,G)$ *is a positive constant depending on* F *and* G*.*

For higher iterated Poisson brackets of two functions the problem of determining whether a given functional is lower semicontinuous or, at least, weakly robust, is, in general, completely open. We have been able to settle only the following special case. Denote by $\mathrm{ad}_F : C_c^\infty(M) \to C_c^\infty(M)$ the operator $G \mapsto \{G,F\}$. For integers $m \ge 1, k \ge 0$ introduce the functional

$$\Phi_{k,m}(F,G) = \mathrm{osc}\,(\mathrm{ad}_H)^m G \,, \text{where } \; H = (\mathrm{ad}_G)^k F \,.$$

Theorem 5 ([4])**.** *For any* $m \ge 1, k \ge 0$ *there exists a constant* $C_{k,m} > 0$ *so that*

$$\Phi_{k,m}(F,G) \ge C_{k,m} \cdot \frac{||\{F,G\}||^{(k+1)m}}{||F||^{km}||G||^{m-1}} \,.$$

In particular, $\Phi_{k,m}$ *is weakly robust.*

References

[1] L.Buhovsky, *The 2/3 - convergence rate for the Poisson bracket*, preprint, arXiv:0802.3792, 2008.

[2] F.Cardin, C.Viterbo, *Commuting Hamiltonians and multi-time Hamilton-Jacobi equations*, preprint, math/0507418, 2005.

[3] M.Entov, L.Polterovich, *C^0-rigidity of Poisson brackets*, preprint, arXiv:0712.2913, 2007.

[4] M.Entov, L.Polterovich, *C^0-rigidity of the double Poisson bracket*, preprint, arXiv:0807.4275, 2008.

[5] D.McDuff, *Geometric variants of the Hofer norm*, J. Symplectic Geom. **1** (2002),197-252.

[6] D.McDuff, *Monodromy in Hamiltonian Floer theory*, preprint, arXiv:0801.1328, 2008.

[7] F.Zapolsky, *Quasi-states and the Poisson bracket on surfaces,* J. of Modern Dynamics, **1**:3, 2007, 465-475.

Manhattan curves and length functions on groups

Richard Sharp

In the early 1990s, M. Burger compared two convex co-compact representations of a group in $\mathrm{Isom}(\mathbb{H}^{n+1})$ by considering an associated object, which he called the Manhattan curve [1]. For example, let $X_1 = \mathbb{H}^2/\Gamma_1$ and $X_2 = \mathbb{H}^2/\Gamma_2$ be two compact hyperbolic surfaces which are homeomorphic, with a fixed homeomorphism $\psi : X_1 \to X_2$. Associated to each conjugacy class γ in $\Gamma_1 \cong \Gamma_2$, are unique closed geodesics on X_1 and X_2, with lengths $l_1(\gamma)$ and $l_2(\gamma)$, respectively. The Manhattan curve is then defined to be

$$\mathfrak{M}(l_1,l_2) = \partial \left\{ (a,b) \in \mathbb{R}^2 : \sum_\gamma e^{-al_1(\gamma)-bl_2(\gamma)} < +\infty \right\}.$$

This is a convex curve, crossing the axes at $(1,0)$ and $(0,1)$, and Burger showed that it is a straight line if and only if X_1 and X_2 are isometric. Furthermore, he showed the normals to the asymptotes have slope $\sup_\gamma l_2(\gamma)/l_1(\gamma)$ and $\inf_\gamma l_2(\gamma)/l_1(\gamma)$, and the normal at $(1, 0)$ has slope equal to the intersection $i(l_1, l_2)$ [1]. Subsequently, the author showed that $\mathfrak{M}(l_1, l_2)$ is real analytic and that $a^* + b^* = \alpha(l_1, l_2)$, where (a^*, b^*) is the unique point on $\mathfrak{M}(l_1, l_2)$ where the normal has slope 1 [7]. Here, $0 < \alpha(l_1, l_2) < 1$ is the correlation number introduced in his work with R. Schwartz and defind as the exponential growth rate of $\#\{\gamma : l_1(\gamma), l_2(\gamma) \in (T, T + \epsilon)\}$, as $T \to +\infty$ (with $\epsilon > 0$ fixed) [6].

Recently, the author showed that a direct analogue of the Manhattan curve may be used to characterize quantities associated to automorphisms of free groups [8]. Let F be a free group on $k \geq 2$ generators $\mathcal{A}$ and let $|\cdot|$ denote the word length. We write Σ^+ for the space of (one-sided) infinite reduced words in $\mathcal{A} \cup \mathcal{A}^{-1}$ and μ_0 for the measure of maximal entropy with respect to the shift map on Σ^+. We say that $\phi \in \mathrm{Aut}(F)$ is simple if it is the product of an inner automorphism and an automorphism induced by a permutation of $\mathcal{A} \cup \mathcal{A}^{-1}$.

We wish to compare $|\cdot|$ with the distorted length $|\phi(\cdot)|$. This may be done through various characteristics, e.g.

(i) the generic stretch (Kaimanovich, Kapovich and Schupp [3]):

$$\lambda(\phi) = \lim_{n \to +\infty} \frac{|\phi(x_0 x_1 \cdots x_n)|}{n} \quad \text{for } \mu_0\text{-a.e. } (x_n) \in \Sigma^+;$$

(ii) the $\mathrm{Curl}(\phi)$ (Myasnikov and Shpilrain [5]):

$$\mathrm{Curl}(\phi) = \limsup_{n \to +\infty} \left(\frac{\#\{x \in F : |x|, |\phi(x)| \leq n\}}{\#\{x \in F : |x| \leq n\}} \right)^{1/n};$$

(iii) the conjugacy distortion spectrum (Kapovich [4]):

$$\mathcal{D}_\phi = \{|\phi(\gamma)|/|\gamma| : \gamma \in \mathcal{C}\},$$

where $\mathcal{C}$ denotes the set of non-trivial conjugacy classes in F and $|\gamma| = \min_{x \in \gamma} |x|$.

It was shown in the above references that the following each hold if and only if ϕ is simple: $\lambda(\phi) = 1$, $\mathrm{Curl}(\phi) = 1$ and $\mathcal{D}_\phi = \{1\}$.

Setting $l_1 = |\cdot|$ and $l_2 = |\phi(\cdot)|$, let $\mathfrak{M}_\phi = \mathfrak{M}(l_1, l_2)$ be defined as above. Then $\mathfrak{M}_\phi$ is convex and crosses the axes at $(\log(2k-1), 0)$ and $(0, \log(2k-1)$. The following result is in [8].

Theorem 1.

(i) *$\mathfrak{M}_\phi$ is real analytic and is a straight line if and only if ϕ is simple;*
(ii) *the set of slopes of normals to $\mathfrak{M}_\phi$ is equal to $\overline{\mathcal{D}_\phi}$;*
(iii) *the normal at $(\log(2k-1), 0)$ has slope $\lambda(\phi)$;*
(iv) *there is a unique point (a^*, b^*) on $\mathfrak{M}_\phi$ where the normal has slope 1 and $\mathrm{Curl}(\phi) = e^{a^*+b^*}/(2k-1)$.*

A version of this result still holds if $|\cdot|$ and $|\phi(\cdot)|$ are replaces by length functions l_1, l_2 in the Culler-Vogtmann Outer space [2]. (Recall that such length functions are obtained from marked metric graphs, with fundamental group F and edge lengths summing to 1.) Each such length function will have a critical exponent δ_i,

$i = 1, 2$, defined to be the abscissa of convergence of $\sum_{\gamma \in \mathcal{C}} e^{-s l_i(\gamma)}$. We define an intersection by

$$i(l_1, l_2) = \lim_{T \to +\infty} \frac{1}{\#\{\gamma \in \mathcal{C} : l_1(\gamma) \leq T\}} \sum_{l_1(\gamma) \leq T} \frac{l_2(\gamma)}{l_1(\gamma)}$$

and a correlation number by

$$\alpha(l_1, l_2) = \lim_{\epsilon \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \# \left\{ \gamma \in \mathcal{C} : l_1(\gamma) \leq T,\ 1 - \epsilon \leq \frac{l_2(\gamma)}{l_1(\gamma)} \leq 1 + \epsilon \right\}.$$

The following result is in [9]. (A result of T. White [10] is used to ensure the existence of (a^*, b^*) in (iv).)

Theorem 2.

(i) $\mathfrak{M}(l_1, l_2)$ *is real analytic and is a straight line if and only if* l_1 *is a constant multiple of* l_2*;*
(ii) *the set of slopes of normals to* $\mathfrak{M}(l_1, l_2)$ *is equal to* $\overline{\{l_2(\gamma)/l_1(\gamma) : \gamma \in \mathcal{C}\}}$*;*
(iii) *the normal at* $(\delta_1, 0)$ *has slope* $i(l_1, l_2)$*;*
(iv) *there is a unique point* (a^*, b^*) *on* $\mathfrak{M}(l_1, l_2)$ *where the normal has slope* 1 *and* $\alpha(l_1, l_2) = a^* + b^*$.

We say that l_1 and l_2 are independent if $\{p l_1(\gamma) + q l_2(\gamma) : \gamma \in \mathcal{C}\} \subset \mathbb{Z}$ implies that $p = q = 0$. Under this condition, the following more precise version of Theorem 2((iv) holds [9].

Theorem 3. *If* l_1 *and* l_2 *independent then there exists a constant* $C > 0$ *such that, for fixed* $\epsilon > 0$,

$$\#\{\gamma \in \mathcal{C} : l_1(\gamma), l_2(\gamma) \in (T, T + \epsilon)\} \sim C \frac{e^{\alpha(l_1, l_2) T}}{T^{3/2}}, \quad \text{as } T \to +\infty.$$

References

[1] M. Burger, *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank* 2, Internat. Math. Res. Notices **7** (1993), 217–225.

[2] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. math. **84** (1986), 91–119.

[3] V. Kaimanovich, I. Kapovich and P. Schupp, *The subadditive ergodic theorem and generic stretching factors for free group automorphisms*, Israel J. Math. **157** (2007), 1–46.

[4] I. Kapovich, *The frequency space of a free group*, Internat. J. Algebra Comput. **15** (2005), 939–969.

[5] A. G. Myasnikov and V. Shpilrain, *Some metric properties of automorphisms of groups*, J. Algebra **304** (2006), 782–792.

[6] R. Schwartz and R. Sharp, *The correlation of length spectra of two hyperbolic surfaces*, Comm. Math. Phys. **153** (1993), 423–430.

[7] R. Sharp, *The Manhattan curve and the correlation of length spectra on hyperbolic surfaces*, Math. Z. **228** (1998), 745–750.

[8] R. Sharp, *Entropy and distortion for automorphisms of free groups*, Discrete and Continuous Dynamical Systems, to appear.

[9] R. Sharp, *Comparing length functions on free groups*, in preparation.

[10] T. White, *The Geometry of Outer Space*, PhD Thesis, UCLA (1991).

Hamiltonian pseudo-representations

Vincent Humilière

Let (M, ω) be any symplectic manifold. Its set of smooth functions $C^\infty(M)$ carries a Lie algebra structure given by the Poisson bracket, defined by

$$\{F, G\} = \omega(X_F, X_G),$$

where X_F, X_G are the Hamiltonian vector fields associated to $F, G \in C^\infty(M)$. We consider the following question.

Question 1. *Suppose we have the following C^0 convergences:*

$$F_n \longrightarrow F,$$
$$G_n \longrightarrow G,$$
$$\{F_n, G_n\} \longrightarrow H,$$

all functions being smooth. Is it true that $\{F, G\} = H$?

In general, the answer to this question is negative. This is not surprising since the Poisson bracket can be locally written in terms of partial derivatives.

Nevertheless, Cardin and Viterbo [2] recently proved that the answer to question 1 is positive in the particular case $H = 0$, discovering in this way the phenomenon of C^0-rigidity of the Poisson brackets. Many studies of this phenomenon have followed this pioneer work [1],[3],[4],[6]. In our work, we generalize the result of Cardin and Viterbo showing that the answer to Question 1 is positive when the considered sequences of functions carry an hidden Lie algebra structure.

We denote by $C_0^\infty(M)$ the set of *normalized* smooth functions, i.e., smooth functions which have zero mean value, if M is compact, and which are compactly supported otherwise. We also fix a *finite-dimensional* Lie algebra $\mathfrak{g}$.

Definition 1. *A* Hamiltonian pseudo-representation *of $\mathfrak{g}$ is a sequence of linear maps $\mathfrak{g} \to C_0^\infty(M)$ such that for any $f, g \in C_0^\infty(M)$, the expression*

$$\{\rho_n(f), \rho_n(g)\} - \rho_n([f, g])$$

converges to 0 in the C^0-sense.

Our following result gives a partial positive answer to Question 1. It also generalizes Cardin-Viterbo's result, which is recovered by taking $\mathfrak{g}$ abelian.

Theorem 2 ([5])**.** *Let ρ_n be a pseudo-representation, such that for any $f \in \mathfrak{g}$, the sequence $(\rho_n(f))$ C^0-converges to some $\rho(f) \in C_0^\infty(M)$. Then, the map $\rho : \mathfrak{g} \to C_0^\infty(M)$ is a Hamiltonian representation, i.e., for all $f, g \in \mathfrak{g}$,*

$$\{\rho(f), \rho(g)\} = \rho([f, g]).$$

The proof of this theorem is based on the existence of special bi-invariant distances on the group of compactly supported Hamiltonian diffeomorphisms, like Hofer's distance (see e.g. [7]). The hypothesis of finite dimension of $\mathfrak{g}$ is also crucial.

Theorem 2 has many surprising applications:

- On some non-compact manifolds (including cotangent bundles of closed manifolds), when we remove the assumption of compact supports, Theorem 2 does not hold anymore: one can construct counter-examples. This leads to the following conclusion:

  On these manifolds, the group of all (not necessarily compactly supported) Hamiltonian diffeomorphisms admits no bi-invariant distance, continuous with respect to the C^0 topology of Hamiltonian functions.

- In $\mathbb{R}^{2n}$, symplectic maps can be characterized in terms of the Poisson brackets of their coordinate functions. As a consequence, the fact that *symplectomorphisms are C^0-closed in diffeomorphisms* (a famous theorem of Gromov and Eliashberg) appears as a corollary of Theorem 2.

- Finally, Theorem 2 also allows to define a notion of C^0 *Hamiltonian representation of a Lie algebra*, as a linear map $\mathfrak{g} \to C^0(M)$ which is the C^0-limit of a pseudo-representation.

References

[1] Lev Buhovski. The 2/3 - convergence rate for the poisson bracket, 2008. arXiv.org:math/0802.3792.

[2] Franco Cardin and Claude Viterbo. Commuting Hamiltonians and multi-time Hamilton-Jacobi equations, 2005. arXiv math/0507418.

[3] Michael Entov, Leonid Polterovich, and Frol Zapolsky. Quasi-morphisms and the Poisson bracket, 2006.

[4] Michael Entov and Leonid Polterovich. C^0-rigidity of poisson brackets, 2007.

[5] Vincent Humiliere. Hamiltonian pseudo-representations, 2007. arXiv math/0703335, to appear in Commentarii Mathematici Helvetici.

[6] Frol Zapolsky. Quasi-states and the Poisson bracket on surfaces. *J. Mod. Dyn.*, 1(3):465–475, 2007.

[7] Leonid Polterovich. *The geometry of the group of symplectic diffeomorphisms*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.

Symplectic quasi-states and applications

Frol Zapolsky

A *homogeneous quasi-morphism* μ on a group Γ is a function $\mu\colon \Gamma \to \mathbb{R}$ which satisfies

$$\sup_{a,b\in\Gamma} |\mu(ab) - \mu(a) - \mu(b)| < \infty \quad \text{and} \quad \mu(a^n) = n\mu(a) \text{ for all } a \in \Gamma,\, n \in \mathbb{Z}\,.$$

It is interesting to look at homogeneous quasi-morphisms in case Γ admits no nonzero homomorphisms to $\mathbb{R}$, in particular if Γ is perfect, that is $\Gamma = [\Gamma, \Gamma]$. If (M, ω) is a closed symplectic manifold, it is a celebrated result of A. Banyaga that the universal cover of the group of Hamiltonian diffeomorphisms $\widetilde{\mathrm{Ham}}(M, \omega)$ of M is perfect. Assume from now on that (M, ω) is a closed $2n$-dimensional symplectic manifold, Γ is the above group, and that the volume $\int_M \omega^n = 1$.

Theorem 1 (Entov, Polterovich and later Ostrover, McDuff)**.** *For certain* (M, ω) *the group* Γ *admits a homogeneous quasi-morphism* μ*, which coincides with the classical Calabi invariant on elements generated by Hamiltonians with support in a displaceable subset. In particular* $\mu \neq 0$.

Remark 2. *The list of symplectic manifolds for which this theorem is proven includes complex projective spaces with the Fubini-Study form, their monotone products, the monotone blowup of* $\mathbb{C}P^2$ *at up to three points,* $\mathbb{T}^4$ *blown up at a positive number of points, and others.*

It should be noted that this μ in addition satisfies a property called stability, which will not be given here, but from which conclusions will be drawn. Below μ denotes the particular quasi-morphism furnished by the theorem.

For $F \in C^\infty(M)$ let $\phi_F \in \Gamma$ be the element generated by the flow of F for time 1. Put

$$\zeta(F) := \int_M F\omega^n - \mu(\phi_F)\,.$$

It follows that ζ satisfies:

(i) $\zeta(1) = 1$;

(ii) ζ is linear on pairs of Poisson-commuting functions (follows from the fact that a homogeneous quasi-morphism is a homomorphism when restricted to abelian subgroups);

(iii) ζ is monotone: $\zeta(F) \leq \zeta(G)$ for $F \leq G$ (follows from stability).

The last property implies that ζ is a Lipschitz functional with respect to the C^0 norm, and hence allows for a unique extension to the space $C(M)$ of all continuous functions on M. A functional on $C(M)$ (M a symplectic manifold) satisfying the above three properties is called a *symplectic quasi-state*. Obviously, if ν is a probability measure on M, then $F \mapsto \int_M F\,d\nu$ is a symplectic quasi-state. These are the trivial symplectic quasi-states because they are linear (in fact, these are the only ones). It is a consequence of the above that the manifolds appearing in the theorem all admit a *nonlinear* symplectic quasi-state. Indeed, since μ coincides with the Calabi invariant on elements generated by Hamiltonians supported in a displaceable subset, it follows that $\zeta(F) = 0$ provided that $\operatorname{supp} F$ is displaceable. If ζ were linear, then, since we can always find C^∞ functions $F_1, \ldots, F_N$ with displaceable supports such that $\sum_i F_i \equiv 1$, we would get $1 = \zeta(\sum_i F_i) = \sum_i \zeta(F_i) = 0$.

The specific symplectic quasi-states constructed above have an additional property:

Theorem 3 (Entov, Polterovich, Z.)**.** *There is a constant* C *such that for* $F, G \in C^\infty(M)$

$$|\zeta(F+G) - \zeta(F) - \zeta(G)| \leq C\|\{F,G\}\|_{C^0}^{1/2}$$

As an application, we obtain the following restriction on partitions of unity:

Theorem 4 (Entov, Polterovich, Z.). *There is a constant $K > 0$ such that if $\{\rho_j\}_{j=1}^N$ is a partition of unity with* $\operatorname{supp}\rho_j$ *displaceable* $\forall j$, *then*

$$\max_{j,k} \|\{\rho_j, \rho_k\}\|_{C^0} \geq \frac{K}{N^3}\,.$$

For another application, consider the following model for simultaneous measurement of observables in classical mechanics. Let $F_1, F_2 \in C^\infty(M)$ be two observables to be measured. Couple the mechanical system (M, ω) with the measuring apparatus $(\mathbb{R}^4(p,q), dp \wedge dq)$ via the coupling Hamiltonian $H(x,p,q) = p_1F_1(x) + p_2F_2(x)$. It follows from Hamilton's equations with initial values $x(0) = y$, $p_i(0) = \varepsilon \geq 0$, $q_i(0) = 0$ that $p_i(t) = \varepsilon$ and $x(t) = g^{\varepsilon t}y$, where g^s is the Hamiltonian flow of the function $G = F_1 + F_2$. By definition, the output of a measurement which has duration $T > 0$ is the pair of functions

$$F_i'(y) := \frac{q_i(T)}{T}\,.$$

Note that if $\varepsilon = 0$ or if $\{F_1, F_2\} = 0$, we obtain $F_i' = F_i$, so the measurement is precise, the fact which also serves as a justification to use the presented model. In any case we can define the *error* of the measurement as the following quantity (which is independent of i):

$$\Delta(F_1, F_2, \varepsilon, T) := \|F_i - F_i'\|_{C^0}\,.$$

Using the above quasi-state ζ, we obtain the following lower bound for this error ($C \equiv C(M, \omega) > 0$ is a constant):

$$\Delta(F_1, F_2, \varepsilon, T) \geq \frac{1}{2}|\zeta(F_1 + F_2) - \zeta(F_1) - \zeta(F_2)| - \sqrt{\frac{C}{\varepsilon T}} \cdot \varphi(F_1, F_2)\,,$$

where φ is some nonnegative function independent of ε, T. Taking $T \to \infty$ we obtain

$$\Delta(F_1, F_2) := \liminf_{T\to\infty} \Delta(F_1, F_2, \varepsilon, T) \geq \frac{1}{2}|\zeta(F_1 + F_2) - \zeta(F_1) - \zeta(F_2)|\,;$$

here the left-hand side is independent of ε as long as $\varepsilon > 0$. $\Delta(F_1, F_2)$ can be interpreted as the error of a measurement performed on a fast-moving system. The inequality gives a (nontrivial) lower bound for this error if ζ is not additive on the pair F_1, F_2. Note that this also implies that F_1, F_2 do not commute.

Rigidity of Lagrangian submanifolds

Octav Cornea
(joint work with Paul Biran)

1. setting

Given a symplectic manifold (M^{2n}, ω), a Lagrangian submanifold $L \subset (M, \omega)$ is an n-dimensional submanifold so that $\omega|_M = 0$. All Lagrangians discussed here are assumed closed. Such a Lagrangian is called monotone if the two morphisms $\omega : \pi_2(M, L) \to \mathbb{R}$ and $\mu : \pi_2(M, L) \to \mathbb{Z}$, the first given by integrating the symplectic form ω and the second by the Maslov class, are proportional with a positive constant of proportionality. All Lagrangians below are supposed to be monotone and we also assume that the minimal Maslov number

$$N_L = \min\{\mu(x) \mid x \in \pi_2(M, L),\ \omega(x) > 0\}$$

verifies $N_L \geq 2$. There are many examples of such Lagrangians: $\mathbb{R}P^n \subset \mathbb{C}P^n$, the Clifford torus $\mathbb{T}^n_{Cliff} \subset \mathbb{C}P^n$ and many others.

An important property of this class of Lagrangians is that Floer homology $HF(L, L)$ is defined (by early work of Oh [6]). It is easy to see that Floer homology is always "smaller" or equal than singular homology (for example, for the 0-section in a co-tangent bundle it equals singular homology) and, at the other end, $HF(L, L)$ can also vanish (for example if L can be disjoined from itself by a hamiltonian isotopy).

In case $HF(L, L)$ equals singular homology (tensored with an appropriate Novikov ring) L is called *wide* and if $HF(L, L)$ vanishes L is called narrow. All known monotone Lagrangians are either wide or narrow.

2. Prototype of Results.

In the work with Paul Biran [1, 2, 3] reviewed here we discovered some systematic forms of rigidity that apply to monotone Lagrangians.

While our results are considerably more general, in this report I will exemplify the types of results we get on some simple cases: monotone Lagrangians in $\mathbb{C}P^n$. First I recall the notion of *Gromov width.* If $U \subset M$ is open, the Gromov width of U is

$$w(U) = \sup_r \{\pi r^2 \mid \exists e : B^{2n}(r) \to M,\ \text{smpl. embedding}\}$$

(where $B^{2n}(r)$ is the ball of radius r endowed with the standard symplectic form) and for a Lagrangian submanifold $L \subset M$ we let the Gromov width of L be defined by:

$$w(L) = \sup_r \{\pi r^2 \mid \exists e : B^{2n}(2r) \to M,\ \text{sympl. embedding},\ e^{-1}(L) = \mathbb{R}^n \cap B^{2n}(r)\}.$$

Theorem 1 ([1, 2, 3]). *Assume that the Lagrangians below are all monotone with $N_L \geq 2$. We have:*

a. *Any two non narrow Lagrangians in $\mathbb{C}P^n$ intersect.*
b. *Any monotone Lagrangian in $\mathbb{C}P^n$ is either a barrier in the sense that $w(\mathbb{C}P^n \backslash L) < w(\mathbb{C}P^n)$ or is small in the sense that $w(L) < w(\mathbb{C}P^n)$.*
c. *For each n there exist monotone narrow Lagrangians in $\mathbb{C}P^n$.*
d. *Any monotone Lagrangian (in any ambient manifold) with singular cohomology generated (as algebra) by classes of degrees strictly less than N_L is either narrow or wide.*

3. Comments

3.1. **Remarks on the results.** Only a very short discussion of the results above is included here. See [1, 2, 3] for full bibliographic references as well as for the strongest forms of these statements.

a. Entov and Polterovich have first noticed that point a. of the Theorem follows from some spectral action estimates in our paper [1] together with some of the results in [5]. There is also a simple, direct proof for this result, again based on spectral invariant estimates, which we present in [2] as well as a more algebraically meaningful one which appears in [3] and which provides more information. A particular case of the statement a. is that there is no symplectomorphism of $\mathbb{C}P^n$ which disjoins the Clifford torus from $\mathbb{R}P^n$. This particular case has also been obtained by Tamarkin [8][1] by completely different methods.

b. Point b. can be strengthened in various ways. In particular, it is possible to show [2] that narrow Lagrangians in $\mathbb{C}P^n$ have width strictly smaller than that of $\mathbb{C}P^n$ and that wide Lagrangians verify the equation:

$$w(L) + 2w(\mathbb{C}P^n \backslash L) \leq 2w(\mathbb{C}P^n) .$$

c. It is remarkable that Lagrangians as at point c. were not known before. The examples we produce are constructed as lifts of Lagrangians in specific hypersurfaces to the associated normal circle bundle. The fact that hey are narrow follows as a consequence of a. and the fact that, by construction, these Lagrangians are disjoint from $\mathbb{R}P^n$.

d. The result at d. can also be strengthened to distinguish between the two cases - narrow or wide - as well as to include yet more general classes of Lagrangian submanifolds.

3.2. **General method of proof.** All our results are based on exploiting the qunatum structures of monotone Lagrangians. To summarize the ideas behind the definition of these structures recall first that all the results in classical algebraic topology can be recovered by Morse theory - more precisely by counting various configurations made up from negative gradient flow lines of one or more Morse-Smale gradient flows (of course, when restrictig to underlying spaces which are

[1] Coincidentally, the preprint of Tamarkin appeared on *Arxiv* precisely the day when I presented this work at Oberwolfach.

manifolds). To construct quantum structures we consider similar configurations but allow for a certain number of points along these flow lines to be replaced by J-holomorphic spheres, or in the Lagrangian case, disks. A classical such example is the quantum product of symplectic manifolds. Another example which is very relevant here is the *pearl* complex and the associated structures. This construction - only valid in the case of monotone Lagrangians - was initially proposed by Oh [7] following an idea of Fukaya and is a particular case of the more recent cluster complex of Cornea-Lalonde [4] which applies to general Lagrangians. There is a considerable amount of work necessary to establish the properties of the pearl complex as well as those of the resulting homology, $QH(L)$, which we call the *quantum homology* of L and the various technical points are treated in [1] (for a more compact presentation see also [2]). The key bridge between the properties of the ambient manifold and those of the Lagrangian is provided by the fact that $QH(L)$ has the structure of an augmented algebra over the quantum homology of the ambient manifold and, with adequate coefficients, is endowed with duality.

These quantum structures have an important property inherited from the fact that they are defined by making use of J-holomorphic objects: they are *positive* in the sense that all algebraic objects admit a filtration so that all the morphisms (or operations) of geometric origin can be written as a sum between a classical object which preserves filtration and a quantum contribution which strictly increases the filtration degree. This allows for a variety of sometimes delicate algebraic arguments which lead to results like those in Theorem 1.

It is important to note that with appropriate coefficients, $QH(L)$ is isomorphic to the Floer homology $HF(L, L)$ and many of the additional algebraic structures also have natural correspondents in Floer theory. However, the positivity mentioned above is lost by this isomorphism and so Floer homology as such is not the appropriate tool to directly approach the proofs of the results in Theorem 1.

References

[1] P. Biran and O. Cornea, *Quantum structures for Lagrangian submanifolds.* Preprint (2007). Can be found at `http://arxiv.org/pdf/0708.4221`.

[2] P. Biran and O. Cornea, *Rigidity and uniruling for Lagrangian submanifolds.* Preprint (2008). Can be found at `http://arxiv.org/pdf/0808.2440`.

[3] P. Biran and O. Cornea, *Lagrangian Quantum Homology.* Preprint (2008) Can be found at `http://arxiv.org/pdf/0808.3989`.

[4] O. Cornea and F. Lalonde, *Cluster homology: an overview of the construction and results*, Electron. Res. Announc. Amer. Math. Soc., **12** (2006), 1–12.

[5] M. Entov and L. Polterovich, *Rigid subsets of symplectic manifolds.* Preprint (2007), can be found at `http://arxiv.org/pdf/0704.0105`.

[6] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I.*, Comm. Pure Appl. Math., **46** (1993), 949–993.

[7] Y.-G. Oh, *Relative floer and quantum cohomology and the symplectic topology of lagrangian submanifolds.* In C. B. Thomas, editor, *Contact and symplectic geometry*, volume 8 of *Publications of the Newton Institute*, pages 201–267. Cambridge Univ. Press, Cambridge, 1996.

[8] D. Tamarkin, *Microlocal condition for non-dsplaceability*, Preprint, Can be found at `hhtp://arxiv.org/pdf/0809.1584`.

A dynamical approach to relatively hyperbolic groups

Victor Gerasimov
(joint work with Léonid Potyagailo)

We generalize the classical Floyd theorem about geometrically finite Kleinian groups to relatively hyperbolic groups. Applying this version of Floyd theorem, we describe the procedure of pulling back of a ̈relatively hyperbolic structureälong a quasiisometric embedding of finitely generated groups. It provides a theorem, generaizing two known results about quasiisometries into relatively hyperbolic groups.

Another application is the equivalence of geometric and dynamical notions of relative quasiconvexity of a subgroup (question of D. Osin).

Instead of geometrical methods we make use of dynamical properties of action by homeomorphisms of compacta such as the convergence property and the expansivity property.

Strong Tits Alternative and Diophantine conditions

Emmanuel Breuillard

1. Two theorems

We show the following strong version of the Tits alternative:

Theorem 1. *(Strong Tits Alternative [3]) For every $d \in \mathbb{N}$ there is $N = N(d) \in \mathbb{N}$ such that if K is any field and F a finite symmetric subset of $GL_d(K)$ containing 1, either the product set $F^N = F \cdot ... \cdot F$ contains two elements which freely generate a non abelian free group, or the group generated by F is virtually solvable (i.e. contains a finite index solvable subgroup).*

Theorem 1 improves earlier results due to Eskin-Mozes-Oh [5] for group growth and to T. Gelander and the author [4] for the Tits alternative. These two papers were concerned with the S-arithmetic setting, namely they were concerned with uniformity of N (the "freeness radius" of F) for sets F with coefficients inside a fixed finitely generated ring R, which in practice amounts to assuming that R is a fixed ring of S-integers in a number field.

Theorem 1 shows that the dependence of N on the field K is in fact superfluous. However the constant $N(d)$ has to tend to $+\infty$ with d as follows from examples constructed by Grigorchuk and de la Harpe arising from the Grigorchuk groups of intermediate growth.

As in Tits' proof or in [4], the proof of Theorem 1 can be divided into two steps : an arithmetic step and a geometric step. While in [4] (as well as in Tits' original theorem) the arithmetic step was the easier one and most of the work lied in showing that a certain geometric configuration (the so-called "ping-pong") did arise, roles are reversed in our proof of Theorem [3] and the arithmetic step is the harder step, while the geometric step routinely follows Tits' original proof.

The arithmetic step in Theorem 1 relies on the following result.

Theorem 2. *(Height Gap Theorem [2]) There is a positive constant $\varepsilon = \varepsilon(d) > 0$ such that if F is a finite subset of $SL_d(\overline{\mathbb{Q}})$ and if we let $\widehat{h}(F)$ be the arithmetic spectral radius of F, then $\widehat{h}(F) > \varepsilon$ as soon as F generates a non virtually solvable group.*

Here $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$ and the *arithmetic spectral radius* (or normalized height) of F, introduced in [2], is the "combined average" over all places of the rate of exponential growth of the largest matrix in F^n. By definition it is the quantity

$$\widehat{h}(F) = \lim_{n\to+\infty} \frac{1}{n} h(F^n),$$

where h is the (absolute) *height* defined for F a finite subset of $M_d(\overline{\mathbb{Q}})$ by :

$$h(F) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in V_K} n_v \log^+ ||F||_v$$

where $\log^+ = \max\{0, \log\}$, K is the number field generated by the matrix coefficients of F, V_K is the set of all places of K, and $||F||_v = \max\{||f||_v, f \in F\}$ is the maximal operator norm of $f \in F$, where $||f||_v = \max_{x\neq 0} ||f(x)||_v/||x||_v$ for the standard norm $||x||_v$ induced on K_v^d by the standard absolute value $|\cdot|_v$ on the completion K_v of K associated to $v \in V_K$. We also set $n_v = [K_v, \mathbb{Q}_v]$, where $\mathbb{Q}_v$ is the field of p-adic numbers if $v|p$ is finite and is $\mathbb{R}$ if v is infinite. The quantities $h(F)$ and $\widehat{h}(F)$ are well defined (i.e. they are independent of the chosen number field). Moreover $\widehat{h}(F)$ is invariant under conjugation by elements from $SL_d(\overline{\mathbb{Q}})$.

Theorem 2 can be seen as a global adelic analogue of the Margulis Lemma about discrete subgroups of isometries in hyperbolic geometry. For instance for SL_2, Theorem 2 shows that either there is a finite place v where F acts without global fixed point on the corresponding Bruhat-Tits tree, or after applying some Galois automorphism of $\mathbb{C}$, the set F, as it acts on the hyperbolic 3-space $\mathbb{H}^3$ via $SL_2(\mathbb{C})$, moves every point away from itself by a positive absolute constant ε.

2. And some consequences for uniform growth, spectral gap and non-commutative diophantine properties.

Corollary 3. *(Uniform exponential growth) There is $\varepsilon = \varepsilon(d) > 0$ such that if F is a finite subset of $GL_d(\mathbb{C})$ containing 1 and generating a non amenable subgroup, then for all $n \geq 1$, $|F^n| \geq (1+\varepsilon)^n$. In particular,*

$$\rho_F = \lim_{n\to+\infty} \frac{1}{n} \log |F^n| \geq \log(1+\varepsilon) > 0$$

Let us remark that it may be the case that ρ_F is bounded away from 0 assuming only that F generates a non virtually nilpotent subgroup of $GL_2(\mathbb{C})$. However we observed in [1] that such an assertion, if true, *would imply the Lehmer conjecture* about the Malher measure of algebraic numbers. We also observed there that although every linear solvable group of exponential growth contains a free semigroup, no analog of Theorem 1 holds for solvable groups, namely one may find sets F_n in $GL_2(\mathbb{C})$ containing 1 and generating a solvable subgroup of exponential growth, such that no pair of elements in $(F_n)^n$ may generate a free semigroup.

Von Neumann showed that groups containing a free subgroup are non amenable, i.e. have a spectral gap in ℓ^2. The uniformity in Theorem 1 implies also a uniformity for the spectral gap:

Corollary 4. *(Uniform Spectral Gap in ℓ^2) There is $\varepsilon = \varepsilon(d) > 0$ with the following property. If F is a finite subset of $GL_d(\mathbb{C})$ containing the identity and generating a non amenable subgroup and if Γ is a countable subgroup of $GL_d(\mathbb{C})$ containing F and $f \in \ell^2(\Gamma)$, then there is $\sigma \in F$ such that*

$$\sum_{x\in\Gamma} \left|f(\sigma^{-1}x) - f(x)\right|^2 \geq \varepsilon \cdot \sum_{x\in\Gamma} |f(x)|^2$$

In particular, if F in $GL_d(\mathbb{C})$ is a finite subset containing the identity and generating a non amenable subgroup, then for every finite subset A in $GL_d(\mathbb{C})$, we have $|FA| \geq (1+\varepsilon)|A|$.

In [8] Lubotzky, Phillips and Sarnak showed that for the compact Lie group $G = SU(2)$, the spectral measure of the "Hecke operators" $T_\mu = \frac{1}{2k}\sum_{1\leq i\leq k} g_i + g_i^{-1}$ acting on $\mathbb{L}^2_0(G)$ is supported on $[-m_\mu, m_\mu]$ where m_μ is the norm of T_μ viewed as an operator on $\ell^2(\Gamma)$, Γ being the abstract group generated by the g_i's. Corollary 4 shows that $m_\mu \leq 1 - \varepsilon(k)$ with $\varepsilon(k) > 0$ independent of T_μ.

Since m_μ is also the exponential rate of decay of the return probability by Kesten's theorem (see [7]), we also have:

Corollary 5. *(Uniform Decay of return probability) There is $\varepsilon = \varepsilon(k,d) > 0$ with the property that for any non amenable k-generated subgroup Γ of $GL_d(\mathbb{C})$ we have*

$$\mathbb{P}(S_n = 1) \leq (1-\varepsilon)^n$$

for all $n \geq 1$, where S_n is the simple random walk on Γ.

Theorem 1 also gives a bound on the co-growth of subgroups of $GL_d(\mathbb{C})$.

Corollary 6. *(Co-growth gap) Given $m \in \mathbb{N}$, there is $n(m) > 0$ such that for every $n \geq n(m)$ the following holds: $F = \{a_1, ..., a_m\} \subset GL_d(\mathbb{C})$ generates a non virtually solvable subgroup, if and only if the number of elements w in the free group F_m of word length n such that $w(a_1, ..., a_m) = 1$ is at most $(2m - 1 - \frac{\varepsilon}{m^D})^n$. Here $\varepsilon, D > 0$ are constants depending on d only.*

This result can be paraphrased by saying that non amenable subgroups of $GL_d(\mathbb{C})$ are very strongly non amenable, i.e. have few relations. This puts a purely group theoretical restriction on a given abstract finitely generated group to admit an embedding in $GL_d(\mathbb{C})$.

The uniformity in Theorem 1 allows to reduce mod p and we obtain a statement giving a lower bound on the girth of subgroups of GL_d in positive characteristic:

Corollary 7. *(Large girth) Given $k, d \geq 2$, there is $N, M \in \mathbb{N}$ and $\varepsilon_0, C > 0$ such that for every prime p and every field K of characteristic p and any finite subset F with k elements generating a subgroup of $GL_d(K)$ which contains no* *solvable subgroup of index at most M, then F^N contains two elements a, b such that $w(a,b) \neq 1$ in $GL_d(K)$ for any non trivial word w in F_2 of length at most $f(p) = C \cdot (\log p)^{\varepsilon_0}$.*

In the same vein, one obtains the following weak diophantine property for subgroups of $GL_2(\mathbb{C})$. Let d be some Riemannian distance on $GL_2(\mathbb{C})$.

Corollary 8. *(Weak diophantine condition) Given d, there is $N_0 \in \mathbb{N}$ and $\varepsilon_1 > 0$ with the following property. For every finite set $F \subset GL_d(\mathbb{C})$ generating a non virtually solvable subgroup, there is $\delta_0(F) > 0$ such that for every $\delta \in (0, \delta_0)$ there are two short words $a, b \in F^N$ such that $d(w(a,b), 1) \geq \delta$ for every reduced word w in the free group F_2 with length $\ell(w)$ at most $(\log \delta^{-1})^{\varepsilon_1}$.*

In [6] Kaloshin and Rodnianski proved that for $G = SU(2, \mathbb{R}) \leq GL_2(\mathbb{C})$ almost every pair $(a, b) \in G \times G$ satisfies $d(w(a,b), 1) \geq \exp(-C(a,b) \cdot \ell(w)^2)$ for all $w \in F_2 \backslash \{e\}$ and some constant $C(a,b) > 0$. Besides it is easy to see that if $a, b \in GL_2(\overline{\mathbb{Q}})$ then the pair (a, b) satisfies the stronger diophantine condition $d(w(a,b), 1) \geq \exp(-C(a,b) \cdot \ell(w))$. It is conjectured by Sarnak that this stronger condition also holds for almost every pair $(a, b) \in SU(2, \mathbb{R})$.

References

[1] Breuillard, E., *On uniform exponential growth for solvable groups,* to appear in the Margulis Volume, Pure and Applied Math. Quart.

[2] Breuillard, E, *An height gap theorem for the finite subsets of $SL_n(\overline{\mathbb{Q}})$ and non amenable linear groups*, preprint April 2008, arXiv:0804.1391.

[3] Breuillard, E, *A Strong Tits alternative*, preprint April 2008, arXiv:0804.1395.

[4] Breuillard, E, and Gelander, T., *Uniform independence in linear groups*, to appear in Invent. Math.

[5] Eskin, A., Mozes, S, Oh, Hee, *On uniform exponential growth for linear groups*, Invent. Math. **160** (2005), no. 1, 1–30

[6] Kaloshin, V. Rodnianski, I, *Diophantine properties of elements of $SO(3)$*, Geom. Funct. Anal. 11 (2001), no. 5, 953–970.

[7] Kesten, H., *Symmetric walks on groups*, Trans. Amer. Math. Soc. **92** (1959) 336–354.

[8] Lubotzky A., Phillips R., Sarnak P., *Hecke operators and distributing points on the sphere I*, CPAM, **39** (1987), 149-186.

A $\mathcal{C}^0$-definition of characteristics in symplectic geometry

Emmanuel Opshtein

The talk concerns a work in progress about a C^0-rigidity result for the characteristic foliations in symplectic geometry. A symplectic homeomorphism (in the sense of Eliashberg-Gromov) which preserves a smooth hypersurface should also preserve its characteristic foliation.

Gromov [2] and Eliashberg [1] theorem on the $\mathcal{C}^0$-closedness of the group of symplectomorphisms inside the diffeomorphisms led to the notion of symplectic homeomorphisms ($\mathcal{C}^0$-limits of symplectic diffeomorphisms in the space of homeomorphisms). It also raises the question of knowing which of the classical invariants survive to this "$\mathcal{C}^0$-symplectic topology". This talk considers what can be said about the characteristic foliation.

Recall that the characteristic foliation of a hypersurface S inside a symplectic manifold (M,ω) is the integral foliation of the (one-dimensional) distribution of the null spaces of the restriction of the symplectic form to S. It is obviously a symplectic invariant of a hypersurface. The talk describes an attempt to answer the following question :

Question 1. *Let S and S' be smooth hypersurfaces of some symplectic manifolds M and M'. If a symplectic homeomorphism between M and M' sends S to S', does it transport the characteristic foliation of S to that of S' ? Equivalently, can one find a $\mathcal{C}^0$-definition of the characteristic foliation ?*

Recall that the characteristics of S are the common trajectories of all Hamiltonian flows whose corresponding Hamiltonians vanish on S. Having in mind Oh's definition of continuous Hamiltonian isotopies [3], the most natural approach would be to characterize the characteristic foliation as the common trajectories of all *continuous* Hamiltonian flows whose Hamiltonians vanish on S. Unfortunately the fundamental property that all these flows travel along common trajectories is very unclear in the $\mathcal{C}^0$-context. I would like to sketch here another possible approach.

A starting point can be the following. There is a link between two things : the characteristic foliation of a hypersurface S on one hand, and the volume preservation of Hamiltonian flows on the other. In order to explain this point, consider a trivalizing flow box T of the characteristic flow on S, and define the flux of a flow across T as the derivative of the algebraic volume swept by T when moved along this flow (the word algebraic here stands for positively counted when T goes up and negatively counted when it goes down). The observation is that the flux of any Hamiltonian flow across T is related to the value of H on the sides s_0, s_1 of this tube by the formula :

$$\text{Flux}\,(T,\Phi_H^t)_{|t=0} = \int_{s_1} H\omega^{n-1} - \int_{s_0} H\omega^{n-1}. \tag{1}$$

Therefore, if H is supported in a small ball centered on S, and if T is a characteristic tube whose ends lie outside this ball, the flux of Φ_H^t across $\Phi_H^t(T)$ vanishes for all time, and the volume preservation of the Hamiltonian flow Φ_H^t localizes on the tube of characteristics. This can be seen as a quantitative difference between Hamiltonians and volume preserving maps.

Now all the above discussion concerns the smooth category, and can by no way provide an answer to question 1. But Oh explained that the notion of continuous Hamiltonian can be defined [3]. The hope is that the quantitative difference between Hamiltonians and volume preserving maps observed above is robust enough to survive in the larger world of continuous Hamiltonians. Then the characteristics would have a purely $\mathcal{C}^0$-definition, leading to their $\mathcal{C}^0$-invariance : they would be the lines of localization of the volume preservation of $\mathcal{C}^0$-Hamiltonians with small enough support.

References

[1] Y. Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology*, Funktsional. Anal. i Prilozhen. **21** (1987), 65–72.
[2] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
[3] Y-G. Oh and S. Müller, *The group of Hamiltonian homeomorphisms and $\mathcal{C}^0$-symplectic topology*, J. Symplectic Geom. **5** 167–219.

On the dynamics of automorphisms of free groups

Gilbert Levitt
(joint work with Martin Lustig)

Given a finitely generated group G, an automorphism α, and an element $g \in G$, one may ask what happens to the sequence $\alpha^n(g)$ when $n \to \infty$. In particular, how does it grow? does it have a limit in a suitable sense ?

The talk focused mostly on the case of a free group F_p. We first pointed out that the growth is linear or exponential when α is geometric, that is α is induced by a homeomorphism of a compact surface with boundary. It is easy to construct an automorphism of F_p for which there are $p-1$ elements growing like $n, n^2, \dots, n^{p-1}$ respectively under iteration of α. Such an automorphism is not geometric for $p \geq 3$.

In general, one can prove that the length of $\alpha^n(g)$ is bi-Lipschitz equivalent to $\lambda^n n^d$ for some $\lambda \geq 1$ and $d \in N$. This fact is not at all obvious, and requires the theory of train tracks in order to control cancellation. One may ask what can be said in a general group G.

One shows that, given α, there are at most $p-1$ different growth types (λ, d). A type is called polynomial if $\lambda = 1$, exponential if $\lambda > 1$. It is possible to have $p-1$ different polynomial growth types, but not $p-1$ exponential ones. In fact, the number of exponential growth types of a given α is bounded by $\frac{3p-2}{4}$. This bound may be achieved by automorphisms induced by homeomorphisms of a compact surface whose Nielsen-Thurston decomposition consists of pseudo-Anosov maps on once-punctured tori and four-punctured spheres.

More precisely, one can determine exactly which couples (t_p, t_e) are possible, where t_p and t_e are the numbers of polynomial and exponential growth types respectively: they must belong to the closed quadrilateral with vertices $(0,0), (0, p-1), (\frac{p-1}{2}, \frac{p-1}{2}), (\frac{3p-2}{4}, 0)$.

At the end of the talk, we explained that the ω-limit set of any sequence $\alpha^n(g)$ always is an α-periodic orbit in F_p or in its boundary ∂F_p. The corresponding statement in free abelian groups is false.

From symplectic embeddings of ellipsoids to Fibonacci numbers

Felix Schlenk
(joint work with Dusa McDuff)

For $0 < a_1 < a_2$ consider the open ellipsoid

$$E(a_1,a_2) \,:=\, \left\{ (z_1,z_2) \in \mathbb{C}^2 = \mathbb{R}^4 \,\middle|\, \frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} < 1 \right\}.$$

We are looking for the smallest ball $B^4(A) := E(A,A)$ into which $E(a_1,a_2)$ symplectically embeds. We can assume that $a_1 = 1$. We therefore would like to understand the function

$$c(a) \,:=\, \inf \left\{ A \mid E(1,a) \text{ symplectically embeds into } B^4(A) \right\}.$$

Since symplectic embeddings are measure preserving, an obvious lower bound for $c(a)$ is $\sqrt{a}$. Upper bounds for $c(a)$ have been found by L. Traynor, [6], and F. Schlenk, [5], via explicit embedding constructions. Recently, in [3], Dusa McDuff has determined $c(k)$ for $k \in \mathbb{N}$ by relating the above problem to the symplectic ball packing problem:

Theorem $\mathbb{N}$ (McDuff). *$E(1,k)$ symplectically embeds into $B^4(A)$ if and only if the disjoint union of k balls $B^4(1) \cup \dots \cup B^4(1)$ symplectically embeds into $B^4(A)$.*

One direction of this theorem is elementary, since it is not hard to symplectically embed $B^4(1) \cup \dots \cup B^4(1)$ into $E(1,k)$.

The symplectic ball packing problem has a long history, starting with Gromov's work [2], followed by McDuff's and Polterovich's work [4], and culminating with Biran's work [1]. Set

$$c(w_1,\dots,w_k) \,=\, \inf \left\{ A \mid \amalg_{i=1}^k B^4(w_i) \text{ symplectically embeds into } B^4(A) \right\}.$$

It is shown in [1] that

$$c(w_1,\dots,w_k) \,=\, \max \left\{ \sqrt{\sum_i w_i^2}, \quad \sup \left\{ \frac{1}{d} \sum_i w_i m_i \right\} \right\}, \tag{1}$$

where the supremum is taken over all tuples $(d, m_1, \dots, m_k)$ of non-negative integers solving the system of Diophantine equations

$$\left\{ \begin{array}{lcl} \sum_{i=1}^k m_i & = & 3d-1, \\ \sum_{i=1}^k m_i^2 & = & d^2+1. \end{array} \right.$$

The numbers $c(1,\dots,1)$ are readily computed, and so $c(k)$ is found.

Since the function $c(a)$ is continuous, it suffices to determine $c(a)$ for $a \in \mathbb{Q}$. There is a version of Theorem $\mathbb{N}$ for $a \in \mathbb{Q}$:

Theorem $\mathbb{Q}$ (McDuff). *To $a \in \mathbb{Q}$ one can associate a weight vector $w(a) = (w_1,\dots,w_k)$ such that $E(1,a)$ symplectically embeds into $B^4(A)$ if and only if the disjoint union of k balls $B^4(w_1) \cup \dots \cup B^4(w_k)$ symplectically embeds into $B^4(A)$.*

Finding $c(a)$ now means computing (1). We have partly succeeded in doing this. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Then the graph of $c(a)$ on $[1, \tau^4]$ is a Jacob's ladder determined by ratios of consecutive odd Fibonacci numbers. For $a \geq 8\frac{1}{36}$ we have $c(a) = \sqrt{a}$. The answer on $[7, 8]$ is more difficult to discribe and not yet fully understood.

References

[1] P. Biran, *Symplectic packing in dimension* 4, Geom. Funct. Anal. **7** (1997), 420–437.

[2] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[3] D. McDuff, *Symplectic embeddings of 4-dimensional ellipsoids*, arXiv:0801.4665

[4] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry. With an appendix by Yael Karshon*, Invent. Math. **115** (1994), 405–434.

[5] F. Schlenk, *Symplectic embeddings of ellipsoids*, Israel J. Math. **138** (2003), 215–252.

[6] L. Traynor, *Symplectic packing constructions*, J. Differential Geom. **41** (1995), 735–751.

On self–similarity for ergodic flows

Krzysztof Fraczek

(joint work with Mariusz Lemańczyk)

Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic measurable flow on a standard probability Borel space $(X, \mathcal{B}, \mu)$. Given $s \in \mathbb{R} \setminus \{0\}$ by $\mathcal{T}_s$ denote the flow $(T_{st})_{t\in\mathbb{R}}$. Let

$$I(\mathcal{T}) = \{s \in \mathbb{R} \setminus \{0\} : \mathcal{T} \text{ and } \mathcal{T}_s \text{ are isomorphic}\}.$$

If there exists $s \in I(\mathcal{T}) \setminus \{-1, 1\}$, the flow is called *self-similar* with the scale of self-similarity s. Another weaker symptom of self-similarity for flows is the existence of pairs of distinct real numbers t, s for which the automorphisms T_t and T_s are isomorphic.

If a flow $\mathcal{T}$ is self-similar we are interested in the study of the size of the set $I(\mathcal{T})$ and

$$I_{aut}(\mathcal{T}) = \{(s, t) \in \mathbb{R}^2 : T_s \text{ and } T_t \text{ are isomorphic}\}$$

in relation to some dynamical properties of $\mathcal{T}$.

If $\mathcal{T}$ has positive and finite entropy then $h_\mu(\mathcal{T}_s) = |s| h_\mu(\mathcal{T}) \neq h_\mu(\mathcal{T})$, and hence $\mathcal{T}_s$ and $\mathcal{T}$ are not isomorphic for $s \in \mathbb{R} \setminus \{-1, 1\}$; similarly $I_{aut}(\mathcal{T}) \subset \{(s, t) : |s| = |t|\}$. In the zero entropy case, there is no universal bound on the size of $I(\mathcal{T})$. A natural example of zero entropy dynamical system which has plenty of self-similarities is the horocycle flow $(\eta_t)_{t\in\mathbb{R}}$ on any finite surface of constant negative curvature M. If $(\gamma_t)_{t\in\mathbb{R}}$ stands for the geodesic flow on M then

$$\gamma_s \circ \eta_t \circ \gamma_s^{-1} = \eta_{e^{-2s}t} \text{ for all } s, t \in \mathbb{R},$$

and hence every positive number s is the scale of self-similarity for the horocycle flow.

The aim of the talk is to present a method which helps to show that $I(\mathcal{T})$ and $I_{aut}(\mathcal{T})$ are small or even trivial. To prove non-isomorphism of flows or automorphisms we apply so-called joining method.

By a *joining* between ergodic flows $\mathcal{T}$ on $(X,\mathcal{B},\mu)$ and $\mathcal{S}$ on $(Y,\mathcal{C},\nu)$ we mean any probability $\{T_t\times S_t\}_{t\in\mathbb{R}}$–invariant measure on $(X\times Y,\mathcal{B}\otimes\mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings $J(\mathcal{T},\mathcal{S})$ between $\mathcal{T}$ and $\mathcal{S}$ is non-empty because the product measure $\mu\otimes\nu$ is a joining. Flows $\mathcal{T}$ and $\mathcal{S}$ are called *disjoint in the sense of Furstenberg* or simply *disjoint* if $J(\mathcal{T},\mathcal{S})=\{\mu\otimes\nu\}$. Disjoint systems are not isomorphic, moreover, they are completely different from dynamical point of view.

Every joining ρ defines an operator $\Phi_\rho : L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ by requiring that

$$\int_{X\times Y} f(x)g(y)\,d\rho(x,y)=\int_Y \Phi_\rho(f)(y)g(y)\,d\nu(y)$$

for each $f\in L^2(X,\mathcal{B},\mu)$ and $g\in L^2(Y,\mathcal{C},\nu)$. This operator has the following Markov property

$$\Phi_\rho 1=\Phi_\rho^* 1=1 \text{ and } \Phi_\rho f\geq 0 \text{ whenever } f\geq 0.$$

Moreover,

$$\Phi_\rho\circ T_t=S_t\circ\Phi_\rho \text{ for each } t\in\mathbb{R},$$

here (T_t) and (S_t) are Koopman representations associated to $\mathcal{T}$ and $\mathcal{S}$. In fact, $\rho\mapsto\Phi_\rho$ is a one-to-one correspondence between $J(\mathcal{T},\mathcal{S})$ and the set of intertwining Markov operators (see e.g. [6] for details). Notice that the product measure corresponds to the Markov operator denoted by $\int$, where $\int(f)$ equals the constant function $\int_X f\,d\mu$. Every Koopman operator T_t can be considered as a Markov operator. The corresponding self-joining μ_{T_t} is determined by $\mu_{T_t}(A\times B)=\mu(A\cap T_t^{-1}B)$ for $A,B\in\mathcal{B}$.

Theorem 1 (see [5] and [3]). *Let $\mathcal{T}=(T_t)_{t\in\mathbb{R}}$ be a weakly mixing flow on a standard Borel space $(X,\mathcal{B},\mu)$. Suppose that there exists a sequence of real numbers (t_n) such that $t_n\to+\infty$ and*

$$T_{t_n}\to\alpha\int_{\mathbb{R}} T_t\,dP(t)+(1-\alpha)J,\ \textit{weakly on } L^2(X,\mathcal{B},\mu) \tag{1}$$

where $\alpha>0$, P is a probability Borel measure on the $\mathbb{R}$ and $J\in J(\mathcal{T},\mathcal{T})$. Then T_t and T_s are disjoint for almost every pair $(t,s)\in\mathbb{R}^2$. In particular, $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint for almost every $s\in\mathbb{R}$.

It turns out that some special flows satisfy (1).

Proposition 2 (see [1]). *Let $T:[0,1)\to[0,1)$ be an ergodic interval exchange transformation and let $f:[0,1)\to\mathbb{R}^+$ be a function of bounded variation. Then the special flow T^f built over T and under the roof function f satisfies (1).*

Since every ergodic translation flow on any translation surface has a special representation over an ergodic interval exchange transformation and under a piecewise constant roof function, as a consequence of Theorem 1 and Proposition 2 we have the following.

Corollary 3. *If a translation flow $\mathcal{F}^\theta$ is weakly mixing, then $\mathcal{F}^\theta$ and $\mathcal{F}^\theta_s$ are disjoint for almost every s. Moreover, F^θ_t and F^θ_s are disjoint for almost every pair $(s,t) \in \mathbb{R}^2$.*

Operator (joining) approach helps also to prove the complete absence of self-similarity for some special flows.

Theorem 4 (see [3]). *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a measure-preserving flow. Suppose that*

- *$T_{t_n} \to \int_{\mathbb{R}} T_t\, dP(t)$ $(t_n \to \infty)$ weakly for some $P \in \mathcal{P}(\mathbb{R})$ and $\mathcal{T}$ is not rigid, or*
- *$T_{t_n} \to a\int_{\mathbb{R}} T_t\, dP(t) + (1-a)J$ for some $0 < a \le 1$, $P \in \mathcal{P}(\mathbb{R})$ and $J \in J(\mathcal{T})$ and $\mathcal{T}$ is not partially rigid.*

Then for every $s \neq \pm 1$ the flow $\mathcal{T}$ is isomorphic to $\mathcal{T}_s$.

The above result allows to prove the absence of self-similarity for so-called von Neumann flows over ergodic interval exchange transformations. Let us consider the special flow lT^f built over an ergodic interval exchange transformation $T : [0,1) \to [0,1)$ and under piecewise absolutely continuous function $f : [0,1) \to \mathbb{R}$. By Proposition 2, T^f satisfies (1). Suppose that the sum of jumps $S(f) = \int_0^1 f'(x)\, dx$ of f is not zero. Then T^f is not partially rigid (this is a more general version of Theorem 7.1 in [2]), and hence T^f is not self-similar.

The absence of self-similarity can be observed also for special flows built over ergodic rotations T on the circle by α satisfying a Diophantine condition and under some piecewise constant roof functions $f : \mathbb{T} \to \mathbb{R}$. Such special flows satisfy (1) but unfortunately they are partially rigid. Nevertheless, as it was shown in [4], under some assumptions on the α and discontinuities of f the flow T^f is mildly mixing, hence not rigid. Consequently, T^f is not self-similar. This allows to construct translation flows which are not self-similar.

Furthermore, more sophisticated joining methods based on so-called minimal self-joining property helps to construct, for every multiplicative subgroup $G \subset \mathbb{R} \setminus \{0\}$, examples of self-similara flows for which $I(\mathcal{T}) = G$.

Open problems

Problem 5. *Is $I(\mathcal{T})$ a Borel group for any measurable flow $\mathcal{T}$?*

Problem 6. *Find a flow $\mathcal{T}$ for which the group $I(\mathcal{T})$ is not countable and has zero Lebesgue measure. Give a classification of multiplicative subgroups of $\mathbb{R}$ that can be obtained as $I(\mathcal{T})$.*

Problem 7. *Is the absence of self-similarity generic in the set of measure preserving flows?*

References

[1] K. Fraczek, M. Lemańczyk, *On disjointness properties of some smooth flows*, Fund. Math. **185** (2005), 117-142.

[2] K. Fraczek, M. Lemańczyk, *On mild mixing of special flows over irrational rotations under piecewise smooth functions*, Ergod. Th. Dynam. Sys. **26** (2006), 719-738.

[3] K. Fraczek, M. Lemańczyk, *On the self–similarity problem for ergodic flows*, preprint available from http://www-users.mat.uni.torun.pl/∼fraczek/prace.html
[4] K. Fraczek, M. Lemańczyk, E. Lesigne, *Mild mixing property for special flows under piecewise constant functions*, Discrete Contin. Dyn. Syst. **19** (2007), 691-710.
[5] V.V. Ryzhikov, *On a connection between the mixing properties of a flow with an isomorphism entering into its transformations*, (Russian) Mat. Zametki **49** (1991), 98–106; translation in Math. Notes **49** (1991), 621–627.
[6] V.V. Ryzhikov, *Polymorphisms, joinings and the tensor simplicity of dynamical systems*, Funct. Anal. Appl. **31** (1997), 109-118.

Locally symmetric subspaces of locally symmetric spaces

Dave Witte Morris
(joint work with Vladimir Chernousov and Lucy Lifschitz)

It is well known that if G is a connected, semisimple Lie group, and $\mathbb{R}$-rank$G \geq 2$, then G contains a closed subgroup that is locally isomorphic to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Passing from semisimple Lie groups to the corresponding symmetric spaces yields the following geometric translation of this observation.

Fact. Let $\widetilde{X}$ be a symmetric space of noncompact type, with no Euclidean factors, such that rank$\widetilde{X} \geq 2$. Then $\widetilde{X}$ contains a totally geodesic submanifold $\widetilde{X}'$, such that $\widetilde{X}'$ is the symmetric space associated to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. In other words, $\widetilde{X}'$ is isometric to either

(1) $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3) \cong \left\{ \begin{matrix} 3 \times 3 \text{ positive-definite symmetric} \\ \text{real matrices of determinant } 1 \end{matrix} \right\}$, or

(2) the product $\mathbb{H}^2 \times \mathbb{H}^2$ of 2 hyperbolic planes.

In short, among all the symmetric spaces of noncompact type with rank ≥ 2, there are only two manifolds that are minimal with respect to the partial order defined by totally geodesic embeddings. Our main theorem provides an analogue of this result for noncompact finite- volume spaces that are locally symmetric, rather than globally symmetric, but, in this setting, the partial order has infinitely many minimal elements.

Theorem. Let X be a finite-volume, noncompact, irreducible, complete, locally symmetric space of noncompact type, with no Euclidean factors (locally), such that rank$X \geq 2$. Then there is a finite-volume, noncompact, irreducible, complete, locally symmetric space X', such that X' admits a totally geodesic, proper immersion into X, and the universal cover of X' is the symmetric space associated to either

(1) $\mathrm{SL}_3(\mathbb{R})$,

(2) $\mathrm{SL}_3(\mathbb{C})$, or

(3) a direct product $\mathrm{SL}_2(\mathbb{R})^m \times \mathrm{SL}_2(\mathbb{C})^n$, with $m + n \geq 2$.

Remark. The symmetric space associated to $\mathrm{SL}_3(\mathbb{R})$ is given above. The others are:

(1) $\mathrm{SL}_3(\mathbb{C})/\mathrm{SU}(3) \cong \left\{\begin{matrix} 3 \times 3 \text{ positive-definite Hermitian} \\ \text{matrices of determinant } 1 \end{matrix}\right\}$, and

(2) the product $(\mathbb{H}^2)^m \times (\mathbb{H}^3)^n$ of m hyperbolic planes and n hyperbolic 3-spaces.

Remark.

(1) Our proof of the theorem is constructive: for a given locally symmetric space X, our methods produce an explicit locally symmetric space X' that embeds in X.

(2) Our theorem assumes X is *not* compact. It would be interesting to obtain an analogous result that assumes X is compact (and X' is also compact).

(3) Our main result actually provides a precise description of X' (modulo finite covers), not only its universal cover. It does this by specifying the fundamental group $\pi_1(X')$.

(4) The Mostow Rigidity Theorem tells us that any locally symmetric space X as discussed above is determined by its fundamental group. This means that the above geometric result can be reformulated in group-theoretic terms, as a classification of the minimal elements in the category of non-cocompact, irreducible lattices in semisimple Lie groups of higher rank.

    Furthermore, the Margulis Arithmeticity Theorem allows us to restate the theorem as a classification of the minimal isotropic simple algebraic $\mathbb{Q}$-groups with $\mathbb{R}$-rank ≥ 2. In fact, the proof is carried out entirely in this algebraic setting, rather than in the geometric setting of symmetric spaces.

References

V. Chernousov, L. Lifschitz, and D. W. Morris, *Almost- minimal nonuniform lattices of higher rank*, Michigan Mathematical Journal (to appear).

Simplicity of Homeo($\mathbb{D}^2, \partial\mathbb{D}^2$, Area) and fragmentation of symplectic diffeomorphisms

Frédéric Le Roux

This talk is concerned with the algebraic study of the group

$$G = \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{Area})$$

of area-preserving homeomorphisms of the 2-disc that are the identity in some neighbourhood of the boundary. The central open question is the following.

Question 1 ([Fa80]). *Is G a simple group?*

Normal subgroups of G have been defined by Müller and Oh ([MO07]), and Ghys ([Gh07], see [Bo08]), but so far no one has been able to prove that these are proper subgroups: they might turn out to be equal to G. In this text, I propose to define still another normal subgroup N of G. I have not been able to prove that this subgroup is proper; however, we can prove that it is a good candidate:

Theorem 2. *The subgroup N is not reduced to the identity element (more precisely, it contains $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$, see below). Furthermore, if $N = G$ then G is simple.*

The present work has its origin in Fathi's paper [Fa80] dealing with higher dimensions, and especially in the proof of the perfectness of the group of volume preserving homeomorphisms of the n-ball which are the identity near the boundary, when $n \geq 3$. Fathi's argument has two steps. The first step is a fragmentation result: any element of the group can be written as a product of two elements which are supported on a topological ball whose volume is $\frac{3}{4}$ of the total volume. The second step shows how this fragmentation result implies the perfectness of the group. While the second step is still valid in dimension 2, the first one fails. The content of the present talk is to show, by defining the normal subgroup N, that Fathi's second step essentially admits a converse in dimension 2: the perfectness (and simplicity) of our group is equivalent to some fragmentation property. Furthermore it is also equivalent to a similar fragmentation property on the subgroup $G^{\mathrm{diff}} = \mathrm{Diffeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{Area})$ consisting of those elements of G that are C^∞-diffeomorphisms. Thus the problem is translated into a problem dealing with Hamiltonian diffeomorphisms. Our fragmentation problem is actually a family of problems depending on a parameter $\rho \in (0, 1]$. Entov-Polterovich quasi-morphisms, coming from Floer homology, immediately give the solution for $\rho \in (\frac{1}{2}, 1]$ ([EP03, EPP]). The problem remains unsolved for the other values of ρ.

Definition of N

For any $g \in G$, let us define the *size* of g to be infimum of the areas of the topological discs that contain the support of g (by a topological disc I mean the image of a euclidean disc under some element of G). The following proposition says that any element can be fragmented into elements having arbitrarily small size.

Proposition 3. *Let $g \in G$, and $\rho \in (0, 1]$. Then there exists some integer m and some elements $g_1, \dots g_m$ of size less than ρ such that*

$$g = g_m \circ \cdots \circ g_1.$$

The proof is essentially the proof of lemma 6.5 in [Fa80]. It allows us to define the *ρ-norm* of any element $g \in G$, denoted by $||g||_\rho$, as the minimum of the numbers m satisfying the above statement. Clearly this norm is invariant under conjugacy, and satisfies the triangular inequality. Thus the formula

$$d_\rho(g_1, g_2) = ||g_1 g_2^{-1}||_\rho$$

defines a bi-invariant metric on G. Now we define the set N of elements whose ρ-norm, as a function of ρ, is essentially bounded by $\frac{1}{\rho}$:

$$N = \left\{ g \in G,\ \exists C > 0,\ \forall \rho > 0, ||g||_\rho < \frac{C}{\rho} \right\}.$$

As an immediate consequence of the above proposition, this set is a normal subgroup of G.

References

[Bo08] Bounemoura, Abed. *Simplicité des groupes de transformations de surfaces.* To appear in *Ensaios Matemáticos.*

[EP03] Entov, Michael; Polterovich, Leonid. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.* 2003, no. 30, 1635–1676.

[EPP] Entov, Michael; Polterovich, Leonid ; Py, Pierre. *Private communication.*

[Fa80] Fathi, Albert. Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. ï¿½cole Norm. Sup.* (4) 13 (1980), no. 1, 45–93.

[Gh07] Ghys, Étienne. Communication at the Symplectic Topology and Measure-Preserving Dynamical Systems Conference, Snowbird, July 2007.

[MO07] Oh, Yong-Geun; Müller, Stefan. The group of Hamiltonian homeomorphisms and C^0-symplectic topology. *J. Symplectic Geom.* 5 (2007), no. 2, 167–219.

Combable functions and the central limit theorem

Koji Fujiwara

(joint work with Danny Calegari)

We define a certain class of functions, which we call bi-combable functions, on a word-hyperbolic group and show that those functions satisfy a central limit theorem with respect to a certain probability measure. Examples of bi-combable functions include word-lengths, homomorphisms and certain quasi-homomorphisms to real numbers.

As a consequence we obtain the following theorem ([1]): let G be a word-hyperbolic group and S_1, S_2 finite generating sets. Let $|g|_i$ be the word length of an element $g \in G$ with respect to S_i. Then, there is a constant $\lambda > 0$ such that for any $\epsilon > 0$, there is a K and an N so that if G_n denotes the set of elements of length $n \geq N$, there is a subset G'_n with $|G'_n|/|G_n| \geq 1 - \epsilon$, so that for all $g \in G'_n$ there is an equality

$$\big||g|_1 - \lambda |g|_2\big| \leq K \cdot \sqrt{n}$$

References

[1] Danny Calegari, Koji Fujiwara. Combable functions, quasimorphisms, and the central limit theorem. preprint, May 2008. arXiv:0805.1755.

Two new families of Cayley graph expanders

Norbert Peyerimhoff
(joint work with Alina Vdovina)

Expander graphs are both highly connected and sparse and they are not only of theoretical importance but also useful in computer science, e.g., for network designs. See, e.g., [3] for an exhaustive recent survey on this topic. We first present the definition:

Definition 1. *Let $\mathcal{G} = (V, E)$ be a finite (undirected) graph with vertex set V and edge set E. The* Cheeger constant *$h(\mathcal{G})$ is defined as follows:*

$$h(\mathcal{G}) = \min_{S \subset V, |S| \leq |V|/2} \frac{|E(S, S^c)|}{|S|},$$

where $E(S, S^c)$ is the set of edges connecting a vertex of S with a vertex of $S^c = V \backslash S$. Let $k \in \mathbb{N}$ be fixed. An infinite family $\mathcal{G}_i = (V_i, E_i)$ of connected, k-regular (i.e., all vertices have degree k), finite graphs is called a family of expanders *if there is an $\epsilon > 0$ such that $|V_i| \to \infty$ and we have*

$$h(\mathcal{G}_i) \geq \epsilon$$

for all i.

The first explicit example of expanders is due to Margulis [6]. By a discrete analogue of the Cheeger-Buser inequality due to Dodziuk and Alon-Milman, expanders can also be described as increasing families of connected, k-regular finite graphs with positive lower bound on the spectral gap (of the corresponding adjacency matrices). If the non-trivial spectrum $\sigma(\mathcal{G})$ of the adjacency matrix of a connected, k-regular, finite graph $\mathcal{G}$ lies in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, we call $\mathcal{G}$ a *Ramanujan graph.* The first explicit expander families of Ramanujan graphs are due to Lubotzky/Phillips/Sarnak [4] and Margulis [7].

We present two new families of Cayley graph expanders of vertex degree 4, which is the minimal vertex degree for Cayley graph expander families. The first example is given by explicit matrix models and defines a tower of coverings, and the second example is given by finite groups with two generators and only four relations. Our results are as follows:

Theorem 1. *There is a family of finite nilpotent groups N_i, generated by two generators x_i, y_i and satisfying $|N_i| = 2^{n_i}$ with strictly monotone increasing $n_i \to \infty$. The corresponding Cayley graphs $\mathcal{G}_i = Cay(N_i, \{x_i^{\pm 1}, y_i^{\pm 1}\})$ are a family of 4-regular expanders forming a tower of coverings*

$$\cdots \to \mathcal{G}_3 \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0. \quad (1)$$

Since every two-group has a nontrivial center, the tower (1) can be "filled up" to a new tower of coverings

$$\cdot \to \widetilde{\mathcal{G}}_3 \to \widetilde{\mathcal{G}}_2 \to \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_0$$

with covering indices *exactly equals two*. The graphs $\widetilde{\mathcal{G}}_i$ define also a family of expanders with the same lower bound on the spectral gap as (1) because of $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$ for every finite covering $\mathcal{G}'$ of $\mathcal{G}$.

Theorem 2. *The groups*

$$G_i = \langle x, y \mid r_1, r_2, r_3, [y, \underbrace{x, \ldots, x}_{i}] \rangle$$

with

$$\begin{aligned} r_1 &= yxyxyxy^{-3}x^{-3}, \\ r_2 &= yx^{-1}y^{-1}x^{-3}y^2x^{-1}yxy, \\ r_3 &= y^3x^{-1}yxyx^2y^2xyx \end{aligned}$$

are finite, satisfy $|G_i| \to \infty$, *and the associated Cayley graphs* $Cay(G_i, \{x^{\pm 1}, y^{\pm 1}\})$ *are a family of* 4*-regular expanders.*

In the second theorem, we use the notation $[a, b] = a^{-1}b^{-1}ab$, and higher commutators are defines recursively by $[a_1, \ldots, a_{n+1}] = [[a_1, \ldots, a_n], a_{n+1}]$.

The underlying finite groups of both families of Cayley graph expanders are quotients of a infinite fundamental group $G = \pi_1(\mathcal{K})$ of a particular finite simplicial complex $\mathcal{K}$ built up by glueing together 14 triangles (see Figure 1). One checks that G has two generators $x = x, y = x$ and is presented by $G = \langle x, y \mid r_1, r_2, r_3 \rangle$. The universal covering $\tilde{\mathcal{K}}$ is a thick Euclidean building of type $\tilde{\mathrm{A}}_2$. Results by [1] (see also [8] or [11] for earlier versions) show that G has Kazdhan property (T). Using investigations by [2] on particular groups associated to buildings of type $\tilde{\mathrm{A}}_2$, we employ a linear representation of G (by upper unitriangular infinite matrices) over the ring $M(3, \mathbb{F}_2)$. Detailed investigations of this linear representation leads to the construction of the two families of expanders described in the two theorems above. The main ingredient is a particular recursive commutator scheme (with a particular 3-periodicity) which allows us to obtain a good understanding of all higher commutators in the generators x, y of the group G.

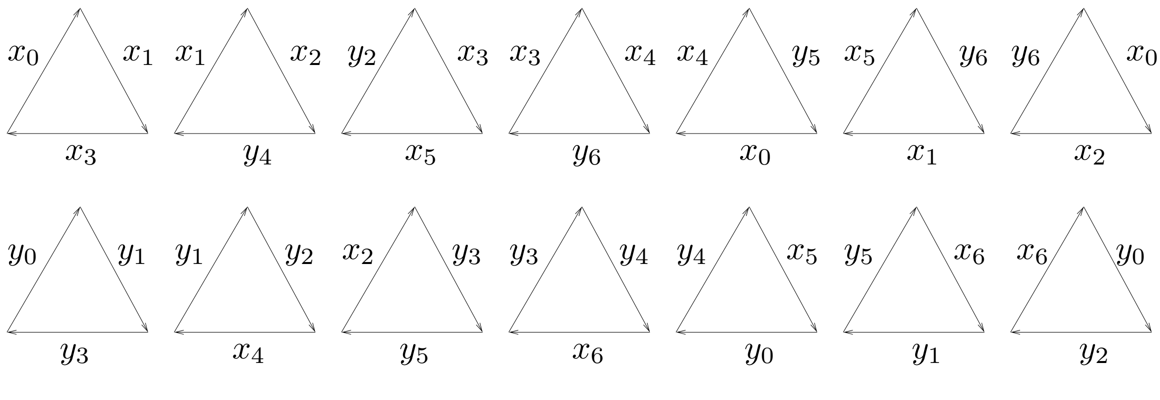

FIGURE 1. Labeling scheme of the simplicial complex $\mathcal{K}$

The appeal of the given expander graph families lies in the properties of them being very simple (only vertex degree 4 and 2-fold covers resp. simple presentations in only four relations) and being very explicit. They are not Ramanujan for high enough vertex degrees. A different construction of Cayley complexes obtained from finite quotients of groups acting cocompactly on buildings of type $\tilde{\mathrm{A}}_n$, considered independently by Lubotzky, Samuels and Vishne [5] and Sarveniazi [10] yields Ramanujan complexes.

Our commutators scheme considerations also suggest the following conjecture.

Conjecture 3. *Let*

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \gamma_2(G) \geq \cdots$$

denote the lower central series of G*, i.e.,* $\gamma_{i+1}(G) = [\gamma_i(G), \gamma_i(G)]$*. Then we have for* $i \geq 2$*:*

$$[\gamma_i(G) : \gamma_{i+1}(G)] = \begin{cases} 8, & \text{if } i \cong 0, 1 \mod 3, \\ 4, & \text{if } i \cong 2 \mod 3, \end{cases}$$

i.e., the group G *is of* finite width 3*.*

For more details on this work, see [9].

References

[1] W. Ballmann, J. Świątkowski, *On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes*, Geom. Funct. Anal. **7** (1997), 615–645.

[2] D. I. Cartwright, A. M. Mantero, T. Steger, A. Zappa, *Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$. I.*, Geom. Dedicata **47** (1993), 143–166.

[3] Sh. Hoory, N. Linial, A. Wigderson, *Expander graphs and their applications*, Bull. Amer. Math. Soc. **43** (2006), 439–561.

[4] A. Lubotzky, R. Phillips, P. Sarnak, *Ramaujan graphs*, Combinatorica **8** (1988), 261–277.

[5] A. Lubotzky, B. Samuels, U. Vishne, *Explicit construction of Ramanujan complexes of type $\tilde{A}_d$*, European J. Combin. **26** (2005), 965–993.

[6] G. A. Margulis, *Explicit constructions of concentrators*, Problems of Information Transmission **9** (1973), 325-332.

[7] G. A. Margulis, *Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators*, Problems Inform. Transmission **24** (1988), 39–46.

[8] P. Pansu, *Formules de Matsushima, de Garland et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles*, Bull. Soc. Math. France **126** (1998), 107–139.

[9] N. Peyerimhoff, A. Vdovina, *Cayley graph expanders and groups of finite width*, arXiv:0809.1560.

[10] A. Sarveniazi, *Explicit construction of a Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph*, Duke Math. J. **139** (2007), 141–171.

[11] A. Żuk, *La propriété (T) de Kazdhan pour les groupes agissant sur les polyèdres*, C. R. Acad. Sci. Paris Sér.I Math. **323** (1996), 453–458.

Arithmeticity of lattices in products of three Kac–Moody groups

Pierre-Emmanuel Caprace
(joint work with Nicolas Monod)

'High rank yields rigidity.' This repeatedly observed phenomenon is illustrated by numerous striking results in various fields, including the rank rigidity for Hadamard manifolds, the arithmeticity of lattices in higher rank semi-simple algebraic groups, or the fact that any irreducible spherical building of rank > 2 (resp. affine building of dimension > 2) is associated to a simple linear algebraic group (possibly over a skew field). We present a new manifestation of this phenomenon in the setting of lattices in products of locally compact Kac–Moody groups. Details may be found in [CMb].

Recall that, besides the standard examples provided by semi-simple linear algebraic groups over local fields and their (S-)arithmetic subgroups, the only known examples of non-trivial products of topologically simple locally compact groups admitting irreducible lattices are provided by complete Kac–Moody groups over finite fields. The latter were constructed by J. Tits [Tit] and O. Mathieu [Mat] and may be viewed as topologically simple closed automorphism groups of locally finite buildings, which are locally compact groups when endowed with the compact-open topology.

Given a complete Kac–Moody group G over a sufficiently large finite field k, it is known that the product $G \times G$ contains at least one (conjugacy class of) finitely generated irreducible lattice Λ called the **Kac–Moody lattice**, see [Rém]. Furthermore, the group Λ (and hence G) does not admit faithful finite-dimensional linear representations over any field provided it is infinite and G is not of affine type [CR]. The following result shows that the existence of such non-linear lattices is necessarily confined to the case of products of at most two factors.

Theorem 1. *Let $G = G_1 \times \cdots \times G_n$ be a product of n infinite complete Kac–Moody groups of simply laced type over finite fields. Assume that G contains a lattice Γ whose projection to each G_i is faithful.*

If $n \geq 3$, then each G_i is a semi-simple linear algebraic group over a local field, and Γ is an S-arithmetic lattice.

The proof relies on the following **arithmeticity vs. non-residual-finiteness alternative**.

Theorem 2. *Let $G = G_1 \times \cdots \times G_n$ be a product of n infinite complete Kac–Moody groups of simply laced type over finite fields. Assume that G contains a lattice Γ which is algebraically irreducible.*

If $n \geq 2$ then either Γ is an S-arithmetic group or Γ is not residually finite.

A group is called **algebraically irreducible** if no finite index subgroup splits as a non-trivial direct product of two subgroups. This condition, as well as the hypothesis of Theorem 1 that Γ has faithful projections, imposes some form of *irreducibility* for the lattice Γ, which prevents Γ from being a direct product of lattices in the respective factors of G. It turns out that, in the above setting, the faithfulness of the projections always implies the algebraic irreducibility of Γ. However the converse fails. An even stronger irreducibility condition is that Γ be **topologically irreducible**, which means that its projection to any proper sub-product of G is dense. As the case may be, this stronger condition indeed yields stronger conclusions, as demonstrated by the following **arithmeticity vs. simplicity alternative**.

Corollary 3. *Let $G = G_1 \times \cdots \times G_n$, where G_i is an infinite irreducible Kac–Moody group of simply-laced type over a finite fields $\mathbf{F}_{q_i}$ and let $\Gamma < G$ be a topologically irreducible lattice. If Γ is not uniform, assume in addition that $q_i \geq \frac{1764^{d_i}}{25}$ for each i, where d_i denotes the maximal rank of a finite Coxeter subgroup of the Weyl group of G_i.*

If $n \geq 2$, then exactly one of the following assertions holds.

(1) *Each G_i is a semi-simple linear algebraic group and Γ is an S-arithmetic lattice.*
(2) *$n = 2$ and Γ is virtually simple.*

In the special case of the aforementioned Kac–Moody lattice $\Lambda < G \times G$, the simplicity was established in [CR] by different methods, taking advantage of the specific structure of Λ.

The results presented here happen to hold for more general types of Kac–Moody groups, including non-Gromov hyperbolic complete Kac–Moody groups of 3-spherical type, provided the ground field has characteristic $\neq 2$. In fact, building upon [CMa], it is shown in [CMb] that Theorems 1 and 2 also hold assuming only that G possesses at least one Kac–Moody factor, while the only condition imposed on the other factors is that they admit proper cocompact actions by isometries on some CAT(0) spaces.

References

[CMa] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups and lattices of non-positively curved spaces*, Preprint, 2008.

[CMb] Pierre-Emmanuel Caprace and Nicolas Monod, *Rank rigidity and simplicity for lattices in product of Kac–Moody groups*, Preprint, 2008.

[CR] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Preprint, 2006.

[Mat] Olivier Mathieu, *Construction d'un groupe de Kac–Moody et applications. (Construction of a Kac-Moody group and applications).*, Compos. Math. **69** (1989), no. 1, 37–60 (French).

[Rém] Bertrand Rémy, *Construction de réseaux en théorie de Kac–Moody*, C. R. Acad. Sc. Paris **329** (1999), 475–478.

[Tit] Jacques Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.

How to solve equations in free groups?

François Dahmani
(joint work with Vincent Guirardel)

Given a group G and a finite set of unknowns X, an equation is a word on the alphabet $G \cup X \cup X^{-1}$. A solution of this equation is a value in G for each unknown, so that substitution gives the trivial element of G.

Here is the general problem we are interested in.

Given a group G and a system of equations and inequations in G, does there exist an algorithm saying whether it has a solution or not ?

This problem is not solvable in general in a finitely presented group. It is solvable for virtually abelian groups, but there exists some nilpotent group where this is not decidable ([Rom79]).

In 1982, Makanin proved the following remarkable result

Theorem 1 ([Mak82]). *There exists an algorithm which decides if a given system of equations and inequations has a solution in a free group F.*

Using canonical representatives, Rips and Sela also reduced the problem of solving equations in torsion free hyperbolic groups to Makanin's result [RS95]. This was one of the main tool for Sela's solution to the isomorphism problem for torsion free rigid hyperbolic groups [Sel95].

Makanin's result was also the main source of inspiration for Rips's study of group actions on $\mathbb{R}$-trees ([BF95, GLP94, Gui98]). The main result of Rips theory is the classification of 2-complexes holding a measured foliation.

We report on the following result, which was independently proved by Lohrey and Senizergues.

Theorem 2 ([DGui]). *Given a virtually free group V, there exists an algorithm which decides if a given system of equations and inequations (with rational constraints) has a solution in V.*

Using a version of canonical representatives, we can deduce the following result:

Theorem 3 ([DGui]). *Given a hyperbolic group G (maybe with torsion), there exists an algorithm which decides if a given system of equations and inequations has a solution in G.*

This was proved by Rips-Sela in the case where G is torsion free and there is no inequation [RS95]. This was generalized, by the speaker, to torsion free hyperbolic groups relative to finitely generated abelian groups, allowing inequations.

A further result is the following:

Theorem 4 ([DGui2]). *The isomorphy problem is solvable in the class of all hyperbolic groups.*

This was proved by Sela for torsion free hyperbolic groups having a finite group of outer autormorphisms ([Sel95]), then generalized by Dahmani-Groves to the class of all torsion free hyperbolic groups relative to finitely generated abelian groups ([DGr]).

Our approach to prove Main Theorem is the following. First, we reduce to the case of *twisted* equations over a free groups.

Then, and this is a new geometrical and dynamical point of view on Makanin's result, we use Rips Theory (on measured laminations on band complexes) to give a new proof for Makanin's algorithm for twisted equations.

Let's start with a general fact. Consider $G_0 \triangleleft G$ a finite index normal subgroup, and let $F = G/G_0$ the finite quotient. For each $f \in F$, choose a lift $\tilde{f} \in G$. Consider the equation $x_1x_2x_3 = 1$ in G. For each of the finitely many possibilites f_i for the image of x_i in F (which should satisfy $f_1f_2f_3 = 1$), the initial equation is equivalent to the equation $\tilde{f}_1y_1\tilde{f}_2y_2\tilde{f}_3y_3 = 1$ where y_i should represent an element of G_0. This equation is equivalent to $\tilde{f}_1\tilde{f}_2\tilde{f}_3 i_{\tilde{f}_3\tilde{f}_2}(y_1) i_{\tilde{f}_3}(y_2)y_3 = 1$ where $i_{\tilde{f}}$ is the automorphism of G_0 induced by the conjugation by $\tilde{f}$. Since $\tilde{f}_1\tilde{f}_2\tilde{f}_3 \in G_0$, we get a *twisted equation* in G_0, where the twisting automorphisms $i_{\tilde{f}}$ generate a finite subgroup of $\mathrm{Out}(G_0)$.

In the case we are interested in, G_0 is a free group, and it will be important that the twisting automorphisms preserve word length. This is done by enlarging G_0 to a larger free group where the twisting automorphisms preserve a free basis. One then has to introduce rational constraints to impose that the variables lie in the initial free group G_0.

Consider now a system of equations in a free group.

Such a system can be encoded in a (family of) 2-complex of squares (that we call complex of bands): define a collection of disjoint oriented segments, one for each unknown, and for each constant in the systemn, and join by bands (squares attached on segments by opposite edges) subwords that are assumed to coincide. For instance, for the equation $xyz = 1$, there are three segments X, Y, Z, and a band joins the end of X to beginning of Y with reverse orientation, another band joins the end of Y to the beginning of Z (with reverse orientation) and similarily for the end of Z and the beginning of X. If, in some other equation of our system, the unknown x appears again, the existing segment X should be used. There are choices to make, about the combinatorics of such a band complex (about the relative position of attaching points), but there are only finitely many, hence yielding a finite family of band complexes.

Solving equations in a free group easily reduces to finding a labelling of the segments of a band complex, so that both sides of each band are labelled by the same reduced word. The constants are encoded by some parts of the labelling which are imposed.

Any solution of a band complex induces a *partial lamination*, encoding which subwords of the labelling correspond to each other.

The algorithm is composed of two machines working simultaneously. A lamination generator generates all the possible laminations by extending the leaves progressively. Laminations which are rejected by the second machine are not extended any more. For each lamination which cannot be extended, it is easy to check whether it corresponds to a solution or not by checking for incompatibilities with the constants. Thus if the band complex has a solution, it will be found by this machine.

The second machine is a lamination analyser which rejects some laminations for which one can prove that they are not induced by the shortest solution: the machine rejects a lamination L if

- there is an incompatibility with constants
- or the lamination has is invariant (combinatorial) transverse measure
- or the algorithm can produce a special kind of certificate proving that if some solution induces the lamination L, then there is a shorter solution.

This machine stops if all laminations have been rejected, in which case one knows that there is no solution.

The proof of the theorem consists in proving that the algorithm always stops by producing appropriate shortening certificates.

This is where we use Rips classification of measured foliations on band complexes. If one assumes that the algorithm does not stop, by extracting a limit in the sequence of prelaminations given by the lamination generator, one can construct a foliation on the band complex with an invariant measure. Technical issues are whether this measure has atoms, and/or not full support. If these situations do not occur, Rips theory gives a decomposition into minimal components of the band complex, and on each component, one can detect criteria for rejection, contradicting that the sequence of prelaminations was never rejected.

References

[BF95] Mladen Bestvina and Mark Feighn. *Stable actions of groups on real trees.* Invent. Math., **121**(2):287–321, 1995.

[DGr] François Dahmani and Daniel Groves. *The Isomorphism Problem for Toral Relatively Hyperbolic Groups.* Publ. Math., Inst. Hautes Études Sci. **107** no.1 (2008) 211-290.

[DGui] François Dahmani, Vincent Guirardel *Solving equations in free groups by Rips theory*, preprint.

[DGui2] François Dahmani, Vincent Guirardel *The isomorphism problem for hyperbolic groups with torsion*, preprint.

[GLP94] D. Gaboriau, G. Levitt, and F. Paulin. *Pseudogroups of isometries of* $\mathbb{R}$ *and Rips' theorem on free actions on* $\mathbb{R}$*-trees.* Israel J. Math., **87**(1-3):403–428, 1994.

[Gui98] Vincent Guirardel. *Approximations of stable actions on* $\mathbb{R}$*-trees.* Comment. Math. Helv., **73**(1):89–121, 1998.

[Mak82] G. S. Makanin. *Equations in a free group.* Izv. Akad. Nauk SSSR Ser. Mat., **46**(6):1199–1273, 1344, 1982.

[Rom79] V. A. Roman'kov. *Universal theory of nilpotent groups.* Mat. Zametki, **25**(4):487–495, 635, 1979.

[RS95] E. Rips and Z. Sela. *Canonical representatives and equations in hyperbolic groups.* Invent. Math., **120**(3):489–512, 1995.

[Sel95] Z. Sela. *The isomorphism problem for hyperbolic groups. I. Ann. of Math. (2)*, 141(2):217–283, 1995.

Problem Session

General Audience

Moderated by A. Katok
Notes taken by Egor Shelukhin and Frol Zapolsky

1. A. Erschler — Isoperimetric inequality in groups

Let G be a finitely generated group. Define the Følner function as

$$\mathrm{Føl}_G(n) = \min\left\{\#V \,\middle|\, V \subset G\,;\ V \neq \varnothing\,;\ \frac{\#\partial V}{\#V} < \frac{1}{n}\right\}.$$

Question 1. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function with $f(n) \geq \exp(n)$. Is any such function asymptotically realizable as the Følner function of some group? That is, is there G such that $\mathrm{Føl}_G(n) \sim f(n)$?*

Question 2. *Let now G be k-solvable. Then there is a constant $S \equiv S(G)$ such that the Følner function satisfies*

$$\exp(n) \leq \mathrm{Føl}_G(n) \leq \underbrace{\exp(\dots(\exp}_{k-1}(n^S)\dots)\,.$$

For $k \geq 3$, is any nondecreasing function between these two bounds asymptotically realizable for some k-solvable G?

In the case $k = 2$, not all nondecreasing function between these bounds can be realized. Nevertheless, which ones can?

Question 3. *Suppose that $\mathrm{Føl}_G(n) \leq \exp(n)$. Is it true that the Poisson boundary of simple random walks on G is trivial?*

2. L. Potyagailo — Floyd boundary of finitely generated convergence groups

Let G be a finitely generated group, S a finite generating set of G and $C(G,S)$ the corresponding Cayley graph. Let $f\colon \mathbb{N} \to \mathbb{R}_+$ be a strictly decreasing function satisfying the following conditions: $\sum_n f(n) < \infty$, and $\forall n \in \mathbb{N} \,:\, \frac{f(n)}{f(n+1)} < \mathrm{const}$ (we assume $f(0) = f(1)$).

Define a new distance on $C(G,S)$ by putting first $d_f(e) := f(d(1,e))$ for an edge e of $C(G,S)$, where $1 \in G$ is the identity element and d is the standard distance on $C(G,S)$. Then

$$d(x,y) = \min_{\gamma\colon\, x \to y} l_f(\gamma)\,,$$

where $l_f(\gamma)$ is the sum of lengths (with respect to d_f) of edges in a path γ running from x to y. Let $\overline{C}_f(G,S)$ denote the completion of the metric space $(C(G,S),d)$. The Floyd boundary of G with respect to S and f is the difference $\partial_f G := \overline{C}_f(G,S) - C(G,S)$.

Theorem 4 (Floyd)**.** *Let G be a discrete and geometrically finite subgroup of the group of isometries* $\text{Isom}\mathbb{H}^3$ *of the hyperbolic space $\mathbb{H}^3$. Then for some polynomial function f there exists a G-equivariant map $F : \partial_f G \to \Lambda(G) \subset \partial\mathbb{H}^3$, where $\Lambda(G) \subset \partial\mathbb{H}^3$ is the limit set of G. Furthermore the map F is one-to-one at each conical point of $Lambda(G)$.*

P. Tukia has generalized the theorem of Floyd to the hyperbolic space $\mathbb{H}^n$ of any dimension n ('88). The following more general result is proven recently :

Theorem 5 (Gerasimov)**.** *Let G be a convergence group of homeomorphisms of a compactum T. Suppose that G is relatively hyperbolic with respect to a system of parabolic subgroups. Then there exists a G-equivariant map $F : \partial_f G \to T$ where $\partial_f G$ is the Floyd boundary with respect to a function f satisfying the above conditions. The map F is one-to-one at conical points for the action of G on T.*

It follows from this theorem that the Floyd boundary $\partial_f G$ of a relatively hyperbolic group is not a point (even in the case if f is exponential). The following question seems to be natural:

Question 6 (Osin, Gerasimov)**.** *Does there exist a non-relatively hyperbolic group G (with respect to any system of finitely generated subgroups) having a nontrivial $\partial_f G$?*

We guess that the answer should be positive. Indeed it was recently proved that any non-uniform arithmetic lattice $G < \mathbb{H}^n$ is non-coherent if $n > 5$, i.e. G contains a subgroup H which is finitely generated but not finitely presented (M. Kapovich, L. Potyagailo, and E. Vinberg, 2008). Note that it follows from a recent preprint of I. Agol that the same result is true for any dimension $n > 3$ and this is best possible estimate as in dimension $n = 3$ the fundamental groups of 3-manifolds are all coherent (P. Scott, '74).

Note that the above infinitely presented group H is not relatively hyperbolic with respect to any finitely presented subgroup. So the following two questions seem to be natural:

Question 7. *Is the Floyd boundary $\partial_f H$ trivial for the above group H ?*

Question 8. *Is H not relatively hyperbolic with respect to any system of finitely generated subgroups ?*

3. J. Franks — Group of interval exchange transformations

Let G be the group of interval exchange transformations. What can be said about its structure? (A question by C. Novak)

Question 9. *What groups are subgroups of G?*

For instance, any finite group and a finite dimensional torus are subgroups of G. Also, many finitely generated torsion-free groups are:

Theorem 10 (C. Novak)**.** *If $g \in G$, $g \neq 1$, then g in not distorted.*

The proof relies on counting discontinuity points of g.

Higher rank lattices contain much distortion, so, for example, $SL(3,\mathbb{Z})$ does not embed into G. What about free groups? Even F_2?

Another property of G is:

Theorem 11. $\mathrm{Out}(G) = \mathbb{Z}/2\mathbb{Z}$.

Question 12. *What about $SO(n)$? Does it embed into G?*

4. L. Polterovich — Lie quasi-states

Definition 1. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $K \subset \mathfrak{g}$ an* ad*-invariant cone. $\zeta\colon \mathfrak{g} \to \mathbb{R}$ is a Lie quasi-state if*

(i) ζ is linear on commutative subspaces;

(ii) $\zeta|_K \geq 0$.

Question 13. *How can one detect all Lie quasi-states for given G and K?*

For example, if $G = U(n)$, $\mathfrak{g}$ is the set of Hermitian matrices, and K is the set of nonnegative matrices, then Gleason's theorem states that for $n \geq 3$ any Lie quasi-state on $\mathfrak{g}$ is linear.

Another example is $G = \widetilde{Sp(2n)}$, $\mathfrak{g}$ is the set of symmetric matrices, and K is the set of nonnegative ones. Let $\mu\colon G \to \mathbb{R}$ be the Maslov quasi-morphism, and $\zeta = \exp^* \mu$ the corresponding Lie quasi-state.

Question 14. *Is this ζ unique, up to isomorphisms?*

A third example is the group $G = \mathrm{Ham}(M,\omega)$, $\mathfrak{g} = C^\infty(M)$, $K = \{F \mid F \geq 0\}$. Here (M,ω) is a closed symplectic manifold.

Question 15. *Does $(\mathbb{T}^4, \omega_{std})$ admit a nonlinear Lie quasi-state?*

Question 16. *How many Lie quasi-states do $(\mathbb{C}P^2, \omega_{std})$, $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ admit?*

Here ω_{std} stands for the standard symplectic form on the above manifolds.

5. F. Le Roux

Question 17. *Which measurable ergodic systems are measurably conjugated to a minimal topological system of the form (M,λ,h)? Here M is a compact manifold, h a uniquely ergodic homeomorphism of M and λ an h-invariant ergodic measure with full support in M.*

Even in the case of $\mathbb{T}^2$, is it true that the adding machine is realizable in the above sense? The Bernoulli shift?

For example, by a result of Jewett and Krieger, every measurable ergodic system is realizable in the above sense on the Cantor set (except that it is not a manifold).

For the torus $\mathbb{T}^2$, the following is true:

Theorem 18 (Béguin, Crovisier, Le Roux). *An ergodic system of the form* $(S^1, \text{Lebesgue measure}, \text{rotation}) \times$ *any system is realizable on the torus* $\mathbb{T}^2$*. More generally, an ergodic system is realizable on the torus provided that its spectrum contains an irrational number.*

Note that this theorem is *not* optimal, since there exist weakly mixing minimal homeomorphisms on $\mathbb{T}^2$.

6. G. Knieper

Let M^n be a compact manifold of hyperbolic type, that is admitting a Riemannian metric of negative curvature. Let g be an arbitrary metric on M, and let ϕ^t denote the geodesic flow with respect to g on the unit tangent bundle of M. It is known that

$$h_{\text{top}}(\phi^t) \geq h_{\text{Vol}}(g) = \lim_{R\to 0} \ln \frac{\text{Vol}(B(p,R))}{R}\,,\ p \in \widetilde{M}\,.$$

Consider in the unit tangent bundle SM the closed subset

$$SM_{\min} = \{v \in SM \,|\, c_v \text{ lifts to a minimal geodesic in } \widetilde{M}\}\,.$$

Question 19. *Is it true that*

$$h_{top}(\phi^t|_{SM_{min}}) = h_{Vol}(g)\,?$$

Question 20. *Let M be a surface of higher genus, and let g be any metric on M. Is it true that*

$$h_{\mu_L}(\phi^t) > 0\,?$$

Here μ_L is the Liouville measure.

7. D. Burago

7.1. Geometric group theory.

Question 21. *Let G be a finitely generated group. Is it true that there is a bi-Lipschitz bijection $G \to G \times \mathbb{Z}_2$? Same for $G_1 \to G$, where G_1 is a subgroup of G of finite index.*

Question 22. *Let L_1, L_2 be two uniform lattices in the same Lie group. Is it true that there is a bi-Lipschitz bijection $L_1 \to L_2$?*

7.2. Hyperbolic dynamics.

Question 23. *Let M be a compact manifold, f a diffeomorphism of M. f is partially hyperbolic if there is a C^0-splitting $TM = E^s \oplus E^c \oplus E^u$, where E^s contracts under f with exponent at most $\lambda < 1$, E^u expands with exponent at least $\mu > 1$, and on E^c dilates with speed at least λ' and at most μ', where $\lambda < \lambda' \leq 1 \leq \mu' < \mu$. In dimension three, let $\dim E^s = \dim E^c = \dim E^s = 1$. What are the restrictions on the topology of M?*

An example would be

Theorem 24 (Burago, Ivanov). *If $\pi_1(M)$ is abelian, then $b_2 \geq 2$.*

Question 25. *It is known that E^s, E^u are uniquely integrable to C^0-foliations with smooth leaves. Is E^c always uniquely integrable?*

7.3. **Ergodic theory.** Consider the following C^1-metric on the sphere S^2: put a flat metric (with constant curvature 0) on the complement of three disks in S^2, and a metric of constant curvature on the disks, such that the total curvature of the disks is 4π. The topological entropy of the geodesic flow is then positive. Is the same true for h_λ, where λ is the Liouville measure?

8. A. Katok

8.1. **Geometric group theory.** Let G be a group and let ϕ be an automorphism; to it one associates the exponential growth $\phi^u(g)$ for an element $g \in G$. Let $h(\phi)$ be the fastest exponential growth.

Question 26. *Which values can h assume:*
(i) for all finitely generated groups;
(ii) for all finitely presented groups?

8.2. **Hyperbolic dynamics.**

8.2.1. *Geometry flavor.* Let M be a compact manifold of hyperbolic type, g an arbitrary Riemannian metric on M (certain smoothness assumptions need to be made, for example, $C^{2+\varepsilon}$ or C^3).

Consider inside the unit tangent bundle SM the set of velocities of globally minimal geodesics $SM^g_{\min}$.

Question 27. *Is there a closed hyperbolic geodesic in $SM^g_{\min}$?*

The answer is not known even in dimension two.

Question 28. *Is there an ergodic hyperbolic invariant measure supported on $SM^g_{\min}$?*

Here, the answer is known in dimension two.

8.2.2. *Rigidity flavor.* Take an Anosov diffeomorphism f on M, $TM = E^s \oplus E^u$; here the splitting is C^∞.

Conjecture 29. *f is C^∞-conjugated to a standard model.*

Theorem 30 (Benoist, Labourie, 90-ies). *The conjecture holds if f is a symplectomorphism.*

8.3. **Symplectic geometry.** Let f be an area-preserving diffeomorphism of M, where M is a surface with an area form.

Question 31. *Can f be measurably conjugated to (S^1, R_β), where $R.$ is rotation, and β is Diophantine?*

Why is this question interesting? The following results point in a similar direction:

Theorem 32 (Anosov, Katok, 70-ies)**.** *True for some Liouvillean* β*.*

Theorem 33 (Fayad, Saprykina, Windsor, 2006)**.** *True for all Liouvillean* β*.*

Reporters: Henrik Köhler, Thomas Stratmann

Participants

Prof. Dr. Emmanuel Breuillard
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Dmitri Y. Burago
Department of Mathematics
Pennsylvania State University
University Park , PA 16802
USA

Prof. Dr. Marc Burger
Forschungsinstitut für Mathematik
ETH-Zürich
ETH Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Leo Butler
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
GB-Edinburgh EH9 3JZ

Prof. Dr. Danny Calegari
Department of Mathematics
California Institute of Technology
Pasadena , CA 91125
USA

Dr. Pierre-Emmanuel Caprace
Institut des Hautes Etudes
Scientifiques
35, route de Chartres
F-91440 Bures-sur-Yvette

Prof. Dr. Octav Cornea
Dept. of Mathematics and Statistics
University of Montreal
Pavillon Andre-Aisenstadt
2920 Chemin de la Tour
Montreal , QC H3T 1J8
CANADA

Prof. Dr. Francois Dahmani
CNRS UMR 5580
Laboratoire Emile Picard
Universite Paul Sabatier
F-31062 Toulouse

Prof. Dr. Michael Entov
Department of Mathematics
Technion - Israel Institute of
Technology
Haifa 32000
ISRAEL

Prof. Dr. Anna Erschler
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Krzysztof Fraczek
Wydzial Matematyki UMK
Nicholas Copernicus University
ul. Chopina 12/18
87 100 Torun
POLAND

Prof. Dr. John Franks
Dept. of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston , IL 60208-2730
USA

Prof. Dr. Koji Fujiwara
Graduate School of Information Science
Tohoku University
Aramaki, Aoba-ku
Sendai 980-8579
JAPAN

Prof. Dr. Victor Gerasimov
Departamento de Matematica
UFMG
Avenida Antonio Carlos 6627
C.P. 702
Belo Horizonte , MG 30161-970
BRAZIL

Dr. Eva Glasmachers
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Dr. Sonja Hohloch
Institut für Algebra u. Geometrie
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. Vincent Humiliere
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Dr. Anders Karlsson
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm

Prof. Dr. Anatole B. Katok
Department of Mathematics
Pennsylvania State University
303 McAllister Building
University Park , PA 16802
USA

Prof. Dr. Gerhard Knieper
Mathematisches Institut
Ruhr Universität Bochum
Universitätsstraße 150
44780 Bochum

Henrik Koehler
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Prof. Dr. Dieter Kotschick
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Frederic Le Roux
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Gilbert Levitt
University of Caen
Department of Mathematics
LMNO
BP 5186
F-14032 Caen Cedex

Prof. Dr. Dave Witte Morris
Department of Mathematics and
Computer Science
University of Lethbridge
4401 University Drive
Lethbridge AB T1K 3M4
CANADA

Gregor Noetzel
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Dr. Emmanuel Opshtein
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Dr. Norbert Peyerimhoff
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Prof. Dr. Mark Pollicott
Mathematics Institute
University of Warwick
Gibbet Hill Road
GB-Coventry CV4 7AL

Prof. Dr. Iosif Polterovich
Dept. of Mathematics and Statistics
University of Montreal
Pavillon Andre-Aisenstadt
2920 Chemin de la Tour
Montreal , QC H3T 1J8
CANADA

Prof. Dr. Leonid V. Polterovich
Tel Aviv University,Dept. of Mathematics
Raymond and Beverly Sackler
Faculty of Exact Sciences
Ramat-Aviv
Tel Aviv 69978
ISRAEL

Prof. Dr. Leonid D. Potyagailo
Universite des Sciences et
Techniques de Lille 1
U.F.R. de Math. Pures et Appl.
F-59655 Villeneuve d' Ascq. Cedex

Prof. Dr. Felix Schlenk
Dept. de Mathematiques
Universite Libre de Bruxelles
Campus de la Plaine, CP 218
Boulevard du Triomphe
B-1050 Bruxelles

Prof. Dr. Matthias Schwarz
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Prof. Dr. Richard J. Sharp
School of Mathematics
Lamb Building
University of Manchester
Oxford Road
GB-Manchester M13 9PL

Egor Shelukhin
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Dr. Vsevolod V. Shevchishin
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Karl Friedrich Siburg
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44221 Dortmund

Thomas Stratmann
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Frol Zapolsky
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL