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Organised by
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Abstract. This conference is one of the few occasions where researchers from many different areas in algebraic and geometric topology are able to meet and exchange ideas. Accordingly, the program covered a wide range of new developments in such fields as geometric group theory, rigidity of group actions, knot theory, and stable and unstable homotopy theory. More specifically, we discussed progress on problems such as Kuhn's realization conjecture, the integral Riemann-Roch theorem, and Simon's conjecture for knots, to mention just a few subjects with a name attached.

Mathematics Subject Classification (2000): 19xxx, 55xxx, 57xxx.

Introduction by the Organisers

This conference was the last of six in the series of topology conferences in Oberwolfach organized by Cameron Gordon and Bob Oliver, joined for the first time by Thomas Schick as successor of Wolfgang Lück, who was organizer for an even longer time. Unfortunately, Lück could not attend the conference for medical reasons.

There were about 45 participants in the meeting, working in many different areas of algebraic and geometric topology.

The 19 talks of the conference covered a wide range of areas such as 3-manifolds and knot theory, geometric group theory, algebraic K- and L-theory, and homotopy theory. One of the goals of the conference is to foster interaction between such different areas and the passage of methods from one to the other.

The following is a summary of some of the highlights.

Karen Vogtmann reported on joint work with Martin Bridson, showing that the automorphism group of a free group on n letters cannot act non-trivially on a Euclidean space of dimension $n-1$, or on any acyclic homology manifold of dimension $n-1$. This uses some delicate refinements of classical Smith theory. In a related direction, Tadeusz Januszkiewicz presented the construction of finitely generated groups with the property that the action on an arbitrary finite dimensional acyclic space always has a fixed point. This construction involves simplices of groups used together with Smith theory and a limit theorem from hyperbolic group theory. In other talks on geometric group theory, Roman Sauer described new results in Mostow rigidity, stating roughly that hyperbolic lattices can only be lattices in the corresponding $\mathrm{SO}(n,1)$; and Clara Löh proved non-vanishing for simplicial volume of certain non-compact locally symmetric spaces.

Simon's conjecture in knot theory asserts that a knot group (i.e., the fundamental group of the complement of a knotted S^1 in S^3) admits surjective homomorphisms onto only finitely many other knot groups. Alan Reid talked about joint work with Michel Boileau, Steve Boyer, and Shicheng Wang, showing that this is true for all 2-bridge knots. The proof involves a skillful use of newly discovered properties of the $\mathrm{SL}(2,\mathbb{C})$ character variety of knot groups due to Kronheimer and Mrowka. In another talk on 3-dimensional topology, Jessica Purcell addressed hyperbolic structures, more precisely the conjecture that if C is a compression body with a single 1-handle attached to a torus cross an interval, then the core of the 1-handle is isotopic to a geodesic in any geometrically finite hyperbolic structure on C. Insight into this problem was also gained via computer aided simulation and visualization.

As one application of homotopy theory, Ib Madsen presented an integral refinement of the Riemann-Roch theorem for curves (with suitable twist bundles). This work was based on precise calculations of the homotopy effect of associated maps on the classifying space level.

Another application of homotopy theory was described by Carles Broto. He presented a purely algebraic result about the p-subgroup structure in finite groups of Lie type, one which was reduced to a statement about their classifying spaces and then proved using the homotopy theory of p-local finite groups. No algebraic proof of this result is available up to now.

Kuhn's realization problem in unstable homotopy theory asks which abstract graded rings over the field $\mathbb{F}_p$, with a compatible action of the Steenrod algebra, are isomorphic to mod p cohomology rings of spaces. Kuhn gives precise conditions which he conjectures to be necessary and sufficient, and his conjecture had already been proven when $p=2$. In Gerald Gaudens' talk, an important special case of the conjecture was proved for odd primes.

Other talks addressed a new homological approach (via so-called blob homology) to topological quantum field theory, the failure of a possible generalization of the Madsen-Weiss theorem to higher dimensions (shown using calculations of family indices), the rigidity of the curve complex of a surface and the construction of exotic smooth structures on 4-manifolds via suitable surgery methods.

The famous Oberwolfach atmosphere made this meeting another wonderful success, and all thanks go to the institute for creating this atmosphere and making the conference possible.

Workshop: Topologie

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Abstracts

Mostow rigidity for locally compact targets and a rigidity theorem for hyperbolic lattices

Roman Sauer
(joint work with Uri Bader, Alex Furman)

Let Γ be a lattice in the isometry group $G = \mathrm{Isom}(\mathbb{H}^n)$ of the hyperbolic n-space, where $n \geq 3$. Our goal is to find all locally compact groups (besides G) that contain Γ as a lattice.

The analogous question for lattices in semisimple Lie groups of higher rank and for cocompact lattices in Lie groups of arbitrary rank was answered in [1]. Furman uses Margulis-Zimmer superrigidity and quasi-isometry rigidity results for that.

It remained to clarify the situation for non-uniform lattice embeddings of hyperbolic lattices. We have the following (partial) answer to the initial question:

Theorem. *Let Γ be a lattice in $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$. Assume Γ embeds as a lattice in some locally compact second countable group H, so that $\Gamma < H$ is an integrable lattice. Then H has one the following forms:*

- *Direct product $G \times K$ or $G^0 \times K$, where K is a compact group;*
- *Semi-direct product $\Gamma' \ltimes K$, where K is a compact group and Γ' is a lattice in G containing Γ: $\Gamma < \Gamma' < G$.*

Moreover, up to conjugation, one may assume that the projection to the first factor maps $\Gamma < H$ to its image in G, G^0, or Γ' respectively.

Next we explain the notion of integrability.

Definition. Let Λ be a finitely generated lattice in a second countable, locally compact group H. For a choice of a word-metric on Λ let $l(\lambda) \in \mathbb{N}$, for $\lambda \in \Lambda$, denote the length of λ. For a Λ-fundamental domain F in H let $\alpha_F : H \times F \to \Lambda$ denote the cocycle given by the condition $\alpha(h,x)xh \in F$. We say that Λ is *integrable* if there is Λ-fundamental domain $F \subset H$ such that

$$\int_F l\big(\alpha(g,x)\big)\, d\mu_{\mathrm{Haar}}(x) < \infty$$

holds for all $g \in G$.

Note that all cocompact lattices are integrable. Furthermore, all lattices in G as above or in any simple Lie groups of higher rank [3] are integrable.

We reduce the theorem above – following the method in [1] – to a (more general than needed for that) measure equivalence rigidity theorem for hyperbolic lattices. We recall the notion of measure equivalence that gained intense attention in recent years by the work of Gaboriau, Furman, Monod, Popa, Shalom and others.

Definition. We say that two countable groups Γ, Λ are *measure equivalent* if there is a standard measure space (Ω, μ) (called a *measure coupling*) such that both actions commute, are μ-preserving, and possess fundamental domains of finite measure.

If Λ and Γ are finitely generated one can impose an analogous integrability condition on fundamental domains of their measure coupling as in the case of lattices. If a measure coupling Ω has integrable fundamental domains we say that Ω is an ℓ^1-*measure coupling.*

Theorem. *Let $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$, let $\Gamma < G$ a lattice, and let Λ be some finitely generated group ℓ^1-measure equivalent to Γ. Then there exists a short exact sequence $1 \to \Lambda_0 \to \Lambda \to \bar{\Lambda} \to 1$, where Λ_0 is finite and $\bar{\Lambda}$ is a lattice in G.*

Moreover, if (Ω, μ) is an ergodic ℓ^1-ME coupling of Γ and Λ, then there exists a measurable map $\Phi : \Omega \to G$ satisfying

$$\Phi(\gamma\omega) = \gamma\,\Phi(\omega), \qquad \Phi(\lambda\omega) = \Phi(\omega)\rho(\lambda)^{-1}$$

and $\Phi_*\mu$ is either the Haar measure on G^0, or the Haar measure on G, or the counting measure on a lattice Γ' containing Γ and a conjugate of $\bar{\Lambda}$ as finite index subgroups.

In the last case one may assume that Γ and $\bar{\Lambda}$ are actually contained in a possibly larger lattice Γ' upon adjusting ρ and Φ by a fixed $g_0 \in G$.

For lattices in semisimple Lie groups of higher rank one has measure equivalence rigidity without imposing an integrability condition [2]. The problem is that we do not have an analog of Margulis-Zimmer superrigidity. For the proof we prove a certain orbit equivalence rigidity theorem that cocyclifies Mostow rigidity rather than superrigidity.

The methods involve bounded cohomology and geometric group theory.

References

[1] A. Furman, *Mostow-Margulis rigidity with locally compact targets*, Geom. funct. anal. **11** (2001), 30–59.

[2] A. Furman, *Gromov's measure equivalence and rigidity of higher rank lattices*, Ann. of Math. **150** (1999), 1059–1081.

[3] Y. Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. of Math. **152** (2000), 113–182.

Why there cannot be a three-dimensional Madsen-Weiss theorem

Johannes Ebert

Let M be any oriented closed smooth n-manifold, let $B_M := B\,\mathrm{Diff}^+(M)$ and let $p : E_M \to B_M$ be the universal oriented M-bundle. Let $\mathrm{MTSO}(n) := \mathbb{T}\mathbf{h}(-L_n)$ be the Madsen-Tillmann spectrum, i.e., the Thom spectrum of the inverse $-L_n$ of the universal n-dimensional oriented vector bundle $L_n \to B\,\mathrm{SO}(n)$. There is a map $\alpha_M : B_M \to \Omega^\infty \mathrm{MTSO}(n)$, the *Madsen-Tillmann map* [4]. The importance of this construction is that all cohomology classes of B_M (alias characteristic classes of smooth M-bundles) which are derived from the vertical tangent bundle $\pi : T_v E_M \to E_M$ are induced from classes on $\Omega^\infty \mathrm{MTSO}(n)$ via the map α_M.

To be more precise, let us compute the rational cohomology of the unit component $\Omega^\infty_0 \operatorname{MTSO}(n)$. There is the Thom isomorphism

$$\tau : H^*\big(B\operatorname{SO}(n)\big) \cong H^{*-n}\big(\operatorname{MTSO}(n)\big)$$

and the isomorphism

$$s : \Lambda H^*_{>0}\big(\operatorname{MTSO}(n);\mathbb{Q}\big) \cong H^*\big(\Omega^\infty_0 \operatorname{MTSO}(n);\mathbb{Q}\big),$$

where Λ is the functor which associates to a graded $\mathbb{Q}$-vector space the free-graded commutative algebra generated by it. On the other hand

$$H^*\big(B\operatorname{SO}(2m+1);\mathbb{Q}\big) = \mathbb{Q}[p_1,\dots,p_m], \text{ and}$$
$$H^*\big(B\operatorname{SO}(2m)\big) = \mathbb{Q}[p_1,\dots,p_m,\chi]/(\chi^2 - p_m).$$

Given any $c \in H^*(B\operatorname{SO}(n))$, then

$$p_!\big(c(T_vE_M)\big) = \alpha_M^* s\tau(c).$$

Another source of characteristic classes of smooth fiber bundles, this time with values in the topological K-theory of the base space, is the index of natural differential elliptic operators. In this talk, we consider only self-adjoint operators. The case of general operators is parallel (and better known). Let D be a family of self-adjoint elliptic differential operators on the universal bundle $E_M \to B_M$. By [2], these data have an index $\operatorname{ind}(D) \in K^1(B_M)$. On the other hand, the Atiyah-Singer family index theorem holds and yields

$$\operatorname{ind}(D) = (p\circ\pi)_!(\operatorname{smb}_D)_{\mathrm{sa}},$$

where $(\operatorname{smb}_D)_{\mathrm{sa}} \in K^1(\mathbb{T}\mathbf{h}(T_vE_M))$ is the self-adjoint symbol class of D [1], and where $(p\circ\pi)_! : K^1(\mathbb{T}\mathbf{h}(T_vE_M)) \to K^1(B_M)$ is the umkehr map in K-theory, which is defined as the composition of the Thom isomorphism $K^1(\mathbb{T}\mathbf{h}(T_vE_M)) \cong K^1(\mathbb{T}\mathbf{h}(-T_vE_M))$ and the map $K^1(\mathbb{T}\mathbf{h}(-T_vE_M)) \to K^1(B_M)$ induced by Pontrjagin-Thom collapse.

If the operator D is natural then there exists an element $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ such that σ_D maps to $(\operatorname{smb}_D)_{\mathrm{sa}}$ under the map $\mathbb{T}\mathbf{h}(T_vE_M) \to \mathbb{T}\mathbf{h}(L_n)$ which comes from the classifying map for the vertical tangent bundle. In this case

$$\operatorname{ind}(D) = \alpha_M^* \operatorname{th}^{-1}\sigma_D, \tag{1}$$

where $\operatorname{th} : K^1(\operatorname{MTSO}(n)) \to K^1(\mathbb{T}\mathbf{h}(L_n))$ is the Thom isomorphism.

Now let M be a $2m+1$-dimensional closed oriented manifold. The even signature operator [1] $D : \bigoplus_{p\ge 0}\mathcal{A}^{2p}(M) \to \bigoplus_{p\ge 0}\mathcal{A}^{2p}(M)$ is defined to be

$$D\phi = i^{m+1}(-1)^{p+1}(*d - d*)\phi$$

whenever $\phi \in \mathcal{A}^{2p}(M)$. It is a self-adjoint, elliptic differential operator, and it is natural. Furthermore, it is related to the Laplace-Beltrami operator on forms by $D^2 = \Delta$. Moreover

$$\ker(D) = \ker(\Delta) = \bigoplus_{p\ge 0} H^{2p}(M;\mathbb{C}) \tag{2}$$

by the Hodge theorem. Now choose a fiberwise smooth metric on the vertical tangent bundle of the universal M-bundle. The even signature operators on the fibers define a family of self-adjoint elliptic differential operators and hence we have an index $\operatorname{ind}(D) \in K^1(B_M)$. Here is our main result.

Theorem 1. [3] *For any odd-dimensional closed oriented manifold M, the family index of the even signature operator* $\operatorname{ind}(D) \in K^1(B_M)$ *is trivial.*

The proof is purely analytic (it uses spectral theory, Kuiper's theorem on the contractibility of the unitary group of a Hilbert space and, crucially, the constancy of the dimension of the kernel, which follows from equation 2.

Because the proof of the vanishing of the index is purely analytical, the Atiyah-Singer index theorem 1 allows us to draw topological conclusions. Apply the Chern character to equation 1. A routine calculation of characteristic classes shows that $\operatorname{ch}(\alpha_M^* \operatorname{th}^{-1} \sigma_D) = \alpha_M^* s\tau\mathcal{L}$, where $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ is the universal symbol for the even signature operator and $\mathcal{L} \in H^{4*}(B\,\mathrm{SO}(2m+1);\mathbb{Q})$ is the Hirzebruch L-class. Therefore Theorem 1 implies:

Theorem 2. [3] *For any closed oriented $2m+1$-manifold M, the Madsen-Tillmann map $\alpha_M : B_M \to \Omega_0^\infty \mathrm{MTSO}(2m+1)$ annihilates*

$$s\tau\mathcal{L} \in H^{4*-2m-1}(\Omega_0^\infty \mathrm{MTSO}(2m+1);\mathbb{Q}).$$

If $m = 1$, then the k-th component $\mathcal{L}_k$ generates $H^{4k}(B\,\mathrm{SO}(3);\mathbb{Q})$. Therefore, the map $B_{M^3} \to \Omega_0^\infty \mathrm{MTSO}(3)$ induces the zero map in rational cohomology. This is in sharp contrast to the 2-dimensional case, where Madsen and Weiss [5] showed that $B_M \to \Omega_0^\infty \mathrm{MTSO}(2)$ induces an isomorphism in integral homology in degrees $* < g/2-1$, whenever M is a connected closed oriented surface of genus g.

References

[1] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry III*, Math. Proc. Cambridge Philos. Soc **79** (1976), 71–99.

[2] M. F. Atiyah, I. M. Singer, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. **37** (1969) 5–26.

[3] J. Ebert, *Madsen-Tillmann spectra and index theory*, in preparation.

[4] S. Galatius, I. Madsen, U. Tillmann, M. Weiss *The homotopy type of the cobordism category*, to appear in Acta Mathematica.

[5] I. Madsen, M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. **165** (2007), 843–941.

On Simon's conjecture for knots

Alan W. Reid
(joint work with Michel Boileau, Steve Boyer, Shicheng Wang)

1. Introduction

Let $K \subset S^3$ be a non-trivial knot. Simon's Conjecture (see [5, Problem 1.12(D)]) asserts the following:

Conjecture 1.1. $\pi_1(S^3 \setminus K)$ *surjects only finitely many distinct knot groups.*

Although this conjecture dates back to the 1970's, and has received considerable attention recently (see [1], [2], [3], [7], [8], and [9] to name a few), little by way of general results appears to be known. Conjecture 1.1 is easily seen to hold for torus knots using elementary Alexander polynomial considerations. In [1] the conjecture is established under the assumption that the epimorphisms are non-degenerate in the sense that a preferred longitude of K is sent to a non-trivial peripheral element under the epimorphism. In particular this holds in the case when the homomorphism is induced by mapping of non-zero degree.

This talk presented some recent progress on Conjecture 1.1, the main result of which is the following, and is the first general result for a large class of hyperbolic knots.

Theorem 1.2. *Conjecture 1.1 holds for all 2-bridge knots.*

Theorem 1.2 will be proved as a consequence of a technical theorem which gives strong control on certain epimorphisms with domain a small knot group. Recall that a knot $K \subset S^3$ is called *small* if $S^3 \setminus K$ contains no closed embedded essential surface (that 2-bridge knots are small is proved in [4]). Before stating this theorem, we introduce the following definition that seems useful in organizing control of epimorphisms between knot groups.

Definition 1.3. Let K be a knot. We will say K has Property L if for any non-trivial knot K' and epimomorphism $\phi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$, the kernel $\ker \phi$ does not contain a longitude of K.

Control of the image of the longitude has featured in other work related to epimorphisms between knot groups; for example Property Q^* of Simon. The motivation for this definition is:

Theorem 1.4. *Let K be a small knot and assume K has Property L. Then Conjecture 1.1 holds for K.*

Simple Alexander polynomial considerations show that any knot group surjects onto only finitely many torus knot groups, so it is when the target is hyperbolic or satellite that the assumption of Property L is interesting.

Although it is easy to construct hyperbolic knots that do not have Property L (see [3]) our proof shows that 2-bridge knots do have Property L, and so Theorem 1.2 will follow from Theorem 1.4.

2. Sketch of the proof of Theorem 1.4 and 1.2

2.1. The proof of Theorem 1.4 proceeds as follows. Assume that K is a small hyperbolic knot and assume that there exist infinitely many distinct knots K_i and epimorphisms

$$\phi_i : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K_i).$$

Let λ be a preferred longitude for K. As discussed, it suffices to deal with the cases when K_i are hyperbolic or satellite knots. The next lemma shows that Property L excludes the satellite knot case.

Lemma. *Let $K \subset S^3$ be a small hyperbolic knot and let K' be a satellite knot. Assume that $\phi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ is an epimorphism. Then $\phi(\lambda) = 1$.*

The proof is completed using [8].

2.2. We now sketch the proof of Theorem 1.2; i.e., we need to show that a 2-bridge hyperbolic knot has Property L. Thus assume that K is a hyperbolic 2-bridge knot and $\phi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ an epimorphism with $\phi(\lambda) = 1$ (we continue to use the notation above).

The main algebraic tool that organizes the proof is the character variety. In particular we make use of the following result of Kronheimer and Mrowka [6].

Theorem 2.1. *Let K be a non-trivial knot and $X(K)$ its $\mathrm{SL}(2, \mathbf{C})$-character variety. Then, $X(K)$ contains a curve of characters of irreducible representations.*

The relevance of this is seen in the following. Suppose that G and H are finitely generated groups and $\phi : G \to H$ is an epimorphism. Then this defines a map at the level of character varieties $\phi^* : X(H) \to X(G)$ by $\phi^*(\chi_\rho) = \chi_{\rho\circ\phi}$. This map is algebraic, Zariski closed and ϕ^* injects $X(H) \hookrightarrow X(G)$. Thus an epimorphism $\phi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$, together with [6] provides "interesting" components in $X(K)$.

In particular, in our setting, Theorem 2.1 provides a curve of characters C of irreducible representations of $\pi_1(S^3 \setminus K')$, which inject in $X(K)$ under ϕ^*. This provides a curve component $D = \phi^*(C) \subset X(K)$, and the proof is completed by showing that every irreducible component $X \subset X(K)$ which contains the character of an irreducible representation, contains the character of a so called p-rep. This discussion, together with the work of Riley can be shown to yield a contradiction.

References

[1] M. Boileau, J. H. Rubinstein and S. Wang, *Finiteness of 3-manifolds associated with non-zero degree mappings*, J. Diff. Geom., to appear.

[2] M. Boileau and S. Boyer, *On character varieties, sets of discrete characters and non-zero degree maps*, preprint (2007), arXiv:math/0701384.

[3] F. González-Acuña and A. Ramirez, *Epimorphisms of knot groups onto free products*, Topology **42** (2003), 1205–1227.

[4] A. Hatcher and W. P. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. math. **79** (1985), 225–246.

[5] R. Kirby, *Problems in low-dimensional topology*, in: Geometric topology (Athens, GA, 1993), Amer. Math. Soc. Publications, (1997), 35–473.

[6] P. B. Kronheimer and T. S. Mrowka, *Dehn surgery, the fundamental group and* SU(2), Math. Res. Lett. **11** (2004), 741–754.
[7] T. Ohtsuki, R. Riley and M. Sakuma, *Epimorphisms between 2-bridge link groups*, in the Zieschang Gedenkschrift, Geometry and Topology Monographs **14** (2008), 417–450.
[8] A. W. Reid and S. Wang, *Non-Haken 3-manifolds are not large with respect to mappings of non-zero degree*, Comm. in Analysis and Geometry **7** (1999), 105–132.
[9] D. S. Silver and W. Whitten, *Knot group epimorphisms, II*, preprint (2008) arXiv:0806.3223.

Infinite groups with fixed point properties

Tadeusz Januszkiewicz
(joint work with G. Arzhantseva, M.R. Bridson, I.J. Leary, A. Minasyan, J. Świątkowski)

For p a prime, one says that a space is *mod-p acyclic* if it has the same mod-p Čech cohomology as a point. Let $\mathcal{X}_{\rm ac}$ be the class of all Hausdorff spaces X of finite covering dimension such that there is a prime p for which X is mod-p acyclic. Let $\mathcal{M}_{\rm ac}$ denote the subclass of smooth manifolds in $\mathcal{X}_{\rm ac}$. Note that the class $\mathcal{X}_{\rm ac}$ contains all finite dimensional contractible spaces and all finite dimensional contractible CW-complexes.

Theorem 1. *There is an infinite finitely generated group Q that cannot act without a global fixed point on any $X \in \mathcal{X}_{ac}$. If $X \in \mathcal{X}_{\rm ac}$ is mod-p acyclic, then so is the fixed point set for any action of Q on X. For any countable group C, the group Q can be chosen to have either the additional properties (i), (ii) and (iii) or (i), (ii) and (iii)′ described below:*

(i) *Q is simple;*
(ii) *Q has Kazhdan's property (T);*
(iii) *Q contains an isomorphic copy of C;*
(iii)′ *Q is periodic.*

Since a countable group can contain only countably many finitely generated subgroups, it follows from property (iii) that there are continuously many (i.e., $2^{\aleph_0}$) non-isomorphic groups Q with the fixed point property described in Theorem 1.

No non-trivial finite group has a fixed point property as strong as the one in Theorem 1. Any finite group not of prime power order acts without a global fixed point on some finite dimensional contractible simplicial complex. Smith theory tells us that the fixed point set for any action of a finite p-group on a finite dimensional mod-p acyclic space is itself mod-p acyclic, but it is easy to construct an action of a non-trivial finite p-group on a 2-dimensional mod-q acyclic space without a global fixed point if q is any prime other than p. Since the fixed point property of Theorem 1 passes to quotients, it follows that none of the groups Q can admit a non-trivial finite quotient. This further restricts the ways in which Q can act on acyclic spaces. For example, if $X \in \mathcal{X}_{\rm ac}$ is a locally finite simplicial complex and Q is acting simplicially, then the action of Q on the successive star neighbourhoods $st_{n+1} := st(st_n(x))$ of a fixed point $x \in X$ must be trivial, because st_n is Q-invariant and there is no non-trivial map from Q to the finite group $\mathrm{Aut}(st_n)$. Since $X = \bigcup_n st_n$, we deduce:

Corollary 1. *The groups Q from Theorem 1 admit no non-trivial simplicial action on any locally-finite simplicial complex $X \in \mathcal{X}_{\mathrm{ac}}$.*

The ideas behind the corollary can be developed further, For example we have.

Corollary 2. *A group Q from the claim of Theorem 1 admits no non-trivial isometric action on any proper metric space $X \in \mathcal{X}_{\mathrm{ac}}$.*

Using similar arguments we can also rule out non-trivial real analytic actions of the groups Q (from Theorem 1) on any acyclic manifold M. However, a stronger result concerning triviality of actions on manifolds can be obtained more directly:

Proposition 2. *A simple group G that contains, for each $n > 0$ and each prime p, a copy of $(\mathbb{Z}_p)^n$ admits no non-trivial action by diffeomorphisms on any $X \in \mathcal{M}_{\mathrm{ac}}$. The group Q in Theorem 1 may be chosen to have this property.*

Finite p-groups have global fixed points whenever they act on compact Hausdorff spaces that are mod-p acyclic, but the groups Q do not have this property. Indeed, if Q is infinite and has property (T) then it will be non-amenable, hence the natural action of Q on the space of finitely-additive probability measures on Q will not have a global fixed point, and this space is compact, contractible, and Hausdorff.

We know of no finitely presented group enjoying the fixed point property described in Theorem 1. However, using techniques quite different from those used to construct the groups Q, we shall exhibit finitely presented groups that cannot act on a range of spaces. In particular we construct groups of the following type.

Theorem 3. *There exist finitely presented infinite groups P that have no non-trivial action by diffeomorphisms on any smooth manifold $X \in \mathcal{M}_{\mathrm{ac}}$.*

Certain of the Higman-Thompson groups can also serve in the role of P.

Theorem 1 answers a question of P. H. Kropholler, who asked whether there exists a countably infinite group G for which every finite-dimensional contractible G-CW-complex has a global fixed point. This question is motivated by Kropholler's study of the closure operator $\mathbf{H}$ for classes of groups, and by the class $\mathbf{H}\mathfrak{F}$ obtained by applying this operator to the class $\mathfrak{F}$ of all finite groups [3].

Our strategy for proving Theorems 1 and 3 is very general. First, we express our class of spaces as a countable union $\mathcal{X} = \cup_{n\in\mathbb{N}}\mathcal{X}_n$. For instance, if all spaces in $\mathcal{X}$ are finite-dimensional, then $\mathcal{X}_n$ may be taken to consist of all n-dimensional spaces in $\mathcal{X}$. Secondly, we construct finitely generated groups G_n that have the required properties for actions on any $X \in \mathcal{X}_n$. Finally, we apply the templates described below to produce the required groups.

Template **FP***: ruling out fixed-point-free actions.* If there is a sequence of finitely generated *non-elementary relatively hyperbolic* groups G_n such that G_n cannot act without a fixed point on any $X \in \mathcal{X}_n$, then there is an infinite finitely generated group that cannot act without a fixed-point on any $X \in \mathcal{X}$.

Template $\mathbf{NA}_{\mathrm{fp}}$*: finitely presented groups that cannot act.* Let $(G_n; \xi_{n,j})$ $(n \in \mathbb{N}, j = 1, \ldots, J)$ be a recursive system of non-trivial groups and monomorphisms $\xi_{n,j} : G_n \to G_{n+1}$. Suppose that each G_{n+1} is generated by $\bigcup_j \xi_{n,j}(G_n)$ and that for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that G_n cannot act non-trivially on any $X \in \mathcal{X}_m$. Then there exists an infinite finitely presented group that cannot act non-trivially on any $X \in \mathcal{X}$.

The engine that drives the first template is the existence of common quotients established in Theorem 4 below. The proof of this theorem is based on a result of Arzhantseva, Minasyan and Osin [1], obtained using small cancellation theory over relatively hyperbolic groups: any two finitely generated non-elementary relatively hyperbolic groups G_1, G_2 have a common non-elementary relatively hyperbolic quotient H.

Theorem 4. *Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable collection of finitely generated non-elementary relatively hyperbolic groups. Then there exists an infinite finitely generated group Q that is a quotient of G_n for every $n \in \mathbb{N}$.*

Moreover, if C is an arbitrary countable group, then such a group Q can be made to satisfy the following conditions

(i) *Q is a simple group;*
(ii) *Q has Kazhdan's property (T);*
(iii) *Q contains an isomorphic copy of C.*

If the G_n are non-elementary word hyperbolic groups, then claim (iii) *above can be replaced by*

(iii)′ *Q is periodic.*

This result immediately implies the validity of the template **FP**. Indeed, if G_n are the hypothesized groups of template **FP**, the preceding theorem furnishes us with a group Q that, for each $n \in \mathbb{N}$, is a quotient of G_n and hence cannot act without a fixed point on any $X \in \mathcal{X}_n$. Now let G_n be the hypothesized groups of template $\mathbf{NA}_{\mathrm{fp}}$. They are not assumed to be relatively hyperbolic. We consider groups $A_n := G_n * G_n * G_n$, which also cannot act non-trivially on any $X \in \mathcal{X}_n$. The group A_n is non-elementary and relatively hyperbolic as a free product of three non-trivial groups. Therefore, Theorem 4 can be applied to the sequence of groups A_n, providing a group Q which, as a quotient of A_n, cannot act non-trivially on any $X \in \mathcal{X}_n$ for any $n \in \mathbb{N}$.

Following the above strategy to prove Theorem 1, we first represent $\mathcal{X}_{ac}$ as a countable union $\mathcal{X}_{ac} = \cup_{n,p}\mathcal{X}_{n,p}$, where, for each prime number p, the class $\mathcal{X}_{n,p}$ consists of all mod-p acyclic spaces of dimension n. Then we construct the groups required by template **FP**, proving the following result.

Theorem 5. *For each $n \in \mathbb{N}$ and every prime p, there exists a non-elementary word hyperbolic group $G_{n,p}$ such that any action of $G_{n,p}$ by homeomorphisms on any space $X \in \mathcal{X}_{n,p}$ has the property that the global fixed point set is mod-p acyclic (and in particular non-empty).*

The mod-p acyclicity of the fixed point for the action of $G_{n,p}$ on the space X is a consequence of the following (n,p)-*generation property*: there is a generating set S of $G_{n,p}$ of cardinality $n+2$ such that any proper subset of S generates a finite p-subgroup.

For certain small values of the parameters examples of non-elementary word hyperbolic groups with the (n,p)-generation property were already known (e.g., when $n = 1$ and $p = 2$ they arise as reflection groups of the hyperbolic plane with a triangle as a fundamental domain). Our construction works for arbitrary n and p. For large n it provides the first examples of non-elementary word hyperbolic groups possessing the (n,p)-generation property.

We construct the groups $G_{n,p}$ as fundamental groups of certain simplices of groups all of whose local groups are finite p-groups. We use ideas related to simplicial non-positive curvature, developed by Januszkiewicz and Świątkowski in [2], to show that these groups are non-elementary word hyperbolic. The required fixed point property is obtained using Smith theory and a homological version of Helly's theorem.

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References

[1] G. Arzhantseva, A. Minasyan, D. Osin, *The SQ-universality and residual properties of relatively hyperbolic groups*, J. of Algebra **315** (2007), 165–177.

[2] T. Januszkiewicz, J. Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. IHES **104** (2006), 1–85.

[3] P.H. Kropholler, *On groups of type* $(\mathrm{FP})_\infty$, J. Pure Appl. Algebra **90** (1993), no. 1, 55–67.

Simplicial volume of non-compact manifolds

Clara Löh

(joint work with Roman Sauer)

A *degree theorem* is a statement of the following type: For all proper, continuous maps $f\colon N \to M$ between Riemannian manifolds of a particular type with finite volume, the degree of f can be bounded in terms of the volume:

$$|\deg f| \leq \mathrm{const}_{\dim M} \cdot \frac{\mathrm{vol}\, N}{\mathrm{vol}\, M}.$$

Gromov successfully applied the following strategy to obtain such degree theorems for negatively curved manifolds [2]: Find a topological replacement v of the Riemannian volume such that

(1) For all proper, continuous maps $f\colon N \to M$ one has $|\deg f| \cdot v(M) \leq v(N)$.
(2) For all suitable domain manifolds N one has $v(N) \leq \mathrm{const}_{\dim N} \cdot \mathrm{vol}\, N$.
(3) For all suitable target manifolds M one has $v(M) \geq \mathrm{const}_{\dim M} \cdot \mathrm{vol}\, M$.

An example of an adequate topological replacement of the Riemannian volume in this sense is the (Lipschitz) simplicial volume: The *simplicial volume* of an oriented, connected n-manifold M is defined as

$$\|M\| := \inf\bigl\{\|c\|_1 \bigm| c \text{ is a locally finite } \mathbb{R}\text{-fundamental cycle of } M\bigr\},$$

where $\|\cdot\|_1$ is the ℓ^1-norm on the locally finite $\mathbb{R}$-chain complex with respect to the basis consisting of all singular simplices.

Based on the methods developed by Besson, Courtois, and Gallot, degree theorems for (most) locally symmetric spaces of non-compact type with finite volume were obtained by Connell and Farb [1].

Generalising Gromov's non-vanishing results to the simplicial volume of *closed* locally symmetric spaces of non-compact type, Lafont and Schmidt derived corresponding degree theorems in the closed case [3].

Simplicial volumes of non-compact manifolds

We study the question to what extent the result of Lafont and Schmidt can be extended to the non-compact case. Our main results in this direction are:

(1) Based on a vanishing result of Gromov [2] we show that the simplicial volume of open locally symmetric spaces of non-compact type of $\mathbb{Q}$-rank at least 3 is zero [4] – in particular, the simplicial volume does not lead to a degree theorem for such targets.

(2) On the other hand, we consider the *Lipschitz simplicial volume*

$$\|M\|^{\mathrm{Lip}} := \inf\bigl\{\|c\|_1 \bigm| c \text{ is a locally finite } \mathbb{R}\text{-fundamental cycle of } M \text{ with } \mathrm{Lip}(c) < \infty\bigr\}$$

of oriented, connected Riemannian manifolds M.

In contrast to the ordinary simplicial volume, the Lipschitz simplicial volume does satisfy a proportionality principle for non-positively curved manifolds with finite volume [4]:

Theorem (Proportionality principle)**.** *Let M and N be oriented, connected, Riemannian manifolds of non-positive sectional curvature with finite volume whose Riemannian universal coverings are isometric. Then*

$$\frac{\|M\|^{\mathrm{Lip}}}{\operatorname{vol} M} = \frac{\|N\|^{\mathrm{Lip}}}{\operatorname{vol} N}.$$

In particular, using the result of Lafont and Schmidt, it follows that the Lipschitz simplicial volume of locally symmetric spaces of non-compact type with finite volume is non-zero (and finite).

A degree theorem for locally symmetric spaces of finite volume

Notice that the Lipschitz simplicial volume satisfies a degree estimate for proper, Lipschitz maps and that it can – by the work of Gromov – be bounded from above in terms of the Riemannian volume. Hence, we obtain the following degree theorem [4], complementing the result of Connell and Farb:

Theorem (Degree theorem). *For every $n \in \mathbb{N}$ there is a constant $C_n > 0$ with the following property: Let $f\colon N \to M$ be a proper Lipschitz map between oriented connected Riemannian n-manifolds of finite volume with the following properties:*

1. *The domain manifold N satisfies $|\sec N| \leq 1$.*
2. *The target manifold M is a locally symmetric space of non-compact type with the standard metric (or more generally, a product of locally symmetric spaces of non-compact type and manifolds with uniformly pinched negative sectional curvature).*

Then

$$|\deg f| \leq C_n \cdot \frac{\operatorname{vol} N}{\operatorname{vol} M}.$$

Open problems

The results stated above give a complete picture of the simplicial volume of locally symmetric spaces of non-compact type, except for the case of $\mathbb{Q}$-rank 1 or 2.

By investigating the metric structure on the Borel-Serre compactification, we showed that Hilbert modular varieties (which are examples of locally symmetric spaces of non-compact type of $\mathbb{Q}$-rank 1) have non-zero simplicial volume [5]; indeed, their simplicial volume coincides with the Lipschitz simplicial volume. However, the general $\mathbb{Q}$-rank 1 case is still open:

Question. Is the simplicial volume of locally symmetric spaces of non-compact type with finite volume non-zero provided that the $\mathbb{Q}$-rank is equal to 1?

On the other hand, our vanishing result only applies if the $\mathbb{Q}$-rank is at least 3.

Question. Does the simplicial volume of locally symmetric spaces of non-compact type with finite volume vanish if the $\mathbb{Q}$-rank is equal to 2?

References

[1] C. Connell, B. Farb, *The degree theorem in higher rank*, J. Differential Geom. **65** (2003), 19–59.

[2] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.

[3] J.-F. Lafont, B. Schmidt, *Simplicial volume of closed locally symmetric spaces of non-compact type*, Acta Math. **197** (2006), 129–143.

[4] C. Löh, R. Sauer, *Degree theorems and Lipschitz simplicial volume for non-positively curved manifolds of finite volume*, arXiv:0710.1635 (2007).

[5] C. Löh, R. Sauer, *Simplicial volume of Hilbert modular varieties*, arXiv:0706.3904 (2007), to appear in Comment. Math. Helv.

The geometry of unknotting tunnels

Jessica Purcell
(joint work with Marc Lackenby)

1. Background

Let M be a 3–manifold with torus boundary components, and let τ be an arc in M with endpoints on ∂M. If $M \setminus N(\tau)$ is a handlebody, then M is called a *tunnel number one* manifold, and τ is called an *unknotting tunnel* for M. Throughout, we will assume that M admits a hyperbolic structure.

In a paper published in 1995 [1], Colin Adams showed that if a tunnel number one manifold M has two torus boundary components, and the tunnel τ stretches between them, then arc τ is isotopic to a geodesic in the hyperbolic structure on M. Moreover, the length of τ is bounded above outside of a maximal horoball neighborhood of the boundary cusps of M. He then asked the following questions.

(1) Is an unknotting tunnel in a hyperbolic tunnel number one manifold always isotopic to a geodesic?
(2) Does an unknotting tunnel always have bounded length?

Since the paper was published, the questions have drawn interest, but seem to be difficult. There have not been many results in answering them. The following results are relevant. Adams and Reid found all the unknotting tunnels in 2–bridge links, and showed these were geodesic [2]. Sakuma and Weeks found that the unknotting tunnels in 2–bridge knots were isotopic to edges of a triangulation of these knot complements [7]. The triangulations they studied were shown to be geometrically canonical by Akiyoshi, Sakuma, Wada, and Yamashita [3], and hence geodesic and "short". Thus for 2–bridge knots and links, unknotting tunnels are known to be geodesic and known to be short. Sakuma and Weeks conjectured that unknotting tunnels were always isotopic to edges of the canonical polyhedral decomposition. This was shown to be false by Heath and Song [6]. However, Adams' two original questions above have remained unanswered.

2. Compression bodies

With Lackenby, we investigate these questions from a new direction. Instead of considering a tunnel number one manifold directly, we consider its pieces. When the handlebody is removed from such a manifold, we are left with a compression body C with $\partial_+ C$ a genus 2 surface, and $\partial_- C$ a torus. The arc τ becomes an arc which runs through the compressible handle of the compression body.

We consider geometrically finite hyperbolic structures on C. The compression body C admits a family of such structures, parameterized by the conformal structures on its genus 2 boundary component. The tool we use to investigate such structures is that of Ford domains.

We have been able to show that the arc τ is isotopic to a geodesic for many large families of geometrically finite hyperbolic structures on a compression body. We conjecture the following.

Conjecture 2.1. *The arc τ is isotopic to a geodesic for any geometrically finite hyperbolic structure on the compression body C.*

3. Lengths of tunnels

We have been able to show the following.

Proposition 3.1. *There exist geometrically finite structures on C for which the arc τ is arbitrarily long.*

Proposition 3.1 may not be very surprising. Perhaps more surprising is the fact that we can use this proposition to settle once and for all Colin Adams' second question above. In fact, we prove:

Theorem 3.2. *There exist tunnel number one manifolds with arbitrarily long unknotting tunnel.*

We prove Theorem 3.2 by showing that a sequence of geometrically finite structures on C with long tunnel converges to a geometrically infinite structure on C with long tunnel. We then find a "maximal cusp" geometrically close to this geometrically infinite structure, using a theorem of Canary–Culler–Hersonsky–Shalen [5]. To the maximal cusp, we glue a maximal cusp structure on a genus 2 handlebody. The result is a hyperbolic structure on a tunnel number one manifold with three extra curves drilled out on the Heegaard surface. Fill in these curves by Dehn filling along a long slope, and the result is the manifold claimed in the theorem.

References

[1] C. Adams, *Unknotting tunnels in hyperbolic 3-manifolds*, Math. Ann. **302** (1995), no. 1, 177–195.

[2] C. Adams and A.W. Reid, *Unknotting tunnels in two-bridge knot and link complements*, Comment. Math. Helv. **71** (1996), no. 4, 617–627.

[3] H. Akiyoshi, M. Sakuma, M. Wada, Y. Yamashita, *Ford domains of punctured torus groups and two–bridge knot groups*, from: "Knot Theory: dedicated to 70th birthday of Prof. Kunio Murasugi", (M. Sakuma, editor) (2000).

[4] B.H. Bowditch, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. **113** (1993), no. 2, 245–317.

[5] R. Canary, M. Culler, S. Hersonsky, P. Shalen, *Approximation by maximal cusps in boundaries of deformationspaces of Kleinian groups*, J. Differential Geom **64** (2003), 57–109.

[6] D.J. Heath and H.-J. Song, *Unknotting tunnels for $P(-2,3,7)$*, J. Knot Theory Ramifications **14** (2005), no. 8, 1077–1085.

[7] M. Sakuma, J. Weeks, *Examples of canonical decompositions of hyperbolic link complements*, Japan. J. Math. (N.S.) **21** (1995), 393–439.

Actions of automorphism groups of free groups on homology spheres and acyclic manifolds

Karen Vogtmann
(joint work with Martin R. Bridson)

This is a report on joint work with Martin Bridson [3].

The group $\mathrm{Aut}(F_n)$ of automorphisms of a finitely-generated free group is not isomorphic to an irreducible lattice in a semisimple Lie group, and is not even commensurable with such a lattice; this follows from deep results of Margulis. Nevertheless, it is possible to show that $\mathrm{Aut}(F_n)$ resembles a lattice in many ways. In particular, $\mathrm{Aut}(F_n)$ has rigidity properties reminiscent of Margulis super-rigidity, which severely restrict the possible homomorphisms from $\mathrm{Aut}(F_n)$ to various types of groups. In this work we study homomorphisms from $\mathrm{Aut}(F_n)$ to groups of homeomorphisms of homology spheres and acyclic manifolds.

More precisely, for $n \geq 3$ let $\mathrm{SAut}(F_n)$ denote the unique subgroup of index two in the automorphism group of a free group. The standard linear action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{R}^n$ induces non-trivial actions of $\mathrm{SAut}(F_n)$ on $\mathbb{R}^n$ and on $\mathbb{S}^{n-1}$, but we prove that any action of $\mathrm{SAut}(F_n)$ by homeomorphisms on an acyclic manifold or sphere of smaller dimension must be completely trivial, fixing every point. For actions on the circle, this follows from our earlier paper [4].

For linear actions, elementary results in the representation theory of finite groups can be combined with an understanding of the torsion in $\mathrm{SAut}(F_n)$ to prove the results; the real challenge lies with non-linear actions. The proof is an induction which combines an analysis of normalizers of finite subgroups of $\mathrm{SAut}(F_n)$ with an understanding of the fixed point sets of prime-order automorphisms. In the talk I outlined this proof, assuming that the action is linear (in which case it is trivial to understand the fixed point sets). Along the way, I pointed out exactly where we encounter difficulties when the action is not linear.

Understanding the fixed point sets of prime-order elements creates a problem in the topological setting because these fixed point sets are not in general manifolds, but only homology manifolds over $\mathbb{Z}_p$. Such homology manifolds can be quite badly behaved; they are not necessarily smooth, not even manifolds, not even integral homology manifolds, and not even ENR's or ANR's. These are well-known difficulties encountered in the theory of transformation groups and much effort has gone into circumventing them. They are overcome using (local and global) Smith theory done in terms of Borel-Moore homology and sheaf cohomology, as developed in [2] and [1].

We derive the facts we need using techniques from Smith theory, and apply them to obtain the following theorems.

Theorem. *If $n \geq 3$ and $d < n-1$, then any action of $\mathrm{SAut}(F_n)$ by homeomorphisms on a generalized d-sphere over $\mathbb{Z}_2$ is trivial, and hence $\mathrm{Aut}(F_n)$ can act only via the determinant map.*

Theorem. *If $n \geq 3$ and $d < n$, then any action of* $\mathrm{SAut}(F_n)$ *by homeomorphisms on a d-dimensional $\mathbb{Z}_2$-acyclic homology manifold over $\mathbb{Z}_2$ is trivial, and hence* $\mathrm{Aut}(F_n)$ *can act only via the determinant map.*

As special cases we obtain the desired minimality result for the standard linear action of $\mathrm{SAut}(F_n)$ on $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$.

Corollary. *If $n \geq 3$, then* $\mathrm{SAut}(F_n)$ *cannot act non-trivially by homeomorphisms on any contractible manifold of dimension less than n, or on any sphere of dimension less than $n-1$.*

We also note that these theorems have as immediate corollaries the analogous statements for $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$. For actions of $\mathrm{SL}(n,\mathbb{Z})$ on spheres, this result was announced in [7]. However the result depended on [6] and there was a flaw in the arguments of that paper.

Corollary. *If $n \geq 3$ and $d < n$, then* $\mathrm{SL}(n,\mathbb{Z})$ *cannot act non-trivially by homeomorphisms on any generalized $(d-1)$-sphere over $\mathbb{Z}_2$, or on any d-dimensional homology manifold over $\mathbb{Z}_2$ that is $\mathbb{Z}_2$-acyclic. Hence* $\mathrm{GL}(n,\mathbb{Z})$ *can act on such spaces only via the determinant map.*

For $p = 3$ there is a subgroup $T \subset \mathrm{SAut}(F_{2m})$ isomorphic to $(\mathbb{Z}_3)^m$ that intersects every proper normal subgroup of $\mathrm{SAut}(F_{2m})$ trivially. This provides a stronger degree of rigidity than is offered by the 2-torsion in $\mathrm{SAut}(F_n)$ and consequently one can deduce the following theorems from Smith theory more readily than is possible in the case of $\mathbb{Z}_2$.

Theorem. *If $n > 3$ is even and $d < n-1$, then any action of* $\mathrm{SAut}(F_n)$ *by homeomorphisms on a generalized d-sphere over $\mathbb{Z}_3$ is trivial.*

Theorem. *If $n > 3$ is even and $d < n$, then any action of* $\mathrm{SAut}(F_n)$ *by homeomorphisms on a d-dimensional $\mathbb{Z}_3$-acyclic homology manifold over $\mathbb{Z}_3$ is trivial.*

We are unsure what happens for primes other than 2 and 3.

We expect that our results concerning $\mathrm{SL}(n,\mathbb{Z})$ should be true for other lattices in $\mathrm{SL}(n,\mathbb{R})$, but our techniques do not apply because we make essential use of the torsion in $\mathrm{SL}(n,\mathbb{Z})$. What happens for subgroups of finite index in $\mathrm{SAut}(F_n)$ is less clear: there are subgroups of finite index in $\mathrm{SAut}(F_n)$ that map non-trivially to $\mathrm{SL}(n-1,\mathbb{R})$ and hence act non-trivially on $\mathbb{R}^{n-1}$, but we do not know whether such subgroups can act non-trivially on contractible manifolds of dimension less than $n-1$.

Our results concerning torsion in $\mathrm{Aut}(F_n)$, can be combined with the application of Smith theory in [5] to imply the following result.

Theorem. *For every compact d-dimensional homology manifold over $\mathbb{Z}_p$, the sum of whose mod p Betti numbers is B, there exists an integer $\nu(d,B)$, depending only on d and B, so that* $\mathrm{Aut}(F_n)$ *cannot act non-trivially by homeomorphisms on M if $n > \nu(d,B)$.*

References

[1] G.E. Bredon, *Sheaf Theory*, second edition. Graduate Texts in Mathematics, 170. Springer-Verlag (1997).

[2] A. Borel, *Seminar on transformation groups*. Annals of Mathematics Studies, no. 46, Princeton University Press, Princeton (1960).

[3] M.R. Bridson and K. Vogtmann, *Actions of automorphism groups of free groups on homology spheres and acyclic manifolds*, to appear in Commentarii Math. Helv.

[4] M.R. Bridson and K. Vogtmann, *Homomorphisms from automorphism groups of free groups*, Bull. London Math. Soc. **35** (2003).

[5] L.N. Mann and J.C. Su, *Actions of elementary p-groups on manifolds*, Trans. Amer. Math. Soc. **106** (1963), 115–126.

[6] K. Parwani, *Actions of* $\mathrm{SL}(n,\mathbb{Z})$ *on homology spheres*, Geom. Dedicata **112** (2005), 215-223.

[7] B.P. Zimmermann, $\mathrm{SL}(n,\mathbb{Z})$ *cannot act on small spheres*, preprint (2006), arXiv:math/0604239.

Blob Homology

Kevin Walker

(joint work with Scott Morrison)

We define a chain complex $\mathcal{B}_*(M,C)$ (the "blob complex") associated to an n-category C and an n-manifold M. For $n=1$, the complex $\mathcal{B}_*(S^1,C)$ is quasi-isomorphic to the Hochschild complex of the 1-category C. So in some sense blob homology is a generalization of Hochschild homology to n-categories. The degree zero homology of $\mathcal{B}_*(M,C)$ is isomorphic to the dual of the Hilbert space associated to M by the TQFT corresponding to C. So in another sense the blob complex is the derived category version of a TQFT.

This is work in progress, so various details remain to be filled in.

We hope to apply blob homology to tight contact structures on 3-manifolds ($n=3$) and the extension of Khovanov homology to general 4-manifolds ($n=4$). In both of these examples, exact triangles play an important role, and the derived category aspect of the blob complex allows this exactness to persist to a greater degree than it otherwise would.

$\mathcal{B}_0(M,C)$ is defined to be finite linear combinations of C-pictures on M. (A C-picture on M can be thought of as a pasting diagram for n-morphisms of C in the shape of M together with a choice of homeomorphism from this diagram to M.) There is an evaluation map from $\mathcal{B}_0(B^n,C)$ (C-pictures on the n-ball B^n) to the n-morphisms of C. Let U be the kernel of this map. Elements of U are called null fields. $\mathcal{B}_1(M,C)$ is defined to be finite linear combinations of triples (B,u,r) (called 1-blob diagrams), where $B\subset M$ is an embedded ball (or "blob"), $u\in U$ is a null field on B, and r is a C-picture on $M\setminus B$. Define the boundary map $\partial:\mathcal{B}_1(M,C)\to\mathcal{B}_0(M,C)$ by sending (B,u,r) to $u\bullet r$, the gluing of u and r. $\mathcal{B}_1(M,C)$ can be thought of as the space of relations we would naturally want to impose on $\mathcal{B}_0(M,C)$, and so $H_0(\mathcal{B}_*(M,C))$ is isomorphic to the generalized skein module (dual of TQFT Hilbert space) one would associate to M and C.

$\mathcal{B}_k(M, C)$ is defined to be finite linear combinations of k-blob diagrams. A k-blob diagram consists of k blobs (balls) $B_0, \ldots, B_{k-1}$ in M. Each pair B_i and B_j is required to be either disjoint or nested. Each innermost blob B_i is equipped with a null field $u_i \in U$. There is also a C-picture r on the complement of the innermost blobs. The boundary map $\partial : \mathcal{B}_k(M, C) \to \mathcal{B}_{k-1}(M, C)$ is defined to be the alternating sum of forgetting the i-th blob.

If M has boundary we always impose a boundary condition consisting of an $n-1$-morphism picture on ∂M. In this note we will suppress the boundary condition from the notation.

The blob complex has the following properties:

- **Functoriality.** The blob complex is functorial with respect to diffeomorphisms. That is, fixing C, the association

$$M \mapsto \mathcal{B}_*(M, C)$$

  is a functor from n-manifolds and diffeomorphisms between them to chain complexes and isomorphisms between them.
- **Contractibility for B^n.** The blob complex of the n-ball, $\mathcal{B}_*(B^n, C)$, is quasi-isomorphic to the 1-step complex consisting of n-morphisms of C. (The domain and range of the n-morphisms correspond to the boundary conditions on B^n. Both are suppressed from the notation.) Thus $\mathcal{B}_*(B^n, C)$ can be thought of as a free resolution of C.
- **Disjoint union.** There is a natural isomorphism

$$\mathcal{B}_*(M_1 \sqcup M_2, C) \cong \mathcal{B}_*(M_1, C) \otimes \mathcal{B}_*(M_2, C).$$

- **Gluing.** Let M_1 and M_2 be n-manifolds, with Y a codimension-0 submanifold of ∂M_1 and $-Y$ a codimension-0 submanifold of ∂M_2. Then there is a chain map

$$\mathrm{gl}_Y : \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \to \mathcal{B}_*(M_1 \cup_Y M_2).$$

- **Relation with Hochschild homology.** When C is a 1-category, then $\mathcal{B}_*(S^1, C)$ is quasi-isomorphic to the Hochschild complex $\mathrm{Hoch}_*(C)$.
- **Relation with TQFTs and skein modules.** $H_0(\mathcal{B}_*(M, C))$ is isomorphic to $A_C(M)$, the dual Hilbert space of the $n+1$-dimensional TQFT based on C.
- **Evaluation map.** There is an 'evaluation' chain map

$$\mathrm{ev}_M : C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \to \mathcal{B}_*(M).$$

  (Here $C_*(\mathrm{Diff}(M))$ is the singular chain complex of the space of diffeomorphisms of M, fixed on ∂M.)

  Restricted to $C_0(\mathrm{Diff}(M))$ this is just the action of diffeomorphisms described above. Further, for any codimension-1 submanifold $Y \subset M$ dividing M into $M_1 \cup_Y M_2$, the following diagram (using the gluing maps described above) commutes.

$$\begin{tikzcd}
C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \arrow[rr, "\mathrm{ev}_M"] & & \mathcal{B}_*(M) \\
C_*(\mathrm{Diff}(M_1)) \otimes C_*(\mathrm{Diff}(M_2)) \otimes \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y^{\mathrm{Diff}} \otimes \mathrm{gl}_Y"] \arrow[rr, bend right, "\mathrm{ev}_{M_1} \otimes \mathrm{ev}_{M_2}"'] & & \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y"']
\end{tikzcd}$$

In fact, up to homotopy the evaluation maps are uniquely characterized by these two properties.

- **A_∞ categories for $n-1$-manifolds.** For Y an $n-1$-manifold, the blob complex $\mathcal{B}_*(Y \times I, C)$ has the structure of an A_∞ category. The multiplication (m_2) is given my stacking copies of the cylinder $Y \times I$ together. The higher m_i's are obtained by applying the evaluation map to $i-2$-dimensional families of diffeomorphisms in $\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$. Furthermore, $\mathcal{B}_*(M, C)$ affords a representation of the A_∞ category $\mathcal{B}_*(\partial M \times I, C)$.
- **Gluing formula.** Let $Y \subset M$ divide M into manifolds M_1 and M_2. Let $A(Y)$ be the A_∞ category $\mathcal{B}_*(Y \times I, C)$. Then $\mathcal{B}_*(M_1, C)$ affords a right representation of $A(Y)$, $\mathcal{B}_*(M_2, C)$ affords a left representation of $A(Y)$, and $\mathcal{B}_*(M, C)$ is homotopy equivalent to $\mathcal{B}_*(M_1, C) \otimes_{A(Y)} \mathcal{B}_*(M_2, C)$.

An integral Riemann-Roch theorem for surface bundles

Ib Madsen

1. Statement of results

Let $\pi\colon E \to B$ be an oriented surface bundle with closed fibers of genus g, and equipped with a fiberwise metric. Let $\mathcal{H}(E)$ be the associated Hodge bundle with fibers $H^1(E_b; \mathbb{R})$. It becomes a g-dimensional complex vector bundle with the complex structure induced from the Hodge star operator, so it is classified by a map from B to $BU(g)$. The element

$$n!\, \mathrm{ch}_n \in H^{2n}\big(BU(g); \mathbb{Z}\big)$$

defines a characteristic class of the Hodge bundle and we set

$$s_n(E) := n!\, \mathrm{ch}_n\big(\mathcal{H}(E)\big) \in H^{2n}(B; \mathbb{Z}).$$

They are torsion classes for even n by [4], but not for odd n where we shall compare them with the standard Miller-Morita-Mumford classes

$$\kappa_n(E) := \pi_*\big(c_1(T^*E)^{n+1}\big) \in H^{2n}(B; \mathbb{Z}).$$

The Grothendieck Riemann-Roch theorem, or the family index theorem of Atiyah and Singer, yields the relation [3, 9]

$$s_{2n-1}(E) = (-1)^{n-1}(B_n/2n)\kappa_{2n-1}(E) \quad \text{in } H^*(B; \mathbb{Q}), \tag{1}$$

where the Bernoulli numbers are defined by the power series

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} B_n/(2n)! z^{2n}.$$

Clearing denominators in (1), T. Akita made the following conjecture in [1]:

Conjecture (Akita)**.** *In $H^*(B;\mathbb{Z})$,*

$$\mathrm{Denom}(B/2n) s_{2n-1}(E) = (-1)^{n-1}\,\mathrm{Num}(B_n/2n)\kappa_{2n-1}(E).$$

The conjecture was verified in [2] for some special values of n, but it turns out to be incorrect in general. A counterexample is given below (for $n = p$, an odd prime), but there are many other counterexamples and the conjecture is incorrect even in $H^*(B;\mathbb{F}_p)$.

On the positive side however, one can replace the standard classes $\kappa_n(E)$ by another set of integral characteristic classes which I shall denote $\bar{\kappa}_n(E)$. In rational cohomology they differ from $\kappa_n(E)$ only by a sign:

$$\bar{\kappa}_n(E) = (-1)^n \kappa_n(E) \quad \text{in } H^*(B;\mathbb{Q}),$$

and moreover, Akita's conjecture becomes correct when we substitute the κ_n classes with $\bar{\kappa}_n$.

Theorem 1.1. (1) *In $H^{4n-2}(B;\mathbb{Z})$,*

$$2\,\mathrm{Denom}(B_n/2n) s_{2n-1}(E) = 2(-1)^n\,\mathrm{Num}(B_n/2n)\bar{\kappa}_{2n-1}(E).$$

(2) *In cohomology with p-local coefficients, the difference $\kappa_n(E) - (-1)^n\bar{\kappa}_n(E)$ is a torsion class of order p for all primes p.*

Remark. *The extra factor 2 is unfortunate, and could possibly be removed with a more detailed consideration.*

The classes $\bar{\kappa}_n(E)$ are not as simple to define as the standard classes $\kappa_n(E)$. I owe the following description to Johannes Ebert. Given an oriented surface bundle (with compact fiber), and equipped with a fiberwise Riemannian metric, one has the fiberwise $\bar{\partial}$-operator

$$\bar{\partial}\colon C^\infty(E;\mathbb{C}) \to C^\infty(E; T^{0,1}_\pi E).$$

The target consists of sections in the conjugate dual of the fiberwise tangent bundle, i.e.,

$$T^{0,1}_\pi E = \mathrm{Hom}_{\mathbb{C}}\big(\overline{T^\pi E}, \mathbb{C}\big).$$

The index bundle of $\bar{\partial}$ is $\mathbb{C} - \mathcal{H}(E)$. More generally, one may twist $\bar{\partial}$ with a complex vector bundle W on E to get

$$\bar{\partial}_W\colon C^\infty(E;W) \to C^\infty(E; T^{0,1}_\pi E \otimes W).$$

The classes $\bar{\kappa}(E)$ are then given by

$$\bar{\kappa}_n(E) = n!\,\mathrm{ch}_n(\mathrm{index}\,\bar{\partial}_W) \in H^{2n}(B,\mathbb{Z}) \tag{2}$$

where $W = \overline{T^\pi E} - \mathbb{C}$ and $\operatorname{index}(\bar\partial_W)$ is the (analytic) index bundle over B. (Its class in $K(B)$ is independent of choice of connection on W when W is not holomorphic.)

Surface bundles with connected fibers of genus g are classified by $B\operatorname{Diff}(F_g) \simeq B\Gamma_g$, where Γ_g is the mapping class group. The proof of Theorem 1.1 depends on the homological description of the stable mapping class group in terms of the cobordism space $\Omega^\infty \mathrm{MT}(2) = \Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1}$ from [8]. In particular the Madsen-Weiss theorem is needed for the counter example to Akita's conjecture.

An oriented surface bundle $\pi\colon E \to B$ induces a "classifying map"

$$\alpha_E\colon B \to \Omega^\infty \mathrm{MT}(2)$$

and the classes $\kappa_n(E)$, $\bar\kappa_n(E)$ and $s_n(E)$ are pull-backs of universal classes κ_n, $\bar\kappa_n$ and s_n in the integral cohomology of $\Omega^\infty \mathrm{MT}(2)$. The analysis of the relationship between these universal classes uses infinite loop space theory and the theory of homology operations associated with this theory.

2. Remarks about proofs

Let $E \subset B \times \mathbb{R}^{2n+2}$ be an embedded, oriented surface bundle and $\pi\colon E \to B$ the projection. Its normal bundle is classified by

$$\begin{array}{ccc} N^\pi E & \xrightarrow{\hat t} & L_n^\perp \\ \big\downarrow & & \big\downarrow \\ E & \xrightarrow{t} & \mathbb{C}\mathrm{P}^n \end{array},$$

and induces the map on Thom spaces (one point compactifications)

$$\hat t\colon \operatorname{Th}(N^\pi E) \to \operatorname{Th}(L_n^\perp).$$

Let $c_E\colon B_+ \wedge S^{2n+2} \to \operatorname{Th}(N^\pi E)$ be the Pontryagin-Thom collapse map, and

$$\omega\colon \operatorname{Th}(L_n^\perp) \to \operatorname{Th}(L_n \oplus L_n^\perp) = \mathbb{C}\mathrm{P}^n_+ \wedge S^{2n+2}$$

the inclusion along the zero section of L_n. The map c_E above together with the Thom isomorphism induces push-forward maps in K-theory and cohomology

$$\pi_*\colon K(E) \to K(B), \quad \pi_*\colon H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z}).$$

The fiberwise index theorem [3] implies that the K-theory element of the index bundle is given by

$$\operatorname{index}(\bar\partial_W) = \pi_*[W] \in K(B).$$

The classes (1), (2) are then alternatively given by

$$s_{2n-1}(E) = -n!\,\mathrm{ch}_n\big(\pi_*(1)\big),$$
$$\bar\kappa_{2n-1}(E) = n!\,\mathrm{ch}_n\big(\pi_*(\overline{T^\pi E} - 1)\big).$$

In terms of the classifying diagram,

$$\pi_*(1) = \Phi_B^{-1} c_E^* \hat{t}^* (\lambda_{L_n^\perp}),$$
$$\pi_*(\overline{T^\pi E} - 1) = \Phi_B^{-1} c_E^* \hat{t}^* \omega^* \Phi_{\mathbb{C}\mathrm{P}^n}(\bar{L}_n),$$

where $\Phi_X \colon K(X) \to \tilde{K}(X_+ \wedge S^{2n+2})$ is the Thom isomorphism for the trivial bundle on X, $\lambda_L^\perp$ the K-theory Thom class, and c_E the collapse map.

Let p be a prime and k an integer prime to p. The "cannibalistic" characteristic class

$$\varrho^k - 1 \colon \tilde{K}(X) \to 1 + \tilde{K}(X) \otimes \mathbb{Z}_{(p)} \xrightarrow{\cong} \tilde{K}(X) \otimes \mathbb{Z}_{(p)}$$

is classified by a map $BU \to BU_{(p)}$. We let r^k be its double loop. Following [7], there is a diagram

$$\begin{tikzcd}
\Omega^\infty \mathrm{MT}(2) \arrow[rr, "\omega"] \arrow[dd, "\lambda_{-L}"'] & & Q(\mathbb{C}\mathrm{P}^\infty_+) \arrow[rr, "Q(\bar{L})"] & & Q(\mathbb{Z} \times BU) \arrow[dl] \arrow[dd, "Q(r^k)"] \\
& B \arrow[ul, "\alpha_E"'] \arrow[dl, "\pi_*(1)"] \arrow[rr] & & \mathbb{Z} \times BU \arrow[dl, "r^k"'] & \\
\mathbb{Z} \times BU \arrow[rr, "k\psi^k - \mathrm{id}"] & & (\mathbb{Z} \times BU)_{(p)} & & Q\big((\mathbb{Z} \times BU)_{(p)}\big) \arrow[ll, "E"']
\end{tikzcd}$$

The first part of Theorem 1.1 follows by evaluating this diagram on cohomology, or more precisely by evaluating the two ways around on the primitive generator $s_{2n-1} = (2n-1)!\,\mathrm{ch}_{2n-1}$ of $H^{4n-2}(BU_{(p)};\mathbb{Z})$. The diagram homotopy commutes, except for the square

$$\begin{tikzcd}
Q(BU) \arrow[r, "Q(r^k)"] \arrow[d, "E"'] & Q(BU_{(p)}) \arrow[d, "E_{(p)}"] \\
BU \arrow[r, "r^k"] & BU_{(p)}
\end{tikzcd}.$$

But this diagram induces a commutative diagram in cohomology when applied to the primitive part of $H^*(BU_{(p)};\mathbb{Z})$. This uses the Artin-Hasse logarithm [10].

Akita's conjecture is false: $\kappa_{p-1} \neq \bar{\kappa}_{p-1}$ in $H^*\big(\Omega^\infty \mathrm{MT}(2); \mathbb{F}_p\big)$, and [8] shows that $\kappa_{p-1}(E) \neq \bar{\kappa}_{p-1}(E)$ for surface bundles of large fiber genus.

References

[1] T. Akita. *Nilpotency and triviality of mod p Morita-Mumford classes of mapping class groups of surfaces*, Nagoya Math. J. **165** (2002), 1–22.

[2] T. Akita and N. Kawazumi. *Integral Riemann-Roch formulae for cyclic subgroups of mapping class groups*, Math. Proc. Cambridge Philos. Soc. **144** (2008), 411–421.

[3] M.F. Atiyah and I.M. Singer. *The index of elliptic operators. IV*, Ann. of Math. (2) **93** (1971), 119–138.

[4] A. Borel. *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1975), 235–272.

[5] S.O. Kochman. *Homology of the classical groups over the Dyer-Lashof algebra*, Trans. Amer. Math. Soc. **185** (1973), 83–136.

[6] I. Madsen. *Higher torsion in SG and BSG*, Math. Z. **143** (1975), 55–80.
[7] I. Madsen and C. Schlichtkrull. *The circle transfer and K-theory*, In Geometry and topology: Aarhus (1998), volume 258 of Contemp. Math., 307–328. Amer. Math. Soc. (2000).
[8] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), 843–941.
[9] S. Morita. *Characteristic classes of surface bundles*, Invent. math. **90** (1987), 551–577.
[10] T.tom Dieck. *The Artin-Hasse logarithm for λ-rings*, In Algebraic topology (Arcata, CA, 1986), volume 1370 of Lecture Notes in Math., Springer (1989), 409–415.

The stable rank of symmetry of $S^{n_1} \times \ldots \times S^{n_k}$

Bernhard Hanke

Transformation group theory investigates symmetries of topological spaces. An important aspect of this program is to define and study invariants that distinguish spaces admitting lots of symmetries from less symmetric ones.

We concentrate on one of these invariants, the so called *p-rank*

$$\mathrm{rk}_p(X) := \max\{r \mid (\mathbb{Z}/p)^r \text{ acts freely on } X\},$$

defined for any topological space X and any prime number p . Here all groups are acting topologically. We recall the following fact from classical Smith theory.

Theorem. *Let $X = S^n$ be the n-dimensional sphere. Then*

$$\mathrm{rk}_p(X) = \begin{cases} 1 \text{ for odd } n\,, \\ 1 \text{ for even } n \text{ and } p = 2\,, \\ 0 \text{ for even } n \text{ and } p > 2\,. \end{cases}$$

In view of this theorem it is natural to look for a corresponding result, if X is not just a single sphere, but a product of spheres,

$$X = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}\,.$$

The following statement appears at several places in the literature either in the form of a question [2, Question 7.2], [13, Problem 809] or as a conjecture [1, Conjecture 2.1], [3, Conjecture 3.1.4].

Conjecture. *If $(\mathbb{Z}/p)^r$ acts freely on X, then $r \leq k$.*

Actually, if p is odd, it is reasonable to conjecture that r is bounded above by the number k_o of odd dimensional spheres in X. The conjecture in this sharper form for odd p has been verified in the following cases:

- $k \leq 2$, see Heller [12]; $k \leq 3$, $p = 2$, see Carlsson [8].
- $n_1 = \cdots = n_k$ and in addition
  - the induced action on integral homology is trivial, see Carlsson [7], or
  - the induced action on integral homology is unrestricted, but if $p = 2$, then $n_i \neq 3, 7$, see Adem-Browder [2] (for $p \neq 2$ or $n_i \neq 1, 3, 7$), and Yalçın [17] (for $p = 2$ and $n_i = 1$).

- assume $n_1 \leq n_2 \leq \cdots \leq n_k$. Then $n_1 \geq 2$ and for all $1 \leq i \leq k-1$ either $n_i = n_{i+1}$ or $2n_i \leq n_{i+1}$. Furthermore, the induced $(\mathbb{Z}/p)^r$-action on $\pi_*(X)$ is trivial and $p > 3 \dim X$, see Sörensen [15].

The following theorem is our main result. It settles a stable form of the above conjecture.

Main Theorem. *If $p > 3 \dim X$, then $\mathrm{rk}_p(X) = k_o$.*

This theorem implies the following estimate for the *toral rank* of X which follows from [11, Theorem T].

Theorem. *If $(S^1)^r$ acts freely on X, then $r \leq k_o$.*

The proof is based on rational homotopy theory applied to the Borel space $X_{(S^1)^r}$.

However, even for very large primes our main theorem cannot be deduced from this result by some sort of limiting process: Browder [5] constructed free actions of $(\mathbb{Z}/p)^r$ on $(S^m)^k$ for each odd $m \geq 3$ and $r \geq 4$, $k \geq r$, $p > km/2$ which are exotic in the sense that they cannot be extended to $(S^1)^r$-actions.

Because the rational homotopy type of $B(\mathbb{Z}/p)^r$ is trivial, it is obvious that rational homotopy theory cannot be applied in a reasonable way to study the Borel space $X_{(\mathbb{Z}/p)^r}$. But *tame homotopy* theory seems a promising approach. This theory was invented by Dwyer [10] and is manufactured along Quillen's rational homotopy theory [14], but without immediately losing p-torsion information for all primes p. One of the first observations in tame homotopy theory may be phrased as follows: If X is an $(r-1)$-connected space ($r \geq 1$) and $\pi_{r+k}(X)$ is a $\mathbb{Z}[p^{-1} \,|\, 2p-3 \leq k]$-module for every $k \geq 1$, then the complexity of the Postnikov invariants of X should be comparable to that of a rational space due to the vanishing of the relevant higher reduced Steenrod power operations in the Postnikov pieces of X.

Nevertheless, in the original setup, Dwyer's theory could only be formulated for $r > 2$, see [10, 1.5].

To any space X one can associate the *Sullivan-de Rham algebra* [6, 16], a commutative graded differential algebra over $\mathbb{Q}$ modeled on de Rham differential forms, which calculates the rational cohomology of X. This construction widens the scope of rational homotopy theory from simply connected to nilpotent spaces. A distinctive feature of this approach is the construction of a *minimal model* for any space X out of the Sullivan-de Rham algebra. The minimal model still calculates the rational cohomology of the underlying space, but in addition its associated vector space of indecomposables can be identified with the dual of $\pi_*(X) \otimes \mathbb{Q}$, if X is nilpotent, $H^*(X; \mathbb{Q})$ is finite dimensional in each degree and $\pi_1(X)$ is abelian.

For the above estimate of the toral rank of X one writes down a minimal model of the Borel space $X_{(S^1)^r}$ and argues that if $r > k_o$, then the cohomology of the minimal model is nonzero in arbitrarily high degrees. But this contradicts the fact that by the freeness of the action, $X_{(S^1)^r}$ is homotopy equivalent to the finite dimensional space $X/(S^1)^r$. The analysis of the minimal model leading to these conclusions is carried out in the fundamental paper [11].

The generalization of the Sullivan-de Rham algebra to the tame setting is realized in the work of Cenkl and Porter [9] and is achieved by considering commutative graded differential algebras over $\mathbb{Q}$ which are equipped with *filtrations* as a new structural ingredient. By definition, elements in filtration q are divisible by any prime $p \leq q$, but not necessarily by larger primes, and filtration degrees are added when elements are multiplied. The construction of the Cenkl-Porter complex is based on differential forms similar to the Sullivan-de Rham algebra. The *Cenkl-Porter equivalence theorem* states that integration of forms yields a cochain map of the Cenkl-Porter complex in filtration q to the singular cohomology with $\mathbb{Z}[p^{-1} \,|\, p \leq q]$-coefficients which in cohomology induces isomorphisms that moreover are compatible with the respective multiplicative structures.

It is now desirable to replace the Cenkl-Porter complex by a smaller filtered commutative graded differential algebra which nicely reflects the homotopy type of the underlying space in a similar manner as the minimal algebra in rational homotopy theory. This uses a *tame Hirsch lemma* which is used to build the desired small filtered cochain algebra along a Postnikov decomposition of the underlying space. The correct formulation and the proof of a tame Hirsch lemma is a nontrivial task, which was performed in the remarkable diploma thesis of Sörensen [15].

With this machinery in hand we construct small approximative commutative $\mathbb{F}_p$-cochain models of Borel spaces associated to $(\mathbb{Z}/p)^r$-spaces X (satisfying certain additional assumptions). The principal idea is to use a simultaneous Postnikov decomposition of the fibre and total space of the Borel fibration $X \hookrightarrow X_{(\mathbb{Z}/p)^r} \to B(\mathbb{Z}/p)^r$. This might be of independent interest, especially when compared to the minimal Hirsch-Brown model of a $(\mathbb{Z}/p)^r$-space (see [4] for a discussion) which is in general neither graded commutative nor associative.

The cochain model resulting from this discussion is the starting point for our proof of the main theorem. The argument is inspired by the paper [11], which provides techniques to simplify the analysis of free commutative graded differential algebras over fields of characteristic zero. We emphasize that in the stated generality, the arguments in *loc. cit.* do not work over fields of prime characteristics. Fortunately, it turns out that the $\mathbb{F}_p$-cochain algebras appearing in our context are special enough so that the proof can be completed.

We believe that our methods are not sufficient to establish the general form of the above conjecture for small primes – this part of the conjecture remains open.

References

[1] A. Adem, *Constructing and deconstructing groups actions*, preprint (2002), arXiv: math.AT/0212280.

[2] A. Adem, B. Browder, *The free rank of symmetry of* $(S^n)^k$, Invent. math. **92** (1988), 431-440.

[3] A. Adem, J. Davis, *Topics in transformation groups*, in: Handbook of geometric topology, R.J. Daverman, R.B. Sher (eds.), Elsevier (2002).

[4] C. Allday, V. Puppe, *Cohomological methods in transformation groups*, Cambridge University Press (1993).

[5] W. Browder, *Homologically exotic free actions on products of S^m*, in: *Topology, Geometry, and Algebra: Interactions and new directions (Stanford, 1999)*, Contemporary Mathematics **279**, AMS (2001).
[6] A. Bousfield, V. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179** (1976).
[7] G. Carlsson, *On the rank of abelian groups acting freely on $(S^n)^k$*, Invent. math. **69** (1982), 393-400.
[8] G. Carlsson, *$(\mathbb{Z}/2)^3$-actions on finite complexes*, in: *Algebraic topology and algebraic K-theory* (Princeton, 1983), Ann. of Math. Stud. **113** (1987), 332-344.
[9] B. Cenkl, R. Porter, *De Rham theorem with cubical forms*, Pacific J. Math. **112** (1984), 35-48.
[10] W. Dwyer, *Tame homotopy theory*, Topology **18** (1979), 321-338.
[11] S. Halperin, *Finiteness in the minimal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173-199.
[12] A. Heller, *A note on spaces with operators*, Ill. J. Math. **3** (1959), 98-100.
[13] J. van Mill, G. M. Reed (eds.), *Open problems in topology*, Elsevier (1990)
[14] D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969), 205-295.
[15] T. Sörensen, *Zahme Homotopietheorie einfacher Kan-Mengen und p-Torus-Operationen*, Diplomarbeit (1992), FU Berlin (unpublished).
[16] D. Sullivan, *Infinitesimal computations in topology*, Publ. IHES **47** (1977), 269-331.
[17] E. Yalçın, *Group actions and group extensions*, Trans. Amer. Math. Soc. **352** (2000), 2689-2700.

Results on N. Kuhn's realization conjectures

Gérald Gaudens

This is a report on the known results about certain conjectures due to N. Kuhn [3].

Let X be a topological space, and consider its singular cohomology groups H^*X with coefficients in $\mathbf{F}_2$. It is functorially in X an object of the category $\mathcal{K}$ of *unstable algebras* [4]. An unstable algebra is the conjunction of:

- a graded commutative $\mathbf{F}_2$-algebra structure,
- a compatible module structure over the Steenrod algebra $\mathcal{A}_2$,
- extra conditions called the *unstability conditions*.

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra, that is the category obtained from $\mathcal{K}$ by neglecting the algebra structure.

Question. How do the objects of the form H^*X look like, as objects of $\mathcal{U}$?

Nota bene. We could ask the same question with $\mathcal{K}$ instead of $\mathcal{U}$. This question turns out to be much more difficult.

A typical result of this kind is due to L. Schwartz [5] and was conjectured by N. Kuhn:

Theorem. *If H^*X is finitely generated as a module over the Steenrod algebra $\mathcal{A}_2$ (that is, as an object of $\mathcal{U}$), then M is finite as an $\mathbf{F}_2$-vector space.*

The analogous statement for cohomology with coefficients in $\mathbf{F}_p$ for any prime p actually holds. This quite striking result can be reformulated in a different way. Unstable modules that are finite dimensional coincide with those finitely generated modules that belong to $\mathcal{U}_0$, the full subcategory of locally finite modules. Recall that an unstable module M is locally finite if for each $m \in M$, the submodule $\mathcal{A}_2.m$ is finite. The subcategory $\mathcal{U}_0$ is the first step in a natural filtration of the category $\mathcal{U}$ by nice Abelian subcategories:

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_n \subset \mathcal{U}_{n+1} \subset \ldots$$

As finitely generated unstable modules are in $\mathcal{U}_n$ for some n, it is natural to conjecture:

Strong realization conjecture (N. Kuhn). *If H^*X is in $\mathcal{U}_n$ for some n, then it is in $\mathcal{U}_0$.*

Nota bene. There are plenty of objects $M \in \mathcal{U}$ that do not belong to $\mathcal{U}_n$ for any n, typically, the cohomology of an Eilenberg-Mac Lane space $K(\mathbf{Z}/2\mathbf{Z}, n)$.

This conjecture was proved by work of L. Schwartz [6], and Dehon-Gaudens [1].

Theorem. *The strong realization conjecture holds.*

The analogous conjecture for the cohomology with coefficients in $\mathbf{F}_p$, where p is an odd prime remains unsolved.

One can sharpen this conjecture. Reflecting the unstability conditions, there is a natural decreasing Hausdorff filtration on any unstable module M:

$$M \supset \mathrm{Nil}_1 M \supset \mathrm{Nil}_2 M \supset \ldots \supset \mathrm{Nil}_s M \supset \mathrm{Nil}_{s+1} M \supset \ldots$$

The notation is due to the fact that the unstable module that is underlying to an unstable algebra K belongs to Nil_1 if and only if it is locally nilpotent as an algebra. An unstable module belongs to $\mathcal{U}_n$ if and only if every subquotient $\mathrm{Nil}_s M/\mathrm{Nil}_{s+1} M$ is in $\mathcal{U}_n$ for any s.

Unbounded strong realization conjecture (N. Kuhn). *If $H^*X \notin \mathcal{U}_0$, then there is an s such that $\mathrm{Nil}_s H^*X/\mathrm{Nil}_{s+1} H^*X$ is not in $\mathcal{U}_n$ for any n.*

This conjecture implies immediately the *strong realization conjecture*.

Provided $H^*X \notin \mathcal{U}_0$, and assuming the conjecture holds, we let $S(H^*X)$ be the smallest s such that $\mathrm{Nil}_s H^*X/\mathrm{Nil}_{s+1} H^*X$ is in not $\mathcal{U}_n$ for any n.

This conjecture has been proved for spaces whose cohomology do have any Bockstein operators in high degrees [2]. In this case, the analogous result holds for odd primes as well, and $S(H^*X)$ is actually 0 or 1. One way to settle all above conjectures at all primes, is to get a better understanding of what is actually going on. A careful analysis of examples yields a conjectural upper bound on $S(H^*X)$.

Consider for $i \geq 0$ the Steenrod operation:

$$\theta_i = \mathrm{Sq}^{2^i} \mathrm{Sq}^{2^i - 1} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1$$

Let $N(H^*X)$ be the integer defined by:

- $N(H^*X) = 0$ if $H^{>0}X$ is not locally nilpotent,
- and otherwise $N(H^*X) = \min\{i \geq 1 \mid \nu(\theta_i \alpha) > 2^{i+1} - 1 + \nu(\alpha), \alpha \in H^*X\}$

where $\nu(\alpha)$ is the nilpotency index of α, i.e., the smallest s such that $\alpha \in \mathrm{Nil}_s H^*X$. We conjecture:

Refined realization conjecture. *The unbounded strong realization conjecture holds, and if $H^*X \notin \mathcal{U}_0$, then $S(H^*X) \leq N(H^*X)$.*

This conjecture has been proved for $N(H^*X) = 0, 1, 2$. The case $N(H^*X) = 1$ is essentially the case of vanishing of Bocksteins over there. In fact this new formulation is strong enough to summarize the known results on the realization conjectures as those known low degree cases of the *refined conjectures*. Moreover, this last conjecture would imply all the conjectures above.

The definition of $N(H^*X)$ might seem quite technical, but is well motivated by examples. Examples also suggest that the *refined realization conjecture* might be accessible via a thorough study of the Bar filtration on spaces.

References

[1] F.-X. Dehon and G. Gaudens, *Espaces profinis et problèmes de réalisabilité*, Algebr. Geom. Topol. **3** (2003), 399-433.

[2] G. Gaudens, *Bocksteins and the nilpotent filtration on the cohomology of spaces*, Geometry & Topology Monographs **11** (2007) 59-79.

[3] N.J. Kuhn, *On topologicaly realizing modules over the Steenrod algebra*, Ann. of Math. **141** (1995), 321-347.

[4] L. Schwartz, *Unstable Modules over the Steenrod Algebra and Sullivan's Fixed Point Set Conjecture*, University of Chicago Press (1994).

[5] L. Schwartz, *A propos de la conjecture de non réalisation due a N. Kuhn*, Invent. math. **134** (1998), 211-227.

[6] L. Schwartz, *La filtration de Krull de la catégorie $\mathcal{U}$ et la cohomologie des espaces*, Algebr. Geom. Topol. **1** (2001), 519-548.

The curve complex is rigid

Saul Schleimer

(joint work with Kasra Rafi)

We take $S = S_{g,b}$ to be the connected, compact, orientable surface of genus g with b boundary components. Define $\xi(S) = 3g - 3 + b$ to be the complexity of S. A simple closed curve $\alpha \subset S$ is essential and non-peripheral if α cannot be isotoped into a collar about ∂S.

The *complex of curves* $\mathcal{C}(S)$, defined by Harvey [4], has as vertices the isotopy classes of essential non-peripheral curves. The simplices span collections of vertices having disjoint representatives. Harvey introduced $\mathcal{C}(S)$ as a combinatorial model for the intersection pattern of the *thin parts* of Teichmüller space.

It is a foundational theorem of Ivanov [5, 7, 8] that, for $S \neq S_{1,2}$, every simplicial automorphism of $\mathcal{C}(S)$ is induced by an element of $\mathcal{MCG}(S)$, the mapping class group. (When $S = S_{1,2}$ the theorem holds for an index five subgroup.) Ivanov's theorem is the combinatorial version of Royden's theorem [11]: every isometry of Teichmüller space is induced by a mapping class element.

We prove a coarse-geometric version of these theorems.

Theorem 1. *Suppose that $\xi(S) \geq 2$. Then every quasi-isometry of $\mathcal{C}(S)$ is bounded distance from a simplicial automorphism.*

This, together with Ivanov's theorem, shows that the quasi-isometry type of $\mathcal{C}(S)$ is enough to recover the surface. In fact we have:

Theorem 2. *If $\mathcal{C}(S)$ and $\mathcal{C}(\Sigma)$ are quasi-isometric then they are simplicially isomorphic. Hence either*

- *S is homeomorphic to Σ,*
- $\{S, \Sigma\} = \{S_{0,6}, S_2\}$,
- $\{S, \Sigma\} = \{S_{0,5}, S_{1,2}\}$,
- $\{S, \Sigma\} \subset \{S_{0,4}, S_1, S_{1,1}\}$, *or*
- $\{S, \Sigma\} \subset \{S_0, S_{0,1}, S_{0,3}\}$.

To prove Theorem 1, we must investigate the coarse geometric properties of $\mathcal{C}(S)$. These are somewhat mysterious as $\mathcal{C}(S)$ is not locally finite.

Theorem 3 (Masur-Minsky [9])**.** *$\mathcal{C}(S)$ is Gromov hyperbolic.*

Theorem 4 (Klarreich [6])**.** *The Gromov boundary of $\mathcal{C}(S)$ is $\mathcal{MCG}(S)$–equivariantly homeomorphic to $\mathcal{EL}(S)$.*

Here $\mathcal{EL}(S)$ is the space of *ending laminations*, obtained from $\mathcal{PML}(S)$ by taking the subset of *filling laminations* and taking a quotient by forgetting the measures. We define $\overline{\mathcal{C}(S)} = \mathcal{C}(S) \cup \mathcal{EL}(S)$. Little is known about the topology of $\mathcal{EL}(S)$. However:

Theorem 5 (Gabai [2])**.** *If $\xi(S) \geq 2$ then $\mathcal{EL}(S)$ is connected.*

Let $B(z, r) \subset \mathcal{C}(S)$ be the ball of radius r about z. Using Theorem 5 and Gromov hyperbolicity we prove:

Proposition 6. *The* shell $B(z, r + 4\delta) \setminus B(z, r - 1)$ *is connected.*

We remark that, due to the locally infinite nature of $\mathcal{C}(S)$, the shell intrinsically has infinite diameter and highly complicated geometry.

Masur and Minsky [10] introduced the notion of *subsurface projection*

$$\pi_X : \overline{\mathcal{C}(S)} \to \mathcal{C}(X).$$

They define $\pi_X(\alpha)$ by intersecting α with the subsurface X and surgering the resulting arcs to obtain curves in X. If $\alpha \cap X = \emptyset$ then $\pi_X(\alpha)$ is left undefined. For $\alpha, \beta \in \overline{\mathcal{C}(S)}$ define

$$d_X(\alpha, \beta) = d_X\big(\pi_X(\alpha), \pi_X(\beta)\big).$$

We say that $\alpha, \beta \in \overline{\mathcal{C}(S)}$ are K*–cobounded* if for all strict subsurfaces $X \subset S$ we have $d_X(\alpha, \beta) \leq K$. Now, there is no reason that a quasi-isometry will preserve the property of K–coboundedness for vertices in $\mathcal{C}(S)$. As usual, better behavior is obtained at infinity.

Theorem 7. *For all S, Σ, q, c there is a c' so that if $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a q–quasi-isometric embedding and $k, \ell \in \mathcal{EL}(S)$ are c–cobounded then $\phi(k), \phi(\ell)$ are c'–cobounded.*

We now turn to another idea of Masur and Minsky: a *marking* of S is any collection of filling curves. We say that a marking $m \subset S$ is K–*small* if $i(m, m) \leq K$. We say that two markings m, n are K'–*near* if $i(m, n) \leq K'$. Then there are constants K, K' so that the *marking graph* $\mathcal{M}(S)$ is quasi-isometric to $\mathcal{MCG}(S)$. Here vertices are K–small markings and edges are given by K'–nearness. There is a canonical map $p\colon \mathcal{M}(S) \to \mathcal{C}(S)$ defined by mapping a marking m to any element $a \in m$. Without being too cautious about quantifiers we have:

Theorem 8. *Suppose that $\xi(S) \geq 2$. Suppose that $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. Then there is an induced map $\Phi\colon \mathcal{M}(S) \to \mathcal{M}(\Sigma)$ so that*

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\Phi} & \mathcal{M}(\Sigma) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{C}(S) & \xrightarrow[\phi]{} & \mathcal{C}(\Sigma) \end{array}$$

and this square commutes up to additive error. Furthermore, Φ is coarsely-Lipschitz.

We are now equipped to prove Theorem 1: suppose that $f\colon \mathcal{C}(S) \to \mathcal{C}(S)$ is a quasi-isometry. Then, using Theorem 8 in both directions, we find a quasi-isometry $F\colon \mathcal{M}(S) \to \mathcal{M}(S)$. We now apply a recent theorem of Behrstock, Kleiner, Minsky, and Mosher [1]: the mapping class group equipped with the word metric is itself rigid. (See also Hamenstaedt [3].) Thus F is close to the action of some mapping class $G \in \mathcal{MCG}(S)$. Pushing this down to the curve complex gives a isometry $g\colon \mathcal{C}(S) \to \mathcal{C}(S)$. Verifying that the given f and the induced g are close completes the proof.

References

[1] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, *Geometry and rigidity of mapping class groups*, preprint (2008), arXiv: 0801.2006.

[2] D. Gabai, *Almost filling laminations and the connectivity of ending lamination space*, preprint (2008), arXiv: 0808.2080.

[3] U. Hamenstaedt, *Geometry of the mapping class groups III: Quasi-isometric rigidity*, preprint (2005), arXiv: math/0512429.

[4] W.J. Harvey, *Boundary structure of the modular group*, in: Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Princeton Univ. Press (1981), 245–251.

[5] N.V. Ivanov, *Mapping class groups*, in: Handbook of geometric topology, North-Holland (2002), 523–633.

[6] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, http://nasw.org/users/klarreich/research.htm.

[7] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology Appl. **95** (1999), 85–111.

[8] F. Luo, *Automorphisms of the complex of curves*, Topology **39** (2000), 283–298.

[9] H.A. Masur and Y.N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. math. **138** (1999), 103–149. Also available at arXiv: math/9804098.

[10] H.A. Masur and Y.N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), 902–974. Also available at arXiv: math/9807150.

[11] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, in: Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, vol. 66, Princeton Univ. Press, (1971), 369–383.

Homotopy theory and fusion systems of finite groups of Lie type

Carles Broto
(joint work with Jesper Møller and Bob Oliver)

It is observed by group theorists that some finite groups of Lie type carry the same p-local structure (see [4] for a discussion of p-local properties of finite groups). The same phenomenon is observed if we look at mod p cohomology computations by several people (cf. [9, 2]).

Our goal is to give a uniform explanation for all of these observations. We use homotopy theoretic methods, and to our best knowledge no alternative proof is known at the moment.

By p-local structure of a finite group we mean its p-fusion system, as defined below.

Definition 1. Let G be a finite group. The p-fusion system of G is a category $\mathcal{F}_p(G)$ with

- objects the p-subgroups P of G, and
- morphisms the group homomorphisms $\varphi\colon P \to Q$ induced by conjugation by elements of G: $\mathrm{Hom}_{\mathcal{F}_p(G)}(P,Q) = \{\, \varphi \in \mathrm{Hom}(P,Q) \mid \exists g \in G\,,\ \varphi = c_g \,\}$.

together with a forgetful functor to the category of groups $\lambda_G\colon \mathcal{F}_p(G) \to \mathrm{Groups}$.

We will say that two finite groups, G and H, have the same p-fusion and will write $\mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$ when there is an *isotypical* equivalence (T,w): $T\colon \mathcal{F}_p(G) \to \mathcal{F}_p(H)$, an equivalence of categories, and $w\colon \lambda_G \to \lambda_H \circ T$, a natural isomorphism of functors. Equivalently, G and H have the same p-fusion if for given Sylow p-subgroups $S_G \in \mathrm{Syl}_p(G)$ and $S_H \in \mathrm{Syl}_p(H)$, there is a fusion preserving isomorphism $S_G \to S_H$.

Now, we can state our main theorem. Here $\mathbb{Z}_p^\times$ stands for the multiplicative group of p-adic units, and for $q \in \mathbb{Z}_p^\times$, $\overline{\langle q\rangle}$ is the closed subgroup generated by q.

Theorem 2 (B-Møller-Oliver)**.** *Fix a prime p, a connected reductive integral group scheme $\mathbb{G}$ and prime powers q, q', both prime to p. Then*

(1) $\overline{\langle q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times \Longrightarrow \mathcal{F}_p(G(q)) \simeq \mathcal{F}_p(G(q'))$.

(2) *for $\mathbb{G}$ of type A_n, D_n, E_6 and the graph automorphism τ,*

$$\overline{\langle q\rangle} = \overline{\langle q'\rangle} \Longrightarrow \mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p({}^\tau\mathbb{G}(q')).$$

(3) *If $\Psi^{-1} \in W(\mathbb{G})$ – the Weyl group of $\mathbb{G}$ contains an element which acts on the maximal torus by inverting all elements –, then $\overline{\langle -1,q\rangle} = \overline{\langle -1,q'\rangle} \le \mathbb{Z}_p^\times$ implies $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ (or $\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p({}^\tau\mathbb{G}(q'))$ for $\mathbb{G}$ and τ as in (2)).*

(4) *For $\mathbb{G}$ of type A_n, D_n with odd n, or E_6, and the graph automorphism τ of order* 2
$$\overline{\langle -q\rangle} = \overline{\langle q'\rangle} \Longrightarrow \mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q')).$$

The same methods of proof apply to other cases that are known or can be checked by more direct methods. This is the case, for a prime p and a prime power q with $q \equiv 1(p)$, of

(1) $p \neq 3$, $\mathcal{F}_p(G_2(q)) \simeq \mathcal{F}_p({}^3D_4(q))$
(2) $p \neq 2$, $\mathcal{F}_p(F_4(q)) \simeq \mathcal{F}_p({}^2E_6(q))$.

Now, we list the main facts that we use in the proof of Theorem 2.

p-completed classifying spaces. By p-completion of a space X we mean the Bousfield-Kan p-completion $X^\wedge_p$ [1].

Theorem 3 ([6, 7, 8])**.** *Let G and H be finite groups. The p-completed classifying spaces $BG^\wedge_p$ and $BH^\wedge_p$ are homotopy equivalent if and only if the fusion systems $\mathcal{F}_p(G)$ and $\mathcal{F}_p(H)$ are isotypically equivalent.*

The aim is to show by homotopy theoretic methods that some p-completed classifying spaces are homotopy equivalent. Then, Theorem 2 implies that they have isotypically equivalent p-fusion systems. This implication was proved by Martino and Pridddy. The converse has been proved by Oliver, based on the classification of finite simple groups.

Finite groups of Lie type. Results of Friedlander [3] relate the homotopy type of p-completed classifying spaces of finite groups of Lie type to that of the corresponding compact connected Lie groups. More precisely:

Theorem 4 ([3])**.** *Let $\mathbb{G}$ be a connected reductive integral group scheme. Fix a prime p and a prime power q with $(q,p) = 1$. There is a homotopy pull-back diagram:*

$$\begin{tikzcd}
B^\tau\mathbb{G}(q)^\wedge_p \arrow[r] \arrow[d] & B\mathbb{G}(\mathbb{C})^\wedge_p \arrow[d, "\Delta"] \\
B\mathbb{G}(\mathbb{C})^\wedge_p \arrow[r, "(1, B\tau\circ\Psi^q)"] & B\mathbb{G}(\mathbb{C})^\wedge_p \times B\mathbb{G}(\mathbb{C})^\wedge_p
\end{tikzcd}$$

Notice that $B\mathbb{G}(\mathbb{C})^\wedge_p$ is homotopy equivalent to $BG^\wedge_p$, for G the corresponding compact form of $\mathbb{G}(\mathbb{C})$. Here, Ψ^q is the Adams map characterized by its restriction to the maximal torus: $x \longmapsto x^q$. We use results of Jackowsky-McClure-Oliver [5] on self maps of classifying spaces of compact Lie groups.

Homotopy fixed points. For a space X and a self homotopy equivalence α of X, one defines the homotopy quotient $X_{h\alpha}$ as the mapping torus $X \times I/\sim$, $(x,0) \sim (\alpha(x),1)$. It comes equipped with a projection to the circle $\mathrm{pr}\colon X_{h\alpha} \to S^1$ with homotopy fibre X. The homotopy fixed points space is defined as the space of sections of the projection $X^{h\alpha} \simeq \Gamma(X_{h\alpha} \xrightarrow{\mathrm{pr}} S^1)$.

An alternative way to express Friedlander's result is $B^{\tau}\mathbb{G}(q)^{\wedge}_p \simeq (BG^{\wedge}_p)^{h\alpha}$ for $\alpha = B\tau \circ \Psi^q$.

For an arbitrary space X, we give a topology to the group $\mathrm{Out}(X)$ of homotopy classes of self–homotopy equivalences of X, by fixing a basis $\mathrm{Out}(X) \supseteq U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \supseteq U_k \supseteq \cdots$ of open neighborhoods of the identity with U_k = self equivalences which induce the identity on $H^*(X;\mathbb{Z}/p^k)$. We define $\widehat{H}^i(X;\mathbb{Z}_p) = \varprojlim H^i(X;\mathbb{Z}/p^k)$ for all i and $\widehat{H}^*(X;\mathbb{Z}_p) = \bigoplus_i \widehat{H}^i(X;\mathbb{Z}_p)$. It turns out that $\mathrm{Out}(X)$ is Hausdorff if and only if it is detected on $\widehat{H}^*(X;\mathbb{Z}_p)$.

Theorem 5 (B-Møller-Oliver)**.** *Fix a prime p. Let X be a connected, p-complete space such that*

(1) $H^*(X;\mathbb{F}_p)$ *is noetherian.*
(2) $\mathrm{Out}(X)$ *is detected on* $\widehat{H}^*(X;\mathbb{Z}_p)$.

If α, β *are self homotopy equivalences of* X *such that* $\overline{\langle\alpha\rangle} = \overline{\langle\beta\rangle} \leq \mathrm{Out}(X)$, *then*

$$X^{h\alpha} \simeq X^{h\beta}.$$

p-completed classifying spaces of compact connected Lie groups. The previous theorem applies to $X = BG^{\wedge}_p$, when G is a compact, connected Lie group:

(1) $H^*(BG;\mathbb{F}_p)$ is well known to be noetherian.
(2) By results of Jackowsky-McClure-Oliver [5] the group $\mathrm{Out}(BG^{\wedge}_p)$ is detected on $\widehat{H}^*(BG^{\wedge}_p, \mathbb{Z}_p)$.

Actually, Jackowsky-McClure-Oliver prove that self maps of $BG^{\wedge}_p$ are detected by restriction to the p-completed maximal torus $BT^{\wedge}_p \simeq K(\mathbb{Z}^n_p, 2)$.

Self equivalences of the maximal torus correspond to matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$. It follows that for q and q' invertible in $\mathbb{Z}_p$, if $\overline{\langle q\rangle} = \overline{\langle q'\rangle} \leq \mathbb{Z}^{\times}_p$ then $\overline{\langle\Psi^q\rangle} = \overline{\langle\Psi^{q'}\rangle} \leq \mathrm{Out}(BG^{\wedge}_p)$, or in case $\Psi^{-1} \in W(G)$, if $\overline{\langle -1, q\rangle} = \overline{\langle -1, q'\rangle} \leq \mathbb{Z}^{\times}_p$ then $\overline{\langle\Psi^q\rangle} = \overline{\langle\Psi^{q'}\rangle} \leq \mathrm{Out}(BG^{\wedge}_p)$. This combines with Theorems 3, 4, and 5 to give the proof of the main theorem.

References

[1] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, 2nd ed., Lecture Notes in Mathematics, 304, Springer (1987)

[2] Z. Fiedorowicz and S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Mathematics, 674, Springer, (1978).

[3] E. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies, vol. 104, Princeton University Press (1982)

[4] G. Glauberman, *Global and local properties of finite groups*, Finite simple groups (Proc. Instructional Conf., Oxford, 1969), 1–64. Academic Press (1971).

[5] S. Jackowski, J. McClure, and B. Oliver, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. **147** (1995), 99–126

[6] J. Martino and S. Priddy, *Unstable homotopy classification of* $BG^{\wedge}_p$, Math. Proc. Cambridge Philos. Soc. **119** (1996), 119–137

[7] B. Oliver, *Equivalences of classifying spaces completed at odd primes*, Math. Proc. Camb. Phil. Soc. **137** (2004), 321–347

[8] B. Oliver, *Equivalences of classifying spaces completed at the prime two*, Amer. Math. Soc. Memoirs 848 (2006)
[9] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586

Group cocycles and the ring of affiliated operators

Andreas Thom
(joint work with Jesse Peterson)

1. Introduction

The computations of ℓ^2-homology have been algebraized through the seminal work of W. Lück, which is summarized and explained in detail in his nice compendium [7]. This extended abstract is a report about results obtained in [9].

Our first theorem gives an identification of dimensions of cohomology groups, where the coefficients vary among the canonical choices LG, ℓ^2G and $\mathcal{U}G$.

Theorem 1.1. *Let G be a countable discrete group. Then*

$$\beta_k^{(2)}(G) = \dim_{LG} H^k(G,\mathcal{U}G) = \dim_{LG} H^k(G,\ell^2G) = \dim_{LG} H^k(G,LG).$$

Moreover, if $\beta_k^{(2)}(G) = 0$ for some k, then $H^k(G,\mathcal{U}G) = 0$.

2. Free subgroups

Throughout this section, we are assuming that G is a torsionfree discrete countable group and most of the time also that it satisfies the following condition:

($\star$) Every non-trivial element of $\mathbb{Z}G$ acts without kernel on ℓ^2G.

Condition ($\star$) is known to hold for all right orderable groups and all residually torsionfree elementary amenable groups. No counterexample is known.

Let G be a discrete group, we use the notation $\dot{G}$ to denote the set $G\setminus\{e\}$. The main result here is the following theorem.

Theorem 2.1. *Let G be a torsionfree discrete countable group. There exists a family of subgroups $\{G_i \mid i \in I\}$, such that*

(1) *We can write G as the disjoint union:*

$$G = \{e\} \cup \bigcup_{i\in I} \dot{G}_i.$$

(2) *The groups G_i are mal-normal in G, for $i \in I$.*
(3) *If G satisfies condition ($\star$), then G_i is free from G_j, for $i \neq j$.*
(4) *$\beta_1^{(2)}(G_i) = 0$, for all $i \in I$.*

Remark 2.2. *It follows from Theorem* 7.1 *in* [9]*, that the set I is infinite if the first ℓ^2-Betti number of G does not vanish.*

Corollary 2.3. *Let G be a discrete countable group satisfying condition $(\star)$. Assume that the first ℓ^2-Betti number does not vanish. Let F be a finite subset of G. There exists $g \in G$, such that g is free from each element in F. In particular, G contains a copy of F_2.*

The following result is a generalization of the main result of J. Wilson in [10] for torsionfree groups which satisfy $(\star)$. For this, note that a group G with n generators and m relations satisfies $\beta_1^{(2)}(G) \geq n - m - 1$.

Corollary 2.4 (Freiheitssatz)**.** *Let G be a torsionfree discrete countable group which satisfies $(\star)$. Assume that $a_1, \ldots, a_n \in G$ generate G and $\lceil \beta_1^{(2)}(G) \rceil \geq k$. There exist $k+1$ elements $a_{i_0}, \ldots, a_{i_k}$ among the generators such that the natural map*

$$\pi \colon F_{k+1} \to \langle a_{i_0}, \ldots, a_{i_k} \rangle \subset G$$

is an isomorphism.

Corollary 2.5. *Let G be a finitely generated torsionfree discrete countable group which satisfies $(\star)$. Then*

$$e_S(G) \geq 2\lceil \beta_1^{(2)}(G) \rceil + 1,$$

for any generating set S. Here, $e_S(G)$ denotes the exponential growth rate w.r.t. the generating set S.

In particular, a torsionfree group satisfying condition $(\star)$ has uniform exponential growth if its first ℓ^2-Betti number is positive.

3. Notions of normality

We now want to review some notions of normality of subgroups which are more or less standard, and introduce some notation. A subgroup $H \subset G$ is called:

(1) normal iff $gHg^{-1} = H$, for all $g \in G$,
(2) s-normal iff $gHg^{-1} \cap H$ is infinite for all $g \in G$, and
(3) q-normal iff $gHg^{-1} \cap H$ is infinite for elements $g \in G$, which generate G.

We say that a subgroup inclusion $H \subset G$ satisfies one of the normality properties from above *weakly*, iff there exists an ordinal number α, and an ascending α-chain of subgroups, such that $H_0 = H$, $H_\alpha = G$, and $\mathcal{U}p_{\beta<\gamma} H_\beta \subset H_\gamma$ has the required normality property.

Example 3.1. The inclusions

$$\mathrm{GL}_n(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Q}), \qquad \mathbb{Z} = \langle x \rangle \subset \langle x, y \mid yx^p y^{-1} = x^q \rangle = \mathrm{BS}_{p,q}$$

are inclusions of s-normal subgroups. The inclusion

$$F_2 = \langle a, b^2 \rangle \subset \langle a, b \rangle = F_2$$

is q-normal but not s-normal.

3.1. **ℓ^2-invariants and normal subgroups.** The two main results in this subsection are Theorem 3.2 and Theorem 3.6. We derive several corollaries about the structure of groups G with $\beta_1^{(2)}(G) \neq 0$.

Theorem 3.2. *Let G be a countable discrete group and suppose H is an infinite weakly q-normal subgroup. We have $\beta_1^{(2)}(H) \geq \beta_1^{(2)}(G)$.*

Corollary 3.3. *Let $H \subset K \subset G$ be a chain of subgroups and assume that $H \subset G$ is weakly q-normal and $[K : H] < \infty$. Then*

$$[K : H] \cdot \beta_1^{(2)}(G) \leq \beta_1^{(2)}(H).$$

Corollary 3.4. *Let G be a torsionfree discrete countable group and let $H \subset G$ be an infinite subgroup. If $\beta_1^{(2)}(H) < \beta_1^{(2)}(G)$, then there exists a proper malnormal subgroup $K \subset G$, such that $H \subset K$.*

Corollary 3.5. *Let G be a countable discrete group and let $H \subset G$ be an infinite weakly q-normal subgroup. Let $K \subset G$ be a subgroup with $H \subset K$ and assume that $\beta_1^{(2)}(G) > n$. Then, K is not generated by n or less elements.*

The second main result in this section is the following.

Theorem 3.6. *Let G be a countable discrete group and suppose H is an infinite index, infinite weakly s-normal subgroup. If $\beta_1^{(2)}(H) < \infty$, then $\beta_1^{(2)}(G) = 0$.*

Corollary 3.7. *Let G be a countable discrete group with $\beta_1^{(2)}(G) > 0$. Suppose that $H \subset G$ is an infinite, finitely generated weakly s-normal subgroup. Then H has to be of finite index.*

Note that the result applies in case H contains an infinite normal subgroup. Hence, this result is a generalization of the classical results by A. Karass and D. Solitar [6], H. Griffiths [5], and B. Baumslag [1]. A weaker statement with additional hypothesis was proved as Theorem 1(2) in [3].

Corollary 3.8 (Gaboriau)**.** *Let G be a group with an infinite normal subgroup of infinite index, which is either finitely generated or amenable. Then $\beta_1^{(2)}(G) = 0$.*

References

[1] B. Baumslag, *Intersections of finitely generated subgroups in free products*, J. London Math. Soc. **41** (1966), 673–679.

[2] M. Bekka, A. Valette, *Group cohomology, harmonic functions and the first L^2-Betti number*, Potential Anal. **6** (1997), 313–326.

[3] M. Bourdon, F. Martin, A. Valette, *Vanishing and non-vanishing of the first L^p-cohomology of groups*, Comment. Math. Helv. **80** (2005), 377–389.

[4] Y. Cornulier, R. Tessera, A. Valette, *Isometric group actions on Banach spaces and representations vanishing at infinity*, to appear in Transformation Groups (2006).

[5] H. Griffiths, *The fundamental group of a surface, and a theorem of Schreier*, Acta Math. **110** (1963), 1–17.

[6] A. Karrass, D. Solitar, *Note on a theorem of Schreier*, Proc. Amer. Math. Soc. **8** (1957), 696–697.

[7] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer (2002).
[8] F. Martin, A. Valette, *On the first L^p-cohomology of discrete groups*, Groups Geom. Dyn. **1** (2007), 81–100.
[9] J. Peterson, A. Thom, *Group cocylces and the ring of affiliated operators*, preprint (2007), arXiv: 0708.4327.
[10] J. Wilson, *On growth of groups with few relators*, Bull. London Math. Soc. **36** (2004), 1–2.

Exceptional Dehn Filling

Steven Boyer
(joint work with Marc Culler, Cameron Gordon, Peter Shalen, Xingru Zhang)

Many of the basic problems in 3-manifold topology can be analysed in terms of the *Dehn filling operation*, that is, the attaching of a solid torus to a 3-manifold along one of its torus boundary components. An important aspect of this operation is that certain geometric and topological properties of the 3-manifold persist in its Dehn fillings, at least generically. This talk reported on work which illustrates this aspect.

In what follows, M will be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus (a *hyperbolic knot manifold*). A *slope* on ∂M is a ∂M-isotopy class of essential simple closed curves. Slopes can be visualized by identifying them with $\pm$-classes of primitive elements of $H_1(\partial M)$ in the *surgery plane* $H_1(\partial M;\mathbb{R})$. The *distance* $\Delta(r_1, r_2)$ between slopes r_1, r_2 is the absolute value of the algebraic intersection number of their associated classes in $H_1(\partial M)$. Given two sets of slopes $\mathcal{S}_1, \mathcal{S}_2$, set $\Delta(\mathcal{S}_1, \mathcal{S}_2) = \sup\{\Delta(r, s) : r \in \mathcal{S}_1, s \in \mathcal{S}_2\}$ and $\Delta(\mathcal{S}_1) = \Delta(\mathcal{S}_1, \mathcal{S}_1)$.

To each slope r on ∂M we associate the *r-Dehn filling* $M(r) = (S^1 \times D^2) \cup_f M$ of M where $f : \partial(S^1 \times D^2) \to \partial M$ is any homeomorphism such that $f(\{*\} \times \partial D^2)$ represents r. Set

$$\mathcal{E}(M) = \{r \mid M(r) \text{ is not hyperbolic}\}$$

and call the elements of $\mathcal{E}(M)$ *exceptional slopes*. It follows from Thurston's hyperbolic Dehn surgery theorem (see [2, Appendix B]) that $\mathcal{E}(M)$ is finite, while Perelman's solution of the geometrisation conjecture (see [16, 17]) implies that $\mathcal{E}(M) = \{r \mid M(r) \text{ is either reducible, toroidal, or small-Seifert}\}$. Here, a small-Seifert manifold is an irreducible, atoroidal manifold which admits a Seifert structure with base orbifold of the form $S^2(a, b, c)$, where $a, b, c \geq 1$. Over the last thirty years, the following two problems have been the focus of intense research:

(A) Understand the structure of $\mathcal{E}(M)$.
(B) Describe the topology of M when $|\mathcal{E}(M)| \geq 2$.

See the survey [11] for instance. Two results which exemplify what can occur when the situation described in problem (B) arises are Gordon's theorem: *if two toroidal filling slopes are of mutual distance at least* 6*, then M is one of four* *specific manifolds* M_1, M_2, M_3, M_4 [12], and Yi Ni's theorem: *if* M *is the exterior of a knot in the* 3*-sphere which has a non-meridinal slope whose associated filling yields a lens space, then* M *fibres over the circle.* In the talk I concentrated on problem (A). Here, one of the key conjectures is the following:

Conjecture (C.McA. Gordon)**.** *For any hyperbolic knot manifold* M*, we have* $\#\mathcal{E}(M) \leq 10$ *and* $\Delta(\mathcal{E}(M)) \leq 8$*. Moreover, if* $M \neq M_1, M_2, M_3, M_4$*, then* $\#\mathcal{E}(M) \leq 7$ *and* $\Delta(\mathcal{E}(M)) \leq 5$*.*

It is shown in [5] that the conjecture holds if the first Betti number of M is at least 2. (Note that it is at least 1.) Lackenby and Meyerhoff have recently announced a proof that the first statement of the conjecture holds in general [14]. See §2 of this paper for a historical survey of results concerning upper bounds for $\#\mathcal{E}(M)$ and $\Delta(\mathcal{E}(M))$. Agol has shown that there are only finitely many hyperbolic knot manifolds M with $\Delta(\mathcal{E}(M)) > 5$ [1], though there is no practical fashion to determine this finite set.

The results which best illustrate the structure of $\mathcal{E}(M)$ are obtained by considering its subsets *red* (slopes whose fillings are reducible), *tor* (slopes whose fillings are toroidal), and *s-sfrt* (slopes whose fillings are small-Seifert). The latter include *cyc*, resp. *fin*, (slopes whose fillings have cyclic, resp. finite, fundamental groups), and the atoroidal filling slopes in *v-small* (slopes whose fillings have fundamental groups which contain no non-abelian free subgroup).

Sharp upper bounds on the distances $\Delta(\mathit{red}, \mathit{red})$, $\Delta(\mathit{red}, \mathit{cyc})$, $\Delta(\mathit{red}, \mathit{tor})$, $\Delta(\mathit{cyc}, \mathit{cyc})$, $\Delta(\mathit{cyc}, \mathit{fin})$, $\Delta(\mathit{fin}, \mathit{fin})$ and $\Delta(\mathit{tor}, \mathit{tor})$ are known (see, resp., [13], [8], [18] and [19], [10], [7], [9], [12]). Moreover, the configuration of these slopes in the surgery plane satisfies further constraints. For instance, there is a basis $\{\alpha, \beta\}$ of $H_1(\partial M)$ such that $\mathit{red} \cup \mathit{fin}$ is contained in $\pm\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta\}$.

The exceptional slopes whose position in $\mathcal{E}(M)$ is least understood are those in *s-sfrt*. The following theorem concerns their relation to *red*. Recall that a *strict boundary slope* is a slope r on ∂M for which there is an essential surface F, properly embedded in M, such that ∂F is non-empty of slope r and $\pi_1(F)$ is not normal in $\pi_1(M)$. For instance, if $r \in \mathit{red}$ and $M(r) \not\cong S^1 \times S^2, P^3 \# P^3$, then r is a strict boundary slope.

Theorem. *Let* M *be a hyperbolic knot manifold.*

(1) [6] *If* $M(r)$ *is reducible and* $\pi_1(M(s))$ *finite, then* $\Delta(r, s) \leq 1$*;*
(2) [4] *If* $M(r)$ *is reducible and* $\pi_1(M(s))$ *very small, then* $\Delta(r, s) \leq 2$*;*
(3) [4] *If* r *is a strict boundary slope,* $M(r)$ *reducible, and* $M(s)$ *small-Seifert, then* $\Delta(r, s) \leq 4$*.*

Parts (1) and (2) represent sharp upper bounds. The expected sharp bound in part (3) is 2 [15], and though the method of [4] does not yield this, it should be useful in showing that any putative knot manifold for which the bound fails is of a special nature.

The proof of the theorem involves a variety of techniques. For part (1), $\mathrm{PSL}_2(\mathbb{C})$-character variety methods are used in [4] to show if $r \in \mathit{red}$ and $s \in \mathit{fin}$, either $\Delta(r, s) \leq 1$ or $\Delta(r, s) = 2, H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2, M(r) \cong P^3 \# L(3, 1)$ and

$\pi_1(M(s)) \cong O^* \times \mathbb{Z}/j$, where O^* is the binary octahedral group. In [6], they are used to show that when $\Delta(r,s) = 2$, M contains a properly embedded, 4-punctured, essential 2-sphere which splits it into two genus 2 handlebodies. Hence, there is an involution on M with quotient a 3-ball. The involution extends over $M(r)$ with quotient branch set the connected sum of a trefoil and a Hopf link, and over $M(s)$ with quotient branch set a Montesinos link of type $(\frac{p}{2}, \frac{q}{3}, \frac{r}{2})$. On the other hand, since $\Delta(r,s) = 2$, the branch sets differ by a crossing change, which is shown to be impossible. Thus (1) holds.

Part (2) is proven using $\mathrm{PSL}_2(\mathbb{C})$-character variety methods, and in particular the geometry of Culler-Shalen seminorms plays a key role. For part (3), a refined version of the JSJ methods of [3] is used. We mentioned above that the latter should show that any putative knot manifold for which the expected minimal upper bound of 2 fails, is of a special nature. For instance, such manifolds should admit interesting symmetries which would be helpful in ruling out these cases. It seems likely that similar methods will yield good upper bounds for $\Delta(tor, fin)$, $\Delta(tor, v\text{-}small)$, and $\Delta(red, s\text{-}sfrt)$, and this is the goal of a current research project with Gordon and Zhang. For the remaining case $\Delta(s\text{-}sfrt, s\text{-}sfrt)$, new ideas appear to be needed.

References

[1] I. Agol, *Bounds on exceptional Dehn filling II*, preprint (2008), arXiv:0803.3088.
[2] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque **272** (2001).
[3] S. Boyer, M. Culler, P. Shalen, and X. Zhang, *Characteristic subsurfaces and Dehn fillings*, Trans. Amer. Math. Soc. **357** (2005), 2389–2444.
[4] ———, *Characteristic subvarieties, character varieties, and Dehn fillings*, Geom. & Top. **12** (2008), 233–297.
[5] S. Boyer, C. McA. Gordon and X. Zhang, *Dehn fillings of large hyperbolic 3-manifolds*, J. Diff. Geom. **58** (2001), 263–308.
[6] ———, *Reducible and finite Dehn fillings*, J. Lond. Math. Soc., to appear.
[7] S. Boyer and X. Zhang, *Finite Dehn surgery on knots*, J. Amer. Math. Soc. **9** (1996) 1005–1050.
[8] ———, *On Culler-Shalen seminorms and Dehn fillings*, Ann. of Math. **148** (1998), 737–801.
[9] ———, *A proof of the finite filling conjecture*, J. Diff. Geom. **59** (2001), 87–176.
[10] M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987) 237–300.
[11] C. McA. Gordon, *Dehn filling: a survey*, in: Knot Theory, Banach Center Publications **42** (1998) 129–144.
[12] ———, *Boundary slopes of punctured tori in* 3*-manifolds*, Trans. Amer. Math. Soc. **350** (1998) 1713–1790.
[13] C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385–410.
[14] M. Lackenby and R. Meyerhoff, *The maximal number of exceptional Dehn surgeries*, preprint (2008), arXiv:0808.1176.
[15] B. Martelli and C. Petronio, *Dehn filling of the "magic"* 3*-manifold*, Commun. Anal. Geom. **14** (2006), 969–1026.
[16] J. Morgan and G. Tian, *Ricci flow and the Poincaré Conjecture*, Clay Mathematics Monographs **3**, American Math. Society (2007).
[17] ———, *Completion of the proof of the Geometrization Conjecture*, preprint (2008), arXiv:math.DG/0809.4040v1.

[18] S. Oh, *Reducible and toroidal manifolds obtained by Dehn filling*, Top. Appl. **75** (1997), 93–104.
[19] Y.-Q. Wu, *Dehn fillings producing reducible manifolds and toroidal manifolds*, Topology **37** (1998), 95–108.

β-Family congruences and the f-invariant

Gerd Laures
(joint work with Mark Behrens)

In [1], J.F. Adams studied the image of the J-homomorphism

$$J : \pi_t(SO) \to \pi_t^S$$

by introducing a pair of invariants

$$d = d_t : \pi_t^S \to \pi_t K,$$
$$e = e_t : \ker(d_t) \to \mathrm{Ext}_{\mathcal{A}}^{1,t+1}(K_*, K_*),$$

where $\mathcal{A}$ is a certain abelian category of graded abelian groups with Adams operations. (Adams also studied analogs of d and e using real K-theory, to more fully detect 2-primary phenomena.) In order to facilitate the study of the e-invariant, Adams used the Chern character to provide a monomorphism

$$\theta_S : \mathrm{Ext}_{\mathcal{A}}^{1,t+1}(K_*, K_*) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

Thus, the e-invariant may be regarded as taking values in $\mathbb{Q}/\mathbb{Z}$. Furthermore, he showed that for t odd, and $k = (t+1)/2$, the image of θ_S is the cyclic group of order $\mathrm{denom}(B_k/2k)$, where B_k is the k-th Bernoulli number.

The d and e-invariants detect the 0 and 1-lines of the Adams-Novikov spectral sequence (ANSS). In [8], I studied an invariant for $T = \mathrm{TMF}$

$$f : \ker(e_t) \to \mathrm{Ext}_{T_*T}^{2,t+2}(T_*, T_*),$$

which detects the 2-line of the ANSS for π_*^S away from the primes 2 and 3. I furthermore used H. Miller's elliptic character to show that, if t is even and $k = (t+2)/2$, there is a monomorphism

$$\iota^2 : \mathrm{Ext}_{T_*T}^{2,t+2}(T_*, T_*) \hookrightarrow D_{\mathbb{Q}}/\big(D_{\mathbb{Z}} + (M_0)_{\mathbb{Q}} + (M_k)_{\mathbb{Q}}\big).$$

where D is Katz's ring of divided congruences and M_k is the space of weight k modular forms of level 1 meromorphic at the cusp. It is natural to ask for a description of the image of the map ι^2 in arithmetic terms.

Attempting to generalize the J fiber-sequence

$$J \to KO_p \xrightarrow{\psi^\ell - 1} KO_p$$

Mark Behrens introduced a ring spectrum $Q(l)$ built from a length two TMF_p-resolution. In [4], it was shown that for $p \geq 5$, the elements $\beta_{i/j,k} \in (\pi_*^S)_p$ of [11] are detected in the Hurewicz image of $Q(l)$. This gives rise to the association of a modular form $f_{i/j,k}$ to each element $\beta_{i/j,k}$. Furthermore, the forms $f_{i/j,k}$ are characterized by certain arithmetic conditions.

The purpose of this talk is to summarize the various aspects of the f-invariant in terms of characteristic numbers of $(U, \mathrm{fr})^2$-manifolds with corners and spectral invariants and to relate the f-invariant to the work of Mark Behrens. It is proven that $\beta_{i/j,k}$ is given by the formula

$$f(\beta_{i/j,k}) = \frac{f_{i/j,k}}{p^k E_{p-1}^j}.$$

In particular, since the 2-line of the ANSS is generated by the elements $\beta_{i/j,k}$, the p-component of the image of the map ι^2 is characterized by the arithmetic conditions satisfied by the Behrens modular forms $f_{i/j,k}$. The comparison between the e-invariant and the f-invariant is summarized in the following table:

	e-invariant	f-invariant
source	$\ker(d)$	$\ker(e)$
target	$\mathrm{Ext}^{1,*}_{K_*K}(K_*, K_*) \subset \mathbb{Q}/\mathbb{Z}$	$\mathrm{Ext}^{2,*}_{T_*T}(T_*, T_*) \subset \frac{D_{\mathbb{Q}}}{D_{\mathbb{Z}}+(M_0)_{\mathbb{Q}}+(M_*)_{\mathbb{Q}}}$
kernel	2^{nd}-AN filtration	3^{rd}-AN filtration
detects	$\alpha_{i,j}$	$\beta_{i/j,k}$
character	Chern character	Miller character
char. numbers	(U, fr)-manifolds	$(U, fr)^2$-manifolds
spectral inv.	η-invariant	parameterized version of η
image	Bernoulli numbers	Behrens modular forms

The right lower corner was the only missing slot until this work with Mark Behrens. J. Hornbostel and N. Naumann [7] computed the f-invariant of the elements $\beta_{i/1,1}$ in terms of Katz's Artin-Schreier generators of the ring of p-adic modular forms. While their result is best suited to describe f-invariants of infinite families, it is difficult to explicitly get one's hands on their output or to characterize the image algebraically. Direct computations with q-expansions are limited by the computability of q-expansions of modular forms, hence are generally not well suited for infinite families of computations. In low degrees, however, the new formula can directly be used to compute with q-expansions. We demonstrate this by giving some sample calculations of some f-invariants at the prime 5.

References

[1] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71.

[2] M. Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402.

[3] ———, *Buildings, elliptic curves, and the $K(2)$-local sphere*, Amer. J. Math. **129** (2007), no. 6, 1513–1563.

[4] ———, *Congruences between modular forms given by the divided β family in homotopy theory*, preprint (2008).

[5] M. Behrens, G. Laures: *β-family congruences and the f-invariant*, preprint (2008), submitted.

[6] M. Behrens and T. Lawson, *Isogenies of elliptic curves and the Morava stabilizer group*, J. Pure Appl. Algebra **207** (2006), no. 1, 37–49.

[7] J. Hornbostel and N. Naumann, *Beta-elements and divided congruences*, Amer. J. Math. **129** (2007), no. 5, 1377–1402.

[8] G. Laures, *The topological q-expansion principle*, Topology **38** (1999), no. 2, 387–425.

[9] ———, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5667–5688.
[10] ——— *K(1)-local topological modular forms.* Invent. math. **157**, (2004) no. 2 , 371–403.
[11] H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), no. 3, 469–516.
[12] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando (1986).
[13] H. von Bodecker, *On the geometry of the f-invariant*, Ph.D. thesis, Ruhr-University Bochum (2008).

Two-vector bundles

Christian Ausoni
(joint work with Bjørn Ian Dundas and John Rognes)

Two-vector bundles, as defined by Baas, Dundas and Rognes [6], are a 2-categorical analogue of ordinary complex vector bundles. A two-vector bundle of rank n over a space X can be thought of as a locally trivial bundle of categories with fibre $\mathcal{V}^n$, where $\mathcal{V}$ is the bimonoidal category of finite dimensional complex vector spaces and isomorphisms. It can be defined by means of an open cover of X by charts with specified trivialisations, gluing data which represents a weakly invertible matrix of ordinary vector bundles on the intersection of two charts, and coherence isomorphisms on the intersection of three charts [6, §2]. Equivalence classes of two-vector bundles of rank n over a finite CW-complex X are in bijective correspondence with homotopy classes of maps from X to $|B\,\mathrm{GL}_n(\mathcal{V})|$ [5]. By group-completing with respect to the direct sum of matrices we obtain the space

$$K(\mathcal{V}) = \Omega B\Big(\coprod_n |B\,\mathrm{GL}_n(\mathcal{V})|\Big)$$

that represents virtual two-vector bundles. Gerbes with band U(1) coincide with two-vector bundles of rank 1.

Very little is known about the geometry of two-vector bundles. However, by a theorem of Baas, Dundas, Richter and Rognes [7], there is a weak equivalence

$$K(\mathcal{V}) \simeq K(ku)\,,$$

where $K(ku)$ is the algebraic K-theory space of the connective complex K-theory spectrum ku (viewed as a ring in a suitable sense). This permits us to study the space $K(\mathcal{V})$ by means of invariants of algebraic K-theory, like the Bökstedt trace map to topological Hochschild homology, or the cyclotomic trace map to topological cyclic homology. In joint work with Rognes [3, 1] we applied trace methods to compute $K(ku)$ with suitable finite p-primary coefficients for $p \geq 5$. We prove that the spectrum $K(ku)$ is of chromatic complexity 2 in the sense of stable homotopy theory. This means that two-vector bundles define a cohomology theory that, from the view-point of stable homotopy theory, is a suitable candidate for elliptic cohomology. In particular, it is a strictly finer invariant than topological K-theory.

The rational information carried by a two-vector bundle is fairly well understood : it is contained in the associated dimension and determinant bundles. Let $\pi : ku \to \mathbb{Z}$ be the unique ring-map that is a π_0-isomorphism, and let

$$\pi : K(ku) \to K(\mathbb{Z})$$

be the induced map in algebraic K-theory. This map represents the forgetful map that associates to a two-vector-bundle its dimension bundle, or decategorification. There is also a rational determinant map [4]

$$\det_{\mathbb{Q}} : K(ku) \to B\,\mathrm{SL}_1(ku)_{\mathbb{Q}}\,.$$

Up to homotopy, this is a rational retraction of the "inclusion of units" map

$$w : B\,\mathrm{SL}_1(ku) \to K(ku)\,.$$

Thus, any virtual two-vector bundle has an associated (rational) determinant bundle. We proved in [4] that the maps π and $\det_{\mathbb{Q}}$ define a rational equivalence

$$K(ku)_{\mathbb{Q}} \simeq B\,\mathrm{SL}_1(ku)_{\mathbb{Q}} \times K(\mathbb{Z})_{\mathbb{Q}}\,.$$

The space of units $\mathrm{SL}_1(ku)$ is equivalent as an infinite loop-space to the space $B\mathrm{U}_{\otimes}$ representing virtual complex line bundles and their tensor product. By a result of Borel [8], the space $K(\mathbb{Z})$ is rationally equivalent to $\mathbb{Z} \times \mathrm{SU}/\mathrm{SO}$.

The map $\pi : K(ku) \to K(\mathbb{Z})$ is 3-connected, from which we deduce that $K_1(ku) \cong \mathbb{Z}/2$ and $K_2(ku) \cong \mathbb{Z}/2$. We expect that in higher degrees, the integral homotopy groups of $K(ku)$ will reflect the high complexity of $K(\mathbb{Z})$ and of some of the v_2-periodic families in the stable homotopy groups of spheres [3, §9]. This is illustrated in the following example. An obvious and meaningful invariant to detect higher dimensional classes in $K_*(ku)$ would be a determinant map $\det : K(ku) \to B\,\mathrm{SL}_1(ku)$ that is an (integral) homotopy retraction of w. However, as observed by Dundas and Rognes, such a map cannot possibly exist : a first obstruction to its existence is an intriguing virtual two-vector bundle ς on the sphere S^3. In effect, we show in [2] that there is an isomorphism of Abelian groups

$$K_3(ku) \cong \mathbb{Z} \oplus \mathbb{Z}/24\,,$$

where the torsion free summand is generated by ς, and the torsion subgroup is generated by a class named ν (the image of the class with the same name in the stable homotopy of S^3). We prove that the U(1)-gerbe μ over S^3 representing the fundamental class in $H^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$ (also known as Dirac's magnetic monopole) decomposes as

$$\mu = 2\varsigma - \nu \in K_3(ku)$$

when viewed as a virtual two-vector bundle of rank one. Therefore, the element ς classifies a virtual two-vector bundle over S^3 that, modulo torsion, is half the magnetic monopole. Its associated dimension bundle is a generator of $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$.

References

[1] Ch. Ausoni, *On the algebraic K-theory of the connective complex K-theory spectrum*, preprint (2008).

[2] Ch. Ausoni, B.I. Dundas and J. Rognes, *Divisibility of the Dirac magnetic monopole as a two-vector bundle over the three-sphere*, preprint (2008), to appear in Doc. Math..

[3] Ch. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), 1–39.

[4] Ch. Ausoni and J. Rognes, *Rational algebraic K-theory of topological K-theory*, preprint (2007), arXiv: math.KT/0708.2160.

[5] N.A. Baas, M. Bökstedt and T.A. Kro, *Two-categorical bundles and their classifying spaces*, preprint (2006), arXiv: math/0612549.

[6] N.A. Baas, B.I. Dundas and J. Rognes, *Two-vector bundles and forms of elliptic cohomology*, in *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser. **308**, Cambridge Univ. Press (2004), 18–45.

[7] N.A. Baas, B.I. Dundas, B. Richter and J. Rognes, *Two-vector bundles define a form of elliptic cohomology*, preprint (2007), arXiv: math.KT/0706.0531.

[8] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272.

Wreath products and representations of p-local finite groups

Assaf Libman

(joint work with Natàlia Castellana)

A classical result which goes back to Eilenberg-MacLane states that $H^*(G;\mathbb{F}_p)$ depends only on the non-trivial p-subgroups of G and the homomorphisms between them which are induced by conjugation by elements of G. This information is contained in a small category known as the *fusion system* $\mathcal{F}_S(G)$ whose objects are the p-subgroups of a fixed Sylow p-subgroup S of G. Its morphisms are the morphisms induced by conjugation in G.

Abstraction of this construction led Puig to consider saturated fusion systems on a p-group S. A fusion system $\mathcal{F}$ on S is a small category whose objects are the subgroups of S. The morphisms in $\mathcal{F}$ are group monomorphisms which contain all the homomorphisms induced by conjugation in S. In addition every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion. Two groups that are isomorphic in $\mathcal{F}$ are called $\mathcal{F}$-conjugate.

Definition 1. Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which are $\mathcal{F}$-conjugate to P. A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which are $\mathcal{F}$-conjugate to P.

A fusion system $\mathcal{F}$ on S is *saturated* if:

(1) Each fully normalized subgroup $P \leq S$ is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(2) For $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$ set
$$N_\varphi = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}.$$
If $\varphi(P)$ is fully centralized then there is $\bar{\varphi} \in \mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

The combinatorial model for $BG_p^\wedge$ was offered by Broto, Levi and Oliver in [2] in the form of a linking system associated to a saturated fusion system. The idea goes back to a work of Puig.

Definition 2. Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers.

Definition 3. Let $\mathcal{F}$ be a fusion system on a p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ and monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(1) π is the identity on objects. For each pair of objects $P, Q \in \mathcal{L}$, the action of $Z(P)$ on $\mathcal{L}(P,Q)$ via precomposition and $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ is free and π induces a bijection $\mathcal{L}(P,Q)/Z(P) \xrightarrow{\cong} \mathcal{F}(P,Q)$.
(2) If $P \leq S$ is $\mathcal{F}$-centric then $\pi(\delta_P(g)) = c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$ for all $g \in P$.
(3) For each $f \in \mathcal{L}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"'] & Q. \end{tikzcd}$$

A *p-local finite group* $(S, \mathcal{F}, \mathcal{L})$ consists of a saturated fusion systems $\mathcal{F}$ on S together with an associated linking system. The *classifying space* of $(S, \mathcal{F}, \mathcal{L})$ is $|\mathcal{L}|_p^\wedge$. The homomorphism $\delta_S\colon S \to \mathrm{Aut}_{\mathcal{L}}(S)$ induces an "inclusion" map $\Theta\colon BS \to |\mathcal{L}|_p^\wedge$.

It turns out that p-local finite groups form a more general framework to study p-local phenomena in finite groups. There are fruitful connections with Representation Theory.

Definition 4. Given a space X and a subgroup $K \leq \Sigma_n$ let $X \wr K$ denote the homotopy orbits space of the action of K by permuting the factors of X^n; In symbols $(X^n)_{hK}$.

Theorem 5. *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S \neq 1$. Let K be a subgroup of Σ_n and let S' be a Sylow p-subgroup of $S \wr K$. Then there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ which is equipped with a homotopy equivalence $|\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ such that the composite*

$$BS' \xrightarrow{B\,\mathrm{incl}} B(S \wr K) \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$$

is homotopic to the natural map $BS' \xrightarrow{\Theta'} |\mathcal{L}'|$. Moreover, $(S', \mathcal{F}', \mathcal{L}')$ satisfying these properties is unique up to an isomorphism of p-local finite groups.

The proof of this theorem uses the ideas in [1, Thorem 4.6]. We call the p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ in the theorem above the *wreath product* of $(S, \mathcal{F}, \mathcal{L})$ with K and denote its fusion system and linking system by $\mathcal{F} \wr K$ and $\mathcal{L} \wr K$

respectively. Let $\Delta\colon |\mathcal{L}| \to |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ denote the diagonal inclusion followed by the homotopy equivalence in Theorem 5.

Definition 6. Let $\mathcal{F}, \mathcal{F}'$ be saturated fusion systems on S, S' respectively. A homomorphism $\varphi\colon S \to S'$ is called *fusion preserving* if for any morphism $\psi \in \mathcal{F}(P,Q)$ there exists some $\psi' \in \mathcal{F}'(\varphi(P), \varphi(Q))$ such that the following square commutes

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\varphi|_P"'] & Q \arrow[d, "\varphi|_Q"] \\ \varphi(P) \arrow[r, "\psi'"'] & \varphi(Q) \end{tikzcd}$$

It is a natural question to ask if $B\varphi$ "extends" to a map between the classifying spaces. This is known to be true stably. We have, however, the following result.

Theorem 7. *Let $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ be p-local finite groups and suppose that $\rho\colon S \to S'$ is a fusion preserving homomorphism. Then there exists some $m \geq 0$ and a map $\tilde{f}\colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}' \wr \Sigma_{p^m}|^\wedge_p$ such that the diagram below commutes up to homotopy*

$$\begin{tikzcd} BS \arrow[r, "\eta\circ\Theta"] \arrow[d, "B\rho"'] & {|\mathcal{L}|^\wedge_p} \arrow[dr, "\tilde{f}"] & \\ BS' \arrow[r, "\eta\circ\Theta'"'] & {|\mathcal{L}'|^\wedge_p} \arrow[r, "\Delta^\wedge_p"'] & {|\mathcal{L}' \wr \Sigma_{p^m}|^\wedge_p.} \end{tikzcd}$$

As a corollary we obtain a p-local version of Cayley's theorem. Recall that a map $f\colon X \to Y$ of topological spaces is called a homotopy monomorphism at p if $H^*(X; \mathcal{F}_p)$ is a finitely generated $H^*(Y; \mathbb{F}_p)$-module via f^*.

Theorem 8. *Any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ admits a map $f\colon |\mathcal{L}| \to (B\Sigma_{p^k})^\wedge_p$ for some k which is a homotopy monomorphism at p.*

References

[1] C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver, *Extensions of p-local finite groups*, Trans. Amer. Math. Soc. **259** (2007), 3791–3858.

[2] C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856

4-Manifolds BIG and small: Reverse engineering smooth 4-Manifolds

Ronald J. Stern
(joint work with Ronald A. Fintushel)

Recent work with Ronald A. Fintushel has suggested a classification scheme for simply-connected smooth 4-manifolds. In its most ambitious form, our work suggests that all simply-connected smooth 4-manifolds are obtained from

$$S^4 \#_k \mathbb{CP}^2 \#_\ell \overline{\mathbb{CP}}^2 \#_s (S^2 \times S^2) \#_t K3$$

via a sequence of surgeries on null-homologous tori. Even more optimistic would be that the Seiberg-Witten invariants distinguish these manifolds up to finite ambiguity. As some evidence for this classification scheme, this talk discussed techniques that go a long way towards proving that if the Seiberg-Witten invariants of a simply-connected smooth 4-manifold X are non-trivial, then X has infinitely many distinct smooth structures. An even more optimistic conjecture would be that every simply-connected topological 4-manifold has either no or infinitely many smooth structures.

This talk discussed a technique which we call *reverse engineering* that can be used to construct infinite families of distinct smooth structures on many 4-manifolds. This is more fully discussed in [7]. Reverse engineering is a three step process for constructing infinite families of distinct smooth structures on a given simply connected 4-manifold. One starts with a model manifold which has non-trivial Seiberg-Witten invariant and the same euler number and signature as the simply connected manifold X that one is trying to construct, but with $b_1 > 0$. The second step is to find b_1 essential tori that carry generators of H_1 and to surger each of these tori in order to kill H_1 and, in favorable circumstances, to kill π_1. The third step is to compute Seiberg-Witten invariants. After each of the first $b_1 - 1$ surgeries one needs to preserve the fact that the Seiberg-Witten invariant is nonzero. The fact that the next to last manifold in the string of surgeries has non-trivial Seiberg-Witten invariant allows the use of the Morgan, Mrowka, Szabó formula [10] to produce an infinite family as was done in [9].

In many instances this procedure can be successfully applied without any computation, or even mention, of Seiberg-Witten invariants. If the model manifold for X is symplectic and $b_1 - 1$ of the tori are lagrangian so that a Luttinger surgery will reduce b_1, then there are infinitely many distinct smooth manifolds with the same cohomology ring as X. If the resulting manifold is simply connected, then one can often show that there are infinitely many distinct smooth structures on X. Aside from finding interesting model manifolds, it seems that the most difficult aspect to the reverse engineering procedure is the computation of fundamental groups.

The conjecture that if the Seiberg-Witten invariants of a simply-connected smooth 4-manifold X are non-trivial, then X has infinitely many distinct smooth structures has been known to be true under the further hypothesis that X contains an embedded torus with trivial normal bundle and simply-connected complement (cf. [8]). It is also known that $\mathbb{CP}^2\#\, k\overline{\mathbb{CP}}^2$ for $k < 9$ contains no such tori, so these manifolds are a good testbed for reverse engineering.

Models for these manifolds are obtained as follows. One can easily check that $X_r \#_{\Sigma_2} X_s$ is a model for $\mathbb{CP}^2\#(r+s+1)\,\overline{\mathbb{CP}}^2$, the fiber sum taken along a genus two surface Σ_2 embedded in X_r and X_s. Here

- $\Sigma_2 \subset X_0 = T^2 \times \Sigma_2$ representing $(0,1)$
- $\Sigma_2 \subset X_1 = T^2 \times T^2\#\overline{\mathbb{CP}}^2$ representing $(2,1) - 2e$
- $\Sigma_2 \subset X_2 = T^2 \times T^2\#2\overline{\mathbb{CP}}^2$ representing $(1,1) - e_1 - e_2$
- $\Sigma_2 \subset X_3 = S^2 \times T^2\#3\overline{\mathbb{CP}}^2$ representing $(1,3) - 2e_1 - e_2 - e_3$
- $\Sigma_2 \subset X_4 = S^2 \times T^2\#4\overline{\mathbb{CP}}^2$ representing $(1,2) - e_1 - e_2 - e_3 - e_4$
- Exception: $X_0\#_{\Sigma_2}X_0 = \Sigma_2 \times \Sigma_2$ is a model for$S^2 \times S^2$

One can check that there are enough lagrangian tori to kill H_1 in $X_r\#_{\Sigma_2}X_s$. The art is to find appropriate tori so that the result has $\pi_1 = 0$.

The first successful implementations of this strategy for $\mathbb{CP}^2\#\,3\overline{\mathbb{CP}}^2$ (i.e., find tori and show surgery on the model manifold results in $\pi_1 = 0$) were obtained by Baldridge-Kirk [5] and Akhmedov-Park [3]. The full implementation (i.e., infinite families) for $\mathbb{CP}^2\#\,3\overline{\mathbb{CP}}^2$ was obtained by Fintushel-Park-Stern [7] using the 2-fold symmetric product $\mathrm{Sym}^2(\Sigma_3)$ as model. Also, Akhmedov-Park [4] have a paper under review to implement this strategy for $\mathbb{CP}^2\#2\,\overline{\mathbb{CP}}^2$ (i.e., show surgery on the model manifold results in $\pi_1 = 0$). The full implementation (i.e., infinite families) for $\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$, $4 \le k \le 9$, can be obtained through a combination of papers by Baldridge-Kirk [6], Akhmedov-Park [3], Akhmedov-Baykur-Baldridge-Kirk-Park [1], and Ahkmedov-Baykur-Park [2].

Here are some possible next steps.

- Find a model for $\mathbb{CP}^2$; also obtain a topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{CP}^2\#\,\overline{\mathbb{CP}}^2$; $\mathbb{CP}^2\#\,2\overline{\mathbb{CP}}^2$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$ obtained by surgery on null-homologous tori in the standard $\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$? (See [9] for first attempts.)
- Are two homeomorphic simply-connected smooth 4-manifolds related by a sequence of logarithmic transforms on (null-homologous) tori?
- Are all 4- manifolds obtained from either $\ell\,\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$ or $nE(2)\#m\,(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

References

[1] A. Akhmedov, S. Baldridge, I. Baykur, P. Kirk, B.D. Park, *Simply connected minimal symplectic 4-manifolds with signature less than* -1, preprint, arXiv: math/0705.0778.

[2] A. Akhmedov, I. Baykur, B.D. Park, *Constructing infinitely many smooth structures on small 4-manifolds*, Journal of Topology **1** (2008), 409–428.

[3] A. Akhmedov and B.D. Park, *Exotic smooth structures on small* 4*-manifolds*, Invent. math. **173** (2008), 209–223.

[4] A. Akhmedov and B.D. Park, *Exotic smooth structures on small* 4*-manifolds with odd signatures*, preprint, arXiv: math/0701829.

[5] S. Baldridge and P. Kirk, *A symplectic manifold homeomorphic but not diffeomorphic to* $\mathbb{CP}^2\#\,3\overline{\mathbb{CP}}^2$, Geom. Topol. **12** (2008), 919–940.

[6] S. Baldridge and P. Kirk, *Constructions of small symplectic 4-manifolds using Luttinger surgery*, preprint, arXiv: math.GT/0703065.
[7] R. Fintushel,B.D. Park and R. Stern, *Reverse engineering small 4-manifolds*, Algebraic & Geometric Topology **7** (2007), 2103–2116.
[8] R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Invent. math. **134** (1998) 363–400.
[9] R. Fintushel and R. Stern, *Surgery on nullhomologous tori and simply connected 4-manifolds with* $b^+ = 1$, Journal of Topology **1** (2008), 1–15.
[10] J. Morgan, T. Mrowka, and Z. Szabó, *Product formulas along* T^3 *for Seiberg-Witten invariants*, Math. Res. Letters **4** (1997) 915–929.

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