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Arbeitsgemeinschaft: Ricci Flow and the Poincare Conjecture

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Abstract. It was the aim of this workshop to introduce the participants to the basic concepts and techniques of Hamilton's Ricciflow programme and cover the main ideas of the proof of the Poincaré conjecture. This was accomplished by having participants present segments of the proof, where an effort was made to include as much detail as possible.

Mathematics Subject Classification (2000): 53C44, 57M40.

Introduction by the Organisers

The Ricciflow, introduced by Richard Hamilton [Ha1], is a geometric evolution equation which deforms the metric on a Riemannian manifold smoothly in the direction of its Ricci curvature. More precisely, the evolution equation for the family of metrics (g_{ij}) on a manifold M is given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{0.1}$$

where R_{ij} denotes the Ricci tensor corresponding to the metric. Written in suitable local coordinates this equation has the form of a nonlinear heat type equation for the metric symbols. Because of this one might naively expect that the equation will try to evolve the geometry on M to one which looks the same at every point on the manifold, a homogeneous geometry. This intuition is correct in dimension two, where the Ricciflow can be used [Ha2], [Ch] to conformally deform any metric on a closed surface to one of constant curvature, which provides a new proof of the famous uniformization theorem.

In higher dimensions, however, the geometry will in general become singular in finite time, i.e. the norm of some of the sectional curvatures will tend to infinity at certain points on the manifold. In three dimensions, if the initial metric has positive Ricci curvature and M is closed and simply connected, Hamilton [Ha1] showed that, after suitable rescaling of the evolving metric, such as to keep the volume of the manifold constant, the metric tends smoothly to the metric on the standard S^3.

Soon after that, Hamilton [Ha3] set up a programme which had the aim of settling Thurston's geometrization conjecture using Ricciflow. This conjecture asks whether any closed 3-manifold can be decomposed along 2-spheres and incompressible tori in such a way that after capping of the 2-sphere boundaries by 3-balls, the resulting finitely many geodesically complete pieces would each admit one out of a list of eight homogeneous geometries formulated by Thurston [Th]. In particular, this would prove the famous Poincaré conjecture, that any simply connected orientable closed 3-manifold had to be topologically equivalent to S^3.

Hamilton himself, but also many others, completed many of the crucial steps in this programme (see [CLN]) but several severe technical difficulties remained unsettled for at least one decade. In 2002, Perelman [P1] - [P3] introduced a number of completely novel ideas and techniques that eventually led to the resolution of the geometrization and hence also the Poincaré conjecture.

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Workshop: Arbeitsgemeinschaft: Ricci Flow and the Poincare Conjecture

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Abstracts

Topology of canonical manifolds

Thomas Schick

The talk's aim is the introduction and investigation of canonical neighborhoods in Riemannian 3-manifolds. Such a canonical neighborhood either is an ϵ-neck, diffeomorphic to $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$, or a neck with a cap (making the neighborhood diffeomorphic to B^3 or to $\mathbb{R}P^3 \setminus \{pt\}$). The metric of the ϵ-neck or ϵ-cap is assumed to be ϵ-close to the standard metric (after rescaling). We classify all manifolds such that each point has a canonical ϵ-neighborhood (provided $\epsilon << 1$). We also derive useful geometric properties of these neighborhoods (again for $\epsilon << 1$). These investigations are relevant as the canonical neighborhoods occur as scaling limits of Ricci flow near the development of singularities. The talk closely follows the discussion in the appendix of [1].

References

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Derivation of evolution equations

Adrian Hammerschmidt

The Ricci Flow is given by the equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \tag{0.1}$$

and in local coordinates

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{0.2}$$

Therefore everything which depends on the metric also depends on time and thus flows by an evolution equation. This lecture mainly derives the evolution equations under Ricci Flow for the Riemannian curvature tensor, curvature operator, and scalar curvature.

The lecture follows Hamilton's sign convention; that is

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z. \tag{0.3}$$

Lemma 1. *The variation of the Christoffel symbols is given by the equation*

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}). \tag{0.4}$$

The variation of inverse metric is given by

$$\frac{\partial}{\partial t} g^{ij} = 2g^{ik} g^{jl} R_{kl}. \tag{0.5}$$

The variation of the Riemannian curvature tensor is given by

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}{}^{l} = g^{lm}(&\nabla_i\nabla_j R_{km} + \nabla_i\nabla_k R_{jm} - \nabla_i\nabla_m R_{jk}\\ &- \nabla_j\nabla_i R_{km} - \nabla_j\nabla_k R_{im} + \nabla_j\nabla_m R_{ik}),\end{aligned} \tag{0.6}$$

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = {}&\nabla_i\nabla_k R_{jl} + \nabla_j\nabla_l R_{ik} - \nabla_i\nabla_l R_{jk}\\ &- \nabla_j\nabla_k R_{il} - R_{ijk}{}^{q}R_{ql} + R_{ijl}{}^{q}R_{qk}.\end{aligned} \tag{0.7}$$

The scalar curvature evolves under Ricci Flow according to the equation

$$\frac{\partial}{\partial t}\operatorname{scal} = \Delta\operatorname{scal} + 2\left|\mathrm{Ric}\right|^2. \tag{0.8}$$

The volume element evolves under Ricci Flow according to

$$\frac{\partial}{\partial t}d\mu = -\operatorname{scal} d\mu. \tag{0.9}$$

Remark 2. *The equation (0.8) can be used to derive the following evolution inequality:*

$$\frac{\partial}{\partial t}\operatorname{scal} \geq \Delta\operatorname{scal} + 2\frac{\operatorname{scal}^2}{n} \tag{0.10}$$

Let's try to rewrite the evolution equation of (0.4) in a simpler form, namely a diffusion-reaction equation.

Lemma 3. *The evolution equation of the Riemannian curvature tensor (0.4) can be rewritten in the form*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = {}&\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\\ &- (R_i^q R_{qjkl} + R_j^q R_{iqkl} + R_k^q R_{ijql} + R_l^q R_{ijkq}),\end{aligned} \tag{0.11}$$

where $B_{ijkl} = g^{rt}g^{su}R_{irjs}R_{ktlu}$.

Remark 4. *Note that there are the following symmetries:*

$$B_{ijkl} = B_{klij} = B_{jilk} \tag{0.12}$$

Next we try to loose the tail in (0.11). This can be done by using Uhlenbeck's trick. The idea is to move an orthonormal frame in such a way, that it stays orthonormal.
So, think of an abstract vector bundle that is isomorphic via φ_0 to TM. Then define a metric $h_0 = \varphi_0^* g_0$. This gives a bundle isometry.
Now, let $\varphi(t)$ be a solution of

$$\begin{aligned}\frac{\partial}{\partial t}\varphi &= \mathrm{Ric}\circ\varphi,\\ \varphi(0) &= \varphi_0.\end{aligned} \tag{0.13}$$

This yields immediately the following:

Corollary 5. $\varphi(t)$ *stays an isometry and* $h(t) := \varphi(t)^* g(t) = h_0 = const.$

Now define a connection

$$\hat{\nabla} := \varphi^*\nabla : \Gamma(TM) \times \Gamma(V) \to \Gamma(V) \quad \text{by} \tag{0.14}$$

$$\hat{\nabla}_X \xi = (\varphi^*\nabla_X)(\xi) = \varphi^{-1}(\nabla_X(\varphi(\xi))). \tag{0.15}$$

Next pull back the Riemannian curvature tensor. Let $\{e_a\}$ denote a basis of V and $\varphi(e_a) = \varphi_a^k \frac{\partial}{\partial x_k}$. We thus have

$$R_{abcd} := (\varphi^* Rm)(e_a, e_b, e_c, e_d) = \varphi_a^i \varphi_b^j \varphi_c^k \varphi_d^l R_{ijkl}. \tag{0.16}$$

Theorem 6. *Let $g(t)$ be a solution of Ricci Flow and $\varphi(t)$ a solution of (0.13). Then we have*

$$\frac{\partial}{\partial t} R_{abcd} = \hat{\Delta} R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{0.17}$$

where

$$\hat{\Delta} := tr_g(\hat{\nabla} \circ \hat{\nabla}) = g^{ij}(\hat{\nabla})_i(\hat{\nabla})_j \text{ and}$$
$$B_{abcd} := h^{eg} h^{fh} R_{aebf} R_{cgdh}.$$

The curvature operator is well defined by

$$\mathcal{R} : \Lambda^2 TM \to \Lambda^2 TM,$$
$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle := R(X, Y, Z, W). \tag{0.18}$$

Thus we can define the square of the curvature operator and also the Lie Algebra square. It can be shown that

$$\mathcal{R}^2 = B_{ijkl} - B_{ijlk}$$
$$\mathcal{R}^\# = B_{ikjl} - B_{iljk},$$

and therefore we obtain the evolution equation

$$\frac{\partial}{\partial t}\mathcal{R} = \Delta \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^\#. \tag{0.19}$$

In dimension three we obtain the associated ODE

$$\frac{\partial}{\partial t} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} = \begin{pmatrix} \lambda^2 + \mu\nu & 0 & 0 \\ 0 & \mu^2 + \lambda\nu & 0 \\ 0 & 0 & \nu^2 + \lambda\mu \end{pmatrix}. \tag{0.20}$$

The maximum principle

Christian Bär

Throughout this talk M denotes a compact manifold without boundary and $T > 0$.

1. Scalar maximum principle

Fix a function $F : \mathbb{R} \times [0, T) \to \mathbb{R}$ satisfying a local Lipschitz condition. Let L be a differential operator acting on sufficiently regular functions $u : M \times [0, T) \to \mathbb{R}$ by

$$Lu = \frac{\partial u}{\partial t} - \sum_{ij} a^{ij}(x,t)\partial_i \partial_j u - \sum_i \xi^i(x,t)\partial_i u - F(u,t),$$

where (a^{ij}) is positive semidefinite. E.g. we can have $(a^{ij}) = (g^{ij})$, the inverse of a time dependent Riemannian metric on M and

$$Lu = \frac{\partial u}{\partial t} - \Delta u - \nabla_X u - F(u,t),$$

where $X(x,t)$ is a time dependent vector field on M.

Theorem 1. *Let $u : M \times [0, T) \to \mathbb{R}$ be a C^2-function satisfying*

$$Lu \geq 0 \text{ and } u(\cdot, 0) \geq c \in \mathbb{R}.$$

Let $v : [0, T) \to \mathbb{R}$ be the solution to the ODE $\frac{dv}{dt} = F(v,t)$ with initial condition $v(0) = c$. Then

$$u(x,t) \geq v(t) \text{ for all } (x,t) \in M \times [0,T).$$

Let us apply this to Ricci flow; i.e., let $\frac{\partial g}{\partial t} = -2\operatorname{Ric}_{g(t)}$. From the previous talk we know that scalar curvature satisfies

$$\frac{\partial R}{\partial t} = \Delta R + 2|\operatorname{Ric}|^2 \geq \Delta R + \frac{2}{n}R^2.$$

Applying theorem 1 with $F(x,t) = \frac{2}{n}x^2$ we get

$$R(x,t) \geq \frac{n\rho_0}{n - 2\rho_0 t}, \tag{1.1}$$

where $\rho_0 = \min_{x \in M} R(x, 0)$. Several remarks on this estimate:

- If the initial metric $g(0)$ is Einstein, then $g(t) = (1 - \frac{2\rho_0}{n}t)g(0)$ and we have equality in (1.1).
- If $R(\cdot, 0) \geq 0$ on M, then $R \geq 0$ on $M \times [0,T)$.
- If $\rho_0 > 0$, then we must have $t < \frac{n}{2\rho_0}$; i.e.,

$$T \leq \frac{n}{2\rho_0} < \infty.$$

- Letting $\rho_0 \to -\infty$ in (1.1) we get the estimate

$$R(x,t) > -\frac{n}{2t}$$

without any assumption on the scalar curvature of the initial metric. The longer Ricci flow extends to the past, the less negative scalar curvature can be.

- In particular, if $g(t)$ is an ancient solution, then $R \geq 0$.

See [1, Ch. 2.3] for the material of this section.

2. Maximum principle for symmetric 2-tensors

If α and β are symmetric 2-tensors (bilinear forms) on a Euclidean vector space, then we write $\alpha < (\leq)\beta$ iff $\beta - \alpha$ is positive (semi) definite.

Let $g(t)$ be a smooth 1-parameter family of Riemannian metrics on M, let X be a time-dependent vector field on M. Let $F : \mathrm{Sym}^2(TM) \times_M \mathrm{Sym}^2(TM) \to \mathrm{Sym}^2(TM)$ be a fiber preserving map which is locally Lipschitz. Here $\mathrm{Sym}^2(TM)$ denotes the bundle of time-dependent symmetric 2-tensors on TM over $M \times [0,T)$. Assume that F satisfies the *null-eigenvector assumption*, i.e., if $\alpha(x,t) \geq 0$ and if $\alpha(x,t)(V,\cdot) = 0$, then

$$F(\alpha(x,t), g(x,t))(V,V) \geq 0,$$

$V \in T_xM$.

Theorem 2. *If α is a smooth 1-parameter family of symmetric 2-tensors such that*

$$\frac{\partial \alpha}{\partial t} \geq \Delta_{g(t)}\alpha + \nabla_{X(t)}\alpha + F(\alpha, g),$$

and if $\alpha(\cdot, 0) \geq 0$ on M, then $\alpha \geq 0$ on $M \times [0,T)$.

In dimension 3 this can be applied to $\alpha = \mathrm{Ric}$ and shows that nonnegativity of Ricci curvature persists under Ricci flow. Using $\alpha = \frac{1}{2}Rg - \mathrm{Ric}$ one can see that nonnegativity of sectional curvature persists in dimension 3.

See [1, Ch. 3.2] for the material of this section.

3. Maximum principle for sections in vector bundles

Let $V \to M$ be a vector bundle with a time-independent Riemannian metric and a smooth 1-parameter family of metric connections $\nabla^{(t)}$. We denote the associated Laplacians by $\Delta^{(t)}$. Let $\mathcal{K} \subset V$ be such that

- $\mathcal{K}_x := \mathcal{K} \cap V_X$ is closed and convex for each $x \in M$.
- $\mathcal{K}$ is invariant under parallel transport with respect to each $\nabla^{(t)}$.

Let X be a time-dependent vector field on M. Let $F : V \times [0,T) \to V$ be a fiber preserving local Lipschitz map.

Theorem 3. *Suppose that any solution $U : [0,T) \to V_x$ of the ODE*

$$\frac{dU}{dt} = F_x(U,t) \text{ with } U(0) \in \mathcal{K}_x$$

exists and satisfies $U(t) \in \mathcal{K}_x$ for all $t \in [0,T)$ and all $x \in M$. Then any C^2-solution $u(x,t)$ of the PDE

$$\frac{\partial u}{\partial t} = \Delta^{(t)}u + \nabla^{(t)}_{X(t)}u + F(u,t)$$

with $u(x,0) \in \mathcal{K}_x$ *for all* $x \in M$ *satisfies* $u(x,t) \in \mathcal{K}_x$ *for all* $(x,t) \in M \times [0,T)$.

See [2, 3]. There are various versions of this theorem. E. g. one can allow $\mathcal{K}$ to vary with time.

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Curvature estimates preserved under Ricci flow

Christian Becker

In this talk, we discuss several curvature estimates that are preserved or even improved during the evolution of the metric under the Ricci flow. All the results presented here appear as applications of the maximum principle for sections in vector bundles, as presented in the previous talk.

The vector bundle in question is the symmetric tensor product of 2-forms $E := \Lambda^2 M \otimes_{sym} \Lambda^2 M$. The Riemannian curvature operator Rm may be considered likewise as a section of E or (using the induced metric on $\Lambda^2 M$) as an operator $Rm : \Lambda^2 M \to \Lambda^2 M$.

As an application of the maximum principle for sections in E, we need to discuss the ODE system

$$\frac{d}{dt}A = A^2 + A^{\#} \tag{0.1}$$

fibrewise in E, i.e. for symmetric $N \times N$ matrices $A \in E_x$, $x \in M$, $N = \frac{n(n-1)}{2}$. The maximum principle then says the following: Let $\mathcal{K} \subset E$ be a closed and fibrewise convex set preserved under parallel transport with respect to a smooth family $\nabla^{(t)}$ of metric connections on E. Suppose that the ODE system (0.1) has the property that any of its solutions A with $A(0) \in \mathcal{K}$ exists on $[0,T)$ and satisfies $A(t) \in \mathcal{K}$ for all $t \in [0,T)$. Then if $g(t)$ solves the Ricci flow equation on $[0,T)$, and if $Rm(0) \in \mathcal{K}$, then $Rm(t) \in \mathcal{K}$ for all $t \in [0,T)$.

1. Static curvature estimates

Throughout this section, let $(M^3, g(t))$ be a solution to the Ricci flow equation on $[0,T)$. We denote by $\lambda_1(Rm) \le \lambda_2(Rm) \le \lambda_3(Rm)$ the eigenvalues of the Riemannian curvature operator Rm.

Note that in dimension $n = 3$, we have $N = 3$, and for a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

the right hand side of the ODE system (0.1) reads

$$A^2 + A^\# = \begin{pmatrix} \lambda_1^2 + \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_1\lambda_2 \end{pmatrix}.$$

Hence if $A(0)$ is a diagonal matrix, then $A(t)$ stays diagonal under the ODE system (0.1). Also the ordering of the eigenvalues is preserved by the ODE system (0.1).

The following examples of closed, fibrewise convex sets $\mathcal{K}$ yield curvature estimates preserved by the Ricci flow. Note that since all those sets are defined in terms of conditions on the eigenvalues of Rm, they are preserved by the parallel transport with respect to the metric connections $\nabla^{(t)}$.

- For $C_0 \in \mathbb{R}$, set $\mathcal{K} := \{A \mid \lambda_1(A)+\lambda_2(A)+\lambda_3(A) \geq C_0\}$. Since the function $A \mapsto \lambda_1(A) + \lambda_2(A) + \lambda_3(A)$ is linear, the sets $\mathcal{K}_x$, $x \in M$, are convex. Clearly, $\mathcal{K}$ is preserved by the ODE system (0.1). Hence a lower bound $R \geq C$ for the scalar curvature is preserved under the Ricci flow.
- For $C_0 \geq 0$, the set $\mathcal{K} := \{A \mid \lambda_1(A) \geq C_0\}$ is closed and fibrewise convex, since $\lambda_1(A) := \min_{|v|=1}(Av, v)$ is a concave function. The relevant equation of the system (0.1) is

$$\frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2\lambda_3 \geq 0\,,$$

which implies that $\mathcal{K}$ is preserved by the solutions of (0.1). Hence a nonnegative lower bound for the Riemannian curvature operator is preserved under the Ricci flow.
- The set $\mathcal{K} := \{A \mid \lambda_1(A)+\lambda_2(A) \geq 0\}$ is fibrewise convex, since the function

$$A \mapsto (\lambda_1(A) + \lambda_2(A)) = \min_{\substack{v_1 \perp v_2 \\ |v_1|=|v_2|=1}} \big((Av_1, v_1) + (Av_2, v_2)\big)$$

is concave. Further,

$$\frac{d}{dt}(\lambda_1 + \lambda_2) = \lambda_1^2 + \lambda_2^2 + \underbrace{(\lambda_1 + \lambda_2)}_{\geq 0} \cdot \lambda_3 \geq 0\,,$$

so that $\mathcal{K}$ is preserved under the ODE system (0.1). Hence the nonnegativity of the Ricci curvature is preserved under the Ricci flow.
- For $C \geq \frac{1}{2}$, set $\mathcal{K} := \{A \mid \lambda_3(A) \leq C \cdot (\lambda_1(A) + \lambda_2(A))\}$. For any $x \in M$, $\mathcal{K}_x$ is convex, since $A \mapsto (\lambda_1 + \lambda_2)(A)$ is a concave function, whereas $A \mapsto \lambda_3(A) = \max_{|v|=1}(Av, v)$ is a convex function. Using $C \geq \frac{1}{2}$, one computes, that $\frac{d}{dt}(\lambda_3 - C\cdot(\lambda_1+\lambda_2)) \leq 0$ at points, where $\lambda_3 = C\cdot(\lambda_1+\lambda_2)$. Thus the condition $\lambda_3 \leq C \cdot (\lambda_1 + \lambda_1)$ is preserved by the ODE system (0.1).
Since M is compact, we can find a $C \geq \frac{1}{2}$ such that

$$\lambda_3(Rm) \leq C \cdot (\lambda_1(Rm) + \lambda_2(Rm))$$

holds for the initial metric $g(0)$, so that $Rm(0) \in \mathcal{K}$. Hence $Rm(t) \in \mathcal{K}$ for $t \in [0,T)$. Thus the Ricci pinching $Ric \geq \frac{1}{C}\lambda_3(Rm) \cdot g \geq \frac{1}{6C} \cdot R\hat{A} \cdot g$ is preserved under the Ricci flow.
By analyzing points (x_0, t_0), where $(\lambda_1 + \lambda_2)(Rm)$ is negative, one can conclude from a similar reasoning, that if $Rm(g(t_0)) \in \mathcal{K}$, then $Ric_{g(t_0)} \geq 0$, unless $g(t_0)$ has negative constant sectional curvature, see [1], p. 134.

- Using a more complicated set $\mathcal{K}$, one can also conclude, that Ricci pinchings improve under the Ricci flow, see [1], p. 134f.

2. The dynamic curvature estimate of Hamilton and Ivey

As another application of the maximum principle, we discuss an estimate due to R. Hamilton and T. Ivey independently. The proof uses a refined version of the maximum principle, where the set $\mathcal{K}$ is allowed to vary with time. As before, the sets $\mathcal{K}(t)_x$ are required to be convex, and the set $\{(v,t) \in E \times [0,T) \,|\, v \in \mathcal{K}(t)\}$ is required to be closed. The reasoning is the same as in the case of a "static" estimate above: Supposed, the ODE system (0.1) preserves the set $\mathcal{K}$, if $Rm(0) \in \mathcal{K}$, then $Rm(t) \in \mathcal{K}$ for all $t \in [0,T)$.

The Hamilton-Ivey theorem now states the following: Let $(M^3, g(t))$ be a solution to the Ricci flow on $[0,T)$. If $\lambda_1(x,0) \geq -1$ for all $x \in M$, then at any point $(x,t) \in M \times [0,T)$ with $\lambda_1(x,t) < 0$, we have

$$R \geq |\lambda_1| \cdot (\log|\lambda_1| + \log(1+t) - 3)\,,$$

or especially $R \geq |\lambda_1| \cdot (\log|\lambda_1| - 3)$.

From the last estimate we conclude that large negative eigenvalues inherit even larger positive eigenvalues: if $\lambda_1 < e^{-C+3} \ll 0$, with $C \gg 0$, then $R > C|\lambda_1|$.

As a consequence of the first estimate, even the sectional curvature tends towards nonnegative under the Ricci flow. For instance, using this estimate, ancient solutions with bounded curvature can be seen to have nonnegative sectional curvature.

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Shi's local estimates

Glen Wheeler

The talk is concerned with some of the analytic results and techniques that are fundamental to the study of the qualitative behavior of solutions of the Ricci flow, later used in singularity analysis.

In particular we focus on derivatives estimates, useful for proving long time existence of solutions and obtaining local control of solutions.

The following result is due to W.-X. Shi:

Proposition 1 ([1])**.** *(Local derivative of curvature estimates). For any* α*,* K*,* r*,* n *and* $m \in \mathbf{N}$*, there exists* C *depending only on* α*,* K*,* r*,* n *and* m *such that if* M^n *is a manifold,* $p \in M$*, and* $g(t)$*,* $t \in [0, T_0]$*,* $0 < T_0 < \frac{\alpha}{K}$*, is a solution to the Ricci flow on an open neighborhood* U *of* p *containing* $\overline{B}_{g(0)}(p, r)$ *as a compact subset, and if*

$$|Rm(x,t)| \leq K \quad for \quad all \quad x \in U \quad and \quad t \in [0, T_0],$$

then

$$|\nabla^m Rm(y,t)| \leq \frac{C(\alpha, K, r, n, m)}{t^{m/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ *and* $t \in (0, T_0]$.

The talk will detail Hamilton's proof of Shi's first derivative estimate, in a slightly better version than the main theorem stated above:

Proposition 2 ([1])**.** *(Interior first derivative estimates) curvatures). There exists a constant* $C(n)$ *depending only on* n *such that if* M^n *is an* n*-manifold,* $p \in M$*, and* $g(t)$*,* $t \in [0, \tau]$*, is a solution to the Ricci flow on an open neighborhood* U *of* p *containing* $\overline{B}_{g(0)}(p, r)$ *as a compact subset, and if*

$$|Rm(x,t)| \leq K \quad for \quad all \quad x \in U \quad and \quad t \in [0, \tau],$$

then

$$|\nabla^m Rm(y,t)| \leq C(n) K (\frac{1}{r^2} + \frac{1}{\tau} + K)^{1/2}.$$

For the proof we follow details of [1] and rely on applying a barrier argument to a quantity containing the first derivative, which has good evolution equation. The main ideas of the proof can be enclosed into 4 steps.

The first of these is based on a simple note about the very estimate we want to obtain. The fact that constant K appears in the right hand side of the estimate allows us to assume without loss of generality in the proof that $\tau \in (0, 1/K]$ and $r \in (0, 1/\sqrt{K}]$.

The next step is finding the good first derivative quantity and derive its evolution equation:

$$G := \frac{c(n)}{K^4}(16K^2 + |Rm|^2)|\nabla Rm|^2$$
$$\frac{\partial G}{\partial t} \leq \Delta G - G^2 + K^2$$

The following step is more technical and consists of construction of the barrier function. This can be done by using the existence of a good cutoff function with bounded first and second derivative on a manifold with bounded curvature, which is our case.

The last step links the estimates of the quantity defined in the second step with those of the first and second derivative of the cutoff function. We prove that as long as our good quantity G is dominated by a certain comparison function, the derivatives of the cutoff function satisfy also good bounds. But on the other hand these are easily obtained from the bounded curvature hypothesis.

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Compactness results for the Ricci flow

Simon Blatt

Before one can perform surgery, it is crucial to understand what singularities of the Ricci flow look like. This is done performing a blowup analysis of such a singularity, that is one rescales the Ricci flow in order to get sequence of flows subconverging to a limit Ricci flow and then analyzes this limit flow.

To be more precise, let us consider a smooth Ricci flow $(M, g(t))$, $t \in [0, T)$ that becomes singular at time T, which means that the curvature tensor does not stay bounded. Then one can choose points x_t such that the norm of the curvature attains nearly its maximum in x_t and let Q_t be this maximum. For a sequence $t_i \to T$ one then considers the rescaled based Ricci flows

$$(M, Q_{t_i} g(Q_{t_i}^{-1}(t - t_i), x_{t_i})$$

on the time interval $[-Q_{t_i} t_i, 0]$.

In this talk, we first make precise in what sense these flows shall converge by introducing the notion of geometric limit of based Ricci flows. After that, we present the Hamilton compactness theorem for Ricci flows, which says that the κ-non collapsing condition and a uniform bound of the curvature tensor on balls imply compactness. In fact, one can show that the sequence above satisfies both conditions. But while the estimates for the curvature tensor follow quite easily from maximum principles, it is one of the big breakthroughs of Perelman to show that these are κ non-collapsing.

To prove the compactness theorem, we first show a version of it for general manifolds under the assumption that not only the curvature tensor itself but also the covariant derivatives of any order of the curvature tensor are uniformly bounded. In the case of Ricci flows, Shi's derivative estimates imply such bounds which proves the original theorem.

In the situation sketched above, Hamilton's compactness theorem shows that the sequence of rescaled Ricci flows converge geometrically to a Ricci flow. Any such limit is called blowup limit and it is an ancient solution. Using the Hamilton-Ivey estimate and the fact that $Q_{t_i} \to \infty$, one can show that any such blowup limit is a manifold of nonnegative section curvature which is a first step towards characterizing all possible limits.

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Perelman's *l*-distance

Valentina Vulcanov

This talk is a preparation of the necessary tools for proving the non-collapsing results. The $\mathcal{L}$-length defined by Perelman is the analog of an energy path, but defined in a Riemannian manifold context. The length is used to define the l reduced distance and later on, the reduced volume. So far the properties of the l-length have two applications in the proof of the Poincaré conjecture. Associated with the notion of reduced volume, they are used to prove non-collapsing results and also to study the κ- solutions.

The first step is introducing the reduced distance by means of the $\mathcal{L}$-length defined by Perelman, [2].

Consider a backward solution of the Ricci flow $(M, g(\tau))$:

$$\frac{\partial}{\partial \tau} g_{ij}(\tau) = 2Ric_{ij}(g(\tau)),$$

where $\tau = T - t$ (T is the final time). Let $\gamma : [\tau_1, \tau_2] \rightarrow M$ be a curve on the manifold parametrized by backward time.

Definition 1 ([2])**.** *The $\mathcal{L}$-length of a curve γ is*

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2) d\tau,$$

where $R(\gamma(\tau))$ is the scalar curvature at the point $\gamma(\tau)$.

Considering a variation of the curve γ, $\tilde{\gamma}(s, \tau)$, $s \in (-\epsilon, +\epsilon)$, $t \in [\tau_1, \tau_2]$ we can define the tangential and variational vector fields by $X = \frac{\partial \tilde{\gamma}}{\partial \tau}$ and $Y = \frac{\partial \tilde{\gamma}}{\partial s}$.

The first properties obtained are the (Euler-Lagrange) equations of $\mathcal{L}$-geodesics:

Proposition 1 ([2])**.** *There holds*

$$\nabla_X X - \frac{1}{2} \nabla R + 2Ric(X, \cdot) + \frac{1}{2\tau} X = 0.$$

The proof comes easily from the first variation formula for the $\mathcal{L}$-length.

Let $\tau_1 = 0$ and $\tau_2 = \overline{\tau}$ we consider furthermore variations of curves on M, connecting points $p, q \in M$ with fixed starting point p and moving end point $q = q(\overline{\tau})$.

Definition 2 ([2])**.** *Denote by $L(q, \overline{\tau})$ the $\mathcal{L}$-length of the $\mathcal{L}$- shortest curve $\gamma(\tau)$, $0 \leq \tau \leq \overline{\tau}$ connecting p and q.*
The reduced length is defined as $l(q, \tau) = \frac{L(q,\tau)}{2\sqrt{\tau}}$.

From the definition one can see that properties of the reduced distance can be easily obtained if we have the corresponding ones for the $\mathcal{L}$-length. We are concentrating the last mentioned ones in two main propositions:

Proposition 2 ([2]). *There holds*

$$L_{\overline{\tau}}(q,\overline{\tau}) = 2\sqrt{\overline{\tau}}R(q) - \frac{1}{\overline{\tau}}K - \frac{1}{2\overline{\tau}}L(q,\overline{\tau}),$$
$$|\nabla L|^2(q,\overline{\tau}) = -4\overline{\tau}R(q) + \frac{2}{\sqrt{\overline{\tau}}}L(q,\overline{\tau}) - \frac{4}{\sqrt{\overline{\tau}}}K.$$

where $K = K(\gamma,\overline{\tau}) = \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} H(X(\tau))d\tau$ *and* $H(X)$ *is the trace of the expression appearing in Hamilton's Harnack inequality,* [4].

Proposition 3 ([2]). *There holds*

$$\Delta L \leq \frac{n}{\sqrt{\overline{\tau}}} - 2\sqrt{\overline{\tau}}R - \frac{1}{\overline{\tau}}\int_0^{\overline{\tau}} H(X)d\tau.$$

For the proof of the last one we have followed the detailed steps of [1, 3]. One starts by computing the second variation and then the Hessian of the $\mathcal{L}$-length. We define the $\mathcal{L}$-Jacobi fields along $\mathcal{L}$-geodesics and prove that they are minimizers of Hessian of the $\mathcal{L}$-length. Then making a special choice of orthonormal basis for $T_{\gamma(\tau)}M$ we obtain the result.

Using the above properties of the $\mathcal{L}$-length we can also obtain the reduced length $l(q,\tau)$ properties, which will be used in the following to prove monotonicity of reduced volume and non-collapsing results:

Proposition 4 ([2]). *One has*

$$l_{\overline{\tau}} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\overline{\tau}} \geq 0$$
$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\overline{\tau}} \leq 0$$
$$\min_{\overline{\tau}} l(\cdot,\overline{\tau}) \leq \frac{n}{2}$$
$$\frac{d}{d\tau}|_{\tau=\overline{\tau}}|\tilde{Y}|^2 \leq \frac{1}{\overline{\tau}} - \frac{1}{\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}} \sqrt{\tau}H(X,\tilde{Y})d\tau,$$

where $\tilde{Y}$ *is any* $\mathcal{L}$*- Jacobi field along* $\gamma(\tau)$*.*

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Monotonicity of reduced volume; local non-collapsing

Valentina Vulcanov

The talk is divided in two parts. In the first one we give the definition of the reduced volume and by means of reduced length we prove it to be non-increasing in backwards time.

Definition 1 ([2]). *The reduced volume of a backwards Ricci flow solution $(M, g(\tau))$ is defined as*

$$\tilde{V}(\tau) = \int_M \tau^{-\frac{n}{2}} e^{-l(q,\tau)} dq.$$

Proposition 1 ([2]). *$\tilde{V}(\tau)$ is non-increasing in τ (non-decreasing in t).*

The proof comes from writing the reduced volume in terms of the exponential map as (details in [3]):

$$\tilde{V}(\tau) = \int_{T_pM} \tau^{-\frac{n}{2}} e^{-l(\mathcal{L}exp_\tau(v),\tau)} \mathcal{J}(v,\tau) \aleph_\tau(v) dv$$

where $\mathcal{J}(v,\tau) = det\ d(\mathcal{L}exp_\tau(v))$ and $\aleph_\tau(v)$ is a cut off function related to the $\mathcal{L}$-cut locus of point $p \in M$.
The tangential injectivity domain and the Jacobian will have the following properties:

$$\Omega^{T_p}(\tau_2) \subset \Omega^{T_p}(\tau_1), \quad \forall \quad \tau_1 \le \tau_2$$
$$\frac{d}{d\tau}|_{\tau=\overline{\tau}} ln \mathcal{J}(v,\tau) \le \frac{n}{2\overline{\tau}} - \frac{1}{2}\overline{\tau}^{-\frac{3}{2}} K(v,\overline{\tau})$$

Here, $K(v,\tau)$ is the analog in the tangential space of the one defined in the previous talk.

The next part concentrates on the local non-collapsing theorem. We start by defining what it means for a solution to be κ-collapsed:

Definition 2 ([2]). *A solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$ is said to be κ-collapsed at (x_0, t_0) on scale $r > 0$ if $|Rm|(x,t) \le \frac{1}{r^2}$ for all (x,t) satisfying $dist_{t_0}(x, x_0) < r$ and $t_0 - r^2 \le t \le t_0$, and the volume of the metric ball $B(x_0, r^2)$ at time t_0 is less than κr^n.*

Our goal is to prove the local non-collapsing theorem.

Theorem 1 ([2]). *For any $A > 0$ there exists $\kappa = \kappa(A)$, $\kappa > 0$ with the following property : if $g_{ij}(t)$ is a smooth solution to the Ricci flow on the time interval $0 \le t \le {r_0}^2$, which has $|Rm|(x,t) \le \frac{1}{{r_0}^2}$ for all (x,t) satisfying $dist_{t_0}(x,x_0) < r_0$, and the volume of the metric ball $B(x_0, r^2)$ at time 0 is at least $A^{-1}{r_0}^n$, then $g_{ij}(t)$ can not be κ-collapsed an the scales less than r_0 at points $(x, {r_0}^2)$ with $dist_{{r_0}^2}(x,x_0) < Ar_0$.*

As an outline, the proof is based on the separation of the region around the problem point in two parts: one with bounded geometry obtained from the assumptions of the theorem and one in which we use the properties of the reduced distance l in the backwards time, previously proved. Following details of [1] we use the Bishop-Gromov inequality to prove that there exists a lower bound for the volume of the geodesic ball at time t from the one at initial time 0. This along with a uniform sectional curvature bound will give us the bounded geometry on the first time region, thus a lower bound on the reduced volume of that region.

For the second part of the argument an effective upper bound on the minimum of the reduced distance will give a lower bound on the reduced volume of the region. This can be computed by using the properties of the $\mathcal{L}$-length and reduced distance l, obtained in the previous talk, localized around the problem point using a ingeniously chosen radial function,depending on the reduced distance l for which we apply a maximum principle. This gives us again a lower bound of the reduced volume also on the second region.

Putting the two bounds together we will get a lower bound of the reduced volume in a region where we have assumed, by contradicting the theorem, that the solution is κ-collapsed at scales less than r_0. This contradict the definition of κ-collapsed and finishes the proof.

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Properties of κ-solutions

Oliver C. Schnürer

A Ricci flow $(M, g(t))$ is κ-non-collapsed if

$$|\mathrm{Rm}| \leq r^{-2} \text{ in } B_r(p,t) \times (t - r^2, t] \quad \Longrightarrow \quad \mathrm{Vol}(B_r(p,t)) \geq \kappa r^n.$$

A κ-solution is defined as a κ-non-collapsed ancient solution. Examples are $\mathbb{S}^2$, $\mathbb{S}^n/\Gamma$, $\mathbb{S}^2 \times \mathbb{R}$. We focus on two or three dimensions, $n \leq 3$. κ-solutions can be rescaled so that they converge to gradient shrinking solitons. Their asymptotic volume $\lim\limits_{r\to\infty} \frac{\mathrm{Vol}(B_r(p,t), g(t))}{r^n}$ is zero. This implies bounds on the scalar curvature R. In two dimensions, κ-solutions are essentially only shrinking round spheres. We establish a compactness theorem for κ-solutions which implies bounds for $|\nabla R|/R^{3/2}$ and $|\frac{d}{dt}R|/R^2$. Locally κ-solutions look like ε-necks, (C,ε)-caps, C-components or are almost round.

These results are due to G. Perelman [2, 3] and can be found in the book [1] by J. Morgan and G. Tian.

References

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Bounded curvature at bounded distance

Mario Listing

We know by Hamilton's work that the curvature of a Ricci flow $(M^3, g(t))$, with $t \in [0, T)$ and M closed, is *pinched towards positive* for all $t \in [0, T)$ if it is pinched towards positive at the initial time $t = 0$. If the curvature is pinched towards positive, there is a decreasing function $\phi : \mathbb{R} \to (0, \infty)$ which tends to zero at infinity and satisfies

$$\mathrm{Rm}(x, t) \geq -\phi(R(x, t)) \cdot R(x, t) \tag{0.1}$$

for all (x, t) where $\mathrm{Rm}(., t) \in \mathrm{End}(\Lambda^2 TM)$ is the Riemannian curvature operator and $R(., t)$ is the scalar curvature of $g(t)$. Perelman proved in [P1, Chp. 12] that for any Ricci flow $(M^3, g(t))$, with $t \in [0, T)$ and M closed, there is some $r > 0$ in such a way that if $(M, g(t))$ satisfies (0.1) and is κ noncollapsed on scales $< r$, then each point $(x_0, t_0) \in M \times [1, T)$ of scalar curvature

$$Q := R(x_0, t_0) \geq \frac{1}{r^2} \tag{0.2}$$

has a small neighborhood which is, after scaling by Q, ϵ–close to the corresponding subset of some ancient κ–solution. In particular, each point (x_0, t_0) satisfying (0.2) has a canonical neighborhood. Moreover, the scalar curvature $R(x, t_0)$ in this small neighborhood stays bounded by $C \cdot R(x_0, t_0)$ for all x with $\mathrm{dist}(x, x_0) < A$ where $C = C(A) > 0$.

Suppose $(M^3, g(t))$, $t \in [0, T)$ is a Ricci flow where M is closed, connected, oriented and the solution becomes singular for $t \to T$ which means that the curvature does not stay bounded for $t \to T$. Let the Ricci flow be κ–noncollapsed on scales $< r$ and assume inequality (0.1). Then each point (x, t) with $R(x, t) \geq r^{-2}$ has a canonical neighborhood: an ϵ–neck, an ϵ–cap or a closed manifold of positive curvature. In the latter case the Ricci flow becomes extinct at T, hence it remains to consider the ϵ–necks and ϵ–caps. Let $\Omega \subset M$ be the set where the curvature stays bounded for $t \to T$, then Ω is open by curvature estimates for ancient κ–solutions. If Ω is empty, the Ricci flow becomes extinct at T and

$$M \in \{S^3, \mathbb{R}\mathrm{P}^3, S^2 \times S^1, \mathbb{R}\mathrm{P}^3 \sharp \mathbb{R}\mathrm{P}^3\}$$

up to diffeomorphism. Thus, it remains to consider the case $\Omega \neq \emptyset$. Due to derivative estimates by Shi, the limit $\overline{g} = \lim_{t \to T} g(t)$ defines a smooth metric on Ω. Choose $\rho < r$ and let $\Omega_\rho \subset \Omega$ be the set of points $x \in \Omega$ where $\overline{R}(x) \leq \rho^{-2}$. Then Ω_ρ is compact and each point $y \in \Omega \setminus \Omega_\rho$ is the center of an ϵ–neck or an ϵ–cap. Considering the bounderies of these ϵ–necks and ϵ–caps we conclude that each ϵ–neck of $(\Omega, \overline{g})$ is contained in one of the following subsets of Ω:

(1) An ϵ–tube (curvature stays bounded) with boundary components in Ω_ρ
(2) An ϵ–cap with boundary in Ω_ρ
(3) An ϵ–horn (curvature goes to infinity at one end) with boundary in Ω_ρ
(4) A capped ϵ–horn
(5) A double ϵ–horn (curvature goes to infinity at both ends)

Since the volume of the subsets (1),(2),(3) is bounded from below, there are only finitely many components of Ω which contain points of Ω_ρ. Since each point of $M \setminus \Omega$ is shortly before T the center of an ϵ–neck or an ϵ–cap, we obtain the topology of M knowing Ω as follows. Let Ω_j, $j = 1 \ldots N$, be the components of Ω with $\Omega_j \cap \Omega_\rho \neq \emptyset$ and define $\overline{\Omega}_j$ to be the one point compactification of Ω_j. Then M is the connected sum of $\overline{\Omega}_j$, $j = 1 \ldots N$, a finite number of $S^2 \times S^1$ and a finite number of $\mathbb{RP}^3$. In particular, if M is simply connected, $\overline{\Omega}_j$ is simply connected for all j and M is the connected sum of $\overline{\Omega}_j$, $j = 1 \ldots N$.

In order to do a suitable Ricci flow with surgery, we have to understand the geometry at the ends of Ω_j. That is why Perelman proved in [P2, Lemma 4.3] the existence of strong δ–necks sufficiently deep in ϵ–horns: Suppose that we have a solution of the Ricci flow with surgery defined on $[0, T)$, $T < +\infty$, normalized initial condition, a finite number of surgery times and satisfying the canonical neighborhood assumption as well as inequality (0.1). Then there is some $h = h(\delta, T)$ with $0 < h < \delta\rho$ such that for each point x with

$$h(x)^{-2} := \overline{R}(x) \geq \frac{1}{h^2}$$

in an ϵ–horn of $(\Omega, \overline{g})$ with boundary in Ω_ρ, the neighborhood

$$B_T(x, h(x)/\delta) = \{y \in \Omega | \mathrm{dist}_T(y, x) < h(x)/\delta\}$$

is the T–slice of the strong δ–neck

$$P(x, T, h(x)/\delta, -h(x)^2) = B_T(x, h(x)/\delta) \times [T - h(x)^2, T]$$

which means that $P(x, T, h(x)/\delta, -h(x)^2)$ is after scaling by $h(x)^{-2}$, δ–close to the corresponding subset of the evolving standard neck.

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The standard solution

Reto Müller

We define a *standard initial metric* to be a complete and rotationally symmetric metric g_0 on $\mathbb{R}^3$ with nonnegative sectional curvature at every point, constant sectional curvature $\frac{1}{4}$ near the origin $p = 0$ and satisfying the condition that there is a compact ball $B = B_{g_0}(p, A_0)$ such that $g_0|_{\mathbb{R}^3 \setminus B}$ is isometric to the half-cylinder $(S^2, h) \times (\mathbb{R}_+, ds^2)$, where h is the round metric on S^2 with scalar curvature 1.

A *partial standard Ricci flow* is a solution $(\mathbb{R}^3, g(t))_{0 \le t < T}$ of the Ricci flow starting from a standard initial metric $g(0) = g_0$ and satisfying the property that its curvature is locally bounded in time. Such a partial standard Ricci flow is a *standard Ricci flow* if T is maximal in the sense that there is no extension of the flow past time T which still has curvature locally bounded in time.

In a first step, we prove (via an explicit construction) that there exists a standard initial metric. We then fix once and for all this (or any other) standard initial metric g_0 and prove the following theorem for the corresponding standard Ricci flow.

Theorem 1. *There exists a standard Ricci flow for some positive amount of time. Let $(\mathbb{R}^3, g(t))_{0 \le t < T}$ be a standard Ricci flow. Then*

- Uniqueness*: If $(\mathbb{R}^3, g'(t))_{0 \le t < T'}$ is a standard Ricci flow, then $T' = T$ and $g'(t) = g(t)$ for all $t \in [0, T)$.*
- Time interval*: $T = 1$.*
- Rotational symmetry*: For all $t \in [0, 1)$, the metric $g(t)$ is invariant under the $SO(3)$-action on $\mathbb{R}^3$.*
- Positive curvature*: For all $t \in (0, 1)$, the metric $g(t)$ is compete of strictly positive curvature.*
- Asymptotics at infinity*: For all $t_0 < 1$ and $\varepsilon > 0$, there is a compact subset $X \subset \mathbb{R}^3$ such that all x in the complement of X have a neighborhood $U \ni x$ with $g(t)|_U$ being ε-close in $C^{[1/\varepsilon]}$ to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), h(t) \oplus ds^2)$, $\forall t \in [0, t_0]$, where $h(t)$ is the round metric on S^2 with scalar curvature $\frac{1}{1-t}$.*
- Curvature bound*: There is $C > 0$ such that for all (x, t) in the standard solution there holds $R(x, t) \ge \frac{C}{1-t}$.*
- Non-collapsing*: There is $r > 0$ and $\kappa > 0$ such that $(\mathbb{R}^3, g(t))$ is κ-non-collapsed on scales less than r for all $t \in [0, 1)$.*
- Canonical neighborhoods*: For all $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that for all (x, t) in the standard solution one of the following holds*
  1. *(x, t) is contained in the core of a (C, ε)-cap.*
  2. *(x, t) is the center of an evolving ε-neck with initial time-slice $t = 0$ and this time-slice is disjoint from the surgery cap $B_{g_0}(p, A_0 + 4)$.*
  3. *(x, t) is the center of an evolving ε-neck defined for rescaled backwards time at least $1 + \varepsilon$.*

Here, we give some remarks about the proof. A complete proof can be obtained by combining Theorem 12.5, Proposition 12.31 and Theorem 12.32 of [3].

Gluing together two copies of $B_{g_0}(p,R)$, $R \geq A_0+1$, at their boundaries yields smooth manifolds (S_R, g_R, p) which converge geometrically to $(\mathbb{R}^3, g_0, p)$ as $R \to \infty$. Since all these manifolds are compact and have the same curvature bounds, existence of a standard solution follows as a geometric limit of the Ricci flows starting from (S_R, g_R, p). Here we used the compactness for Ricci flows from [2]. This argument also implies completeness of the standard solution. Similarly, letting $y_k \to \infty$, we obtain $(S^2, h) \times (\mathbb{R}, ds^2)$ as the geometric limit of $(\mathbb{R}^3, g_0, y_k)$ and the claimed asymptotics at infinity follow. Now, positive curvature follows easily by localizing the standard result in dimension three.

One can obtain uniqueness form [1]. This immediately implies rotational symmetry, too. On the other hand, rotational symmetry also follows from the fact that Killing fields are stationary under the flow and stay Killing fields. With this symmetry, one can then reduce the equation to a one-dimensional problem for which uniqueness is obtained from a slightly modified version of DeTurck's trick for compact manifolds. This gives a much simpler proof than the general one in [1].

Non-collapsing of the standard solution is a direct application of the general non-collapsing result, cf. [4], section 8. All the remaining results ($T=1$, curvature bounds, and the canonical neighborhoods statement) can then be proven by contradiction via a blow-up argument. Indeed, if one of the statements fails to hold, one finds a sequence of points for which the blow-up limit is a κ-solution. Using the fact that κ-solutions have canonical neighborhoods and asymptotically vanishing volume, we then obtain the desired contradictions.

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Surgery

Sebastian Goette

This talk explains how surgery in the Ricci flow is actually done. In particular, we describe the new Riemannian metric after surgery. For this purpose, we stop the Ricci flow at the first singular time. From Listing's talk we know that the manifold decomposes into a regular and a singular part, and that the end of the regular part consists of ϵ-horns. Inside an ϵ-horn, one finds a δ-neck for a constant δ that will be fixed later. An algorithm to find a δ-neck at the right spot will be given in Böhm's talk. We can now cut the δ-neck through its center, throw away the piece pointing toward the end, and replace it by a cap. On the cap, we construct a metric that interpolates between the original metric of the δ-neck and a properly rescaled initial metric for the standard solution from Müller's talk.

We have learned in previous talks that the Ricci flow "improves" the local geometry. In particular, there is the Hamilton-Ivey pinching towards positive curvature that depends on a time parameter t, see Becker's talk. To prove finite extinction of the Ricci flow in the case of the Poincaré conjecture we also need a control on the growth of the metric under the flow, see Hanisch's talk. When constructing the surgery metric, we will take care of the following.

(1) If the original metric on the regular part of M was pinched towards positive curvature with parameter t, then the same holds for the surgery metric.
(2) The surgery metrics have positive sectional curvature near the tip and is δ'-close to the initial metric for the standard solution.
(3) There is a distance decreasing map from the original δ-neck to the surgery cap that is the identity on the remaining half of the original δ-neck.

This ensures that the estimates mentioned above persist after surgery. Thus, one can perform surgery in a similar way at the next singular time. Moreover, one can prove that singular times do not accumulate and that the Ricci flow becomes extinct after finite time in the case of a finite fundamental group.

Our talk closely follows chapter 13 in the book [1] by Morgan and Tian, with a focus on curvature computations and the careful choice of several parameters involved in the construction.

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Canonical neighborhood theorem

Miles Simon

In this talk we explain the strong canonical neighbourhood theorem of Perelman, and some of the content of the proof thereof. We also explain how this helps us to show that surgery times do not accumulate. We follow closely the paper of Morgan/Tian. The statement of the canonical neighbourhood theorem of Perelman is:

Proposition: Suppose that for some $i \geq 0$ we have surgery parameters $\delta_0 \geq \delta_1 \geq \dots \delta_i > 0$, $\epsilon = r_0 \geq r_1 \geq \dots \geq r_i > 0$ and $\kappa_0 \geq \kappa_1 \geq \dots \kappa_i > 0$. For any $r_{i+1} \leq r_i$ let $\delta(r_{i+1}) > 0$ be the constant in Proposition 16.1 associated to these three sequences and to r_{i+1}. Then there are positive constants $r_{i+1} \leq r_i$ and $\delta_{i+1} \leq \delta(r_{i+1})$ such that the following holds. Suppose (M, G) is a Ricci Flow with surgery defined for $0 \leq t < T$ for some $T \in (T_i, T_{i+1}]$ with surgery control parameter $\bar{\delta}(t)$. Suppose that the restriction of the Ricci Flow with surgery to $t^{-1}([0, T_i))$ satisfies assumptions (1)-(7) and also the five properties given in the hypothesis of Theorem 15.9 with respect to the given sequences. Suppose also that $\bar{\delta}(t) \leq \delta_{i+1}$ for all $t \in [T_{i-1}, T]$. Then (M, G) satisfies the strong (C, ϵ) - canonical nbhd. assumption with parameter r_{i+1}.

The proof is by contradiction. One important reoccurring theme in the proof is to examine neighborhoods. (in space time) of points (p,t) which DO satisfy the strong canonical nbhd. assumption, but for which all points at later times do not. This leads in most cases (by simple continuity arguments) to a contradiction, and leaves us with the case that (p_0,t_0) is the centre of a strong evolving neck. Then there are two possibilities (after scaling appropriately in time and space and shifting so that $t_0 = 0$): either this strong evolving neck lives backwards in time for a time interval $[-1,0]$ (instead of just $(-1,0]$) (in which case we obtain a contradiction again) or it does not. If we can't then we see that a point on the neck is coming out of surgery region which occurs at time -1. This also leads to a contradiction by comparing with the standard solution and kappa solutions in the set up of the theorem.

Ricci flow with surgery

Christian Böhm

In this talk we defined Ricci flow with surgery following chapter 15 of [MT], which in itself is a detailed exposition of Perelman's work.

Before going into details we will sketch Ricci flow with surgery and show how to deduce topological consequences. For an initial metric g_0 on a compact, connected 3-manifold M_0^3 one runs the Ricci flow and assumes that after finite time a singularity occurs. This will for instance be the case for M_0^3 with finite fundamental group. If at this first singular time at all points of M_0^3 the scalar curvature tends to $+\infty$ the diffeomorphism type of M_0^3 can be determined, since any point in M_0^3 with large scalar curvature has a canonical neighborhood. If by contrast, at the first singular time there are points where the scalar curvature remains bounded one can remove from M_0^3 the high energy regions of the scalar curvature in a controlled manner; one can perform surgery along finitely many S^2's and glue in 3-balls to obtain a possible different manifold M_1^3. Since again every point in the disappearing region has a canonical neighborhood, the topology of this region can be determined. As a consequence, M_0^3 is the connected sum of the connected components of M_1^3 and known compact 3-manifolds. Also, the surgery process can be geometrically described very well. On the manifold M_1^3 one obtains a new initial metric g_1 which can be thought of as an extension of the singular limit metric of the Ricci flow on M_0^3 at the first singular time. Now one iterates this process until after a finite number of surgeries the above described first case occurs. Going backwards in time the topology of the initial manifold can be reconstructed. For instance, if M_0^3 is simply connected, then M_0^3 must be diffeomorphic to S^3.

Now let us explain in greater detail how Ricci flow with surgery has been defined by Perelman. First of all universal constants $C, \epsilon > 0$ have to be chosen which are related to several structure results, for instance:

(1) For ϵ' small enough a complete positively curved 3-manifold does not contain ϵ'-necks of arbitrarily small scale (Prop. 2.19 [MT]).
(2) Structure results on κ-solutions (Thm. 9.93, Cor. 9.94 [MT]).

(3) Results on the standard solution (Lemma 12.3, Thm. 13.32 [MT]).
(4) Classification of connected complete Riemannian manifolds for which every point is either contained in the core of a (C', ϵ')-cap or in the center of an ϵ'-neck (Prop. A.25 [MT]).

Secondly, one considers a family $(M_t^3, g(t))_{t\in[0,\infty)}$ of compact Riemannian 3-manifolds containing no embedded $\mathbb{R}P^2$ with trivial normal bundle. The manifold M_t^3 may be disconnected and one even allows $M_t^3 = \emptyset$. The initial manifold $(M_0^3, g(0))$ is assumed to have normalized initial conditions, that is M_0^3 is connected, the norm of the Riemann curvature tensor $\mathrm{Rm}_{g(0)}$ is bounded from the above by one and the volume of unit balls in $(M^3, g(0))$ is bounded from the below by one half of the volume of the unit ball in flat $\mathbb{R}^3$.

For notational reasons one sets $T_0 := 0$. There might be (possibly infinitely many) singular, non-accumulating times $0 < T_1 < T_2 < \cdots$, such that $(M_t^3, g(t))_{t\in[T_i,T_{i+1})}$ is a maximal Ricci flow for $i \geq 0$. In particular, for $t \in [T_i, T_{i+1})$ the manifolds M_t^3 are diffeomorphic and it holds

$$\lim_{t\nearrow T_{i+1}} \max_{x\in M_t^3} \mathrm{scal}_x(g(t)) = +\infty.$$

One says that the Ricci flow goes extinct at time $t = T$ if

$$\lim_{t\nearrow T} \min_{x\in M_t^3} \mathrm{scal}_x(g(t)) = +\infty.$$

In this case one sets $M_t^3 = \emptyset$ for $t \geq T$.

Next, we have to explain how the topology and the geometry of the Riemannian metric $(M_{T_i}^3, g(T_i))$ is defined using the limiting behavior of $(M_t^3, g(t))$ for $t \nearrow T_i$. To this end, for $i \geq 1$ let us set $\Omega_{T_i} := \{x \in M_{T_{i-1}}^3 : \lim_{t\nearrow T_i}(\mathrm{scal}_x(g(t)) < \infty\}$. Then Ω_{T_i} is open and the metrics $g(t)|_{\Omega_{T_i}}$ converge uniformly for $t \nearrow T_i$ to a limit metric $\bar{g}(T_i)|_{\Omega_{T_i}}$ in the C^∞-topology on every compact subset of Ω_{T_i}. The scalar curvature of the limit metric $\bar{g}(T_i)|_{\Omega_{T_i}}$ is a proper function bounded below. Now it is possible to describe the ends of those connected components of Ω_{T_i}, which admit points of bounded scalar curvature. This upper bound is given by an explicit parameter depending on the canonical neighborhood parameter r and the surgery control parameter δ, both parameters to be defined below. Notice that all surgery parameters will depend on time. One can show that there are only finitely many such ends each one containing a strong $\delta(T_i)$-neck. On these necks surgery is performed in a very precise manner:

On the topological level one cuts along an S^2 in the neck, removes the outer region of the neck and glues in a 3-ball. When going back in time, the disappearing region consists of points with large scalar curvature, which have a strong (C, ϵ)-canonical neighborhood for all $t < T_i$ with $|T_i - t|$ sufficiently small. To ensure this one must know that the strong (C, ϵ)-canonical neighborhood assumption (CN) with parameter r is satisfied; that is, if $\mathrm{scal}_x(g(t)) \geq \frac{1}{r(t)^2}$, then x has a strong (C, ϵ)-canonical neighborhood. By the above mentioned result (4) the topology of the disappearing region is known.

On a geometrical level the geometry of these necks is also very well understood. The above mentioned S^2's are central 2-spheres of strong $\delta(T_i)$-necks. On the attached 3-balls one can therefore glue in a standard solution to the Ricci flow in a geometrically very controlled way. Using this one defines the new initial metric $g(T_i)$. The gluing process can be performed in such a way that the Hamilton-Ivey estimates still hold after having performed surgery; that is the pinching towards positive assumption (PTP) if fulfilled.

In order to prove that Ricci flow with surgery exists - for instance to rule out accumulating surgery times - one has to introduce one further parameter, called the non collapsing parameter κ. One says that $(M_t^3, g(t))_{t\in[0,\infty)}$ satisfies the non-collapsing condition (NC), if for any $x \in M_t^3$, which does not lie in a connected component of M_t^3 with positive sectional curvature, the following holds: If for some $s \leq \epsilon$ the backward parabolic neighborhood $P(x, t, s, -\frac{1}{s^2})$ exists and if on this parabolic neighborhood the norm of the Riemann curvature tensor is bounded from above by $\frac{1}{s^2}$, then $\mathrm{Vol}(B_s^{g(t)}(x)) \geq \kappa(t) \cdot s^3$.

The key problem in showing the existence of Ricci flow with surgery is to prove that surgery control parameters r, δ and κ exist, such that for any normalized initial metric $(M_0^3, g(0))$ a Ricci flow with surgery can be defined, which also satisfies the assumptions (PTP), (CN) and (NC).

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Finite extinction time for simply connected 3-manifolds

Florian Hanisch

The aim of this talk was to present the finite time extinction result for the Ricci flow on certain 3-manifolds, following the work of Colding and Minicozzi (see [1], [2]). A Ricci flow with surgery is said to be extinct at finite time, if the time slices M_t are empty for all sufficiently large t. For Ricci flow without surgery, this is proven for closed, orientable 3-manifolds M satisfying $\pi_3(M) \neq 0$. Colding and Minicozzi consider sweepouts β of M; that is, continuous maps $\varphi : S^2 \times [0,1] \to M$ such that $(s \mapsto \varphi(\cdot, s)) \in C^0([0,1], C^0(M) \cap W^{1,2}(M))$ and $\varphi(S^2, 0), \varphi(S^2, 1)$ are points. If $E_g(f)$ and $A_g(f)$ denote the energy and the area of a map $f : S^2 \to M$ with respect to some (possibly time-dependent) Riemannian metric g on M, the energy/area width associated to any homotopy class $[\beta]$ of sweepouts of M is given by

$$W_E([\beta], g) = \inf_{\varphi\in[\beta]} \max_{s\in[0,1]} E(\varphi(\cdot, s)) \qquad W_A([\beta], g) = \inf_{\varphi\in[\beta]} \max_{s\in[0,1]} A(\varphi(\cdot, s)).$$

It may be shown (see [2], proposition 1.5) that here, both concepts yield the same value, $W_A = W_E =: W$. More importantly , in case $[\beta]$ is induced by a nontrivial element in $\pi_3(M)$, Jost has proven ([4], p.125) that $W([\beta], g) > 0$. Under this assumption, Colding and Minicozzi established the following estimate for the width $W(g(t)) = W([\beta], g(t))$ on a closed, orientable 3-manifold M equipped with a solution $g(t)$ of Ricci flow without surgery:

$$\frac{\overline{d}}{dt}W(g(t)) := \limsup_{h \searrow 0} \frac{W(g(t+h)) - W(g(t))}{h} \leq -4\pi + \frac{3}{4(t+C)}W(g(t)) \tag{0.1}$$

This estimate may be used to prove finite time extinction by integrating it on some interval $[0,T]$ to obtain

$$W(g(T))(T+C)^{-\frac{3}{4}} \leq W(g(0))C^{-\frac{3}{4}} - 16\pi\left[(T+C)^{\frac{1}{4}} - C^{\frac{1}{4}}\right]$$

Since the left hand side is non-negative, this yields a contradiction if T can be arbitrary large.

It was pointed out by Burkhard Wilking that additional assumptions are needed to obtain finite time extinction for Ricci flow with surgery, because the connected sum of $S^1 \times S^2$ with a hyperbolic 3-manifold provides an example that does not become extinct in finite time but satisfies all other assumptions. To be able to apply (0.1) to the restarted flow after surgery, it is in fact necessary that the surgery procedure is distance-decreasing (see [5] 15.12) and that each component of the modified manifold satisfies the condition $\pi_3(M_k) \neq 0$. This is in particular true if M is assumed to be prime (which is not a restriction with regard to the proof of the Poincaré conjecture). In that case, it is sufficient to suppose in addition, that M is non-aspherical or simply connected, since this already implies $\pi_3(M) \neq 0$ (see [1] if M is non-aspherical, a similar argument works in the simply connected case). Thus, the Ricci flow with surgery on a prime 3-manifold becomes extinct in finite time if it is non-aspherical or simply connected.

The strategy for the proof of (0.1) given by Colding and Minicozzi is as follows: The width $W([\beta], g)$ of a nontrivial class $[\beta]$ may be approximated by the area of certain "good" sweepouts, which are closed to a collection of minimal spheres with respect to the varifold distance (see below). Thus, the problem is reduced to estimate the area of minimal spheres in $(M, g(t))$ for a Ricci flow solution $g(t)$.

To introduce the varifold distance, denote by Gr_2M the Grassmannian of 2-planes (without orientation) which may be identified with their unit normals $\pm\nu$. Any immersed surface $\varphi : \Sigma \looparrowright M$ gives rise to a 2-varifold on M, that is, a Radon measure on Gr_2M, by virtue of the following functional on $C^0(Gr_2M, \mathbb{R})$:

$$h \mapsto \int_\Sigma h(d\varphi(T_\sigma\Sigma))J_\varphi$$

Here, J_φ denotes the Jacobian of φ. Choosing a countable, dense subset $\{h_k\}$ of $\{f \in C^0(Gr_2M, \mathbb{R}) | \ |f|, Lip(f) \leq 1\}$, the metric on the set of such 2-varifolds on

M may be defined by

$$d_V(\varphi_1,\varphi_2) := \sum_k 2^{-k} \left| \int_{\Sigma_1} h_k \circ d\varphi_1 J_{\varphi_1} - \int_{\Sigma_2} h_k \circ d\varphi_2 J_{\varphi_2} \right|$$

The topology induced by d_V is the usual topology of convergence of Radon measures and hence independent of the choice of $\{h_k\}$. In particular, by choosing $h_1 = 1$ and $h_2 = Ric/\|Ric\|_{C^1}$ respectively, $d_V(\varphi_1,\varphi_2) < \epsilon$ implies

$$|A_g(\varphi_1) - A_g(\varphi_2)| \leq C d_V(\varphi_1,\varphi_2) \tag{0.2}$$

$$\left| \int_{\Sigma_1} (R - Ric(\nu,\nu)) - \int_{\Sigma_2} (R - Ric(\nu,\nu)) \right| \leq C d_V(\varphi_1,\varphi_2) \|Ric\|_{C^1} A_g(\varphi_1)$$

for some constant C, since $Ric(d\varphi(T_\sigma\Sigma)) = R(\varphi(\sigma)) - Ric(\nu_\sigma,\nu_\sigma)$.

The following approximation result holds: For any sweepout β, inducing a nontrivial class $[\beta] \in \pi_3(M)$, there exist an approximating sequence $\{\gamma_j\} \subset [\beta]$ satisfying

(a) $\lim_{j\to\infty} \max_{s\in[0,1]} E(\gamma_j(s)) = W([\beta], g)$

(b) For all $\epsilon > 0$, there exist $J \in \mathbb{N}, \delta > 0$ satisfying:
If $j > J$ and $A_g(\gamma_j(s)) > W(g) - \delta$, there are branched minimal maps $u_i : S^2 \to M$ $(i = 1,\ldots,N)$, which approximate γ_j:

$$d_V(\gamma_j(s), \{u_i\}) < \epsilon$$

See [2], theorem 1.14 and chapter 2 for a proof. Jost (see [4], 4.2.1) has shown a similar statement using bubble instead of varifold convergence.

If $A^{(min)}_{g(t)}$ denotes the area of a possibly branched (minimal) immersion $S^2 \looparrowright M$ and $g(t)$ is a Ricci flow on M, the variation of area $g(t)$ may be explicitly computed (see [3]):

$$\frac{d}{dt} A_{g(t)} = -\int_{S^2} (R - Ric(\nu,\nu)) \qquad \frac{d}{dt} A^{min}_{g(t)} = -\int_{S^2} K_{S^2} - \frac{1}{2}\int_{S^2} (|II|^2 + R) \tag{0.3}$$

On the right hand side, II denotes the second fundamental form; this equation follows from the left one using the Gauss formula and the additional assumption, that the immersion is minimal. Applying the Gauss-Bonnet theorem for branched immersions (with branching orders b_i) and the previous estimates yields

$$\begin{aligned} \frac{d}{dt} A_{g(t)} &\leq -\int_{S^2} K_{S^2} - \frac{1}{2}\int_{S^2} R = -4\pi - 2\pi \sum_i b_i - \frac{1}{2}\int_{S^2} R \\ &\leq -4\pi - A_{g(t)} \min_M R \end{aligned} \tag{0.4}$$

The change of $A_{g(t)}(\gamma_j)$ may be estimated taking into account the approximation by spheres, and equations (0.3), (0.2) and (0.4):

$$\left.\frac{d}{dt}\right|_{\tau} A_{g(t)}(\gamma_j(s,\tau)) \leq -4\pi k - \frac{1}{2}A_{g(\tau)}(\{u_j\})\min_M R(\tau) + C'\epsilon$$

Using the curvature estimate $R(\tau) \geq -3/2(\tau + C)$ obtained from maximum principle (where C is some constant) and (0.2) yields

$$\left.\frac{d}{dt}\right|_{\tau} A_{g(t)}(\gamma_j(s,\tau)) \leq -4\pi + \frac{3}{4(\tau+C)}\max_{s_0} A_{g(\tau)}(\gamma_j(s_0,\tau)) + C'\epsilon$$

The term $\max_{s_0} A_{g(\tau)}(\gamma_j(s_0,\tau))$ converges to $W(g(\tau))$ by property (a) of the approximating sequence $\{\gamma_j\}$ and $W_E = W_A$. Taylor expansion of the preceeding inequality may then be used to show

$$\frac{W(g(\tau+h)) - W(g(\tau))}{h} \leq -4\pi + C'\epsilon + \frac{3}{4(\tau+C)}W(g(\tau)) + C'h$$

which implies (0.1), by taking the limits $\epsilon, h \searrow 0$.

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Proof of the Poincaré conjecture

Klaus Ecker

We discuss some of the history of the Ricci flow, notably Hamilton's work, by presenting various (by now almost explicit) ways singularities can form. We give examples of collapsing solutions and show how Perelman's ingenious non-k-collapsing result rules these out as “blow-ups” of closed solutions on finite time intervals. We furthermore give an intuitive explanation of the behavior for $t \to \infty$ and geometrization.

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