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Infinite Dimensional Random Dynamical Systems and Their Applications

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Abstract. The theory of infinite dimensional random dynamical systems shares many of the concepts and results for finite dimensional random dynamical systems, but with considerably more technical complications. Many examples are generated by stochastic partial differential equations (SPDE) which are used to model climate dynamics, turbulence, porous media, random surface motions and many other systems of physical interest, etc. The workshop covered broad spectrum of issues ranging from the theoretical to applications and numerics.

Mathematics Subject Classification (2000): 37Hxx, 37Lxx, 60Kxx.

Introduction by the Organisers

The theory of random dynamical systems presented in the monograph of Ludwig Arnold (*Random Dynamical Systems*, Springer-Verlag, 1998) has given rise to completely new understanding of the dynamics of stochastic differential equations and has been particularly successful for finite dimensional systems when applied to specific examples. Many of the ideas and results can be generalized to infinite dimensional random dynamical systems. Here the theory meets the field of stochastic partial differential equations (SPDE) which is an active area of recent research with strong connections with relevant problems of physics. SPDEs are used to model climate dynamics, turbulence, porous media, random surface motions and many other systems of physical interest. Asymptotic properties analyzed by the theory of random dynamical systems like invariant measures, attractors, stability, are of clear importance for the understanding of the long time behavior of these systems. Numerical simulation, system reduction and parameter estimations are other related topics of interest.

The aim of this workshop was to gather experts of SPDE and random dynamical systems, some of them more physically or numerically oriented, other with more theoretical experience, in order to exchange their ideas about the state of the art and the main open problems they investigate.

1. Structure and style of the talks

We had 25 one-hour lectures. Most of the participants gave a talk and three speakers gave very short series of talks.

The short series, devoted to Lévy noise driven equations (Zabczyk), stochastic porous media equations (Röckner), stabilization by noise and random attractors (Crauel) have been devised as an introduction to new emerging research directions or fundamental topics of general interest for the participants.

All the other single talks were allowed to be one-hour long in order to offer each speaker the possibility to introduce more smoothly the subjects and give more details. The result of this choice, as generally acknowledged by the participants, has been that it was possible to learn more than usual from the talks and ask freely a lot of questions. Having a good 10 minutes of time at the end of each talk meant that an unusually large number of questions could be asked and the speakers had adequate time to answer them with the necessary details.

2. Topics of the lectures

The full list of talks is reported below in detail, so here we limit ourselves to group them following different conceptual lines, with some repetition.

Applications in Physics. Analysis of models and questions of interest for Physics has been part of the presentations of several authors. *Röckner* devoted his lectures to porous media equations and proved, for instance, an extinction property as an example of self-organized criticality. *Kuksin* considered KdV equations perturbed by noise and the zero-noise limit problem, proved an averaging result for action/angle variables and identified the physical invariant measure of the deterministic KdV equation. *Kotelenez* revised Einstein theory of Brownian motion in the case of two or more large particles suspended in a medium and quantified their correlation at short distance, justifying in this way the use of correlated noise in many examples of SPDEs. *Kondratiev* considered a stochastic birth and death system of infinitely many particles in the continuum and described a general mathematical framework for their analysis. *Duan* considered SPDEs related to geophysical models, in the framework of the problem of model uncertainty, and described a way to estimate Hurst parameter and coefficients of the noise to fit data as best as possible. *Imkeller*, again in the field of climate modelling, considered temperature data of the last millions of years and compared the performances of equations based on Levy and Brownian noise to produce qualitative features similar to those of these data. Sharp interface random motion described by a stochastic version of Allen-Cahn equation have been presented by *Romito* who analyzed birth and annihilation of random interfaces.

Porus media. As we already mentioned, porous media equations received particular attention. The talks of *Röckner* ranged from the foundational results expressed in very large generality, to the analysis of special properties like the extinction and the existence of random attractors. *Russo* gave a probabilistic representation formula for solutions of the deterministic porous media equations in very singular situations by means of a nonlinear Fokker-Planck equation, which gives particular insight in the structure and behavior of solutions and provides new numerical methods. *Wei Liu* proved dimension-free Harnack inequalities in the sense of Wang for the stochastic porous media equations and obtained as a consequence regularity of transition kernel and ergodicity.

Lévy, fractional Brownian and other noise perturbations. While the main body of past and present investigation in SPDE is concerned with Brownian noise, several recent investigations are concerned with Lévy noise and fractional Brownian noise. *Zabczyk* gave two introductory talks on infinite dimensional stochastic equations driven by Lévy noise, providing both foundational results of the properties of infinite dimensional Lévy processes and advanced results of time-regularity, support and regularity of the law for the solutions to the equations. *Imkeller* considered equations with multiple-well potentials driven by α-stable processes, estimated the probability of tunnelling and described the relation between this phenomenon and observations of temperature changes in climate data. Fractional Brownian motion entered the SPDEs of Schmalfuss and Duan. *Schmalfuss* described foundational results on infinite dimensional equations driven by fractional Brownian motion and proved a result of existence of local unstable manifold. *Duan* showed how to detect the parameters of fractional Brownian noise to fit experimental data. Both the works of Duan and Imkeller show that interest of these noise terms in real applications. *Gubinelli* gave an introduction to the less traditional formulation of differential equations driven by rough paths, both in finite and infinite dimensions, which covers a large class of fractional Brownian motions in particular, and reviewed recent results on SPDE including the tree expansion approach to nonlinear problems.

Numerics. Numerical analysis and simulation of SPDE is a rather recent and very important subject for the future of the field and its applications. *Kloeden* reviewed recent results about error estimates and showed particular improvements of more classical results. Simulations of stochastic Burgers type equations have been showed by *Blömker* to illustrate the stabilization by noise associated to amplitude equations, described below. *Romito* described the problems emerging in the numerical approximation of sharp interface stochastic equations due to the presence of steep gradients and space-time white noise.

Stabilization by noise. Noise is usually associated to the idea of perturbation and destabilization, but in certain circumstances it may have a stabilizing effect. *Crauel* showed how noise can stabilize an unstable deterministic system by means of rotations which average stable and unstable directions, applied this technique to a nonlinear stochastic parabolic equation and showed other stabilizability results by random and non random sources. *Blömker* proved by means of amplitude equations that an additive noise in the second Fourier component of the stochastic Burgers equation may have the effect of a multiplicative noise acting on the first component and stabilizes it.

Stochastic flows. Stochastic flows are at the core of the theory of random dynamical systems: they are the random analog of deterministic flows and allow to analyze properties which depend on the simultaneous action of the dynamic on different initial conditions. *Le Jan* considered stochastic flows associated to isotropic Brownian motion, both in the regular case of flows of maps and in the generalized case of random Markov kernels, and proved a pathwise central limit theorem with respect to randomness in the initial conditions. *Scheutzow* considered the problem of existence and possible non existence of stochastic flows for stochastic equations which have unique global solutions for each individual initial condition and gave examples of non existence of flows in certain classes of regularity of coefficients.

Random and deterministic attractors and invariant manifolds. The existence and the structure of attractors and other invariant sets are some of the most basic questions to ask for dynamical systems. Several talks mentioned this topic. In particular, *Crauel* revised the concepts of strong and weak random attractor, their main properties and presented new sufficient conditions for their existence based on probabilistic estimates. *Schmalfuss* applied a stochastic version of Lyapunov-Perron transformation to construct local unstable manifold for SPDE driven by fractional Brownian motion. *Brzeźniak* proved the existence of a compact random attractor for the 2D stochastic Navier-Stokes equations in unbounded domain, where classical arguments of compactness do not apply. *Maier-Paape* considered Cahn-Hilliard equations or more generally gradient type systems and gave elements of the theory of Conley index theory for the purpose of resolving the fine structure of invariant sets.

Finally, *Hairer* presented a full picture of classical ergodic theory and the recent improvements based on the concept of asymptotic Strong Feller property, with application to a general class of SPDE which includes 2D stochastic Navier-Stokes equations, while *Johnson* described the recent theory of Sturm-Liouville problems with algebro-geometric potentials, related to solutions of the Camassa-Holm equation and with analogies with the theory of Schrödinger and KdV equations.

Workshop: Infinite Dimensional Random Dynamical Systems and Their Applications

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Abstracts

Stabilization due to additive noise

Dirk Blömker

Amplitude Equations describe essential dynamics of a complicated (stochastic) partial differential equation near a change of stability. The approximations are derived using the natural separation of time-scales near a bifurcation for a multi-scale analysis. Here we focus on results for equations with locally quadratic nonlinearity and show various applications (stabilization, random invariant manifolds, modulated pattern)

1. General setting - examples

Consider an Equation of the type

$$du = (Lu + B(u,u) + \epsilon^2 Au + B(u,u))dt + \epsilon^2 dW \ ,$$

where L is a non-positive operator with non-empty kernel $\mathcal{N}$, B is a bilinear operator and Au a linear perturbation. The noise $\partial_t W$ is Gaussian and white in time. Typical examples are

- Burgers: $\partial_t u = \partial_x^2 u + \nu u + u\partial_x u + \sigma\xi$
- Kuramoto Shivashinsky: $\partial_t u = -\Delta^2 u - \nu\Delta u + |\nabla u|^2 + \sigma\xi$
- Surface Growth: $\partial_t u = -\Delta^2 u - \nu\Delta u - \Delta|\nabla u|^2 + \sigma\xi$

or the Rayleigh Benard system (3D-Navier-Stokes coupled to a heat equation).

2. The general approximation result

One aim of amplitude equations is to show on large time scales:

$$u(t) = \epsilon a(\epsilon^2 t) + \mathcal{O}(\epsilon^2)$$

where on the slow time-scale $a \in \mathcal{N}$ solves an equation of the type

$$da = (\nu a + \mathcal{F}(a))dT + d\beta \ ,$$

where $\mathcal{F}$ is a cubic, β Brownian motion in $\mathcal{N}$ given by the projection $\beta = P_c W$ of W onto $\mathcal{N}$. Note that in the case $P_c W = 0$, the dynamics is essentially deterministic, as the dominant modes are only driven by noise acting directly on the dominant modes. We could consider larger noise then.

3. Stabilization by degenerate additive noise

In a special case [10], rigorously verified in [6], for highly degenerate noise the amplitude equation for $u(t) \approx a(\epsilon^2 t)\sin$ (in Stratonovic sense) is

(A) $$da = (\nu - \tfrac{\sigma^2}{88})a\ dT - \tfrac{1}{12}a^3 dT + \tfrac{\sigma}{6} a \circ d\beta,$$

In the SPDE $\sigma\epsilon$ is strength of the noise and ν distance from bifurcation. For sufficiently large additive noise in the SPDE the dominating mode is stabilized (i.e. the bifurcating pattern destabilized).

As numerical example consider the following Burgers-type SPDE

$$\text{(B)} \qquad \partial_t u = (\partial_x^2 + 1)u + \epsilon^2 u + u\partial_x u + \sigma\epsilon\xi \ ,$$

where $u(t,x) \in \mathbb{R}$ for $t > 0$, $x \in [0,\pi]$ subject to Dirichlet b.c. Set $\epsilon = 0.1$ and use highly degenerate noise $\xi(t,x) = \partial_t\beta(t)\sin(2x)$, which acts only on the second Fourier mode, where $\beta(t)$ is a standard one-dimensional Brownian motion.

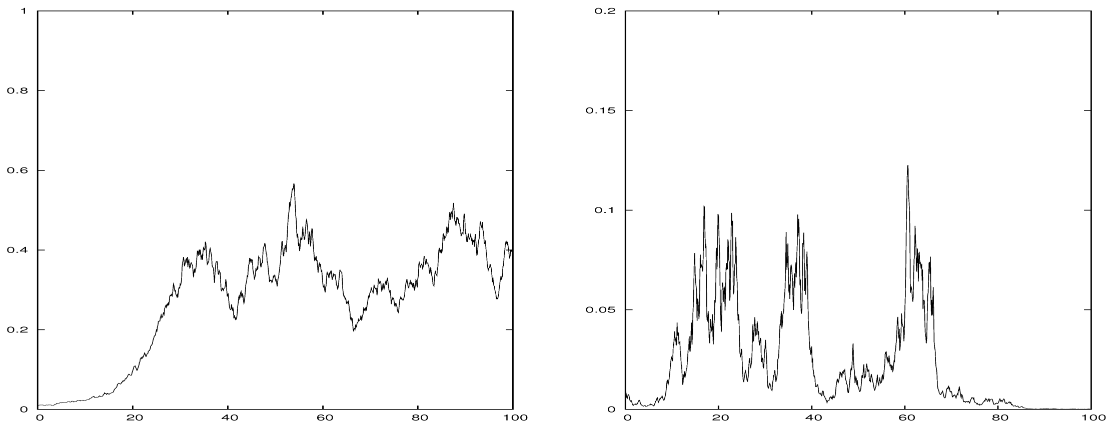

Figure 1. Time evolution of the first Fourier mode of the solution of a Galerkin truncation of (B) for $\sigma = 2$ (left) and for $\sigma = 10$ (right) for a single typical realization. Stabilization by additive noise on that mode is clearly seen (cf. [8]).

4. Large domains - modulated pattern

Amplitude or modulation equations are a standard tool for spatially extended deterministic PDEs, which help to overcome the lack of center manifold theory in such a setting. Despite long use in physics (cf. [9]), starting from [12, 11] (see also [13]) the rigorous theory of modulation equations has only been developed in the last two decades.

Little is known about modulation equations for extended SPDEs. There is some work in this direction, for the example the study of the motion of solitons in the stochastic Korteweg-de-Vries [1] and the derivation of the stochastic Ginzburg-Landau equation as the amplitude equation for the Swift-Hohenberg equation [7].

Consider as an example an equation of Kuramoto Sivashinsky type:

$$\partial_t u = -(\partial_x^2 + 1)^2 u - \nu\epsilon^2\partial_x^2 u + |\partial_x u|^2 + \epsilon^{3/2}\xi$$

with $u = u(t,x)$, $t > 0$, $x \in \mathbb{R}$, and ξ is space-time white noise.

The expected result is a modulated pattern

$$u(t,x) \approx \epsilon A(\epsilon^2 t, \epsilon x)e^{ix} + \text{compl.conj.} + \mathcal{O}(\epsilon^2) \ ,$$

where $A \in \mathbb{C}$ solves a complex Ginzburg-Landau equation of the type

$$\partial_T A = \partial_X^2 A + \nu A - cA|A^2 + \eta \ ,$$

and η is complex valued space-time white noise. See [7, 14].

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Random attractors for stochastic Navier-Stokes equations in some unbounded domains

Zdzisław Brzeźniak

In this talk I will present new developments in the theory of infinite dimensional random dynamical systems. The starting point is my recent paper with Y. Li [4]. In that paper we constructed a RDS for the stochastic Navier-Stokes equations in some unbounded domains $\mathcal{O} \subset \mathbb{R}^2$. We proved that that RDS is asymptotically compact, what roughly means that if a sequence x_n of initial data is bounded in the energy Hilbert space H and the sequence of initial times $(-t_n)$ converges to $-\infty$, then the sequence $u(0, -t_n, x_n)$, where $u(t, s, x)$, $t \geq s$ is a solution of the SNSEs such that $u(s, s, x) = x$, is relatively compact in H. A RDS satisfying this condition is called an asymptotically compact one. We also proved that for any asymptotically compact RDS on a separable Banach space, the Ω-limit set of any bounded *deterministic* set B is non-empty, compact and forward invariant with respect to the RDS (and attracting the set B). We were not able to show existence of a random attractor, as such a proof would require that there exists a family of closed and bounded random sets such that each element of it is absorbed but another element form the same family. In our case we only showed that each deterministic bounded set is absorbed by some *random* bounded set. Nevertheless, our proof of the existence of a non-empty compact invariant set implied, as a byproduct, the existence of an invariant measure for the SNSEs, a non-trivial question as the domain $\mathcal{O}$ is unbounded.

In a recent paper [3] we overcame she shortcomings of [4] and, motivated by a recent paper by [7], we found a family of random sets having the properties mentioned above. The fundamental observation is that in the proof of the absorption property, see the proof of Theorem 8.8 in [4] it is not necessary to assume that the sequence (x_n) of initial data is bounded but it is enough that it satisfies that random growth condition. This random growth condition can be expresses in terms of large time behaviour of an auxiliary Ornstein-Uhlenbeck process. This suggest a choice of a ceratin family of closed random sets. A modification of the proof of Theorem 8.8 in [4] yields that this family behave well with respect to the absorption property described above. Moreover, one can naturally modify a notion of an asymptotically compact random dynamical system with respect to such a family. We can then show that such a generalised asymptotically compact RDS has a minimal attractor, see Theorem 7. In the last section we discuss why the 2-D stochastic NSEs are asymptotically compact in this generalised sense, see Theorem 9.

We will use the framework from [4]. Let us begin with recalling some definitions and main results from that paper.

Definition 1. *A triple $\mathfrak{T} = (\Omega, \mathcal{F}, \vartheta)$ is called a* ***measurable dynamical system*** *(DS) iff $(\Omega, \mathcal{F})$ is a measurable space and $\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t\omega \in \Omega$ is a measurable map such that for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$.*
A quadruple $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called ***metric DS*** *iff $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathfrak{T}' := (\Omega, \mathcal{F}, \vartheta)$ is measurable DS such that for each $t \in \mathbb{R}$, the map $\vartheta_t : \Omega \to \Omega$ is $\mathbb{P}$-preserving.*

Definition 2. *Suppose that X is a Polish space, i.e. a metrizable complete separable topological space, $\mathcal{B}$ is its Borel $\sigma-$field and $\mathfrak{T}$ is a metric DS. A map $\varphi : \mathbb{R}_+ \times \Omega \times \mathrm{X} \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in \mathrm{X}$ is called a* ***measurable random dynamical system*** *(RDS) (on X over $\mathfrak{T}$), iff*

(i) φ is $(\mathcal{B}(\mathbb{R}_+) \bigotimes \mathcal{F} \bigotimes \mathcal{B}, \mathcal{B})$-measurable;
(ii) ***(Cocycle property)*** *for all $\omega \in \Omega$ and all $s, t \in \mathbb{R}_+$, $\varphi(0, \omega) = id$ and*

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega).$$

A RDS φ is said to be ***continuous*** *iff for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $\varphi(t, \cdot, \omega) : \mathrm{X} \to \mathrm{X}$ is continuous.*

For two non-empty sets $A, B \subset X$, we put

$$d(A, B) = \sup_{x \in A} d(x, B) \quad \text{and} \quad \rho(A, B) = \max\{d(A, B), d(B, A)\}.$$

According to [8] the latter restricted to the family $\mathfrak{C}(X)$ of all non-empty closed subsets of X, is a metric and is called the Hausdorff metric. From now on, let $\mathcal{X}$ be the $\sigma-$field on $\mathfrak{C}(X)$ generated by open sets with respect to the Hausdorff metric ρ, e.g. [2], [8] or [9].

Definition 3. *Let us assume that $(\Omega, \mathcal{F})$ is a measurable space and* X *a Polish space. A set valued map $C : \Omega \to \mathfrak{C}(X)$ is said to be measurable iff C is $(\mathcal{F}, \mathcal{X})$-measurable. Such a map C will often be called a* ***closed random set*** *on X. A random set C on X will be called a* ***compact random set*** *on X iff for each $\omega \in \Omega$, $C(\omega)$ is a compact subset of X.*

The following definition is a generalization of a definition of an asymptotically compact RDS from [4] which in turn was motivated by works of Ladyzhenskaya [13], see also [12] and [15].

Definition 4. *Assume that φ is a RDS defined on a separable Banach space* X. *Assume that $\mathfrak{D}$ is a nonempty class of closed random sets on X. We say that φ is $\mathfrak{D}$-* ***asymptotically compact*** *iff for each $D \in \mathfrak{D}$, for every $\omega \in \Omega$, for any positive sequence (t_n) such that $t_n \to \infty$ and any sequence $\{x_n\}_n$ such that $x_n \in D(\vartheta_{-t_n}\omega)$ for all $n \in \mathbb{N}$, the set $\{\varphi(t_n, \vartheta_{-t_n}\omega)x_n :\in \mathbb{N}\}$ is relatively compact in* X.

Remark 5. *The asymptotical compactness introduced in* [4] *is simply $\mathfrak{B}$-asymptotical compactness, where by definition $B \in \mathfrak{B}$ iff there exists a closed bounded set $B_0 \subset X$ such that $B(\omega) = B_0$ for all $\omega \in \Omega$. When later we use the notion of asymptotical compactness we mean the $\mathfrak{B}$-asymptotical compactness as above.*

Throughout this section we will assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS, X is a separable Banach space, φ is a continuous RDS on X (over $\mathfrak{T}$) and $\mathfrak{D}$ is a nonempty class of closed random sets on X. We begin our discussion with the following fundamental property.

Definition 6. *A compact random set A on X is said to be a* ***random $\mathfrak{D}$-attractor*** *iff (i) A is φ-invariant and (ii) A is $\mathfrak{D}$-attracting.*

A random $\mathfrak{D}$-attractor A is said to be minimal if and only if for any compact random set C on X satisfying conditions (i) and (ii), i.e. C is φ-invariant and $\mathfrak{D}$-attracting, one has $A(\omega) \subset C(\omega)$, for every $\omega \in \Omega$.

The first of our two main results is about the existence of a random $\mathfrak{D}$-attractor. It generalises Theorem 3.5 from [11] and Theorems 3.3 and 3.4 from [4].

Theorem 7. *Assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS,* X *is a separable Banach space, $\mathfrak{D}$ is a nonempty class of closed random sets on X and φ is a continuous, $\mathfrak{D}$-asymptotically compact RDS on* X *(over $\mathfrak{T}$). Assume that*

(ii) *there exists a $\mathfrak{D}$-absorbing closed random set B on X.*

Then, the Ω-limit set Ω_B of B defined by

$$\Omega(B,\omega) = \Omega_B(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega)}. \tag{1}$$

is a minimal random $\mathfrak{D}$-attractor.

The next results are about applications to stochastic Navier-Stokes equations

$$(2) \qquad \begin{cases} du + \left[\nu \mathrm{A} u + B(u)\right] dt = f\, dt + dW(t), & t \geq 0 \\ u(0) = x, \end{cases}$$

where we assume that $x \in \mathrm{H}$, $f \in \mathrm{V}'$ and $W(t), t \in \mathbb{R}$ is a two-sided K-cylindrical Wiener process satisfying Assumption A.1 below (see Remark 6.1 in [4]) defined on some probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{O} \subset \mathbb{R}^2$ is an open set, not necesarily bounded, such that the Stokes operator A, which is a self-adjoint linear operator in a Hilbert space $\mathrm{H} := \{u \in \mathbb{L}^2(\mathcal{O}) : \operatorname{div} u = 0, u \cdot \vec{n}|_{\partial \mathcal{O}} = 0\}$, where $\vec{n}$ denotes the external normal vector field to ∂D, satisfies the following inequalities (for some $\lambda_1 > 0$)

(3)

$$\|u\|^2 := (\mathrm{A}u, u) \geq \lambda_1 |u|^2, \quad u \in \mathrm{V} = D(\mathrm{A}^{1/2}), \qquad |\mathrm{A}u|^2 \geq \lambda_1 \|u\|^2, \quad u \in D(\mathrm{A}).$$

Assumption A.1. $\mathrm{K} \subset \mathrm{H} \cap \mathbb{L}^4$ *is a Hilbert space such that for some* $\delta \in (0, 1/2)$,

$$(4) \qquad \mathrm{A}^{-\delta} : K \to \mathrm{H} \cap \mathbb{L}^4 \text{ is } \gamma\text{-radonifying.}$$

We define now a class $\mathfrak{R}$ of functions $r : \Omega \to (0, \infty)$ such that

$$(5) \qquad \limsup_{t\to\infty} r(\vartheta_{-t}\omega)^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu} \int_{-t}^0 |z(\omega)(s)|^2_{L^4}\, ds} \leq 1/2.$$

The class $\mathfrak{R}$ is closed with respect to sum, multiplication by a constant and if $r \in \mathfrak{R}$, $0 \leq \bar{r} \leq r$, then $\bar{r} \in \mathfrak{R}$ and, e.g. a function $r_4 : \Omega \to (0, \infty)$ defined below belongs to $\mathfrak{R}$.

$$r_4^2(\omega) \quad := \quad \int_{-\infty}^0 |z(\omega)(s)|^4_{\mathbb{L}^4} e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, ds$$

For a set $B \subset H$, let the radius $r(B)$ of B be defined by $r(B) := \sup\{|x|_H : x \in B\}$. We will say that a function $D : \Omega \to \mathfrak{C}(H)$ belongs to class $\mathfrak{D}$ iff the function $\omega \mapsto r(D(\omega))$ belongs to the class $\mathfrak{R}$ defined earlier. The results proved earlier allow us state the following fundamental result.

Theorem 8. *Assume that* φ *is the RDS generated by the 2D Stochastic NSEs* (2). *Then the following properties hold.*

(i) φ *is* $\mathfrak{D}$*-asymptotically compact;*
(ii) *a random set* B *defined by* $B(\omega) = \bar{B}(0; r_0(\omega))$, $\omega \in \Omega$, *where* $(r_0 - r_1)^2 = 2 + 2r_2^2 + \frac{3}{\nu}\left(|f|^2_{V'} + 2\alpha^2 r_3^2 + 2C r_4^2\right)$, *belongs to* $\mathfrak{D}$*;*
(iii) *the random closed set* B *defined in (ii) is* $\mathfrak{D}$*-absorbing.*

Moreover, φ *has a minimal* $\mathfrak{D}$*-attractor.*

Now we are ready to state our main result.

Theorem 9. *Suppose that a domain* $\mathcal{O} \subset \mathbb{R}^2$ *is such that the Stokes operator* A *satisfies the Poincaré inequality* (3). *Then the RDS* φ *over the metric DS* $\mathfrak{T} = (\Omega(\xi, \mathrm{E}), \mathcal{F}, \mathbb{P}, \vartheta)$ *generated by the 2D stochastic Navier-Stokes equations* (2) *with additive noise satisfying Assumption A.1 is* $\mathfrak{D}$*-asymptotically compact.*

The following result is the main technical one.

Proposition 10. *Assume that for each random set B belonging to $\mathfrak{D}$, there exists a closed random set K belonging to $\mathfrak{D}$ such that K absorbs B. Then the RDS φ is $\mathfrak{D}$-asymptotically compact.*

Remark 11. *In the framework of Remark 5 the proof of the fact that for each B random set from the class $\mathfrak{B}$ there exists a bounded random set K absorbing B, is actually due to Crauel and Flandoli* [10].

Remark 12. *In a recent paper with J Zabczyk* [5] *we proved that the stochastic Burgers equation with an additive Lévy process is well posed. We consider a class of Lévy processes of the form $Y(t) = W(\beta(t))$, $t \geq 0$, where $\beta \in \mathrm{Sub}(p)$, $p \in (1, 2]$ and W is a $H^{\theta,2}(0, 1)$-cylindrical Wiener process with $\theta > 0$. The case $\theta = 0$ is an open problem. We believe that the results described in this paper can be extended to both Burgers and Navier-Stokes equations driven by the above class Lévy processes. This is a work in progress.*

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Random attractors

Hans Crauel

First we briefly recall some notions for deterministic dynamical systems $\varphi(t) : X \to X$, $t \in [0,\infty)$, given by a semigroup of continuous mappings on a Polish space X, i.e. $\varphi(t+s) = \varphi(t) \circ \varphi(s)$ for all $t, s \in [0,\infty)$, where $\varphi(0)$ is the identity. In particular, for a family $\mathcal{B}$ of subsets of X a $\mathcal{B}$-*attractor* is a compact $A \subset X$, such that $\varphi(t)A = A$ for all t (invariance), and such that $d\big(\varphi(t)B, A\big)$ converges to zero for t tending to infinity for every $B \in \mathcal{B}$, where $d\big(B, A\big) = \sup_{b\in B} d(b, A)$, and d is some complete metric on X metrizing the topology of X.

Random dynamical systems (RDS) The notion of a random dynamical system is recalled, consisting of the basic 'noise', given by a measurable dynamical system $\big(\Omega, \mathscr{F}, P, (\vartheta_t)_{t\in\mathbb{R}}\big)$ on a probability space leaving the probability measure P invariant, and a skew product semiflow given by continuous fibre maps $\varphi(t,\omega) : X \to X$ on the Polish space X, $t \in [0,\infty)$. That means that $\varphi(t+s,\omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s,\omega)$ for all s, t and (P-almost) all ω, and $\varphi(0,\omega)$ is the identity. We note that often one is interested in RDS induced by stochastic ordinary and partial differential equations (SDE and SPDE), driven by finite-dimenensional or by infinite-dimensional – possibly cylindrical – Wiener processes, but also Lévy processes or fractional Brownian motion may be considered.

Random attractors Several generalizations of the notion of an attractor for an RDS φ are described. They all amount to specifying a random subset $\omega \mapsto A(\omega)$ of X such that $A(\omega)$ is compact and such that $\varphi(t,\omega)A(\omega) = A(\vartheta_t\omega)$, both almost surely, and, in addition, $d\big(\varphi(t, \vartheta_{-t}\omega)B, A(\omega)\big)$ converges to zero for t tending to infinity either almost surely (*strong* random attractor) or in probability (*weak* random attractor). Again this is assumed to hold for every $B \in \mathcal{B}$, where now also random B may be taken into consideration. We restrict ourselves to $\mathcal{B}$ consisting of either all bounded subsets $B \subset X$ (which depends on the choice of the particular metric d on X) or of all compact subsets $C \subset X$ (which gives a notion of attractors independent of d). Thus we consider four classes of attractors, referred to as *weak* or *strong* B-attractors and C-attractors, respectively. What existence of strong B-attractors and strong C-attractors, respectively, is concerned it is well established that this is equivalent to the existence of a compact random $\omega \mapsto K(\omega)$ attracting every bounded or compact, respectively, subset of X almost surely. Furthermore, this strong attractor is unique almost surely, and it is measurable "with respect to the past" (which implies, in case of an RDS induced by an S(P)DE, measurability with respect to the past of the noise). In several applications it is possible to verify an even stronger property, which is the existence of an *absorbing* random compact set. The attractor may be constructed by taking first the union of all Ω-limit sets of bounded or compact sets, respectively, and then taking the closure of the resulting set. The arguments used to obtain these results all proceed ω-wise. Therefore they may also be applied to general evolution semigroups, as they are often induced by general non-autonomous systems. There is one drawback, insofar here uniqueness of the attractor gets lost due to the lack of an invariant measure for the base flow.

Probabilistic existence criteria for random attractors We finally present recent results from Crauel, Dimitroff and Scheutzow [1]:

THEOREM A: An RDS φ has a strong B- or C-attractor, respectively, if and only if for every $\varepsilon > 0$ there exists a compact $C_\varepsilon \subset X$ such that, for every $\delta > 0$ and every $B \subset X$ bounded or compact, respectively, one has

$$P\Big\{\omega \in \Omega : \bigcup_{t\geq 0} \bigcap_{\tau \geq t} \varphi(\tau, \vartheta_{-\tau}\omega)B \subset C_\varepsilon^\delta\Big\} \geq 1 - \varepsilon,$$

where C^δ denotes the δ-neighbourhood of a subset $C \subset X$. Of course this is, in turn, equivalent to the existence of a compact almost surely attracting $\omega \mapsto K(\omega)$.

THEOREM B: An RDS φ has a weak B- or C-attractor, respectively, if and only if for every $\varepsilon > 0$ there exists a compact $C_\varepsilon \subset X$ such that, for every $\delta > 0$ and every $B \subset X$ bounded or compact, respectively, there exists a $t_0 = t_0(B)$ such that

$$P\{\omega \in \Omega : \varphi(t, \omega)B \subset C_\varepsilon^\delta\} \geq 1 - \varepsilon$$

for every $t \geq t_0$. Furthermore, this is equivalent to the existence of a compact $\omega \mapsto K(\omega)$ which attracts in probability.

The main ingredient of the proofs of these results consists of the definition of the random set

$$A(\omega) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \varphi(t_k, \vartheta_{-t_k}\omega) C_{2^{-k}}^{\gamma_k}} \tag{1}$$

with suitably chosen $\gamma_n > 0$ and times $(t_n)_{n\in\mathbb{N}}$ tending to infinity.

From (1) one obtains that also the weak attractors constructed in the above theorems are measurable with respect to the past.

We finally discuss a characterization of the difference between weak and strong $\mathcal{B}$-attractors, where $\mathcal{B}$ general is allowed, given by

PROPOSITION: If $\omega \mapsto A(\omega)$ is a weak $\mathcal{B}$-attractor for an RDS φ, then A is a strong $\mathcal{B}$-attractor if and only if the Ω-limit set of every $B \in \mathcal{B}$ is a subset of $A(\omega)$ for P-almost every ω.

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Stabilization

Hans Crauel

The effect of 'noise' on a trivial solution of a differential equation $\dot{u} = F(u)$ is discussed under various angles of view.

Artificial stabilization Stabilization induced by the Itô correction term is briefly mentioned. This is the simplest example: Perturbing the real valued differential equation $\dot{x} = \alpha x$ by multiplicative white noise gives $dx = \alpha x\,dt + \sigma x\,dW(t)$. If this is understood as an Itô stochastic differential equation (SDE), its zero solution is almost surely asymptically stable as soon as $\sigma^2 > 2\alpha$. This stabilization effect, though artificial, has been re-discovered several times in the literature for various stochastic ordinary and partial differential equations interpreted in the Itô sense. For instance, the artificial stabilization of the Chafee-Infante scalar reaction diffusion equation

$$du = \left(\Delta u + \beta u - u^3\right) dt + \sigma u\,dW(t) \tag{1}$$

on a smooth bounded domain $D \subset \mathbb{R}^d$ with Dirichlet boundary condition $u|_{\partial D} = 0$ induced by the interpretation of (1) as an Itô stochastic partial differential equation (SPDE) has been observed by Caraballo *et al.* [3].

We therefore consider only Stratonovich S(P)DE in the following.

Stabilization by rotation The following result of Arnold *et al.* [1] is recalled: The linear stochastic differential equation

$$dx = Ax\,dt + \sigma \sum_{j=1}^{m} A_j x \circ dW_j(t) \tag{2}$$

in $\mathbb{R}^d$ with $\{A_j\}$ spanning the space of skew symmetric $(d \times d)$-matrices is stable for σ sufficiently large if and only if the trace of A is negative. After giving a brief sketch of the proof the following two problems arising with an implementation of this result in applications are pointed out:

(i) In general perturbations of a deterministic system do not activate 'all possible directions of rotation'. Therefore the condition of a perturbation spanning the space of all skew symmetric matrices is often not satisfied.

(ii) The intensity of the perturbation, which depends on the size of σ, can usually not be increased arbitrarily, but rather the perturbation is given and it cannot be influenced, or its intensity can only be increased up to some bound.

We briefly mention that problem (i) has been investigated carefully, and conditions on the relation of A and $\{A_j\}$ have been formulated yielding stabilization of (2) by Arnold *et al.* [2]. See also Wihstutz [6] for a survey of what can happen for a system of the form (2) with arbitrary $\{A_j\}$.

The idea of stabilization by stochastic rotations of sufficient richness has been shown to apply to the (infinite dimensional and nonlinear) Chafee-Infante equation

$$du = \left(\Delta u + \beta u - u^3\right) dt + \sum_{j=1}^{m^2} B_j u \circ dW_j(t). \tag{3}$$

There one obtains that for every $\beta \in \mathbb{R}$ there exists an $m = m(\beta)$ (increasing with β) and finite-dimensional skew-symmetric operators B_j, $1 \leq j \leq m^2$, yielding the trivial solution of (3) almost surely asymptotically stable, see Caraballo *et al.* [3].

Noise assisted stabilization For deterministic control systems the notion 'adaptive stabilization' is used for situations where only certain structural properties of a control system are known, but the particular data are unknown. A simple linear deterministic control problem which allows for adaptive stabilization is given by $\dot{x} = Ax + Bu$, $y = C^T x$, where $x \in \mathbb{R}^d$, and the control u as well as the output y are real. Assuming $C^T B > 0$ together with an additional algebraic condition on the system parameters, the control $u = -ky$ stabilizes the system provided that k is sufficiently large. Restricting to the simplest case $d = 2$ we show that the presence of a rotational noise of arbitrarily small intensity implies that stabilization for sufficiently large k takes place also in case that the algebraic condition for the corresponding deterministic system is not satisfied, see [5].

Stabilization with degenerate rotation The previous sections have described stabilization results resting upon or at least using sufficient richness of rotation brought into the system by a stochastic perturbation. In this final section we turn to the question whether a deterministic linear system $\dot{x} = Ax$ with A having negative trace can be stabilized by activating just *one* direction of rotation, i.e. whether there exists a skew symmetric Σ such that $\dot{x} = (A + k\Sigma)x$ is stable for k sufficiently large. This question can be answered positively, and we finish by sketching a way how to achieve stabilization by choosing $t \mapsto k(t)$ increasingly, thus increasing the intensity of the rotation, see Crauel *et al.* [4].

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Quantifying Model uncertainty by correlated noises

Jinqiao Duan
(joint work with Baohua Chen)

Multiscale complex systems are often subject to uncertainties, since some mechanisms are not represented due to the lack of scientific understanding for these mechanisms. These uncertainties are sometimes also called *unresolved scales*, as they are not represented or not resolved in in the models. Although these unresolved processes or mechanisms may be very small or very fast, their long time impact on the system evolution may be delicate (i.e., may be negligible, or may have significant effect, or in other words, uncertain [7]). Thus, to take the effects of unresolved mechanisms on the resolved one into account, representations of these effects are desirable [3, 4].

We consider a spatially extended system modeled by a partial differential equation (PDE):

$$u_t = Au + N(u), \tag{1}$$

where A is a linear (unbounded) differential operator, and N is a nonlinear function of $u(x,t)$ with $x \in D$ and $t > 0$, and satisfies a local Lipschitz condition. In fact, N may also depend on the gradient of u.

If this (deterministic) model is accurate, i.e., its prediction on the field u matches with the observational data $\tilde{u}$ on a certain period of time $[0,T]$, then there is no need for a stochastic approach. However, when the prediction u deviates from the observational data $\tilde{u}$, we then need to modify the model (1). In this case, the observational data $\tilde{u}$ may be thought to satisfy a modified model:

$$\tilde{u}_t = A\tilde{u} + N(\tilde{u}) + F(\tilde{u}), \tag{2}$$

where the model uncertainty $F(\tilde{u})$ is usually a fluctuating (i.e., random) process, as the observational data $\tilde{u}$ is so (i.e., has various samples or realizations).

The model discrepancy or model uncertainty $F(\tilde{u})$ may have various causes, such as missing physical mechanisms (not represented in the deterministic model (1)). Sometimes, the model uncertainty $F(\tilde{u})$ is smaller in magnitude than other terms in the model (2) and thus is often ignored in the deterministic modeling. However, being small and being fluctuating may not necessarily imply that its impact on the overall system evolution to be small [1]. To take this impact into account, we would like to model or approximate $F(\tilde{u})$ by a stochastic process.

We first calculate the model uncertainty $F(\tilde{u})$ via observational data $\tilde{u}$. By discretizing (2) and using data samples for $\tilde{u}$, we obtain (discretized) samples for F.

The mean, variance, variation, and time correlation may then be calculated using the samples of F.

$$F = \sigma(x) u \frac{dB_t^H}{dt}, \tag{3}$$

where B_t^H is the fractional Brownian motion with Hurst parameter H, and $\sigma(x) \geq 0$ is the (deterministic) noise intensity which usually depends on space.

The multiplicative noise $\sigma(x)\ u\ \dot{B}_t^H$ is a non-Gaussian colored (time-correlated) process.

Estimation of the Hurst parameter H:

As in [6], we use the data F to estimate H and believe/take this as the same H for B_t^H.

Estimation of the noise intensity $\sigma(x)$:

The equation (3) may be written as, on the observational time interval $[0, T]$:

$$\int_0^t F ds = \sigma(x) \int_0^t u(x,s) dB_s^H,\ 0 < t < T. \tag{4}$$

Here the integration is in the sense of Riemann-Stieltjes (for $H > \frac{1}{2}$).

Let us denote $Z_t = \int_0^t F ds$. Here σ may be computed via stochastic calculus [2], Theorem 1:

For a given $p < \frac{1}{1-H}$, as $n \to \infty$, we have the (uniform) convergence in probability:

$$n^{-1+pH}\ V_p^n(Z)_T \to c_p\ \sigma^p(x) \int_0^T |u(x,s)|^p ds, \tag{5}$$

where $c_p = \mathbb{E}(|B_1^H|^p) = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}$ with Γ the Gamma function, and $V_p^n(Z)_T$ is defined as

$$V_p^n(Z)_T = \sum_{i=1}^{[nT]} |Z_{i/n} - Z_{(i-1)/n}|^p. \tag{6}$$

Thus we have the estimator for the noise intensity for large n:

$$\sigma(x) \approx \{ \frac{n^{-1+pH}\ V_p^n(Z)_T}{c_p \int_0^T |u(x,s)|^p ds} \}^{\frac{1}{p}}. \tag{7}$$

In particular, for $H > \frac{1}{2}$, we could take $p = \frac{1}{H}$.

With this approximation, we obtain the following stochastic partial differential equation (SPDE) as a modified model for the original deterministic model (1): $U \approx \tilde{u}$

$$U_t = AU + N(U) + \sigma\ U\dot{B}_t^H, \tag{8}$$

with the appropriately filtered boundary condition and filtered initial condition.

This strategy for quantifying model uncertainty is then tested in an advection-diffusion-condensation equation [5] for relative humidity. We are currently working with David Nualart and Frederi Viens to get a stronger convergence result in the estimation of the noise intensity, and to study related theoretical issues.

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Some infinite dimensional rough paths

Massimiliano Gubinelli

Given a smooth driving function x_t the solution to the driven differential equation (DDE) $dy_t = f(y_t)dx_t$, $y_0 = \xi$ for $t \in [0,1]$ has a series expansion given at lowest order in $t-s$ by

$$y_t = y_s + f(y_s)(x_t - x_s) + f'(y_s)f(y_s)\int_s^t \int_s^u dx_v dx_u + \cdots \tag{1}$$

where *iterated integrals* of the control x appears. These integrals are at the core of the *rough path* approach to generalize and solve DDEs with non-smooth controls developed by Lyons [4] and inspired by the fundamental work of Chen on interated integrals [3]. In the more basic but nontrivial situation a rough path is a couple X^1_{ts}, X^2_{ts} of functions satisfying the algebraic relations

$$X^1_{ts} = X^1_{tu} + X^1_{us}, \qquad X^2_{ts} = X^2_{tu} + X^2_{us} + X^1_{tu}X^1_{us}, \qquad 1 \le t \le u \le s \le 0$$

and some analytic conditions of the form $|X^n|_{ts} \le C|t-s|^{n\gamma}$ with $\gamma > 0$ and $n = 1, 2$. In the smooth setting these objects are given by

$$X^1_{ts} = x_t - x_s, \qquad X^2_{ts} = \int_s^t \int_s^u dx_v dx_u$$

The basic observation is that eq. (1) is *equivalent* to require that the function y_t satisfy

$$(y_t - y_s) - f(y_s)X^1 ts - f'(y_s)f(y_s)X^2_{ts} \in \mathcal{R} \tag{2}$$

where $\mathcal{R}$ is the set of all functions h_{ts} such that $|h_{ts}| \le C|t-s|^\mu$ for some $\mu > 1$. Solution of this problem is unique and, under suitable assumptions on f, it exists by a fixed-point argument if $3\gamma > 1$ [8].

First in [10] (in collaboration with A. Lejay and S. Tindel) and then in [9] (together with S. Tindel) we developed an extension of this theory suitable for dealing with SPDEs in mild form

$$y_t = S_{t-s} y_s + \int_s^t S_{t-u} dw_u f(y_u) \tag{3}$$

where the solution y_t lives in some Hilbert space $\mathcal{B}$, S is an analytic semi-group in $\mathcal{B}$, $f : \mathcal{B} \to \mathcal{V}$ some nonlinear function with values another Hilbert space $\mathcal{V}$ and w a Gaussian stochastic process with values in the space of linear operators from $\mathcal{V}$ to $\mathcal{B}$ (possibly unbounded). When the non-linear term is polynomial this equation can be easily expanded like in eq. (2). For example with $f(\varphi) = B(\varphi, \varphi)$ for some symmetric bilinear operator B we get

$$y_t = S_{t-s} y_s + X^{\bullet}_{ts}(y_s^{\otimes 2}) + X^{[\bullet]}_{ts}(y_s^{\otimes 3}) + X^{[\bullet\bullet]}_{ts}(y_s^{\otimes 4}) + X^{[[\bullet]]}_{ts}(y_s^{\otimes 4}) + \cdots \tag{4}$$

where the multilinear operators X^τ are indexed by *rooted trees* τ which are either the trivial tree $\bullet$ or a tree $[\tau_1 \cdots \tau_n]$ made of a root attached to the n subtrees $\tau_1, \cdots, \tau_n$. The explicit form of the operators is given inductively by iterated convolutional integrals on the stochastic process w:

$$X^{\bullet}_{ts}(\varphi^{\otimes 2}) = \int_s^t S_{t-u} dw_u B(S_{u-s}\varphi, S_{u-s}\varphi)$$

$$X^{[\tau^1]}_{ts}(\varphi^{\otimes(d(\tau^1)+1)}) = \int_s^t S_{t-u} dw_u B(X^{\tau^1}_{us}(\varphi^{\otimes d(\tau^1)}), \varphi)$$

and

$$X^{[\tau^1\tau^2]}_{ts}(\varphi^{\otimes(d(\tau^1)+d(\tau^2))}) = \int_s^t S_{t-u} dw_u B(X^{\tau^1}_{us}(\varphi^{\otimes d(\tau^1)}), X^{\tau^2}_{us}(\varphi^{\otimes d(\tau^2)}))$$

for some suitable degree function d on the trees. In the case where w is an Hilbert space valued Brownian motion these operators can be defined using standard Ito integration and moreover it is possible to choose some version that has the path-wise regularity properties needed in the theory. By construction they also satisfy algebraic relations like the following

$$X^{[\bullet\bullet]}_{ts} = X^{[\bullet\bullet]}_{tu} S^{\otimes 4}_{u-s} + S_{u-s} X^{[\bullet\bullet]}_{us} + X^{[\bullet]}_{tu}(X^{\bullet}_{us} \otimes S^{\otimes 2}_{u-s}) + X^{[\bullet]}_{tu}(S^{\otimes 2}_{u-s} \otimes X^{\bullet}_{us}) + X^{\bullet}_{tu}(X^{\bullet}_{us} \otimes X^{\bullet}_{us})$$

which are a crucial component to prove that Eq. (4) has a unique local solution which depends nicely on the initial condition. This allows the constuction of the flow map for some 1d SPDEs driven by noise which is distributional in space and brownian in time.

The same approach has been applied in [7] to solve the periodic deterministic KdV equation in negative Sobolev spaces with and without an additive Brownian noise. In [5] we discuss the expansion over trees from the point of view of the finite-dimensional rough path theory proving that the algebraic relations between the tree-indexed rough paths are governed by the Hopf algebra on rooted trees introduced by Connes and Kreimer [2]. Moreover we also show how the tree expansion for the 3d Navier-Stokes equation studied in [6] can be interpreted in the context of this generalization of rough path theory.

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Ergodic theory for infinite-dimensional stochastic processes

Martin Hairer

The aim of this note is to provide a very short overview of a number of recent results that aim at supplementing the theory of Harris chains [10] by an alternative theory yielding weak instead of strong convergence results. This turns out to be much more suitable in many infinite-dimensional situations where strong convergence simply doesn't take place.

Let us first recall the basic concepts of the theory of Harris chains. Throughout all of this note, we fix a Polish space $\mathcal{X}$ (the 'state space' of our system) and a stochastically continuous semigroup of Feller Markov transition kernels $\{\mathcal{P}_t\}_{t\geq 0}$ on $\mathcal{X}$. One of the most basic notions in the theory of Harris chains is that of a small set:

Definition 1. *A set $A \subset \mathcal{X}$ is 'small' (for the semigroup $\{\mathcal{P}_t\}_{t\geq 0}$) if there exists $\delta > 0$ and $t \geq 0$ such that $\|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{\mathrm{TV}} \leq 2-\delta$ for every pair $x, y \in A$.*

This is actually slight weakening of the usual notion of smallness (see [10]) which is somewhat more suitable for the purpose of this note. Another important notion is that of a *strong Feller* semigroup:

Definition 2. *The semigroup $\{\mathcal{P}_t\}$ is strong Feller if there exists $t \geq 0$ such that $\mathcal{P}_t\phi$ is continuous for every bounded Borel measurable function $\phi\colon \mathcal{X} \to \mathbb{R}$.*

Finally, we say that a point $x \in \mathcal{X}$ is *accessible* if there exists $t \geq 0$ such that $\mathcal{P}_t(y, U) > 0$ for every $y \in \mathcal{X}$ and every neighbourhood U of x. With these definitions at hand, we have [10, 2]:

Theorem 3. *If a Markov semigroup is strong Feller and has an accessible point, then every compact set is small. Furthermore, it can have at most one invariant probability measure.*

If one knows furthermore that the process remains in 'small' regions of the phase space for most of the time, then one can say much more. In order to measure the stability of a process, the concept of a Lyapunov function is useful:

Definition 4. *A measurable function $V\colon \mathcal{X} \to \mathbb{R}_+$ is a Lyapunov function if there exist strictly positive constants C, γ and K such that the bound*

$$\mathcal{P}_t V(x) \le C e^{-\gamma t} V(x) + K \;, \tag{1}$$

holds for every $x \in \mathcal{X}$ and for every $t \ge 0$.

Given a Lyapunov function V, one can define a weighted total variation norm on the space of all signed measures that integrate V by

$$\|\mu\|_V = \int_{\mathcal{X}} \bigl(1 + V(x)\bigr)\, |\mu(dx)| \;. \tag{2}$$

With this definition at hand, we have (see [10]; see also [7] for a simple proof):

Theorem 5 (Harris)**.** *If a Markov semigroup admits a Lyapunov function V such that the level sets $\{x : V(x) \le C\}$ are all small, then there exist constants $\tilde{C}$ and $\tilde{\gamma}$ such that*

$$\|\mathcal{P}_t \mu - \mathcal{P}_t \nu\|_V \le \tilde{C} e^{-\tilde{\gamma} t} \|\mu - \nu\|_V \;, \tag{3}$$

for every $t \ge 0$ and every pair of probability measures μ and ν on $\mathcal{X}$. In particular, $\mathcal{P}_t$ admits exactly one invariant probability measure.

The problem with the theory of Harris chains is that it is not very well adapted to infinite-dimensional problems. While probability measures on finite-dimensional spaces are 'often' absolutely continuous with respect to some reference measure (usually Lebesgue measure), this is not so often the case in infinite dimensions due to the lack of a 'natural' reference measure. It is therefore relatively 'rare' for a Markov semigroup on an infinite-dimensional space to have the strong Feller property. (Basically, some rather strong form of invertibility is typically required of the covariance operator of an infinite-dimensional diffusion for it to generate a strong Feller semigroup, see for example [1, 2, 3].) In particular, one does often not expect to have convergence in a total variation norm as in Theorem 5.

This suggests that one should look for an alternative 'weak' (i.e. dealing with weak convergence rather than total variation convergence) theory. This can be achieved by making use of the following notion. An increasing sequence d_n of continuous pseudo-metrics on $\mathcal{X}$ is called *totally separating* if one has $d_n(x, y) \nearrow 1$ as $n \to \infty$ for any $x \neq y$. Each of these metrics can be lifted in a natural way to the space of probability measures on $\mathcal{X}$ by

$$d(\mu, \nu) = \sup_{Lip_d\, \phi = 1} \int_{\mathcal{X}} \phi(x) \bigl(\mu - \nu\bigr)(dx) \;, \tag{4}$$

where we denote by $Lip_d\, \phi$ the (minimal) Lipschitz constant of $\phi\colon \mathcal{X} \to \mathbb{R}$ with respect to the metric d. With this definition, it turns out [5] that for any system of totally separating pseudo-metrics, one has the identity

$$\|\mu - \nu\|_{\mathrm{TV}} = 2 \sup_{n \geq 0} d_n(\mu, \nu) . \tag{5}$$

On the other hand, it is known [11, 4] that the strong Feller property is equivalent to the continuity of transition probabilities in the total variation distance. Therefore, a Markov semigroup is strong Feller if and only if there exists $t \geq 0$ such that, for every $x \in \mathcal{X}$, one has

$$\inf_{U \ni x} \sup_{y \in U} \sup_{n \geq 0} d_n(\mathcal{P}_t(x, \cdot\,), \mathcal{P}_t(y, \cdot\,)) = 0 , \tag{6}$$

where the first infimum runs over all neighbourhoods of x. This motivates the following natural extension to the strong Feller property [5]:

Definition 6. *A Markov semigroup $\mathcal{P}_t$ satisfies the asymptotic strong Feller property if there exists a sequence $t_n \nearrow \infty$ and a system d_n of totally separating continuous pseudo-metrics such that, for every $x \in \mathcal{X}$, one has*

$$\inf_{U \ni x} \sup_{y \in U} \sup_{n \geq 0} d_n(\mathcal{P}_{t_n}(x, \cdot\,), \mathcal{P}_{t_n}(y, \cdot\,)) = 0 . \tag{7}$$

This definition is not only natural, it is also useful as can be seen by [5]:

Theorem 7. *If a Markov semigroup is asymptotically strong Feller and has an accessible point, then it can have at most one invariant probability measure.*

Under slightly stronger assumption, it is also possible to give a generalisation of Harris' theorem on exponential convergence to this setting [9]. To conclude, let us point out that the asymptotic strong Feller property can be verified in a number of situations where the strong Feller property is either known to fail (stochastic delay equations with fixed delay in the diffusion coefficient [9]) or conjectured to fail (stochastic PDEs satisfying a Hörmander-type condition, but driven only by finitely many Wiener processes [8]).

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Simple SDE dynamical models interpreting climate data and their meta-stability

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(joint work with Claudia Hein, Ilya Pavlyukevich)

Simple models of the earth's energy balance are able to interpret some qualitative aspects of the dynamics of paleo-climatic data. In the 1980s this led to the investigation of periodically forced dynamical systems of the reaction-diffusion type with small Gaussian noise, and a rough explanation of glacial cycles by Gaussian meta-stability. A spectral analysis of Greenland ice time series performed at the end of the 1990s representing average temperatures during the last ice age suggest an $\alpha-$stable noise component with an $\alpha \sim 1.75$. ([3], [4]). Based on this observation, papers in the physics literature attempted an interpretation featuring dynamical systems perturbed by small Lévy noise. The typical 1-dimensional system is given by the stochastic differential equation

$$X^\epsilon(t) = x - \int_0^t U'(X^\epsilon(s-))\,ds + \epsilon L(t), \quad t \geq 0, \epsilon > 0. \tag{1}$$

Here L is symmetric α-stable Lévy process, with $\alpha \in (0, 2)$, U a potential function, at least C^2, typically with finitely many local minima where U has positive curvature, separated by saddles where U has negative curvature. We study the exit time and transition asymptotics in the small noise limit $\epsilon \to 0$ between the meta-stable local minima of the potential function for the solution trajectories of (1). Due to the heavy-tail nature of its α-stable component, the results for Lévy noise differ strongly from the well known case of purely Gaussian perturbations. For a comparison of Gaussian and α-stable noise asymptotics in a simple case, consider a potential function with exactly one well at 0, where $U(0) = 0$, assume that $U' < 0$ on $]-\infty, 0[$, and $U' > 0$ on $]0, \infty[$, consider the exit times from an interval $[-b, a]$ containing 0, and let $h = \max\{U(-b), U(a)\}$. Denote by σ resp. τ the exit time of the trajectories of X^ϵ resp. the analogue of (1) with a Wiener process W replacing L. Then the following type of asymptotic behavior results, for which the Gaussian part is classical.

Theorem 1. *For any $\delta > 0$, $x \in]-b, a[$ with respect to the probability measure $\mathbf{P_x}$ of X^ϵ perturbed by W starting at x*

$$\mathbf{P_x}(e^{(2h-\delta)/\epsilon^2} < \tau < e^{(2h+\delta)/\epsilon^2}) \to 1 \quad (\epsilon \to 0) \quad \textit{(Freidlin-Wentzell)},$$

$$\mathbf{E_x}\tau \approx \frac{\epsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\epsilon^2} \quad \textit{(Kramers, Williams, Bovier et al.)},$$

$$\mathbf{P_x}(\frac{\tau}{\mathbf{E_x}\tau} > u) \sim \exp(-u) \quad \textit{(Day, Bovier et al.)}.$$

Theorem 2. *([8]) For any $\delta > 0$, $x \in]-b,a[$ with respect to the probability measure $\mathbf{P_x}$ of X^ϵ perturbed by L starting at x*

$$\mathbf{P_x}(\frac{1}{\epsilon^{\alpha-\delta}} < \sigma < \frac{1}{\epsilon^{\alpha+\delta}}) \to 1 \quad (\epsilon \to 0),$$

$$\mathbf{E_x}\sigma \approx \frac{1}{\epsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]} \frac{dy}{|y|^{1+\alpha}}\Big)^{-1},$$

$$\mathbf{P_x}(\frac{\sigma}{\mathbf{E_x}\sigma} > u) \sim \exp(-u).$$

Ditlevsen's ([3], [4]) interpretation of paleo-climatic time series by simple dynamical systems with noise presents a typical statistical model selection problem. In the parametric version involved, one needs an efficient testing method for the parameter α corresponding to the best fitting α-stable noise component. We develop a statistical testing method based on the p-variation of the solution trajectories of SDE with Lévy noise which have been used before in model selection problems for high frequency financial time series (Corcuera et al. [2], Jacod [9], Barndorff-Nielsen, Shephard [1]). If $X^\epsilon = Y^\epsilon + L^\epsilon$ denotes the solution of (1), where Y^ϵ is the absolutely continuous part related to the potential gradient, and $L^\epsilon = \epsilon L$, and if for some stochastic process Z we let

$$V_t^{p,n}(Z) = \sum_{i=1}^{[nt]} |Z(\frac{i}{n}) - Z(\frac{i-1}{n})|^p, \quad V_t^p(Z) = \lim_{n\to\infty} V_t^{p,n}(Z), \quad t \geq 0,$$

we take $V^p(X^\epsilon)$ as a test statistic for α. In fact, it is well known that the stability index α of a stable process L can be identified with the least p for which $V^p(L) = 0$. Our elementary testing results for the real data from the Greenland ice core are based on the following limit theorems. Special cases of the first one have been treated in Greenwood [5] and Greenwood, Fristedt [6].

Theorem 3. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process. If $p > \alpha/2$ then*

$$\big(V_p^n(L)_t - nt\, B_n(\alpha,p)\big)_{t\geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t\geq 0} \quad \textit{as } n \to \infty, \tag{2}$$

where L' is an independent $\frac{\alpha}{p}$-stable Lévy process, and $\xrightarrow{\mathcal{D}}$ denotes convergence in the Skorokhod topology. The normalizing sequence $(B_n(\alpha,p))_{n\geq 1}$ is deterministic and given by

$$B_n(\alpha,p) = \begin{cases} n^{-p/\alpha}\mathbb{E}|L_1|^p, & p \in (\alpha/2,\alpha), \\ \mathbb{E}\sin\big(n^{-1}|L_1|^\alpha\big), & p = \alpha, \\ 0, & p > \alpha. \end{cases} \tag{3}$$

Intuitively, adding absolutely continuous processes to a Lévy process L should not alter its power variations. This intuition is confirmed by the following theorems.

Theorem 4. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process, and $(Y_t)_{t\geq 0}$ be another stochastic process that satisfies*

$$V_p^n(Y) \to 0, \quad n \to \infty, \tag{4}$$

uniformly on compacts of $\mathbb{R}_+$ in probability for some $p \in (\alpha/2, 1) \cup (\alpha, \infty)$. Then

$$(V_p^n(L+Y)_t - nt\, B_n(\alpha, p))_{t\geq 0} \stackrel{\mathcal{D}}{\to} (L'_t)_{t\geq 0} \quad \text{as } n \to \infty, \tag{5}$$

with L' an independent $\frac{\alpha}{p}$-stable process, $(B_n(\alpha,p))_{n\geq 1}$ defined in (3).

Theorem 5. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process, $\alpha \in (1,2)$ and let $(Y_t)_{t\geq 0}$ be another stochastic process. Let $p \in (1, \alpha]$ and $T > 0$. If Y is such that for every $\delta > 0$ there exists $K(\delta) > 0$ that satisfies*

$$\mathbb{P}(|Y_s(\omega) - Y_t(\omega)| \leq K(\delta)|s-t| \text{ for all } s,t \in [0,T]) \geq 1 - \delta, \tag{6}$$

the process Y does not contribute to the limit of $V_p^n(L+Y)$, i.e.

$$(V_p^n(L+Y)_t - nt\, B_n(\alpha, p))_{0\leq t\leq T} \stackrel{\mathcal{D}}{\to} (L'_t)_{0\leq t\leq T}, \quad \text{as } n \to \infty, \tag{7}$$

with L' an independent $\frac{\alpha}{p}$-stable process, $(B_n(\alpha,p))_{n\geq 1}$ defined in (3).

We apply these limit theorems to test the real data from the Greenland ice core for the best fitting stability index α. To this end, we use the Kolmogorov-Smirnov distance between the empirical law of $V^{2\alpha,n}(X^\epsilon)$ and the $\frac{1}{2}$-stable law, stipulating that this distance has to be minimal in case the noise in the data contains an α-stable component. We obtain $\alpha = 0.73$ as best fit, which is nicely confirmed in a comparison with simulated data, and surprisingly differs from the result obtained by Ditlevsen ([3], [4]) by a quantity very close to 1.

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Questions concerning the algebro-geometric Sturm-Liouville potentials

Russell Johnson

In what follows, we first indicate how the algebro-geometric Sturm-Liouville potentials are derived. Then we discuss their connection with the Camassa-Holm equation of shallow water-wave theory. Finally we consider more general classes of Sturm-Liouville potentials and some questions concerning them.

Our starting point is the Sturm-Liouville equation

$$-(p\varphi')' + q\varphi = \lambda y \varphi \qquad ' = \frac{d}{dx} \tag{1}$$

where p, q and y are taken to be bounded, uniformly continuous, real-valued functions of $x \in \mathbb{R}$, and λ is a complex parameter. We also suppose that $p' = \frac{dp}{dx}$ is bounded and uniformly continuous, and that there exists a positive number δ such that $p(x) \geq \delta$, $y(x) \geq \delta$ for all $x \in \mathbb{R}$. Introduce the notation $a = (p, q, y)$.

We define a linear differential expression L_a by

$$L_a(\varphi) = -(p\varphi')' + q\varphi.$$

Then L_a determines a self-adjoint operator on the weighted L^2-space $L^2(\mathbb{R}, y(x)dx)$. Its spectrum Σ_a is a closed subset of $\mathbb{R}$ which is bounded below and unbounded above.

Let us now describe without much detail the construction of the algebro-geometric Sturm-Liouville potentials $a = (p, q, y)$. Introduce the Weyl m-functions $m_\pm(\lambda)$ for the operator L_a; these are well-defined because, under our assumptions, L_a is in the limit-point case at $x = \pm\infty$. The functions $m_\pm(\lambda)$ are defined and holomorphic in the set $\mathbb{C}_* = \{\lambda \in \mathbb{C} \mid \Im\lambda \neq 0\}$, and one has

$$sgn \frac{\Im m_\pm(\lambda)}{\Im\lambda} = \pm 1 \qquad (\lambda \in \mathbb{C}_*).$$

Next we impose the following conditions on L_a and on the Weyl functions $m_\pm(\lambda)$.

(i) The spectrum Σ_a is a finite union of non degenerate intervals:

$$\Sigma_a = [\lambda_0, \lambda_1] \cup \ldots \cup [\lambda_{2g}, \infty);$$

(ii) The Weyl functions $m_\pm(\lambda)$ glue together to form a single meromorphic function M_a on the Riemann surface of the relation

$$y^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_{2g});$$

A Sturm-Liouville potential $a = (p, q, y)$ which satisfies (i) and (ii) is by definition of algebro-geometric type. Such potentials can be described in a fairly detailed way [5, 6]. When condition (i) holds and a is T-periodic, then condition (ii) is automatically satisfied. One can construct non-periodic, "ergodic" algebro-geometric potentials by assuming that condition (i) holds and that the Lyapunov exponent β_a vanishes on Σ_a.

One obtains the Schrödinger potentials by setting $p = y = 1$; then $L_a = L_q$ takes the form

$$L_q(\varphi) = -\varphi'' + q(x)\varphi$$

and acts in $L^2(\mathbb{R}, dx)$. The algebro-geometric Schrödinger potentials were discussed by Dubrovin-Matweev-Novikov in [3] and have a quite explicit description. Moreover, they have an important role in the theory of the Korteweg-de Vries equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = 3u\dfrac{\partial u}{\partial x} - \dfrac{1}{2}\dfrac{\partial^3 u}{\partial x^3} \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

Namely, if the initial datum $u_0(x)$ is an algebro-geometric Schrödinger potential, then (2) admits a global solution with an explicit description.

An analogous situation arises when $p = q = 1$ and the operator $L = -\dfrac{d^2}{dx^2} + 1$ acts in the Hilbert space $L^2(\mathbb{R}, y(x)dx)$. Then the algebro-geometric potentials $(1, 1, y)$ determine solutions of the Camassa-Holm equation

$$\begin{cases} 4\dfrac{\partial u}{\partial t} = \dfrac{\partial^3 u}{\partial^2 \partial t} + 2u\dfrac{\partial^3 u}{\partial x^3} + 4\dfrac{\partial u}{\partial x}\dfrac{\partial^2 u}{\partial x^2} - 24u\dfrac{\partial u}{\partial x} \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

The details have been worked out by Zampogni [10]; see also previous papers by Alber-Fedorov [1] and Gesztesy-Holden [4].

There are two more general classes of Sturm-Liouville potentials which are of interest, namely the reflectionless potentials and the Segal-Wilson (SW) potentials.

The first class is defined as follows. Let $a = (p, q, y)$ determine an operator L_a whose spectrum is Σ_a. Introduce the so-called diagonal Green's function

$$g_a(\lambda) = \frac{y(0)}{m_-(\lambda) - m_+(\lambda)} \qquad (\Im\lambda > 0).$$

Thus g_a is obtained by evaluating the Kernel $g_a(x, y, \lambda)$ of $(L_a - \lambda)^{-1}$ at $x = y = 0$, for complex values of λ with $\Im\lambda > 0$. The nontangential limit $g_a(\lambda + i0) = \lim_{\varepsilon \to 0^+} g_a(\lambda + i\varepsilon)$ is defined for Lebesgue a.a. $\lambda \in \Sigma_a$.

Definition 1. *The potential $a = (p, q, y)$ is said to be reflectionless if Σ_a has locally positive Lebesgue measure and if $\Re g_a(\lambda + i0) = 0$ for a.a. $\lambda \in \Sigma_a$.*

The reflectionless Sturm-Liouville potentials have been studied in [6]; motivation for that paper came from previous papers on reflectionless Schrödinger potentials (e.g., [2]). It turns out that certain kinds of Cantor sets $\Sigma \subset \mathbb{R}$ can be realized as the spectrum Σ_a of an operator L_a determined by a reflectionless potential $a = (p, q, y)$.

The Segal-Wilson potentials a are determined by the following conditions:

(i) The spectrum Σ_a contains a half-interval (λ_*, ∞) where $\lambda_* \in \mathbb{R}$;
(ii) The Weyl functions $m_\pm(\lambda)$ ramify near $z = \infty$ where $z = \sqrt{\lambda}$.

The second condition means that, if

$$M(z) = \begin{cases} m_+(z^2) & \Im z > 0 \\ m_-(z^2) & \Im z < 0, \end{cases}$$

then for sufficiently large λ_*, $M(\cdot)$ extends holomorphically to $\{z \in \mathbb{C} \mid |z| > \sqrt{\lambda_*}\}$ and M admits a simple pole at $z = \infty$.

In the Schrödinger case, an analogous class $\{q\}$ of potentials was introduced and studied by Segal and Wilson [9], who interpreted ideas of Sato [8]. It turns out that the K-dV equation can be solved if the initial datum u_0 is of SW type.

Both the reflectionless class and the SW class of Strum-Liouville potentials merit further study. We indicate some basic questions concerning them.

(1) Is a Segal-Wilson potential always reflectionless?
(2) Does the Camassa-Holm equation (3) admit a global solution if the initial datum is derived from reflectionless potential $a = (1, 1, y)$?

We finally say a few words about potentials which satisfy a Novitskii condition [7]. Let $(\mathcal{A}, \{\tau_x\}, \mu)$ be an ergodic family of Sturm-Liouville potentials $a(\cdot) = (p(\cdot), q(\cdot), y(\cdot))$. Here $\mathcal{A}$ is a compact subset of an appropriate functional space of triples $a(\cdot)$, and $\tau_x : \mathcal{A} \to \mathcal{A}$ is translation by x, thus $\tau_x a(\cdot) = a(\cdot + x)$. Moreover, μ is a $\{\tau_x\}$-ergodic probability measure on $\mathcal{A}$.

Introduce the Floquet exponent of $(\mathcal{A}, \{\tau_x\}, \mu)$:

$$w(\lambda) = \int_{\mathcal{A}} \frac{m(a, \lambda)}{p(0)} d\mu(a) \qquad (\Im\lambda > 0).$$

This is a remarkable function. One of its properties is that its boundary value $w(\lambda + i0) = \lim_{\varepsilon \to 0} w(\lambda + i\varepsilon)$ equals $-\beta(\lambda) + i\alpha(\lambda)$ $(\lambda \in \mathbb{R})$, where β (resp. α) is the μ-Lyapunov exponent (resp. the μ-rotation number).

Suppose now that $a(\cdot) \in C^\infty(\mathbb{R})$ for μ-a.a. $a \in \mathcal{A}$. Then w admits a formal asymptotic expansion:

$$w(\lambda) \sim i\sqrt{\lambda} + \frac{i}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \frac{I_n}{\lambda^n}.$$

Novitskii considers conditions on $(\mathcal{A}, \{\tau_x\}, \mu)$ which ensure that the integrals $\{I_n\}$ uniquely determine $w(\lambda)$.

We formulate one more question.

(3) If Novitskii's conditions are satisfied, is $a(\cdot)$ reflectionless for μ-a.a. $a \in \mathcal{A}$?

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The exponential Euler scheme for stochastic partial differential equations

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(joint work with Arnulf Jentzen)

The numerical approximation of stochastic partial differential equations (SPDEs), by which we mean stochastic evolution equations of the parabolic or hyperbolic type, encounters all of the difficulties that arise in the numerical solution of both deterministic PDEs and finite dimensional stochastic ordinary differential equations (SODEs) plus more due to the infinite dimensional nature of the driving noise processes. The state of development of numerical schemes for PDEs compares with that for SODEs in the early 1970s. Most of the numerical schemes that have been proposed to date have a low order of convergence, especially in terms of an overall computational effort, and only recently has it been shown how to construct higher order schemes. The break through for SODEs started with the Milstein scheme and continued with the systematic derivation of stochastic Taylor expansions and the numerical schemes based on them. These stochastic Taylor schemes are based on an iterated application of the Ito formula. The crucial point is that the mulptiple stochastic integrals which they contain provide more information about the noise processes within discretization subintervals and this allows an approximation of higher order to be obtained.

Davie & Gaines [1] showed that a numerical scheme involving only increments of the driving Wiener process applied to a stochastic reaction diffusion equation with additive noise has maximal computational convergence rate $\frac{1}{6}$ (for a one-dimensional domain). This was improved in Jentzen & Kloeden [5] who considered a general parabolic SPDE with additive noise in a Hilbert space $(H, |\cdot|)$,

$$dU_t = [AU_t + f(U_t)]\,dt + dW_t, \qquad U_0 = u_0, \tag{1}$$

where A is an in general unbounded operator (for example, $A = \Delta$), f is a nonlinear continuous function and W_t is a cylindrical Wiener process. They interpeted this in the mild sense, i.e. as satisfying the integral equation

$$U_t = e^{At}u_0 + \int_0^t e^{A(t-s)} f(U_s)\,ds. + \int_0^t e^{A(t-s)}\,dW_s \tag{2}$$

They introduced the *exponential Euler scheme*

$$\begin{aligned} V_{k+1}^{(N,M)} \quad &= \quad e^{A_N\Delta}V_k^{(N,M)} + A_N^{-1}\left(e^{A_N\Delta} - I\right) f_N(V_k^{(N,M)}) \\ &\qquad + \int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\,dW_s^N \end{aligned} \tag{3}$$

with time–step $\Delta = \frac{T}{M}$ for some $M \in \mathbb{N}$ and discretization times $t_k = k\Delta$ for $k = 0, 1, \ldots, M$, to the N-dimensional Ito-Galerkin SODE in the space $H_N := P_N H$ (or, equivalently, in $\mathbb{R}^N$). This scheme is in fact easier to simulate than may seem on the first sight it can be rewritten componentwise as

$$\begin{aligned} V_{k+1,1}^{(N,M)} &= e^{-\lambda_1\Delta}V_{k,1}^{(N,M)} + \frac{(1-e^{-\lambda_1\Delta})}{\lambda_1} f_N^1(V_k^{(N,M)}) + \left(\frac{q_1}{2\lambda_1}(1-e^{-2\lambda_1\Delta})\right)^{\frac{1}{2}} R_k^1 \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ V_{k+1,N}^{(N,M)} &= e^{-\lambda_N\Delta}V_{k,N}^{(N,M)} + \frac{(1-e^{-\lambda_N\Delta})}{\lambda_N} f_N^N(V_k^{(N,M)}) + \left(\frac{q_N}{2\lambda_N}(1-e^{-2\lambda_N\Delta})\right)^{\frac{1}{2}} R_k^N \end{aligned}$$

where the R_k^i for $i = 1, \ldots, N$ and $k = 0, 1, \ldots, M-1$ are independent, standard normally distributed random variables. They obtained the convergence rate:

Theorem 1. *Under Assumptions (A1-(A4) given in* [5] *. Then, there is a constant* $C_T > 0$ *such that*

$$\sup_{k=0,\ldots,M}\left(\mathbb{E}\left|U_{t_k} - V_k^{(N,M)}\right|^2\right)^{\frac{1}{2}} \le C_T\left(\lambda_N^{-\gamma} + \frac{\log(M)}{M}\right) \tag{4}$$

holds for all $N, M \in \mathbb{N}$*, where* U_t *is the solution of SDE* (1)*,* $V_k^{(N,M)}$ *is the numerical solution given by* (3)*,* $t_k = T\dfrac{k}{M}$ *for* $k = 0, 1, \ldots, M$*, and* $\gamma > 0$ *is the constant given in Assumption (A2).*

In fact, the exponential Euler scheme (3) converges in time with a strong order $1-\varepsilon$ for an arbitrary small $\varepsilon > 0$ since $\log(M)$ can be estimated by M^ε, so

$$\frac{\log(M)}{M} \sim h\log\frac{1}{h} \approx h^{1-\varepsilon}$$

Importantly, the error coefficient C_T does not depend on the dimension N of the Ito-Galerkin SDE.

An essential point is that the integral $\int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\, dW_s^N$ includes includes more information about the noise on the discretization interval. Such additional information was the key to the higher order of stochastic Taylor schemes for SODE, but is included there in terms of simple multiple stochastic integrals rather than integrals weighted by an exponential intergand. Taylor expansions of solutions of SPDE in Hilbert spaces are the basis for deriving higher order Taylor schemes for SPDE, just as for SODE. However, there is a major difficulty for SPDE. Although, the SPDE (1) is driven by a martingale Brownian motion, the solution process is not a semi-martingale any more and a general Itô formula does not exist for its solutions, just special cases. Hence stochastic Taylor-expansions for the solutions of the SPDE (1) cannot be derived as for the solutions of finite dimensional SODE. However Jentzen & Kloeden [6] showed how can such expansions be constructed by taking advantage of the mild form representation of the solutions. Moreover, these expansions are robust with respect to the noise, i.e. hold for other types of stochastic processes with Hölder continuous paths such as fractional Brownian motion.

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Glauber dynamics in a continuum

Yuri Kondratiev

The continuous Glauber type dynamics may be characterized as birth-and-death Markov processes on the configuration space (CS) in continuum which have given grand canonical Gibbs equilibrium state as a symmetrizing measure. Markov generators of these processes are related with (non-local) Dirichlet forms of Gibbs measures. The latter gives a possibility to construct so-called equilibrium Glauber dynamics associated with these forms [4]. There are many possibilities to choose a particular form the square field expression inside of the Dirichlet forms leading to different birth and death intensities in the generator of the Glauber dynamics. For example, we can use a constant death rate including all the information about the interaction in the system in the birth rate (G^+ Glauber dynamics). Another extremal possibility is to take a constant birth rate and to assure reversibility of the dynamics via proper death coefficient (G^- Glauber dynamics). In the case of G^+ equilibrium dynamics and a positive interaction potential, there exists a spectral gap for the generator in high temperature and low density regime [1], [4], [5]. This gives an exponential L^2 ergodicity for the corresponding Markov semigroup.

Coming to the problem of non-equilibrium Glauber dynamics, we note that only recently were constructed Markov processes with some classes of initial distributions (i.e., Markov functions) for G^+ [3] and G^- [2] dynamics. These papers use a constructive approach to the dual Kolmogorov equation describing the evolution of the initial distributions in the Glauber stochastic dynamics. Let us stress that, in the infinite particle case, a class of admissible initial conditions should be considered as an essential parameter in the study of dynamical properties. Depending on the initial conditions, stochastic dynamics may have very different behaviors including a possibility to be explosive in a finite time. Roughly speaking, a choice of initial states defines the level of deviation from the equilibrium dynamics.

In the talk we analyze ergodic properties of such non-equilibrium random evolutions. Namely, we consider the case of G^- stochastic dynamics for interacting potentials which satisfy stability and strong integrability properties (but admit a possible negative part). Main result about ergodicity gives a convergence of the time evolution for a class of initial measures to the Gibbs invariant measure under natural restrictions on parameters of the system. More precisely, we need to consider the equilibrium state in the high temperature and low density regime to assure the uniqueness of the limiting Gibbs measure. Then depending on these parameters we define explicitly the set of admissible initial states. Let us stress that this set forms a ball in a proper metric in the space of all probability measures on the CS. This ball includes the invariant Gibbs measure but also all probability measures on the CS with a common Ruelle bounds. The convergence of measures on the CS is defined in the sense of their correlation functions convergence. We show an exponential rate of such convergence in the Ruelle type norm on correlation functions.

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Brownian noise and the depletion phenomenon

Peter Kotelenez
(joint work with M.J. Leitman and J.A. Mann)

Kotelenez [5, 6] introduced a model of correlated Brownian motions as a perturbation of as system of coupled stochastic ordinary differential equations (SODEs). The associated measure process is a solution of a stochastic partial differential equation (SPDE), similar to the transition from the description of N point vortices by a system of coupled ordinary differential equations (ODEs) to a first order partial differential equation (PDE - the Euler equation) for the vorticity in 2D fluid mechanics, [2, 11]. We briefly review the derivation of a system of N correlated Brownian motions as a kinematic mesoscopic limit from a system of nonlinear deterministic oscillators consisting of N large (solute) particles and infinitely many small (solvent) particles in space dimension $d \geq 2$, cf. [7]. The oscillators are coupled by a mean field force between the large and the small particles. The correlated Brownian motions are represented as the convolution of a suitable d-dimensional kernel G_ε with standard Gaussian space-time white noise $w(dr, dt)$. The kernel G_ε is a properly re-scaled version of the mean field force from the system of oscillators where ε is the correlation length. We analyze these correlated Brownian motions in the context of modeling the depletion effect in colloids, cf. [1, 9, 10]. In particular we show that two correlated Brownian particles when close have an initial tendency to attract each other further. In space dimension $d \geq 2$ for large times they behave like independent Brownian motions. The key to this short time result is a generalization of the one-dimensional probability flux, as defined by van Kampen [4] to $d \geq 2$ dimensions. The flux is obtained by decomposing the associated SODE for the separation of the two Brownian particles into two differential equations. The first differential equation is a deterministic ordinary differential equation, describing the flux. The second differential equation is an SODE which describes a symmetric diffusion. Employing fractional steps can reverse the decomposition, cf. [3]. We also refer to Kotelenez [8] for more details on most of the results mentioned above.

The following unresolved problem arises in our analysis:

For a Maxwellian kernel G_ε (and similar unimodal kernels) there is an attractive zone for very short distances between the Brownian (solute) particles and a repulsive zone at moderate distances. At large distances there are no visible correlations. The presence of the repulsive zone is a mathematical fact. The problem is whether the physics requires such a zone and, if not, whether it is possible to define correlated Brownian motions without such a repulsive zone.

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Random Perturbations of KdV

Sergei B. Kuksin

Consider the damped-driven KdV equation

$$(1) \qquad \dot{u} - \nu u_{xx} + u_{xxx} - 6uu_x = \sqrt{\nu}\,\eta(t,x),\ x \in S^1,\ \int u\,dx \equiv \int \eta\,dx \equiv 0\,,$$

where $0 < \nu \leq 1$ and the random process η is smooth in x, white in t and satisfies some mild non-degeneracy assumptions. Solution $u(t,x)$ either has a prescribed initial value

a) $\qquad u(0,x) = u_0(x)$,

or

b) $u(t,x)$ is stationary in time t.

For any periodic function $u(x)$ let $I = (I_1, I_2, \dots)$ be the vector, formed by the KdV integrals of motion, calculated for the potential $u(x)$. The vector I belongs to the positive octant $\mathbb{R}^\infty_+$ and is called the vector of actions. The actions I may be supplemented by angles $\phi \in \mathbb{T}^\infty$ in such a way that action-angles (I,ϕ) form a coordinate system in the function phase-space for eq. (1).

Let u_ν be a solution of (1)+(a) or of (1)+(b). Let $\tau = \nu t$ be the fast time. We are concerned with the following

Problem. How the vector $I(u_\nu(\tau))$, $0 \leq \tau \leq 1$, behave when $\nu \to 0$?

Applying Ito formula to the map $u \mapsto I$ we get for $I(\tau) = I(u(\tau))$ a stochastic differential equation (in $\mathbb{R}^\infty_+$) with coefficients, depending on I and ϕ. Averaging the drift and the diffusion in ϕ (note that the diffusion has to be averaged, using the rules of the stohastic calculus) we get for $I(\tau)$ the new equation, called *the Whitham equation.*

In [1] we proved

Theorem 1. Every sequence $\nu'_j \to 0$ contains a subsequence $\nu_j \to 0$ such that the process $I(u_{\nu_j}(\tau)), 0 \le \tau \le 1$, converges in distribution to a limiting process $I(\tau)$ which satisfies the Whitham equation. In the case a) we have $I(0) = I(u_0)$. In the case b) the process $I(\tau)$ is stationary. Moreover, $\mathcal{D}u_{\nu_j}(0)$ (the law of $u_{\nu_j}(0)$) converges to a limiting measure γ which in the action-angle variables (I, ϕ) may be written as $\gamma = \mu \times d\phi$, where $d\phi$ is the Haar measure on $\mathbb{T}^\infty$ and $\mu = \mathcal{D}I(0)$ (this is a stationary measure for the Whitham equation).

It is unknown if the Whitham equation has a unique solution and a unique stationary measure. Therefore, apriori the limiting process $I(\tau)$ may depend on the sequence $\nu_j \to 0$. Fortunately this does not happen in the case a) and does not happen in the case b) if the force η is not big:

Theorem 2 (see [2]). In the case a) we have

$$\mathcal{D}I(u_\nu(\cdot)) \to \mathcal{D}I(\cdot) \qquad \text{as} \quad \nu \to 0 \tag{2}$$

(the first arrow indicates the weak convergence of measures).

Denote by B "the energy of the force η", $B = \mathbf{E}|\int_0^1 \eta(s,\cdot)\,ds|^2_{L_2}$.

Theorem 3 (see [3]). In the case b) there exists a constant $\rho > 0$ such that if $B \le \rho$, then (2) holds.

Corollary. If $B \le \rho$, then any solution $u_\nu(t,x)$ of eq. (1) satisfies

$$\lim_{\nu\to 0}\lim_{t\to\infty} \mathcal{D}u_\nu(t,\cdot) = \gamma,$$

where γ is the same measure as in Theorem 1.

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A Central Limit Theorem for Isotropic Flows

Yves Le Jan

(joint work with M. Cranston)

A general problem in fluid dynamics is to describe the behavior of a body of passive tracers being carried by a random current or turbulent action of a fluid. Real world examples of a body of passive tracers would be an oil slick on the surface of the ocean or a mass of plankton, both of which would be dispersed in some way by the random current of the ocean. Since turbulent actions are approximated by stochastic flows, a reasonable approach is to consider how a body of passive tracers behaves under stochastic flows. As is usually the case, the advantage of the model using stochastic flows over that of random currents lies in the availability of the powerful tools of stochastic differential equations. The objective of this paper is to establish an *a.s.* Central Limit Theorem for the asymptotic distribution of a body of passive tracers carried by some canonical stochastic flows. In more precise terms, we establish an *a.s.* Central Limit Theorem for the flow of transition probabilities defined by the Kraichnan model of isotropic flows as well as for the more 'traditional' isotropic Brownian flows on $\mathbf{R^d}$. See [2], [3], [5], [1] and [6] for related work on these flows.

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Dimension-free Harnack inequality and applications for SPDE

Wei Liu

The (dimension-free) Harnack inequality was first introduced by Wang in [6] for the diffusion on Riemannian manifolds, we refer to the survey article [7] for it's applications in the study of diffusion semigroups in recent years. Recently, the Harnack inequality was established in [8] for the stochastic porous media equations and in [2] for the stochastic fast-diffusion equations, where the proofs are mainly based on a coupling argument and Girsanov transformation.

In [1] the Harnack inequality and strong Feller property are established for the transition semigroups associated to a large class of stochastic evolution equations with additive noise under the variational framework (cf. [3]). Moreover, it has been used to derive the ergodic, contractive and compact properties for the corresponding Markov semigroups. Recently, the exponential convergence to the equilibrium and the existence of the spectral gap has also been obtained. The main results can be applied to different type of SPDE in Hilbert space such as the stochastic porous media equation in [8], stochastic reaction-diffusion equation and stochastic p-Laplace equation in [3, 4].

Let $V \subset H \subset V^*$ be a Gelfand triple, i.e. H is a separable Hilbert space and V is a reflexive and separable Banach space such that $V \subset H$ is continuous and dense. Suppose ${}_{V^*}\langle \cdot, \cdot \rangle_V$ is the dualization between V^* and V, W_t is a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. In this report for simplicity we only consider the following stochastic evolution equation (with time independent coefficient)

$$dX_t = A(X_t)dt + BdW_t, \; X_0 = x \in H, \tag{1}$$

where $A : V \to V^*$ and B is a Hilbert-Schmidt operator from U to H. We first assume the following conditions for the well-posedness of (1)(cf.[3, 4]).

For a fixed exponent $\alpha > 1$, there exist constant $\delta > 0, \gamma$ and K such that the following conditions hold for all $v, v_1, v_2 \in V$.

$(A1)$ Hemicontinuity: The map $\lambda \mapsto {}_{V^*}\langle A(v_1 + \lambda v_2), v \rangle_V$ is continuous on $\mathbb{R}$.

$(A2)$ Strong monotonicity:

$${}_{V^*}\langle A(u) - A(v), u - v \rangle_V \le -\delta \|u - v\|_V^\alpha + \gamma \|u - v\|_H^2.$$

$(A3)$ Growth condition: $\|A(v)\|_{V^*} \le K(1 + \|v\|_V^{\alpha-1})$.

Then for any $x \in H$ (1) has a unique solution $\{X_t(x)\}$ which is an adapted continuous process on H. Now we consider the associated transition semigroup

$$P_t F(x) := \mathbf{E} F(X_t(x)), \; t > 0, \; x \in H.$$

We assume that B is non-degenerate(i.e. $Bx = 0$ implies $x = 0$) and define

$$\|x\|_B := \begin{cases} \|y\|_U, & \text{if } y \in U, By = x, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 1. *Suppose* $(A1) - (A3)$ *hold and there exist constant* $\sigma \ge 2, \sigma > \alpha - 2$ *and* $\xi > 0$ *such that*

$$\|u\|_V^\alpha \ge \xi \|u\|_B^\sigma \|u\|_H^{\alpha - \sigma}, \; v \in V. \tag{2}$$

Then for any $t > 0$, P_t *is strong Feller operator and for any positive measurable function* F *on* H, $p > 1$ *and* $x, y \in H$,

$$(P_t F)^p(y) \le P_t F^p(x) \exp\left[\frac{p}{p-1} C(t, \sigma) \|x - y\|_H^{2 + \frac{2(2-\alpha)}{\sigma}}\right], \tag{3}$$

where

$$C(t, \sigma) = \frac{2 t^{\frac{\sigma - 2}{\sigma}} (\sigma + 2)^{2 + \frac{2}{\sigma}}}{(\sigma + 2 - \alpha)^{2 + \frac{2}{\sigma}} (\delta \xi)^{\frac{2}{\sigma}} \left[\int_0^t \exp\left(\frac{(\alpha - 2 - \sigma)\gamma}{2\sigma} s\right) ds\right]^2}.$$

Theorem 2. *Suppose* $(A1) - (A3)$ *hold and the embedding* $V \subseteq H$ *is compact.*

(i) Suppose $\gamma \le 0$ *if* $\alpha \le 2$, *then* $\{P_t\}$ *has an invariant probability measure* μ *satisfying* $\mu\left(\|\cdot\|_V^\alpha + e^{\varepsilon_0 \|\cdot\|_H^\alpha}\right) < \infty$ *for some* $\varepsilon_0 > 0$.

(ii) If $\alpha > 2$ *and* $\gamma \le 0$, *then* $\{P_t\}$ *has a unique invariant measure* μ *and for any Lipschitz continuous function* F *on* H *we have*

$$\sup_{x \in H} |P_t F(x) - \mu(F)| \le C Lip(F) t^{-\frac{1}{\alpha - 2}}, \; t > 0, \tag{4}$$

where $C > 0$ is a constant and $Lip(F)$ is the Lipschitz constant. In particular, if $B = 0$ and Dirac measure at 0 is the unique invariant measure of $\{P_t\}$, then we can take $F(x) = \|x\|_H$ in (4) and obtain

$$\sup_{x \in H} \|X_t(x)\|_H \leq Ct^{-\frac{1}{\alpha-2}}, \ t > 0.$$

It gives the decay of the solution to a large class of deterministic evolution equations, which coincide with some well-known decay estimates in PDE theory.

Theorem 3. *Suppose all assumptions in the Theorem 1 hold.*

(i) $\{P_t\}$ is (topologically) irreducible (i.e. $P_t 1_M(\cdot) > 0$ on H for any $t > 0$ and nonempty open set M) and has a unique invariant measure μ with full support.

(ii) For any $x \in H$, $t > 0$ and $p > 1$, the transition density $p_t(x, y)$ of P_t w.r.t μ satisfies

$$\|p_t(x,\cdot)\|_{L^p(\mu)} \leq \left\{ \int_H \exp\left[-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}} \right] \mu(dy) \right\}^{-\frac{p-1}{p}}.$$

(iii) If $\alpha = 2$ and $\gamma \leq 0$, then P_t is hyperbounded (i.e.$\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$) and compact on $L^2(\mu)$ for some $t > 0$.

(iv) If $\alpha > 2$ and $\gamma \leq 0$, then P_t is ultrabounded (i.e.$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty$) and compact on $L^2(\mu)$ for any $t > 0$. Moreover, there exists a constant $C > 0$ such that

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq \exp\left[C(1 + t^{-\frac{\alpha}{\alpha-2}}) \right], \ t > 0.$$

Let $\mathcal{L}_p$ be the generator of the semigroup $\{P_t\}$ in $L^p(\mu)$. We say that $\mathcal{L}_p$ has the spectral gap in $L^p(\mu)$ if there exists $\gamma > 0$ such that

$$\sigma(\mathcal{L}_p) \cap \{\lambda : Re\lambda > -\gamma\} = \{0\}$$

where $\sigma(\mathcal{L}_p)$ is the spectrum of $\mathcal{L}_p$. The largest constant γ with this property is denoted by $gap(\mathcal{L}_p)$(cf. [5]).

Theorem 4. *Suppose all assumptions in Theorem 1 hold and μ denotes the unique invariant measure of $\{P_t\}$.*

(i) If $\alpha = 2$, then $\{P_t\}$ is V-uniformly ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \geq 0$ and $x \in H$

$$\sup_{\|F\|_V \leq 1} |P_t F(x) - \mu(F)| \leq CV(x)e^{-\eta t},$$

where $V(x) = 1 + \|x\|_H^2$ and

$$\|F\|_V := \sup_{x \in H} \frac{|F(x)|}{V(x)} < \infty.$$

(ii) If $\alpha > 2$, then $\{P_t\}$ is uniformly exponential ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \geq 0$ and $x \in H$

$$\sup_{\|F\|_\infty \leq 1} |P_t F(x) - \mu(F)| \leq Ce^{-\eta t}.$$

And for $p \in (1, \infty)$ *we have*

$$gap(\mathcal{L}_p) \geq \frac{\eta}{p},$$

and for each $F \in L^p(\mu)$

$$\|P_t F - \mu(F)\|_{L^p(\mu)} \leq C_p e^{-\eta t/p} \|F\|_{L^p(\mu)}, \ t \geq 0.$$

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Resolving dynamical fine structure of invariant sets using Conley index theory

Stanislaus Maier-Paape

In this talk we report on recent accomplishments of resolving dynamical fine structure of isolated invariant sets with the help of computer–assisted proofs and Conley index theory.

In particular we report on several subproblems like

- rigorous path following of equilibria (cf. eg. [3])
- rigorous resolution of bifurcation points
- calculation of Conley indices of (nontrivial) isolated invariant sets
- calculation of connection matrices and the software *conley* (cf. [2, 1])
- rigorous completeness of the equilibria set ,

how far they already have been automatized, and in which frameworks (certain Galerkin systems with gradient structure like for instance the Cahn–Hilliard equation) already mayor parts have been worked out.

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Stochastic porous media equations

Michael Röckner

In the three lectures I presented a selection of recent results on stochastic porous media equations which were obtained in joint work with a number of colleagues (see the references).

1. Existence and uniqueness

Stochastic porous media equations are stochastic partial differential euquations (SPDE) of the following type

$$dX(t) = [L\Psi(t, X(t)) + \Phi(t, X(t))]dt + B(t, X(t))dW(t) \tag{1}$$

where $W = (W(t))_{t\geq 0}$ is a cylindrical Wiener process in $L^2(m) := L^2(E, \mathcal{B}, m)$ on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(E, \mathcal{B}, m)$ is an arbitrary σ-finite measure space (e.g. $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), d\xi)$ with $d\xi$ = Lebesgue measure). L with domain $\mathcal{D}(L)$ is a nonnegative definite self-adjoint operator on $L^2(m)$ such that Ker $L = \{0\}$. Furthermore, for $T > 0$

$$\Psi, \Phi : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \tag{2}$$

are progressively measurable, satisfying certain monotonicity conditions (see below and [1]). To exploit these the following state space for the solution $X = (X(t))_{t\in[0,T]}$ turns out to be convenient:

$$H := \mathcal{F}_e^*$$

where $\mathcal{F}_e^*$ is the dual space of $\mathcal{F}_e$ which is defined as the completion of $D(\sqrt{-L})$ with respect to the norm $\|\cdot\|_{\mathcal{F}_e} := \langle\cdot,\cdot\rangle_{\mathcal{F}_e}^{\frac{1}{2}}$ where

$$\langle u, v\rangle_{\mathcal{F}_e} := \int \sqrt{-L}u\sqrt{-L}v \, dm; \quad u, v \in D(\sqrt{-L}).$$

Here, we assume that the following conditions hold

(L1) ("Transience", cf [8]) There exists $\varrho \in L^1(m) \cap L^\infty(m)$, $\varrho > 0$ m-a.e. such that $\mathcal{F}_e \subset L^1(\varrho \cdot m)$ continuously.

(L2) $T_t := e^{tL}$, $t > 0$, is sub-Markovian, i.e. $0 \leq u \leq 1 \Rightarrow 0 \leq T_t u \leq 1$ for all $u \in L^2(m)$, $t > 0$.

H is a Hilbet space with inner product $\langle\cdot,\cdot\rangle_H$ dual to $\langle\cdot,\cdot\rangle_{\mathcal{F}_e}$. We apply the so-called variational approach to solve (1). To this end we need to find a reflexive Banach space V continuously and densely embedded into H so that we get a Gelfand triple

$$V \subset H(\equiv H^*) \subset V^*.$$

The minimal condition on the noise is then that

$$B : [0,T] \times V \times \Omega \to L_2(L^2(m), H)$$

where $L_2(L^2(m), H)$ denotes the set of all Hilbert-Schmidt operators from $L^2(m)$ to H. Before we explain how to find V, we here make some simplifying assumptions and refer to [1] for the general case. Assume $\Phi \equiv 0$ and that Ψ, B are independent of t and ω and there exists $c \in (0,\infty)$ such that

$$\|B(u) - B(v)\|_{L_2(L^2(m),H)} \leq c\,\|u - v\|_H^2 .$$

Here are the crucial assumptions on Ψ:

(Ψ1) $\Psi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and continuous.

(Ψ2) There exists a Δ_2-regular Young function $N : \mathbb{R} \to \mathbb{R}$ with Δ_2-regular dual N^* (cf. [7]) and c_1, $c_2 \in (0,\infty)$ such that

$$c_1^{-1} N(s) - c_2 1_{\{m(E<\infty)\}} \leq s\Psi(s) \leq c_1 N(s) + c_2 1_{\{m(E)<\infty\}}.$$

Remark 1.1. *We emphasize that the only restrictive nature of* (Ψ2) *is the* Δ_2*-regularity of* N*, since this implies that* Ψ *can be of at most polynomial growth. Otherwise, we can always take* N *as the symmetrization of the integral of* Ψ*. So,* N *can be directly calculated from* Ψ*.*

Now we can choose V as follows:

$$V := H \cap L_N := \{u \in L_N | L_{N^*} \cap \mathcal{F}_e \ni v \mapsto \int uv\,dm \text{ is continuous on } \mathcal{F}_e\}$$

where L_N, L_{N^*} is the Orlicz space determined by N and N^* respectively (see e.g. [7]). Then there is a natural extension $\bar{L}$ of L such that $\bar{L}\Psi : V \to V^*$ and $\bar{L}$ is maximal monotone.

The main existence and uniqueness result from [1] (in simplified form, see above) now reads as follows:

Theorem 1.2. *Suppose* L, H, V, Ψ *and* B *are as above and that (L1), (L2), (*Ψ*1) and (*Ψ*2) hold. Let* $T > 0$ *and* $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$*. Then there exists a unique continuous adapted* H*-valued process* $X(t)$, $t \in [0,T]$*, such that:*

(1)

$$\mathbb{E}\int_0^T \|X(t)\|_H^2\,dt < \infty$$

(2)

$$X \in L_N([0,T] \times \Omega \times E, dt \otimes \mathbb{P} \otimes m) \cap L^2([0,T] \times \Omega, dt \otimes \mathbb{P}; H)$$
$$\big(\subset L^1([0,T] \times \Omega, dt \otimes \mathbb{P}; V)\big)$$

(3)

$$X(t) = X_0 + \int_0^t \bar{L}\Psi(X(s))ds + \int_0^t B(X(s))dW(s), \quad t \in [0,T]\ \mathbb{P}\text{-}a.s..$$

Example 1.3. *The above theorem in particular applies to the case where*

$$(E, \mathcal{B}, m) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), d\xi),\ L = -(-\Delta)^\alpha,\ \alpha \in \left(0, \frac{d}{2}\right) \cap (0, 1].$$

Remark 1.4. *Suppose $L = \Delta$ and $(E, \mathcal{B}, m) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), d\xi)$ where $\mathcal{O} \subset \mathbb{R}^d$, $\mathcal{O}$ open, bounded. Then:*

1. *(1) The above results extend to non-continuous Ψ of arbitrary growth (see* [2]*).*
2. *(2) In case $V \ni x \mapsto B(x)$ is linear, it can be proved that positivity of the initial condition is preserved, i.e. if $X_0 = x \in H$ such that $x \geq 0$ (in the sense of distributions) then so is $X(t)$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. (see* [3] *for details).*

Remark 1.5. *In* [4] *among other things spatial regularity of the solution $X(t)$ was studied and under certain conditions it was proved that if $X_0 \in L^2(m)$ $\mathbb{P}$-a.s., then so is $X(t)$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. and $[0,T] \ni t \mapsto X(t) \in L^2(m)$ is right continuous $\mathbb{P}$-a.s.*

2. Extinction

Assume that $\mathcal{O} \subset \mathbb{R}^d$, open and bounded, $d \leq 3$, and that $(E, \mathcal{B}, m) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), d\xi)$, $L = \Delta$. Suppose

$$\Psi(r) := \Psi_0(r) + c,\ r \in \mathbb{R},$$

where

$$\Psi_0(r) := \begin{cases} \alpha_1 r, & r > 0, \\ [-\varrho, \varrho], & r = 0, \\ \alpha_2 r, & r < 0, \end{cases}$$

and $\varrho, \alpha_1,\ \alpha_2 \in (0, \infty)$, $c \in (-\varrho, \varrho)$. That is, we allow Ψ to have a jump at zero. Then (1) becomes the multivalued (SPDE)

$$\begin{aligned} dX(t) - \Delta\Psi(X(t))dt &\ni B(X(t))dW(t). \\ X(0) &= x \in H. \end{aligned} \tag{3}$$

Furthermore, we assume that for $\mu_k \in \mathbb{R}$, $k \in \mathbb{N}$,

$$B(x) := \sum_{k=1}^\infty \mu_k x \langle e_k, \cdot \rangle_{L^2(\mathcal{O})} e_k, \quad \sum_{k=1}^\infty \mu_k^2 \lambda_k^2 < \infty, \tag{4}$$

where $\{e_k | k \in \mathbb{N}\}$ is an eigenbasis of $-\Delta$ on $L^2(\mathcal{O}) := L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), d\xi)$ with corresponding eigenvalues λ_k, $k \in \mathbb{N}$.

Then by [5] there exists a unique continuous adapted H-valued solution to (3) provided $x \in L^4(\mathcal{O})$. If $d = 1$ we have proved that we have extinction of X in finite time with strictly positive probability (see [5] and also [6]). Before we formulate the corresponding result in the concrete case when $\mathcal{O} = (0, \pi)$, we define

$$\gamma := \inf \left\{ \frac{|x|_{L^1(\mathcal{O})}}{|x|_H} \,\middle|\, x \in L^1(\mathcal{O}) \right\}.$$

Note that $\gamma > 0$ only if $d = 1$. Furthermore, let

$$\tau := \inf\{t \geq 0 | \; |X(t)|_H = 0\}.$$

Theorem 2.1. *Define $\tilde{\varrho} := \varrho - |c|$ (> 0) and assume that $\mathcal{O} = (0, \pi)$, $x \in L^2(0, \pi)$ such that*

$$|x|_H < \tilde{\varrho}\gamma C_\infty^{-1}$$

where $C_\infty := \frac{\pi}{4} \sum_{k=1}^{\infty} (1 + k)^2 \mu_k^2$. Then $|X(t)|_H = 0 \; \forall t \geq \tau$ and

$$\mathbb{P}[\tau \leq t] \geq 1 - \frac{|x|_H}{\tilde{\varrho}\gamma \int_0^\infty e^{-C_\infty s} ds}.$$

In particular, $\mathbb{P}[\tau < \infty] > 0$ and if $C_\infty = 0$

$$\tau \leq \frac{|x|_H}{\tilde{\varrho}\gamma}.$$

Remark 2.2. *We have proved a similar result for the stochastic fast diffusion equation (see* [9]*).*

3. Random attractors

Let $(E, \mathcal{B}, m) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), d\xi)$ with $\mathcal{O} \subset \mathbb{R}^d$, $\mathcal{O}$ open, bounded, $d \geq 1$, and $L = \Delta$. Let $p \in (1, \infty)$ and

$$\Psi(s) := s|s|^{p-1}, \; s \in \mathbb{R}.$$

Let $m \in \mathbb{N}$ be fixed and $\varphi_1, \ldots, \varphi_m \in C_0^\infty(\mathcal{O})$. Define $B \in L_2(L^2(\mathcal{O}), H)$ by

$$Bh := \sum_{i=1}^{m} \langle e_i, h \rangle_{L^2(\mathcal{O})} \varphi_i$$

where $\{e_i, i \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\mathcal{O})$. Consider the corresponding stochastic porous media equation, but with the Wiener process indexed by the whole real line, i.e. $W = (W(t))_{t \in \mathbb{R}}$, with the usual corresponding filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$, i.e. for $t_0 \in \mathbb{R}$ consider

$$dX(t) = \Delta(|X(t)|^{p-1} X(t))dt + BdW(t), \; t \geq t_0, \tag{5}$$

and denote its solution starting at x at $t = t_0$ by $X(t, t_0, x)$, $t \geq t_0$. Then

$$\varphi(t, \omega)x := X(t, 0, x)(\omega), \quad t \geq 0, \, \omega \in \Omega, \, x \in H,$$

defines a stochastic flow. Then the main result in [10] is the following:

Theorem 3.1. *The stochatsic flow associated with* (5) *admits a global compact stochastic attractor $A(\omega)$, $\omega \in \Omega$.*

Here the terminology of stochastic attractors is meant in the sense of [11] (see also [12]).

Remark 3.2. *In the talk it was conjectured that $\#A(\omega) = 1$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, i.e. the random attractor is just a (random) point. This, however, has not been confirmed yet.*

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Random interfaces and the numerical discretization of the 1D stochastic Allen-Cahn problem

Marco Romito
(joint work with Omar Lakkis and Georgios T. Kossioris)

We aim to describe a work which is currently in progress and is developed in collaboration with O. Lakkis (Sussex) and G. Kossioris (Crete) on the numerical discretization of the one dimensional stochastic Allen-Cahn governed by a small *interface thickness* parameter ϵ,

$$\dot{u} - u_{xx} + \frac{1}{\epsilon^2} F'(u) = \delta\eta, \qquad t > 0, x \in (-1, 1), \tag{1}$$

where $F(x) = \frac{1}{4}(x^2 - 1)^2$ is the double-well potential and η is space-time white noise.

The asymptotic behaviour of solutions to (1) in the limit $\epsilon \to 0$ has been thoroughly analysed by Brassesco, De Masi & Presutti [3] and by Funaki [7]. As

$\epsilon \to 0$ the potential term tends to be dominant and the solution converges, with an appropriate rescaling, to a step-like function that takes only the two stable values ± 1. The *interface* between the two *phases* performs a Brownian motion.

In the non-zero regime for ϵ, which is more suitable for the numerical discretization, the dynamics is more complicated, and several phenomena may occur (see for example Fatkullin & Vanden-Eijnden [6]),

- annihilations,
- exit phenomena,
- nucleations,

and the *strength* of these phenomena (their probability) is governed by the parameter δ (and its value with respect to ϵ). Here we shall consider $\delta = \epsilon^\gamma$.

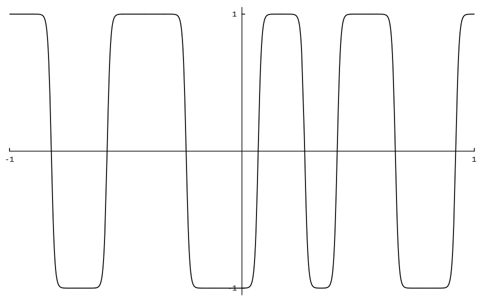

FIGURE 1. A *multikink* limiting object

The numerical scheme can capture these phenomena only if the value of all parameters relative to the discretization of the noise and to the space-time mesh are small enough with respect to ϵ. Smallness here has to be understood in dependence of the value of γ.

The main result we are going to explain is that solutions, at any level of the approximation, spend a characteristic time close to a *metastable* solution made of kinks and this turns out to be crucial from two points of view,

(1) the convergence analysis improves dramatically, from exponential to polynomial in ϵ (following the use of spectral estimates of [4] made for instance in [9]),
(2) it opens the possibility of using adaptive (in space) methods, in spite of the presence of the noise.

The one dimensional case is indeed a toy problem, in view of the more challenging case of larger dimension, where already the stochastic model has proved to be crucial (see [10] and [5] for instance) and where the finite elements approach is going to be more effective in the description of the geometry of interfaces.

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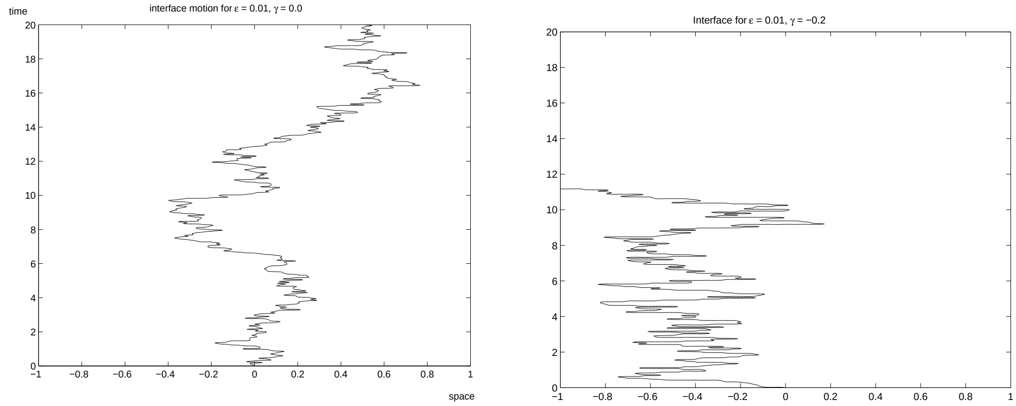

FIGURE 2. Motion of one interface

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Probabilistic representation for a porous media equation

FRANCESCO RUSSO

We consider a (generalized) porous media type equation over all of $\mathbb{R}^d$ with $d = 1$, with monotone discontinuous coefficients with linear growth and prove a probabilistic representation of its solution in terms of an associated microscopic diffusion. This equation is motivated by some singular behaviour arising in complex self-organized critical systems. One of the main analytic ingredients of the proof, is a new result on uniqueness of distributional solutions of a linear PDE on $\mathbb{R}^1$ with non-continuous coefficients. The porous media type equation is given by

$$\left\{ \begin{array}{rcl} \partial_t u & = & \frac{1}{2}\partial^2_{xx}(\beta(u)), \\ u(0,x) & = & u_0(x) \end{array} \right. \tag{1}$$

in the sense of distributions, where u_0 is an initial probability density. We look for a solution of (2) with time evolution in $L^1(\mathbb{R})$.

We will always suppose that $\beta : \mathbb{R} \to \mathbb{R}$ is monotone, $0 \in \text{intD}(\beta)$, with

$$|\beta(u)| \leq \text{const}|u| \tag{2}$$

In particular, $\beta(0) = 0$ and β is continuous at zero. Moreover we will suppose the existence of λ positive such that $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$, $id(x) \equiv x$. Since β is monotone, (2) implies $\beta(u) = \Phi^2(u)u$, Φ being a non-negative bounded Borel function. We recall that when $\beta(u) = |u|u^{m-1}$, $m > 1$, (1) is nothing else but the classical *porous media equation.*

A peculiar case that we wish consists in Φ as continuous with a possible jump at one positive point, say $e_c > 0$. A typical example is

$$\Phi(u) = H(u - e_c), \tag{3}$$

H being the Heaviside function. This can be done in the framework of monotone partial differential equations (PDE) allowing multi-valued coefficients. In this abstract, for simplicity, we restrict our presentation the single-valued case.

We will say that equation (1) or Φ is **non-degenerate** if there is a constant $c > 0$ such that $\Phi \geq c > 0$.

Of course, Φ in (3) is not non-degenerate. In order to have Φ non-degenerate, one needs to add a postive constant to it.

Several contributions were made in this framework starting from [5] for existence, [9] for uniqueness in the case of bounded solutions and [8] for continuous dependence on the coefficients. The authors consider the case where β is continuous, even if their arguments allow some extensions for the discontinuous case. Important contributions related with similar problems are for instance contained in [3].

As mentioned above, the first motivation of this presentation was to discuss a continuous time model of self-organized criticality (SOC), which are described by equations of type (3).

SOC is a property of dynamical systems which have a critical point as an attractor, see [1] for a significant monography on the subject. SOC is typically observed in slowly-driven out-of-equilibrium systems with threshold dynamics relaxing through a hierarchy of avalanches of all sizes. We, in particular, refer to the interesting physical papers [2] and [10]. The second makes reference to a system whose evolution is similar to the evolution of a “snow layer” under the influence of an “avalanch effect” which starts when the top of the layer attains a critical value e_c. Adding a stochastic noise should describe other contingent effects. For instance, an additive perturbation by noise could describe the regular effect of “snow falling”. Taking inspiration from the discrete models, a possible representation of that phenomenon is equation (1) perturbed by a space time noise $\xi(t, x)$.

Since the “avalanche effect” happens on a much shorter time scale than the “snow falling effect” it seems to be reasonnable to analyze the phenomenon in two separate phases, i.e. avalanche and snow falling. This presentation concentrates on the avalanche phase and therefore it investigates the (unperturbed) equation discussing existence, uniqueness and a probabilistic representation.

The singular non-linear diffusion equation (1) models the *macroscopic* phenomenon for which we try to give a *microscopic* probabilistic representation, via a non-linear stochastic differential equation (NLSDE) modeling the evolution of a single point on the layer.

Even if the irregular diffusion equation (1) can be shown to be well-posed, up to now we can only prove existence (but not yet uniqueness) of solutions to the corresponding NLSDE. On the other hand if $\Phi \geq c > 0$, then uniqueness can be proved. For our applications, this will solve the case $\Phi(u) = H(x - e_c) + \varepsilon$ for some positive ε. The main novelty with respect to the literature is the factthat Φ can be irregular with jumps. A probabilistic interpretation of (1) when $\beta(u) = |u|u^{m-1}, m > 1$, was provided for instance in [6].

Some considerations about the case of equation (1) with some forcing term will be finally discussed.

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Strong and weak completeness for stochastic differential equations on Euclidean spaces

Michael Scheutzow

(joint work with Xue-Mei Li)

Consider a stochastic differential equation (SDE)

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sum_{i=1}^{m} \sigma_i(X(t))\,\mathrm{d}W_i(t),\ X(s) = x \in \mathbf{R}^d, \tag{1}$$

where $W_1, ..., W_m$ are standard Wiener processes. It is well-known that the SDE has a unique global solution for each initial condition x in case b and $\sigma_1, ..., \sigma_m$ satisfy a local Lipschitz condition and are of at most linear growth at infinity (see [7] for even more general conditions). It is also well-known that the solutions generate a stochastic flow of homeomorphisms ϕ in case b and $\sigma_1, ..., \sigma_m$ satisfy a global Lipschitz condition (see [3]), i.e. $\phi : \{0 \le s \le t\} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ satisfies

i) $\phi_{st}(x, .)$, $t \ge s$ solves (1) for every $x \in \mathbf{R}^d, s \ge 0$,
ii) $(s, t, x) \mapsto \phi_{st}(., \omega)$ is continuous for all $\omega \in \Omega$,
iii) $\phi_{st}(., \omega)$ is a homeomorphism on $\mathbf{R}^d$ for all s, t, ω,
iv) $\phi_{su}(., \omega) = \phi_{tu}(., \omega) \circ \phi_{st}(., \omega)$ for all $0 \le s \le t \le u$, $\omega \in \Omega$.

Kunita also showed that under a local Lipschitz condition on the coefficients the solutions always generate a *stochastic flow of local homeomorphisms*, see [3], Section 4.7. We will call an SDE *strongly complete* if there exists a map ϕ with properties i),ii) and iv) and *weakly complete* in case it admits a unique *global* solution for each fixed s and x.

Strong completeness results without global Lipschitz assumptions on the coefficients can be found in [1, 4, 6], where the results in [6] even allow for delays in the coefficients and the proof is based on spatial estimates of stochastic flows contained in [2, 5].

One can ask whether a local Lipschitz condition plus a linear growth condition which – as pointed out above – implies weak completeness actually implies strong completeness. In dimension $d = 1$ this is clearly true. In the talk, we show that the answer is however negative in dimensions $d \ge 2$. It turns out that the answer is negative even under the additional constraint that the coefficients of (1) are globally bounded (and C^∞). It spite of the boundedness of the coefficients it is possible that due to the lack of a growth condition on the local Lipschitz constant, initially nearby trajectories become almost independent after a short time and as soon as there is enough independence, there may exist trajectories (with a random initial condition) which blow up in finite time. One might think that in order to create a sufficient amount of independence in order for strong completeness to fail, one needs an infinite number of driving Brownian motions but it turns out that a single Brownian motion is sufficient.

In the talk we provide an example of an SDE in the plain ($d = 2$) with $b = 0$, $m = 1$ and with a bounded and C^∞ function σ such that there exist (random) initial conditions for which the solution blows up in finite time. The example is of the following type:

$$\begin{aligned} \mathrm{d}X_1(t) &= \sigma(X_1(t), X_2(t))\, \mathrm{d}W(t) \\ \mathrm{d}X_2(t) &= 0, \end{aligned}$$

where the function σ takes values in the interval $[1, 2]$.

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Unstable manifolds for a stochastic partial differential equation driven by a fractional Brownian motion

Björn Schmalfuss
(joint work with M.J. Garrido and K. Lu)

A metric dynamical system which is a model for a noise is a quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a measurable mapping:

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F})$$

where the *flow* property holds for the partial mappings $\theta_t = \theta(t, \cdot)$:

$$\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1}\theta_{t_2} = \theta_{t_1+t_2}, \qquad \theta_0 = \mathrm{id}_\Omega$$

for all t_1, $t_2 \in \mathbb{R}$, and $\mathbb{P}$ is supposed to be θ–ergodic.

For our purpose let V be a separable Hilbert space. A random dynamical system (rds) is a mapping φ

$$\varphi : \mathbb{R}^+ \times \Omega \times V \to V$$

which is $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V))$–measurable and satisfies the cocycle property

$$\varphi(t, \theta_\tau\omega, \cdot) \circ \varphi(\tau, \omega, \cdot) = \varphi(t + \tau, \omega, \cdot),$$
$$\varphi(0, \omega, \cdot) = \mathrm{id}_V \quad \text{for } t,\, \tau \in \mathbb{R}^+,\, \omega \in \Omega.$$

For $H \in (0, 1)$ a continuous centered Gauß process $\beta^H(t)$, $t \in \mathbb{R}$, with the covariance

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \qquad t,\, s \in \mathbb{R}$$

is called a *two–sided one-dimensional fractional Brownian motion* (fBm), and H is the *Hurst parameter*.

Let $(e_i)_{i\in\mathbb{N}}$ be a complete orthonormal system in V. We introduce

$$\omega = \sum_{i=1}^{\infty} \alpha_i \beta_i e_i$$

where $(\beta_i)_{i\in\mathbb{N}}$ is an iid family of fBm and the sequence $(\alpha_i)_{i\in\mathbb{N}}$ grows to 0 with a particular speed such that this process has almost surely Hölder continuous trajectories with Hölder exponent less than H. Then the above sum gives us an infinite fBm. In particular, for $\Omega = C_0(\mathbb{R}; V)$ the continuous functions in V with value zero for $t = 0$, $\mathcal{F}$ the associated Borel–σ–algebra,

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R},\ \omega \in \Omega$$

and $\mathbb{P}$ the distribution of this infinite dimensional fBm defines a metric dynamical system.

Let $V = V^+ \oplus V^-$ be a splitting of V, V^+ finite dimensional. We consider an rds φ such that $\varphi(t, \omega, 0) = 0$ for $t \in \mathbb{R}^+$, $\omega \in \Omega$.

Let m be a mapping such that

- $m : \Omega \times V^+ \to V^-$ and $m(\omega, 0) = 0$.
- $m(\cdot, x^+) : \Omega \to V^-$ is measurable for every $x^+ \in V^+$.
- $m(\omega, \cdot) : V^+ \to V^-$ is Lipschitz for every $\omega \in \Omega$.

A set $M(\omega)$ defined by $\{x^+ + m(\omega, x^+) : x^+ \in V^+\}$ is called unstable random manifold at 0 if

(1) for every $x \in M(\omega)$, $t \in \mathbb{R}^+$, $\omega \in \Omega$ we can find an $x_{-t} \in M(\theta_{-t}\omega)$ such that $\phi(t, \theta_{-t}\omega, x_{-t}) = x$ and

$$\lim_{t\to\infty} x_{-t} = 0$$

with exponential speed,

(2) $\phi(t, \omega, M(\omega)) \subset M(\theta_t\omega)$ for $t \in \mathbb{R}^+$, $\omega \in \Omega$.

Similarly we can define local manifolds if we assume that the above properties hold when x^+ is contained in a closed random neighborhood $U(\omega)$ of 0. Such a kind of random problems is studied in [2].

We consider the following evolution equation:

$$du = Audt + G(u)d\omega, \qquad u(0) = x \in V. \tag{1}$$

Here A is the generator of an analytic semigroup on V with spectrum $(\lambda_i)_{i\in\mathbb{N}}$ of finite multiplicity such that such that these eigenvalues tend to $-\infty$. The eigenfunctions $(e_i)_{i\in\mathbb{N}}$ form a complete ONS in V. $u \to G(u) \in L(V)$ and $u \to G'(u) \in L^2(V)$ are supposed to be Lipschitz continuous in some sense. The associated stochastic integral is related to stochastic fractional derivatives, see [4]. Then it is known from [3] that the above equation has an ω–wise mild solution. Hence the above equation defines an rds on V, see [1].

We will also assume that $G(0) = 0$ such that $u = 0$ is a stationary solution to (1). In addition we need $G'(0) = 0$.

We now introduce the space V^+ spanned by the first eigenfunctions $\{e_1, \cdots, e_i\}$ of A such that for the associated eigenvalues we have

$$\frac{\lambda_i + \lambda_{i+1}}{2} =: \kappa > 0$$

and $V^- = V \ominus V^+$. For the following we will write $S^\pm$, x^+ for the projections on V^+, V^-. Let

$$\begin{aligned} J(\omega, U)(s) = &\int_{-\infty}^{s} S^-(s-\tau)G(U(\tau))d\omega(\tau) \\ &+ S^+(s)u_0^+ - \int_0^s S^+(s-\tau)G(U(\tau))d\omega(\tau). \end{aligned}$$

be the Lyapunov–Perron transform defined on a class of functions $U : \mathbb{R}^- \to V$ such that

$$\sup_{t\in\mathbb{R}^-} e^{-\kappa s}\|U(s)\| < \infty.$$

We note if $\Gamma(\omega, x^+)(s)$ is the fixed point of this operator then

$$m(\omega, x^+) = \Gamma^-(\omega, x^+)(0)$$

gives us the random Lipschitz graph of the unstable manifold $M(\omega)$ related to an exponential growth of order κ. However, to obtain these fixed points Γ we have to cut off the coefficients in a special way such that the contraction condition, which is nothing but the gap condition is satisfied. Then the original and the cut off Lyapunov Perron transform coincide if $x^+ \in U(\omega)$ where $U(\omega) \subset V^+$ is a ball with center 0 and random radius $r(\omega)$ chosen sufficiently small. In particular, $m(\omega, \cdot)$ restricted to $U(\omega)$ gives us a local random unstable Lipschitz manifold for the random dynamical system generated by (1)

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Semilinear SPDEs driven by cylindrical stable processes

Jerzy Zabczyk
(joint work with Enrico Priola)

We are concerned with structural properties of solutions to nonlinear stochastic equations

$$dX_t = AX_t dt + F(X_t)dt + dZ_t, \quad t \geq 0, \quad X_0 = x \in H, \tag{1}$$

in a real separable Hilbert space H driven by an infinite dimensional jump process $Z = (Z_t)$, with independent increments. We present results on properties like irreducibility, stochastic continuity and integrability of trajectories and the strong Feller property of the associated transition semigroup, which are obtained in [4]. The main results are gradient estimates for the associated transition semigroup (see Theorem 4 when $F = 0$ and Theorem 7 in the general case), from which the strong Feller property follows, and a theorem on time regularity of trajectories (see Theorem 2). To cover interesting cases, we consider processes Z which take values in a Hilbert space U usually greater than H(see [2] and the references therein).

We assume that $Z = (Z_t)$ is a *cylindrical α-stable process* , $\alpha \in (0, 2)$, defined by the orthogonal expansion

$$Z_t = \sum_{n \geq 1} \beta_n Z_t^n e_n, \quad t \geq 0, \tag{2}$$

where (e_n) is an orthonormal basis of H and (Z_t^n) are independent, real valued, normalized, symmetric α-stable processes defined on a fixed stochastic basis. Moreover, (β_n) is a given, possibly unbounded, sequence of *positive* numbers.

Let us recall that a real valued process $L_t,\ t \geq 0,\ L_0 = 0,$ with independent increments, is said to be $\alpha-$ stable, if it has càdlàg (i.e., right-continuous with left-limits) trajectories and time homogeneous increments, with the characteristic function

$$\mathbb{E}[e^{i\lambda(L_t - L_s)}] = e^{-(t-s)|\lambda|^\alpha}, \quad \lambda \in \mathbb{R},\, t \geq s \geq 0. \tag{3}$$

We assume that F is a bounded and Lipschitz transformation on H, A is a linear operator and the vectors (e_n), from the representation (2), are eigenvectors of A, corresponding to negative eigenvalues $-\gamma_n,\ n = 1, 2, \ldots,$

$$Ae_n = -\gamma_n e_n, \tag{4}$$

such that $\gamma_n \to +\infty$.

Our results apply to stochastic heat equations with Dirichlet boundary conditions

$$\begin{cases} dX(t,\xi) = \big(\triangle X(t,\xi) + f(X(t,\xi))\big)\,dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \ \ t > 0, \quad \xi \in \partial D, \end{cases} \tag{5}$$

in a given bounded domain $D \subset \mathbb{R}^d$ having Lipschitz-continuous boundary ∂D. Here $x(\xi) \in H = L^2(D)$, $f : \mathbb{R} \to \mathbb{R}$ is bounded and Lispchitz continuous and the noise Z is a cylindrical α-stable process of the form (2), where (e_n) is a basis of eigenfunctions for the Laplace operator $\triangle$ (with Dirichlet boundary conditions).

1. The linear stochastic PDE

We consider first the linear equation

$$dX_t = AX_t dt + dZ_t, \quad x \in H, \tag{6}$$

where Z is a cylindrical α-stable process, $\alpha \in (0,2)$, given by (2). Then

$$X_t^x = e^{tA}x + Z_A(t), \tag{7}$$

where

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s.$$

Proposition 1. *The process X given in (7) takes values in H if and only if*

$$\sum_{n \geq 1} \frac{\beta_n^\alpha}{\gamma_n} < \infty \tag{8}$$

If the cylindrical Lévy process Z in (6) takes values in the Hilbert space H then, by the Kotelenez regularity result (see [2, Theorem 9.20]) trajectories of the process X which solves (6) are càdlàg with values in H. However $Z_t \in H$, for any $t > 0$, if and only if

$$\sum_{k \geq 1} \beta_k^\alpha < \infty. \tag{9}$$

Compared with (8), this is a very restrictive assumption. We conjecture that the càdlàg property holds under much weaker conditions but, at the moment, we are able to establish a weaker time regularity of the solutions.

Theorem 2. *If (8) holds, then*

(i) for any $x \in H$, X^x is stochastically continuous,

(ii) for any $x \in H$, $T > 0$, $0 < p < \alpha$, X^x has trajectories in $L^p(0,T;H)$, $\mathbb{P}$-a.s..

We have also the following result about the support of the solution.

Theorem 3. *Assume (8) and fix $T > 0$, $x \in H$ and $p \in (0,\alpha)$. The support of the random variable $(X, X_T) : \Omega \to L^p(0,T;H) \times H$ is $L^p(0,T;H) \times H$.*

Let (R_t) be the Markov semigroup associated to (X_t^x), i.e. $R_t : B_b(H) \to B_b(H)$,

$$R_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \ f \in B_b(H), \quad t \geq 0.$$

The next result shows not only that (R_t) is strong Feller, but also that it has a smoothing effect, i.e., that gradient estimates hold for it. The gradient (Fréchet derivative) of a function $f : H \to \mathbb{R}$ at point x will be denoted by $Df(x)$ and the set of all functions whose gradients are continuous and bounded on H, by $C_b^1(H)$.

Theorem 4. *Assume (8) and*

$$C_t = \sup_{n\geq 1} \frac{e^{-\gamma_n t}\,\gamma_n^{1/\alpha}}{\beta_n} < +\infty, \quad t > 0. \tag{10}$$

Then the operator (R_t) maps Borel and bounded functions into $C_b^1(H)-$functions. Moreover we have

$$\sup_{x\in H} |DR_t f(x)|_H \leq 8c_\alpha\, C_t \sup_{x\in H} |f(x)|, \quad f \in B_b(H), \tag{11}$$

were c_α is a constant depending only on α.

In the proof one first establishes a formula for the density of $R_t(x,\cdot)$ with respect to $R_t(0,\cdot)$ and then justifies, in several steps, a formula for the gradient of $R_t f$ under less and less requirements on f.

Remark If dim $H < +\infty$, the strong Feller property, established in the theorem, holds under much weaker conditions, see [3] and references therein. Therefore, some generalizations of the theorem should be possible.

2. Nonlinear stochastic PDEs

We pass now to nonlinear SPDEs of the form

$$dX_t = AX_t dt + F(X_t)dt + dZ_t, \quad X_0 = x \in l^2 = H, \tag{12}$$

where $Z = (Z_t)$ is a cylindrical α-stable Lévy process. Throughout the section, we will assume that (8) and also that

$$F : H \to H \text{ is Lipschitz continuous and bounded.} \tag{13}$$

We say that a predictable H-valued stochastic process $X = (X_t^x)$, depending on $x \in H$, is a *mild solution* to equation (1) if, for any $t \geq 0$, $x \in H$, it holds:

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^x)ds + Z_A(t), \quad \mathbb{P}-a.s., \quad \text{where}$$
$$Z_A(t) = \int_0^t e^{(t-s)A}dZ_s, \tag{14}$$

see (7). In formula (14) we are considering a predictable version of the process $(Z_A(t))$.

Note that, since F is bounded, the deterministic integral in (14) is a well defined continuous process. Moreover, as far as the regularity of trajectories is concerned, the mild solution will have the same regularity as $(Z_A(t))$. In particular, according to Theorem 2, any mild solution X will be stochastically continuous. We have the following existence result. Since the trajectories of the solution are not regular in time, the proof of markovianity requires some additional care.

Theorem 5. *Assume (8) and that $F : H \to H$ is Lipschitz continuous and bounded. Then there exists a unique mild solution (X_t^x) to the equation (14). Moreover (X_t^x) is a Markov process and its transition semigroup is Feller.*

We finally formulate a result on strong Feller property.

Let (P_t) be the Markov semigroup associated to $X = (X_t^x)$, i.e. $P_t : B_b(H) \to B_b(H)$,

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \;\; f \in B_b(H), \;\; t \geq 0. \tag{15}$$

We will need the following hypothesis.

Hypothesis 6. *There exists* $\gamma \in (0,1)$ *and* $C > 0$ *such that*

$$\beta_n \geq C\, \gamma_n^{(\gamma - \frac{1}{\alpha})}, \quad n \geq 1. \tag{16}$$

This assumption is stronger than (10). Indeed one can check, see [4], that (16) holds if and only if for some $\hat{C} > 0$,

$$C_t = \sup_{n \geq 1} \frac{e^{-\gamma_n t}\, \gamma_n^{1/\alpha}}{\beta_n} \leq \frac{\hat{C}}{t^\gamma}, \qquad t > 0. \tag{17}$$

Theorem 7. *Assume that (8) and (10) hold and that Hypothesis 6 is satisfied. Then the operator* (P_t) *maps Borel and bounded functions into Lipschitz continuous functions. Moreover, there exists a constant* $K = K(\gamma, c_\alpha, \hat{C}, F)$*, such that, for any* $x, y \in H$*, we have*

$$|P_t f(x) - P_t f(y)| \leq K \sup_{z \in H} |f(z)|\, \frac{1}{\min(t^\gamma, 1)} |x - y|, \quad t > 0, \;\; f \in B_b(H). \tag{18}$$

To prove the result one first investigates existence of a solution to the so called mild Kolmogorov equation associated to (P_t) (or to (X_t^x)) as in [1, Section 9.4.2].

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