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Hyperbolic Conservation Laws

Organised by
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Abstract. The subject of this workshop concerned the analysis and numerical investigation of hyperbolic systems. In particular the interactions between theoretical and numerical contributions were in the focus of this meeting. Many theoretical results have been initiated by numerical methods and experiments and rigorous numerical results are based on fundamental theorems. In particular during this workshop recent results about hyperbolic Monge Ampere systems, elastodynamics, dissipation in systems of conservation laws, Boussinesq, Boltzmann and (error control for the) Navier Stokes equations, mixture of fluids, anelastic and weakly compressible models for atmospherical flows, hydrodynamic limits, wetting and drying for shallow water problems, potential flows, divergence free transport equations, initial boundary Riemann problems, various Discontinuous Galerkin and operator splitting methods, IMEX schemes, entropy stable methods, well balancing schemes, convergence rate for the Glimm scheme and reduced basis methods for conservation laws were discussed.

Mathematics Subject Classification (2000): 35L65, 37K05, 35Q30, 76Nxx, 76Txx.

Introduction by the Organisers

The workshop *Hyperbolic Conservation Laws*, organized by Constantine M. Dafermos (Providence), Dietmar Kröner (Freiburg) and Randall J. LeVeque (Seattle) was held December 7th – 13th, 2008. We had 44 participants from eight different countries. The atmosphere in the Oberwolfach Research Institute was very stimulating and has initiated many fruitful discussions and exchange of ideas. The time schedule was organized as follows: We had three 30-min-lectures in the morning and in the afternoon and for each lecture at least 15 min for discussion. This time was actively used for many questions and remarks from people in the audience. As usual, on Wednesday afternoon we walked to a restaurant in Oberwolfach Kirche.

In the evening on Wednesday D. Serre (Lyon) chaired a problem session for all participants. Many of them spontaneously described some open problems and initiated discussions about some ideas for solving them. The time between the lectures, in the lunch breaks and in the evenings were intensively used for starting new and to continue old cooperations. The participants of the workshop thank the administration of the institute for the possibility for organizing this meeting and the staff for the perfect service.

For this workshop we invited worldwide leading specialist and younger researcher who work on theoretical or numerical questions for hyperbolic conservation laws.

In the field of theoretical contributions the speakers reported on the following results. Existence of solutions for the linear transport equation, if the transport velocity is in L^∞, was the main subject of Bianchini's contribution. Brenier considered a coupled system consisting of the Monge-Ampère equation and a conservation law that occurs as a "high field limit" of the Vlasov-Poisson system or, alternately, as a simplified model for chemotaxis. The objective of the work of Christoforou was to construct an entropy weak solution of bounded variation for the Riemann problem for systems of conservation laws in one-space dimension via a vanishing viscosity method, for which the viscosity coefficient varies with time. Feldman described in his talk recent results on regular shock reflection for potential flow equation in dimension two. Klingenberg presented a relaxation system for ideal MHD system and derived approximate Riemann solvers with three, five or seven waves, that generalize the HLLC solver for gas dynamics. They satisfy discrete entropy inequalities, and preserve positivity of density and internal energy. T.P. Liu discussed different approaches to study the Boltzmann equations: the probability approach and the CFD approach from continuums mechanics. The purpose of Pan's contribution was to prove the existence of unique global smooth solutions for smooth initial and boundary values for the Boussinesq system. Ruggeri considered different models of a mixture of compressible fluids, and in the case of Euler fluids, the local and global well-posedness of the relative Cauchy problem for smooth solutions. In particular he presented a classical approach of mixture of compressible fluids when each constituent has its own temperature. Serre proved that entropy dissipative viscous extensions of nonlinear systems of conservation laws satisfy under some structure condition for the dissipation tensor the main assumption of Kawashima in his fundamental study of the hyperbolic-parabolic Cauchy problem. The objective of the work of Trivisa was to develop a rigorous mathematical framework based on the principles of continuum physics and to analyze the global in time existence, stability and asymptotic behavior of multicomponent reactive flows. Tzavaras studied the mechanism of shear band formation, in particular the development of a quantitative criterion explaining the onset of instabilities. Westdickenberg proposed a time discretization for the isentropic Euler equations that consists of a sequence of minimization problems and

analyzed the convergence of the approximations towards a measure-valued solution. Riemann problems for the two dimensional compressible Euler system were considered in the talk of Zheng.

The contributions concerning numerical results are the following: Ancona investigated the rate of convergence of an approximate solution of general nonlinear hyperbolic systems constructed by the Glimm scheme for a suitable choice of an equidistributed sampling sequence. A fast scheme for multi-layer shallow water equations, including elliptic regions and drying was considered by Bochut. Feistauer developed a Discontinuous Galerkin scheme for the solution of the initial boundary value problem for the compressible Euler equations which is unconditionally stable and allows the solution of compressible flows for practically all Mach numbers. Kurganov studied convection dominated diffusion problems by an hyperbolic parabolic operator splitting technique. The parabolic solver is partially based on a dicretization of the convolution with the Green function formula for the exact solution of the heat equation. Lukáčová-Medviďová proved entropy-stability for a scheme, based on a Roe-type linearization coupled with the multidimensional Finite Volume evolution Galerkin method. Actually this problem was mentioned by Tadmor during the Oberwolfach conference on hyperbolic conservation laws in 2004. A framework within which existing well-balanced schemes may be rederived and reinterpreted, and new ones may be developed more easily was presented in the talk of Noelle. In particular this framework refers to Finite Volume and Discontinuous Galerkin schemes. Makridakis highlighted the main structure of an algorithm for solving the incompressible Navier Stokes equations which permits mesh redistribution with time and the nontrivial characteristics associated with it. Munz introduced a framework based on Riemann problems for diffusion equations to define suitable numerical diffusion fluxes at grid cell interfaces and developed on this basis Discontinuous Galerkin methods. A completely new problem is treated by Ohlberger. He has developed a reduced basis method for parameterized nonlinear conservation laws, discretized by finite volume schemes. Numerical schemes for the BGK model, as a simplified model for the Boltzmann equation, were studied by Puppo.

The contribution by Klein was concerned with the mathematical modeling of atmospheric flows. Most models are characterized by low Mach numbers and Strouhal numbers of order one and lead to zero Mach number incompressible flow models. However, due to the strong pressure and density variations in the vertical direction the incompressible flow models are not sufficient to describe all important features of the flow. The details of this problem were discussed in his contribution.

Workshop: Hyperbolic Conservation Laws

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Abstracts

On the convergence rate of the Glimm scheme for general nonlinear hyperbolic systems

Fabio Ancona
(joint work with Andrea Marson)

Consider a general strictly hyperbolic, quasilinear system, in one space dimension

$$u_t + A(u)\, u_x = 0\,, \tag{1}$$

where $u \mapsto A(u)$, $u \in \Omega \subset \mathbb{R}^N$, is a smooth matrix-valued map. Given an initial datum $u(0,\cdot)$ with small total variation, let $u(t,\cdot)$ be the corresponding (unique) vanishing viscosity solution of (1) obtained as limit of solutions to the viscous parabolic approximation $u_t + A(u)u_x = \mu u_{xx}$, as $\mu \to 0$ (cf. [6]). We wish to investigate the rate of convergence of an approximate solution u^ε of (1) constructed by the Glimm scheme (cfr. [2]), with mesh size $\Delta x = \Delta t = \varepsilon$, and with a suitable choice of an equidistributed sampling sequence $\{\theta_\ell\}_{\ell\in\mathbb{N}} \subset [0,1]$. For conservative systems (1) (where $A(u)$ is the Jacobian matrix of a flux function $F(u)$) with *genuinely nonlinear* (GNL) or *linearly degenerate* (LD) characteristic fields, it was shown in [7] that the $\mathbb{L}^1$ convergence rate of Glimm approximate solutions is $o(1)\cdot\sqrt{\varepsilon}\,|\ln\varepsilon|$ if one employs a sampling sequence whose discrepancy

$$D_{m,n} \doteq \sup_{\lambda\in[0,1]} \left|\lambda - \frac{1}{n-m}\sum_{m\le\ell<n} \chi_{[0,\lambda]}(\theta_\ell)\right| \tag{2}$$

satisfies the estimate

$$D_{m,n} \le \mathcal{O}(1)\cdot\frac{1+\log(n-m)}{n-m} \qquad \forall\ n > m \ge 1\,. \tag{3}$$

This error estimate was recently extended in [3, 8] to *non genuinely nonlinear* (NGNL) quasilinear systems (1) satisfying the assumption:

(H) For each $k \in \{1,\dots,N\}$-th characteristic family, letting $\lambda_k(u)$, $r_k(u)$ denote the k-th eigenvalue and a corresponding eigenvector of $A(u)$, respectively, the linearly degenerate manifold

$$\mathcal{M}_k \doteq \big\{u\in\Omega\ :\ \nabla\lambda_k(u)\cdot r_k(u) = 0\big\} \tag{4}$$

is either empty (GNL characteristic field), or it is the whole space (LD characteristic field), or it consists of a *finite* number of smooth, connected, hypersurfaces, and there holds

$$\nabla(\nabla\lambda_k\cdot r_k)(u)\cdot r_k(u) \neq 0 \qquad \forall u\in\mathcal{M}_k\,. \tag{5}$$

Notice that the Liu admissible solution of a Riemann problem for a system of conservation laws $u_t + F(u)_x = 0$ satisfying the assumption (H) consists of centered rarefaction waves, compressive shocks or *composed waves* made of a finite number of Liu admissible contact-discontinuities adjacent to rarefaction waves. On the

contrary, the solution of a Riemann problem for a general hyperbolic system may well be a composed wave containing a *countable* number of rarefaction waves and (one or several) Liu admissible contact-discontinuities with increasing speeds.

The key step of the proof of the convergence rate for a Glimm approximate solution consists in producing a Glimm functional whose variation provides a bound on the change in strength and on the product of strength times the variation in speeds of the primary waves selected by a wave tracing algorithm. For example, if we consider an interaction between two (primary) shock waves of a k-th NGNL family, say s', s'', with speeds λ', λ'', respectively, then letting λ denote the shock speed of the outgoing wave of the k-th family, it will be crucial to show that the decrease of the Glimm functional is of the same order as the term

$$[s\Delta\lambda] \doteq |s'|\big|\lambda-\lambda'\big| + |s''|\big|\lambda-\lambda''\big|\,. \tag{6}$$

There are several Glimm type functionals for NGNL systems available in the literature which work perfectly well to establish uniform a-priori bounds on the total variation of the solution, but are not truly effective to control the type of errors $[s\Delta\lambda]$ arising in a wave tracing analysis of the Glimm scheme. On the other hand, in the case of systems satisfying the assumption (H), were recently introduced in [3, 8] two type of potential interaction functionals whose decrease actually bounds the products of strength times the variation in speeds $[s\Delta\lambda]$. The Glimm functional defined in [3] is the sum of a quadratic term Q_q and of the cubic interaction potential defined in [5] concerning waves of the same family, that takes the form $\mathcal{Q} = \sum_{k_\alpha=k_\beta} \int_0^{|s_\alpha|} \int_0^{|s_\beta|} \big|\sigma_\alpha(\tau) - \sigma_\beta(\tau')\big|\; d\tau d\tau'$. Here, in presence of interactions between waves of the same families and strength smaller than some threshold parameter δ_0, Q_q behaves as the interaction functional introduced in [1] for systems with a single connected hypersurface (4), while the decrease of $\mathcal{Q}$ controls the possible increase of Q_q at interactions involving waves of the same family and strength larger than δ_0. The cubic part of the functional proposed in [8] corresponding to waves of the same family instead depends globally on the wave patterns of the solution. It is defined as $\sum_{k_\alpha=k_\beta} \big(|s_\alpha, s_\beta|[\Theta(s_\alpha, s_\beta)]^-\big)/V_{k_\alpha}(s_\alpha, s_\beta)$, where $\Theta(s_\alpha, s_\beta)$ represents the effective angle between s_α and s_β, computed taking into account all the k_α-waves lying between s_α and s_β, $[\,\cdot\,]^-$ denotes the negative part, while $V_{k_\alpha}(s_\alpha, s_\beta)$ is the total strength of all k_α-waves between s_α and s_β (including s_α and s_β). Employing these interaction potentials it is shown in [3, 8] that, for systems (1) satisfying the assumption (H), one can produce a simplified wave partition pattern whose errors are controlled by the total decrease of the corresponding Glimm functional in the time interval taken in consideration, and thus yield the error estimate

$$\big\|u^\varepsilon(T,\cdot) - u(T,\cdot)\big\|_{\mathbb{L}^1} = o(1)\cdot\sqrt{\varepsilon}\,|\log\varepsilon|\,. \tag{7}$$

Unfortunately, the decreasing properties of both functionals strongly rely on the assumption that the linearly degenerate manifold (4) is a finite union of hypersurfaces transversal to the characteristic vector fields, and thus are of no use to establish an accurate convergence rate for general systems (1).

In order to construct a more effective potential interaction functional for general NGNL systems let's consider again an interaction occurring between two waves s', s'' of the same family k-th NGNL family, and with the same sign. Observe that, by the interaction estimates in [5, Theorem 3.7] there holds

$$[s\Delta\lambda] = \mathcal{O}(1) \cdot \frac{\big|s's''\big|\big|\lambda' - \lambda''\big|}{\big|s' + s''\big|}\,. \tag{8}$$

Moreover, using the wave-speed maps $\sigma'(\cdot), \sigma''(\cdot)$ associated to the waves s', s'' (cfr. [2, 6]), one can rewrite the term on the right-hand side of (8) as

$$\mathcal{I}(s', s'') \doteq \frac{1}{\big|s'\big| + \big|s''\big|} \cdot \int_0^{|s'|} \int_0^{|s''|} \big|\sigma'(\tau) - \sigma''(\tau')\big|\, d\tau d\tau'\,. \tag{9}$$

Thus, a natural suggestion of the above estimate would be to define the cubic part of a Glimm functional related to the potential interaction of waves of the same family as the sum of terms as (9) corresponding to all pair of waves s', s'' of each characteristic family. In fact, in [4] we have introduced a Glimm functional defined by

$$Q(t) \doteq \sum_{\substack{k_\alpha < k_\beta \\ x_\alpha(t) > x_\beta(t)}} \big|s_\alpha s_\beta\big| + \sum_{k_\alpha = k_\beta} \mathcal{I}(s_\alpha, s_\beta) \tag{10}$$

where, as usual, $x_\alpha(t)$ denotes the position of the wave s_α in the approximate solution $u^\varepsilon(t)$, and k_α its characteristic family, while the second summation extends to all pair of waves s_α, s_β of the $k_\alpha \in \{1, \dots, N\}$ family (including $s_\alpha = s_\beta$). The main result in [4] shows that the potential interaction Q defined in (10) is actually decreasing in time at any interaction, and that the products $[s\Delta\lambda]$ of strength times the variation in speeds of the primary waves are bounded by $\mathcal{O}(1) \cdot |\Delta Q|$. Relying on such functional, and exploiting the same strategy followed in [3, 7, 8], one then finds that the estimate (7) holds for general non conservative, strictly hyperbolic systems (1), with no assumptions on the matrix-valued function A, beyond the $\mathcal{C}^2$ regularity.

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Transport equation for divergence free vector fields in the plane

Stefano Bianchini
(joint work with Giovanni Alberti, Gianluca Crippa)

Let us consider an autonomous vector field $b \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ in the plane such that $\mathrm{div} b = 0$. It is well-known that in this situation it is possible to find a Hamiltonian function $H \in \mathrm{Lip}(\mathbb{R}^2)$ such that

$$b(x) = \nabla^\perp H(x) = \left(-\frac{\partial H(x)}{\partial x_2}, \frac{\partial H(x)}{\partial x_1}\right) \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in \mathbb{R}^2. \tag{1}$$

The starting point for all the two-dimensional well-posedness results is the heuristic remark that the value of the Hamiltonian is constant on the trajectories. Indeed, if $\dot\gamma(t) = b(\gamma(t))$, then we can compute

$$\frac{d}{dt} H(\gamma(t)) = \nabla H(\gamma(t)) \cdot \dot\gamma(t) = \nabla H(\gamma(t)) \cdot b(\gamma(t)) = \nabla H(\gamma(t)) \cdot \nabla^\perp H(\gamma(t)) = 0\,.$$

This means that the trajectories "follow" the level sets of the Hamiltonian. Heuristically, one can try to implement the following strategy:

(a) Localize the equation to each level set, thanks to the fact that the level sets are invariant under the action of the flow;
(b) Understand the structure of the level sets, trying to prove that generically they are "one-dimensional sets";
(c) See the equation on each level set as a one-dimensional problem and show uniqueness for it;
(d) Deduce uniqueness for the problem in $\mathbb{R}^2$ from the uniqueness of all the problems on the level sets.

Since we can hope for uniqueness on the level sets under quite general hypotheses, the reduced equation being one-dimensional, we expect stronger well-posedness results in this case: it is natural to imagine that no regularity of b (in terms of weak derivatives) would be needed.

We first indicate the essential literature on this subject. Previous results by Bouchut and Desvillettes [8], Hauray [21] and Colombini and Lerner ([10] and [11]) show that uniqueness holds for the transport equation relative to an autonomous bounded divergence-free vector field, under an additional condition on the local direction of the vector field. A first extension to the non-divergence-free case is due to Colombini and Rauch [12]: they are able to show that the uniqueness holds in the case of autonomous bounded vector fields with bounded divergence for which there exists a positive Lipschitz function θ, bounded and bounded away from zero, such that

$$\mathrm{div}(\theta b) = 0\,. \tag{2}$$

In our talk we present a more recent result, still in progress, in collaboration with Alberti and Crippa [2].

The strategy is a bit different: we do not perform a local change of variable according to the Hamiltonian, but we rather split the equation on the level sets of the Hamiltonian, using the coarea formula. Then we would like to look at the equation level set by level set. It turns out that, where $\nabla H \neq 0$, the level sets are in fact nice rectifiable curves, and this will allow to consider the PDE in the parametrization. The interesting point is that, in order to separate the evolution in $\{\nabla H = 0\}$ from the evolution in $\{\nabla H \neq 0\}$, we need again a regularity condition, regarding again the "amount of the critical points of H". This is precisely the weak Sard property:

$$H_{\#}\big(\mathscr{L}2 \llcorner \{\nabla H = 0\}\big) \perp \mathscr{L}1\,. \tag{3}$$

However we notice that condition (3) is much weaker than the previous one; moreover, two examples (for which we refer to [2]) indicate that the weak Sard property is necessary in order to obtain uniqueness.

The main theorem presented is the following:

Theorem 0.1. *Let $b \in L^\infty(\mathbb{R}2;\mathbb{R}2)$ with compact support and assume that* $\mathrm{div} b = 0$. *Let $H \in \mathrm{Lip}_c(\mathbb{R}2)$ be as in* (1) *and assume that H satisfies the weak Sard property* (3). *Then, for every initial data $\bar u \in L^\infty(\mathbb{R}2)$, the Cauchy problem*

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0 & \\ u(0,\cdot) = \bar u & \end{cases} \qquad \text{in } \mathcal{D}'([0,T]\times\mathbb{R}2), \tag{4}$$

has a unique solution $u \in L^\infty([0,T]\times\mathbb{R}2)$.

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A fast scheme for multi-layer shallow water equations, including elliptic regions and drying

François Bouchut
(joint work with Vladimir Zeitlin)

The multi-layer shallow water system reads, in one space dimension,

$$
\begin{aligned}
&\partial_t h_j + \partial_x(h_j u_j) = 0,\\
&\partial_t(h_j u_j) + \partial_x(h_j u_j^2 + g h_j^2/2) + g h_j \partial_x \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) = 0,
\end{aligned}
\tag{1}
$$

where h_j, $j = 1, \dots, n$ are the fluid depths, u_j are the velocities, and $z(x)$ is the topography. The constants g, $\rho_1 \leq \cdots \leq \rho_n$ are respectively the gravity and the densities of the fluids. The fluids $1, \dots, n$ are labeled from top to bottom.

This system admits a convex entropy, and thus we are looking for entropy solutions, satisfying

$$
\begin{aligned}
&\partial_t \left(\sum_j \rho_j \left(h_j u_j^2/2 + g h_j^2/2 + h_j g z \right) + g \sum_{j,k,k<j} \rho_k h_k h_j \right) \\
&+\partial_x \left(\sum_j \rho_j u_j \left(h_j u_j^2/2 + g h_j^2 + h_j g \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) \right) \right) \leq 0.
\end{aligned} \tag{2}
$$

This system has the steady states at rest

$$
u_j = 0, \qquad \partial_x \left(h_j + z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) = 0, \text{ for } j = 1, \ldots, n. \tag{3}
$$

Notice that if $\rho_1 < \cdots < \rho_n$, this reduces to

$$
u_j = 0, \qquad \partial_x (z + h_n) = 0, \qquad \partial_x h_j = 0 \text{ for } j < n, \tag{4}
$$

while if $\rho_1 = \cdots = \rho_n$, (3) reduces to

$$
u_j = 0, \qquad \partial_x (z + h_1 + \cdots + h_n) = 0. \tag{5}
$$

As for the one-layer shallow water system ($n = 1$), the numerical difficulties related to this system are positivity of the depths h_j, with the possibility of treating drying, the exact preservation of the steady states at rest (well-balanced property), and the property to have a discrete entropy inequality. Overall, the multi-layer system has extra difficulties which are the nonconservativity of the system (even for smooth topography z), and the possibility of having complex eigenvalues (the system is not everywhere hyperbolic).

Several attempts have been made in order to solve this system. In [4] and subsequent papers of the Malaga-Sevilla school, the bilayer case is treated by a Roe type method. A special treatment is done in order to recover positivity, and a special treatment is performed for complex eigenvalues, making the scheme unconsistant in this case. A relaxation method is proposed in [5], with similar properties, but which is not able to treat drying, nor complex eigenvalues. In [3], the n-layer system is treated in the case $\rho_1 = \ldots \rho_n$, without restrictions on the eigenvalues and including drying. However, topography is not included, and it does not extend to general densities.

Here, we adopt the splitting approach of [1], that enables to treat separately each of the layers without computing the eigenvalues of the whole system. In [1], some difficulties of getting wrong solutions were found. In this talk, I shall explain that they were due to the failure of the conservation of total momentum. I shall give a method to correct this, based on a source-centered hydrostatic scheme for the one-layer shallow water, a variant of the hydrostatic scheme proposed in [2]. The final method enables to treat an arbitrary number n of layers, with arbitrary densities $\rho_1, \ldots, \rho_n$, and arbitrary topography. It has no restriction concerning complex eigenvalues, it is well-balanced and it is able to treat vacuum, it satisfies

a semi-discrete entropy inequality. The scheme is fast to execute, as is the one-layer hydrostatic method.

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A coupled Monge-Ampère system generalizing one-dimensional scalar conservation laws

Yann Brenier

A one dimensional scalar conservation laws such as:

$$\partial_t u + \partial_x(G(u)) = 0 \ ,$$

where G is a given smooth function, admits many possible multidimensional generalizations. One of particular interest is given by the following coupled system

$$\partial_t \rho + \nabla \cdot (\rho w) = 0 \ , \quad w = g(\nabla \phi) \ , \quad \Delta \phi = \rho \ ,$$

where $\rho = \rho(t,x)$, $\phi = \phi(t,x)$ are unknown, x varies in R^d and $g : R^d \to R^d$ is a given smooth function with bounded derivatives. (This, of course, has to be supplemented by suitable boundary conditions.) This model, that we call coupled Poisson system, or CP in short, occurs as a "high field limit" of the Vlasov-Poisson system (as studied by Nieto, Poupaud and Soler in the early 2000s), or, alternately, as a simplified model for chemotaxis. In one space dimension ($d = 1$) the scalar conservation law

$$\partial_t u + \partial_x(G(u)) = 0 \ ,$$

is recovered just by setting $g = G'$, $\rho(t,x) = \partial_x u(t,x)$.

Another possible extension of the inviscid Burgers equation is obtained by substituting the Monge-Ampère equation for the Poisson equation, namely:

$$\partial_t \rho + \nabla \cdot (\rho w) = 0 \ , \quad w = g(\nabla \phi) \ , \quad det(D_x^2 \phi) = \rho \ .$$

Notice that for $d = 1$, there is no difference between the Poisson and the Monge-Ampère equations!

The coupled Monge-Ampère system, CMA in short, has severable features, that are not shared by the coupled Poisson system.
First, it is easily integrable for short times, by the method of characteristics and particle trajectories are just straight lines (just as for the inviscid Burgers equations), at least when $g = G'$ for some smooth scalar function $G : R^d \to R$ with bounded second order derivatives.
Next, the initial value problem always has a (possibly non unique) global solution in a suitable sense (which is not known for the coupled Poisson system).
Finally, it has a physical interpretation and can be derived from the Navier-Stokes Boussinesq (NSB) system

$$\epsilon(\partial_t + v \cdot \nabla - \Delta)v + \nabla p = f \ , \quad (\partial_t + v \cdot \nabla)f = g \ , \ \nabla \cdot v = 0,$$

as ϵ goes to zero, where the unknowns $f = f(t,x) \in R^d$, $v = v(t,x) \in R^d$, $p = p(t,x) \in R$ depend on t and $x \in R^d$ and $g = g(x) \in R^d$ is given. The limit system, that we call Hydrostatic Boussinesq (HB) model,

$$\nabla p = f \ , \quad (\partial_t + v \cdot \nabla)f = g \ , \ \nabla \cdot v = 0,$$

is locally well posed under the requirement that $p = p(t,x)$ is a uniformly strictly convex smooth function in x. The HB and CMA systems are formally equivalent once we set

$$p(t,x) = \sup_{\tilde{x}} \ \ x \cdot \tilde{x} - \phi(t,\tilde{x}),$$

which means that p and ϕ are Legendre dual. For both HB and CMA systems, we suggest the following weak formulation: $f(t,x) = \nabla p(t,x)$ is a weak solution if it belongs to $C^0_t(L^2_x)$, $p(t,x)$ is convex in x, and

$$\frac{d}{dt}\int z(f(t,x))dx = \int g(x) \cdot (\nabla z)(f(t,x))dx,$$

for all smooth test functions z with quadratic growth at infinity. Then, it is easy to prove the existence of a (possibly non unique) solution for each compatible initial condition.

For more details, we refer to the paper "Optimal transport, convection, magnetic relaxation and generalized Boussinesq equations", arXiv:0801.1088.

The initial-boundary Riemann problem and the time-variant vanishing viscosity method

Cleopatra Christoforou
(joint work with Laura V. Spinolo)

We study the initial-boundary Riemann problem for systems of conservation laws in one-space dimension:

$$\begin{array}{ll} \partial_t u + \partial_x f(u) = 0, & \\ u(0,x) = u_0, & x > 0, \\ u(t,0) = \bar{u}, & t > 0 \ . \end{array} \tag{1}$$

Here, $x \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth function and u_0, $\bar{u}$ are constant states in $\mathbb{R}^n$. We assume that (i) the system is strictly hyperbolic, i.e. $A(u) = Df(u)$ has n real and distinct eigenvalues $\lambda_i(u)$, and therefore, n linearly independent eigenvectors $r_i(u)$, $i = 1, \ldots, n$ and (ii) the boundary is non-characteristic, i.e. $det(A) \neq 0$.

The boundary problem is very interesting since one faces the additional challenge of imposing appropriate conditions on the boundary data $\bar{u}$ to expect a well-posed problem and in general, this is a difficult task. This problem has been studied extensively, especially using the standard vanishing viscosity method:

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u \, . \tag{2}$$

Results of existence, stability, uniqueness of solutions, stability of boundary layer profiles and convergence have been established under various conditions; cf. [1, 5, 7, 8, 9, 11, 12]. We refer the reader to the books [4, 10] and the references therein.

The objective of this work is to construct an entropy weak solution of bounded variation (BV) to system (1) via a different vanishing viscosity method, for which we let the viscosity coefficient to vary with time. Namely, we consider

$$\begin{array}{ll} \partial_t u + \partial_x f(u) = \varepsilon \, t \, \partial_x^2 u, & \\ u(0,x) = u_0, & x > 0, \\ u(t,0) = \bar{u}_b, & t > 0 \, . \end{array} \tag{3}$$

It is easy to check that the solution u_ε to (3) is given by $u_\varepsilon(x,t) = V_\varepsilon(\frac{x}{t})$ if and only if $V_\varepsilon(\xi)$ is a solution of the ODE

$$\begin{array}{l} (A(V_\varepsilon(\xi)) - \xi \, I)\dot{V}_\varepsilon(\xi) = \varepsilon \ddot{V}_\varepsilon(\xi), \qquad \xi > 0 \, , \\ V_\varepsilon(0) = \bar{u}_b, \quad V_\varepsilon(+\infty) = \bar{u}_0 \, . \end{array} \tag{4}$$

Thus, the problem reduces to establishing BV solution to system (1) as a limit of viscous self-similar solutions to ODE (4).

The notion of self-similar viscous limits was first studied in the context of Cauchy problems by Dafermos [3], Kalasnikov [6] and Tupciev [13, 14] independently. Dafermos initiated the use of self-similar limits as an admissibility criterion, *the viscous wave fan criterion*, which also serves as an alternative approach for constructing Riemann solutions. Tzavaras [15] implemented this program for constructing Riemann solutions to general strictly hyperbolic systems assuming that the Riemann data are sufficiently close. Moreover, he studied the structure of the emerging solution and proved that it is the same one with the classical Riemann solution as constructed by Lax and Liu. It should be added that recently Dafermos [4] revisited this problem and constructed the viscous wave fan curves in the spirit of Bianchini and Bressan [2].

We consider this program in the setting of the boundary problem with the additional challenges that arise while treating the boundary layer. Following the ideas and techniques of Tzavaras [15] and introducing additional tools to treat carefully the waves that correspond to negative characteristic speeds, we prove that if $|\bar{u}_b - u_0|$ is sufficiently small, then for each $\varepsilon > 0$, there exists a unique

solution $V_\varepsilon(\xi)$ to (4) defined on $\xi \in (0,\infty)$, that satisfies the decomposition

$$\dot{V}_\varepsilon(\xi) = \sum_{j=1}^{n} \left[\tau_j \varphi_j(\xi) + \vartheta_j(\xi,\tau)\right] r_j(V_\varepsilon(\xi)) \,, \tag{5}$$

where φ_j solves $\varepsilon\dot{\varphi}_j + [\xi - \lambda_j(V_\varepsilon)]\varphi_j = 0$, τ_j corresponds to the strength of the j-wave and $\theta_j(\cdot,\tau)$ is the contribution on the j–wave from the interaction of waves. Studying carefully the interaction of waves, we show that the total variation of V_ε on $(0,\infty)$ is small and uniformly bounded. Hence, by Helly's compactness theorem, we can extract a convergent subsequence $\{V_{\varepsilon_m}\}$ and the limit V induces a solution $u(x,t) = V(x/t)$ to the boundary problem (1).

The second part of our project is to characterize the hyperbolic trace $\bar{u}$, the boundary data of the viscous self-similar limit $u(x,t)$. Namely, we describe $\bar{u}$ in terms of the data $\bar{u}_b$ and u_0 of the viscous approximation (3). To achieve this, we construct a locally invertible map ϕ^B that connects the initial data u_0 and the boundary data $\bar{u}_b$

$$\begin{aligned} \bar{u}_b &= \phi^B(s_1, s_2, \ldots, s_n, u_0) \\ &= F(s_1, \ldots, s_{n-p}, \circ T^{n-p+1}_{s_{n-p+1}} \circ \cdots \circ T^n_{s_n} u_0) \end{aligned} \tag{6}$$

using the viscous waves fan curves $T^j_{s_j}$ that correspond to positive characteristic speeds and the boundary layer profiles F that correspond to all negative characteristic speeds. This allows us to establish implicitly a characterization of the hyperbolic data and get

$$\bar{u} \doteq \lim_{x\to 0+} u(x,t) = T^{n-p+1}_{s_{n-p+1}} \circ \cdots \circ T^n_{s_n} u_0 \,. \tag{7}$$

An interesting remark that follows through this work is that we can compare the hyperbolic trace obtained via the self-similar viscous limits (3) with the one via the standard vanishing viscosity limit (2) and show that it is the same. This is very interesting and useful since it is known that the hyperbolic trace depends in general on the approximate scheme.

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On Discontinuous Galerkin Schemes for Compressible Flow

Miloslav Feistauer

In the numerical solution of compressible flow, it is necessary to overcome a number of obstacles. Let us mention the necessity to resolve accurately shock waves, contact discontinuities and (in viscous flow) boundary layers, wakes and their interaction. All these phenomena are connected with the simulation of high speed flow with high Mach numbers. However, it appears that the solution of low Mach number flow is also rather difficult. This is caused by the stiff behavior of numerical schemes and acoustic phenomena appearing in low Mach number flows at incompressible limit. In this case, standard finite volume schemes fail. This led to the development of special finite volume techniques allowing the simulation of compressible flow at incompressible limit, which is based on modifications of the Euler or Navier-Stokes equations. These techniques are adequate for low Mach number flows, but cannot be applied to transonic and supersonic flows with shock waves and contact discontinuities.

In the Department of Numerical Mathematics at the Faculty of Mathematics and Physics of Charles University Prague together with Vít Dolejší and Václav Kučera we are concerned with the development of a new technique for the numerical solution of compressible flow.

Here we are concerned with the treatment of two-dimensional flow, but the method can be applied to 3D flow as well. The system of the Euler equations describing 2D inviscid flow can be written in the form ([5])

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0 \quad \text{in } Q_T = \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain occupied by gas, $T > 0$ is the length of a time interval,

$$\boldsymbol{w} = (w_1, \dots, w_4)^T = (\rho,\, \rho v_1,\, \rho v_2,\, E)^T \tag{2}$$

is the so-called state vector and

$$\boldsymbol{f}_s(\boldsymbol{w}) = (\rho v_s,\, \rho v_s v_1 + \delta_{s1} p,\, \rho v_s v_2 + \delta_{s2} p,\, (E+p)\, v_s)^T \tag{3}$$

are the inviscid (Euler) fluxes of the quantity $\boldsymbol{w}$ in the directions x_s, $s = 1, 2$. We use the following notation: ρ – density, p – pressure, E – total energy, $\boldsymbol{v} = (v_1, v_2)$ – velocity, δ_{sk} – Kronecker symbol. The equation of state implies that

$$p = (\gamma - 1)\,(E - \rho|\boldsymbol{v}|^2/2). \tag{4}$$

Here $\gamma > 1$ is the Poisson adiabatic constant. The system (1) – (4) is *diagonally hyperbolic*. It is equipped with the initial condition

$$\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x), \quad x \in \Omega, \tag{5}$$

and suitable boundary conditions.

We developed a numerical method for the solution of the above initial-boundary value problem. Our technique is based on the application of the discontinuous Galerkin finite element method (DGFEM), which employs piecewise polynomial approximations without any requirement on the continuity on interfaces between neighboring elements. The DGFEM was first used for the solution of inviscid compressible flow in [1].

The discontinuous Galerkin space semidiscretization is combined with a semi-implicit time discretization proposed in [3]. In this way we obtain a numerical scheme requiring the solution of only one linear system on each time level solved either by the direct UMFPACK technique ([2]) or the GMRES iterations with a block diagonal preconditioning. An important ingredient is a special characteristic treatment of boundary conditions in inviscid convective terms, transparent for acoustic effects coming from inside of the computational domain ([6]). In case of high-speed flow with discontinuous solutions the limiting procedure avoiding the Gibbs phenomenon manifested by spurious overshoots and undershoots in the vicinity of discontinuities is applied. It is based on the use of the discontinuity indicator proposed and tested in [4].

The described method is unconditionally stable and allows the solution of compressible flow for practically all Mach numbers, from very low Mach number flow up to hypersonic regimes, without any modification of the Euler equations.

The efficiency, accuracy and robustness of the presented technique has been demonstrated by numerical examples.

Recently, we have adapted the method to the solution of compressible flow in time-dependent domains and applications to fluid-structure interaction problems ([7]).

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Shock Reflection and Free Boundary Problems

Mikhail Feldman
(joint work with Gui-Qiang Chen)

One of the important problems in mathematical fluid dynamics is reflection of shock by a wedge. It arises in many physical application, and in the study of multidimensional conservation laws since its solutions are building blocks and asymptotic attractors for the general solutions of Euler equations for compressible fluids. The reflection picture was first described by Ernst Mach in 1878. In later works, experimental, computational, and asymptotic analysis have shown that various patterns of reflected shocks may occur, including regular and Mach reflection [2, 5, 6, 7, 8, 9]. However, there has been no rigorous mathematical results on the global existence and structural stability of shock reflection, especially for potential flow equation, which has been used in aerodynamics. Such problems involve several difficulties in the analysis of nonlinear partial differential equations including equations of elliptic-hyperbolic mixed type, free boundary problems, degenerate ellipticity along the sonic line.

In the talk I describe recent results on regular shock reflection for potential flow equation in dimension two. For potential flow, velocity $\mathbf{u}$ is $D_{\mathbf{x}}\Phi$, where Φ is the potential.

A plane shock in the $(\mathbf{x}, t)$–coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left state $(\rho, D_{\mathbf{x}}\Phi) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0), u_1 > 0, \rho_0 < \rho_1$, hits a symmetric wedge $W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_w, x_1 > 0\}$ at time zero. We can consider only upper half-plane $\mathbb{R}^2_+ = \{x_2 > 0\}$. We are looking for a solution in $\Lambda = \mathbb{R}^2_+ \setminus W$ of the time-dependent potential flow system satisfying initial data

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \text{for } |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (\rho_1, u_1 x_1) & \text{for } x_1 < 0, \end{cases} \tag{1}$$

and boundary condition

$$\nabla\Phi \cdot \nu|_{\partial\Lambda} = 0. \tag{2}$$

Since self-similar solutions are expected, we rewrite this as a quasi-static problem in self-similar plane.

Potential flow equation for self-similar solutions, in self-similar variables $(\xi,\eta)=(\frac{x}{t},\frac{y}{t})$, is

$$\operatorname{div}\left(\rho(|D\varphi|^2,\varphi)D\varphi\right)+2\rho(|D\varphi|^2,\varphi)=0, \tag{3}$$

with $\rho(|D\varphi|^2,\varphi)=\left(\rho_0^{\gamma-1}-(\gamma-1)(\varphi+\frac{1}{2}|D\varphi|^2)\right)^{\frac{1}{\gamma-1}}$, where $\varphi(\xi,\eta)$ is the pseudo-velocity potential, ρ is density, and $\gamma>1,\rho_0>0$ are constants. Equation is elliptic-hyperbolic mixed, which is elliptic (resp. hyperbolic) if and only if

$$|D\varphi|<c(|D\varphi|^2,\varphi),\quad \left(\text{resp. } |D\varphi|>c(|D\varphi|^2,\varphi)\right),$$

where $c(|D\varphi|^2,\varphi)$ is the sonic speed defined by $c^2=\rho^{\gamma-1}$. Solution is called subsonic (resp. supersonic) in elliptic (resp. hyperbolic) regions. Shocks are discontinuities in the pseudo-velocity $D\varphi$. That is, if Ω^+ and $\Omega^-:=\Omega\setminus\overline{\Omega^+}$ are two nonempty open subsets of $\Omega\subset\mathbb{R}^2$ and $S:=\partial\Omega^+\cap\Omega$ is a C^1–curve where $D\varphi$ has a jump, then $\varphi\in W^{1,1}_{loc}(\Omega)\cap C^1(\Omega^\pm\cup S)\cap C^2(\Omega^\pm)$ is a global weak solution of (3) in Ω if and only if φ satisfies equation (3) in $\Omega^\pm$ and the Rankine-Hugoniot conditions on S:

$$[\varphi]_S=0,\qquad \left[\rho(|D\varphi|^2,\varphi)D\varphi\cdot\nu\right]_S=0. \tag{4}$$

The plane incident shock solution in the $(\mathbf{x},t)$–coordinates with states $(\rho,\nabla_{\mathbf{x}}\Psi)=(\rho_0,0,0)$ and $(\rho_1,u_1,0)$ corresponds to a weak solution φ of (3) of the form:

$$\varphi_0(\xi,\eta)=-\frac{1}{2}(\xi^2+\eta^2)\qquad \text{for } \xi>\xi_0, \tag{5}$$

$$\varphi_1(\xi,\eta)=-\frac{1}{2}(\xi^2+\eta^2)+u_1(\xi-\xi_0)\qquad \text{for } \xi<\xi_0, \tag{6}$$

respectively, where $S_0=\{\xi=\xi_0\}$ is the incident shock. Here ξ_0 is uniquely determined by (ρ_0,ρ_1,γ) through (4). Denote by P_0 the point of intersection of S_0 with the wedge boundary, that is, $P_0=(\xi_0,\xi_0\tan\theta_w)$. Shock reflection problem is now reduced to the following problem in self-similar plane:

Problem 1. *Seek a solution φ of equation* (3) *in the self-similar domain Λ with the slip boundary condition (2) and the asymptotic boundary condition at infinity:*

$$\varphi\to\bar{\varphi}:=\begin{cases}\varphi_0 & \text{for } \xi>\xi_0,\eta>\xi\tan\theta_w,\\ \varphi_1 & \text{for } \xi<\xi_0,\eta>0,\end{cases}\qquad \text{when } \xi^2+\eta^2\to\infty,$$

where the convergence holds in the sense that $\lim\limits_{R\to\infty}\|\varphi-\overline{\varphi}\|_{C(\Lambda\setminus B_R(0))}=0.$

Since φ_1 does not satisfy the slip boundary condition (2), the solution must differ from φ_1 in $\{\xi<\xi_0\}\cap\Lambda$ and thus a shock diffraction by the wedge occurs.

Denote by $P_0=(\xi_0,\xi_0\tan\theta_w)$ the point of intersection of the incident shock S_0 with the wedge boundary. There exists an angle $\theta_{sonic}\in(0,\pi/2)$ determined by

ρ_0, ρ_1, γ such that for the wedge angles $\theta_w \in (\theta_{sonic}, \pi/2)$ there exists a uniform state

$$\varphi_2(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_2(\xi-\xi_0) + (\eta - \xi_0 \tan\theta_w)u_2 \tan\theta_w, \tag{7}$$

which satisfies (2) on the wedge boundary $\{\eta = \xi \tan\theta_w\}$, and satisfies Rankine-Hugoniot conditions (4) with φ_1 at P_0 and thus along the line $S_1 = \{\varphi_1 = \varphi_2\}$. Constant velocity $(u_2, u_2 \tan\theta_w)$ and density ρ_2 are determined by $(\theta_w, \rho_0, \rho_1, \gamma)$ from the two algebraic equations expressing the conditions above. Moreover $\rho_2 > \rho_1$, and φ_2 is supersonic(hyperbolic) at the point P_0. For such wedge angles $\theta_w \in (\theta_{sonic}, \pi/2)$ the structure of global solution φ to Problem 1 is expected to be regular reflection which described as following:

Let B be the sonic circle for state (2) with center $(u_2, u_2 \tan\theta_w)$ and radius $c_2 = \rho_2^{(\gamma-1)/2} > 0$ (the sonic speed of φ_2). Denote by P_1 (resp P_4) the point of intersection of ∂B with S_1 (resp. with the wedge boundary $\{\eta = \xi \tan\theta_w\}$). It is expected that the solutions φ and φ_1 differ within $\{\xi < \xi_0\}$ only in the domain $P_0P_1P_2P_3P_4$, where $P_2 \in \{\xi < 0,\ \eta = 0\}$ and $P_3 = (0,0)$. The curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. Then, within $P_0P_1P_2P_3P_4$, solution φ differs from φ_2 in the domain $\Omega = P_1P_2P_3P_4$, where the equation (3) is elliptic. The boundary of Ω consists of the sonic arc P_1P_4, line segments P_2P_3 and P_3P_4 and the curved part of the reflected shock P_1P_2, which is apriori unknown (the free boundary).

Theorem 1 ([3]). *For any $\gamma > 1$ and $\rho_1 > \rho_0 > 0$ there exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0,1)$ such that, when $\theta_w \in [\theta_c, \frac{\pi}{2})$, there exists a weak solution of Problem 1, which satisfies the following:*

(i)

$$\varphi \in C^{0,1}(\Lambda), \qquad \varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega}),$$
$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan\theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflection shock } P_0P_1P_2, \\ \varphi_2 & \text{in } P_0P_1P_4. \end{cases}$$

(ii) *equation (3) is elliptic in Ω;*
(iii) $\varphi \geq \varphi_2$ *in* Ω;
(iv) *the reflected shock $P_0P_1P_2$ is C^2 at P_1 and C^∞ elsewhere;*
(v) *φ is $C^{1,1}$ across the part $\Gamma_{sonic} = P_1P_4$ of the sonic circle.*

Theorem 2 ([4]). *Let $\gamma > 1$ and $\rho_1 > \rho_0$ satisfy the condition $u_1 < c_1$, where $c_1^2 = \rho^{\gamma-1}$. Then solution of Problem 1 satisfying properties (i)-(v) of Theorem 1 exists for all $\theta_w \in (\theta_{sonic}, \pi/2)$.*

The condition in Theorem 2 is an explicit algebraic condition in terms of γ, ρ_0, ρ_1.

Next we show that $C^{1,1}$ regularity near and across sonic arc $\Gamma_{sonic} = P_1P_4$ where ellipticity degenerates is optimal:

Theorem 3 ([1]). *Let φ be a solution of Problem 1 satisfying properties (i)-(v) of Theorem 1. Then:*

(i) *φ is $C^{2,\alpha}$ in Ω up to Γ_{sonic} away from the point P_1 for any $\alpha \in (0,1)$,*
(ii) *φ is $C^{1,1}$ but not C^2 across Γ_{sonic}, specifically $D^2\varphi$ has a jump across Γ_{sonic},*
(iii) *The limit $\lim_{\substack{(\xi,\eta)\to P_1 \\ (\xi,\eta)\in\Omega}} D^2\varphi$ does not exist.*

For the proofs, we reformulate Problem 1 as a free boundary problem for the free boundary Γ_{sonic} and φ in the elliptic region Ω. Free boundary conditions are Rankine-Hugoniot conditions on Γ_{sonic}. We solve this problem by method of continuity, which involves deriving some regularity estimates for degenerate elliptic equations, and controlling geometry of free boundary using maximum principle.

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Anelastic and weakly compressible motions in the atmosphere

Rupert Klein
(joint work with Didier Bresch)

Most atmospheric motions of meteorological interest are characterized by low Mach numbers and Strouhal numbers of order one, where the Strouhal number describes the ratio of a typical advection and the characteristic flow time scales. As a consequence, the assumption of complete pressure equilibration through rapid acoustic wave propagation is justified, and this leads to "sound-proof" model equations in the spirit of the zero Mach number incompressible flow models in engineering fluid mechanics. However, due to the strong pressure and density variations in the vertical over distances comparable to and larger than about 10 km, the structure of sound-proof models for atmospheric flows differs from the incompressible flow models. Instead of the velocity itself, the velocity weighted by the background

density $\overline{\rho}(z)$ or by a certain power of the background pressure $\overline{P}(z) = \overline{p}(z)^{\frac{1}{\gamma}}$ is divergence-controlled, i.e., one has

$$\nabla \cdot (\overline{\rho}(z)\vec{v}) = 0 \qquad \text{or} \qquad \nabla \cdot (\overline{P}(z)\vec{v}) = 0.$$

In addition, the ubiquitous background stratification of entropy (or potential temperature θ) enables internal gravity wave propagation, and these waves are of crucial importance for a range of meteorological phenomena. Ogura and Phillips [1] systematically derived, from the full compressible flow equations, a reduced "sound-proof" system that combined the above-mentioned divergence constraint based on the density stratification with a Boussinesq-type approximation for the effects of gravity. This model, as desired, does not support acoustic modes but it maintains advection and internal gravity waves. Yet, it is based on the assumption that internal waves and advection act on comparable characteristic time scales, which implies unrealistically weak stratification of the potential temperature.

Later extensions of the Ogura-Phillips model by, e.g., Lipps and Hemler [2] or Durran [3] yielded more satisfactory results in simulations, yet their derivations were partially inconsistent as revealed recently by the author. The physical set-up targeted by Lipps-Hemler, Durran and others corresponds, mathematically speaking, to a *three-scale* asymptotic problem, where the sound propagation time is much shorter than the characteristic internal gravity wave time scale, which in turn is much shorter than the time scale of advection. The goal in designing sound-proof models now is to eliminate the fastest of these three modes while retaining simultaneously the internal wave and advection time scales.

In this presentation I have explained this problem set-up, pointed out that the desired type of model is not in reach of classical single or multiple scales asymptotics, and have summarized recent attempts at constructing a sound-proof two-scale model from the full compressible flow equations which involve three separated time scales.

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New relaxation solvers for Hydro- and Magnetohydrodynamics applied to astrophysical flow simulations

Christian Klingenberg
(joint work with François Bouchut, Wolfram Schmidt, Knut Waagan)

We present a relaxation system for ideal MHD that is an extension of the Suliciu relaxation system for the Euler equations of gas dynamics. From it one can derive approximate Riemann solvers with three, five or seven waves, that generalize the HLLC solver for gas dynamics [1]. Under some subcharacteristic conditions, the solvers satisfy discrete entropy inequalities, and preserve positivity of density and internal energy. The subcharacteristic conditions are nonlinear constraints on the relaxation parameters relating them to the initial states and the intermediate states of the approximate Riemann solver itself. The 7-wave version of the solver is able to resolve exactly all material and Alfven isolated contact discontinuities.

Next we consider the practical implementation, and derive explicit wave speed estimates satisfying the stability conditions [2]. We present a 3-wave solver, and a 5-wave solver that resolves accurately the cases when characteristic speeds coincide. For the full 7-wave solver we make some simplifications when deriving the explicit speed estimates, but we still get accurate and robust results. We test the solvers on one-dimensional shock tube data and smooth shear waves.

We put this into an astrophysical application by comparing our new positive and entropy stable approximate Riemann solver with state-of the-art algorithms for astrophysical fluid dynamics [3]. We implemented the new Riemann solver into an astrophysical PPM-code, the Prometheus code. We present shock tube tests, two-dimensional instability tests and forced turbulence simulations in three dimensions. We find differences between the codes in the shock tube tests, and in the statistics of the turbulence simulations. The new Riemann solver increases the computational speed without significant loss of accuracy.

These 3-dimensional turbulence simulations are part of a plan to develop, implement, and apply a new numerical scheme for modeling turbulent, multiphase astrophysical flows such as galaxy cluster cores and star forming regions [4] [5]. The method combines the capabilities of adaptive mesh refinement and large - eddy simulations to capture localized features and to represent unresolved turbulence, respectively; we therefore refer to it as **F**luid m**E**chanics with **A**daptively **R**efined **L**arge- **E**ddy **S**imulation**S** or FEARLESS. We shall present advances in this ongoing project.

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Fast explicit operator splitting method for convection-dominated problems

Alexander Kurganov
(joint work with Alina Chertock, Charles R. Doering, E. Kashdan, and Guergana Petrova)

We study numerical methods for (systems of) convection-diffusion equations that arise in a variety of applications and represent mathematical models for a number of (physical) processes in fluid mechanics, astrophysics, meteorology, multiphase flow in oil reservoirs, polymer flow, financial modeling, and many other areas.

We consider the initial value problem

$$\mathbf{q}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{q}) = D\Delta\mathbf{q}, \qquad \mathbf{q}(\mathbf{x},0) = \mathbf{q}_0(\mathbf{x}), \tag{1}$$

where, $\mathbf{q}(\mathbf{x},t) = (q_1(\mathbf{x},t),\dots,q_l(\mathbf{x},t))^T$ is an unknown l-vector, $\mathbf{f}$ is a nonlinear convection flux, and $D = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_l)$ is a constant diagonal matrix with positive entries. In the general multidimensional case, $\mathbf{q}$ is a vector function of a time variable t and d-dimensional spatial variable $\mathbf{x} = (x_1,\dots,x_d)$ with corresponding fluxes $\mathbf{f} = (f^1,\dots,f^d)$. We also consider a closely related viscous Hamilton-Jacobi (HJ) equation

$$\varphi_t + H(\varphi_{x_1},\dots,\varphi_{x_d}) = \varepsilon\Delta\varphi, \tag{2}$$

where $\varphi(\mathbf{x},t)$ is an unknown function, H is a nonlinear Hamiltonian, and ε is a positive constant.

It is well-known that the considered models are parabolic and thus they admit global smooth solutions even for discontinuous initial data. This makes it easy to design stable and convergent numerical methods for (1). Their resolution, however, will depend on the size of the diffusion coefficients. The convection dominated regime ($\varepsilon \ll 1$) is the most challenging one from a numerical perspective since utilizing the grid refinement strategy for small ε may be computationally unaffordable, especially in the multidimensional case. In practice, one is forced to use underresolved methods (with $\Delta x \gg \varepsilon$) and therefore may want to ignore the right-hand side of (1) by taking $\varepsilon_i = 0\ \forall i$, and applying a shock-capturing method to the resulting hyperbolic system. This, however, may produce unphysical shocks and boundary layers solely determined by numerical diffusion.

One way to overcome this difficulty is to use an operator splitting method, which can be briefly described as follows. Consider the system (1) and denote by $\mathcal{S}_{\mathcal{H}}$ the *exact* solution operator associated with the corresponding hyperbolic system:

$$\mathbf{q}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{q}) = 0, \tag{3}$$

and by $\mathcal{S}_\mathcal{P}$ the *exact* solution operator associated with the (linear) parabolic system

$$\mathbf{q}_t = D\Delta\mathbf{q}. \tag{4}$$

Let us assume that the solution of the original convection-diffusion system (1) is available at time t. We then introduce a (small) time step Δt and evolve the solution of (1) from t to $t + \Delta t$ according to the Strang splitting method, which consists of three substeps:

$$\mathbf{q}(\mathbf{x}, t + \Delta t) = \mathcal{S}_\mathcal{H}(\Delta t/2)\mathcal{S}_\mathcal{P}(\Delta t)\mathcal{S}_\mathcal{H}(\Delta t/2)\mathbf{q}(\mathbf{x}, t). \tag{5}$$

Our fast explicit operator splitting method is based on the Strang splitting algorithm (5).

In practice, the exact solution operators $\mathcal{S}_\mathcal{H}$ and $\mathcal{S}_\mathcal{P}$ are to be replaced by their numerical approximations. Note that the hyperbolic, (3), and the parabolic, (4), subproblems, which are of different nature, can be solved by different numerical methods—this is one of the main advantages of the operator splitting technique.

Hyperbolic Solvers. The choice of a discrete hyperbolic solution operator is typically motivated by the properties of the flux function in (3) or the Hamiltonian in (2). If $\mathbf{f}(\mathbf{q})$ is nonlinear, then (3) is a hyperbolic system of conservation laws, whose solutions are generically discontinuous. In this case, the system (3) should be solved by a shock-capturing scheme. If H is nonlinear, the corresponding inviscid HJ equation should be solved by an appropriate high-resolution method. If $\mathbf{f}$ (H) is linear, then the shock-capturing techniques may be overly diffusive, so that one may prefer to use either a spectral or particle method, or the method of characteristics.

Parabolic Solvers. Using the method of lines, the parabolic subproblem (4) can be reduced to a system of ODEs, which can be efficiently and accurately integrated by either an appropriate implicit, large stability domains explicit, or implicit-explicit ODE solver. As an alternative, (4) may be solved exactly using the heat kernel solution formula, as proposed in [2, 5, 4], or using a pseudo-spectral method. In the latter cases, there is no stability restriction on the size of the "parabolic" substep, which is the key point in designing an efficient explicit method.

In the recent paper [3], we have provided a detailed description of three different versions of a **fast explicit operator splitting method**:

- *Version I*: **FV-GF method**, which is based on the finite-volume (FV) hyperbolic solver and the exact parabolic solver implemented by discretizing the convolution with the Green function formula for the exact solution of the heat equation;
- *Version II*: **MC-GF method**, in which the hyperbolic solver is the method of characteristics (MC) while the parabolic solver is the same as in the FV-GF method;
- *Version III*: **FD-PS method**, which is based on a high-order finite-difference (FD) scheme and the exact parabolic solver implemented in the pseudo-spectral manner.

We note that the FV-GF and the MC-GF methods have been proposed in [5, 4] and [2], respectively.

Depending on the convection-diffusion model at hand, one of the above versions may be particularly advantageous. When the system (1) with a nonlinear flux function $\mathbf{f}$ is considered, the FV-GF method seems to be a natural choice since its hyperbolic solver is designed to treat (systems of) hyperbolic conservation laws. Our particular choice of the FV method is the second-order Godunov-type central-upwind schemes [6, 7, 8]. We have successfully applied the FV-GF method to the Burgers equation and to a polymer system modeling flooding processes in enhanced oil recovery.

The MC-GF method, on the other hand, seems to be optimal when the hyperbolic problem is linear and thus can be easily solved by the method of characteristics, which is diffusion-free. The latter guarantees that the only diffusion present in the splitting method is the physical one because it comes from the parabolic part. The MC-GF method has been applied to a linear convection-diffusion equation as well as to a model describing the propagation of a passive pollutant in shallow water.

Finally, the FD-PS method seems to be preferable in the case of a viscous HJ equation with periodic boundary conditions, in which one may take advantage of the FFT algorithm to significantly speed up the implementation of the exact parabolic solver. We have applied the FD-PS method to the vorticity formulation of the two-dimensional incompressible Navier-Stokes equations written in the transport form, which thus can be viewed as a HJ equation with a global Hamiltonian. Our particular choice for the hyperbolic solver is a fourth-order FD scheme based on the central-upwind numerical Hamiltonian from [7] and the fifth-order Weighted Power-ENO reconstruction [1, 9].

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Kinetic Theory and Gas Dynamics

Tai-Ping Liu

There are two general approaches to the study of Boltzmann equation in the kinetic theory. There is the more probabilistic approach, which is generally practiced by those coming from the field of statistical mechanics. Such an approach relies primarily on the equilibrating mechanism of the collision operator as exemplified by the Boltzmann H-Theorem. The approach yields strong results for space homogeneous solutions. The other approach is practiced by those coming from the fluid dynamics. The second approach has the advantage of being able to study the dynamics around the equilibrium manifolds of local Maxwellians. The around the equilibrium manifold the fluid nonlinearity plays a basic role for the Boltzmann solutions. With Shih-Hsien Yu, we have been working on the quantitative aspect of the Boltzmann equation using the Green's function approach. Such a quantitative study allows us to consider the Boltzmann boundary layers through our recent works on invariant manifolds for stationary Boltzmann equation. We show that when one of the Euler characteristics for the flow around the boundary is close to zero, there are boundary layers containing both the Knudsen layer and gas dynamics layers such as the compression, expansion, and thermal layers. There are direct correspondences between our results and those studied by Yoshio Sone and others in the Kyoto School of physicists and engineers, particularly on the bifurcation phenomena of transition between subsonic and supersonic evaporation and condensation. We believe that the Green's function approach is useful for the study of other subtle relation between kinetic theory and gas dynamics.

On the entropy stability of some finite volume schemes

Mária Lukáčová-Medvid'ová
(joint work with Eitan Tadmor)

It is a well-known fact that numerical schemes based on a linearization strategy can produce solutions violating the entropy condition. In particular, such an phenomenon is profound on sonic rarefaction. In [2] we have studied the **entropy stability** of a class of finite volume methods for systems of hyperbolic conservation laws. The methods under consideration are based on a Roe-type linearization coupled with the multidimensional FV evolution Galerkin (FVEG) method [1].

In order to prove entropy-stability of these FV schemes we follow the lines of Tadmor [3], see also [4] for a related work. Applying the local analysis of entropy-stable schemes and comparing the numerical viscosity of our FV schemes with the

entropy-conservative schemes we derived in [2] an entropy stable variant of the FVEG scheme as well as the Roe-type scheme.

Let $\{\boldsymbol{r}^j_{\nu+1/2}\}_{j=1}^N$, $\{\boldsymbol{\ell}^j_{\nu+1/2}\}_{j=1}^N$ be given orthonormal systems, such that $\langle \boldsymbol{r}^j_{\nu+1/2}, \boldsymbol{\ell}^k_{\nu+1/2}\rangle = \delta_{jk}$ for all $j,k = 1,\dots,N$. Construct the Riemann path $\{\boldsymbol{u}^j_{\nu+1/2}\}_{j=1}^N$ connecting the left state $\boldsymbol{u}_\nu$ with the right state $\boldsymbol{u}_{\nu+1}$,

$$
\begin{aligned}
\boldsymbol{u}^1_{\nu+1/2} &= \boldsymbol{u}_\nu, \\
\boldsymbol{u}^{j+1}_{\nu+1/2} &= \boldsymbol{u}^j_{\nu+1/2} + \alpha^j_{\nu+1/2}\boldsymbol{r}^j_{\nu+1/2}, \quad j = 1,\dots,N
\end{aligned} \tag{1}
$$

with $\alpha^j_{\nu+1/2} := \langle \boldsymbol{\ell}^j_{\nu+1/2}, \Delta\boldsymbol{u}_{\nu+1/2}\rangle$ and $\Delta\boldsymbol{u}_{\nu+1/2} = \boldsymbol{u}_{\nu+1} - \boldsymbol{u}_\nu$.

Let us assume now that the path (1) consists solely of shocks. Thus the Rankine-Hugoniot shock condition is satisfied:

$$
\boldsymbol{f}(\boldsymbol{u}^{j+1}_{\nu+1/2}) - \boldsymbol{f}(\boldsymbol{u}^j_{\nu+1/2}) = \overline{s}^j_{\nu+1/2}(\boldsymbol{u}^{j+1}_{\nu+1/2} - \boldsymbol{u}^j_{\nu+1/2}),
$$

where $\overline{s}^j_{\nu+1/2}$ denotes the speed of discontinuity.

For one-dimensional systems of hyperbolic conservation laws we can show that the following numerical flux yields an entropy-stable FVEG scheme, see [2],

$$
\boldsymbol{H}_{\nu+1/2}(\boldsymbol{u}_\nu, \boldsymbol{u}_{\nu+1}) := \boldsymbol{f}(\boldsymbol{u}^*_{\nu+1/2}) - J_{\nu+1/2}, \tag{2}
$$

where $\boldsymbol{u}^*_{\nu+1/2} := \boldsymbol{u}_\nu + \sum_{j=1;\overline{s}^j\le 0}^N \alpha^j_{\nu+1/2}\boldsymbol{r}^j_{\nu+1/2}$. The jump term controlling the entropy production reads

$$
J_{\nu+1/2} := \frac{\kappa}{2}\sum_{j=1}^N \left[\lambda^j_{\nu+1/2}\right]^+_{j+1/2} \alpha^j_{\nu+1/2}\boldsymbol{r}^j_{\nu+1/2}, \qquad \kappa \ge 1/4,
$$

where $\left[\lambda^j_{\nu+1/2}\right]^+_{j+1/2} := \max\left(\lambda^j\left(\boldsymbol{A}(\boldsymbol{u}^{j+1}_{\nu+1/2})\right) - \lambda^j\left(\boldsymbol{A}(\boldsymbol{u}^j_{\nu+1/2})\right), 0\right)$.

For the Roe-type scheme an entropy stable variant has the following form

$$
\boldsymbol{H}_{\nu+1/2}(\boldsymbol{u}_\nu, \boldsymbol{u}_{\nu+1}) := \frac{1}{2}\left(\boldsymbol{f}(\boldsymbol{u}_\nu) + \boldsymbol{f}(\boldsymbol{u}_{\nu+1})\right) - \frac{1}{2}\sum_{j=1}^N |\overline{s}^j_{\nu+1/2}|\alpha^j_{\nu+1/2}\boldsymbol{r}^j_{\nu+1/2} - J_{\nu+1/2}. \tag{3}
$$

More precisely, we are able to show that the numerical viscosity of FVEG scheme (2) as well as of the Roe-type scheme (3) is larger in leading order terms than the numerical viscosity of entropy stable scheme. Our numerical experiments for the Euler equations of gas dynamics as well as for the shallow water equations indeed demonstrate the entropy stability of modified FV schemes and show, for example, a correct resolution of sonic rarefaction wave, see [2] for details.

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Mesh redistribution and error control for Navier-Stokes solvers

Charalambos Makridakis
(joint work with Eberhard Bänsch, Fotini Karakatsani)

We consider certain issues related to the combination mesh redistribution algorithms with known efficient solvers for the nonstationary-incompressible Navier-Stokes equations,

$$\begin{aligned} \mathbf{u}_t - \nu\,\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\times[0,T], \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega\times[0,T], \end{aligned} \tag{1}$$

with Dirichlet boundary conditions, on a bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$ with a sufficiently smooth boundary. For the space discretization one can consider *finite elements* or *finite volumes.* Time-discretization schemes that are popular for Navier-Stokes are e.g., the Crank-Nicolson method, the $\theta-$fractional step method of Glowinski, and the class of projection methods introduced by Chorin and Temam and further developed by several others. It is well known that Computational/Numerical Analysis Challenges for Navier - Stokes are mainly due to the incompressibility condition and to the highly not trivial physical effects that are present at high Raynolds numbers (i.e. for very small viscosity coefficients ν). The effect of the incompressibility condition and its involved numerical approximation is present in the saddle point formulation and the corresponding *inf-sup* conditions related to the discrete velocity and pressure spaces, the difficulties associated to the solution of the resulting linear systems etc. In certain cases, the need of decoupling velocity and pressure in the linear systems leads to non-standard time discretizations, such as the methods mentioned above.

Self adjusted meshes have important benefits approximating PDEs with solutions that exhibit nontrivial characteristics. When appropriately chosen, they lead to efficient, accurate and robust algorithms. Error control is also important, since appropriate analysis can provide guarantees on how accurate the approximate solution is through *a posteriori* estimates. Error control may lead to appropriate adaptive algorithms by identifying areas of large errors and adjusting the mesh accordingly. Error control and associated adaptive algorithms for Navier-Stokes is an open problem. Main obstacles are due at one hand to open *methodological* issues and on the other hand to the lack of techniques applicable to popular *efficient schemes.*

In this talk we highlight the main structure of an algorithm which permits mesh redistribution with time and the nontrivial characteristics associated with it. The simplest such algorithm has roughly the form: Given the approximation u_n at the time step n, which belongs to a finite dimensional space V_n (reflecting the spatial discretization method)

1a : choose the next space V_{n+1},
1b : project u_n to the new space V_{n+1} to get $\tilde{u}_n$,
1c : use $\tilde{u}_n$ as starting value to perform the evolution step in V_{n+1} resulting the new approximation u_{n+1} .

Standard schemes involve only step (1c) (uniform or nonuniform mesh). The presence of (1a) and (1b) are in most of the cases neglected in the analysis. It should be noted though that, on one hand, such algorithms can accumulate the nodes of the computational mesh in the areas of interest, as expected, and on the other hand (1a) and (1b) have fundamental influence on the qualitative behavior of the schemes. For schemes related to the proper approximation of (1) we mention three interesting such cases: 1). *Refinement can spoil Crank-Nicolson schemes.* Indeed, in [3] we present examples where recursive refinement can spoil standard Crank-Nicolson schemes. This is reflected in the a posteriori estimate by the presence of a term of the type $k_n\|(A_h^{n+1} - A_h^n)u_n\|$ which might grow without control (A_h^n denotes the discrete elliptic operator corresponding to the space V_n and k_n the local time step.) A version of the Crank-Nicolson scheme consistent with mesh redistribution is introduced and analyzed in [3]. Similar effects appear in $\theta-$fractional schemes. 2). *Severe pressure pollution in Navier Stokes solvers.* Mesh redistribution can pollute in a severe way the pressure approximation. In [4] examples based on van Karman vortex shedding highlighting this effect are presented. On the other hand, this problem is addressed, [4], by appropriate modifications which lead to consistent with mesh redistribution algorithms for (1). 3). *Geometric mesh redistribution can stabilize unstable schemes.* For very small viscosity ν one should use schemes with *artificial numerical diffusion* (upwinding). The right selection of such schemes is a nontrivial task. Recent results obtained by our group, see e.g., [1, 2] and their references, show that when steps (1a) and (1b) are based on geometric information on u_n they effectively stabilize schemes even without additional terms reflecting artificial diffusion or upwinding.

Towards error control the approach summarized in [4] is presented. This is based on the appropriate definition of an auxiliary function $\hat{U}$ which we call *Reconstruction* of the approximation U. Then the error estimate relies on the separate control of $u - \hat{U}$ and $\hat{U} - U$. A key ingredient of this approach is the fact that $\hat{U}$ should satisfy the same PDE with the exact solution, but perturbed with an a posteriori term which we would like to have in the final estimate. Once this goal is achieved the final result relies on PDE estimates. Regarding Navier-Stokes solvers the error control of space discrete approximations (by finite elements or finite volumes) was based in [5] in the introduction of *Stokes Reconstruction.* This

is the weak solution of a stationary Stokes problem for each fixed $t \in [0,T]$,

$$-\Delta \hat{U} + \nabla \hat{P} = g_h(t),$$
$$\operatorname{div} \hat{U} = 0,$$

where $g_h := -\tilde{\Delta}_h u_h - \mathbf{f}_h + \mathbf{f}$. Here u_h denotes the space-discrete approximation to the time-dependent problem, and $\tilde{\Delta}_h$ denotes the projection of the discrete Laplacian into the space of discrete divergence free functions. A technical obstacle for the error control of space discrete approximations of Navier-Stokes solvers that this definition addresses is the fact that the discrete approximations are almost never divergence free; thus comparing with u_h directly would lead to error equations with nonzero divergence. As another example of application of the above framework we presented a posteriori error estimates for projection time-stepping schemes. PDE estimates and appropriate error equations through a *time*-reconstruction can lead to a posteriori error control of *rotational pressure correction* time-stepping schemes. Such schemes can be formulated in a compact form, [6],

$$\frac{1}{k}(\mathbf{u}^{n+1} - \mathbf{u}^n) - \nu P_{\mathrm{J}} \Delta \mathbf{u}^n = \nu \Delta(\mathbf{u}^{n+1} - \mathbf{u}^{n+1}) + \nu \nabla \operatorname{div} \mathbf{u}^n + P_{\mathrm{J}} \mathbf{f}(t^{n+1}).$$

The a posteriori bounds follow by using this formulation, an appropriate error equation which the time-reconstruction satisfies, an important commutator estimate for $P_{\mathrm{J}}\,\Delta - \Delta\, P_{\mathrm{J}}$ proved in [6], and energy estimates. The case of error control of fully discrete Navier-Stokes solvers combined with mesh redistribution with n is of course more technical and involved. Such estimates are derived in [4] by combining ideas presented in this talk.

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A Numerical Diffusion Flux Based on the Generalized Diffusive Riemannproblem

Claus-Dieter Munz
(joint work with Gregor Gassner, Frieder Lörcher)

Overview. We propose numerical approximations for diffusion fluxes for both, finite volume and discontinuous Galerkin schemes. These methods share the property that the approximate solutions belong to the space of piecewise polynomials, which may be discontinuous across grid cell interfaces, even for the approximation of diffusion terms containing second order derivatives. To take this into account a framework based on Riemann problems for diffusion equations is introduced to define suitable numerical diffusion fluxes at grid cell interfaces. For convection diffusion equations this gives the possibility to approximate both convection and diffusion in the same spirit. We start from the one-dimensional scalar case and extend this exact Riemann solutions to an approximative Riemann solver for multi-dimensional diffusion systems. The proposed fluxes are validated within the discontinuous Galerkin framework for several test cases including the compressible Navier-Stokes equations.

Introduction. In finite volume (FV) or discontinuous Galerkin (DG) schemes the approximate solution may jump at the grid cell interface. Any physical phenomena which can not be resolved on the given grid will result in such a jump. If the time evolution of these jumps can be approximated in a stable and consistent way, then the numerical scheme does not generate spurious oscillations and gives meaningful mean values of the under-resolved phenomena. In his pioneering work Godunov proposed to approximate the convection flux between grid cells by solving the break down of the jumps into different waves.

We propose to follow Godunov's way also for diffusion and to base the flux on the local diffusion of a discontinuity. We look at the exact solution of the so called diffusive generalized Riemann problem. We start with a scalar linear diffusion equation and extent the results to systems of diffusion equations. A linearization leads to the general nonlinear case. For finite volume schemes this diffusion flux is based on the same reconstruction as the convection flux and establishes a self-consistent treatment of convection and diffusion.

The Generalized Riemann Problem for Diffusion. Motivated by the success of Riemann problem based schemes for hyperbolic problems, we will make use of ideas within this Godunov type methodology to construct accurate and robust discretizations for parabolic problems.

0.1. **Scalar Diffusion.** For a scalar diffusion diffusion equation

$$u_t = (\mu(x)u_x)_x =: f^d(u,u_x)_x \quad \text{with} \quad \mu(x) = \begin{cases} \mu^+, & \text{for } x > 0, \\ \mu^-, & \text{for } x < 0, \end{cases} \tag{1}$$

where $\mu(x)$ denotes the positive diffusion coefficient the pure initial value problem with piecewise linear data

$$u(x,0) = \begin{cases} u^+ + x\, u_x^+, & \text{for } x > 0, \\ u^- + x\, u_x^-, & \text{for } x < 0 \end{cases} \tag{2}$$

is considered. We call this initial value problem the diffusive *generalized* Riemann problem (dGRP). We note that it is important to allow the diffusion coefficient to be discontinuous at $x = 0$, because this is the basis to extend the consideration for the linear case to nonlinear diffusion problems. In this case the diffusion coefficients will be discontinuous, if the data are discontinuous.

The exact solution of this problem can be obtained, e.g. by Laplace transformation and solving the linear second order ordinary differential equations. This can be done separately for the right and left part of the x-axis. The right and left solutions are then put together by imposing the boundedness of the solution and the continuity of the solution together with the heat flux. The numerical diffusion flux is then defined to be the physical flux at the point of the initial jump of the data $x = 0$ averaged over the time step.

The dGRP diffusion flux can be extended to linear systems by a diagonalization of the diffusion matrix. The scalar flux is then applied to every equation of the uncoupled system. The transformation back then gives the desired diffusion flux for the system. For nonlinear systems the solution of the dGRP (1),(2) are takes as the state for linearization. The diffusion coefficients in (1) are defined to be the limits of the nonlinear diffusion function from the left and from the right to the initial jump. In such a way the diffusion flux is defined for the compressible Navier-Stokes equations. More details can be found in [1] and [2].

A Discontinuous Galerkin Scheme Based on a Space-Time Expansion. The class of DG schemes is quite interesting for practical calculations in complex geometries, because the DG schemes reproduce the order of accuracy even on distorted unstructured grids. The approximate solution in the discontinuous Galerkin schemes may be discontinuous at grid cell interfaces which allows the approximation of strong gradients on coarse grids or even discontinuities. Usually, high order time discretization of DG schemes for unsteady computations is done separately from space discretization by the method of lines approach using a Runge Kutta scheme (RK-DG).

An actual topic of research are explicit discontinuous Galerkin (DG) schemes which provide the time approximation in one single step. Using ideas of the so called ADER finite volume approach (Arbitrary order using DERivatives) we propose the space-time expansion discontinuous Galerkin scheme which are based on a space-time Taylor expansion about the barycenter $\vec{x}_i$ of a grid cell Q_i and about the old time level t_n:

$$U(\vec{x},t) = U(\vec{x}_i,t_n) + \sum_{j=1}^{N_i} \frac{1}{j!}((t-t_n)\frac{\partial}{\partial t} + (\vec{x}-\vec{x}_i)\cdot\vec{\nabla})^j\, U(\vec{x}_i,t_n). \tag{3}$$

With this space time Taylor series it is possible to approximate U at all space time points $(\vec{x}, t) \in Q_i \times [t_n; t_{n+1}]$, which is needed for the evaluation of the volume and surface integrals. While the space derivatives are known from the approximate solution at time t_n, the time and mixed space-time derivatives are the problem. These are replaced by pure space derivatives using the differential equation several times. This is called the Cauchy-Kovalevskaya or Lax-Wendroff procedure, see, e.g. [5] for more details. This expansion gives the values for the Gaussian quadrature in the case of nonlinear problems by which the surface and volume space-time integrals in the variational formulation are approximated. In the linear case the polynomials may be integrated analytically resulting in quadrature-free DG schemes.

The so-called space-time expansion discontinuous Galerkin scheme (STE-DG) scheme may be locally adapted to the behavior of the solution. This includes h-adaptivity, which is especially efficient due to the possibility of using nonconforming mesh nodes, and p-adaptivity allowing the degree of the local approximation space to vary from grid cell to grid cell. The STE-DG scheme is also locally adapted with respect to the time approximation, see [3], [4]. Each grid cell may evolve in time with a local time step corresponding to the local stability restrictions. This technique strongly increases the performance for multi-scale problems or distorted grids where the local time steps strongly vary in the computational domain. The local time stepping guarantees efficiency of the simulation on the entire domain. Large grid cells in the far field allow larger time steps there.

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A systematic approach to high resolution well-balancing

Sebastian Noelle

(joint work with Chi-Wang Shu, Yulong Xing)

Over the last decade, the development of high resolution *well-balanced* schemes was a central topic in the numerical analysis of hyperbolic systems. Indeed, many applications in continuum mechanics lead to systems of balance laws. For such

flows, the source terms are often in near-perfect equilibrium with the convective forces. A numerical scheme which does not respect these equilibria at the discrete level may produce spurious oscillations, and hence convergence may slow down.
Most numerical approximations are only well-balanced for certain subclasses of equilibria. Often, these stationary states are characterized by constant momentum and energy, together with further simplifying assumptions. The more complex the equilibria are, the more involved well-balancing becomes. For example, compared with balancing still water (Audusse, Bouchut, Bristeau, Klein, Perthame 2004), the moving water case solved by Noelle, Xing and Shu [4] was a tremendous technical challenge. More and more, these publications are only accessible to a small circle of experts.

Unifying framework

Therefore, we see a strong need for a unifying and at the same time simplifying framework within which existing schemes may be rederived and reinterpreted, and new ones may be developed more easily. Here we will present such a framework for finite volume and discontinuous Galerkin schemes.
When deriving a well-balanced scheme, we first select a class of stationary states which the scheme should preserve. Examples are the lake at rest, river flows, jets in a rotational frame, or multi-layer shallow water flows. For each component of the algorithm

- reconstruction
- quadrature
- flux-source-computation for the singular layer at the cell interface

we define appropriate notions of well-balancing, always tailored to the class of stationary states under consideration. Once these three balancing-properties are fulfilled, well-balancing of the overall scheme follows immediately. As an example, we demonstrate that the scheme based on hydrostatic reconstruction of Audusse et al., the scheme based on a more general equilibrium reconstruction by Noelle, Xing and Shu [4] and the schemes based on a general hydrostatic reconstruction by Castro, Pares et al (2007) fall into this framework.

Applications, non-standard schemes, and open questions

Instabilities of the gulf stream. Together with Pankratz, Natvig and Gjevik [2] we extended our high-order-accurate well-balanced scheme [1] to the atlantic shelf off the Norwegian coast, and verified a theory of Gjevik on the onset of horizontal vortex-instabilities for the gulf stream. We derived a new non-reflecting inflow-boundary condition. Non-reflecting outflow boundary conditions are still an issue. The main computational issue here is whether fourth- and higher-order codes are more efficient that second order central differences (when there are no discontinuities).

Bi-characteristic FVEG (Finite Volume Evolution Galerkin) schemes. In [3] we constructed a well-balanced scheme on bicharacteristic theory. In a current joint project with M. Lukáčová-Medvidová, A. Bollermann and A. Zauskova we are studying low Froude number flows.

Non-Uniquenes. Beyond the technical difficulties contained in some of these steps (which are now clearly recognized and separated from each other), there is a major issue concerning hyperbolic balance laws which is not yet resolved, neither analytically nor numerically. This concerns the non-uniqueness of the solution to the Riemann problem when both the conservative variables and the source term exhibit discontinuities. From another point of view, the choice of paths defining non-conservative products of measures in state space is not unique, and different paths may define different weak solutions (Dal Maso, Lefloch, Murat 1995; Castro, Chacón, Fernández-Nieto, Parés 2007). For the shallow water equations, two different choices, each with a sound physical justification, have recently been proposed. They both concern the choice of the auxiliary, intermediate height at an interface. Audusse et al. choose it to be the maximum of the neighboring heights, which preserves positivity at the shore. In [4] we show that for waterfalls, which occur frequently in rivers, the natural height is the minimal one. Preliminary studies of entropy production (with J.A. Lopez-Garcia and T. Morales) did not reveal a general principle how to choose the intermediate height.

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Reduced Basis Methods for Non-Linear Conservation Laws

Mario Ohlberger

(joint work with Bernard Haasdonk)

Reduced basis methods [9] are increasingly popular methods for complexity reduction in problems, where parameterized partial differential equations are to be solved repeatedly for varying parameters. This means that high-dimensional finite element or finite volume approximations $u_H(\boldsymbol{\mu}) \in \mathcal{W}_H$ are to be computed for varying parameter vectors $\boldsymbol{\mu} \in \mathcal{P}$ from some polygonal parameter domain. Examples for such applications are design, control, optimization, inverse modeling based on PDEs, etc. Instead of repeated computation of these expensive detailed

simulations, a problem-specific low dimensional subspace $\mathcal{W}_N \subset \mathcal{W}_H$ is chosen in a preprocessing step, which captures the solution variety under parameter changes. Based on this *reduced basis space* $\mathcal{W}_N$, a reduced model is devised, which inexpensively computes $u_N(\boldsymbol{\mu}) \in \mathcal{W}_N$ as approximation of the unknown $u_H(\boldsymbol{\mu})$ for any new parameter vector.

In this contribution we focus on a reduced basis method for parameterized non-linear conservation laws, discretized by finite volume schemes. Such conservation laws can be formulated as

$$\partial_t u(\boldsymbol{\mu}) + L_{\boldsymbol{\mu}}[u(\boldsymbol{\mu})] = 0 \qquad \text{in } \Omega \times [0,T],$$

where $\Omega \subset \mathbb{R}^d$ is a spatial domain, $\boldsymbol{\mu} \in \mathcal{P} \subset \mathbb{R}^p$ a parameter vector and $L_{\boldsymbol{\mu}}[u(\boldsymbol{\mu})]$ denotes a parameter dependent space operator in divergence form, i.e

$$L_{\boldsymbol{\mu}}[v] := \nabla \cdot f(\boldsymbol{\mu}; v)$$

with a non-linear flux function $f : \mathcal{P} \times \mathbb{R} \to \mathbb{R}^d$.

The foundation for the reduced basis method consists of a monotone finite volume discretization of the conservation law, which can be expressed in abstract form as

$$u_H^{k+1}(\boldsymbol{\mu}) = u_H^k(\boldsymbol{\mu}) - \Delta t L_H^k(\boldsymbol{\mu})[u_H^k(\boldsymbol{\mu})]$$

where $u_H^k(\boldsymbol{\mu}) \in \mathcal{W}_H$ is an approximation of $u(\boldsymbol{\mu})$ at time t^k. The approximation space $\mathcal{W}_H$ is the space of piecewise constant functions on a given triangulation of Ω with dimension $H := \dim(\mathcal{W}_H)$. The principal idea of the reduced basis method for such discrete evolution equations is to construct a reduced basis space $\mathcal{W}_N \subset \mathcal{W}_H$ with low dimension $N := \dim(\mathcal{W}_N) \ll H$ and to approximate $u_H^k(\boldsymbol{\mu}) \in \mathcal{W}_H$ further by a reduced basis solution $u_N^k(\boldsymbol{\mu}) \in \mathcal{W}_N$ defined through a Galerkin projection of the finite volume scheme into the space $\mathcal{W}_N$. i.e.

$$\int_\Omega u_N^{k+1}(\boldsymbol{\mu})\varphi = \int_\Omega \left(u_N^k(\boldsymbol{\mu}) - \Delta t L_H^k(\boldsymbol{\mu})[u_N^k(\boldsymbol{\mu})]\right)\varphi, \quad \forall \varphi \in W_N.$$

For details concerning this method in the case of linear problems we refer to [7]. Having this general approach in mind, there are at least two open problems. First, we have to construct a suitable subspace $\mathcal{W}_N$, and second, we have to ensure that our method is efficient and gives good approximations.

The efficiency of the method is usually obtained by a so called offline-online decomposition, such that all computations depending on the complexity H can be done in an offline phase, while in the online-phase only problems with complexity polynomial in N have to be solved. While online-offline decomposition is easy to achieve for linear equations with affine parameter dependence (see [9]), it is not at all straight forward for non-linear problems (see [2]). In [8] we therefore introduced the concept of empirical interpolation for localized operators, which we apply here

in order to approximate the non-linear operator $L_H^k(\boldsymbol{\mu})$ by its empirical interpolation $\mathcal{I}_M[L_H^k(\boldsymbol{\mu})]$. This again results in an affine parameter dependence. Hence, offline-online decomposition can be done as in the linear, affine case. For the empirical interpolation it is necessary to construct a second approximation space which is called collateral reduced basis space $\mathcal{W}_M$ with low dimension M. This space is defined as a span of discrete functions given by the operator $L_H^k(\boldsymbol{\mu})$ applied to a snapshot $u_H^k(\boldsymbol{\mu})$ for thoroughly chosen parameters $\boldsymbol{\mu}$ and time instances t^k. For details in the context of finite volume approximation for conservation laws we refer to [6], where also numerical results are discussed.

Finally, it remains to discuss the efficient construction of the reduced basis space $\mathcal{W}_N$. This should be done in such a way that we can guarantee a certain approximation quality of the resulting method. The key to this issue are *a posteriori* error estimates for the error between the finite volume approximation u_H and the reduced basis approximation u_N. Having such an error estimate at hand, we may define an algorithm for adaptive basis enrichment in order to construct $\mathcal{W}_N$ with a prescribed error tolerance. For details we refer to [4, 5]. In recent work we also advised a concept of adaptively choosing the reduced basis space in time using *a posteriori* error estimates. The concept and preliminary results are given in [3].

In our ongoing work on reduced basis methods we are currently also working on an application of the technology to evolution equations on parameterized geometries. Such problems for instance appear in shape optimization [10]. By mapping the physical geometry to a fixed reference geometry, the partial differential equations are transformed to quite general non-linear evolution equations that can be again treated with the concept of empirical interpolation for localized operators. We refer to [1] for first results for such applications.

So far, the concept of empirical interpolation for localized operators is restricted to explicit evolution schemes. In a new project we will also start to look at a generalization of this technique to implicit and semi-implicit discretizations.

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2D Viscous Boussinesq Equations on a bounded domain

Ronghua Pan
(joint work with M. Lai and K. Zhao)

This talk reports my recent joint work with M. Lai and K. Zhao on the initial boundary value problem for 2D Boussinesq equations on a bounded domain. More precisely, we consider the following problem

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \rho \mathbf{e}_2 \\ \rho_t + U \cdot \nabla \rho = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{1}$$

where $U = (u, v)$ is the velocity vector field, P is the scalar pressure, ρ is the scalar density, the constant $\nu > 0$ models viscous dissipation, and $\mathbf{e}_2 = (0, 1)^{\mathrm{T}}$. Here we consider (1) in a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$. The system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U, \rho)(\mathbf{x}, 0) = (U_0, \rho_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ U|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

The Boussinesq system is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as other astrophysical situations where rotation and stratification play a dominant role (see e.g. [4] and [5]). In fluid mechanics, system (1) is used in the field of buoyancy-driven flow. It describes the motion

of incompressible inhomogeneous viscous fluid under the influence of gravitational force (c.f. [3]). In addition to its own physical background, the Boussinesq system was known by its close connection to the fundamental models, such as Euler and Navier-Stokes equations, for 3D incompressible flows by sharing the vortex stretching effect.

Recent progress is made for the Cauchy problem by [2] and [1] for the global regularity. Our purpose is to prove the existence of unique global smooth solution if the data is smooth. We propose the following compatibility conditions

$$\begin{cases} \nabla \cdot U_0 = 0, \quad U_0|_{\partial\Omega} = 0, \\ \nu\Delta U_0 + \rho_0 \mathbf{e}_2 - \nabla P_0 = 0, \quad \mathbf{x} \in \partial\Omega,\ t = 0, \end{cases} \tag{3}$$

where $P_0(\mathbf{x}) = P(\mathbf{x}, 0)$ is the solution to the Neumann boundary problem

$$\begin{cases} \Delta P_0 = \nabla \cdot [\rho_0 \mathbf{e}_2 - U_0 \cdot \nabla U_0], \quad \mathbf{x} \in \Omega, \\ \nabla P_0 \cdot \mathbf{n}|_{\partial\Omega} = [\nu\Delta U_0 + \rho_0 \mathbf{e}_2] \cdot \mathbf{n}|_{\partial\Omega}, \end{cases} \tag{4}$$

with $\mathbf{n}$ the unit outward normal to $\partial\Omega$.

Our main results are stated in the following theorem.

Theorem 4. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary. If $(\rho_0(\mathbf{x}), U_0(\mathbf{x})) \in H^3(\Omega)$ satisfies the compatibility conditions (4)–(5), then there exists a unique solution (ρ, U) of (1)–(2) globally in time such that $\rho(\mathbf{x}, t) \in C([0, T); H^3(\Omega))$ and $U(\mathbf{x}, t) \in C([0, T); H^3(\Omega)) \cap L^2([0, T]; H^4(\Omega))$ for any $T > 0$. Moreover, there exists a constant $\bar{C} > 0$ independent of t such that*

$$\|U(\cdot, t)\|_{L^2}^2 \le \max\left\{ \|U(\cdot, 0)\|_{L^2}^2 \ , \ \frac{\bar{C}^2}{\nu^2} \|\rho(\cdot, 0)\|_{L^2}^2 \right\}, \quad \forall\ t \ge 0. \tag{5}$$

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Numerical schemes for the BGK kinetic model

Gabriella Puppo
(joint work with Sandra Pieraccini)

Kinetic models describe fluid flows at rarefied regimes. These flows include gas flows in the upper layers of the atmosphere, but also microflows in MEMS, where the dimensions of the device are of the same order as the mean free path of a gas molecule. The main tool of kinetic theory is Boltzmann equation, which describes the evolution of the flow, taking into account the impact of the distribution of the microscopic velocities of fluid particles. We are interested in the computational issues arising from kinetic models. The reference scheme for kinetic flows is the DSMC (Direct Simulation Monte Carlo) method [3], which however becomes extremely costly close to hydrodynamic regimes. This is one reason to motivate the study of simplified models for Boltzmann equation. In this work, we concentrate on the BGK model, [9].

This model has several characteristics which are of interest from a computational point of view. It is a microscopic model, which takes into account the microscopic effects of a gas not in equilibrium. Thus it can be applied also in kinetic regimes, and approximating macroscopic models, such as Grad moment method or Burnett equations can be derived from BGK [10]. It has a strong theoretical background, see for instance [6] where the existence of solutions for the BGK model is proved. Moreover, it is known that the BGK model is endowed with an H-theorem describing the approach towards equilibrium for an isolated system, [9].

Several extensions have been proposed, enabling the application of the BGK model to different regimes. Among these we consider the ES-BGK model [2], for flows approaching the Navier Stokes regimes, BGK models for mixtures [1] and for gas mixtures undergoing chemical reactions [5]. The widening of applications of the BGK model has prompted a parallel development of numerical schemes designed to integrate the BGK equation. In the following, we will outline the main difficulties arising in the discretization of the BGK equation, and describe the ideas we have developed to overcome them.

The BGK model describes the evolution of the probability density $f(x, v, t)$ of finding a gas molecule in the volume element dV centered at the point (x, v) in phase space at time t:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\tau}(f_M - f) \qquad t \geq 0, \quad x \in R^3, v \in R^3 \tag{1}$$

Here f and f_M are functions of (x, v, t), i.e. space, microscopic velocity and time, respectively; τ is the relaxation time, which is a function of (x, t), and close to the hydrodynamic regime is $\tau << 1$. The function f_M is the Maxwellian obtained from the moments of f, namely:

$$f_M(x, v, t) = \frac{\rho(x, t)}{(2\pi RT(x, t))^{3/2}} \exp\left(-\frac{\|v - u(x, t)\|^2}{2RT(x, t)}\right). \tag{2}$$

Here R is the constant of a perfect gas, and we are considering a monoatomic gas, with 3 translational degrees of freedom. The quantities ρ, u and T are respectively the macroscopic density, velocity and temperature of the gas, and they are obtained from the moments of f, which are defined as

$$\begin{pmatrix} \rho \\ m \\ E \end{pmatrix} = \left\langle f \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} \right\rangle \equiv \int_{R^3} f \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} dv. \tag{3}$$

Here m is momentum, so that the macroscopic velocity is simply $u = m/\rho$, while E is the total energy, and the temperature is obtained from the relation: $3\rho RT/2 = E - \frac{1}{2}\rho u^2$.

The macroscopic moments of f are conserved, in the sense that:

$$\partial_t < f > + \nabla_x \cdot < fv > \; = 0, \tag{4}$$

$$\partial_t < fv > + \nabla_x \cdot < v \otimes vf > \; = 0, \tag{5}$$

$$\partial_t < \frac{1}{2}\|v\|^2 f > + \nabla_x \cdot < \frac{1}{2}\|v\|^2 vf > \; = 0. \tag{6}$$

A numerical scheme for (1) should be able not only to yield an accurate solution to equation (1), but also to satisfy the conservation equations and the entropy principle in some discretized form. Moreover, the BGK equation is stiff when the relaxation time τ is small, and the convective part of the equation (left hand side of (1)) gives a restrictive CFL, if the quadrature in velocity space, needed to compute the macroscopic moments of f, involves large values of the microscopic velocity, which can be much larger than the local sound speed.

In [7] we have constructed a scheme which is explicit in the convective part and implicit in the stiff source term, exploiting the fact that the distribution f and the corresponding Maxwellian f_M have the same moments defined in (3). Thus, taking moments of (1), the source term disappears, and the updated values of the macroscopic moments do not depend on the values of f at the new time step, and can be computed starting from known values of f. Once the updated values of the moments are known, the Maxwellian at the new time level can be computed from (2). In this fashion, the stiff source term is implicit, but it remains linear in f. The resulting scheme is conservative, if the discrete Maxwellian is computed as in [4]. However, the scheme is explicit in the stiff fast velocity modes.

To overcome this difficulty, we are currently working on a new scheme, for which the Maxwellian at the new time step is computed integrating explicitly the macroscopic conservation equations, (4)-(6). This system of equations in fact has a CFL depending on macroscopic quantities, independently of the fast microscopic modes. From the updated moments, the new Maxwellian function can be computed. Finally, this quantity is substituted in the time discretized equation obtained from the BGK equation (1), which now becomes linear in the unknown values of f at the new time level. At this point, the equation can be integrated implicitly with respect to both the fast velocity modes and the source term, see [8], with no need to trigger the CFL to the fast microscopic velocities.

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Identification of an average temperature and a dynamical pressure in a multi-temperature mixture of fluids

Tommaso Ruggeri

We first present the different models of a mixture of compressible fluids and we discuss in the case of Euler fluids the local and global well-posedness of the relative Cauchy problem for smooth solutions. Then we present a classical approach of mixture of compressible fluids when each constituent has its own temperature. The introduction of an *average temperature* together with the entropy principle dictates the classical Fick law for diffusion and also new constitutive equations associated with the difference of temperatures between the components. The constitutive equations fit with results recently obtained through the *Maxwellian iteration* procedure in extended thermodynamics theory of multi-temperature mixtures. The differences of temperatures between the constituents imply the existence of a new *dynamical pressure* even if the fluids have a zero bulk viscosity. The non-equilibrium dynamical pressure can be measured and may be convenient in several physical situations as for example in cosmological circumstances where - as many authors assert - a dynamical pressure played a major role in the evolution of the early universe.

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The nature of viscous dissipation in conservation laws

Denis Serre

Let us consider a first-order system of conservation laws

$$\partial_t u + \mathrm{div} f(u) = 0, \qquad (x \in \mathbb{R}^d,\ u(x,t) \in \mathcal{U}), \tag{1}$$

where $\mathcal{U}$ is a convex, open subset of $\mathbb{R}^n$. We assume that (1) admits a *strongly* convex entropy η of flux Q. Strong convexity just means that $\mathrm{D}^2\eta(u)$ is positive definite for every state $u \in \mathcal{U}$.

We are interested in viscous extensions of (1):

$$\partial_t u + \mathrm{div} f(u) = \mathrm{div}(B(u)\nabla u) = \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u). \tag{2}$$

We say that (2) is *strongly entropy-dissipative* if it implies, at least for smooth solutions, the inequality

$$\partial_t \eta(u) + \mathrm{div} Q(u) + \omega \sum_\alpha \left| \sum_\beta B^{\alpha\beta}(u)\partial_\beta u \right|^2 \leq \mathrm{div}(\mathrm{d}\eta(u) B(u) \nabla u), \tag{3}$$

where $\omega = \omega(u)$ is strictly positive and continuous. This amounts to saying that

$$\begin{aligned} \sum_{\alpha,\beta} \mathrm{D}^2\eta(u)(X_\alpha, B^{\alpha\beta}(u)X_\beta) \quad &\geq \quad \omega \sum_\alpha \left| \sum_\beta B^{\alpha\beta}(u) X_\beta \right|^2 \\ &=: \omega |B(u)\mathbf{X}|^2, \qquad \forall u \in \mathcal{U},\ \forall X_1, \dots, X_d \in \mathbb{R}^n. \end{aligned} \tag{4}$$

This definition can be weakened by asking only that (here, we drop the word *strongly* and keep only *entropy-dissipative*)

$$\int_{\mathbb{R}^d} \sum_{\alpha,\beta} \mathrm{D}^2\eta(U)(\partial_\alpha U, B^{\alpha\beta}(U)\partial_\beta U)\, dx \geq \omega_0 \int_{\mathbb{R}^d} \sum_\alpha \left| \sum_\beta B^{\alpha\beta}(U)\partial_\beta U \right|^2 dx. \tag{5}$$

We notice that (5) implies the algebraic condition

$$\mathrm{D}^2\eta(u)(X, B(\xi;u)X) \geq \omega(u) \sum_\alpha |B^\alpha(\xi;u)X|^2, \qquad \forall u \in \mathcal{U},\ \forall \xi \in \mathbb{R}^d,\ \forall X \in \mathbb{R}^n, \tag{6}$$

where the symbol B is defined by

$$B^\alpha(\xi) := \sum_\beta \xi_\beta B^{\alpha\beta}, \qquad B(\xi) = \sum_\alpha \xi_\alpha B^\alpha(\xi) := \sum_{\alpha,\beta} \xi_\alpha \xi_\beta B^{\alpha\beta}, \qquad (\xi \in \mathbb{R}^d).$$

We prove the following result.

Theorem. Assume that (2) is entropy dissipative. Assume moreover that the dissipation tensor has the structure

$$B(\xi;u) = \begin{pmatrix} 0_{p\times n} \\ b(\xi;u) \end{pmatrix},$$

with b of full rank $n-p$ when $\xi \neq 0$.

Then the dissipative terms can be rewritten in the following way

$$B^{\alpha\beta}(u)\partial_\beta u = \sum_{i=p+1}^{n} Y^{\alpha\beta}(u)\partial_\beta z_i,$$

with

$$z_i := \frac{\partial \eta}{\partial u_i}.$$

At last, one has

$$Y(\xi,u) = \begin{pmatrix} 0_p \\ Z(\xi;u) \end{pmatrix},$$

where Z satisfies the Legendre–Hadamard condition

$$X^T Z(\xi;u)X \geq \omega_1 |X|^2|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d,\ \forall X \in \mathbb{R}^n.$$

If the viscous system is strongly entropy-dissipative, we have the stronger property that

$$\sum_{i,j}\sum_{\alpha\beta} F_{i\alpha}F_{j\beta}Z_{ij}^{\alpha\beta} \geq \omega_2 \|F\|^2$$

for every matrix $F \in \mathbf{M}_{(n-p)\times d}(\mathbb{R})$.

♣

This theorem is well illustrated by various models of fluids, but it not limited to this area. The assumption about the block structure of B is a natural one. For instance, in the Navier-Stokes equations, we have $p = 1$, and the derivatives, $z_2, \ldots$ can be rewritten in terms of those of the velocity and the temperature. The structure found in the theorem is one of the main assumptions of Kawashima in his fundamental study of the hyperbolic-parabolic Cauchy problem [2].

Details can be found in [3, 4]. The assumption of strong entropy dissipation is taken from [1].

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Mixtures

Konstantina Trivisa

Multicomponent reactive mixtures arise daily in science and engineering. Many practical applications such as modeling pollutant formation, chemical vapor deposition reactors, laminal flame extinction limits, astrophysical plasma and atmospheric modeling, for instance, require us to take into account complex chemistry mechanisms and detailed transport phenomena. The objective of this work is to develop a rigorous mathematical framework based on the principles of continuum physics and in particular to analyze the global in time existence, stability and asymptotic behavior of multicomponent reactive flows.

The state of such flows is, in general, characterized by the macroscopic variables: the total mass density $\varrho = \varrho(t,x)$, the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, and the species mass fractions $Y_k = Y_k(t,x)$, $k = 1, ..., N$, depending on the time $t \in (0,T)$ and the Eulerian spatial coordinate $x \in \Omega \subset R^3$.

The primitive conservation equations governing multicomponent flows, express the conservation of mass, momentum, energy, and conservation of species mass (cf. Feireisl, Petzeltová and Trivisa [6], Giovangigli [7, Chapter 2, Section 2.2]):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}, \tag{2}$$

$$\partial_t(\varrho E) + \mathrm{div}_x\Big((\varrho E + p)\mathbf{u}\Big) + \mathrm{div}_x\Big(\mathbf{q} - \mathbb{S}\mathbf{u} - \sum_{k=1}^{N} h_k \mathcal{F}_k\Big) = 0, \tag{3}$$

$$\partial_t(\varrho Y_k) + \mathrm{div}_x(\varrho Y_k \mathbf{u}) = \mathrm{div}_x(\mathcal{F}_k) + \varrho \omega_k, \ k = 1, ..., N. \tag{4}$$

Here, p denotes the pressure, $\mathbb{S}$ stands for the viscous stress tensor, E represents the total energy per unit mass, $\mathbf{q}$ is the heat flux, $\mathcal{F}_k$ denotes the diffusion flux, ω_k is the production rate of the k-th species, while h_k are the species formation enthalpies.

The objective of this research activity is the investigation and mathematical analysis of a wide spectrum of physical systems in compressible fluids, nonlinear materials sciences, combustion, multi–phase flows, astrophysics and other applied areas. The systems under investigation are governed by the Navier–Stokes equations in Eulerian coordinates often coupled with additional equations depending on the specific physical context.

The articles [3, 4] deal with multidimensional models which generalize the Navier–Stokes equations for compressible reacting fluids, in the sense that they involve the coupling of the hyperbolic-parabolic system (Navier-Stokes-Equations) with one chemical kinetics equation modeling one irreversible chemical reaction. The goal of this work is to analyze the dynamics of mixtures of compressible reacting fluids. In these articles, global existence results have been obtained by

extending the framework and techniques introduced by P.L. Lions [10] for the nonisentropic Navier-Stokes equations and by Feireisl [5] in the framework of *variational solutions*. More precisely, the articles [3], [4] focus on multidimensional models for the combustion of a viscous, compressible, radiative–reactive gas. The motion of the gas is governed by the Navier–Stokes equations, which represent the balance of mass, momentum and energy and the two–species chemical kinetics equation for higher–order kinetics. The *gas* is viewed here as a *continuum* occupying at a given time t a bounded domain $\Omega \subset \mathbf{R}^3$. This work establishes the global existence of weak solutions to an initial–boundary–value problem with large initial data.

The article [11] deals with general multicomponent models formulated by systems governed by the Navier Stokes equations in Eulerian coordinates coupled with the species concentration balance. The system of equations takes now a new form due to the choice of rather complex constitutive relations that can accommodate appropriately the physical context. The transport fluxes satisfy rather general constitutive laws, the viscosity and heat conductivity depend on the temperature, the pressure law is a nonlinear function of the temperature depending on the *species concentration* as well as the *molecular weights* of the individual species. In accordance the heat flux depends also on the density of the individual species and contains additional terms accounting for the *enthalpy*. The constitutive laws presented here are in agreement with the fundamental principles of continuum physics. The dependence of the *pressure* and the *heat flux* on the species concentration captures quite accurately the physical setting offering a better description of the dynamic behavior of fluid mixtures. This addition in the pressure law, due to the underlying physics, complicates the mathematical analysis since it affects both the constitutive relations and the equations of the system in a significant way. The main ingredients of the approach in this work can be formulated as follows:

- A suitable *variational formulation* of the underlying physical principles based on the second law of thermodynamics, in particular, replacing the energy balance by the corresponding equation for the *total entropy* of the system.
- Physically grounded structural hypotheses imposed on the thermal equation of state for the pressure p. In particular, the effect of radiation, significant in the high temperature regime, as well as the concentration of the individual components in the mixture are taken into account.
- *A priori* estimates based solely on boundedness of the initial energy and entropy of the system. As a matter of fact, this step requires the transport coefficients μ, κ, and D_k to be effective functions of the absolute temperature.
- The weak stability property of the effective viscous pressure combined with the approach based on the oscillation defect measures (see also [5, 10]).

The main contribution of this work to the existing theory, and the principal new difficulties to be dealt with can be characterized as follows:

- The approximation scheme used to construct the solution is based, on one hand, on the Faedo-Galerkin type approximation to deal with the "fluid" part of the system while the "reaction" part requires uniform estimates based on the invariant regions technique (cf. Chueh et al.). These two approaches being rather incompatible, some extra terms must be introduced in the approximate system.
- In order to accommodate the presence of individual components in the mixture and therefore the dependence of the state equation for the pressure on the concentrations of the individual species a new technique based on weighted oscillations defect measures must be used.
- The standard entropy with the corresponding balance equation must be considerably modified (cf. Giovangigli [Chapter 2, Section 2.6] [97]) in order to accommodate the presence of multiple individual components in the mixture.

• Motivated by several recent studies devoted to the scale analysis as well as numerical experiments related to multicomponent models (see Klein et al. [119]), article [6] deals with multicomponent reactive flows governed by the Navier-Stokes system coupled with a given set of N *reversible* chemical reactions. Several additional difficulties appear in the coupling of the hyperbolic-parabolic Navier-Stokes system with the reaction diffusion part for the chemistry consisting of $N > 1$ reversible chemical reactions (see for instance Bose [Chapter 6]). Therefore, there is a fundamental need for developing a relevant existence theory. The main objective of this work is to undertake a first step in this direction.

This rich and complex system has been investigated by Giovangili in [7] where an existence result was presented under the assumption that the initial state is near an equilibrium. For general *large data*, however, the pertubative arguments of [7] no longer apply and a new machinery is required.

The main contribution of this work to the existing theory, and the principal new difficulties to be dealt with can be characterized as follows:

- In order to accommodate realistic growth conditions imposed on the transport coefficients, a new technique based on weighted oscillations defect measures must be used. Moreover, the velocity does not (is not known to) belong to the standard "energy space" $W^{1,2}(\Omega; R^3)$. This fact calls for rather delicate energy estimates. In particular, a generalized version of Korn's inequality is shown that may be of independent interest.
- In order to ensure strict positivity of the absolute temperature, a singular source term must be added to the approximate thermal energy equation as well as to the corresponding total energy balance, which makes the analysis quite delicate.
- The standard entropy with the corresponding balance equation must be considerably modified (cf. Giovangigli [Chapter 2, Section 2.6][97]) in order to handle the reversibility of one or several chemical reactions.

Results on the stability and large time behavior for multicomponent flows are presented in [9]. The current research activity of the PI extends the earlier work by Chen, Hoff, Trivisa [1], [2] on one-dimensional models.

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Effective equations for shear band formation at high strain rates

Athanasios Tzavaras

(joint work with Th. Katsaounis)

One striking instance of material instability is observed in the course of deformations of metals at high strain-rates. It appears as an instability in shear and leads to regions of intensely concentrated shear strain, called shear bands. In experimental investigations of high strain-rate deformations of steels, observations of shear bands are typically associated with strain softening response – past a critical strain – of the measured stress-strain curve [1]. It was early recognized that the effect of the deformation speed is twofold: First, an increase in the deformation speed changes the deformation conditions from isothermal to nearly adiabatic. Second, strain rate has an effect *per se*, and needs to be included in the constitutive modeling.

Under isothermal conditions, metals, in general, strain harden and exhibit a stable response. As the deformation speed increases, the heat produced by the plastic work causes an increase in the temperature. For certain metals, the tendency for thermal softening may outweigh the tendency for strain hardening and deliver net softening. A destabilizing feedback mechanism is then induced, which operates

as follows ([1]): Nonuniformities in the strain rate result in nonuniform heating. Since the material is softer at the hotter spots and harder at the colder spots, if heat diffusion is too weak to equalize the temperatures, the initial nonuniformities in the strain rate are, in turn, amplified. This mechanism tends to localize the total deformation into narrow regions. On the other hand, there is opposition to this process by "viscous effects" induced by strain-rate sensitivity. The outcome of the competition depends mainly on the relative weights of thermal softening, strain hardening and strain-rate sensitivity, as well as the loading circumstances.

This qualitative scenario is widely accepted as the mechanism of shear band formation. On the other hand, this qualitative picture is somewhat imprecise in terms of what determines (or rules out) the onset of localization. The goal of this work is to develop a quantitative criterion explaining the onset of instability. We use the model

$$
\begin{aligned}
v_t &= \frac{1}{r}\,\sigma_x,\\
\theta_t &= \kappa\theta_{xx} + \sigma\gamma_t,\\
\gamma_t &= v_x,
\end{aligned}
\tag{1}
$$

where r, κ are non-dimensional constants, and the stress is given by an empirical power law in the normalized form

$$
\sigma = \theta^{-\alpha}\gamma^m\gamma_t^n\,,
\tag{2}
$$

appropriate for the flow rule of a viscoplastic material exhibiting thermal softening, strain hardening and strain-rate sensitivity.

The model (1) admits a class of special solutions describing uniform shearing

$$
\begin{aligned}
v_s &= x,\\
\gamma_s &= t+\gamma_0,\\
\theta_s &= \left[\theta_0^{1+\alpha} + \frac{1+\alpha}{m+1}\left[(t+\gamma_0)^{m+1} - \gamma_0^{m+1}\right]\right]^{\frac{1}{1+\alpha}},\\
\sigma_s &= \theta_s^{-\alpha}(t)(t+\gamma_0)^m\,,
\end{aligned}
\tag{3}
$$

Much of the previous analysis on (1) has centered on the issue of their stability. The form of (3) suggests the change of variables

$$
\begin{aligned}
&\theta(x,t) = (t+1)^{\frac{m+1}{\alpha+1}}\Theta(x,\tau(t)), \quad \gamma(x,t) = (t+1)\Gamma(x,\tau(t)),\\
&\sigma(x,t) = (t+1)^{\frac{m-\alpha}{\alpha+1}}\Sigma(x,\tau(t)), \quad v(x,t) = V(x,\tau(t)), \quad \tau = \ln(1+t).
\end{aligned}
\tag{4}
$$

that transforms the problem into the study of the asymptotic behavior for a reaction-diffusion system.

In the special case of a fluid with temperature dependent viscosity ($m = 0$) the kinematic equation $(1)_3$ decouples from the remaining equations, and the problem reduces to the study of a simplified system of two equations. This simpler system

has been analyzed in most detail [2, 3, 5]. Its rescaled variant can be rewritten as

$$
\begin{aligned}
\Sigma_\tau &= \frac{\ell}{r}\, e^{\frac{1}{1+\alpha}\,\tau}\Theta^{-\frac{\alpha}{\ell}}\ \Sigma^{\frac{\ell-1}{\ell}}\Sigma_{xx} + \left(-\alpha\Theta^{\frac{\alpha}{\ell}-1}\Sigma^{\frac{\ell+1}{\ell}} + \frac{\alpha}{1+\alpha}\right)\Sigma,\\
\Theta_\tau &= \left(\Theta^{\frac{\alpha}{\ell}-1}\ \Sigma^{\frac{\ell+1}{\ell}} - \frac{1}{1+\alpha}\right)\Theta\,.
\end{aligned}
\tag{5}
$$

The system (5) admits invariant rectangles in the parameter range $q = -\alpha+n > 0$ but misses this property in the range $q = -\alpha + n < 0$. It is this dichotomy that provides a quantitative threshold to stability: In the parameter range $q > 0$ the invariant rectangles yield asymptotic stability of the uniform shearing solution. By contrast, in the complementary region $q < 0$ moderate perturbations of the uniform solutions can lead to instability and formation of shear bands.

We present a connection with the theory of relaxation systems that turns out to be instrumental for understanding the onset of localization. This connection motivates the derivation of an effective equation for the onset of localization. The result is the following [6]: Let T be a parameter describing a time-scale, and consider a change of variables of the form

$$
\begin{aligned}
\theta(x,t) &= (t+1)^{\frac{m+1}{\alpha+1}}\Theta(x,\frac{s(t)}{T})\,, \quad \gamma(x,t) = (t+1)\Gamma(x,\frac{s(t)}{T})\,,\\
\sigma(x,t) &= (t+1)^{\frac{m-\alpha}{\alpha+1}}\Sigma(x,\frac{s(t)}{T})\,, \quad v_x(x,t) = V_x(x,\frac{s(t)}{T})\,,
\end{aligned}
\tag{6}
$$

where T is a parameter representing a change of time-unit and $s(t) : [0,\infty) \to [0,\infty)$ is selected as a monotone increasing, surjective map that represents a change of time-scale. The new functions $(U^T, \Theta^T, \Gamma^T, \Sigma^T)$ with $U^T = V_x^T$ satisfy the system

$$
\begin{aligned}
\partial_s U &= \Sigma_{xx},\\
\frac{1}{T}(\beta s+1)\Theta_s &= \Sigma U - \frac{m+1}{1+\alpha}\Theta,\\
\frac{1}{T}(\beta s+1)\Gamma_s &= U - \Gamma,\\
\Sigma &= \Theta^{-\alpha}\Gamma^m U^n.
\end{aligned}
\tag{7}
$$

If $(U^T, \Theta^T, \Gamma^T, \Sigma^T)$ stabilizes as $T \to \infty$ then its limiting profile will describe the asymptotic form of (v_x, θ, γ) as $t \to \infty$. The asymptotic behavior problem is then reduced to studying the large T behavior of (7), a problem lying within the realm of relaxation theory. Using ideas analogous to the Chapman-Enskog expansion in kinetic theory of gases, one shows that for large $T >> 1$ and $r = O(T)$ the quantity $U^T = V_x^T$ satisfies the effective equation

$$
\partial_s U = \partial_{xx}\Big(c\,U^p + \frac{\lambda c^2}{T}(\beta s+1)U^{p-1}\partial_{xx}U^p\Big),
\tag{8}
$$

within order $O(\frac{1}{T^2})$. The parameters are $p = \frac{q}{1+\alpha} = \frac{-\alpha+m+n}{1+\alpha}$, $\beta = \frac{m+1}{1+\alpha}$, $c = \beta^{\frac{\alpha}{1+\alpha}}$ and the coefficient of the fourth order term is $\lambda = \dfrac{\alpha(1+m+n) - m(m+1)}{(m+1)(1+\alpha)}$.

The leading order term in the effective equation (8) changes type from forward parabolic for $q = -\alpha + m + n > 0$ to backward parabolic for $q = -\alpha + m + n < 0$. This change of type captures the parameter regime associated with the onset of localization. We note that in the region of instability $q < 0$ the coefficient $\lambda > 0$ and thus the fourth order term has a regularizing effect. Numerical comparisons between the effective equation (8) and the system (7) indicate good agreement between the effective equation and the response of the original problem for $T >> 1$.

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Optimal Transport for the System of Isentropic Euler Equations

Michael Westdickenberg

(joint work with Wilfrid Gangbo, Jon Wilkening)

It is well-known that for the one-dimensional system of isentropic Euler equations, there exists a large family of convex entropies. Control over the entropy dissipation of all entropies of this family is the crucial ingredient for the proof of global existence of weak solutions. In the multidimensional case, on the other hand, the only convex entropy seems to be the total energy. Consequently, the global existence of weak solutions is an open problem.

In joint work [1] with Wilfrid Gangbo, we try to make better use of the convex entropy provided by the total energy, by adding a variational flavor to the entropy condition: As advocated by Dafermos, we try to construct weak solutions for which the total energy is not only nonincreasing in time, but for which the total energy is in fact dissipated as fast as possible. We proposed a time discretization for the isentropic Euler equations that consists of a sequence of minimization problems: In each timestep the energy is minimized subject to a constraint that measures the deviation of particles from their characteristic paths. This constraint is realized in terms of a new functional, called the Minimal Acceleration Cost, which is modeled after the Wasserstein distance. The intuition is that the particles prefer to stay on their free flight paths, but may deviate from a straight line in order to decrease the energy. We analyze the convergence of the approximations towards a measure-valued solution.

In joint work [2] with Jon Wilkening, we developed a fully discrete version of the variational time discretization described above, for the one-dimensional case. Experiments show that the method captures very well the nonlinear features of the flow, such as rarefaction waves and shocks. While our scheme is too expensive to be competitive, it clearly demonstrates that our interpretation of the isentropic Euler equations as a "steepest descent" is plausible.

There is intense research activity on flows on spaces of probability measures. By interpreting the isentropic Euler equations in this framework (suitably modified), we can use techniques from this body of work, as well as from convex analysis and the calculus of variations.

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Two-Dimensional Riemann Problems for Conservation Laws

Yuxi Zheng

Consider the two-dimensional compressible Euler system

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + pI) = 0, \\ (\rho E)_t + \nabla \cdot (\rho E \mathbf{u} + p\mathbf{u}) = 0, \end{cases} \tag{1}$$

where ρ is the density, $\mathbf{u} = (u, v)$ is the velocity, p is the pressure, and $E = p\rho^{-1}/(\gamma - 1) + (u^2 + v^2)/2$ where $\gamma > 1$ is the gas constant. The other variables we use are speed of sound c such that $c^2 = \gamma p/\rho$ and entropy $S = p\rho^{-\gamma}$. Cauchy problems for (1) are open. Riemann problems for (1) are a current research topic, as they are reducible to involve fewer independent variables.

Riemann problems are Cauchy problems with special initial data that are constant along each ray from the origin. The four-wave Riemann problems are special Riemann problems whose initial data yield single waves along the interfaces of the four quadrants. A list of all possible configurations is available in [12, 10, 7, 14]. Numerical solutions to these configurations have been done in [1, 3, 4, 5, 10]. In particular, paper [3], being the latest, concentrates on Configurations A and B, the simplest two of the many cases, and reveals new details of the solutions.

In Config. A, the initial data $\{p_i, \rho_i, u_i, v_i\}$ in the ith-quadrant ($i = 1, 2, 3, 4$) are such that a forward planar rarefaction wave R_{ij}^+ is there to connect the neighboring states i and j for each interface $(i, j) \in \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. The entropy turns out to be constant $S_i = S_j$, while (u, v, c) are related by

$$\begin{aligned} &u_i - u_j = 2(c_i - c_j)/(\gamma - 1), \quad v_i = v_j, \qquad (i, j) \in \{(1, 2), (3, 4)\}, \\ &v_i - v_j = 2(c_i - c_j)/(\gamma - 1), \quad u_i = u_j, \qquad (i, j) \in \{(2, 3), (4, 1)\}. \end{aligned} \tag{2}$$

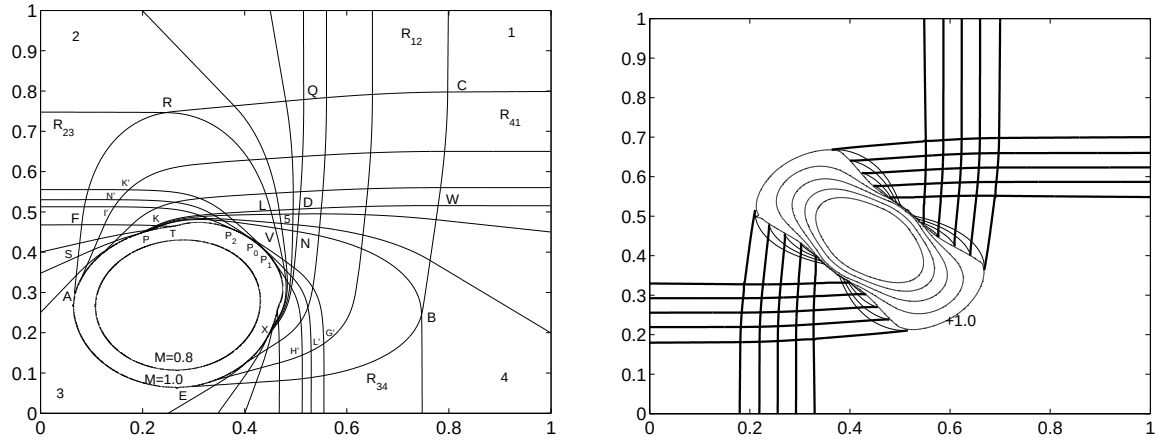

FIGURE 1. Left: Configuration A from [3] shows some pseudo-characteristic curves (light) and Mach number contours (bold) marked with $M = 1.0$ and $M = 0.8$, in which a shock occurs along curve AP. Right: Configuration B from [3] shows contour curves of pseudo-Mach number (light closed curves), some pseudo-characteristic curves (bold, short, and light curves), and shocks with large data.

For simplicity, we assume $\rho_2 = \rho_4$ and $u_1 = v_1 (= 0)$. Connections (2) are possible iff we have

$$2c_2 = c_1 + c_3. \tag{3}$$

For any fixed $\{\rho_1, p_1, u_1, v_1, \rho_3\}$, we find c_2 from compatibility condition (3) and other variables from (2) and the symmetry. We use $\gamma = 1.4$. We draw both families of (pseudo) characteristic curves corresponding to $\lambda_\pm$ by (see e.g. [7]),

$$\frac{d\eta}{d\xi} = \lambda_\pm(\xi, \eta) \equiv \frac{(u-\xi)(v-\eta) \pm c[(u-\xi)^2 + (v-\eta)^2 - c^2]^{1/2}}{(u-\xi)^2 - c^2}, \tag{4}$$

where $\xi = x/t, \eta = y/t$. The pseudo-Mach number is $M = [(u-\xi)^2+(v-\eta)^2]^{1/2}/c$. In numerical simulation in which $\rho_1 = 1.0, p_1 = 0.444, u_1 = v_1 = 0.0, \rho_3 = 0.15, \Delta x = \Delta y = 1/3200$, shock formation is found in paper [3] which has not been expected or seen in earlier work, see Figure 1.

In Config. B, the initial data (p_i, ρ_i, u_i, v_i) in the i-th quadrants $(i = 1, 2, 3, 4)$ are such that states 1 and 2 form a forward rarefaction wave R_{12}^+, states 2 and 3 form a backward rarefaction wave R_{23}^-, states 3 and 4 form a forward rarefaction wave R_{34}^+, and states 4 and 1 form a backward rarefaction wave R_{41}^-. These requirements on the data force the speed of sound to satisfy $c_2 = c_4, c_1 = c_3$, and $S_i = S_1 (i = 2, 3, 4)$. For data $p_1 = 0.444, \rho_1 = 1.0$, $u_1 = v_1 = 0.00$, $\rho_2 = 0.5197, \gamma = 1.4$, shock formation occurs, see Fig. 1.

The difficulty to a rigorous proof lies in the shortage of effective methods of analysis. Using methods that we have developed in recent years [8, 2, 6], we are able to construct a class of analytic solutions to Configuration B. We have

Theorem ([13]) *Consider Config. B for system (1). Let* $\gamma > 1 + \sqrt{2}$. *Then, there exists a number* $c_2^*(\gamma) \in (0,1)$ *such that Config. B has a global continuous solution, provided* $0 < c_2 < c_2^*(\gamma)c_1$. *The solution has a vacuum at the center.*

Additionally, if the waves R_{12}^+, R_{23}^- etc. are not large, then shock waves form internally for Config. B. In our paper [9] we construct solutions for Config. B that show shock wave formation and other waves which we call semi-hyperbolic wave patches. For the pressure gradient system, which is a very interesting model of the Euler system, the semi-hyperbolic wave patches are constructed in paper [11].

For future simulations of Config. B, we suggest to use normalization $c_1 = 1$, and the only two free parameters are $c_2 \in (0, c_1)$ and $\gamma > 1$. The other variables are given by $u_1 = (c_1 - c_2)/(\gamma - 1) > 0,\ v_1 = v_2 = u_4 = u_1,\ u_2 = u_3 = v_3 = v_4 = -u_1,\ c_3 = c_1,\ c_4 = c_2$ with constant entropy $S_i = S_1 (i = 2, 3, 4)$. The set-up is symmetric w.r.t. both lines $\xi \pm \eta = 0$.

We hope that the properties of these solutions of Config. A and B are useful in applying these problems as testing cases for various numerical schemes.

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