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**Mini-Workshop: Product Systems and Independence in  
Quantum Dynamics**

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ABSTRACT. Quantum dynamics, both reversible (i.e., closed quantum systems) and irreversible (i.e., open quantum systems), gives rise to product systems of Hilbert spaces or, more generally, of Hilbert modules. When we consider reversible dynamics that *dilates* an irreversible dynamics, then the product system of the latter is equal to the product system of the former (or is contained in a unique way). Whenever the dynamics is on a proper subalgebra of the algebra of all bounded operators on a Hilbert space, in particular, when the open system is classical (commutative) it is indispensable that we use Hilbert modules.

The product system of a reversible dynamics is intimately related to a filtration of subalgebras that are independent in a state or conditionally independent in a conditional expectation of the reversible system. This has been illustrated in many concrete dilations that have been obtained with the help of quantum stochastic calculus. Here the underlying Fock space or module determines the sort of quantum independence underlying the reversible system.

The mini-workshop brought together experts from quantum dynamics, product systems and quantum independence who have contributed to the general theory or who have studied intriguing examples. As the implications of the tight relationship between product systems and independence had so far been largely neglected, we expect from our mini-workshop a strong innovative impulse to this field.

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## Introduction by the Organisers

The analysis of dynamical systems, classical or quantum, reversible or irreversible, is a central issue when applying mathematics to natural sciences, economy, sociology, computer and information sciences, and others.

Since William Arveson's fundamental work on product systems of Hilbert spaces (*Arveson systems*) we know that product systems play an outstanding role in the classification of quantum dynamics. Boris Tsirelson's large classes of examples of *spatial* product systems of Arveson systems showed that product systems are intimately related to stochastic independence. If classical dynamics should be included, then Hilbert modules are an indispensable tool.

The Mini-Workshop *Product Systems and Independence in Quantum Dynamics*, concentrated on these connections, with the goal to open a new promising line of research. To achieve this goal we brought together leading experts in the relevant fields. In the first two days, the state of the art was presented in a series of lectures on selected important topics. During last three days more specific aspects were developed in one-hour research talks. The files of some lectures and talks can be found on the web page

<http://www.math-inf.uni-greifswald.de/algebra/q-dyn/>

of the workshop.

Particularly exciting were the open problems sessions. Trying to open a new line of research, it appeared natural to us scheduling two open problem sessions. As a matter of surprise to us, each open problem session took almost two hours. We think it is not possible to underline better the resonance our concept had. It is our wish to share these open problems, and a detailed description can be found in the Open Problems Section.

We wish to thank all participants, whose participation made the workshop a success. We would also like to express our deep gratitude to the MFO, its Director Prof. Dr. Greuel, the *Gesellschaft für Mathematische Forschung e.V.*, and last but surely not least, the staff of the MFO, for offering to us the opportunity to organize this Mini-Workshop.

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## Mini-Workshop: Product Systems and Independence in Quantum Dynamics

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## Abstracts

### Introductory Lectures — Brief Summary

The lectures give an up-to-date introduction to some aspects of the workshop which are already well established. Emphasis is on type II Arveson systems (Tsirelson, Liebscher, and Floricel), on axiomatic quantum independence (Schürmann), and on distributional symmetries leading to conditional quantum independence by quantum De Finetti theorems (Köstler). The lectures are preceded by an overall introduction to the concept of the workshop.

### An Introduction

For the organizers: MICHAEL SKEIDE

**Idea of the Workshop.** All basic (microscopic) laws of nature are reversible, while everyday's experience concerns almost always irreversible phenomena. Understanding irreversible behaviour from *first principles* is, therefore, a fundamental task. We explain briefly the connection of that task with product systems and independence, claimed in the abstract.

**Dynamics, dilations, and modules.** An irreversible (quantum) evolution is modelled as a *Markov semigroup* (that is, a unit-preserving semigroup of completely positive maps)  $T = (T_t)_{t \geq 0}$  on a unital  $C^*$ - or von Neumann algebra  $\mathcal{B}$ .<sup>[a]</sup> A reversible quantum evolution is modelled as an automorphism group  $\alpha = (\alpha_t)$  on some  $\mathcal{B}(H)$  ( $H$  a Hilbert space).<sup>[b]</sup> Often one restricts attention to a unital  $C^*$ - or von Neumann subalgebra  $\mathcal{A} \subset \mathcal{B}(H)$  that is left invariant by  $\alpha_t$  for  $t \geq 0$ . That is, one considers an  $E_0$ -*semigroup* (that is, a semigroup of unital endomorphisms)  $\vartheta = (\vartheta_t)_{t \geq 0}$  on  $\mathcal{A}$ .

Even if the evolution  $\vartheta$  of the algebra  $\mathcal{A}$  of observables is reversible, usually not all of  $\mathcal{A}$  is accessible to measurement but only a rather small subalgebra  $\mathcal{B}$  of  $\mathcal{A}$ . In particular, if  $\mathcal{B}$  consists of *macroscopic parameters* of the system, then  $\mathcal{B}$  is *classical*, that is,  $\mathcal{B}$  is commutative. The evolution  $\vartheta$ , usually, evolves  $\mathcal{B}$  into a subset  $\vartheta_t(\mathcal{B}) \subset \mathcal{A}$  which is no longer a subset of  $\mathcal{B}$ . However, since we can only measure the observables in  $\mathcal{B}$ , what we actually can measure is  $\mathfrak{p} \circ \vartheta_t(\mathcal{B})$  where  $\mathfrak{p}: \mathcal{A} \rightarrow \mathcal{B}$  is a conditional expectation onto  $\mathcal{B}$ . Clearly, the maps

$$T_t = \mathfrak{p} \circ \vartheta_t \upharpoonright \mathcal{B}$$

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<sup>[a]</sup>The case of a classical Markov semigroup  $P = (P_t)_{t \geq 0}$  of transition probabilities on a state space  $S$  is included, if we define the positive maps  $[T_t(f)](x) := \int_S f(y)P_t(y, dx)$  on the commutative algebra  $L^\infty(S)$  of bounded random variables.

<sup>[b]</sup>*Wigner's theorem* tells us that such an automorphism group is *inner*, that is, there is a unitary group  $u = (u_t)$  in  $\mathcal{B}(H)$  such that  $\alpha_t = u_t \bullet u_t^*$ . The unitary group is related to its strong generator, the *Hamiltonian* of the system, by the *Schrödinger equation*.

are CP.<sup>[c]</sup> If the  $T_t$  form a Markov semigroup  $T$  on  $\mathcal{B}$ , then it is justified to say that we can understand the irreversible evolution of the measurable subsystem  $\mathcal{B}$  as projection from the reversible evolution of the whole system. Irreversibility, in that case, is due to the fact that we can never measure all the information about the actual *status* of the whole system.

In general, starting from a Markov semigroup  $T$  on  $\mathcal{B}$ , one tries to construct a **dilation** of  $T$  that allows to understand  $T$  as a projection from a reversible evolution on a bigger system. More precisely, a dilation is a quadruple  $(\mathcal{A}, \vartheta, i, \mathfrak{p})$  where  $\vartheta$  is an  $E_0$ -semigroup on  $\mathcal{A}$ , where  $i: \mathcal{B} \rightarrow \mathcal{A}$  is an embedding and  $\mathfrak{p}: \mathcal{A} \rightarrow \mathcal{B}$  is such that  $i \circ \mathfrak{p}$  is a conditional expectation, and where the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\ i \downarrow & & \uparrow \mathfrak{p} \\ \mathcal{A} & \xrightarrow{\vartheta_t} & \mathcal{A} \end{array}$$

commutes for all  $t \geq 0$ . A dilation is **unital**, if  $i$  is unital. A dilation is **weak** if, like in the situation of Footnote [c],  $i(\mathcal{B})$  is a corner of  $\mathcal{A}$  and  $i \circ \mathfrak{p} = i(\mathbf{1}) \bullet i(\mathbf{1})$ .

In this setting, Hilbert modules enter in several ways by Paschke's **GNS-construction** for CP-maps [Pas73]: If  $T: \mathcal{A} \rightarrow \mathcal{B}$  is a CP-map between unital  $C^*$ -algebras, then we find a **correspondence**<sup>[d]</sup>  $E$  from  $\mathcal{A}$  to  $\mathcal{B}$  and a vector  $\xi \in E$  such that

$$T(a) = \langle \xi, a\xi \rangle.$$

Moreover, if  $\xi$  generates  $E$  as correspondence (that is, if  $E = \overline{\text{span}}\mathcal{A}\xi\mathcal{B}$ ), then the pair  $(E, \xi)$  is unique up to bilinear unitary equivalence.<sup>[e]</sup>

Applying the GNS-construction to each  $T_t$  is at the heart of the construction of product systems for CP-semigroups in [BS00] and, further, a weak dilation. We come back to this, later. Applying, in a dilation of  $T$ , the GNS-construction to the CP-map  $\mathfrak{p}$  we obtain a correspondence  $E$  from  $\mathcal{A}$  to  $\mathcal{B}$  and a **unit vector**  $\xi \in E$  (that is,  $\langle \xi, \xi \rangle = \mathbf{1}$ ) such that  $\mathfrak{p}(a) = \langle \xi, a\xi \rangle$  for all  $a \in \mathcal{A}$ . The left action of  $\mathcal{A}$  induces a canonical homomorphism from  $\mathcal{A}$  into the adjointable operators  $\mathcal{B}^a(E)$  on  $E$ . Experience shows: Firstly, if the canonical homomorphism is not faithful, then normally we may divide out its kernel (the  $E_0$ -semigroup on  $\mathcal{A}$  gives rise to

<sup>[c]</sup>If  $\mathcal{B}$  contains the unit of  $\mathcal{A}$ , then the  $T_t$  are unital, automatically. Also if  $\mathcal{B} = p\mathcal{A}p$  is a corner of  $\mathcal{A}$  for some projection  $p \in \mathcal{A}$  and  $\mathfrak{p} = p \bullet p$ , then as soon as the  $T_t$  form a semigroup,  $T_t$  must be unital.

<sup>[d]</sup>*Correspondence* is the fashionable name for Hilbert bimodule. A Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule is a (right, of course) Hilbert  $\mathcal{B}$ -module  $E$  with a **nondegenerate** left action of  $\mathcal{A}$ . Recall that a correspondence  $E$  from  $\mathcal{A}$  to  $\mathcal{B}$  and a correspondence  $F$  from  $\mathcal{B}$  to  $\mathcal{C}$  have as (**internal**) **tensor product** a correspondence  $E \odot F$  from  $\mathcal{A}$  to  $\mathcal{C}$  which is determined uniquely (up to a bilinear unitary) by the requirement that it is generated by elementary tensors  $x \odot y$  with inner product  $\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$ .

<sup>[e]</sup>Note that if  $\mathcal{B} \subset \mathcal{B}(G)$  is represented by operators on a Hilbert space, then it is easy to obtain the *Stinespring representation* [Sti55] of  $T$ . Indeed, if we put  $H := E \odot G$  (see Footnote [d] for the tensor product), then the induced representation  $\rho(a): x \odot g \mapsto ax \odot g$  of  $\mathcal{A}$  on  $H$  is the Stinespring representation, and  $\xi: g \mapsto \xi \odot g$  is an element in  $\mathcal{B}(G, H)$  such that  $T(a) = \xi^* \rho(a) \xi$ .

an  $E_0$ -semigroup on the quotient still dilating  $T$ ). In other words, we may assume  $\mathcal{A} \subset \mathcal{B}^a(E)$ . Secondly, even if  $\mathcal{A} \subsetneq \mathcal{B}^a(E)$ , then normally the structure (that is, the  $E_0$ -semigroup and, of course, the conditional expectation) extends to all of  $\mathcal{B}^a(E)$  so that we may assume  $\mathcal{A} = \mathcal{B}^a(E)$ . (We do not know of a single concrete example where it would be known that this does not work.) This is very convenient, because, though sufficiently general for all (known) applications, the algebra  $\mathcal{B}^a(E)$  preserves much of the simplicity that makes  $\mathcal{B}(H)$  more tractable than general von Neumann algebras. On the other hand, we know lots of examples where  $\mathcal{B}^a(E)$  is contained canonically in some  $\mathcal{B}(H)$ , and where the  $E_0$ -semigroup on  $\mathcal{B}^a(E)$  **does not** extend to  $\mathcal{B}(H)$ . Also, [Sto72] shows that conditional expectations onto a unital commutative von Neumann subalgebra  $\mathcal{B}$  of  $\mathcal{B}(H)$  exist if and only if  $\mathcal{B}$  is purely atomic. This means, almost never we may hope to dilate a classical irreversible evolution to an  $E_0$ -semigroup on  $\mathcal{B}(H)$ . But algebras of observables that are measurable **simultaneously**, are always commutative.

We think that for these and other reasons the use of Hilbert modules in irreversible dynamics is unavoidable. But often the experience in studying Markov semigroups on  $\mathcal{B}(G)$  (leading to dilations on  $\mathcal{B}(H)$ !) can serve as an optimal preparation for the more general theory.

The objects occurring in dilations, Markov semigroups and  $E_0$ -semigroups, are related in several ways to product systems and independence. Independence itself is intimately related to a large subclass of product systems, the *spatial* product systems. On the other hand, also the dynamics leading to spatial product systems have intrinsic characterizations as spatial  $E_0$ -semigroups and spatial Markov semigroups.

There are lots of open problems in both the intrinsic classification of product systems — even for Hilbert spaces far from being close to a stable situation, while for modules not more than just started — and classification of dynamics (reversible and irreversible) in terms of their associated product systems. The lectures of Tsirelson (Page 505), of Liebscher (Page 508), and of Floricel (Page 509) deal with spatial Arveson systems (see the next section), and Izumi's talk (Page 517) with nonspatial ones. Apart from making progress in these general questions on classification, it is in the scope of the mini-workshop to make progress where there is a relation with independence.

**Spatial reversible dynamics and noises.**  $E_0$ -semigroups on  $\mathcal{B}^a(E)$  give rise to product systems. For the case  $\mathcal{B}(H)$ , in [Arv89a] this marked the beginning of the modern theory of product systems of Hilbert spaces or, for short, **Arveson systems**; see the monograph [Arv03]. Even in the Hilbert space case the number of new Arveson systems is still increasing every day; see, for instance, [Tsi00a, Tsi00b, Lie03, Pow03, BS05]. The relation between  $E_0$ -semigroups and product systems was generalized to  $\mathcal{B}^a(E)$  first in [Ske02] (adapting to the module case the approach from [Bha96] to Arveson systems in presence of a unit vector). The most general form is in [Ske04b]. By a **product system** we mean a family  $E^\odot = (E_t)_{t \geq 0}$

of correspondences over  $\mathcal{B}$  that factor in an associative way as

$$E_{s+t} = E_s \odot E_t$$

under tensor product.  $E^\odot$  is the product system of the  $E_0$ -semigroup  $\vartheta$  on  $\mathcal{B}^a(E)$ , if we have identifications  $E = E \odot E_t$  such that  $\vartheta_t(a) = a \odot \text{id}_t$ .<sup>[f]</sup> Depending on the circumstances, there are also technical conditions on both sides, which we do not mention here. Arveson systems classify  $E_0$ -semigroups  $\mathcal{B}(H)$  up to *cocycle conjugacy* (provided  $H$  is infinite-dimensional and separable) and for every Arveson system there exists an  $E_0$ -semigroup having that Arveson system; see [Arv89a, Arv90a, Arv89b, Arv90b, Lie03, Ske06a, Arv06]. The (very recent) classification in the module case is not up to cocycle conjugacy but up to *stable cocycle conjugacy*; see [Ske09a]. The existence results are in [Ske07, Ske09b].

For an  $E_0$ -semigroup on  $\mathcal{B}(H)$  with an invariant vector state  $\varphi = \langle \Omega, \bullet \Omega \rangle$  (a so-called **noise** on  $\mathcal{B}(H)$ ) the factorization  $H = H \otimes E_t$  gives rise to a filtration of subalgebras  $\mathcal{A}_t := \text{id}_H \otimes \mathcal{B}(E_t)$  of  $\mathcal{B}(H)$  that is **tensor independent** in the state  $\varphi$ . (The increment algebras  $\mathcal{A}_{[s,t]} := \vartheta_s(\mathcal{A}_{t-s})$  to disjoint intervals commute and the monomials factor in the state.) This is the notion of independence underlying *quantum Lévy processes* in the sense of [Sch93] that generalizes the notion of classical Lévy processes with values in groups; see [Fra06, FS99] and also Schürmann's talk (Page 536). Also **Tsirelson noises** in the sense of [Tsi98, Tsi03] give rise to  $E_0$ -semigroups with invariant vector states and *vice versa*; see the lecture of Tsirelson on Page 505. The critical property of an Arveson system  $E^\otimes$  in order that it be derived from a noise, classical or on  $\mathcal{B}(H)$ , is that  $E^\otimes$  is **spatial**.  $E^\otimes$  is spatial, if it possesses a **unit**, that is, a family  $u^\otimes = (u_t)_{t \geq 0}$  of elements  $u_t \in E_t$  that composes as  $u_s \otimes u_t = u_{s+t}$ . The  $E_0$ -semigroups having spatial Arveson system are precisely those that are spatial in the sense of [Pow88]. Not all  $E_0$ -semigroups that are spatial possess an invariant vector state. However, [Pow88] showed that they are always cocycle conjugate to a noise. Recently, in [Ske08] this fact has been used to prove that all *spatial* Markov semigroups on  $\mathcal{B}(G)$  admit a dilation that is a cocycle perturbation of a noise in a very specific way, a so-called *Hudson-Parthasarathy dilation*. We come back to spatial Markov semigroups, later.

Starting again with an  $E_0$ -semigroup on  $\mathcal{B}(H)$  with an invariant vector state  $\varphi = \langle \Omega, \bullet \Omega \rangle$ , instead of the filtration of subalgebras  $\mathcal{A}_t := \text{id}_H \otimes \mathcal{B}(E_t)$  and their time shifts, we may use the subalgebras  $\widehat{\mathcal{A}}_t := \Omega \Omega^* \otimes \mathcal{B}(E_t)$  and their time shifts. This filtration is **monotone independent** in the sense of [Mur01]. While tensor independence may not be generalized to modules ( $\text{id}_E \odot \mathcal{B}^a(E_t)$  does not make sense, in general), monotone independence does generalize. Indeed, if we start with an  $E_0$ -semigroup on  $\mathcal{B}^a(E)$  with an invariant **vector expectation**  $\varphi = \langle \Omega, \bullet \Omega \rangle$  for a unit vector  $\Omega \in E$  (that is, if we start with a **noise** on  $\mathcal{B}^a(E)$  in the sense of [Ske06b]), then the filtration of subalgebras  $\widehat{\mathcal{A}}_t = (\Omega \odot \text{id}_t) \mathcal{B}^a(E_t) (\Omega^* \odot \text{id}_t)$  and their time shifts is **conditionally monotone independent** in the vector

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<sup>[f]</sup>This is nothing but the representation theory of  $\mathcal{B}^a(E)$ . The most general form is in [MSS06].



expectation  $\varphi$  in the sense of [Ske04a]. Also here there is a relation between noises and *spatial* product systems. Just that the elements of the unit that makes the product system a spatial one, must be *central* (that is, they must commute with the algebra  $\mathcal{B}$  over which the  $E_t$  are bimodules). Again, spatial Markov semigroups on  $\mathcal{B}$  admit Hudson-Parthasarathy dilations; see [Ske09a].

Other types of independence might lead even to new types of product systems. For instance, as indicated in [Ske06b], *conditionally free independence* in the sense of [Voi95] leads to *free product systems* (the tensor product of correspondences replaced by the free product). The relation between the tensor product system of such *free flows* [Fow95, Ske06b] and their free product system is subject to present research, and has been discussed in Skeide's talk, see Page 528.

Voiculescu's free independence [Voi87] is just one, though probably the most fundamental, independence different from monotone or tensor independence. Today there is an impressive (and still increasing) number of quantum independences; see the lectures of Schürmann on Page 511 and of Köstler on Page 512. Next to independence is *exchangeability*. Köstler explains in his lecture (see Page 512) that such and other distributional symmetries, like in the classical De Finetti theorem, lead to conditional independence.

Many independences have found realizations as noises in quantum dynamics. There are quantum stochastic calculi with respect to these noises that allow to construct dilations of Markov semigroups as cocycle perturbation of the underlying noises. Still today most known examples have been constructed with a calculus for tensor independence based on the fundamental work [HP84]; see the monograph [Par92] and the up-to-date survey [Lin05]. But there are also calculi on other Fock spaces [BSW82, KS92] or modules [GS99, Ske00] or even representation free versions [AFQ92, Kös00]. For all of them a program like the one indicated in the preceding paragraph for the free case still has to be carried out. A detailed study of the underlying product systems promises to provide us with many concrete examples. Stimulating the analysis of how these examples fit into the classification of product systems ([Arv89a, Tsi00a, Lie03, Pow03] for Arveson systems and [LS01, Ske03, BLS04, Ske06b] for modules) was in the scope of the workshop.

**Irreversible dynamics, spatial and not.** Clearly, everything that has to do with *Markov properties*, actually, regards possible notions of *conditional independence*. Quantum stochastic calculus allowed to construct explicit dilations of Markov semigroups as cocycle perturbations of noises on Fock spaces or Fock modules; see above. These dilations are unital and related to independence in another way through their noises. The construction of a (unique minimal) *weak* dilation of a Markov semigroup on  $\mathcal{B}$  is due to [Bha96] for  $\mathcal{B} = \mathcal{B}(G)$  and [Bha99] for a general  $C^*$ -algebra. However, the dilation of [Bha99] is to a  $C^*$ -subalgebra  $\mathcal{A}$  of  $\mathcal{B}(H)$  and does, in general, not extend to  $\mathcal{B}(H)$ .

Starting from a CP-semigroup  $T$  on  $\mathcal{B}$ , [BS00] constructed a product system  $E^\odot$  and a unit  $\xi^\odot = (\xi_t)$  such that  $T_t(b) = \langle \xi_t, \bullet \xi_t \rangle$  and such that  $E^\odot$  is generated by  $\xi^\odot$ , the *GNS-system* of  $T$ . In general, whenever  $E^\odot$  is a product system with a *unital* unit  $\xi^\odot$  (that is, all  $\xi_t$  are unit vectors) so that  $T_t(b) := \langle \xi_t, \bullet \xi_t \rangle$  defines a

Markov semigroup  $T$ , one may construct a weak dilation  $\vartheta$  of  $T$  to some  $\mathcal{B}^a(E)$  (in a vector expectation  $\langle \xi, \bullet \xi \rangle$  for some unit vector  $\xi \in E$ ), which has  $E^\odot$  as product system.

$\mathcal{B}^a(E)$  contains the  $C^*$ -algebra  $\mathcal{A}$  of the minimal dilation from [Bha99], and  $\vartheta$  restricts to  $\mathcal{A}$ . If  $E^\odot$  is the GNS-system of  $T$ , then  $\mathcal{A}$  acts cyclicly on the unit vector  $\xi$ . (The dilation in [Bha99] extends to  $\mathcal{B}^a(E) \subset \mathcal{B}(H)$ , but not necessarily to  $\mathcal{B}(H)$ .)

On the other hand, if  $E^\odot$  can be chosen spatial, then by the classification of  $E_0$ -semigroups, it is stably cocycle conjugate to a noise. This can be used to prove that the GNS-system of a Markov semigroup  $T$  can be embedded into a spatial product system if and only if  $T$  admits a dilation that is a cocycle perturbation of a noise; see [Ske09a]. This property is equivalent to that  $T$  is spatial in a sense generalising naturally the definition of [Arv97] for  $\mathcal{B}(G)$ .

In [BLS08] it is shown that spatiality of a Markov semigroup on a  $C^*$ -algebra does not imply that the GNS-system is already spatial. But, spatial Markov semigroups on  $C^*$ -algebras are a bit boring, in that they are all uniformly continuous. The von Neumann case allows for strongly continuous spatial Markov semigroups with unbounded generator and all of them have spatial product systems. In fact, we do not know a single *nontrivial* example of a nonspatial Markov semigroup on  $\mathcal{B}(G)$ . (Nontrivial means, it is not a nonspatial  $E_0$ -semigroup nor derived from such by trivial tensor product constructions.) On the other hand, we know that the Brownian semigroup and the Ornstein-Uhlenbeck semigroup (on  $\mathcal{B} = L^\infty(\mathbb{R})$ ) are nonspatial; see [FLS09]. They admit, however, spatial *quantum extensions* to  $\mathcal{B}(L^2(\mathbb{R}))$ ; see [CFL00]. This raises several natural questions which are discussed in the open problem; see Page 543.

A large class of (spatial) Markov semigroups on  $\mathcal{B}(G)$ , the largest class we know, have *form generators* in the sense of [CF98], which resemble the Lindblad form [GKS76, Lin76] of bounded generators; see Fagnola's talk on Page 526. Also here several questions are postponed to the open problems; see Page 543.

**Field versions.** It is natural to ask for generalizations of the discussion of one-parameter semigroups to more general semigroups or even to fields (that is, without stationarity, therefore, without any semigroup action). There are results in the dilation of two-parameter Markov semigroups in the discrete case by [Sol06] and in the continuous case (under an additional condition that still has to be understood better) by [Sha07]; see the talks by Shalit (Page 519) and by Skalski (Page 531). Accardi describes in his talk (Page 538) a nonstationary construction for classical Markov fields. There are several related open problems; see the Section on Open Problems on Page 539.

**Nonspatial dynamics.** Nonspatial product systems and the related reversible dynamics fall a bit outside, as far as the direct relation with independence is not given. Nevertheless, we think that at least on two stages there are clear connections. Firstly, once we find (nontrivial) nonspatial Markov semigroups, the dilation problem relates the associated nonspatial product system to conditional

independence. Secondly, the latest constructions [Tsi00b, BS05] of nonspatial product systems arise by perturbing Fock systems (that is, perturbing not only noises, but even *white noises*). We think that the question how independence can be perturbed, clearly, should be interesting also for understanding independence. Results like [IS07, Izu07] about the structure of such nonspatial product systems are discussed in Izumi's talk on Page 517.

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## Brownian Motion and Arveson Systems

BORIS TSIRELSON

Elaborate constructions (especially, counterexamples) in a Hilbert space often use a coordinate system (orthonormal basis). In other words, the sequence space  $l_2$  is used rather than an abstract Hilbert space. An Arveson system consists of Hilbert spaces, but we cannot choose their bases without sacrificing the given tensor product structure. Instead, we can choose maximal commutative operator algebras, which leads to the probabilistic approach. Especially, the white noise (or Brownian motion) will be used rather than an abstract type  $I_1$  Arveson system.

I do not reproduce here the definition of an Arveson system [1, 3.1.1], since we only need the special case

$$(1) \quad H_t = L_2(\Omega, \mathcal{F}_{0,t}, P)$$

corresponding to a noise, for now, to the white noise generated by the increments of the one-dimensional Brownian motion  $(B_t)_{-\infty < t < \infty}$ ,  $B_t : \Omega \rightarrow \mathbb{R}$ . The relation  $H_s \otimes H_t = H_{s+t}$  for Hilbert spaces (or rather a unitary operator  $H_s \otimes H_t \rightarrow H_{s+t}$ ) emerges naturally from the similar relation for probability spaces.

We specialize the definition of a unit [1, 3.6.1] to systems of the form (1).

**Definition 1.** A *unit* (of the system (1)) is a family  $(u_t)_{t>0}$  of non-zero vectors  $u_t \in H_t = L_2(\mathcal{F}_{0,t}) \subset L_2(\mathcal{F})$  such that  $t \mapsto u_t$  is a Borel measurable map  $(0, \infty) \rightarrow L_2(\mathcal{F})$ , and

$$u_s u_t = u_{s+t} \quad \text{for all } s, t > 0.$$

(In other words, the given unitary operator  $H_s \otimes H_t \rightarrow H_{s+t}$  maps  $u_s \otimes u_t$  to  $u_{s+t}$ .) The unit is *normalized*, if  $\|u_t\| = 1$  for all  $t$ . (In general,  $\|u_t\| = \exp(ct)$  for some  $c \in \mathbb{R}$ .)

Here is the general form of a unit in our  $(H_t)_t$ :

$$u_t = \exp(zB_t + z_1 t); \quad z, z_1 \in \mathbb{C};$$

it is normalized iff  $(\operatorname{Re} z)^2 + \operatorname{Re} z_1 = 0$ . The units generate  $(H_t)_t$  in the following sense: for every  $t > 0$ ,  $H_t$  is the closed linear span of vectors of the form  $(u_1)_{\frac{t}{n}} (u_2)_{\frac{t}{n}} \dots (u_n)_{\frac{t}{n}}$ , where  $u_1, \dots, u_n$  are units,  $n = 1, 2, \dots$ . Indeed,  $L_2(\mathcal{F}_{0,t})$  is spanned by random variables of the form  $\exp(i \int_0^t f(s) dB_s)$  where  $f$  runs over step functions  $(0, t) \rightarrow \mathbb{R}$  constant on  $(0, \frac{1}{n}t), \dots, (\frac{n-1}{n}t, t)$ .

We specialize two notions, ‘type I’ and ‘automorphism’, to systems of the form (1).

**Definition 2.** A system of the form (1) is of *type I*, if it is generated by its units.

We see that our  $(H_t)_t$  is of type I.

**Definition 3.** An *automorphism* (of the system (1)) is a family  $(\Theta_t)_{t>0}$  of unitary operators  $\Theta_t : H_t \rightarrow H_t$  such that  $\Theta_{s+t}(XY) = (\Theta_s X)(\Theta_t Y)$  for all  $X \in H_s$ ,  $Y \in H_t$ ,  $s > 0$ ,  $t > 0$ , and the function  $t \mapsto \langle \Theta_t X_t, Y_t \rangle$  is Borel measurable whenever  $t \mapsto X_t$  and  $t \mapsto Y_t$  are Borel measurable maps  $(0, \infty) \rightarrow L_2(\mathcal{F})$  such that  $X_t, Y_t \in L_2(\mathcal{F}_{0,t}) \subset L_2(\mathcal{F})$ .

Basically,  $\Theta_s \otimes \Theta_t = \Theta_{s+t}$ . The group  $G$  of all automorphisms is called the *gauge group*. Clearly,  $G$  acts on the set of normalized units,  $(u_t)_t \mapsto (\Theta_t u_t)_t$ .

Normalized units  $(u_t)_t$  and  $(e^{i\lambda t} u_t)_t$  will be called *equivalent*. The gauge group  $G$  acts on the set of all equivalence classes of normalized units.

We turn to the gauge group  $G$  of our type I system  $(H_t)_t$ . Equivalence classes of normalized units of  $(H_t)_t$  are parametrized by numbers  $z \in \mathbb{C}$ , since each class contains exactly one unit of the form

$$u_t = \exp(zB_t - (\operatorname{Re} z)^2 t).$$

The scalar product corresponds to the distance:

$$|\langle u_t^{(1)}, u_t^{(2)} \rangle| = \exp(-\frac{1}{2}|z_1 - z_2|^2 t)$$

for  $u_t^{(k)} = \exp(z_k B_t - (\operatorname{Re} z_k)^2 t)$ ,  $k = 1, 2$ . The action of  $G$  on equivalence classes boils down to its action on  $\mathbb{C}$  by isometries. The orientation of  $\mathbb{C}$  is preserved, since

$$\frac{\langle u_t^{(1)}, u_t^{(2)} \rangle \langle u_t^{(2)}, u_t^{(3)} \rangle \langle u_t^{(3)}, u_t^{(1)} \rangle}{|\langle u_t^{(1)}, u_t^{(2)} \rangle \langle u_t^{(2)}, u_t^{(3)} \rangle \langle u_t^{(3)}, u_t^{(1)} \rangle|} = \exp(itS(z_1, z_2, z_3)),$$

where  $S(z_1, z_2, z_3) = \Im((z_2 - z_1)\overline{(z_3 - z_1)})$  is twice the *signed* area of the triangle. So,  $G$  acts on  $\mathbb{C}$  by motions (see [1, 3.8.4]).

Shifts of  $\mathbb{C}$  along the imaginary axis,  $z \mapsto z + i\lambda$  (for  $\lambda \in \mathbb{R}$ ) emerge from automorphisms

$$\Theta_t = \Theta_t^{\text{shift}(i\lambda)} = \exp(i\lambda B_t);$$

here the random variable  $\exp(i\lambda B_t) \in L_\infty(\mathcal{F}_{0,t})$  is treated as the multiplication operator,  $X \mapsto X \exp(i\lambda B_t)$  for  $X \in L_2(\mathcal{F}_{0,t})$ .

Shifts of  $\mathbb{C}$  along the real axis,  $z \mapsto z + \lambda$  (for  $\lambda \in \mathbb{R}$ ) emerge from less evident automorphisms

$$(2) \quad \Theta_t^{\text{shift}(\lambda)} X = D_t^{1/2} \cdot (X \circ \theta_t^\lambda);$$

here  $\theta_t^\lambda : C[0, t] \rightarrow C[0, t]$  is the drift transformation  $(\theta_t^\lambda b)(s) = b(s) - 2\lambda s$  (for  $s \in [0, t]$ ),  $D_t$  is the Radon-Nikodym derivative of the Wiener measure shifted by  $\theta_t^\lambda$  w.r.t. the Wiener measure itself,

$$(3) \quad D_t = \exp(2\lambda B_t - 2\lambda^2 t),$$

and  $X \in L_2(\mathcal{F}_{0,t})$  is treated as a function on  $C[0, t]$  (measurable w.r.t. the Wiener measure). Thus,

$$(\Theta_t^{\text{shift}(\lambda)} X)(b) = \exp(\lambda b(t) - \lambda^2 t) X(\theta_t^\lambda b).$$

By the way, these two one-parameter subgroups of  $G$  satisfy Weyl relations

$$\Theta_t^{\text{shift}(\lambda)} \Theta_t^{\text{shift}(i\mu)} = e^{-2i\lambda\mu t} \Theta_t^{\text{shift}(i\mu)} \Theta_t^{\text{shift}(\lambda)};$$

that is,  $\Theta_t^{\text{shift}(\lambda)} \Theta_t^{\text{shift}(i\mu)} = \Theta_t^{\text{trivial}(-2\lambda\mu)} \Theta_t^{\text{shift}(i\mu)} \Theta_t^{\text{shift}(\lambda)}$ .

Rotations of  $\mathbb{C}$  around the origin,  $z \mapsto e^{i\lambda} z$  (for  $\lambda \in \mathbb{R}$ ) emerge from automorphisms  $\Theta_t^{\text{rotat}(\lambda)}$ . These will not be used, but are briefly described anyway. They preserve Wiener chaos spaces  $H_n$ ,

$$\Theta_t^{\text{rotat}(\lambda)} X = e^{in\lambda} X \quad \text{for } X \in H_n \cap L_2(\mathcal{F}_{0,t});$$

the  $n$ -th chaos space  $H_n \subset L_2(\mathcal{F})$  consists of stochastic integrals

$$X = \int \cdots \int_{-\infty < s_1 < \cdots < s_n < \infty} f(s_1, \dots, s_n) dB_{s_1} \cdots dB_{s_n}$$

where  $f \in L_2(\mathbb{R}^n)$  (or rather, the relevant part of  $\mathbb{R}^n$ ). One may say that  $\Theta_t^{\text{rotat}(\lambda)}$  just multiplies each  $dB_s$  by  $e^{i\lambda}$ .

Combining shifts and rotations we get all motions of  $\mathbb{C}$ . Accordingly, all automorphisms of  $(H_t)_t$  are combinations of  $\Theta_t^{\text{shift}(i\lambda)}$ ,  $\Theta_t^{\text{shift}(\lambda)}$ ,  $\Theta_t^{\text{rotat}(\lambda)}$  and  $\Theta_t^{\text{trivial}(\lambda)}$ . More generally, the  $N$ -dimensional Brownian motion leads to the (unique up to isomorphism) Arveson system of type  $I_N$  and motions of  $\mathbb{C}^N$ . We need  $N = 1$  only;  $(H_t)_t$  is the Arveson system of type  $I_1$ .

See also Sect. 1 of [2].

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**Random Sets and Invariants for Type II Product Systems**

VOLKMAR LIEBSCHER

This lecture summarises some results from [1], [3]. Any closed set  $Z \subseteq [0, 1]$  is characterised by the values  $X_{s,t} = \chi_{\{Z': Z' \cap [s,t] = \emptyset\}}(Z)$  for  $0 \leq s < t \leq 1$ . For random closed sets,  $X_{s,t}$  are random variables fulfilling the relation

$$X_{r,s} X_{s,t} = X_{r,t}.$$

The distribution of these random variables determines the distribution of a random closed set  $Z$ . This is manifested by the fact that a probability measure  $\mu$  is uniquely determined by the function  $c_\mu : \{G \subseteq [0, 1], : G \text{ open}\} \mapsto [0, 1]$ ,  $c_\mu(G) = \mu(\{Z : Z \cap G \neq \emptyset\})$ , which has to be a Choquet capacity of infinite order:

$$\sum_{S \subseteq \{1, \dots, n\}} (-1)^{n-\#S} c_\mu(G \cup \bigcup_{i \in S} G_i) \geq 0$$

for all  $G, G_1, \dots, G_n$ .

Let  $(\mathcal{E}_t)_{t \geq 0}$  be a product system of Hilbert spaces, i.e.  $\mathcal{E}_{s+t} \cong \mathcal{E}_s \otimes \mathcal{E}_t$  in a consistent and measurable manner. To any product subsystem  $(\mathcal{F}_t)_{t \geq 0}$  one can associate the family  $(P_{s,t}^{\mathcal{F}})_{0 \leq s < t \leq 1} \in \mathcal{B}(\mathcal{E}_1)$ ,

$$P_{s,t}^{\mathcal{F}} = 1_{\mathcal{E}_s} \otimes \text{Pr}_{\mathcal{F}_{t-s}} \otimes 1_{\mathcal{E}_{1-t}}$$

These commuting projection fulfil the similar relation

$$P_{r,s}^{\mathcal{F}} P_{s,t}^{\mathcal{F}} = P_{r,t}^{\mathcal{F}}.$$

The reported result is that we can fix in a sense a distribution of these (quantum stochastic) random variables and even a distribution of a random closed set. Namely, for all normal states  $\eta$  on  $\mathcal{B}(\mathcal{E}_1)$  there is a unique probability measure  $\mu_\eta$  on  $\{Z \subseteq [0, 1] : Z \text{ closed}\}$  with

$$\int X_{s_1,t_1} \cdots X_{s_k,t_k} d\mu_\eta = \eta(P_{s_1,t_1}^{\mathcal{F}} \cdots P_{s_k,t_k}^{\mathcal{F}})$$

for all  $0 \leq s_i < t_i \leq 1, i = 1, \dots, k$ . Further, if  $\omega$  is faithful, there is a unique isomorphism  $j_\omega : L^\infty(\mu_\omega) \mapsto \{P_{s,t}^{\mathcal{F}} : 0 \leq s < t \leq 1\}''$  extending

$$j_\omega(X_{s,t}) = P_{s,t}^{\mathcal{F}}$$

for all  $0 \leq s < t \leq 1$ .

If  $\mathcal{F}$  is the product system generated by all units  $\xi$  of  $\mathcal{E}$ , i.e. factorising vectors

$$\xi_{s+t} = \xi_s \otimes \xi_t,$$



then  $\{\mu_\omega : \omega \text{ faithful normal state on } \mathfrak{B}(\mathcal{E}_1)\}$  is an invariant of the product system. The range of this invariant can be fully characterised.

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**CP-Flows and  $E_0$ -Semigroups of Type II**

REMUS FLORICEL

Our purpose in this expository note is to give an account of R.T. Powers' approach to classification of  $E_0$ -semigroups of type II ([4], [3], [1]).

As shown by Bhat [2],  $E_0$ -semigroups can be obtained as dilations of semigroups of unital completely positive maps ( $CP_0$ -semigroups). More precisely, if  $\alpha$  is a  $CP_0$ -semigroup of  $\mathfrak{B}(H)$ , then there is a Hilbert space  $H_1$ , an  $E_0$ -semigroup  $\alpha^d$  of  $\mathfrak{B}(H_1)$ , and an isometry  $W : H \rightarrow H_1$  such that  $\alpha_t = \text{Ad}[W^*] \circ \alpha_t^d \circ \text{Ad}[W]$ ,  $t \geq 0$ . Moreover, if the projection  $E = WW^*$  is minimal, i.e.,

$$\overline{\text{sp}}\left\{\prod_{i=1}^n \alpha_{t_i}^d(EA_iE)W\xi \mid \xi \in H, A_i \in \mathfrak{B}(H), t_i \geq 0, n \in \mathbb{N}\right\} = H_1,$$

then  $\alpha^d$  is unique up to conjugacy. The  $E_0$ -semigroup  $\alpha^d$  is called the minimal dilation of  $\alpha$ .

Minimal dilations of  $CP_0$ -semigroups can be described and classified up to cocycle conjugacy in terms of the associated corners ([3]). Recall that if  $\alpha$  and  $\beta$  are  $CP_0$ -semigroups of  $\mathfrak{B}(H_1)$  and  $\mathfrak{B}(H_2)$ , then a corner from  $\alpha$  to  $\beta$  is a family  $\gamma = \{\gamma_t \mid t \geq 0\}$  of mappings  $\gamma_t : \mathfrak{B}(H_1, H_2) \rightarrow \mathfrak{B}(H_1, H_2)$  such that  $\Theta = \{\Theta_t \mid t \geq 0\}$  is a  $CP$ -semigroup of  $\mathfrak{B}(H_1 \oplus H_2)$ , where

$$\Theta_t \left( \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right) = \left( \begin{bmatrix} \alpha_t(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta_t(X_{22}) \end{bmatrix} \right),$$

for all  $t \geq 0$ ,  $X_{ij} \in \mathfrak{B}(H_i, H_j)$ .  $\gamma$  is said to be a hyper-maximal corner, if for every subordinate  $CP$ -semigroup  $\Theta' \leq \Theta$  of the form

$$\Theta'_t \left( \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right) = \left( \begin{bmatrix} \alpha'_t(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta'_t(X_{22}) \end{bmatrix} \right),$$

one has  $\alpha' = \alpha$  and  $\beta' = \beta$ . The minimal dilations  $\alpha^d$  and  $\beta^d$  of  $\alpha$  and  $\beta$  are then cocycle conjugate if and only if there is a hyper-maximal corner  $\gamma$  from  $\alpha$  to  $\beta$  ([3]).

The simplest objects that can be dilated to spatial  $E_0$ -semigroups are the  $CP$ -flows ([4]): if  $K$  is a Hilbert space,  $H = K \otimes L^2(\mathbb{R}_+) = L^2(\mathbb{R}_+, K)$ , and  $U = \{U_t\}_{t \geq 0}$  is the right translation semigroup of  $H$ , then a  $CP$ -flow over  $K$  is a  $CP$ -semigroup  $\alpha$  of  $\mathfrak{B}(H)$  which is intertwined by  $U$ , i.e.,  $\alpha_t(A)U_t = U_tA$ , for all

$A \in \mathfrak{B}(H)$  and  $t \geq 0$ .

As shown by Powers ([4]), the problem of constructing  $CP$ -flows is equivalent to constructing  $q$ -positive boundary weight maps, objects that are defined as follows. Consider the contraction  $\Lambda : \mathfrak{B}(K) \rightarrow \mathfrak{B}(H)$ ,  $(\Lambda(A)f)(x) = e^{-x} Af(x)$ ,  $A \in \mathfrak{B}(K)$ ,  $f \in H$ ,  $x \geq 0$ , and denote  $\Lambda = \Lambda(I)$ . The boundary algebra  $\mathcal{A}(H)$  of  $\mathfrak{B}(H)$  is the algebra of all operators of the form  $A = (I - \Lambda)^{\frac{1}{2}} B (I - \Lambda)^{\frac{1}{2}}$ ,  $B \in \mathfrak{B}(H)$ . We say that  $\omega$  is a boundary weight on  $\mathcal{A}(H)$  (write  $\omega \in \mathcal{A}(H)_*$ ), if  $\omega$  is a linear functional on  $\mathcal{A}(H)$  (a possibly unbounded weight on  $\mathfrak{B}(H)$ ), and there is  $\mu \in \mathfrak{B}(H)_*$  such that  $\omega((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}}) = \mu(A)$ ,  $A \in \mathfrak{B}(H)$ . A linear completely bounded map  $\mathfrak{B}(K)_* \ni \rho \mapsto \omega(\rho) \in \mathcal{A}(H)_*$  is said to be a  $q$ -positive boundary weight map if the mappings

$$\widehat{\pi}_t := \omega_t \circ (I + \widehat{\Lambda}\omega_t)^{-1}$$

are completely positive. Here  $\omega_t \in \mathfrak{B}(H)_*$  is the truncated boundary weight  $\omega_t(A) = \omega(U_t U_t^* A U_t U_t^*)$ ,  $A \in \mathfrak{B}(H)$ , and the symbol “ $\widehat{\phantom{x}}$ ” denotes the extension to the predual.

The family  $\pi^\# = \{\pi_t^\# \mid t > 0\}$  of completely positive contractions  $\pi_t^\# = \pi_t$  of  $\mathfrak{B}(H)$  into  $\mathfrak{B}(K)$ , called the generalized boundary representation of the  $q$ -positive boundary weight map  $\rho \mapsto \omega(\rho)$ , provides information about the structure of the minimal dilation  $\alpha^d$  of a  $CP$ -flow  $\alpha$  over  $K$  associated to  $\rho \mapsto \omega(\rho)$  ([4], [3]). Indeed, the index of  $\alpha^d$  is the rank of the normal spine  $\pi_0^\# = \lim_{t \rightarrow 0^+} \pi_t^\#$  of the generalized boundary representation  $\pi^\# = \{\pi_t^\# \mid t > 0\}$  of  $\rho \mapsto \omega(\rho)$ , and  $\alpha^d$  is of type I if and only if  $\alpha$  is the minimal  $CP$ -flow derived from  $\pi_0^\#$ .

If the Hilbert space  $K$  is 1-dimensional, then any  $q$ -positive boundary weight map is specified by a single boundary weight  $\omega$  on the boundary algebra  $\mathcal{A}(H)$ . Moreover, if  $\omega$  is infinite, then the minimal dilation  $\alpha^d$  of the  $CP$ -flow  $\alpha$  associated to  $\omega$  is of type  $\text{II}_0$ , and the corners from  $\alpha$  to  $\alpha$  can be described explicitly ([1], [4]).

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## Classification of Notions of Non-commutative Independence

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A natural product for linear functionals on algebras assigns to each pair of linear functionals  $(\varphi_1, \varphi_2)$ ,  $\varphi_i : \mathcal{A}_i \rightarrow \mathbb{C}$ ,  $i = 1, 2$ , a linear functional  $\varphi_1 \bullet \varphi_2$  on the free product  $\mathcal{A}_1 \sqcup \mathcal{A}_2$  of the algebras  $\mathcal{A}_i$  such that

- (1) 
$$\begin{aligned} (\varphi_1 \bullet \varphi_2) \circ \iota_1 &= \varphi_1 \\ (\varphi_1 \bullet \varphi_2) \circ \iota_2 &= \varphi_2 \end{aligned}$$
- (2) 
$$(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 = \varphi_1 \bullet (\varphi_2 \bullet \varphi_3)$$
- (3) 
$$(\varphi_1 \circ j_1) \bullet (\varphi_2 \circ j_2) = (\varphi_1 \bullet \varphi_2) \circ (j_1 \sqcup j_2)$$

where  $\iota_i$  are the canonical embeddings of  $\mathcal{A}_i$  into  $\mathcal{A}_1 \sqcup \mathcal{A}_2$ , and  $j_i : \mathcal{B}_i \rightarrow \mathcal{A}_i$  are \*-algebra homomorphisms. Then by a result of N. Muraki [Mur1, Mur2], under a normalization assumption, there are exactly 5 possibilities for a natural product: the tensor, Boolean, free and monotone/anti-monotone products.

A natural product is equivalent to a family  $\sigma_{\mathcal{A}_1, \mathcal{A}_2}$  of linear mappings

$$\sigma = \sigma_{\mathcal{A}_1, \mathcal{A}_2} : \mathcal{A}_1 \sqcup \mathcal{A}_2 \rightarrow S(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong S(\mathcal{A}_1) \otimes S(\mathcal{A}_2).$$

via

$$\varphi_1 \bullet \varphi_2 = (S(\varphi_1) \otimes S(\varphi_2)) \circ \sigma$$

where  $S(V)$  is the unital symmetric tensor algebra over the vector space  $V$ . The axioms (1)-(3) can be translated to the family  $\sigma_{\mathcal{A}_1, \mathcal{A}_2}$ :

- (4) 
$$\sigma_{\mathcal{A}_1, \mathcal{A}_2} \circ \iota_{1,2} = \iota_{\mathcal{A}_1/\mathcal{A}_2}$$
- (5) 
$$(S(\sigma_{\mathcal{A}_1, \mathcal{A}_2}) \otimes \text{id}) \circ \sigma_{\mathcal{A}_1 \sqcup \mathcal{A}_2, \mathcal{A}_3} = (\text{id} \otimes S(\sigma_{\mathcal{A}_2, \mathcal{A}_3})) \circ \sigma_{\mathcal{A}_1, \mathcal{A}_2 \sqcup \mathcal{A}_3}$$
- (6) 
$$\sigma_{\mathcal{B}_1, \mathcal{B}_2} \circ (j_1 \sqcup j_2) = (S(j_1) \otimes S(j_2)) \circ \sigma_{\mathcal{A}_1, \mathcal{A}_2}$$

We have the functor  $S$  from the category of algebras to the category of unital commutative algebras, and a universal product is a family of maps

$$\sigma_{\mathcal{A}_1, \mathcal{A}_2} : \mathcal{A}_1 \sqcup \mathcal{A}_2 \rightarrow S(\mathcal{A}_1) \otimes S(\mathcal{A}_2)$$

from coproduct to coproduct satisfying (4)-(6); see [BGSch2].

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## Noncommutative De Finetti Theorems and Noncommutative Independence

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The classical de Finetti theorem characterizes an exchangeable infinite sequence of random variables to be identically distributed and conditionally independent over its tail algebra. It is a foundational result for the subject of distributional symmetries and invariance principles. Since we have still an insufficient understanding of noncommutative independence (including its conditional versions), a transfer of this result to noncommutative probability is of great interest. We introduce to related progress recently made in [20, 12, 21, 22].

Our starting point is a sequence of random variables  $\mathcal{J}$ , given by the embeddings

$$(\iota_n)_{n \geq 0} : (\mathcal{A}_0, \varphi_0) \rightarrow (\mathcal{A}, \varphi).$$

Here  $(\mathcal{A}_0, \varphi_0)$  and  $(\mathcal{A}, \varphi)$  are two (noncommutative) probability spaces, each consisting of a von Neumann algebra and a faithful normal state. (We suppress some modular conditions needed if  $\varphi$  is non-tracial.) It is immediate that such a sequence is identically distributed. Also the notions of the distributional symmetries ‘exchangeability’, ‘spreadability’ (a.k.a. ‘contractibility’ or ‘subsymmetry’) and ‘stationarity’ transfer straightforward from probability to present noncommutative setting (see [20] for details). Now the main result of [20] is a noncommutative extended de Finetti theorem which shows that these symmetries yield noncommutative conditional independence, more general than those permitted by natural universality rules (see the talk of M. Schürmann). Actually the following notion of noncommutative conditional independence can be seen to emerge from these distributional symmetries.

**Definition 1** (Noncommutative conditional independence [20]). The sequence  $\mathcal{J}$  is said to be *full  $\mathcal{T}$ -independent* if

$$E_{\mathcal{T}}(xy) = E_{\mathcal{T}}(x)E_{\mathcal{T}}(y)$$

for  $x \in \text{vN}\{\mathcal{T}, \iota_i(\mathcal{A}_0) \mid i \in I\}$  and  $y \in \text{vN}\{\mathcal{T}, \iota_j(\mathcal{A}_0) \mid j \in J\}$  whenever  $I$  and  $J$  are disjoint subsets of  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

Here  $\text{vN}\{\mathcal{T}, \iota_i(\mathcal{A}_0) \mid i \in I\}$  means the von Neumann algebra generated by  $\mathcal{T}$  and  $\{\iota_i(\mathcal{A}_0) \mid i \in I\}$ . Moreover  $E_{\mathcal{T}}$  denotes the  $\varphi$ -preserving conditional expectation from  $\mathcal{A}$  onto a von Neumann subalgebra  $\mathcal{T}$  of  $\mathcal{A}$ . In the context of de Finetti type results,  $\mathcal{T}$  is given by the tail algebra of the considered sequence  $\mathcal{J}$ :

$$\mathcal{T} = \bigcap_{n \geq 0} \text{vN}\{\iota_k(\mathcal{A}_0) \mid k \geq n\}.$$

We remark that  $\mathcal{T}$  may be not commutative and, of course,  $\mathcal{T}$  may be not contained in the von Neumann algebra generated by the ranges of finitely many random variables  $\iota_i$ .

**Theorem 2** (Noncommutative extended de Finetti theorem [20]). *Given the sequence  $\mathcal{J}$  consider the following conditions:*

- (E)  $\mathcal{J}$  is exchangeable;
- (S)  $\mathcal{J}$  is spreadable;
- (b)  $\mathcal{J}$  is stationary and full  $\mathcal{T}$ -independent;
- (a)  $\mathcal{J}$  is identically distributed and full  $\mathcal{T}$ -independent.

*Then it holds (E)  $\Rightarrow$  (S)  $\Rightarrow$  (b)  $\Rightarrow$  (a), but each of the converse implications fails.*

These four conditions are equivalent if the ranges of the random variables commute and, in this case, one recovers a dual version of E. Størmer's pioneering results on symmetric states on tensor products of  $C^*$ -algebras [29]. If the underlying von Neumann algebra  $\mathcal{A}$  is commutative, i.e. comes from a classical probability space, then Theorem 2 reduces to the usual extended de Finetti theorem, rewritten in an algebraic language.

One might object that a quantized result of de Finetti type should provide the equivalence of all four conditions (E) to (a), in particular since the result of Ryll-Nadzewski [27] ensures the equivalence of (E) and (S) in the classical setting. But such common folklore from classical probability would be misleading in the quantum world. It is a well known phenomena in physics that one-dimensional quantum statistical models correspond to twodimensional classical ones [10]. So Theorem 2 should better be compared to results on classical random arrays, where exchangeability and spreadability are known to be inequivalent [18]. On the mathematical side and as explicated in the introduction of [20], each of the three inverse implications in Theorem 2 fails due to a deep structural reason (see also [20], Theorems 5.1 & 9.2). The implication (a)  $\Rightarrow$  (b) fails because noncommutative independence in the sense of Definition 1 permits examples which are *not* captured by universality rules (compare the talk of M. Schürmann). A much better understanding of these broken equivalences comes from the work [12] on braided random objects which I address next.

A new way towards a probabilistic interpretation of braid groups is taken in [12]. There we look at braided structures from the perspective of distributional symmetries and invariance principles. The guiding idea is that, as representations of the symmetric group  $\mathbb{S}_{\infty}$  are connected to exchangeability, the representations of the braid group  $\mathbb{B}_{\infty}$  should be connected to a new symmetry which we call *braidability*.

The realization of this idea rests on two important pillars, the noncommutative extended de Finetti theorem, Theorem 2, and product representations of endomorphisms in operator theory [13].

**Definition 3** (Braidability [12]). The random sequence  $\mathcal{J}$  is *braidable* if there exists a representation of the braid group,  $\rho: \mathbb{B}_\infty \rightarrow \text{Aut}(\mathcal{A}, \varphi)$ , such that

$$\begin{aligned} \iota_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) \iota_0 & (n \in \mathbb{N}) \\ \iota_0 &= \rho(\sigma_n) \iota_0 & (n \geq 2) \end{aligned}$$

Here denotes  $\sigma_i$  the Artin generators of  $\mathbb{B}_\infty$  satisfying the relations  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  and  $\sigma_i \sigma_j = \sigma_j \sigma_i$  if  $|i - j| > 1$ . Moreover,  $\text{Aut}(\mathcal{A}, \varphi)$  denotes the  $\varphi$ -preserving \*-automorphisms of  $\mathcal{A}$ . Note also that Definition 3 provides an extension of exchangeability:  $\mathcal{J}$  is exchangeable if and only if  $\mathcal{J}$  is braidable and  $\rho(\sigma_i^2) = \text{id}$  for  $i \in \mathbb{N}$ .

Our key result inserts braidability between the distributional symmetries (E) and (S):

**Theorem 4** (Braided extended de Finetti theorem [12]). *Consider the following additional property in Theorem 2:*

(B)  $\mathcal{J}$  is braidable.

Then one has (E)  $\Rightarrow$  (B)  $\Rightarrow$  (S).

This refinement makes it clear that (E) and (S) are inequivalent in a ‘quantized’ framework, since otherwise representations of  $\mathbb{B}_\infty$  would automatically be representations of  $\mathbb{S}_\infty$ . Most importantly, since exchangeability and spreadability are ‘purely distributional’ symmetries, Theorem 4 endows braid groups with a new intrinsic probabilistic interpretation. We emphasize this by a braided Hewitt-Savage Zero-One Law (see [12, Theorem 2.5]).

The results in [12] shed fresh light onto braid group representations [5, 9] as well as on Jones subfactor theory [17, 11]. We illustrate this by means of the *braid group von Neumann algebra*  $L(\mathbb{B}_\infty)$ , a non-hyperfinite  $II_1$ -factor non-isomorphic to all free group von Neumann algebras  $L(\mathbb{F}_n)$ , and study an irreducible subfactor inclusion of *infinite Jones index* (see [12, Theorem 5.2 and Corollary 5.3]). Our investigation reveals that the left regular representation of Artin generators provides a sequence which is neither braidable nor spreadable, but identically distributed and full  $\mathbb{C}$ -independent (see [12, Theorem 5.6]). This result prompts our search for another presentation which we call the *square root of free generator presentation* (see [12, Theorem 4.1]). We show that these new generators give rise to braidable sequences (see [12, Theorem 5.9]) and that their squares are a free system of Haar unitaries in the sense of Voiculescu. Additional considerations of random walks on groups [19] lead us to the speculation that combinatorial free probability [26] has a braided extension.

Some concrete examples from subfactor theory, among them Hecke algebras, show how this new approach allows to simplify some arguments. For this purpose we invoke *noncommutative Bernoulli shifts* (see [20, 12]). They correspond to sequences of ‘quantized’ random variables which are conditionally independent over

their tail algebra<sup>1</sup>. Together with product representations of endomorphisms, these shifts provide us with powerful, flexible tools to analyze Galois type structures, in particular coming from braidable random sequences.

The long list of parallel results in classical probability and free probability [32, 31] suggests that there should also be a free version of de Finetti's theorem<sup>2</sup>. It is immediate from the noncommutative extended de Finetti theorem, Theorem 2, that many different models of 'quantized' conditionally i.i.d. sequences enjoy exchangeability. So exchangeability as an invariance principle is unable to distinguish between them in the full generality of the present 'quantized' setting.

A new idea of how to strengthen exchangeability comes from the fact that exchangeability is not only a *distributional* symmetry (by its very definition), but also a *dynamical* symmetry: for all  $n \in \mathbb{N}$ , there exists a (state-preserving) action of the symmetric group  $\mathbb{S}_n$ . Now this action of  $\mathbb{S}_n$  can equally well be expressed as a coaction of the commutative Hopf C\*-algebras  $C(\mathbb{S}_n)$  of continuous functions on  $\mathbb{S}_n$ . Doing so one can reformulate exchangeability as an invariance property of this coaction. This observation is a starting point of [22] and the crucial idea is now to replace 'permutations' by 'quantum permutations':

Wang introduces in [33] the *quantum permutation group*  $A_s(n)$  as the universal quantum automorphism group acting on the C\*-algebra  $C(\{1, \dots, n\})$  of continuous functions on  $n$  points, answering a question of Connes. The C\*-algebra  $A_s(n)$  is a compact quantum group in the sense of Woronowicz [35] and contains  $C(\mathbb{S}_n)$  as ordinary automorphisms on  $C(\{1, \dots, n\})$ , i.e. as permutation matrices.

As exchangeability is an invariance property under the coaction of all  $C(\mathbb{S}_n)$ 's, *quantum exchangeability* is introduced in [22] as an invariance property under the coaction of all  $A_s(n)$ 's. This leads us to a new characterization of freeness with amalgamation:

**Theorem 5** (Free de Finetti theorem [22]). *The following are equivalent for an infinite sequence of random variables  $\mathcal{J}$ :*

- (a)  $\mathcal{J}$  is quantum exchangeable.
- (b)  $\mathcal{J}$  is identically distributed and free over its tail algebra.

To our knowledge this is the first result where a quantum group is considered as a quantum probabilistic invariant. Free independence has so far mainly been used as a tool in the study and classification of certain quantum groups [3, 2, 6]. On the other hand, actions of quantum groups on noncommutative spaces appear in subfactor theory (e.g. [34, 14, 1, 30]), random walks on noncommutative spaces (e.g. [4, 7, 15, 16]) and in examples for noncommutative Bernoulli shifts [28]. Together with Theorem 2 and Theorem 4, the characterization result in Theorem 5

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<sup>1</sup>Due to the broad concept of independence, our notion of a noncommutative Bernoulli shift is more general than shifts on tensor products or free products, in particular the one considered for the Connes-Størmer dynamical entropy [8, 25].

<sup>2</sup>A first attempt in this direction is undertaken by F. Lehner, based on a cumulant approach via exchangeability systems [23, 24]. But the investigations in [21] reveal that weak freeness (as introduced in [24]) implies already freeness with amalgamation in a stationary setting.

hints at a very deep connection between quantum symmetries and noncommutative conditional independence.

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## Research Talks — Brief Summary

While the lectures served to give an overview over the state of the art about some well specified aspects relevant to the workshop, the talks gave occasion to present also recent and tentative news. The talks by Izumi and by Sinha deal with type III Arveson systems. The talks by Shalit, by Skalski, and by Accardi discuss versions different from one-parameter semigroups (the former two more-parameter semigroups, the latter a general nonstationary situation). The talks by Köstler, by Skeide, by von Waldenfes, and by Schürmann are in some sense related to independence. The talks by Tsirelson and by Bhat deal with specific problems of type II Arveson systems. The talk by Fagnola discusses the role of form generators.

### Generalized CCR Flows

MASAKI IZUMI

(joint work with R. Srinivasan)

An  $E_0$ -semigroup is a weakly continuous semigroup of unital  $*$ -endomorphisms of  $\mathbb{B}(H)$ , where  $H$  is a separable infinite dimensional Hilbert space. The notion of a product system was introduced by W. Arveson in his study of  $E_0$ -semigroups. He showed, on one hand, that the product system associated with an  $E_0$ -semigroup completely determines the cocycle conjugacy class of the  $E_0$ -semigroup, and on the other hand, that every product system arises from an  $E_0$ -semigroup (see [1]).

Therefore classification of  $E_0$ -semigroups up to cocycle conjugacy is equivalent to that of product systems up to isomorphism.

Product systems, and hence  $E_0$ -semigroups, are classified into three categories, type I, type II, and type III, according to abundance of units. Type I  $E_0$ -semigroups are completely classified, and they are cocycle conjugate to so called CCR flows. By definition, type III  $E_0$ -semigroups have no units, and it is not so easy to construct them. Indeed, before Tsirelson's construction of uncountably many type III product systems [8], Powers' type III  $E_0$ -semigroup [7] was the only example. The main purpose of this talk is to report on the recent development of type III  $E_0$ -semigroups.

In [5], we showed that there exist uncountably many mutually non-isomorphic type III product systems that are not distinguished from type I product systems by Tsirelson's invariant. The invariant we employ is the von Neumann algebra  $\mathcal{A}_U$  associated with an open subset  $U$  of  $[0, 1]$ , which is either a type I factor or a type III factor in our case.

Inspired by Tsirelson's construction of infinitely many mutually non-isomorphic type III product systems [8], Bhat and Srinivasan [2] introduced the notion of a sum system, which is a sort of "logarithm" of a product system giving rise to a product system via the Bosonic second quantization procedure. They obtained a dichotomy result about types, which says that every product system arising from a *divisible* sum system is either of type I or of type III.

One of the purposes of this talk is to show that every sum system is indeed divisible (see [4] for details). The proof goes through the notion of generalized CCR flows [5], which include the  $E_0$ -semigroups corresponding to Tsirelson's type III product systems [3]. In particular, every generalized CCR flow is either of type I or type III.

I also announce our recent result [6] showing that Powers' CAR construction also produces uncountably mutually non-cocycle conjugate type III  $E_0$ -semigroups.

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## Subproduct Systems and Dilation Theory of CP-Semigroups

ORR MOSHE SHALIT

(joint work with Baruch Solel)

The classical theory of isometric dilations deals with the following problem: *given a semigroup of contractions  $T = \{T_s\}_{s \in \mathcal{S}}$  acting on a Hilbert space  $H$  ( $\mathcal{S}$  is some semigroup), find a semigroup  $V = \{V_s\}_{s \in \mathcal{S}}$  of isometries acting on a Hilbert space  $K \supseteq H$  such that for all  $s \in \mathcal{S}$ ,*

$$T_s = P_H V_s|_H.$$

This theory is highly developed [16] and well known. It is desirable to obtain a parallel theory for  $*$ -endomorphisms dilations of semigroups of completely positive maps.

Let  $\mathcal{S}$  be some sub-semigroup of  $\mathbb{R}_+^k$ , and let  $\mathcal{M} \subseteq B(H)$  be a von Neumann algebra. A *CP-semigroup* (over  $\mathcal{S}$ ) is a semigroup of CP maps, that is, a family  $\Theta = \{\Theta_s\}_{s \in \mathcal{S}}$  of completely positive, contractive and normal maps on  $\mathcal{M}$  such that

$$\Theta_{s+t}(a) = \Theta_s(\Theta_t(a)) , \quad s, t \in \mathcal{S}, a \in \mathcal{M}$$

and

$$\Theta_0(a) = a , \quad a \in \mathcal{M}.$$

When  $\mathcal{S}$  has a topology we also require that the functions  $s \mapsto \langle \Theta_s(a)h, g \rangle$  be continuous for all  $a \in \mathcal{M}, h, g \in H$ . A *CP<sub>0</sub>-semigroup* is a semigroup of unital CP maps. An *E-semigroup* is a semigroup of  $*$ -endomorphisms. An *E<sub>0</sub>-semigroup* is a semigroup of unital  $*$ -endomorphisms.

An *E-dilation* of  $\Theta$  is a triple  $(\alpha, K, \mathcal{R})$  consisting of a Hilbert space  $K \supseteq H$  (with orthogonal projection  $P_H : K \rightarrow H$ ), a von Neumann algebra  $\mathcal{R} \subseteq B(K)$  that contains  $\mathcal{M}$  as a corner  $\mathcal{M} = P_H \mathcal{R} P_H$ , and an E-semigroup  $\alpha = \{\alpha_s\}_{s \in \mathcal{S}}$  on  $\mathcal{R}$  such that for all  $T \in \mathcal{R}, s \in \mathcal{S}$ ,

$$\Theta_s(P_H T P_H) = P_H \alpha_s(T) P_H.$$

The assertion that for every CP-semigroup there exists an E-dilation when  $\mathcal{S} = \mathbb{N}$  or  $\mathbb{R}_+$  is called *Bhat's Theorem*, and appears in this form in [4] (under the additional restrictions that the semigroup is unital and that  $\mathcal{M} = B(H)$ ). This results was reproved in [9, 6, 8, 3] (in greater generality).

In [5] and [15] appeared the result that for every CP-semigroup over the semigroup  $\mathcal{S} = \mathbb{N}^2$  there exists an E-dilation.

The purpose of my lecture is to report on my (together with Baruch Solel) contributions to dilation theory of CP-semigroups, and to describe the general framework by which these results were obtained, and which I believe can underly a general dilation theory of CP-semigroups.

The reported contributions are these:

- (1) For the case  $\mathcal{S} = \mathbb{R}_+^2$ : under a technical assumption of *strong commutativity*, every CP-semigroup on  $B(H)$  has an E-dilation, and every

CP<sub>0</sub>-semigroup on an arbitrary von Neumann algebra has an E<sub>0</sub>-dilation [11, 12].

- (2) For the case  $\mathcal{S} = \mathbb{N}^k$ ,  $k > 2$ : in general there does not have to exist an E-dilation. We have an example of three commuting CP maps that have no (commuting) \*-endomorphoric dilation [13].
- (3) For the case  $\mathcal{S} = \mathbb{N}^k$ ,  $k > 2$ : (under the annoying technical assumption that the CP maps are of finite index and act on  $B(H)$ ) if the semigroup is unit preserving, or if the generators are “small enough in norm”, then there exist an E-dilation [13].

As a framework for obtaining the above results, we used *subproduct systems* and their representations (these notions have been formalized in [13], but they originate from Arveson’s works [1] and [2]).

A *subproduct system* (over a von Neumann algebra  $\mathcal{M}$ ) is a bundle of  $\mathcal{M}$ -correspondences  $X = \{X(s)\}_{s \in \mathcal{S}}$ , with  $X(0) = \mathcal{M}$ , such that there exist coisometric bimodule maps

$$U_{s,t} : X(s) \otimes X(t) \rightarrow X(s+t)$$

that compose associatively, i.e.,

$$U_{s+t,r} (U_{s,t} \otimes I_{X(r)}) = U_{s,t+r} (I_{X(s)} \otimes U_{t,r}).$$

A *representation* of a subproduct system  $X$  on a Hilbert space  $G$  is a family of completely contractive maps  $T = \{T_s\}_{s \in \mathcal{S}}$ ,  $T_s : X(s) \rightarrow B(G)$  such that

- (1)  $T_0$  is a nondegenerate \*-representation,
- (2) For all  $a, b \in \mathcal{M}, x \in X(s)$ , we have  $T_s(axb) = T_0(a)T_s(x)T_0(b)$ .
- (3) For all  $X \in X(s), y \in X(t)$ , we have  $T_{s+t}(U_{s,t}(x \otimes y)) = T_s(x)T_t(y)$ .

We have shown in [13] that there is an essentially 1-1 correspondence between CP-semigroups and pairs  $(X, T)$  where  $X$  is a subproduct system and  $T$  is a representation of  $X$ . This correspondence respects the notions of dilations, so the problem of dilating CP-semigroups to E-semigroups is equivalent to the problem of constructing “isometric dilations” to subproduct system representations. The latter problem may be handled using methods from classical isometric dilation theory, thus providing a foundation for a theory of dilation of CP-semigroups, which runs parallel to the classical theory of isometric dilations.

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## Noncommutative Factorizations

CLAUS KÖSTLER

Homogeneous measure factorizations (or noises) canonically induce Arveson systems of Hilbert spaces (see [9, 8, 7, 1]). If the measure factorization comes from Lévy-Khinchin processes, the obtained Arveson product system are known to be of type I. Starting from non-classical noises as models for homogeneous measure factorizations, B. Tsirelson has established a rich source of Arveson systems of type II.

This talk reports about ongoing research on a noncommutative analogue of measure factorizations and their connection to Arveson systems [6]. The formulation of such an analogue is straightforward, based on noncommutative independence as it appears in [5, 2] or in [4].

**Definition 1.** Let  $\mathcal{A}$  be a von Neumann algebra (with separable predual) equipped with the faithful normal state  $\varphi$ . A *noncommutative factorization* of the probability space  $(\mathcal{A}, \varphi)$  over a (complete) Boolean algebra  $\mathbb{B}$  is given by a family  $(\mathcal{A}_B)_{B \in \mathbb{B}}$  of  $\varphi$ -conditioned von Neumann subalgebras  $\mathcal{A}_B$  of  $\mathcal{A}$  satisfying the following conditions:

- (i)  $\mathcal{A}_{B_1 \wedge B_2} = \mathcal{A}_{B_1} \wedge \mathcal{A}_{B_2}$ ;
- (ii)  $\mathcal{A}_{B_1 \vee B_2} = \mathcal{A}_{B_1} \vee \mathcal{A}_{B_2}$ ;
- (iii)  $\mathcal{A}_{B_1}$  and  $\mathcal{A}_{B_2}$  are  $\mathbb{C}$ -independent if  $B_1 \wedge B_2 = 0_{\mathbb{B}}$ ;
- (iv) if  $B_n \nearrow 1_{\mathbb{B}}$  then  $\bigvee_n \mathcal{A}_{B_n} = \mathcal{A}$ .

Further a continuity (or measurability) condition is required if  $\mathbb{B}$  is non-atomic.

We are especially interested in the following choices of the Boolean algebra  $\mathbb{B}$ : Lebesgue measurable sets in  $\mathbb{R}$  or all subsets of  $\mathbb{Z}$  (or their restrictions to  $\mathbb{R}_0$  resp.  $\mathbb{N}_0$ ). The noncommutative factorization is said to be *homogeneous* if there exists a group of  $\varphi$ -preserving automorphisms  $(\alpha_t)_t$  of  $\mathcal{A}$  such that  $\alpha_t(\mathcal{A}_B) = \mathcal{A}_{B+t}$  for all  $t \in \mathbb{R}$  (or  $\mathbb{Z}$ ) and  $B \in \mathbb{B}$ . Note that the setting in Definition 1 is such that the usual notion of a (homogeneous) measure factorization can be recovered as soon as the von Neumann algebra  $\mathcal{A}$  is commutative. In this case  $\mathbb{C}$ -independence reduces to stochastic independence in probability theory.

The underlying notion of noncommutative independence is very general and presently almost nothing is known about noncommutative factorizations in the sense of Definition 1. First investigations give strong evidence that there exists a rich theory of noncommutative factorizations waiting to be explored further.

Candidates for homogeneous noncommutative factorizations (over  $\mathbb{R}$ ) are  $\mathbb{C}$ -expected white noises in the sense of [4, Definition 6.5.2]. Examples for the latter are, aside of classical Poisson or Gaussian white noise, CCR white noises, CAR white noises and  $q$ -Gaussian white noises (see Examples 6.6.3 to 6.6.5 in [4]). In these examples the von Neumann algebra  $\mathcal{A}$  is generated by unitary (adapted) cocycles with respect to the time shift; and each of these unitary cocycles provides a normalized unit in the GNS Hilbert space of these noises.

**Conjecture 2.** *Every  $\mathbb{C}$ -expected white noise (in the sense of [4]) gives rise to a noncommutative factorization.*

These quantum white noises may be regarded as noncommutative analogues of Fock factorizations.

**Question 3.** *Suppose  $\mathcal{A}$  is a type  $II_1$  or type  $III$  factor. Does there exist a homogeneous noncommutative factorization of  $(\mathcal{A}, \varphi)$  (over  $\mathbb{R}$ ) which leads to an Arveson system of type  $II$ ?*

Due to the underlying notion of independence, already the study of noncommutative factorizations over  $\mathbb{Z}$  or  $\mathbb{N}_0$  is highly non-trivial.

Very recently we have found the following example in the context of exchangeability. Let  $\mathbb{S}_\infty$  be the group of all finite permutations on  $\mathbb{N}_0$ . We consider as probability space the group von Neumann algebra  $L(\mathbb{S}_\infty)$  equipped with the normalized trace  $\text{tr}_\infty$ . Let  $\mathcal{A}_I = \text{vN}\{L_{(0,k)} \mid k \in I \subset \mathbb{N}\}$ , where  $L_{(0,k)}$  is the left shift on the Hilbert space  $\ell^2(\mathbb{S}_\infty)$  associated to the transposition  $(0, k)$  on  $\mathbb{N}_0$ .

**Theorem 4.** *The family  $(\mathcal{A}_I)_{I \subset \mathbb{N}_0}$  is a noncommutative factorization of  $(L(\mathbb{S}_\infty), \text{tr}_\infty)$ .*

The proof of this result is based on exchangeability in a noncommutative setting [3]. We expect more generally:

**Conjecture 5.** *Every minimal exchangeable random sequence with trivial tail algebra (in the sense of [5]) gives rise to a noncommutative factorization over  $\mathbb{N}_0$ .*

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Nonequivalent Units

BORIS TSIRELSON

**Theorem 1.** There exists an Arveson system of type  $II_1$  such that the action of the group of automorphisms on the set of normalized units is not transitive.

In order to prove the theorem it is sufficient to construct a noise, extending the white noise and *drift sensitive* in the following sense: for every  $\lambda \in \mathbb{R} \setminus \{0\}$  the extension obtained by the drift  $\lambda$  is non-isomorphic to the original extension on the level of Arveson systems (that is, the corresponding extensions of the type  $I_1$  Arveson system are non-isomorphic).

Here is why it is sufficient. In the group of all motions of the complex plane we consider the subgroup  $G$  of motions that correspond to automorphisms of the type  $I_1$  Arveson system  $(H_t^{\text{white}})_t$  extendable to the type  $II_1$  Arveson system  $(H_t)_t$ . Real shifts  $z \mapsto z + \lambda$  (for  $\lambda \in \mathbb{R} \setminus \{0\}$ ) do not belong to  $G$  by the drift sensitivity. Imaginary shifts  $z \mapsto z + i\lambda$  (for  $\lambda \in \mathbb{R}$ ) belong to  $G$ , since the operators  $\Theta_t^{\text{shift}(i\lambda)}$  of multiplication by  $\exp(i\lambda B_t)$  act naturally on  $H_t$ . It follows that  $G$  contains no rotations (except for the rotation by  $\pi$ ) and therefore is not transitive.

The drift sensitive extension is achieved by attaching a random function  $S : (\tau, \infty) \rightarrow \{-1, +1\}$  to each (random) point  $\tau$  of local minimum of the Brownian motion. The function  $S$  is constant on  $[\tau + 2 \cdot 3^{-n-1}, \tau + 2 \cdot 3^{-n})$  for each  $n$  and such that

$$(1) \quad \frac{S(\tau + 2 \cdot 3^{-n})}{S(\tau + 2 \cdot 3^{-n-1})} = f_n(B(\tau + 2 \cdot 3^{-n}) - B(\tau))$$

for all  $n$ . Here  $f_n : \mathbb{R} \rightarrow \{-1, +1\}$  are nonrandom functions chosen appropriately.

Definitions and statements given below are not really used in the proof of the theorem, but probably help to understand the idea. The phenomenon of a non-extendable isomorphism (as well as nonisomorphic extensions) is demonstrated here by a toy model, — a kind of product system of Hilbert spaces, simpler than Arveson system.

**Definition 1.** A *toy product system of Hilbert spaces* is a triple  $(H_1, H_\infty, U)$ , where  $H_1, H_\infty$  are Hilbert spaces (over  $\mathbb{C}$ , separable), and  $U : H_1 \otimes H_\infty \rightarrow H_\infty$  is a unitary operator.

We treat it as a kind of product system, since

$$H_\infty \sim H_1 \otimes H_\infty \sim H_1 \otimes H_1 \otimes H_\infty \sim \dots$$

where ‘ $\sim$ ’ means: may be identified naturally (using  $U$ ).

An evident example:  $H_\infty = (H_1, \psi_1)^{\otimes \infty}$  is the infinite tensor product of (an infinite sequence of) copies of  $H_1$  relatively to (the copies of) a given vector  $\psi_1 \in H_1$ ,  $\|\psi_1\| = 1$ .

A more interesting example:  $H_\infty = (H_1, \psi_1)^{\otimes \infty} \oplus (H_1, \psi_2)^{\otimes \infty}$  is the direct sum of two such infinite tensor products, one relative to  $\psi_1$ , the other relative to another vector  $\psi_2 \in H_1$ ,  $\|\psi_2\| = 1$ ,  $\psi_2 \neq \psi_1$ .

**Definition 2.** Let  $(H_1, H_\infty, U)$  and  $(H'_1, H'_\infty, U')$  be toy product systems of Hilbert spaces. An *isomorphism* between them is a pair  $\Theta = (\Theta_1, \Theta_\infty)$  of unitary operators  $\Theta_1 : H_1 \rightarrow H'_1$ ,  $\Theta_\infty : H_\infty \rightarrow H'_\infty$  such that the diagram

$$\begin{array}{ccc} H_1 \otimes H_\infty & \xrightarrow{U} & H_\infty \\ \downarrow \Theta_1 \otimes \Theta_\infty & & \downarrow \Theta_\infty \\ H'_1 \otimes H'_\infty & \xrightarrow{U'} & H'_\infty \end{array}$$

is commutative.

Thus,

$$\Theta_\infty \sim \Theta_1 \otimes \Theta_\infty \sim \Theta_1 \otimes \Theta_1 \otimes \Theta_\infty \sim \dots$$

A unitary operator  $\Theta_1 : H_1 \rightarrow H_1$  leads to an automorphism of  $(H_1, \psi_1)^{\otimes \infty}$  (that is, of the corresponding toy product system) if and only if  $\Theta_1 \psi_1 = \psi_1$ . Similarly,  $\Theta_1$  leads to an automorphism of  $(H_1, \psi_1)^{\otimes \infty} \oplus (H_1, \psi_2)^{\otimes \infty}$  if and only if either  $\Theta_1 \psi_1 = \psi_1$  and  $\Theta_1 \psi_2 = \psi_2$ , or  $\Theta_1 \psi_1 = \psi_2$  and  $\Theta_1 \psi_2 = \psi_1$ .

Taking  $\Theta_1$  such that  $\Theta_1 \psi_1 = \psi_1$  but  $\Theta_1 \psi_2 \neq \psi_2$  we get an automorphism of  $(H_1, \psi_1)^{\otimes \infty}$  that cannot be extended to an automorphism of  $(H_1, \psi_1)^{\otimes \infty} \oplus (H_1, \psi_2)^{\otimes \infty}$ .

**Definition 3.** A *toy product system of probability spaces* is a triple  $(\Omega_1, \Omega_\infty, \alpha)$ , where  $\Omega_1, \Omega_\infty$  are probability spaces (standard), and  $\alpha : \Omega_1 \times \Omega_\infty \rightarrow \Omega_\infty$  is an isomorphism mod 0 (that is, an invertible measure preserving map).



Every toy product system of probability spaces  $(\Omega_1, \Omega_\infty, \alpha)$  leads to a toy product system of Hilbert spaces  $(H_1, H_\infty, U)$  as follows:

$$H_1 = L_2(\Omega_1); \quad H_\infty = L_2(\Omega_\infty);$$

$$(U\psi)(\cdot) = \psi(\alpha^{-1}(\cdot)).$$

Here we use the canonical identification

$$L_2(\Omega_1) \otimes L_2(\Omega_\infty) = L_2(\Omega_1 \times \Omega_\infty)$$

and treat a vector  $\psi \in H_1 \otimes H_\infty$  as an element of  $L_2(\Omega_1 \times \Omega_\infty)$ .

An evident example:  $\Omega_\infty = \Omega_1^\infty$  is the product of an infinite sequence of copies of  $\Omega_1$ . It leads to  $H_\infty = (H_1, \mathbb{1})^{\otimes \infty}$  where  $H_1 = L_2(\Omega_1)$  and  $\mathbb{1} \in L_2(\Omega_1)$  is the constant function,  $\mathbb{1}(\cdot) = 1$ .

Here is a more interesting example. Let  $X_1 : \Omega_1 \rightarrow \{-1, +1\}$  be a random variable (not a constant). We define  $\Omega_\infty$  as the set of all double sequences  $(\begin{smallmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_1, & s_2, & s_3, & \dots \end{smallmatrix})$  such that  $\omega_k \in \Omega_1$ ,  $s_k \in \{-1, +1\}$  and  $s_k = s_{k+1}X_1(\omega_k)$  for all  $k$ . Sequences  $(\omega_1, \omega_2, \dots) \in \Omega_1^\infty$  are endowed with the product measure. The conditional distribution of the sequence  $(s_1, s_2, \dots)$ , given  $(\omega_1, \omega_2, \dots)$ , must be concentrated on the two sequences obeying the relation  $s_k = s_{k+1}X_1(\omega_k)$ . We give to these two sequences equal conditional probabilities, 0.5 to each. Thus,  $\Omega_\infty$  is endowed with a probability measure. The map  $\alpha : \Omega_1 \times \Omega_\infty \rightarrow \Omega_\infty$  is defined by

$$\alpha\left(\omega_1, \begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right) = \begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_2X_1(\omega_1), & s_2, & s_3, & \dots \end{pmatrix}.$$

Clearly,  $\alpha$  is measure preserving.

This system  $(\Omega_1, \Omega_\infty, \alpha)$  leads to a system  $(H_1, H_\infty, U)$  of the form  $(H_1, \psi_1)^{\otimes \infty} \oplus (H_1, \psi_2)^{\otimes \infty}$  (up to isomorphism), as explained below. We have

$$H_1 = L_2(\Omega_1), \quad H_\infty = L_2(\Omega_\infty),$$

$$(U\psi)\left(\begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_1, & s_2, & s_3, & \dots \end{pmatrix}\right) = \psi\left(\omega_1, \begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right).$$

The function  $S : \Omega_\infty \rightarrow \{-1, +1\}$  defined by

$$S\left(\begin{pmatrix} \omega_1, & \omega_2, & \omega_3, & \dots \\ s_1, & s_2, & s_3, & \dots \end{pmatrix}\right) = s_1$$

may be identified with  $X_1^{\otimes \infty}$ , since

$$S\left(\alpha\left(\omega_1, \begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right)\right) = s_2X_1(\omega_1) = X_1(\omega_1)S\left(\begin{pmatrix} \omega_2, & \omega_3, & \dots \\ s_2, & s_3, & \dots \end{pmatrix}\right).$$

See also [2].

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## Inclusion Systems and Amalgamated Products of Product Systems

B.V.R. BHAT

Inclusion systems (or subproduct systems) are families of Hilbert spaces  $\{E_t : t \geq 0\}$  with an associative family of *isometries*  $\{\beta_{s+t} : s, t \geq 0\}$ :

$$\beta_{s,t} : E_{s+t} \rightarrow E_s \otimes E_t.$$

Every inclusion system gives rise to a product system through an inductive limit procedure. We show that structures such as units and morphisms of these product systems can be determined at the level of inclusion systems. We use this idea to study Powers problem and Skeide product of spatial product systems. We determine the product system of 'Powers' sum through arbitrary corners' of two CP semigroups. This is a joint work with Mithun Mukherjee.

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## Quantum Markov Semigroups and Flows Arising from Form Generators on $\mathcal{B}(\mathfrak{h})$

FRANCO FAGNOLA

A form generator  $\mathcal{L}$  on  $\mathcal{B}(\mathfrak{h})$  is given by a family  $\mathcal{L}(x)$  of quadratic forms

$$\mathcal{L}(x)[v, u] = \langle Gv, xu \rangle + \sum_{\ell \geq 1} \langle L_\ell v, xL_\ell u \rangle + \langle v, xGu \rangle$$

where  $x \in \mathcal{B}(\mathfrak{h})$ ,  $G$  is the generator of a strongly continuous semigroup  $(P_t)_{t \geq 0}$  on  $\mathfrak{h}$ ,  $v, u \in \text{Dom}(G)$ , the  $L_\ell$  are defined on  $\text{Dom}(G)$  and the algebraic condition for identity preservation  $\mathcal{L}(\mathbb{1})[v, u] = 0$  holds.

This is an extension of the well-known Gorini-Kossakowski-Sudarshan-Lindblad representation of generators norm-continuous completely positive Markov semigroups on  $\mathcal{B}(\mathfrak{h})$  also called quantum dynamical semigroups.

A semigroup  $(\mathcal{T}_t)_{t \geq 0}$  associated with the above quadratic should satisfy

$$(1) \quad \langle v, \mathcal{T}_t(x)u \rangle = \langle v, xu \rangle + \int_0^t \mathcal{L}(\mathcal{T}_s(x))[v, u] ds$$

for all  $u, v \in \text{Dom}(G)$  and  $x \in \mathcal{B}(\mathfrak{h})$ . Computing the derivative of the function  $s \rightarrow \langle P_{t-s}v, \mathcal{T}_s(s)P_{t-s}u \rangle$  it is easy to see that a strongly continuous family  $(\mathcal{T}_t(x))_{t \geq 0}$  in  $\mathcal{B}(\mathfrak{h})$  solves (1) if and only if it satisfies

$$\langle v, \mathcal{T}_t(x)u \rangle = \langle P_t v, x P_t u \rangle + \sum_{\ell=1}^{\infty} \int_0^t \langle L_{\ell} P_{t-s} v, \mathcal{T}_s(x) L_{\ell} P_{t-s} u \rangle ds.$$

This leads naturally to the iteration scheme  $\mathcal{T}_t^{(0)}(x) = P_t^* x P_t$  and

$$\langle v, \mathcal{T}_t^{(n+1)}(x)u \rangle = \langle v, P_t^* x P_t u \rangle + \sum_{\ell=1}^{\infty} \int_0^t \langle L_{\ell} P_{t-s} v, \mathcal{T}_s^{(n)}(x) L_{\ell} P_{t-s} u \rangle ds$$

It can be shown that, for  $x$  positive,  $\mathcal{T}_t^{(n+1)}(x) \uparrow \mathcal{T}_t(x)$  and  $(\mathcal{T}_t)_{t \geq 0}$  is a weakly\*-continuous semigroup on  $\mathcal{B}(\mathfrak{h})$  (see e.g. [5] Sect.3). It is called the minimal semigroup because, if  $(\mathcal{T}'_t)_{t \geq 0}$  is another semigroup solving (1), then  $\mathcal{T}_t(x) \leq \mathcal{T}'_t(x)$  for all positive  $x$ . As a consequence, a priori we know only that  $\mathcal{T}_t(\mathbb{1}) \leq \mathbb{1}$ . Indeed, the semigroup solving (1) is unique if and only if  $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ .

This construction, and its pre-dual version of E.B. Davies in [4], generalises the minimal solution of Kolmogorov equations for classical time-continuous Markov chains (see e.g. [3]).

When the minimal semigroup  $\mathcal{T}$  is identity preserving and the unbounded operators  $G, L_{\ell}$  are well behaved (e.g. the intersection of their domains and the domains of their adjoints is an essential domain for  $G$  and  $G^*$ , one can compose any pair of them on a certain dense domain and similar technical conditions) a dilation of  $\mathcal{T}$  to a inner quantum flow of the form  $j_t(X) = U_t^* X U_t$  ( $X$  operator on  $\mathfrak{h} \otimes \mathcal{F}$ ) can be constructed by means of Hudson-Parthasarathy quantum stochastic calculus. The operators  $U_t$  are unitary cocycles with respect to the natural right shift on the Fock space  $\mathcal{F}$  over  $L^2(\mathbb{R}_+; \mathfrak{k})$  and are the unique solution of a quantum stochastic differential equation driven by creation, annihilation and number processes whose coefficients are partially determined by the operators  $G, L_{\ell}$ .

In this talk we present concrete examples of form generators with “good” or singular operators  $G, L_{\ell}$ . Singular examples appear in the study of extensions of simple classical Markov semigroup on an abelian algebra to quantum Markov semigroups on a  $\mathcal{B}(\mathfrak{h})$  that contains it as a subalgebra (see e.g. [2] for the semigroup of a Brownian motion on a closed set with some boundary condition). We also illustrate the analytical obstructions that one can find in the construction of the unitary cocycle  $(U_t)_{t \geq 0}$  and distinguish good cases from singular ones.

By their own construction minimal semigroups lead to a product system with units. One expects that, when the operators  $G, L_{\ell}$  are well behaved, this product system is type I as it happens when  $G$  and the  $L_{\ell}$  are bounded and the  $L_{\ell}$  are chosen to be linearly independent as proved by Bhat [1]. It would be interesting to know whether there are type II product system arising in the dilation of minimal semigroups.

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## Free Product Systems

MICHAEL SKEIDE

For terminology, notations, and references we refer to the introduction, which starts on Page 497.

Let  $\mathcal{F}(F) := \bigoplus_{n=0}^{\infty} F^{\odot n}$  denote the **full Fock module** over the Hilbert  $\mathcal{B}$ – $\mathcal{B}$ –bimodule  $F$ . The full Fock module to the direct sum  $F_1 \oplus F_2$  factors as

$$\mathcal{F}(F_1 \oplus F_2) = \mathcal{F}(F_1) \odot \left( \omega\mathcal{B} \oplus (F_2 \odot \mathcal{F}(F_1 \oplus F_2)) \right).$$

Let us define  $E_S := L^2(S, F)$  for any (measurable) subset  $S$  of  $\mathbb{R}_+$ , and  $E := E_{\mathbb{R}_+}$ . For every  $S$  and  $T \supset S + t$  define the second quantized right shift isometry  $s_t: \mathcal{F}(E_S) \rightarrow \mathcal{F}(E_T)$ . Then the factorization

$$\begin{aligned} \mathcal{F}(E) &= \mathcal{F}(E_{[t, \infty)}) \odot \left( \omega\mathcal{B} \oplus (E_{[0, t)} \odot \mathcal{F}(E)) \right) \\ &\cong s_t \mathcal{F}(E_{[t, \infty)}) \odot \left( \omega\mathcal{B} \oplus (E_{[0, t)} \odot \mathcal{F}(E)) \right) \\ &= \mathcal{F}(E) \odot \left( \omega\mathcal{B} \oplus (E_{[0, t)} \odot \mathcal{F}(E)) \right) \end{aligned}$$

induces an  $E_0$ –semigroup  $\vartheta_t(a) := a \odot \text{id}_t$  on  $\mathcal{B}^a(E)$ , the **free flow** of **free index**  $F$ . We immediately read off from this factorization that the product system of  $\vartheta$  is

$$\mathcal{F}(F)^{\odot} = \left( \omega\mathcal{B} \oplus (E_{[0, t)} \odot \mathcal{F}(E)) \right)_{t \in \mathbb{R}_+}.$$

When  $F = K$  is a (separable) Hilbert space, Fowler [Fow95] showed that  $\vartheta$  is cocycle conjugate to a CCR flow of index  $\infty$ , independently of the dimension of  $K$ . In Skeide [Ske06b] we showed for arbitrary Hilbert  $\mathcal{B}$ – $\mathcal{B}$ –bimodules  $F$  that  $\vartheta$ , actually, is a CCR-flow of index  $F \odot \mathcal{F}(E)$ .

However, in [Ske06b] we also mentioned that the free flow on  $\mathcal{F}(E)$  can be suitably interpreted in terms of **free product systems**. In our talk we give the first concise definition of **free product systems**, we discuss their relation with  $E_0$ –semigroups (all noises; see the introduction), and we discuss how spatial tensor product systems can be blown up to obtain free product systems. We raise several natural questions that have to be answered.

Let us recall that the **free product** of two Hilbert  $\mathcal{B}$ - $\mathcal{B}$ -bimodules  $E_i$  with respect to unital central reference vectors  $\omega_i \in F_i$  is

$$E_1 \otimes E_2 := \bigoplus_{n \in \mathbb{N}_0} \bigoplus_{\varepsilon \in A_n(\{1,2\})} E_\varepsilon = \bigoplus_{n \in \mathbb{N}_0} \bigoplus_{\varepsilon \in A_n(\{1,2\})} \omega_{\varepsilon_1}^\perp \odot \dots \odot \omega_{\varepsilon_n}^\perp, \text{ where}$$

$A_n(S) := \{\varepsilon \in S^n : \varepsilon_i \neq \varepsilon_{i+1}\}$  and  $E_{()} := \omega\mathcal{B}$  with  $A_0(S) := \{()\}$ . Note that  $E_1 \otimes E_2 \supset \omega\mathcal{B} \oplus \omega_i^\perp = E_i$ . Using this, we find

$$E_1 \otimes E_2 = E_1 \odot \left( \omega\mathcal{B} \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{\varepsilon \in A_n(\{1,2\}), \varepsilon_1 \neq 1} E_\varepsilon \right),$$

so that there is a unital embedding of  $\mathcal{B}^a(E_1)$  as  $\mathcal{B}^a(E_1) \odot \text{id}$  into  $\mathcal{B}^a(E_1 \otimes E_2)$ .

Apart from the above mentioned tensor factorization, the full Fock module also satisfies the factorization

$$\mathcal{F}(F_1 \oplus F_2) = \mathcal{F}(F_1) \otimes \mathcal{F}(F_2)$$

with respect to the respective vacua. If we put  $\mathcal{F}_t(F) := \mathcal{F}(E_{[0,t)})$ , then the family  $\mathcal{F}^\otimes(F) := (\mathcal{F}_t(F))_{t \in \mathbb{R}_+}$  forms a free product system in the sense of the following definition sketched already in [Ske06b]:

**Definition.** Algebraically, a **free product system** is a family of Hilbert  $\mathcal{B}$ - $\mathcal{B}$ -bimodules  $E^\otimes = (E_t)_{t \in \mathbb{S}}$  with a family  $\omega^\otimes = (\omega_t)_{t \in \mathbb{S}}$  of central unit vectors  $\omega_t \in E_t$  and bilinear unitaries

$$u_{s,t}: (E_s, \omega_s) \otimes (E_t, \omega_t) \longrightarrow (E_{s+t}, \omega_{s+t})$$

that compose associatively, and  $E_0 = \omega_0\mathcal{B}$  such that  $u_{0,t}, u_{t,0}$  are the canonical identifications.

Technical conditions (continuity, measurability) on the bundle structure may be put as in in the tensor product case. Notice that  $E_s \odot E_t \subset E_s \otimes E_t$ , so that it makes also sense to require that the product of continuous (measurable) sections is continuous (measurable) in any version we prefer.

Since  $E_t \cong \omega_s \otimes E_t \subset E_s \otimes E_t \cong E_{s+t}$ , we obtain an inductive limit  $E_\infty \ni \omega$ , which is a Hilbert  $\mathcal{B}$ - $\mathcal{B}$ -bimodule and has a unital central reference vector  $\omega$ .

**Theorem.** The inductive limit comes with a **free** left dilation

$$v_t: E_\infty \otimes E_t \longrightarrow E_\infty$$

that iterates associatively with the free product system structure. The unital embeddings  $\mathcal{B}^a(E_\infty) \rightarrow \mathcal{B}^a(E_\infty \otimes E_t) \cong \mathcal{B}^a(E_\infty)$  induce, therefore, an  $E_0$ -semigroup  $\vartheta$  on  $\mathcal{B}^a(E_\infty)$ .

Since  $E_\infty \otimes E_t = E_\infty \odot \left( \omega\mathcal{B} \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{\varepsilon \in A_n(\{\infty, t\}), \varepsilon_1 \neq \infty} E_\varepsilon \right)$ , the (tensor!) product system of  $\vartheta$  is

$$\omega\mathcal{B} \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{\varepsilon \in A_n(\{\infty, t\}) \\ \varepsilon_1 \neq \infty}} E_\varepsilon.$$

In the case of the free product system of the free flow on a full Fock module we get back the free flow we started with.

An **open problem** is to characterize all  $E_0$ -semigroups (necessarily noises) that arise in this way.

Unlike the  $E_0$ -semigroups described by their tensor product system, the free flows come along with unital embeddings of  $\mathcal{B}^a(E_t)$  into  $\mathcal{B}^a(E_\infty)$  arising from the identification  $E_\infty = E_\infty \otimes E_t = E_t \otimes E_\infty$ . (The tensor product system description  $E = E \odot E_t$  allows only nonunital embeddings as  $(\omega \odot \text{id})\mathcal{B}^a(E_t)(\omega^* \odot \text{id})$ ; see the introduction.) Moreover, these subalgebras generate a filtration that is conditionally free independent in the vacuum expectation  $\langle \omega, \bullet \omega \rangle$ . (The filtration from the nonunital embedding in the tensor case is only conditionally monotone independent.)

An **open problem** is to find out what it means for a cocycle to be adapted to the free filtration. We hope that in the Fock situation a solution of that problem might enable us to write down adapted unitary cocycles directly from the (well-known) units of the tensor product system picture.

We mentioned already that in a free product system we have  $E_s \odot E_t \subset E_{s+t}$ . (Contrary to what has been considered in Bhat's talk on Page 526 and in Shalit's talk on Page 519, respectively, this would not be an inclusion system and a sub-product system, respectively, but rather a superproduct system. This is also closely related to a situation described in Köstler's talk on Page 521.) Clearly, the intersection  $\bigcap_{\mathbf{t}} E_{t_n} \odot \dots \odot E_{t_1}$  over all  $\mathbf{t} = (t_n, \dots, t_1)$  with  $t_1 + \dots + t_n = t$  is the biggest tensor product subsystem of the free product system. An interesting **open problem** is, whether this subsystem generates  $E^\otimes$  as a free product system. In a minute we see that in the Fock situation the answer is affirmative. But, it is not to be excluded that, in the worst case, the intersection consists only of the vacuum sector  $\omega_t \mathcal{B}$ .

Let  $(E^\odot, \omega^\odot)$  be a spatial tensor product system, see the introduction. For each  $\mathbf{t} = (t_n, \dots, t_1)$  with  $t_1 + \dots + t_n = t$  define

$$\mathcal{E}_{\mathbf{t}} := E_{t_n} \otimes \dots \otimes E_{t_1}$$

By the inclusion  $E_t = E_{t_n} \odot \dots \odot E_{t_1} \subset \mathcal{E}_{\mathbf{t}}$  and since free products respect inclusions, we obtain an inductive limit  $\mathcal{E}_t$  of  $\mathcal{E}_{\mathbf{t}}$  over  $\mathbf{t}$ .

**Theorem.** The  $\mathcal{E}_{\mathbf{t}}$  form a free product system  $\mathcal{E}^\otimes$ . It contains  $E^\odot$  as a tensor product subsystem and is generated by  $E^\odot$  as a free product system.

An **open problem** is, if we get back  $E^\odot$  by the intersection procedure as described before. In general, we ask when a free product system be obtained in that way from a spatial tensor product subsystem.

## Isometric Dilations of Representations of Product Systems of $C^*$ -Correspondences over $\mathbb{N}_0^k$

ADAM SKALSKI

Classical multi-dimensional dilation theory ([SzF]) for Hilbert space operators is concerned with dilating tuples of contractions to tuples of isometries or unitaries, preserving some specific properties of the original family. Celebrated examples of S. Parrott, N. Varopoulos and others show that commuting isometric dilations of more than two commuting contractions need not exist. In general we cannot expect a characterisation of these tuples for which the commuting dilation exists, however if one requests a specific form of the dilation then precise answers can be obtained. In particular the existence of so-called regular or  $*$ -regular dilations (i.e. dilations satisfying additional conditions with respect to products of the original contractions and their adjoints, see [Bre], [Tim]) can be detected via simple conditions corresponding to positive-definiteness of certain operator-valued functions associated with the initial tuple.

In this talk we discuss analogous results for dilations of representations of product systems of representations of product systems of  $C^*$ -correspondences over  $\mathbb{N}_0^k$ . For us a  $C^*$ -correspondence  $E$  over a  $C^*$ -algebra  $A$  is a  $C^*$ -Hilbert module (a right module over  $A$  with  $A$ -valued scalar product) equipped additionally with the structure of a left module over  $A$ . Formally: there is a nondegenerate  $*$ -homomorphism  $\phi : A \rightarrow \mathcal{L}(E)$ .

**Definition.** A product system of  $C^*$ -correspondences over  $\mathbb{N}_0^k$ , denoted by  $\mathbb{E}$ , is a family of  $k$   $C^*$ -correspondences  $\{E_1, \dots, E_k\}$  over a  $C^*$ -algebra  $A$  together with the unitary isomorphisms  $t_{i,j} : E_i \otimes E_j \rightarrow E_j \otimes E_i$  ( $i > j$ ) satisfying a natural associativity condition.

For the purpose of this talk we think of each  $E_i$  describing the type of the  $i$ -th element of the tuple we intend to dilate (a contraction, a row contraction, a family of contractions associated to a graph) and of  $t_{i,j}$  as means of encoding the commutation relations between different elements of the tuple.

Definitions of completely contractive representations of  $\mathbb{E}$  and their isometric dilations can be found in [Sol] or [Sk]. In [Sol] Solel characterised the existence of regular isometric dilation of a given representation via Brehmer-type conditions. Here we focus on  $*$ -regular dilations. Contrary to the classical context of dilating commuting tuples of contractions regular dilations cannot be straightforwardly transformed into  $*$ -regular ones. We still however have the following result:

**Theorem.** *A minimal isometric dilation  $\overrightarrow{V}$  of a representation  $\overrightarrow{T}$  of  $\mathbb{E}$  is  $*$ -regular if and only if it is doubly commuting.*

Under a certain technical condition on the product system  $\mathbb{E}$  and the assumption that a given representation  $\overrightarrow{T}$  of  $\mathbb{E}$  on a Hilbert space  $H$  satisfies a so-called Popescu condition (or condition ‘P’),  $*$ -regular isometric dilation of  $\overrightarrow{T}$  can be constructed via so-called generalised Poisson transform. Generalised Poisson transform is a completely positive map  $R_{\overrightarrow{T}} : \mathcal{T}_{\mathbb{E}} \rightarrow B(H)$ , where  $\mathcal{T}_{\mathbb{E}}$  is the Toeplitz-type algebra

associated with  $\mathbb{E}$ . The map  $R_{\overrightarrow{T}}$  was earlier studied in a simpler context for example in [Pop]. Sufficient conditions for the construction of  $R_{\overrightarrow{T}}$  are summarised in the following theorem.

**Theorem.** *Let  $\mathbb{E}$  have a normal ordering property and let  $\overrightarrow{T}$  be a representation of  $\mathbb{E}$  on  $\mathbf{H}$  satisfying the Popescu condition. Then there exists a unique continuous linear map  $R_{\overrightarrow{T}} : \mathcal{T}_{\mathbb{E}} \rightarrow B(\mathbf{H})$  satisfying*

$$R_{\overrightarrow{T}}(L_e L_f^*) = T(n)(e)(T(m)(f))^*, \quad n, m \in bn_0^k, e \in \mathbb{E}(n), f \in \mathbb{E}(m).$$

*The map  $R_{\overrightarrow{T}}$  is completely positive and contractive, unital if  $\mathcal{T}_{\mathbb{E}}$  is unital.*

The Stinespring dilation for  $R_{\overrightarrow{T}}$  provides in a natural way an isometric dilation for  $\overrightarrow{T}$ .

In the second part of the talk we present applications of the above results to families of contractions associated to a given higher-rank graph  $\Lambda$  ([KuPa], [Rae]). There is a natural way of associating to  $\Lambda$  a product system  $\mathbb{E}(\Lambda)$  ([RaS]) and it can be showed that  $\mathbb{E}(\Lambda)$  has the normal ordering property if and only if  $\Lambda$  is *finitely aligned* ([Rae]) and if and only if  $\mathbb{E}(\Lambda)$  is *compactly aligned* ([Fow]). We have the following result:

**Theorem.** *There is a 1-1 correspondence between (completely contractive) representations of  $\mathbb{E}(\Lambda)$  on a Hilbert space  $\mathbf{H}$  and  $\Lambda$ -contractions in  $B(\mathbf{H})$ . The representation is isometric if and only if the corresponding  $\Lambda$ -contraction forms a Toeplitz family, isometric and doubly commuting if and only if the corresponding  $\Lambda$ -contraction forms a Toeplitz-Cuntz-Krieger family.*

The above correspondence can be used to transform our and Solel's results on dilation of representations of product systems to the context of dilating  $\Lambda$ -contractions. As an example we present the following theorem, first proved in [SkZ]:

**Theorem.** *Let  $\Lambda$  be finitely aligned and let  $\mathcal{V}$  be a  $\Lambda$ -contraction on a Hilbert space  $\mathbf{H}$  which satisfies the Popescu condition. Then there exists a Hilbert space  $\mathbf{K} \supset \mathbf{H}$  and a  $\Lambda$ -contraction  $\mathcal{W}$  on  $\mathbf{K}$  consisting of partial isometries forming a Toeplitz-Cuntz-Krieger family such that*

$$W_\lambda^*|_{\mathbf{H}} = V_\lambda^*, \quad \lambda \in \Lambda.$$

*One may assume that  $\mathbf{K} = \overline{\text{Lin}}\{W_\lambda \mathbf{H} : \lambda \in \Lambda\}$ ; under this assumption the family  $\mathcal{W}$  is unique up to unitary equivalence.*

Most of the results presented in the talk can be found in [Sk].

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## An Approach to the Mathematical Theory of Pressure Broadening of Spectral Lines

WILHELM VON WALDENFELS

One of the three “natural” independences of quantum probability is called “Boolean independence” and the authors cite my old paper [1]. As this paper is not well known, I was asked to present it at the workshop.

Pressure broadening is a stochastic Stark effect: due to fluctuating electrical fields a spectral line coming from a plasma appears broadened. One considers an atom and the transition between two energy levels, which might be degenerate. The electrical field is generated by the electrons passing by. The electrons are assumed to be independent, they form a Poisson point process, they move on straight lines with constant velocity and their directions are uniformly distributed on the unit sphere. The mathematical problem comes from the fact, that the interactions with the atom overlap in time. The time of nearest approach to the atom is called impact time, the vector from the atom to the electron at the nearest approach is called impact parameter. If the distance between the impact times is very big, then the interactions are independent and the overlapping can be neglected. These considerations suggest a cluster expansion.

It is not the point, to present here the main features and the result of the theory. Instead I want to describe the principle of the cluster expansion and a typical example.

We consider intervals of the form

$$A = (k, k + 1, \dots, l - 1, l)$$

in  $\mathbb{N}$ . We say that  $A_1, \dots, A_K$  form an *interval partition* of  $A$  and write

$$A = A_1 \circ A_2 \circ \dots \circ A_K$$

if

$$\begin{aligned} A_1 &= (k, \dots, l_1), \\ A_2 &= (l_1 + 1, \dots, l_2), \\ &\vdots \\ A_K &= (l_{K-1} + 1, \dots, l_K = l). \end{aligned}$$

We form the free algebra  $\mathfrak{A}$  generated of all the intervals of the form  $A$  and define the *cumulants* recursively by the formula

$$A = \sum_{A_1 \circ \dots \circ A_K = A} [A_1] \cdots [A_K].$$

That implies

$$[A] = \sum_{A_1 \circ \dots \circ A_K = A} (-1)^{K+1} A_1 \cdots A_K.$$

We define the splitting operator

$$P_m(k, \dots, l) = \begin{cases} (k, \dots, m)(m+1, \dots, l) & \text{for } k \leq m < l \\ (k, \dots, l) & \text{otherwise} \end{cases}$$

and extend it to an homomorphism  $\mathfrak{A} \rightarrow \mathfrak{A}$ . We obtain

$$P_m[k, \dots, l] = \begin{cases} 0 & \text{for } k \leq m < l \\ [k, \dots, l] & \text{otherwise} \end{cases}.$$

These equations are special cases of relations holding in partially ordered sets [2], related to the names  $\zeta$ -function and Moebius function.

Assume, that the electrical field strength due to an electron with impact time 0 and impact parameter  $p$  and direction  $v$  is given by  $\varphi(\zeta)(t)$  with  $\zeta = (p, v)$ . We truncate and assume, that  $\varphi(\zeta)(t) = 0$  for  $|t| > \tau/2, p > \varrho$ . The Hilbert space is the line space  $\mathfrak{H}$ , the tensor product of the two eigenspaces of the Hamiltonian between which the transition takes place. The effect of electrons with impact parameters  $\zeta_k, \dots, \zeta_l$  and impact times  $t_k < \dots < t_l$  is at time  $t$

$$h(\varphi(\zeta_k)(t - t_k) + \dots + \varphi(\zeta_l)(t - t_l))$$

where  $h$  is a function from  $\mathbb{R}^3$  into the space of hermitian operators in the finite dimensional line space  $\mathfrak{H}$  with the property, that  $h(0) = 0$ , e.g. a linear function or a quadratic function. If  $k \leq m < l$  and  $t_{m+1} - t_m > \tau$  then

$$\begin{aligned} h(\varphi(\zeta_k)(t - t_k) + \dots + \varphi(\zeta_l)(t - t_l)) &= h(\varphi(\zeta_k)(t - t_k) + \dots + \varphi(\zeta_m)(t - t_m)) \\ &\quad + h(\varphi(\zeta_{m+1})(t - t_{m+1})) + \dots + \varphi(\zeta_l)(t - t_l). \end{aligned}$$

Due to the truncation of the impact parameter  $p$  there exists a constant  $c$  such that the difference between two impact times is distributed with respect to an exponential law with parameter  $c$ , i.e.

$$\text{probability}\{t_{k+1} - t_k \in (t, t + dt)\} = ce^{-ct} dt.$$

The time differences are independent and the  $\zeta_k$  are independent identical distributed on  $\mathbb{R}^3 \times \mathbb{S}^2$ .

We want now to discuss a part of the problem, which is relevant for the line center. Assume a sequence of independent random variables  $u_1, u_2, \dots$  exponentially distributed with parameter  $c$ . Denote by  $\Theta$  the right shift, so  $(\Theta(s)\varphi(\zeta))(s) = \varphi(\zeta)(t - s)$ . Define for  $A = (k, \dots, l)$

$$\begin{aligned} U(A) &= u_k + \dots + u_l \\ \Phi(A) &= h(\varphi(\zeta_k) + \Theta(u_k)\varphi(\zeta_{k+1}) + \dots + \Theta(u_k + \dots + u_{l-1})\varphi(\zeta_l)) \\ S(A) &= \prod_{-\infty}^{+\infty} (1 + i\Phi(A)(t)dt) \\ \alpha(A) &= \delta_{U(A)}S(A) \end{aligned}$$

where  $\delta_{U(A)}$  is the point measure in the point  $U(A)$  and the product integral  $\prod$  has to be read from the right to the left with increasing time. If  $u_m > \tau$  for  $k \leq m < l$ , denote  $A_1 = (k, \dots, m), A_2 = (m + 1, \dots, l)$  and obtain

$$\begin{aligned} U(A) &= U(A_1) + U(A_2) \\ \Phi(A) &= \Phi(A_1) + \Theta(U(A_1))\Phi(A_2) \\ S(A) &= S(A_2)S(A_1) \\ \alpha(A) &= \alpha(A_2) * \alpha(A_1), \end{aligned}$$

where  $*$  denotes the usual convolution. Extending  $\alpha$  to the algebra  $\mathfrak{A}$  one gets in this case  $\alpha[A] = 0$ . After some straight forward calculations one obtains

$$\begin{aligned} \mathbb{E}(A(1) + A(1, 2) + A(1, 2, 3) + \dots) &= \frac{\alpha}{\delta - \alpha} = \alpha * (\delta - \alpha)^{* - 1} \\ \alpha &= \alpha_1 + \alpha_2 + \alpha_3 + \dots \\ \alpha_k &= \mathbb{E}\alpha[1, 2, \dots, k]. \end{aligned}$$

As  $\alpha[1, 2, \dots, k] = 0$  if  $u_i > \tau$  for any of the  $u_i, i = 1, \dots, k - 1$  the  $L_1$ - norm of  $\alpha_k$  is  $\leq 2^{k-1}(1 - e^{-c\tau})^{k-1}$  and the sum defining  $\alpha$  converges for  $c\tau \leq \ln 2$ .

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## Transformation of Quantum Lévy Processes

MICHAEL SCHÜRMAN

Let  $(\mathcal{B}, \Delta, \delta)$  and  $(\mathcal{C}, \Lambda, \lambda)$  be two dual semigroups in the sense of D. Voiculescu [Voi] with comultiplications  $\Delta, \Lambda$  and counits  $\delta, \lambda$ . Suppose that there is also given a  $*$ -algebra homomorphism  $\kappa : \mathcal{C} \rightarrow \mathcal{B}$  such that

$$\delta \circ \kappa = \lambda.$$

Then, if  $\psi$  is the generator of a quantum Lévy process (QLP)  $j_{st}$  on  $\mathcal{B}$  in the sense of [BGSch2],  $\psi \circ \kappa$  is the generator of a QLP  $k_{st}$  on  $\mathcal{C}$ . The QLP  $k_{st}$  can be constructed from the QLP  $j_{st}$  by a ‘convolution product integral’.

Applying this construction to the dual semigroups  $(T(\mathcal{B}_0), T(\Delta_0), T(0))$  and  $(T(\mathcal{B}_0), T(\iota_1 + \iota_2), T(0))$ , it can be seen that  $j_{st}$  can be realized on a Fock space (the type of which depends on the chosen independence) and that the vacuum vector is cyclic for this realization.

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**Theory of Integration in Izumi-Srinivasan Generalized CCR Flows**

KALYAN B. SINHA

(joint work with R. Srinivasan)

This is an account of a preliminary attempt to formulate the concepts of adaptedness and integration in the context of the theory of generalized CCR flow, constructed by Izumi and Srinivasan [1] as examples of a type III product systems.

In this set-up, we shall work in  $H = L^2(\mathbb{R}_+)_{\mathbb{R}}$  (for our purposes, the multiplicity space  $K$  is not so important) and we are given two  $C_0$ -semi groups  $T = (T_t)_{t \geq 0}$ ,  $S = (S_t)_{t \geq 0}$  in  $H$  (of which  $S$  is the right shift in  $L^2(\mathbb{R}_+)_{\mathbb{R}}$ , i.e  $(S_t f)(x) = f(x - t)$  for  $x > t$ ,  $= 0$  for  $x \leq t$ ) satisfying,

- (i)  $T_t^* S_t = I$  and
- (1)  $(ii) T_t - S_t \in \mathbf{B}_2(H)$  for all  $t \geq 0$ .

Then the flow  $\varphi$  defined by:

(2) 
$$\varphi_t(W(f)) = W(S_t f_R + iT_t f_I)$$

with  $f = f_R + if_I$ , gives a generalized CCR flow which extends to a  $E_0$ -semi group on  $\mathbf{B}(\Gamma(H^{\mathbb{C}}))$ .

Thus there are two different “shifts”-one for the real part and the other for the imaginary part; hence, we can expect that the definition of ‘past’ and ‘future’ will be different for real and imaginary parts, unlike the traditional Hudson-Parthasarathy theory.

Set  $p_t = I - S_t T_t^*$ ,  $t \geq 0$  Then one has .

- LEMMA 1:** (i)  $p_t$  is a real idempotent, not necessarily self-adjoint,  
 (ii)  $\text{Range}(p_t) = \text{ker}(T_t^*)$  and  $\text{Range}(p_t^*) = \text{ker}(S_t^*)$ ,  
 (iii)  $p_{t+s} = p_s + S_s p_t T_t^*$  (and we write  $p_t^s = S_s p_{t-s} T_s^*$ , for  $0 \leq s < t$ ),  
 (iv)  $p_s p_t^s = p_t^s p_s = 0$  for  $0 \leq s < t$ ,  
 (v)  $p_s p_t = p_t p_s = p_s \wedge t$ ,  
 (vi)  $p_t - p_t^* = 2i \text{Im} (T_t - S_t) S_t^* \in \mathbf{B}_2(H)$  and  $p_t = p_t^*$  if and only if  $T_t = S_t$ .

As has been argued in [1] we can assume without loss of generality that  $p_{\infty} = I$ . We write  $f_t = p_t f_R + ip_t^* f_I$  and  $f^t = p^t f_R + ip^{t*} f_I$

(3) 
$$\text{where } p^t = I - p_t = S_t T_t^*.$$

so that  $f = f_t + f^t$  in  $H^{\mathbb{C}}$

The Weyl operators behave well with respect to this splitting.

- LEMMA 2:** (i)  $W(f) = W(f_t)W(f^t) = W(f^t)W(f_t)$  for  $f \in H^{\mathbb{C}}$  and every  $t \geq 0$ .  
 (ii) More generally, if we set  $f_t^s = p_t^s f_r + ip_t^{s*} f_I$ ,  $W(f_t^s)$  and  $W(f_t^{s'})$  commute if

(4) 
$$[s, t] \cap [s', t'] = \emptyset.$$

If we now set  $B_{s,t}$  = the von Neumann algebra generated by  $\left\{ W(f_t^s) \mid f \in L^2(\mathbb{R}_+)_{\mathbb{C}} \right\}$ , then it is clear that  $B'_t = B^t$  where we have set  $B_t = B_{[0,t]}$  and  $B^t = B_{[t,\infty]}$ .

Because of the structure given above, it seems that the integrators here would be the “position” and “momentum” operators rather than the annihilation and creation operators in  $\Gamma(H^{\mathbb{C}})$ . We define  $P(f_R)$  and  $Q(f_I)$  as

$$W(\lambda f_R) = e^{i\lambda P(f_R)} \text{ and } W(i\lambda f_I) = e^{i\lambda Q(f_I)} \text{ and note the commutation relation:}$$

$$(5) \quad [P(f_R), Q(g_I)] = 2i \left\langle f_R, g_I \right\rangle,$$

with the other P-P and Q-Q commutators vanishing. With this definition, it is clear that both  $P(p_t^s f_R)$  and  $Q(p_t^{s*} f_I)$  commute with  $B[s', t']$  whenever  $[s, t] \cap [s', t'] = \emptyset$

We say that a family  $X \equiv \{X_t, t \geq 0\}$  of bounded operators in  $\Gamma(H^{\mathbb{C}})$  adapted with respect to the filtration  $B_t$  if  $X_t \in B_t \forall t \geq 0$  and it is said to be regular if  $t \rightarrow X_t$  is strongly continuous.

With these concepts, one can next define the integrals of an adapted process with respect to the integrators as above. However, the convenient expressions are yet to be derived and suitable “quantum Ito formula” has to emerge.

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### Some Comments on Product Systems and Markov Processes

LUIGI ACCARDI

Following our suggestion, in the conference talks [1, 2] it has been pointed out how the product system of Hilbert modules associated with the Markov semigroup of a classical stationary Markov process over  $\mathbf{R}_+$  can be obtained directly from the process.

In this talk, as a motivation, we first describe a version for the nonstationary case, and then generalize this to Markov fields. The field version requires a concise definition of a *family of admissible subsets* of  $\mathbf{R}^d$  which allows, in some sense, to resemble that some subsets are “further outside” than others. After giving such a definition, we obtain a far reaching generalization of product systems indexed by subsets of  $\mathbf{R}^d$  with an interior and an exterior boundary which consists of Hilbert bimodules over the two local algebras related to the two boundaries.

In the end, we make some considerations about time reflection. We suspect a deep interrelation of time reflection with the commutant of von Neumann bimodules, and pose it as an **open problem** to investigate this connection.

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## Open Problems

In this section we reproduce several open problems that were discussed at the Mini-Workshop.

### B.V.R. BHAT: TYPE OF INDUCTIVE LIMITS OF FINITE-DIMENSIONAL INCLUSION SYSTEMS

Let  $\{E_t : t \in \mathbb{R}_+\}$  be a finite-dimensional inclusion systems, i.e. a family of finite-dimensional Hilbert spaces with isometries

$$\beta_{s,t} : E_{s+t} : E_{s+t} \rightarrow E_s \otimes E_t, \quad , s, t \geq 0,$$

satisfying the usual assumptions of associativity, etc. Then  $\{E_t : t \in \mathbb{R}_+\}$  can be embedded into a product system  $\{\hat{E}_t : t \in \mathbb{R}_+\}$ .

Question: Is it possible to get product systems of type II or III in this way?

It may be noted that finite dimensional inclusion systems of CP semigroups on  $M_n$  give rise to type I product systems. For more information about the context of the problem and references see the abstract of the Bhat's talk on page 526.

### CLAUS KÖSTLER: DOES SPREADIBILITY IMPLY BRAIDABILITY?

In [1, 2], exchangeability, braidability, and spreadibility for sequences of quantum random variables were introduced. It was shown that they imply tail independence. Furthermore, exchangeability implies braidability, and braidability implies spreadability. We have constructed examples that prove that these notions are stronger than tail independence and that spreadability does not imply exchangeability. In view of these results, it remains still open if spreadability implies already braidability.

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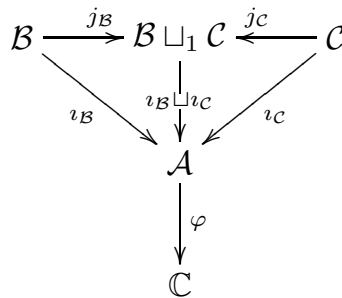
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VOLKMAR LIEBSCHER: ON QUANTUM INDEPENDENCE

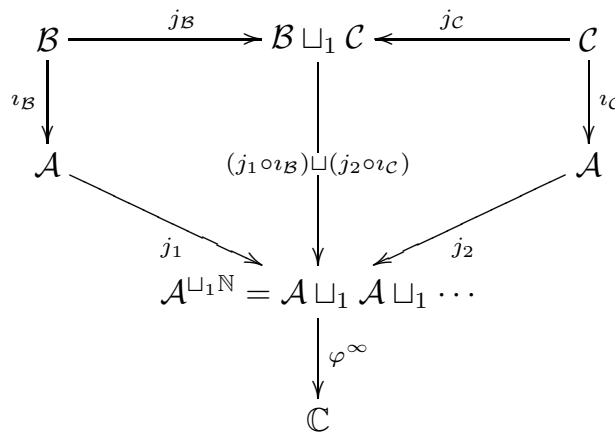
One might argue, that de Finetti's Theorem (see talk of Claus Köstler) may serve as a basis to define independence of algebras under certain states.

To make the idea clear, we propose the following definition for two subalgebras  $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$  in an algebraic probability space  $(\mathcal{A}, \varphi)$ .

Note that the state  $\varphi$  on  $\mathcal{A}$  induces a state  $\varphi \circ (\iota_{\mathcal{B}} \sqcup \iota_{\mathcal{C}})$  on the free product  $\mathcal{B} \sqcup_1 \mathcal{C}$  of the two subalgebras, as a consequence of the universal property of the free product.



If we embed the subalgebras into two different (“independent”) copies of  $\mathcal{A}$  inside  $\mathcal{A} \sqcup_1 \mathcal{A} \sqcup_1 \dots$ , e.g., the first and the second, then any state  $\varphi^\infty$  on  $\mathcal{A} \sqcup_1 \mathcal{A} \sqcup_1 \dots$  induces a state  $\varphi^\infty \circ ((j_1 \circ \iota_{\mathcal{B}}) \sqcup (j_2 \circ \iota_{\mathcal{C}}))$  on the free product  $\mathcal{B} \sqcup_1 \mathcal{C}$ .



We will call  $\mathcal{B}$  and  $\mathcal{C}$  *independent*, if there exists an extremal exchangeable (i.e.  $S^\infty$ -invariant) state on  $\mathcal{A} \sqcup_1 \mathcal{A} \sqcup_1 \dots$  such that the two states

$$\varphi \circ (\iota_{\mathcal{B}} \sqcup \iota_{\mathcal{C}}) \quad \text{and} \quad \varphi^\infty \circ ((j_1 \circ \iota_{\mathcal{B}}) \sqcup (j_2 \circ \iota_{\mathcal{C}}))$$

coincide. In the classical (i.e. commutative) case, exchangeability and extremality imply that  $\varphi^\infty$  is a product state (therefore corresponding to a product measure), and we recover the usual notion of independence.

This definition leaves several open questions behind:

- (1) We break the dependence on the embedding algebra  $\mathcal{A}$  explicitly. The reason is that possibly the embedding  $\mathcal{A} \hookrightarrow \mathcal{A}'$  need not map the set of extremal  $S^\infty$ -invariant (or exchangeable) states on  $\mathcal{A}'^{\sqcup_1 \mathbb{N}}$  onto a superset of the extremal  $S^\infty$ -invariant states on  $\mathcal{A}^{\sqcup_1 \mathbb{N}}$ . Is this caution necessary?



- (2) Is the above definition natural, i.e. associative and invariant under algebra morphisms?
- (3) Do the 5 natural products obey this definition? The tensor product should do due to [1]. The monotone products are not symmetric, so it seems likely that they are not covered by the above definition. For the free and boolean products, extremality is open but expected.
- (4) Are there examples besides the 5 natural products?
- (5) What changes, if we substitute the condition “extremal  $S^\infty$ -invariant” by “ $S^\infty$ -invariant with trivial tail field”? In the tensor case, both conditions are equal.
- (6) It is not clear to me, whether something changes in the category of von Neumann algebras (only there the tail-field makes full sense) and normal states or  $C^*$ -algebras.

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#### ORR SHALIT: EMBEDDING A SUBPRODUCT SYSTEM IN A PRODUCT SYSTEM

Let  $\mathcal{S}$  be a semigroup. A *subproduct system* (over  $\mathcal{S}$ ) is a bundle of  $W^*$ -correspondences  $X = \{X(s)\}_{s \in \mathcal{S}}$  over the  $W^*$ -algebra  $\mathcal{M} := X(0)$ , such that there exist coisometric bimodule maps

$$U_{s,t} : X(s) \otimes X(t) \rightarrow X(s+t)$$

that compose associatively, i.e.,

$$U_{s+t,r} (U_{s,t} \otimes I_{X(r)}) = U_{s,t+r} (I_{X(s)} \otimes U_{t,r}).$$

This notion has been formalized in [8], but it originates from Arveson’s works [1] and [2].

A subproduct system is called a *product system* when each map  $U_{s,t}$  is a unitary. Product systems of Hilbert spaces (over the semigroup  $\mathbb{R}_+$ ) have been introduced in [1] in relation to noncommutative dynamics. See [3] for a detailed account of the theory of product systems of Hilbert spaces, or [9] for a survey of product systems of Hilbert  $C^*$ -correspondences (over the semigroup  $\mathbb{R}_+$ ). Product systems over other semigroups were probably first studied by Fowler, see [5].

**Definition 1.** Let  $X$  and  $Y$  be subproduct systems of  $\mathcal{M}$  correspondences ( $\mathcal{M}$  a  $W^*$ -algebra) over the same semigroup  $\mathcal{S}$ . Denote by  $U_{s,t}^X$  and  $U_{s,t}^Y$  the coisometric maps that make  $X$  and  $Y$ , respectively, into subproduct systems.  $X$  is said to be a *subproduct subsystem of  $Y$*  (or simply a *subsystem of  $Y$*  for short) if for all  $s \in \mathcal{S}$  the space  $X(s)$  is a closed subspace of  $Y(s)$ , and if the orthogonal projections  $p_s : Y(s) \rightarrow X(s)$  are bimodule maps that satisfy

$$(1) \quad p_{s+t} \circ U_{s,t}^Y = U_{s,t}^X \circ (p_s \otimes p_t), \quad s, t \in \mathcal{S}.$$

Embedding a subproduct system as a subsystem of a product system plays an essential role in the dilation theory of CP-semigroups [8]. It is easy to see that every subproduct system  $X$  over  $\mathbb{N}$  is a subproduct subsystem of the product system  $\{X(1)^{\otimes n}\}_{n \in \mathbb{N}}$ . It is harder to show that every subproduct system over  $\mathbb{R}_+$  is a subsystem of a product system - this has been done in [4, 6, 7]. It seems that the same result for  $\mathbb{N}^2$  follows from [10]. In [8] we showed that every subproduct system of finite dimensional Hilbert spaces over  $\mathbb{N}^k$  is a subproduct subsystem of a product system over  $\mathbb{N}^k$ . Can this always be done?

Explicitly, let  $\mathcal{S}$  be either  $\mathbb{N}^k$  or  $\mathbb{R}_+^k$ . Is every subproduct system over  $\mathcal{S}$  a subproduct subsystem of a product system? A positive solution will immediately lead to new dilation theorems for CP-semigroups over  $\mathbb{N}^k$  and  $\mathbb{R}_+^k$ . A negative answer will be fantastically interesting.

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#### MICHAEL SKEIDE: SOME PROBLEMS

**On the Universality of the Module Approach.** Let  $T$  be a Markov semigroup on a unital  $C^*$  or von Neumann algebra  $\mathcal{B}$ . Suppose we have a general dilation of  $T$  to an  $E_0$  or automorphism semigroup  $\vartheta$  on a  $C^*$  or von Neumann algebra  $\mathcal{A}$ . In the introduction we proposed to pass, via GNS-construction for the expectation  $\mathfrak{p}: \mathcal{B} \rightarrow \mathcal{A}$ , to a dilation where the semigroup  $\vartheta$  acts on a  $\mathcal{B}^a(E)$  and where  $\mathfrak{p} = \langle \xi, \bullet \xi \rangle$  for some unit vector  $\xi \in E$ .

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\
 \downarrow i & & \uparrow \mathfrak{p} \\
 \mathcal{A} & \xrightarrow{\vartheta_t} & \mathcal{A}
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\
 \downarrow i & & \uparrow \langle \xi, \bullet \xi \rangle \\
 \mathcal{B}^a(E) & \xrightarrow{\vartheta_t} & \mathcal{B}^a(E)
 \end{array}$$

To do so, it is necessary and sufficient that: Firstly,  $\vartheta$  respects the kernel of the GNS-representation of  $\mathcal{A}$  on  $E$ , so that it gives rise to an  $E_0$ -semigroup on the image of  $\mathcal{A}$  in  $\mathcal{B}^a(E)$ . Secondly, the  $E_0$ -semigroup on the subset of  $\mathcal{B}^a(E)$  extends to all of  $\mathcal{B}^a(E)$ .

We conjecture that this is always possible, at least if the algebra  $\mathcal{A}$  is not too big. By this we mean, for instance, that  $\mathcal{A}$  is generated by the image of  $\mathcal{B}$  and its time shifts. (If  $\mathcal{A}$  has parts that are essentially unrelated to the dilation, then it should be possible to violate our requirements.) Our conjecture is supported by the observation, formulated very carefully, that there is not a single example of a dilation where it was explicitly known that our conjecture is false.

Our first open problem is: Prove the conjecture or, otherwise, give a counter example. For that is indispensable to find a “good” formulation for that  $\mathcal{A}$  is not “too big”.

Suppose we have an automorphic dilation  $\vartheta$  (after the first part, acting on  $\mathcal{B}^a(E)$ ). In all existing cases, the automorphism semigroup  $\vartheta$  will be inner, that is, will be implemented by a unitary semigroup in  $\mathcal{B}^a(E)$ . Unfortunately, the product system of an inner automorphism semigroup is the trivial one.

Our second open problem asks if every such inner automorphic dilation (fulfilling suitable minimality conditions) may be restricted to a proper  $E_0$ -semigroup on the “future part” of the algebra  $\mathcal{B}^a(E)$ . More precisely, suppose that  $E_+ = \overline{\text{span}} \vartheta_{\mathbb{R}_+} \circ i(\mathcal{B})\xi \subset E$ , so that  $\vartheta$  gives rise to an  $E_0$ -semigroup  $\vartheta'$  on  $\overline{\text{alg}} \vartheta_{\mathbb{R}_+} \circ i(\mathcal{B}) \subset \mathcal{B}^a(E_+)$ . Does this  $E_0$ -semigroup extend to all of  $\mathcal{B}^a(E_+)$ ? Moreover, does there exist a unital embedding of  $\mathcal{B}^a(E_+)$  into  $\mathcal{B}^a(E)$  such that the extension of  $\vartheta'$  may be identified with the restriction of  $\vartheta$  to  $\mathcal{B}^a(E_+) \subset \mathcal{B}^a(E)$ ? Note that by the representation theory [MSS06], this means that  $E$  factors as  $E = E_+ \odot F$  for some multiplicity bimodule  $F$ . As a case study, we propose to investigate the two-sided time shift on the  $q$ -Fock space over the whole axis  $\mathbb{R}$ .

A positive answer to the first question will show that all dilations maybe discussed conveniently in the language of Hilbert modules. A positive answer to the second question shows that even if we start with an automorphic dilation, then we always find a proper  $E_0$ -semigroup on a subalgebra  $\mathcal{B}^a(E_+)$ . Both together would show that our approach proposed as the main theme of the workshop — discuss dilations in terms of Hilbert modules and classify them by product systems — is universal.

**Problems around Spatial Markov Semigroups.** A Markov semigroup is *spatial*, if it *dominates* and *elementary* CP-semigroup, that is, if there is a (sufficiently continuous) semigroup  $c = (c_t)_{t \in \mathbb{R}_+}$  of elements in  $\mathcal{B}$  such that the difference  $T_t - c_t^* \bullet c_t$  is CP for all  $t$ . Spatial Markov semigroup on a  $C^*$ -algebra are

always uniformly continuous with a Christensen-Evans generator

$$\mathcal{L}(b) = \langle \zeta, b\zeta \rangle + b\beta + \beta^*b,$$

where  $\zeta$  is an element in a Hilbert bimodule  $F$  and  $\beta \in \mathcal{B}$ . Their product systems always embed into a *time ordered product system* of Fock modules (which is even completely spatial), but they need not coincide; see [BLS08]. Spatial Markov semigroups on von Neumann algebras have a richer structure. Their product systems are always spatial. But only if the semigroup is uniformly continuous we can show that it embeds into and, therefore, coincides with a time ordered product system; see [BLS04]. There is a huge literature on spatial Markov semigroups on  $\mathcal{B}(H)$ .

We collect some open questions. Are there nontrivial nonspatial Markov semigroups on  $\mathcal{B}(H)$ ? (By nontrivial we mean that they are not nonspatial  $E_0$ -semigroups or just tensor products of such with a spatial Markov semigroup.) Are there spatial Markov semigroups on  $\mathcal{B}(H)$  whose product system is non-Fock (unlike quantum Ornstein-Uhlenbeck)? Which nonspatial Markov semigroups on a von Neumann algebra  $\mathcal{B} \subset \mathcal{B}(H)$  admit spatial extensions to  $\mathcal{B}(H)$ ? (Recall that the product system of the Markov semigroups on  $L^\infty(\mathbb{R})$  of classical Brownian motion and classical Ornstein-Uhlenbeck processes are all nonspatial, but their quantum extensions to  $\mathcal{B}(L^2(\mathbb{R}))$  are even type I.) More generally, if a Markov semigroup  $T$  on  $\mathcal{B}$  admits an extension to  $\mathcal{C} \supset \mathcal{B}$ , then the product system of  $T$  sits inside the product system of its extension. In how far such an embedding does change the basic classification of the product system?

There is the theory of form generators for Markov semigroups on  $\mathcal{B}(H)$ ; see [CF98]. They are all spatial. Are these all spatial Markov semigroups, or are there other spatial ones? Some of them could be dilated to a symmetric Fock space, others not. Are the latter possibly candidates for Markov semigroups that are not type I? Form generators of Markov semigroups on  $\mathcal{B}(H)$  resemble the Lindblad form of the uniformly continuous case. Do generators of spatial Markov semigroups on general von Neumann algebras resemble the Christensen-Evans form [CE79] of bounded generators? Is it possible to adapt the theory of [CF98] to general von Neumann algebras?

**$n$ -Dimensional Free Flows and Their Units.** We mentioned already several open problems about free flows and their free and tensor product systems in our talk; see Page 528. Here we wish to propose a generalization to  $n$ -parameter free flows. This is closely related to the talks of Shalit (Page 519) and Skalski (Page 531).

The idea is the following: Once we have an  $n$ -parameter  $E_0$ -semigroup  $\vartheta$  on  $\mathcal{B}^a(E)$ , one may try to find all (unital) units of its product system. Such a unit gives, then, necessarily rise to an  $n$ -parameter Markov semigroup. Question: Are these semigroups products of strongly commuting (see Shalit's talk) one-parameter semigroups or not?

To follow this idea it is, therefore, necessary to construct  $n$ -parameter  $E_0$ -semigroups as candidates. We propose a general procedure starting from free product systems, and pose the open problem to put the above program into practise.

Suppose, we have  $n$  one-parameter free product systems  $E^{i*}$ . (We described in our talk, how to obtain them starting with  $n$  spatial product systems.) Then the free products

$$E_{(t_1, \dots, t_n)} := E^1 * \dots * E^n$$

form an  $n$ -parameter free product system. Doing the inductive limit over the reference vector (as discussed for the one-parameter case in our talk), we obtain an  $E_0$ -semigroup (a noise, actually). In the case of the full Fock module over  $E = L^2(\mathbb{R}_+^n, F)$ , we obtain the  $E_0$ -semigroup induced by the  $n$ -dimensional right shift. We know all the units in the one-dimensional case (see [Ske06b]), and it appears natural to adapt the same methods to more parameters. We do not exclude the possibility that (due to the high degree of noncommutativity in free products) there are no units but the trivial one (consisting of the reference vectors). We simply do not know. That is why we pose it as an open problem.

BORIS TSIRELSON: CAN WE GET TYPE II ARVESON SYSTEMS FROM LÉVY PROCESSES?

The Arveson system of a Lévy process on  $\mathbb{R}$  is always of type I, i.e. the units are total. The same holds for Lévy processes with values in Lie groups, since they can be constructed from an equivalent noise in the tangent space, i.e. Lie algebra. A similar result was shown for Brownian motions in the unitary group  $\mathcal{U}(H)$  of an infinite-dimensional separable Hilbert  $H$ , which is continuous in the strong topology, cf. [1]. Lévy processes with values in diffeomorphisms groups also give always rise to type I Arveson systems. On the other hand, it is possible to construct Lévy processes with values in semigroups, which give type II Arveson systems, cf. [2], Sections 2c, 2d, 4d.

It is unknown if this can happen for Lévy processes with values in groups. E.g., do there exist Lévy processes with values in a group of homeomorphisms (e.g. of the circle) whose Arveson system is of type II? This question was asked in 1998, see Conjecture 1.11 in [1], and is still open, see Sect. 8a in [2].

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