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Hilbert Modules and Complex Geometry

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Abstract. The major topics discussed in this workshop were Hilbert modules of analytic functions on domains in $\mathbb{C}^n$, Toeplitz and Hankel operators, the interplay of commutative algebra, complex analytic geometry and multivariable operator theory, coherent and quasi-coherent sheaves as localizations of Hilbert modules, Hilbert bundles and Jordan varieties on Cartan domains.

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Introduction by the Organisers

Much of the progress in multivariable spectral theory during the last decades was made possible by the use of methods from several complex variables, complex analytic and algebraic geometry. The language of Hilbert modules has become an effective tool and a unifying framework for the systematic development of this side of operator theory. The purpose of the meeting was to bring together researchers from these areas to assess the current state of the field and to identify crucial problems whose solution will be of central importance for future progress.

The main topics included Hilbert modules of analytic functions on different types of domains in $\mathbb{C}^n$, spectral properties of the associated Toeplitz and Hankel operators, in particular Schatten - von Neumann properties of cross commutators and trace formulas in the spirit of Helton and Howe, classification of homogeneous vector bundles and homogeneous operators in the Cowen-Douglas class, applications of commutative algebra and complex analytic geometry to multivariable Fredholm theory, coherent and quasi-coherent sheaves as a tool to solve global problems by localization, non-commutative spectral theory and geometry for Banach algebras and the non-commutative disc algebra, homogeneous vector bundles over Cartan domains and group representations.

M. Putinar discussed the solvability of division problems of the form $Fu = f$ in Bergman spaces over domains G in $\mathbb{C}^n$ with smooth strictly convex boundary, where F is a matrix-valued analytic function defined in a neighbourhood of $\overline{G}$. Basic tools are the theory of Toeplitz operators and quasi-coherent localizations of the Bergman space. J. McCarthy studied pairs of commuting isometries that are algebraic in the sense that they are annihilated by a non-trivial polynomial. One of the main results shows that two cyclic algebraic pairs of commuting isometries with the same minimal annihilating polynomial are almost unitarily equivalent. G. Zhang studied Schatten - von Neumann, in particular trace and weak trace-class properties, of quotient modules of Hardy or Bergman spaces on the bidisc and indicated trace-class formulas for suitable self-commutators.

Let Z be an irreducible hermitian Jordan triple of rank r with associated symmetric domain B. H. Upmeier indicated how suitable boundary measures can be used to construct $\mathrm{Aut}(B)$-invariant Hilbert space bundles $(\mathcal{H}_z)_{z\in B}$ on B having a canonical connexion which is projectively flat. K.-H. Neeb described some recent results on semi-bounded representations of infinite-dimensional Lie groups. Representations on spaces of holomorphic vector bundles provide natural examples.

F.-H. Vasilescu gave a new definition of a Cayley transform, using the algebra of quaternions, and used this quaternionic Cayley transform to characterize 2×2-operator matrices which admit unbounded normal extensions. St. Richter showed that the extremals (in the sense of Agler) for the families of spherical contractions are given by a direct sum of a backward shift on a Drury-Arveson space and a spherical unitary. This and partial results on the extremals of the family of d-contractions is based on joint work with C. Sundberg. B. Wick explained recent results, obtained with S. Treil, on the connection between the operator-valued corona problem and the existence of bounded analytic projection-valued maps.

Let B be a finite Blaschke product with n zeros counting multiplicity and let M_B be the induced multiplication operator on the Bergman space $L^2_a(\mathbb{D})$. R. G. Douglas reported on recent joint work with S. Sun and D. Zheng on the reducing subspaces of M_B. The largest C^*-algebra in the commutant of M_B is finite dimensional and its dimension equals the number of connected components of the "Riemann surface" of the function $B^{-1} \circ B$ over $\mathbb{D}$. For $n \leq 8$, this number coincides with the number of non-trivial minimal reducing subspaces for M_B. Whether the same is true in general remains open. R. Yang proposed a new notion of joint spectrum for (non-commuting) elements in a Banach algebra and discussed the complex analytic and topological properties of the associated resolvent set.

M. Andersson discussed an analytic proof of a uniform Briançon-Skoda theorem for an analytic variety $Z \subset \Omega \subset \mathbb{C}^n$ obtained with H. Samuelsson and J. Sznajdman. A sharper result holds for ideals with few generators. The basic tool is a residue current associated with Z that is obtained from a free resolution of the structure sheaf $\mathcal{O}_Z = \mathcal{O}/I_Z$. X. Fang illustrated the applicability of methods from commutative algebra to multivariable operator theory in a few illuminating examples. J. Eschmeier showed how a base change and comparison theorem from commutative algebra can be used to deduce a limit formula for the generic values of the cohomology dimensions of the Koszul complex of a commuting tuple $T \in L(X)^n$ on its Fredholm domain. This limit formula calculates at the same time the Samuel multiplicity of the stalks of the cohomology sheaves of the induced complex of analytic sheaves and allows to estimate the growth of the cohomology groups $H^p(T^k, X)$ of the powers of T.

J. Arazy considered analytic continuations of Toeplitz operators and forms on Besov-type spaces over Cartan domains. In the case of the unit ball the analytic continuation is obtained via an integration by parts in the radial direction. M. Englis presented recent results on the Dixmier trace of Hankel operators on Hardy and Bergman spaces over strictly pseudoconvex domains. The resulting trace formulas are reminiscent of classical results of Helton and Howe. The ball case is due to joint work with R. Rochberg, the general case was studied by M.Englis, K. Guo and G. Zhang. R. Rochberg indicated a decomposition of the space of bilinear forms on the Hardy space over the unit disc into Hankel forms, obtained with S. Ferguson, and its connection with a variant of the Rankin Cohen Bracket operations.

K. R. Davidson discussed the classification and representation theory of semi-crossed products of the non-commutative disc algebra by an automorphism. Among other things one can show that two semi-crossed products given by automorphisms ϕ and ψ of the n-ball $\mathbb{B}_n$ are algebraically isomorphic if and only if ϕ and ψ are biholomorphically conjugate. M. T. Jury presented an operator space approach to Schur-Agler norms on convex balanced domains in $\mathbb{C}^n$. Examples include the Agler norm on the polydisc, the Drury-Arveson multiplier norm and many new cases.

C. Sundberg explained methods which allow the construction of rank-one perturbations of self-adjoint operators whose spectral measures possess prescribed properties. G. Misra reported on joint results with S. Biswas on localizations of analytic Hilbert modules over suitable domains Ω in $\mathbb{C}^n$. The aim is to replace the given Hilbert module M by a coherent subsheaf of $\mathcal{O}_\Omega$ which gives new information on the properties of M. A. Koranyi discussed a recent joint result with G. Misra which gives a complete list of all homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$. The classification is obtained via an explicit realization of all homogeneous Hermitian holomorphic vector bundles on the unit disc under the action of the universal covering group of the biholomorphic automorphism group of the unit disc. The workshop ended with a problem session organized by R. G. Douglas.

The varied background of the 23 participants from about ten different countries lead to a number of new joint projects started in Oberwolfach. The unique atmosphere of the research institute and a wonderful week of beautiful early spring contributed to the success of the meeting.

Workshop: Hilbert Modules and Complex Geometry

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Abstracts

Division in Bergman space of a convex domain

Mihai Putinar

Let Ω be a strictly convex domain, domain of $\mathbb{C}^d$, $d \geq 1$, with smooth boundary. Let $L^2_a(\Omega)$ denote the Bergman space of analytic functions in Ω which are square summable with respect to the $2d$-dimensional Lebesgue measure on Ω. Fix two positive integers m, n and consider a given vector $f \in L^2_a(\Omega) \otimes_{\mathbb{C}} \mathbb{C}^n$ and a matrix A of analytic functions defined in a neighborhood of $\overline{\Omega}$: $A \in M_{m,n}(\mathcal{O}(\overline{\Omega}))$. We study in this note the solvability of the linear equation

$$Au = f, \quad u \in L^2_a(\Omega) \otimes_{\mathbb{C}} \mathbb{C}^m,$$

in conjunction with an earlier work [11] dealing with the disk algebra instead of the Bergman space, and within the general concept of "privilege" introduced by Douday more than forty years ago [4, 5].

To fix notation, for an open set $G \subset \mathbb{C}^d$ we denote by $\mathcal{O}(G)$ the space of complex analytic functions in G and analogously for a closed set $F \subset \mathbb{C}^d$ the space of germs of analytic functions defined in a neighborhood of F is denoted by $\mathcal{O}(F)$. The sheaf of analytic functions is denoted by $\mathcal{O}$. The commutative Banach algebra $\mathcal{O}(\Omega) \cap C(\overline{\Omega})$ of analytic functions in Ω, which are continuous on the closure is denoted by $\mathcal{A}(\Omega)$. By a widely accepted abuse of terminology, we call $\mathcal{A}(\Omega)$ the *disk algebra* associated to the domain Ω.

Two independent steps are necessary to understand the nature of the above equation: first, the solution u may not be unique, simply due to the non-trivial relations among the columns of the matrix A. This difficulty is clarified by homological algebra: at the level of coherent analytic sheaves, $\mathcal{F} = \operatorname{coker}(A : \mathcal{O}|^m_{\overline{\Omega}} \longrightarrow \mathcal{O}|^n_{\overline{\Omega}})$ admits a finite free resolution:

$$0 \to \mathcal{O}|^{n_p}_{\overline{\Omega}} \xrightarrow{d_p} \cdots \to \mathcal{O}|^{n_1}_{\overline{\Omega}} \xrightarrow{d_1} \mathcal{O}|^{n_0}_{\overline{\Omega}} \to \mathcal{F} \to 0, \tag{1}$$

where $n_1 = m, n_0 = n$ and $d_1 = A$. The existence of such a resolution is assured by the analogue of Hilbert syzygies theorem in the analytic context, see for instance [8].

The second step, of circumventing the non-existence of boundary values for Bergman space functions, is resolved by a canonical quantization method, that is by passing to the algebra of Toeplitz operators with continuous symbol on $L^2_a(\Omega)$. We import below from the well understood theory of Toeplitz operators on domains of $\mathbb{C}^d$ is a crucial criterion for a matrix of Toepliz operators to be Fredholm , cf. [12, 8].

Following Douady [4, 5], the $\mathcal{A}(\Omega)$-module $F = \operatorname{coker}(A : L^2_a(\Omega)^m \longrightarrow L^2_a(\Omega)^n)$ is called *privileged with respect to the Bergman space* if it is a Hilbert module in the quotient metric and there exists a resolution

$$0 \to L^2_a(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to L^2_a(\Omega)^{n_1} \xrightarrow{d_1} L^2_a(\Omega)^{n_0} \to F \to 0, \tag{2}$$

where $d_q \in M(n_{q+1}, n_q;\ \mathcal{A}(\Omega))$. Note that implicitly in the statement is assumed that the range of the operator A is closed at the level of Bergman space.

Assume that the analytic matrix $A(z)$ is defined on a neighborhood of $\overline{\Omega}$. One proves by standard homological techniques that every free, finite type resolution of the analytic coherent sheaf $\mathcal{F} = \text{coker}(A : \mathcal{O}|_{\overline{\Omega}}^m \longrightarrow \mathcal{O}|_{\overline{\Omega}}^n)$ induces at the level of the Bergman space $L_a^2(\Omega)$ an exact complex, see [4]. The similarity between the two resolutions above is not accidental, as it will be revealed in the sequel.

The interest for Hilbert space privilege arose from the recent work on analytic Hilbert modules of R. G. Douglas and G. Misra. The author thanks them both for many illuminating discussions.

1. Main result

After understanding the disk-algebra privilege on a strictly convex domain [11], the main result of this note, stated below, is not surprising.

Theorem. *Let $\Omega \subset \mathbb{C}^d$ be a strictly convex domain with smooth boundary, let m, n be positive integers and let $A \in M_{m,n}(\mathcal{A}(\Omega))$ be a matrix of analytic functions belonging to the disk algebra of Ω. The following assertions are equivalent:*

a). The analytic module $\text{coker}(A : L_a^2(\Omega)^m \longrightarrow L_a^2(\Omega)^n)$ *is privileged with respect to the Bergman space;*

b). The function

$$\zeta \mapsto \text{rank } A(\zeta), \quad \zeta \in \partial\Omega,$$

is constant;

c). Let $f \in L_a^2(\Omega)^n$. The equation

$$Au = f$$

has a solution $u \in L_a^2(\Omega)^m$ if and only if it has a solution $u \in \mathcal{O}(\Omega)^m$.

Sketch of proof. Assume that the resolution 2 exists and that the last arrow has closed range. The exactness at each degree of the resolution is equivalent to the invertibility of the Hodge operator:

$$d_k^* d_k + d_{k+1} d_{k+1}^* : L_a^2(\Omega)^{n_k} \longrightarrow L_a^2(\Omega)^{n_k}, \quad 1 \le k \le p,$$

where we put $d_{p+1} = 0$. To be more specific: the condition $\ker[d_k^* d_k + d_{k+1} d_{k+1}^*] = 0$ is equivalent to the exactness of the complex at stage k, implying hence that $\text{ran}(d_{k+1})$ is closed. In addition, if the range of d_k is closed, then, and only then, the self-adjoint operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible.

Since the boundary of Ω is smooth, the commutator $[T_f, T_g]$ of two Toeplitz operators acting on the Bergman space and with continuous symbols $f, g \in C(\overline{\Omega})$ is compact, see for details and terminology [2, 12, 13]. Consequently for every k, $d_k^* d_k + d_{k+1} d_{k+1}^*$ is, modulo compact operators, a $n_k \times n_k$ matrix of Toeplitz operators with symbol

$$d_k(x)^* d_k(x) + d_{k+1}(x) d_{k+1}(x)^*, \quad x \in \overline{\Omega},$$

where the adjoint is now taken with respect to the canonical inner product in $\mathbb{C}^{n_k}$. According to the main result of [13], or [12], if the Toeplitz operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is Fredholm, then its matrix symbol is invertible. Hence

$$\ker[d_k(x)^* d_k(x) + d_{k+1}(x) d_{k+1}(x)^*] = 0, \quad 1 \leq k \leq p.$$

Thus, for every $x \in \partial\Omega$,

$$\text{rank} A(x) = \dim \text{coker}(d_1(x)) = n_0 - n_1 + n_2 - ... + (-1)^p n_p.$$

To prove the other implication, we rely on the disk algebra privilege criterion obtained in the note [11]. Namely, in view of Theorem 2.2 of [11], if the rank of the matrix $A(x)$ does not jump for x belonging to the boundary of Ω, then there exists a resolution of $E = \text{coker } A : \mathcal{A}(\Omega)^m \longrightarrow \mathcal{A}(\Omega)^n$ with free, finite type $\mathcal{A}(\Omega)$-modules:

$$(3) \qquad 0 \to \mathcal{A}(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{A}(\Omega)^{n_1} \xrightarrow{d_1} \mathcal{A}(\Omega)^{n_0} \to E \to 0.$$

As before, we denote $d_1 = A$. We have to prove that the induced complex (2), obtained after applying (3) the functor $\otimes_{\mathcal{A}(\Omega)} L_a^2(\Omega)$, remains exact and the boundary operator d_1 has closed range.

To this aim, we "glue" local resolutions of coker$A$ with the aid of Cartan's lemma of invertible matrices, as explained in [11]. For points close to the boundary of Ω such a resoltuion exists by the local freeness assumption, while in the interior, in neighborhoods of the points where the rank of the matrix A may jump, they exist by Douady's privilege on polydisks. this will prove that the Hilbert analytic module Coker$A : L_a^2(\Omega)^m \longrightarrow L_a^2(\Omega)^n$ is privileged with respect to the Bergman space.

As for assertion c), we simply remark that it is equivalent to the injectivity of the restriction map

$$\text{coker}(A : L_a^2(\Omega)^m \longrightarrow L_a^2(\Omega)^n) \longrightarrow \text{coker}(A : \mathcal{O}(\Omega)^m \longrightarrow \mathcal{O}(\Omega)^n).$$

The last co-kernel is always Hausdorff in the natural quotient topology as the global section space of a coherent analytic sheaf.

It is worth mentioning that for non-smooth domains Ω in $\mathbb{C}^d$ the above result is not true. For instance $\mathcal{A}(\Pi)$-privilege for a poly-domain Π was fully characterized by Douady [5]. On the other hand, even for smooth boundaries, the privilege with respect to the Fréchet algebra $\mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$ seems to be quite intricate and definitely different than the Bergman space or disk algebra privileges, as indicated by an observation of Amar [1].

Besides the expected relaxations of the main result above, for instance from convex to pseudoconvex domains, a natural problem to consider at this stage is the classification of the analytic Hilbert modules $F = \text{coker}(A : L_a^2(\Omega)^m \longrightarrow L_a^2(\Omega)^n)$ appearing in the Theorem above. This question fits into the framework of quasi-free Hilbert modules introduced in [7]. That the resulting parameter space is wild, there is no doubt, as all Artinian modules M (over the polynomial algebra) supported by a fix point $a \in \Omega$ enter into discussion. Specifically, we can take

$$M = \operatorname{coker}((f_1, ..., f_n) : L^2_a(\Omega)^n \longrightarrow L^2(\Omega)),$$

where $f_1, ..., f_n$ are polynomials with the only common zero $\{a\}$. Then in virtue of Theorem 2.1, the analytic module M is finite dimensional and privileged with respect to the Bergman space $L^2_a(\Omega)$. An algebraic reduction of the classification of all finite co-dimension analytic Hilbert modules of the Bergman space associated of a smooth, strictly convex domain can be found in [9, 10].

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Algebraic pairs of isometries

John E. McCarthy
(joint work with Jim Agler)

The purpose of the talk is to discuss a restricted class of pairs of commuting isometries $V = (V_1, V_2)$, namely ones that satisfy an algebraic relation: $q(V) = 0$ for some polynomial q of two variables. We shall call such a pair an *algebraic isopair*, and we shall say that an isopair is *pure* if both isometries are pure. Pure algebraic isopairs turn out to have a rich structure.

It is easy to find an algebraic isopair annihilated by the polynomial $z^2 - w^2$, but a moment's thought shows that none can be annihilated by $z^2 - 2w^2$. The polynomial $1 - zw$ can annihilate an isopair, but only if this is a pair of unitaries whose joint spectrum is contained in

$$\mathbb{T}^2 \cap \{(z, w) \ : \ 1 - zw = 0\}.$$

(We shall use the notation that $\mathbb{D}$ is the open unit disk $\{z : |z| < 1\}$, $\mathbb{T}$ is the unit circle $\{z : |z| = 1\}$, and E_λ is the exterior of the closed disk $\{z : |z| > 1\}$.) No pure isopair is annihilated by $1 - zw$.

What polynomials q can be the minimal annihilating polynomial for some pure isopair?

Theorem 1: *Let $V = (V_1, V_2)$ be a pure algebraic isopair on a Hilbert space $\mathcal{H}$. Then there exists a square-free inner toral polynomial q that annihilates V. Moreover, if p is any polynomial that annihilates V, then q divides p.*

A polynomial q is called an *inner toral polynomial* if its zero set lies in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup E_\lambda^2$; the zero set of an inner toral polynomial is called a *distinguished variety.*

Theorem 1 gives a way to construct algebraic isopairs. Start with an inner toral polynomial q; put a nice measure μ on $Z_q \cap \mathbb{T}^2$; construct the Hardy space $H^2(\mu)$ that is the closure in $L^2(\mu)$ of the polynomials; and look at the pair of operators on $H^2(\mu)$ given by multiplication by the coordinate functions. In a way that can be made precise, this construction in some sense gives you all cyclic algebraic isopairs.

However, they also arise in another setting. In [1, 2], it is shown that on every finitely connected planar domain R there is a pair of inner functions (u_1, u_2) that map the domain conformally onto some distinguished variety intersected with the bidisk. If ν is a measure on ∂R that is a log-integrable weight times harmonic measure, one can form a Hardy space $H^2(\nu)$ (provided every component in the complement of R has interior, this is just the closure in $L^2(\nu)$ of all functions analytic in a neighborhood of $\overline{R}$). Multiplication by u_1 and u_2 on $H^2(\nu)$ then give a pure cyclic algebraic isopair.

A q-isopair (an isopair annihilated by $q \in \mathbb{C}[z, w]$) can almost be broken up into a direct sum of isopairs corresponding to each of the irreducible factors of q. Specifically, we have:

Theorem 2: *Let $V = (V_1, V_2)$ be a pure algebraic isopair with minimal polynomial q, and let $q_1, q_2, \ldots, q_N$ be the (distinct) irreducible factors of q. If both V_1 and V_2 have finite dimensional cokernels, then V has a finite codimension invariant subspace $\mathcal{K}$ on which*

$$V\mid_{\mathcal{K}} = W_1 \oplus W_2 \oplus \cdots \oplus W_N$$

where W_j is a q_j-isopair, $j = 1, \ldots, N$.

The restriction to $\mathcal{K}$ is essential. Our main result says that any two pure cyclic algebraic isopairs are nearly unitarily equivalent if and only if they have the same minimal polynomial. "Nearly" means after restricting to a finite codimensional invariant subspace. So we say that two pairs are nearly unitarily equivalent if and only if each one is unitarily equivalent to the other restricted to a finite codimensional invariant subspace. We say a pair is nearly cyclic if, when restricted to a finite codimensional invariant subspace, it becomes cyclic. We have:

Theorem 3: *Any two nearly cyclic pure isopairs are nearly unitarily equivalent if and only if they have the same minimal polynomial.*

There is a function-theoretic consequence of the operator theory. Given a polynomial q, one can ask when $Y = Z_q \cap \mathbb{T}^2$ is polynomially convex. Apart from the trivial case of when q has factors of $(z - e^{i\theta})$ or $(w - e^{i\theta})$, the answer is that Y fails to be polynomially convex if and only if q has an inner toral factor.

Theorem 4: *Let q be a polynomial in two variables with no linear factors. Then $Y = Z_q \cap \mathbb{T}^2$ is polynomially convex if and only if q has no inner toral factor.*

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Trace formulas for quotient modules on the bidisk

Genkai Zhang

(joint work with Kunyu Guo, Kai Wang)

There is a rich theory of Schatten - von Neumann $\mathcal{L}^p$-properties of Toeplitz and Hankel operators on the unit ball in $\mathbb{C}^n$. On the qualitative side there is the characterization of the membership in $\mathcal{L}^p$ of Hankel operators with anti-holomorphic symbols f (studied by e.g. Arazy-Fisher-Janson-Peetre, Rochberg, Zhu.) On the quantitative side the Hilton-Howe formula computes the anti-commutator of the Toeplitz operators $\{T_{f_1}, T_{f_1}^*, \dots, T_{f_n}, T_{f_n}^*\}$ as an integration of the product of differentials df_j; in one variable case it is the Dirichlet norm of f, which is a prototype of the Berger-Shaw inequality. The Hankel operators (defined similarly as for the ball) with anti-holomorphic symbols on the polydisks are never compact and there is thus no such trace formula. However the quotient modules of the Hardy space $H^2 = H^2(D^2)$ on the bidisk D^2 behave in certain cases as Bergman spaces on the unit disk. It is therefore a natural question to classify those quotient modules with trace class properties.

Let M be an invariant subspace generated by homogeneous polynomials of H^2 of the multiplication operators M_z and M_w. We denote by S_f the compression on $M^\perp$ of the Toeplitz operator M_f. We call $M^\perp$, *p-essentially normal* if all the cross commutators among the operators $\{S_z^*, S_w^*, S_z, S_w\}$ are in the Schatten - von Neumann class $\mathcal{L}^p$. For quotient modules on the unit ball this property has been studied by, among others, Averson and Guo-Wang. One may refine the definition above by using the Macaev class $\mathcal{L}^{p,\infty}$, or the weak $\mathcal{L}^p$ class instead of $\mathcal{L}^p$. There arises then also the question of computing the Dixmier trace of the (power of) the commutators $[S_f^*, S_f]$.

It has been proved by R. Yang that up to a finite dimensional subspace, M is of the form $M = [p]$ for a single homogeneous polynomial p with $p = p_1 p_2$, where the zero sets $Z(p_1)$ and $Z(p_2)$ have the properties that

$$Z(p_1) \cap \partial D^2 = Z(p_1) \cap T^2 \tag{1}$$

and respectively

$$Z(p_2) \cap \partial D^2 = Z(p_2) \cap (\partial D^2 \setminus T^2), \tag{2}$$

where ∂D^2 is the topological boundary of D^2, so that $\partial D^2 \setminus T^2 = (T \times D) \cup (D \times T)$. The compact property has been studied by Guo-Wang. They proved that the quotient $[p]^\perp$ is compact if and only if $p = p_1 p_2$, with p_2 being one of the following polynomials:

$$1,\ (z - \alpha w),\ (z - \alpha w)(w - \beta z),\ \text{ for } |\alpha| < 1, |\beta| < 1.$$

In the present work we prove

Theorem 1. (1) The quotient module $[p]^\perp$ is trace class if and only if p is one of the following polynomials:

$$(z - \alpha_1 w)^{n+1}, \quad z - \alpha w, \quad (z - \alpha w)(w - \beta z),$$

where $|\alpha_1| = 1$, $|\alpha| < 1, |\beta| < 1$.

(2) The quotient module is in the weak trace class if and only if p is one of the following polynomials:

$$\prod_{j=1}^{k} (z - \alpha_j w)^{n_k+1}, \quad z - \alpha w, \quad (z - \alpha w)(w - \beta z),$$

where $|\alpha_j| = 1$, $\forall j$, and $|\alpha| < 1, |\beta| < 1$.

To prove the theorem we study first the quotient module $[(z - \alpha w)^n]^\perp$. For $|\alpha| = 1$ it has a decomposition in terms of vanishing degrees and Ferguson-Rochberg computed the matrix entries of the operator S_z under the decomposition. When there are more than one factors we prove certain p-orthogonal property (defined in terms of the product of the corresponding projections) of the quotient modules, and we estimate the eigenvalues of related operators between different subspaces.

Theorem 2. Let $F(z, w)$ be a polynomial.

(1) If $p = (z - \alpha w)^{N+1}$ for some $|\alpha| = 1$, then

$$\operatorname{Tr}[S_F^*, S_F] = (N + 1) \int_D |f'(w)|^2 dm(w), \quad f(w) = F(\alpha w, w).$$

(2) If $p = z - \alpha w$ for some $|\alpha| < 1$, then

$$\operatorname{Tr}[S_F^*, S_F] = \int_D |f'(w)|^2 dm(w), \quad f(w) = F(\alpha w, w).$$

(3) If $p = (z - \alpha w)(w - \beta z)$ for some $|\alpha| < 1, |\beta| < 1$, then

$$\operatorname{Tr}[S_F^*, S_F] = \int_D |f_1'(w)|^2 dm(w) + \int_D |f_2'(z)|^2 dm(z),$$

where $f_1(w) = F(\alpha w, w)$, $f_2(z) = F(z, \beta z)$.

We consider also the Dixmier trace of Hankel type operators and obtain similar equality as above.

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Hilbert bundles and flat connexions on Jordan varieties

Harald Upmeier

Let Z be an irreducible hermitian Jordan triple of rank r. Its open unit ball B is a Cartan domain (bounded symmetric domain), realized as $B = G/K$, where $G = \mathrm{Aut}(B)$ is the holomorphic automorphism group of B and $K \subset \mathrm{GL}(Z)$ is the maximal compact subgroup. The Jordan determinant varieties

$$Z_\ell := \{z \in Z | \mathrm{rank}(z) = \ell\}$$

are the $K^{\mathbb{C}}$-orbits corresponding to the boundary G-orbits $\partial_\ell B$ of B $(1 \leq \ell \leq r)$ under the well-known Matsuki duality.

The polynomial algebra over Z has a Peter-Weyl decomposition

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m}} \mathcal{P}_{\boldsymbol{m}}(Z)$$

under the natural K-action, where $\boldsymbol{m} = (m_1 \geq \ldots \geq m_r \geq 0)$ runs over all integer partitions of length r. The restricted sum

$$\mathcal{P}_\ell(Z) = \sum_{m_{\ell+1}=0} \mathcal{P}_{\boldsymbol{m}}(Z)$$

for partitions of length ℓ admits Z_ℓ as a set of uniqueness. We construct a measure μ on Z_ℓ and a G-action on Z_ℓ with the following properties:

(i) The completion $\mathcal{H}_\ell$ of $\mathcal{P}_\ell(Z)$ under the measure μ has an orthogonal projection given by the Jordan theoretic Bessel function [1] for parameter $\ell a/2$, where a is the characteristic multiplicity.

(ii) The G-invariant Hilbert space bundle $(\mathcal{H}_z)_{z\in B}$ over B, constructed via the G-action on Z_ℓ and the fibre $\mathcal{H}_\ell$ at the origin $0 \in B$, has a canonical connexion which is projectively flat [2].

(iii) The corresponding parallel transport (generalized Bogolyubov transformations) can be expressed in Jordan theoretic terms, via the Bergman operator and the Jordan-Bessel function.

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Convexity and Complex Geometry in Unitary Representation Theory

Karl-Hermann Neeb

We are interested in a systematic approach to unitary representations of infinite-dimensional Lie groups in terms of boundedness conditions on spectra in the derived representation.

Let G be a Lie group (modeled on a locally convex space) with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$ and a smooth exponential function $\exp_G \colon \mathfrak{g} \to G$. See [Ne06] for details on infinite dimensional Lie theory. For any unitary representation $\pi \colon G \to \mathrm{U}(\mathcal{H})$ which is smooth in the sense that the subspace

$$\mathcal{H}^\infty := \{v \in \mathcal{H} \colon \pi^v \in C^\infty(G, \mathcal{H})\}, \quad \pi^v(g) := \pi(g)v,$$

of smooth vectors in $\mathcal{H}$ is dense (which is automatic for continuous unitary representations of finite-dimensional groups), we consider on the projective space

$$\mathbb{P}(\mathcal{H}^\infty) := \{[v] = \mathbb{C}v \colon 0 \neq v \in \mathcal{H}^\infty\}$$

the *momentum map*

$$\Phi_\pi \colon \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}', \quad \Phi_\pi([v])(x) := \frac{-\langle i\mathtt{d}\pi(x)v, v\rangle}{\langle v, v\rangle},$$

where

$$\mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v, \quad v \in \mathcal{H}^\infty, x \in \mathfrak{g}$$

denotes the derived representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}^\infty$ and $\mathfrak{g}'$ is the topological dual of the locally convex space $\mathfrak{g}$. We now define the momentum set of the representation by

$$I_\pi := \overline{\mathrm{conv}(\mathrm{im}(\Phi_\pi))},$$

where the closure is taken in the weak-$*$-topology on $\mathfrak{g}'$. Since the momentum map Φ_π is equivariant with respect to the G-action on $\mathbb{P}(\mathcal{H}^\infty)$ by $g[v] := [\pi(g)v]$ and the coadjoint action on $\mathfrak{g}'$, given by $\mathrm{Ad}^*(g)f := f \circ \mathrm{Ad}(g)^{-1}$, the momentum set is a convex weak-$*$-closed subset invariant under the coadjoint action. From the invariance of $\mathcal{H}^\infty$ under $\pi(G)$ one derives that the operators $i\mathtt{d}\pi(x)$ on $\mathcal{H}^\infty$ are essentially selfadjoint, and this in turn leads to the relation

$$\sup\langle I_\pi, x\rangle = -\inf \mathrm{Spec}(i\mathtt{d}\pi(x)) \in \mathbb{R} \cup \{\infty\}$$

for $x \in \mathfrak{g}$, which shows that the information encoded in I_π is precisely the lower bounds of the spectra of the operators $i\mathtt{d}\pi(x)$.

We call π *bounded* if I_π is equicontinuous and *semi-bounded* if I_π has a weaker property which we call *semi-equicontinuity*, i.e., its *semipolar set*

$$\tilde{I}_\pi := \{x \in \mathfrak{g} \colon \inf\langle I_\pi, x\rangle \geq -1\}$$

has interior points. The semiboundedness condition implies in particular that the convex cone of all elements in $\mathfrak{g}$ for which the spectrum of $i\mathtt{d}\pi(x)$ is bounded from below has interior points. This leads to an invariant open convex cone in the Lie algebra $\mathfrak{g}$.

The following characterization of bounded representations makes this condition more accessible.

Theorem 1. ([Ne09]) *For a smooth representation $(\pi, \mathcal{H})$ of G the following are equivalent:*

(a) *π is bounded.*
(b) *$\mathcal{H}^\infty = \mathcal{H}$ and $\mathtt{d}\pi \colon \mathfrak{g} \to B(\mathcal{H})$ is a continuous morphism of topological Lie algebras.*
(c) *$\pi \colon G \to \mathrm{U}(\mathcal{H})$ is smooth as a map from the Lie group G to the Banach–Lie group $\mathrm{U}(\mathcal{H})$.*

One can show that each semi-equicontinuous set is in particular locally compact with respect to the weak-$*$-topology. For abelian groups, this leads to the following spectral theorem ([Ne09]):

Theorem 2. *If $G = (E, +)$ is the additive group of a locally convex space E and $(\pi, \mathcal{H})$ is a semibounded unitary representation of G, then there exists a Borel spectral measure P on the locally compact space I_π with*

$$\pi(v) = \int_{I_\pi} e^{i\alpha(v)}\, dP(\alpha).$$

If, conversely, $C \subseteq E'$ is a convex closed semi-equicontinuous subset and P is a $B(\mathcal{H})$-valued Borel spectral measure on C, then

$$\pi(v) := \int_C e^{i\alpha(v)}\, dP(\alpha)$$

defines a semibounded unitary representation of G with $I_\pi \subseteq C$.

In general it is hard to determine the momentum set I_π in concrete terms. However, if the representation $(\pi, \mathcal{H})$ can be realized in a space of holomorphic functions on a complex manifold or, more generally, a space of holomorphic sections of a line bundle, then we have the following variant of [Ne09, Thm. 2.7].

Theorem 3. *Let G be a Fréchet–Lie group acting smoothly by holomorphic maps on the complex manifold M and $(\pi, \mathcal{H})$ be a semibounded representation of G for which there exists a G-equivariant holomorphic map $\eta\colon M \to \mathbb{P}(\mathcal{H}^\infty)$ whose image spans a dense subspace of $\mathcal{H}$. If, for each $x \in \tilde{I}_\pi^0$, the action $(t, m) \mapsto \exp_G(tx)m$ of $\mathbb{R}$ on M extends to a holomorphic action of the upper half plane $\mathbb{C}_+$, then*

$$I_\pi = \overline{\mathrm{conv}}(\Phi_\pi(\eta(M))).$$

The preceding theorem is of particular interest for the case where G acts transitively on M because in this case $\mathcal{O}_\pi := \Phi_\pi(\eta(M))$ is a coadjoint orbit in $\mathfrak{g}'$.

For finite-dimensional groups, the semi-bounded representations are precisely the unitary highest weight representations and only groups with compact Lie algebras have bounded representations (cf. [Ne00]).

For infinite-dimensional groups, the picture is much more colorful. There are many interesting bounded representations, in particular all those coming from representations of C^*-algebras, and most of the unitary representations appearing in physics are semibounded (cf. [PS86] for the case of affine Kac–Moody groups, where semiboundedness is equivalent to the positive energy condition).

Representations of a C^*-algebra $\mathcal{A}$ are called *physically equivalent* if they lead to the same momentum set of the unitary group $\mathrm{U}(\mathcal{A})$. This property is equivalent to having the same kernel.

Complex analysis, resp., geometry shows up in this theory in many ways. First of all, representations in spaces of holomorphic vector bundles provide natural sources of semibounded representations for which the momentum set can often be described as the closed convex hull of a single coadjoint orbit, carrying a compatible Kähler structure (cf. Theorem 3; [Ne00b]). This is in particular the case for all irreducible representations of C^*-algebras (cf. [Ne02]). Furthermore, the spectral bounds permit us to use holomorphic extensions of one-parameter groups, which in many cases combine to holomorphic representations of complex semigroups, such as the compression semigroups of symmetric Hilbert domains ([Ne01]). Presently, we are interested in convexity properties of the coadjoint representation of a Lie group G on the dual space $\mathfrak{g}'$ which are necessary to understand the geometry of the momentum sets of unitary representations (cf. [Ne98] for applications of similar methods to some classes of bounded representations).

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Normal Extensions via Quaternionic Cayley Transforms

Florian-Horia Vasilescu

The classical Cayley transform $\kappa(t) = (t-i)(t+i)^{-1}$, $t \in \mathbb{R}$, can be extended to more general situations, as for instance that of (not necessarily bounded) symmetric operators in Hilbert spaces, which yields a homonymic transform due to von Neumann (see, for instance, [3]). A Cayley type transform may be actually defined for larger classes of operators, which are no longer symmetric, as well as for other objects, in particular for some linear relations (see, for example, [2]).

In order to find a formula of this type, valid for (generally unbounded) normal operators, one is leaded to consider a quaternionic framework, as made in [6]. If we slightly modify the basic definitions from [6], we get (in a simpler way) the properties of the quaternionic Cayley transform directly from those of von Neumann's Cayley transform, and refine some results from the quoted work. Moreover, this new construction does not require densely defined operators, and it applies to larger classes of operators.

Characterizations of those Hilbert space (bounded) operators having normal extensions are well known (results going back to Halmos and Bram). The corresponding problem, stated for unbounded operators (see, for instance, [4]), is more intricate (see [1], [5], [6], etc.). In fact, the main motivation of the introduction of the quaternionic Cayley transform in [6] was precisely to try to give an answer to this extension problem, with applications to some moment problems. Unlike in [1] and [5], we use the quaternionic Cayley transform to obtain normal extensions in a given Hilbert space. Moreover, we do not require the invariance of the domain of definition under the given operator, and get results for both densely defined operators and not necessarily densely defined ones.

Let $\mathcal{H}$ be a complex Hilbert space and let $D(T), N(T), R(T)$ be the domain, the null-space and the range of a linear operator T in $\mathcal{H}$, respectively

If I is the identity on $\mathcal{H}$, we set on $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ the operator matrices

$$\mathbf{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

and $\mathbf{E} = i\mathbf{J}, \mathbf{F} = i\mathbf{L}$. We have the relations

$\mathbf{J}^* = \mathbf{J}$, $\mathbf{K}^* = -\mathbf{K}$, $\mathbf{L}^* = \mathbf{L}$, $\mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I}, \mathbf{JK} = \mathbf{L} = -\mathbf{KJ}$, $\mathbf{KL} = \mathbf{J} = -\mathbf{LK}$, $\mathbf{JL} = \mathbf{K} = -\mathbf{LJ}, \mathbf{E}^* = -\mathbf{E}$, $\mathbf{E^2} = -\mathbf{I}$, $\mathbf{F}^* = -\mathbf{F}$, $\mathbf{F^2} = -\mathbf{I}$.

Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric. We may correctly define the operator $V : R(S{+}\mathbf{E}) \mapsto R(S{-}\mathbf{E}),\quad V(S{+}\mathbf{E})x = (S{-}\mathbf{E})x,\quad x \in D(S)$, which is a partial isometry. In other words, $V = (S - \mathbf{E})(S + \mathbf{E})^{-1}$, defined on $D(V) = R(S + \mathbf{E})$.

The operator V will be called the $\mathbf{E}$–*Cayley transform* of S.

Similarly, we may define an $\mathbf{F}$–*Cayley transform* of S.

Because the two Cayley transforms are alike, we mainly deal with the $\mathbf{E}$–Cayley transform.

Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry. Then the inverse V^{-1} is defined on the subspace $D(V^{-1}) = R(V)$. Now, suppose that the operator $\mathbf{I} - V$ is injective. Then the operator $S : R(\mathbf{E}(V - \mathbf{I})) \mapsto \mathcal{H}^2$, given by $S(\mathbf{E}(V - \mathbf{I})x) = (V + \mathbf{I})x$, $x \in D(V)$, is well defined and will be called the *inverse* $\mathbf{E}$*–Cayley transform* of the partial isometry V. In other words, $S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}$ on $D(S) = \mathbf{E}R(\mathbf{I} - V)$.

We may define, in a similar way, the *inverse* $\mathbf{F}$*–Cayley transform*, having similar properties.

Let $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ the set of those operators $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ with the properties (i) $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$; (ii) $\mathbf{J}T$ is symmetric; (iii) $T\mathbf{K} = \mathbf{K}T$; (iv) $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ for all $x \in D(T)$.

Let also $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ be the set of those partial isometries $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that (a) $V^{-1} = -\mathbf{K}V\mathbf{K}$; (b) $\mathbf{I} - V$ is injective; (c) $\mathbf{E}R(\mathbf{I} - V) = R(\mathbf{I} - V)$ and $(\mathbf{I} - V)^{-1}\mathbf{E}(\mathbf{I} - V)$ is an isometry on $D(V)$.

It can be shown that the $\mathbf{E}$–Cayley transform is a bijective map from $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ onto $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

If $\mathcal{U}(\mathcal{H}^2)$ is the set of all unitaries in $\mathcal{H}^2$, we put $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \{U \in \mathcal{U}(\mathcal{H}^2); U^* = -\mathbf{K}U\mathbf{K}, N(\mathbf{I}{-}U) = \{0\}, (U{+}U^*)\mathbf{E} = \mathbf{E}(U{+}U^*)\}$, that is, those unitary operators whose inverse $\mathbf{E}$–Cayley transform is a normal operator. Therefore, if $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) = \{S : D(S) \subset \mathcal{H}^2 \to \mathcal{H})^2; S \text{ normal}, (\mathbf{J}S)^* = \mathbf{J}S, \mathbf{K}S = S\mathbf{K}\}$, then the map $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) \ni S \mapsto (S - \mathbf{E})(S + \mathbf{E})^{-1} \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ is bijective.

The question concerning the existence of an extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ of an operator $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ is equivalent to the description of those partial isometries in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ having extensions in the family $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$.

Theorem. *Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be densely defined. The operator T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with $D(W) = R(T + \mathbf{E})^\perp$.*

For not necessarily densely defined operators we have:

Corolary. *Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be closed and let V be the $\mathbf{E}$–Cayley transform of T. The operator T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with the properties $D(W) = R(T + \mathbf{E})^\perp$ and $R(\mathbf{I} - V) \cap R(\mathbf{I} - W) = \{0\}$.*

Let us finally remark that if A, B is a pair of symmetric operators having a joint domain of definition in $\mathcal{H}$, setting $T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, we can explicitly describe the membership $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$, which leads to a characterization of the existence of a normal extension of $A + iB$, via the previous results.

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Extremals for the families of commuting spherical contractions and their adjoints.

Stefan Richter

(joint work with Carl Sundberg)

It is fair to say that the Sz. Nagy dilation theorem is of central importance for the theory of contraction operators on Hilbert spaces. One version of this theorem states that every contraction on a Hilbert space can be extended to a co-isometric operator acting on a larger Hilbert space. Because of the known structure of the co-isometric operators, this means that one can use the function theory of the Hardy space of the unit disc to study arbitrary contractions.

Partial extensions of Sz. Nagy's theorem are available for the study of tuples of operators. The best known result is Ando's theorem which says that for any pair of commuting contraction operators S and T acting on a Hilbert space $\mathcal{H}$, there is a pair U, V of commuting co-isometric operators acting on a larger space $\mathcal{K} \supseteq \mathcal{H}$ such that U extends S and V extends T, [2]. It is also known that a direct analogue of Ando's theorem fails for three or more commuting contractions. Ando's theorem relates the study of commuting contractions to function theory on the bidisc, while it remains an open problem to find an effective model for three or more commuting contractions. The spherical contractions and the row contractions are collections of operator tuples which have been studied recently and which can be associated with function theory in the unit ball of $\mathbb{C}^d$. A convenient way to approach many such theorems is through J. Agler's model theory (see [1]). In this note we will present some examples of this model theory for the multivariable context.

The following definition is from [1]. We will assume that all our Hilbert spaces are separable.

Definition 1. Let $d \geq 1$. A family $\mathcal{F}$ is a collection of d-tuples $T = (T_1, .., T_d)$ of Hilbert space operators, $T_i \in \mathcal{B}(\mathcal{H})$, such that:

(a) $\mathcal{F}$ is bounded, i.e. there exists $c > 0$ such that for all $T = (T_1, .., T_d) \in \mathcal{F}$ we have $\|T_i\| \leq c$ for all $i = 1, ..., d$,

(b) $\mathcal{F}$ is preserved under restrictions to invariant subspaces, i.e. whenever $T \in \mathcal{F}$ and $\mathcal{M} \subseteq \mathcal{H}$ such that $T_i\mathcal{M} \subseteq \mathcal{M}$ for all i, then $T|\mathcal{M} \in \mathcal{F}$,

(c) $\mathcal{F}$ is preserved under direct sums, i.e. whenever $T_n \in \mathcal{F}$ is a sequence of tuples, then $\oplus_n T_n \in \mathcal{F}$,

(d) $\mathcal{F}$ is preserved under unital * -representations, i.e. if $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a *-homomorphism with $\pi(I) = I$ and if $T = (T_1, .., T_d) \in \mathcal{F}$, then $\pi(T) = (\pi(T_1), .., \pi(T_d)) \in \mathcal{F}$.

For $d = 1$ some examples are given by the families of contractions, isometries, subnormal contractions, and hyponormal contractions. For $d > 1$ we will be interested only in families which consist of commuting tuples of operators. The family of commuting contractions has already been mentioned. The spherical contractions $\mathcal{F}$ are those commuting d-tuples $T = (T_1, .., T_d)$ of Hilbert space operators satisfying $\sum_{j=1}^{d} T_j^* T_j \leq I$. The collection of adjoint tuples $\mathcal{F}^*$ consists of the row contractions. They satisfy $\| \sum_{j=1}^{d} T_j x_j \|^2 \leq \sum_{j=1}^{d} \|x_j\|^2$ for all $x_1, ..., x_d$ in the Hilbert space. It is easy to check that both $\mathcal{F}$ and $\mathcal{F}^*$ form a family. An example of a row contraction is the d-shift $M_z = (M_{z_1}, ..., M_{z_d})$ acting on the Drury-Arveson space H_d^2. A tuple $U = (U_1, ..., U_d)$ of commuting operators is called a spherical unitary, if $\sum_{i=1}^{d} U_i^* U_i = I$, and if each U_i is a normal operator.

Suppose T is an operator tuple acting on a Hilbert space $\mathcal{H}$ and R is a tuple acting on $\mathcal{K}$. We will write $R \geq T$, if R is an extension of T, i.e. if $\mathcal{H} \subseteq \mathcal{K}$ is a subspace which is invariant for each R_i, and if $T_i = R_i|\mathcal{H}$ for all i.

Definition 2. Let $\mathcal{G}$ be a family. An operator tuple $T \in \mathcal{G}$ acting on $\mathcal{H}$ is called an extremal for $\mathcal{G}$, if and only if whenever $R \in \mathcal{G}$ satisfies $R \geq T$, then $\mathcal{H}$ reduces R, i.e. if and only if the only way to extend T to a tuple $R \in \mathcal{G}$ is by taking direct sums.

We shall write ext $\mathcal{G}$ for the extremals of the family $\mathcal{G}$. It is a theorem of J. Agler that every operator tuple in a family can be extended to an extremal [1] (also see [4]).

Thus it is an important question to identify the extremals of families of interest. We note that it is easy to see that the extremals for the family of contractions are the co-isometric operators, the extremals for the isometric operators are the unitary operators, and the extremals for the subnormal contractions are the normal contractions. It is unknown what the extremals for the hyponormal contractions are.

Next we discuss some examples for $d > 1$. Ando's theorem can be used to show that the pairs of two commuting co-isometric operators are extremal for the pairs of commuting contractions, and it is an open problem to identify the extremals for the d-tuples of commuting contractions if $d > 2$. On the other hand the extremals for the family of commuting isometries are easily identified as the tuples of commuting unitary operators.

Theorem 1. *Let $\mathcal{F}$ be the family of commuting spherical contractions, and let $T = (T_1, .., T_d)$ be a commuting operator tuple.*

Then the following are equivalent:

(1) $T \in \text{ext } \mathcal{F}$
(2) $T = S^* \oplus U$, *where U is spherical unitary and S is a direct sum of d-shifts,*
(3) (a) $\sum_{i=1}^d T_i^* T_i = P =$ *a projection,*
 (b) $\sum_{i=1}^d T_i T_i^* \geq I$,
 (c) *If $x_1, .., x_d \in \mathcal{H}$ with $T_i x_j = T_j x_i$, then there is an $x \in \mathcal{H}$ with $x_i = T_i x$ for all i.*

Note that (3c) says that the Koszul complex for T is exact at the second stage. The resulting extension theorem (i.e. that any $R \in \mathcal{F}$ has an extension T of the type as in (2)) had been known and is due to Müller-Vasilescu [5] and to Arveson [3].

For the family of row contractions we have partial results.

Theorem 2. *Let $\mathcal{F}^*$ be the family of commuting row contractions. Let $T \in \mathcal{F}^*$ and write $D_* = (I - \sum_{i=1}^d T_i T_i^*)^{1/2}$.*

(1) *If $D_* = 0$, then $T \in \text{ext } \mathcal{F}^*$.*
(2) *If D_* is onto, then $T \notin \text{ext } \mathcal{F}^*$.*
(3) *If D_* is a projection, then $T \notin \text{ext } \mathcal{F}^*$ if and only if there are $x_1, .., x_d \in \text{ran } D_*$ with $\sum_{i=1}^d \|x_i\|^2 > 0$ and $T_i x_j = T_j x_i$ for all i, j.*
(4) *If D_* has rank one, i.e. if $D_* = u \otimes u$ for some $u \neq 0$, then $T \in \text{ext } \mathcal{F}^*$ if and only if $\dim \text{span } \{u, T_1 u, .., T_d u\} \geq 3$.*

If $d = 1$, then part (1) of Theorem 2 describes all extremals (the co-isometric operators). For $d > 1$ the d-shift is an example of an extremal with $D_* \neq 0$. For the d-shift one verifies that D_* is a projection of rank 1, so its extremality can be derived either from part (3) or part (4) of Theorem 2.

If $S = (M_z, H_d^2)$ is the d-shift, and if $\mathcal{M} \subsetneqq H_d^2$ is invariant for S, then $T = P_{\mathcal{M}^\perp} S | \mathcal{M}^\perp \in \mathcal{F}^*$ and D_* has rank 1. Because of this one can use Theorem 2 to verify the following Corollary.

Corollary 3. *If $\mathcal{M} \neq H_d^2$ is an invariant subspace for the d-shift $S = (M_z, H_d^2)$, and if $\mathcal{L} = \{a + \sum_{i=1}^d b_i z_i : a, b_1, .., b_d \in \mathbb{C}\}$ denotes the collection of polynomials of degree less than or equal to one, then $T = P_{\mathcal{M}^\perp} S | \mathcal{M}^\perp \in \text{ext } \mathcal{F}^*$ if and only if $\dim \mathcal{M} \cap \mathcal{L} < d - 1$.*

Corollary 3 implies that there are extremals T for $\mathcal{F}^*$ whose defect operators are not projections. In light of Theorem 1 one might consider this fact a little surprising. In any case, it follows that part (3) of Theorem 2 does not cover all extremals.

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Bounded Analytic Projections and the Corona Problem

Brett D. Wick
(joint work with Sergei Treil)

The main results in this talk originally appeared in the paper [7] by the authors.

The Operator Corona Problem is to find a (preferably local) necessary and sufficient condition for a bounded operator-valued function $F \in H^\infty_{E_*\to E}$ to have a left inverse in $H^\infty_{E_*\to E}$, i.e., a function $G \in H^\infty_{E\to E_*}$ such that

$$\text{(B)} \qquad G(z)F(z) \equiv I \qquad \forall z \in \mathbb{D}.$$

In the literature, such equations are sometimes called Bezout equations, and "B" here is for Bezout. The simplest necessary condition for (B) is

$$\text{(C)} \qquad F^*(z)F(z) \geq \delta^2 I, \qquad \forall z \in \mathbb{D} \qquad (\delta > 0)$$

(the tag "C" is for Carleson). If condition (C) implies (B), we say that the Operator Corona Theorem holds. In the particular case when F is a column $F = (f_1, f_2, \dots, f_n)^T$ the Operator Corona Theorem is just the classical Carleson Corona Theorem.

The Operator Corona Theorem plays an important role in different areas of analysis; in particular, in Operator Theory (angles between invariant subspaces, unconditionally convergent spectral decompositions, see [1, 2, 5]) as well as in Control Theory and other applications.

The main motivation is that in the matrix case, all the information about the Corona Problem is encoded in the analytic family of subspaces (a holomorphic vector bundle) $\operatorname{Ran} F(z)$, $z \in \mathbb{D}$. So, a natural question arises: Is it possible to characterize condition (C) (or (B)) in purely geometric terms, i.e., in terms of the family of subspaces $\operatorname{Ran} F(z)$, $z \in \mathbb{D}$? It turns out that the answer is "yes", and such a characterization is given below.

The following surprising lemma of N. Nikolski provides the connection between the Corona Problem for F and the family of subspaces $\operatorname{Ran} F(z)$. Let Ω be a domain in $\mathbb{C}^n$ (in fact we can let Ω be a manifold).

Lemma 4 (Nikolski's Lemma)**.** Let $F \in H^\infty_{E_*\to E}(\Omega)$ satisfy

$$F^*(z)F(z) \geq \delta^2 I, \qquad \forall z \in \Omega.$$

Then F is left invertible in $H^\infty_{E_*\to E}(\Omega)$ (i.e., there exists $G \in H^\infty_{E\to E_*}(\Omega)$ such that $GF \equiv I$) if and only if there exists a function $\mathcal{P} \in H^\infty_{E\to E}(\Omega)$ whose values are projections (not necessarily orthogonal) onto $F(z)E$ for all $z \in \Omega$.

Moreover, if such an analytic projection $\mathcal{P}$ exists, one can find a left inverse $G \in H^\infty_{E\to E_*}(\Omega)$ satisfying $\|G\|_\infty \le \delta^{-1}\|\mathcal{P}\|_\infty$.

Instead of considering families of subspaces, we consider more "analytic" objects; namely, the families of orthogonal projections $\Pi(z)$ onto these subspaces. The function $\Pi(z)$ is not analytic, except in the trivial case of a constant function. The fact the family of subspaces $\operatorname{Ran}\Pi(z)$ is an analytic family (a holomorphic vector bundle) is expressed by the identity $\Pi\partial\Pi = 0$.

Let us now list the main results. Here Δ is the "normalized" Laplacian, $\Delta := \partial\overline{\partial} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

Theorem 5 (Main Result)**.** *Let $\Pi : \mathbb{D} \to B(E)$ be a $\mathcal{C}^2$ function whose values are orthogonal projections in E satisfying $\Pi\partial\Pi = 0$. Assume that there exists a bounded non-negative subharmonic function φ such that*

$$(1) \qquad \Delta\varphi(z) \ge |\partial\Pi(z)|^2 \qquad \forall z \in \mathbb{D}.$$

Then there exists a bounded analytic projection onto $\operatorname{Ran}\Pi(z)$, i.e., a function $\mathcal{P} \in H^\infty_{E\to E}$ such that $\mathcal{P}(z)$ is a projection onto $\operatorname{Ran}\Pi(z)$ for all $z \in \mathbb{D}$.

Moreover, if $0 \le \varphi(z) \le K$ for all $z \in \mathbb{D}$, then one can find $\mathcal{P}$ satisfying

$$\|\mathcal{P}\|_\infty \le 1 + 2\sqrt{(Ke^{K+1}+1)Ke^{K+1}}.$$

We note that, if there is a bounded analytic projection, both $\dim\operatorname{Ran}\Pi(z)$ and $\operatorname{codim}\operatorname{Ran}\Pi(z)$ are constant for all $z \in \mathbb{D}$. One of the main corollaries of the above results is the following theorem.

Theorem 6 (Operator Corona Theorem)**.** *Let $F \in H^\infty_{E_*\to E}$ satisfy the Corona Condition $F^*F \ge \delta^2 I$. Assume also that the orthogonal projections $\Pi(z)$ onto $\operatorname{Ran}F(z)$ satisfy assumption (1) of Theorem 5. Then F has a holomorphic left inverse $G \in H^\infty_{E\to E_*}$.*

Moreover, if the function φ from condition (1) satisfies

$$0 \le \varphi(z) \le K \qquad \forall z \in \mathbb{D},$$

then one can find the left inverse G satisfying

$$\|G\|_\infty \le \tilde{\delta}^{-1}\left(1 + 2\sqrt{(Ke^{K+1}+1)Ke^{K+1}}\right),$$

where $\tilde{\delta} := \operatorname{essinf}\{|F(z)e| : z \in \mathbb{T}, e \in E_*, |e| = 1\}$.

Theorem 6 with δ instead of $\tilde{\delta}$ in the estimate of $\|G\|_\infty$ is an immediate corollary of Lemma 4 and Theorem 5. The estimate with $\tilde{\delta}$ requires slightly more analysis.

If $F \in H^\infty_{E_*\to E}$ satisfies $F^*F \ge \delta^2 I$ and $\dim E_* < \infty$, an easy computation shows that the orthogonal projection $\Pi(z)$ satisfies condition (1). Therefore, the Operator Corona Theorem in the case $\dim E_* < \infty$ follows immediately from

Theorem 6. However, this result has been known for a long time as the Fuhrmann–Vasyunin Theorem, see [3]; see also [4] or [6] for the modern treatment with better estimates.

More genreally, the following proposition shows that in the case of finite dimension, or finite codimension, the above condition (1) of Theorem 5 is necessary, and so this presents no real restriction.

Proposition 7. *Suppose there exists a bounded analytic projection $\mathcal{P}(z)$ onto* $\operatorname{Ran}\Pi(z)$, $z \in \mathbb{D}$. *Assume also that either* $\dim \operatorname{Ran}\Pi(z) < \infty$ *or* $\operatorname{codim}\operatorname{Ran}\Pi(z) < \infty$. *Then condition* (1) *of Theorem 5 holds.*

Probably the most important new and non-trivial corollary in this direction is the following theorem, solving the operator Corona Problem in the case of finite codimension.

Theorem 8 (Finite Codimension Operator Corona Problem)**.** *Let $F \in H^\infty_{E_*\to E}$ satisfy the Corona Condition $F^*F \geq \delta^2 I$, and let* $\operatorname{codim}\operatorname{Ran}F(z) < \infty$. *Then F has a bounded analytic left inverse if and only if the orthogonal projections* $\Pi(z)$ *onto* $\operatorname{Ran}F(z)$ *satisfy assumption* (1) *of Theorem 5.*

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Complex geometry and reducing subspace

Ronald G. Douglas

Specific questions about concrete operators often reveal unexpected connections between operator theory and other parts of mathematics. Such is the case in studying the reducing subspaces for the Toeplitz-like operator M_B defined to be multiplicative by B on the Bergman space $L^2_a(\mathbb{D})$ for a finite Blaschke product B. In joint work with S. Sun and D. Zheng, the structure of the reducing subspaces is shown to be intimately connected with that of the "Riemann surface" R_B for the rational function $B(z) - B(w)$ or for the factorization of the numerator $P(z, w)$ of $B(z) - B(w)$ after the denominators are cleared.

If one considers the Toeplitz operator T_B on the Hardy space $H^2(\mathbb{D})$ instead, one knows it is unitarily equivalent to $T_z \otimes I_{\mathbb{C}^N}$, where N is the number of zeros of $B(z)$ counting multiplicity. The analogous result is not valid for M_B although it is now known that M_B and $M_z \otimes I_{\mathbb{C}^N}$ are similar.

A straightforward argument shows that M_B^* is in the $\mathcal{B}_N(\mathbb{D})$ class, introduced by M. Cowen and this researcher about thirty years ago, and hence defines a hermitian holomorphic, rank N vector bundle E_B over $\mathbb{D}$. (Here we use the dual bundle which is holomorphic.) As a result one knows that operators in the commutant of M_B can be expressed in terms of bundle maps. Equivalently, if one specifies a holomorphic frame for E_B over some open subset U of $\mathbb{D}$, an operator X in the commutant is determined by an $N \times N$ matrix of holomorphic functions on U. One way to obtain such a local frame is in terms of the local inverses introduced in this area by J. Thomson for the Riemann surface R_B.

Operators that doubly commute with M_B have a simpler structure and, in particular, unitaries that commute with M_B are determined by a unit vector in $\mathbb{C}^N$, where components corresponding to "disks in the stack" describing R_B, which can be connected by analytic continuation, are equal. As a result one shows that the dimension of the algebra $\mathfrak{A}_B$ of operators on $L_a^2(\mathbb{D})$ that doubly commute with M_B has dimension less than or equal to q, the number of connected components of R_B. The argument involving the unit vectors shows that the dimension of $\mathfrak{A}_B$ is no greater than q. One also defines a set of q operators in $\mathfrak{A}_B$ which are linearly independent to complete the proof of this main result.

A much stronger relation between M_B and R_B is suggested by the proofs. In particular, the local inverses define a metric on R_B which enables one to define a Bergman space on R_B, which we'll call $L_a^2(R_B)$. Bounded holomorphic functions of B can be seen to act on this space. One would like to show that M_B on $L_a^2(R_B)$ and M_B on $L_a^2(\mathbb{D})$ are unitarily equivalent but the natural proof doesn't quite work. A more specific unresolved question concerns the structure of $\mathfrak{A}_B$. In particular, is $\mathfrak{A}_B$ commutative and what is the multiplicities of its representation on $L_a^2(\mathbb{D})$. Finally, how does the action of the Galois group for the polynomial $P(z, w)$ determined by $B(z) - B(w)$ relate to the covering group of $\mathfrak{A}_B$ and the algebra $\mathfrak{A}_B$?

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Projective spectrum in Banach algebras

Rongwei Yang

For a collection of n elements in a unital algebra $\mathcal{B}$ over $\mathbb{C}$, how they interact with each other is an important subject of multivariable operator theory. If the n elements are commutative, then joint spectrum is a well-established good measurement and has been an important topic in multivariable operator theory (Curto [Cu], Hörmander [Hö] Ch3, Taylor [Ta], Vasilescu [Va]). Study of non-commuting cases, on the other hand, is relatively insufficient. In this talk we introduce a simple spectrum, namely *projective spectrum*, for tuples of elements (commuting or non-commuting), and report on some of its properties.

The classical spectrum $\sigma(A)$ of an element A in $\mathcal{B}$ is defined through the invertibility of $A - \lambda I$. In a certain sense $\sigma(A)$ can be viewed as a measurement of how $A = (A_1,\ A_2,\ ...,\ A_n)$ interacts with the unit I. The idea of projective spectrum is to set I free, and consider the invertibility of $z_1A_1 + z_2A_2$, or more generally, the invertibility of $A(z) := z_1A_1 + z_2A_2 + \cdots + z_nA_n$. This is a measurement of how the elements interact with each other. In literature, $A(z)$ is called a multiparameter pencil for the tuple A. The invertibility of $A(z)$ has been studied in various fields, most notably in Differential Equations (cf. [At]). Unlike classical notions of joint spectra, projective spectrum is valid for all tuples, not just commutative ones. This talk will report on some general properties of projective spectrum. Results reported in this paper are preliminary, but they serve the purpose of establishing connections with some other branches of mathematics.

Geometric properties of projective spectrum. When $\mathcal{B}$ is the matrix algebra $M_k(\mathbb{C})$, projective spectrums are degree k projective hypersurfaces. When A is a commutative tuple, its projective spectrum is a union of hyperplanes. The main result in this direction is regarding the complement of projective spectrum $\mathbb{C}^n \setminus P(A)$(which we shall call projective resolvent set) — when $\mathcal{B}$ is of certain type, for instance C^*, the complement is made of domains of holomorphy.

Topology of $P^c(A)$. Since the tuple A in general is of infinite dimensional nature, its projective resolvent can be very complicated. Nonetheless, with the aid of the Maurer-Cartan type form $A^{-1}(z)dA(z)$ and multilinear functionals on $\mathcal{B}$, we can establish a Chern-Weil type homomorphism (cf. [Ch]) from the algebra of *invariant* multilinear functionals to the de Rham cohomology algebra $H_d^*(P^c(A),\ \mathbb{C})$.

Many example will be given in this talk.

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Residue currents with prescribed annihilator ideals and applications to algebraic geometry

Mats Andersson

Let $\mathcal{O}$ be the sheaf of holomorphic functions (on a complex manifold) and let $\mathcal{J}$ be a coherent ideal sheaf. Together with E Wulcan we introduced a couple of years ago in [2] a residue current $R^{\mathcal{J}}$ with the property that its annihilator sheaf is precisely $\mathcal{J}$. The current is obtained from a free resolution of $\mathcal{O}/\mathcal{J}$ and it is essentially unique. One can consider R as an analytic representation of the ideal $\mathcal{J}$, and it is explicit enough to admit a transformation of various geometric and algebraic questions into analytic considerations. To exemplify the utility we present an analytic proof, see [1], of (a partly sharpened version of) a Briançon-Skoda type theorem due to Huneke -92.

Let now $\mathcal{O} = \mathcal{O}_0$ be the local ring of holomorphic functions at $0 \in \mathbb{C}^d$ and let $(a) = (a_1, \ldots, a_m)$ be an ideal in $\mathcal{O}$. Notice that $|a| = \sum |a_j|$ essentially only depends on the ideal. The classical Briançon-Skoda theorem, [4], is:

If $|\phi| \leq C|a|^{\mu+\ell-1}$*, then* $\phi \in (a)^\ell$*, where* $\mu = \min(m, d)$.

It can be reformulated algebraically as the inclusion $\overline{(a)^{\mu+\ell-1}} \subset (a)^\ell$ where the bar means integral closure. This theorem was first proved by L^2 methods -74, and only after several years algebraic proofs were obtained, [7] and [6]. In -94, in [3], a proof based on multivariable residue calculus appeared. Such a proof can be described in the following way: Given the ideal (a) one forms a certain residue current R^a of Bochner-Martinelli type. It turns out that if the holomorphic function ϕ annihilates R^a, i.e., $\phi R^a = 0$, then $\phi \in (a)$. This can be done by solving a sequence of $\bar\partial$-equations or by an explicit integral representation of the membership. One then verifies that the hypothesis in the theorem indeed implies that $\phi R^a = 0$.

If instead $\mathcal{O} = \mathcal{O}_{Z,x}$ is the local ring at a (non-regular) point x on an analytic space Z, then the Briançon-Skoda theorem is not true with $\mu = \min(m, n)$ in general even if $m = 1$. However, Huneke, [5], proved in -92 algebraically that there is a number μ such that the statement holds uniformly in (a) and ℓ. Assume that Z is embedded in $\mathbb{C}^n$ and let $\mathcal{J}$ be the radical ideal associated to Z so that $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}$. If then $R^{\mathcal{J}}$ is the current mentioned above, one can form the "product" $R^a \wedge R^J$ and prove that $\phi \in (a) + \mathcal{J}$, i.e., $\phi \in (a)$ in $\mathcal{O}_Z$, if $\phi R^a \wedge R^J = 0$. Finally one confirms that if μ is large enough the hypothesis implies that indeed $\phi R^a \wedge R^{\mathcal{J}} = 0$. The number μ so obtained is directly related to the complexity of the free resolution and thus an invariant of the ring $\mathcal{O}$. We also prove that if (a) has "few" generators in relation to the complexity of the singularities of Z in a certain way, then the statement holds with the same μ as in the regular case, see [1].

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Noetherian modules of Hilbert modules

Xiang Fang

0. Introduction

For a tuple of a commuting operators $T = (T_1, \cdots, T_n)$ acting on a separable Hilbert space H, there is an immediate algebraic framework of Douglas-Paulsen [2] to put the study of commuting operator tuples in a ring-module setting. There they consider general rings of function algebras. The simplest case is probably to let $A = \mathbb{C}[z_1, \cdots, z_n]$ be the polynomial ring in n complex variables. Then H is endowed with an A-module structure by

$$(p(z_1, \cdots, z_n), h) \in A \times H \to p(T_1, \cdots, T_n)h \in H.$$

A difficulty in the study of Hilbert modules is that the module H is essentially never Noetherian, that is, not finitely generated as a module over A, hence hard to apply fruitful machineries in commutative algebra.

The purpose of this note is to summarize a few ways to define Noetherian modules related to a Hilbert module. In particular, for any point $\lambda \in \rho_F(T)$ in the Fredholm domain of the tuple T, one can define $(2n + 3)$ finitely generated modules over Noetherian rings.

For the rest of this note we assume $\lambda = 0 \in \rho_F(T) = \sigma(T) \setminus \sigma_e(T)$.

1. The graded module

The first one we present here is the graded module first considered by Douglas-Yan [3]. We consider the polynomial ring A as a graded ring, graded naturally according to the degrees of homogeneous polynomials. Let $I = (z_1, \cdots, z_n)$ be the maximal ideal at the origin. Then one define a graded module $gr(H)$ by

$$gr(H) = (H/IH) \oplus (IH/I^2H) \oplus \cdots \oplus (I^{k-1}H/I^kH) \oplus \cdots .$$

Note that the above direct sum is taken in an algebraic sense, so for each element only finite many components are non-zero.

Since the ring A, regarded as a graded ring, is the coordinate ring of the tangent space at the origin, we can consider the module $gr(H)$ is, in a sense, representing the tangent space of the Hilbert module at the origin.

2. At the sheaf level

The second module $\mathcal{H}$ we will talk about is indeed the stalk of the sheaf model $\tilde{H}$ of the tuple T at the origin. The sheaf model is thoroughly explored in the monograph of Eschmeier-Putinar. The sheaf $\tilde{H}$ under consideration is a coherent analytic sheaf on the Fredholm domain of T. According to a result of Markoe, $\mathcal{H}$ is finitely generated over the Noetherian ring $\mathcal{O}_0$.

Recall that the sheaf model is defined as

$$\mathcal{O}(H)/(z-T)\mathcal{O}(H).$$

We observe that it can be regarded as the nth homology group of the Koszul complex of $(z-T)$ on $\mathcal{O}(H)$, hence it is reasonable to define homological version of the sheaf model.

Definition We call the ith homology sheaf $\tilde{H}_i$ of the Koszul complex of $(z-T)$ on $\mathcal{O}(H)$ to be the ith homological sheaf model of T. In particular, $\mathcal{H} = \tilde{H}_{n,0}$.

Then we have $(n+1)$ many Noetherian modules $\tilde{H}_{i,0}$ by looking at the stalks of the sheaves $\tilde{H}_i$, $i = 0, 1, \cdots, n$.

3. I-adic modules

Inspired by the Grothendieck's local cohomology theory we also define

$$\hat{H}_i = \operatorname{inj\,lim} H_i(T_1^k, \cdots, T_n^k),$$

the injective limit of Koszul homology of powers of T. Then $\hat{H}_i$ is a finitely generated module over the Noetherian ring of power series $\mathbb{C}[[z_1, \cdots, z_n]]$ for each $i = 0, 1, \cdots, n$.

In general it is easier for things to be equal when taking I-adic completion. For instance, one can look at the theorem of formal schemes in algebraic geometry. We conjecture that

Conjecture: The I-adic completion of $\tilde{H}_{i,0}$ is equal to $\hat{H}_i$ for each i.

This conjecture has recently been affirmatively verified by J. Eschmeier.

4. Hilbert polynomials and neighborhoods

Lastly we observe that $\tilde{H}_0$ represent a neighborhood of the Hilbert module in a small neighborhood of the origin and $\hat{H}_0$ in an even smaller neighborhood of the origin. When considering their tangent spaces by the graded module construction as considered by Douglas-Yan in [3], one should end up with the same object. This suggests that the Hilbert polynomials of H, $\tilde{H}_0$, and $\hat{H}_0$ should be the same, which is indeed true.

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Samuel multiplicity and Fredholm theory

Jörg Eschmeier

A commuting tuple $T = (T_1, \dots, T_n) \in L(X)^n$ of bounded operators on a complex Banach space X is Fredholm if all the cohomology groups $H^p(T, X)$ $(p = 0, \dots, n)$ of its Koszul complex $K^\bullet(T, X)$ are finite dimensional. The Fredholm index of T is defined as the Euler characteristic

$$\operatorname{ind}(T) = \sum_{p=0}^{n} (-1)^p \dim H^p(T, X).$$

It is well known that the essential resolvent set $\rho_e(T)$, that is, the set of all points $z \in \mathbb{C}^n$ such that $z - T$ is Fredholm, is open and that the index of $z - T$ is locally constant on $\rho_e(T)$, while the individual dimensions of the cohomology groups $H^p(z - T, X)$ are only upper semicontinuous as functions in z. The discontinuity points of these functions form proper analytic subsets of $\rho_e(T)$ (see [3]). The observation that T is Fredholm if and only if all cohomology sheaves $\mathcal{H}^p = H^p(z - T, \mathcal{O}^X_{\mathbb{C}^n})$ of the induced Koszul complex of Banach-free analytic sheaves are coherent near $0 \in \mathbb{C}^n$ allows the application of methods from complex analytic geometry. For instance, the Fredholm spectrum $\sigma(T) \cap \rho_e(T)$ of T is an analytic subset of the essential resolvent set, since it is the support of the direct sum of the coherent sheaves $\mathcal{H}^p | \rho_e(T)$.

Suppose that T is Fredholm. Then the stalks of the cohomology sheaves $\mathcal{H}^p$ at $z = 0$ are finitely generated modules over the Noetherian local ring $\mathcal{O}_0$ of all convergent power series at the origin. Hence there are rational polyomials q_p in one variable of degree $\leq n$, the Hilbert-Samuel polynomials of $\mathcal{H}_0^p$ with respect to the maximal ideal $m = (z_1, \ldots, z_n) \subset \mathcal{O}_0$, such that

$$\dim(\mathcal{H}_0^p / m^k \mathcal{H}_0^p) = q_p(k)$$

for sufficiently large k. The limits

$$e(z, \mathcal{H}_0^p) = n! \lim_{k\to\infty} \dim(\mathcal{H}_0^p / m^k \mathcal{H}_0^p)/k^n$$

define natural numbers which are called the multiplicities of z on $\mathcal{H}_0^p$ (see Chapter 7 in [6]).

Using the fact that the Koszul complex $K^\bullet(z-T, X)$ is quasi-isomorphic to an analytically parametrized complex $L^\bullet = (u^p(z), L^p)_{p=0}^n$ of finite-dimensional vector spaces on a small zero neighbourhood, and by applying a suitable base change theorem, one obtains vector-space isomorphisms

$$H^p(T^k, X) \cong H^p(z-T, \mathcal{O}_0^X/(z^k)\mathcal{O}_0^X) \cong H^p(u^\bullet, \mathcal{O}_0^{L^\bullet}/(z^k)\mathcal{O}_0^{L^\bullet})$$

for $p = 0, \ldots, n$ and $k \geq 1$ (Corollary 1.3 and Lemma 2.1 in [1]).

By Lech's limit formula the multiplicities $e(z, \mathcal{H}_0^p)$ can also be computed as

$$e(z, \mathcal{H}_0^p) = \lim_{k\to\infty} \dim\big(H^p(u^\bullet, \mathcal{O}_0^{L^\bullet})/(z^k)H^p(u^\bullet, \mathcal{O}_0^{L^\bullet})\big)/k^n.$$

A variant of a comparison theorem due to Grothendieck (Theorem 2.2 in [1] and Theorem 1.2 in [2]) and the observation that the multiplicities $e(z, \mathcal{H}_0^p)$ calculate the rank of the coherent sheaves $\mathcal{H}^p$ at $z = 0$, allow one to deduce that, for each connected open neighbourhood $U \subset \rho_e(T)$ of $0 \in \mathbb{C}^n$ and each $p = 0, \ldots, n$, there is a proper analytic subset $S_p \subset U$ such that the limit formula

$$\dim H^p(z-T, X) = \lim_{k\to\infty} \frac{\dim H^p(T^k, X)}{k^n} < \dim H^p(w-T, X)$$

holds for $z \in U \setminus S_p$ and $w \in S_p$ (Theorem 2.4 in [1]). In this sense the above limits calculate the generic values of the cohomology dimensions of the Koszul complex of $z - T$ near the origin $z = 0$.

A well-known result in commutative algebra says that, for an arbitrary Noetherian module E over a unital commutative ring R and an arbitrary n-tuple $x = (x_1, \ldots, x_n) \in R^n$, the vanishing conditions

$$\varprojlim_k H^p(x^k, E) = 0 \qquad \text{for } p = 0, \ldots, n-1$$

hold. Here the inverse limit is formed over the cohomology modules of the Koszul complexes $K^\bullet(x^k, E)$ of the powers $x^k = (x_1^k, \ldots, x_n^k)$. Suppose that, in addition, the tuple $x \in R^n$ is a multiplicity system on E, that is, the quotient module $E/\Sigma_{i=1}^n x_i E$ has finite length. Then by a result of Kirby the cohomology modules of x^k satisfy the growth conditions

$$L_R\big(H^p(x^k, E)\big) = \mathrm{O}(k^p) \qquad \text{as } k \to \infty$$

for $p = 0, \ldots, n$.

We use the above mentioned comparison theorem to show that, for a Fredholm tuple $T = (T_1, \ldots, T_n) \in L(X)^n$ of commuting bounded operators on a complex Banach space X, there are vector-space isomorphisms

$$\varprojlim_k H^p(T^k, X) \cong \varprojlim_k H^p(z - T, \mathcal{O}_0^X)/(z^k)H^p(z - T, \mathcal{O}_0^X)$$

for $p = 0, \ldots, n$. This answers a question posed by X. Fang in [5]. By Krull's intersection theorem and a suitable closed range theorem obtained in [4], for a Fredholm tuple $T \in L(X)^n$, the vanishing conditions

$$\varprojlim_k H^p(T^k, X) = 0 \quad \text{for } p = 0, \ldots, n-1$$

hold if and only if the tuple T satisfies a spectral property known as Bishop's property (β) locally at the origin $z = 0$. We show that, for a graded Fredholm tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ on a complex Hilbert space H, there exists a finitely generated graded $\mathbb{C}[z]$-module M such that the coordinate tuple $z = (z_1, \ldots, z_n)$ is a multiplicity system on M and such that there are cohomology isomorphisms

$$H^p(z^k, M) \cong H^p(T^k, H) \qquad (p = 0, \ldots, n,\ k \geq 1).$$

By applying the above mentioned results from commutative algebra, we obtain that graded Fredholm tuples $T \in L(H)^n$ on Hilbert spaces possess Bishop's property (β) at $z = 0$ and that their powers T^k satisfy the growth conditions

$$\dim_{\mathbb{C}} H^p(T^k, H) = \mathrm{O}(k^p) \quad \text{as } k \to \infty$$

for $p = 0, \ldots, n$. If $T \in L(H)^n$ is graded with respect to the orthogonal decomposition $H = \oplus_{k=0}^{\infty} H_k$ (but not necessarily Fredholm), then by a classical result of Hilbert, there is a polynomial $q \in \mathbb{Q}[x]$ of degree at most $n - 1$ such that $\dim H_k = q(k)$ for sufficiently large k. We show that $\deg(q) + 1$ is the dimension of the inessential right spectrum of T at $z = 0$.

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Analytic continuation of Toeplitz operators

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Let D be a Cartan domain of genus p in $\mathbb{C}^d$. For any Wallach point $\nu \in W(D)$ let $\mathcal{H}_\nu$ be the associated quasi-invariant Hilbert space of holomorphic functions on D. If $\nu > p-1$ then $\mathcal{H}_\nu$ coincides with the weighted Bergman space $L^2_a(D,\mu_\nu)$, where μ_ν is the associated quasi-invariant weighted probability measure on D. In this case, one can define and study the *Toeplitz operator* with symbol $\varphi \in L^\infty(D)$ on $L^2_a(D,\mu_\nu)$ in the usual way: $T^{(\nu)}_\varphi(f) := P_\nu(\varphi f)$, where $P_\nu : L^2(D,\mu_\nu) \to L^2_a(D,\mu_\nu)$ is the orthogonal projection.

We discuss the analytic continuation of the map $\nu \mapsto T^{(\nu)}_\varphi$ and present some partial results. Generally speaking, for $\nu \le p-1$ the space $\mathcal{H}_\nu$ need not be a Bergman space, and in most cases it is a Besov-type space. Hence, in order that the operator $T^{(\nu)}_\varphi$ (obtained by analytic continuation) be bounded on $\mathcal{H}_\nu$ one needs the boundedness of certain derivatives of the symbol φ in addition to its boundedness. In the case where D is the open Euclidean unit ball in $\mathbb{C}^d$ the analytic continuation is very explicit, since it is obtained by integration by parts in the radial parameter.

Hankel operators and the Dixmier trace

Miroslav Engliš
(joint work with K. Guo, R. Rochberg, G. Zhang)

Let $A^2(\Omega)$ denote the Bergman space of all holomorphic functions in $L^2(\Omega)$, where Ω is a domain in $\mathbf{C}^n$. The Toeplitz and the Hankel operator with symbol ϕ, $\phi \in L^\infty(\Omega)$, are defined by

$$T_f : A^2(\Omega) \to A^2(\Omega), \quad u \mapsto P(fu);$$
$$H_f : A^2(\Omega) \to L^2(\Omega) \ominus A^2(\Omega), \quad u \mapsto (I-P)(fu),$$

where $P : L^2(\Omega) \to A^2(\Omega)$ is the orthogonal (Bergman) projection.

For $\Omega = \mathbf{D}$, the unit disc in $\mathbf{C}$, and f holomorphic, it was shown by Arazy, Fisher, Janson and Peetre [1] that $H_{\overline{f}}$ belongs to the Schatten class $\mathcal{S}^p$, $1<p<\infty$, if and only if f is in the diagonal Besov space B^p; while for $0<p\le 1$, $H_{\overline{f}} \in \mathcal{S}^p$ only if $H_{\overline{f}} = 0$. Thus there is a *cut-off* at $p=1$. Similarly, for $\Omega = \mathbf{B}^n$, the unit ball of $\mathbf{C}^n$, $n \ge 2$, and f holomorphic, $H_{\overline{f}} \in \mathcal{S}^p$ if and only if $f \in B^p$ if $2n < p < \infty$, and $H_{\overline{f}} \in \mathcal{S}^p$ if and only if $H_{\overline{f}} = 0$ for $0 < p \le 2n$; thus there is a cut-off cut-off at $p = 2n$. The same result, with the cut-off again at $p=2n$, turns out to hold for any smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$, $n \ge 2$ (Li and Luecking [8]).

The aim of this talk is to present a supplement to these results involving the *Dixmier ideal* and the *Dixmier trace.*

Recall that a compact operator T on a Hilbert space belongs to the Schatten class $\mathcal{S}^p$ if and only if $\sum_j s_j(T)^p < \infty$, where $\{s_j(T)\}_{j=0}^\infty$ are the eigenvalues of

$(T^*T)^{1/2}$ (counting multiplicities) arranged in decreasing order. The *Dixmier ideal* $\mathcal{S}^{\mathrm{Dixm}}$ consists of all T such that

$$\sum_{j=1}^{N} s_j(T) = O(\log N).$$

Equipped with the norm $\|T\|_{Dixm} := \sup_{N\geq 1} \frac{1}{1+\log N} \sum_{j=1}^{N} s_j(T)$, $\mathcal{S}^{\mathrm{Dixm}}$ becomes a Banach space, lying strictly between $\mathcal{S}^1$ and any $\mathcal{S}^p$, $p > 1$. Let $\omega : l^\infty \to \mathbf{C}$ be a Banach limit, i.e. a bounded linear functional on l^∞ of norm 1 extending the usual limit on the subspace $c \subset l^\infty$ of all convergent sequences. For any positive operator $A \in \mathcal{S}^{\mathrm{Dixm}}$, the *Dixmier trace* of $A \in \mathcal{S}^{\mathrm{Dixm}}$ is then defined as

$$\mathrm{Tr}_\omega(A) := \omega\Big(\frac{1}{1+\log n} \sum_{j=1}^{n} s_j(A)\Big).$$

If ω satisfies a certain technical condition, then $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$, and Tr_ω can thus be unambiguously extended by linearity from the positive operators to all of $\mathcal{S}^{\mathrm{Dixm}}$. The value of $\mathrm{Tr}_\omega(A)$ in general depends on the choice of ω; the operator A is called *measurable* if $\mathrm{Tr}_\omega(A)$ is in fact the same for all ω. A good source for further information on the Dixmier trace is the book of Connes [3].

Our first main result is the following.

Theorem 4. If $f \in C^\infty(\overline{\mathbf{D}})$, then

$$\mathrm{Tr}_\omega(|H_f|) = \int_{\mathbf{T}} |\overline{\partial} f| \, d\sigma,$$

where $\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ for $z = x + yi \in \mathbf{C}$, and $d\sigma$ is the normalized arc-length measure on the unit circle $\mathbf{T}$. In particular, H_f is measurable.

To state the analogue of Theorem 4 for domains in higher dimension, we need to review some notions from analysis of several complex variables. Recall that a real-valued function Φ defined on a domain in $\mathbf{C}^n$ is called *strictly-plurisubharmonic* (*strictly-PSH* for short) if for any $z, v \in \mathbf{C}^n$, the function of one complex variable $t \mapsto \Phi(z + tv)$, $t \in \mathbf{C}$, is strictly subharmonic where defined. A bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary is called *strictly pseudoconvex*[1] if there exists a function r, strictly-PSH in a neighbourhood of the closure of Ω, such that

$$r < 0 \quad \text{on } \Omega, \quad \text{and} \quad r = 0, \ \|\nabla r\| > 0 \quad \text{on } \partial\Omega.$$

(One calls r a strictly-PSH *defining function* for Ω.)

Consider the *anti-holomorphic complex tangent space* $\mathcal{T}''$ to $\partial\Omega$; its fiber $\mathcal{T}''_x$ at a point $x \in \partial\Omega$ thus consists of all vectors X of the form $X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial \overline{z}_j}$, where $X_j \in \mathbf{C}$ and $\sum_j X_j \frac{\partial r}{\partial \overline{z}_j}(x) = 0$. The *Levi form* is the Hermitian form L'' on $\mathcal{T}''$ defined by

$$L''(X, Y) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \overline{z}_j} X_j \overline{Y}_k.$$

[1]This is not the usual definition, but it will do for our purposes.

It is a consequence of the strict pseudoconvexity that L'' is positive definite (independently of the choice of the defining function r). The *boundary d-bar operator* $\overline{\partial}_b$ from $C^\infty(\overline{\Omega})$ into the space $C^\infty(\partial\Omega, \mathcal{T}''^*)$ of smoothly varying linear functionals (forms) on $\mathcal{T}''$ is defined by[2]

$$\overline{\partial}_b f(X) := Xf.$$

(That is, $\overline{\partial}_b f$ is the restriction of the differential df to $\mathcal{T}''$.)

Recall finally that given any positive definite Hermitian form B on a vector space V, there is a canonically defined *dual form* B^* on the dual V^* of V. Namely, for any $\phi \in V^*$, there exists unique $v_\phi \in V$ (the predual of ϕ under B) such that

$$\phi(\cdot) = B(\cdot, v_\phi).$$

The dual form is then defined as

$$B^*(\phi, \psi) := B(v_\psi, v_\phi) = \phi(v_\psi) = \overline{\psi(v_\phi)}.$$

In particular, applying this construction to the Levi form L'' on $\mathcal{T}''$, we have the dual Levi form $\mathcal{L}$ on $\mathcal{T}''^*$. For any functions f, g smooth on the closure $\overline{\Omega}$ of Ω, the expression $\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b g)$ is thus a smooth function on the boundary $\partial\Omega$.

Our second main result is as follows.

Theorem 5. Let $\Omega \subset \mathbf{C}^n$ be smoothly bounded and strictly pseudoconvex. Then for any $2n$ functions $f_1, g_1, \dots, f_n, g_n \in C^\infty(\overline{\Omega})$, the product of the corresponding Hankel operators

$$H^*_{f_1} H_{g_1} \dots H^*_{f_n} H_{g_n} =: H$$

belongs to the Dixmier class, is measurable, and

$$\mathrm{Tr}_\omega(H) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \prod_{j=1}^{n} \mathcal{L}(\overline{\partial}_b g_j, \overline{\partial}_b f_j)\, d\mu,$$

where $d\mu := \frac{1}{2i^n} (\partial r - \overline{\partial} r) \wedge (\overline{\partial}\partial r)^{n-1}$.

For the special case of the unit ball $\mathbf{B}^n$, this result was originally proved by the speaker with K. Guo and G. Zhang in [4], using the so-called *pseudo-Toeplitz operators* of Howe [7]. For the disc and the strictly pseudoconvex domains, however, a different method was used in [5] and [6], respectively, based on the reduction to the boundary and application of the theory, due to Boutet de Monvel and Guillemin, of Toeplitz operators with pseudodifferential symbols [2]. For the disc, the full power of the Boutet de Monvel-Guillemin machinery can in fact be avoided by using the calculus of *discrete* or (*periodic*) pseudodifferential operators on the circle, developed by several authors (see e.g. [9]).

[2]Again, $\overline{\partial}_b f$ in fact depends only on the restriction of f to $\partial\Omega$, so that $\overline{\partial}_b$ can be defined as an operator from $C^\infty(\partial\Omega)$ into $C^\infty(\partial\Omega, \mathcal{T}''^*)$; but we will not need this refinement.

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Higher Order Hankel Forms

Richard Rochberg

For $\gamma > -1$ let A^2_γ be the weighted Bergman space, the spaces of holomorphic functions in $L^2(\mathbb{D}, (1-|z|^2)^\gamma dxdy)$, and let H^2 be the Hardy space, formally the limiting case A^2_{-1}. These are Hilbert spaces with reproducing kernels, the reproducing kernel for A^2_γ is $k_\alpha(z,w) = (1-\bar{w}z)^{-\alpha}$ with $\alpha = \gamma + 2$. (Here and throughout we will ignore normalizing numerical factors.) A Hankel form on one of these spaces is a bilinear form B with the property that $B(f,g)$ is a linear function of the pointwise product fg; for example $B(f,g) = f(0)g(0)$.

In a 1987 paper Janson and Peetre [JaPe] introduced a generalization of Hankel forms on the spaces A^2_γ, $\gamma = -1, 0, 1, \ldots$. They viewed $\mathcal{H}_\gamma$, the space of Hilbert Schmidt bilinear forms on A^2_γ as elements of the Hilbert space tensor product $A^2_\gamma \otimes A^2_\gamma$. The irreducible action of the Mobius group on A^2_γ induces an action on the tensor product. That action is not irreducible and when it is decomposed we obtain (1.1) below and an irreducible action of the Mobius group on each summand. The first summand corresponds in a natural way to the Hankel forms. Janson and Peetre named the forms associated with the other summands *Hankel forms of higher weight,* here referred to as *higher order.* Using Fourier theory and representation theory they obtained explicit descriptions of these classes of forms. Associated to $b \in A^2_{2\gamma+2k}$ is the bilinear form $H_{\gamma,k}(b)$ acting on A^2_γ given by $H_{\gamma,k}(b) = \langle B_{\gamma,k}(f,g), b\rangle_{A^2_{2\gamma+2k}}$ with

$$B_{\gamma,k}(f,g) = \sum_{r+s=k} \frac{(-1)^r\, k!}{r!s!(\gamma+2)_r(\gamma+2)_s} f^{(r)}(z)g^{(s)}(z) \tag{0.1}$$

where $(\alpha)_\beta = \alpha(\alpha+1)....(\alpha+\beta-1)$. There are similar but slightly more complicated formulas for bilinear forms on $A^2_{\gamma_1} \otimes A^2_{\gamma_2}$; again involving bidifferential operators, now $B_{\gamma_1,\gamma_2,k}$. It was pointed out in [JaPe] that the operators $B_{\gamma_1,\gamma_2,k}$ are essentially the transvectant operators of classical invariant theory.

1. Higher Order Hankel Forms Without Representation Theory

The papers [PeRo], [Ro], and [FR] include attempts to reformulate these ideas in ways that did not require an underlying group action.

Here is one approach. For B in $\mathcal{H}_\gamma$ define the bilinear form ΔB by $\Delta B(f,g) = B(zf,g) - B(f,zg)$. Classical Hankel forms are the B for which $\Delta B = 0$. With this as a starting point we say that B is a *Hankel*$_1$ *form of order* n if $\Delta^{n+1}B = 0$. We call such a form *special* if it is also orthogonal in $\mathcal{H}_\gamma$ to the Hankel$_1$ forms of order $n-1$.

Here is a second approach. Using the identification of $f \otimes g \in A^2_\gamma \otimes A^2_\gamma$ with the holomrphic function $f(z_1)g(z_2)$ we can identify $A^2_\gamma \otimes A^2_\gamma$ with $A^2_{\gamma,\gamma} = A^2_{\gamma,\gamma}(\mathbb{D}^2)$; the Hilbert space of holomorphic functions on the bidisk with reproducing kernel $k_\alpha(z_1,w_1)k_\alpha(z_2,w_2)$. Hence we can identify $\mathcal{H}_\gamma$ with $A^2_{\gamma,\gamma}$. With this identification the classical Hankel forms correspond to functions which are orthogonal to V_D, the subspace of $A^2_{\gamma,\gamma}$ consisting of functions that vanish on the diagonal $D = \{(\zeta,\zeta)\} \subset \mathbb{D}^2$. We have a splitting $A^2_{\gamma,\gamma} = V_D^\perp \oplus V_D$ and we continue this decomposition as follows. Let $V_D^n \ominus V_D^{n+1}$ be the orthocomplement of V_D^{n+1} in V_D^n. We then have the decomposition in (1.2)

$$\mathcal{H}_\gamma \approx A^2_\gamma \otimes A^2_\gamma \approx \bigoplus_{k=0}^{\infty} A^2_{2\gamma+2k} \tag{1.1}$$

$$\approx A^2_{\gamma,\gamma} \approx V_D^\perp \oplus \left(V_D^1 \ominus V_D^2\right) \oplus ... \oplus \left(V_D^n \ominus V_D^{n+1}\right) \oplus .... \tag{1.2}$$

We define *special Hankel*$_2$ *forms of order* n to be those whose "symbol functions" are in the summand $V_D^n \ominus V_D^{n+1}$ of the decomposition (1.2).

Using tools from the theory of Hilbert spaces with reproducing kernel and specific formulas for the reproducing kernels it was shown in [FR] that these approaches agree; the special Hankel$_1$ forms of order n, the special Hankel$_2$ forms of order n, and the Hankel forms of order n introduced by Janson and Peetre are the same.

There is also an explicit description of the map of $A^2_{\gamma,\gamma}$ to $\bigoplus_{k=0}^{\infty} A^2_{2\gamma+2k}$ which sets up this equivalence. Up to normalizing numerical factor the map of functions on the bidisk to functions on the disk given by

$$F(z,w) \to \sum_{r+s=k} \frac{(-1)^r\, k!}{r!s!(\gamma+2)_r(\gamma+2)_s} \partial_z^r \partial_w^s F|_{z=w=\zeta} \tag{1.3}$$

is zero on $\left(V_D^n \ominus V_D^{n+1}\right)^\perp$ and takes $V_D^n \ominus V_D^{n+1}$ isometrically onto $A^2_{2\gamma+2n}$.

2. Transvectants and Rankin Cohen Brackets

The transvectant operators of (0.1) and the intimately allied bidifferential operators, the Rankin Cohen bracket operators introduced in the theory of modular forms, have been an active area of research in recent years; the papers [OS], [EG] and [BTY] give some indication. One of the reasons for this activity is the insight of Zagier that these operators satisfy algebraic identities which allow them to be used to define a product on $\bigoplus_{k=0}^{\infty} A^2_{2\gamma+2k}$ making that sum into a graded associative algebra [CMZ].

3. Questions

My continuing interest in these questions is driven in part by wanting to find satisfying answers to two questions.

First, how do these ideas play out for general Hilbert spaces of holomorphic functions. Both of the approaches in the second section generalize. However in those approaches the explicit formulas such as (0.1) and (1.3) are derived from the combinatorics of the Taylor coefficients of the kernel functions. Hence the nature, in fact the existence, of similarly explicit formulas for more general space is unclear. The theory of quotient Hilbert modules [DMV] may be an appropriate framework for analysis of those more general cases.

Second, the associativity of the multiplication induced by the Rankin Cohen brackets has been "explained" by viewing the product as induced by a symbol calculus for operators, with the associativity of the product being a consequence of associativity of operator composition. A recent version and earlier references are in [Pe]. However such considerations seem unnatural in discussions of the space $\mathcal{H}_\gamma$ of bilinear forms or the space or the space $A^2_{\gamma,\gamma}$ of holomorphic functions on the bidisk. So far it seems unclear how to view this new product structure in those contexts.

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Semicrossed products of the non-commutative disc algebra

Kenneth R. Davidson

The results mentioned are contained in [2, 3], joint work with Elias Katsoulis.

The non-commutative disk algebra $\mathfrak{A}_n$ is the norm closed nonself-adjoint operator algebra generated by n isometries $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ with pairwise orthogonal ranges. In particular, we can consider $\mathfrak{A}_n$ as an algebra on the Fock space $\ell^2(\mathbb{F}_n^+)$ generated by the left creation operators, which are defined by $L_i\xi_w = \xi_{iw}$. A character α is determined by $\lambda = (\alpha(\mathfrak{s}_1), \dots, \alpha(\mathfrak{s}_n))$, which is an arbitrary point in the closed ball $\overline{\mathbb{B}_n}$ of $\mathbb{C}^n$. The Gelfand map carries $A \in \mathfrak{A}_n$ to a function $\hat{A}$ in the algebra generated by $\hat{\mathfrak{s}}_i = z_i$, and thus is analytic on $\mathbb{B}_n$ [4].

If φ is an automorphism of $\mathfrak{A}_n$, it induces a map $\hat{\varphi}$ of $\overline{\mathbb{B}_n}$ which is readily seen to be a biholomorphic automorphism. Conversely, given any conformal automorphism $\hat{\varphi} \in \mathrm{Aut}(\mathbb{B}_n)$ of the ball, Voiculescu [8] constructed a unitary U_φ on Fock space so that $\varphi(A) = U_\varphi^* A U_\varphi$. These are precisely the isometric automorphisms of $\mathfrak{A}_n$, so we have the identification $\mathrm{Aut}(\mathfrak{A}_n) \simeq \mathrm{Aut}(\mathbb{B}_n)$ [4].

Given an automorphism φ of $\mathfrak{A}_n$, we consider all *covariant representations* consisting of a completely contractive representation π of $\mathfrak{A}_n$ and a contraction K such that

$$\pi(A)K = K\pi(\varphi(A)) \quad \text{for all} \quad A \in \mathfrak{A}_n.$$

The universal algebra for such representations is called the semicrossed product, $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$. This is an operator algebra generated by $\mathfrak{A}_n$ and an element $\mathfrak{u}$ with the norm

$$\Big\| \sum_n \mathfrak{u}^n A_n \Big\| = \sup_{(\pi,K)} \Big\| \sum_n K^n \pi(A_n) \Big\|.$$

To understand the structure of this algebra, one needs to consider dilations of covariant representations. We show that one can always simultaneously dilate $\pi(L)$ to a row isometry and dilate K to a unitary. With some special techniques, we can further dilate so that the row isometry has Cuntz type (ranges summing to the identity). The automorphism φ extends to a $*$-automorphism of the Cuntz algebra $\mathcal{O}_n$. This yields structure theory for the *maximal dilations.*

Theorem 9. *If (π, K) is a covariant representation of $(\mathfrak{A}_n, \varphi)$ on a Hilbert space $\mathcal{H}$, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, n isometries $S_, \dots, S_n$ on $\mathcal{K}$ such that $\sum_{i=1}^n S_i S_i^* = I$ and a unitary operator U so that $\sigma(\mathfrak{s}_i) = S_i$ determines a $*$-representation of $\mathcal{O}_n$ and $\sigma(A)U = U\sigma(\varphi(A))$ for all $A \in \mathcal{O}_n$ such that $\pi(A) = P_\mathcal{H}\sigma(A)|_\mathcal{H}$ for $A \in \mathfrak{A}$ and $K^n = P_\mathcal{H} U^n|_\mathcal{H}$ for $n \geq 0$.*

Arveson [1] defined the C*-envelope of an operator algebra $\mathcal{A}$ as the C*-algebra generated by a completely isometrically isomorphic copy of $\mathcal{A}$ with the property that any other such C*-algebra has a canonical homomorphism onto $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$. The existence of this envelope was established by Hamana [6]. Recently, Dritschel and McCullough [5] characterized the C*-envelope in terms of the maximal dilations of $\mathcal{A}$, which always extend uniquely to a $*$-representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$.

This allow us to prove:

Theorem 10. $\mathrm{C}^*_{env}(\mathfrak{A}_n \times_\varphi \mathbb{Z}^+) \simeq \mathcal{O}_n \times_\varphi \mathbb{Z}$.

In addition, the unique extension of the automorphism φ to $\mathcal{O}_n$ is always outer (except $\varphi = \mathrm{id}$). It then follows from a result of Kishimoto [7] that $\mathcal{O}_n \times_\varphi \mathbb{Z}$ is simple whenever φ is aperiodic. In this case, there is an explicit representation of the crossed product as $\mathrm{C}^*(\mathcal{E}_n, U_\varphi)/\mathfrak{K}$.

Two automorphisms $\hat\varphi$ and $\hat\psi$ in $\operatorname{Aut}\mathbb{B}_n$ are biholomorphically conjugate if there is an element $\hat\rho \in \operatorname{Aut}\mathbb{B}_n$ such that $\hat\varphi\hat\rho = \hat\rho\hat\psi$. Since $\hat\rho$ induces the automorphism ρ of $\mathfrak{A}_n$ which is completely isometric, it is easy to see that $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+_n \simeq \mathfrak{A}_n \times_\varphi \mathbb{Z}^+_n$ is a completely isometric isomorphism.

To establish a strong form of the converse, we wish to determine φ from the algebraic structure of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+_n$, up to conjugacy. The first step is to identify the character space of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+_n$. One defines an *analytic subset* $\mathcal{O}$ of the character space $\mathfrak{M}_\mathcal{A}$ of a Banach algebra $\mathcal{A}$ as the image of a non-constant map $h : \Omega \to \mathcal{O}$, where Ω is a domain in $\mathbb{C}^k$, such that $\hat{A} \circ h$ is analytic for every $A \in \mathcal{A}$. Also let $\mathrm{Fix}(\hat\varphi)$ denote the set of fixed points of $\hat\varphi$, and set $F_0(\varphi) = \mathrm{Fix}(\varphi) \cap \mathbb{B}_n$ and let $F_1(\varphi) = \mathrm{Fix}(\varphi) \cap \partial\mathbb{B}_n$.

Lemma 11. $\mathfrak{M}_{\mathfrak{A}_n \times_\varphi \mathbb{Z}^+} = (\overline{\mathbb{B}}_n \times \{0\}) \cup (\mathrm{Fix}(\varphi) \times \overline{\mathbb{D}})$. *The maximal analytic sets in* $\mathfrak{M}_{\mathfrak{A}_n \times_\varphi \mathbb{Z}^+}$ *are* $\mathbb{B}_n \times \{0\}$, $F_0(\varphi) \times \mathbb{D}$, $F_0(\varphi) \times \{\lambda\}$ *for* $\lambda \in \mathbb{T}$, *and* $\{x\} \times \mathbb{D}$ *for* $x \in F_1(\varphi)$.

The key technical step is to establish that the ideal $\mathfrak{J}_\varphi = \mathfrak{u}(\mathfrak{A}_n \times_\varphi \mathbb{Z}^+)$ is intrinsically defined within $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$. We study representations of the semicrossed product onto the $k \times k$ upper triangular matrices $\mathcal{T}_k$, known as *nest representations*. The main issue is that $\mathfrak{A}_n$ itself has many nest representations, and they interfere with the analysis of representations of the semicrossed product. The reason for considering a nest representation ρ is that the diagonal entries are characters, and thus restrict on $\mathfrak{A}_n$ to evaluation at points in $\overline{\mathbb{B}}_n$. If these points lie in the open ball, say $z_1, \dots, z_k$, then the ij entry of $\rho(\mathfrak{u})$ must be 0 if $\hat\varphi(z_j) \neq z_i$. Combining this with explicitly constructed representations, one is able to filter out the 'noise' from representations of $\mathfrak{A}_n$ and 'see' the ideal $\mathfrak{J}$. This leads to our second main result.

Theorem 12. *Let* φ *and* ψ *be automorphisms of the non-commutative disc algebra* $\mathfrak{A}_n$ *for* $n \geq 2$. *Then the following are equivalent:*

(1) $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ *and* $\mathfrak{A}_n \times_\psi \mathbb{Z}^+$ *are isomorphic as algebras;*

(2) $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ *and* $\mathfrak{A}_n \times_\psi \mathbb{Z}^+$ *are completely isometrically isomorphic;*

(3) $\hat{\varphi}$ *and* $\hat{\psi}$ *are biholomorphically conjugate;*
(4) φ *and* ψ *are conjugate via an automorphism of* $\mathfrak{A}_n$.

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An operator space approach to Schur-Agler norms on convex balanced domains

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To each convex balanced domain in $\mathbb{C}^n$ (i.e., each open ball for a norm on $\mathbb{C}^n$) we associate a family of operator algebra norms on the space of n-variable polynomials. These norms are in one-to-one correspondence with operator space structures over the underlying n-dimensional Banach space. Examples include the Agler norm on the polydisk, the Drury-Arveson multiplier norm on the ball, but many other apparently new examples as well (even on the ball and polydisk). As is the case for the well-known examples, each norm is characterized by a Nevanlinna factorization and a transfer function realization.

More precisely: let $V = (\mathbb{C}^n, \|\cdot\|)$ be a finite-dimensional Banach space and fix an isometric embedding $\varphi : V \to B(H)$. Let E be the resulting operator space structure over V.

Theorem 13. *Let E be a finite dimensional operator space (with underlying Banach space V) and q an analytic M_N-valued polynomial. Then the following are equivalent:*

1) Agler-Nevanlinna factorization. *There exists a Hilbert space K, a completely contractive map $\psi : V \to B(K)$, and an analytic function $F : \Omega \to B(K, \mathbb{C}^N)$ such that*

$$1 - q(z)q(w)^* = F(z)\left[I_K - \rho(z)\rho(w)^*\right]F(w)^* \tag{0.1}$$

2) Transfer function realization. *There exists a Hilbert space K', a unitary transformation $U : K' \oplus \mathbb{C}^N \to K' \oplus \mathbb{C}^N$ of the form*

$$\begin{array}{cc} & \begin{array}{cc} K' & \mathbb{C}^N \end{array} \\ \begin{array}{c} K' \\ \mathbb{C}^N \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{0.2}$$

and a completely contractive map $\rho : V \to B(K')$ so that

$$q(z) = D + C(I - \rho(z)A)^{-1}\rho(z)B. \tag{0.3}$$

3) von Neumann inequality. *For every commuting, completely contractive map S of E^* on a Hilbert space K,*

$$\|q(S)\|_{M_N \otimes B(K)} \leq 1. \tag{0.4}$$

To clarify the von Neumann inequality 3), the operators S are n-tuples of commuting operators $(S_1, \dots S_n)$ with the property that the map

$$(z_1, \dots z_n) \to \sum_{j=1}^{n} z_j S_j$$

is completely contractive for the dual operator space E^*.

The case $E = R_n$ (n-dimensional row space) corresponds to the Drury-Arveson multiplier algebra over the unit ball in $\mathbb{C}^n$. In this case the operators S are all commuting n-tuples with the property that

$$I - \sum_{j=1}^{n} S_j S_j^* \geq 0.$$

and the above theorem is proved by [1]. When $E = MIN(\ell_n^\infty)$, the operators S run over all commuting contractions, and the resulting norm is the Agler norm on the polydisk. The theorem essentially reduces to a result of Agler [2]. However even over these classical domains the above theorem provides many new examples of operator algebra norms on the space of polynomials. In particular there is always a "minimal Schur-Agler operator algebra" which corresponds to choosing $E = MAX(V)$ above (here $MAX(V)$ denotes the maximal operator space over V) so that $E^* = MIN(V^*)$. This point of view sheds some new light on the counterexample to von Neumann's inequality due to Varopoulos [3]. It can be checked that the triple of commuting contractions produced by Kaijser and Varopoulos are in fact completely contractive for $MIN(\ell_n^1)$, so that the minimal Schur-Agler norm on the polydisk strictly dominates the supremum norm; the classical understanding of this example shows only that the maximal Schur-Agler norm dominates the supremum norm.

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Clark measures with prescribed behavior and rank-one perturbations of self-adjoint and unitary operators

Carl Sundberg

Let φ be an analytic map of the unit disc $\mathbb{D} \subset \mathbb{C}$ to itself for which $\varphi(0) = 0$. For each $\alpha \in \partial\mathbb{D}$ the function

$$\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}$$

has positive real part and hence a Herglotz integral representation

$$\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int \frac{\zeta + z}{\zeta - z} d\mu_\alpha(\zeta)$$

for a uniquely determined probability measure μ_α on $\partial\mathbb{D}$. The measures $\{\mu_\alpha\}_{\alpha\in\partial\mathbb{D}}$ are called the *Clark measures* associated to φ after the paper [1] by D. N. Clark, though other names are common in the literature.

If we define the unitary operators $\{U_\alpha\}_{\alpha\in\partial\mathbb{D}}$ to be multiplication by the coordinate function z on $L^2(\mu_\alpha)$ then a calculation shows

$$U_\alpha f(z) = Uf(z) + (\alpha - 1)\int f d\mu$$

where $U = U_1$. Thus $\{U_\alpha\}_{\alpha\in\partial\mathbb{D}}$ is a family of unitary rank-one perturbations of the unitary operator U; with spectral measures μ_α. Such families and analogous families of rank-one perturbations of self-adjoint operators have been much-studied in the mathematical physics community, see e.g [6]. A question of interest is the extent to which properties of the spectral measures μ_α can vary with α. A standard calculation shows that the absolutely continuous parts of the μ_α's are all mutually absolutely continuous, so we focus on the singular parts σ_α. In this context we note that if φ is inner then all μ_α's are singular.

It came as a bit of a surprise that properties of the singular measures σ_α can be very sensitive to changes in α. The first result along this line was due to W. Donoghue [3]. Translated into our language his result says:

Theorem 14. *There exists an inner function φ whose Clark measures $\{\mu_\alpha\}$ satisfy*

$$\begin{array}{rl} \mu = \mu_1 & \textit{is purely atomic} \\ & \textit{but} \\ \mu_\alpha & \textit{is continuous singular for } \alpha \in \partial\mathbb{D} \setminus \{1\} \end{array}$$

The measure μ in Donoghue's example has all of $\partial\mathbb{D}$ for its support. The following result of R. Del Rio, N. Makarov, and B. Simon and, independently, A. Ya. Gordon, shows that the "largeness" of the set of α's for which μ_α is singular continuous is to be expected (we translate their result into our language):

Theorem 15 ([2], [4], [5])**.** *Let φ, $\{\mu_\alpha\}$ be as above and let $I \subset \partial\mathbb{D}$ be a closed interval, not a singleton, such that $I \subset spt\mu$ and $\mu|_I$ is singular. Then for all α in a dense G_δ-subset of $\partial\mathbb{D}$ $\mu_\alpha|_I$ is singular continuos.*

We prove the following converse:

Theorem 16. *Let I be a closed subinterval of $\partial\mathbb{D}$, not a singleton, and let G be a G_δ-subset of $\partial\mathbb{D}$. Then there exists an inner function φ whose associated Clark measures μ_α satisfy:*

$$\begin{aligned} &spt\mu \subset I \\ &\mu_\alpha | I \text{ is singular if } \alpha \in G \\ &\mu_\alpha \text{ is purely atomic if } \alpha \in \partial\mathbb{D} \setminus G \end{aligned}$$

If G is dense, then of necessity $spt\mu = 1$.

The function φ is produced by constructing a Riemann surface $\mathcal{R}$ lying over $\mathbb{D}$ with projection $\pi : \mathcal{R} \to \mathbb{D}$, then setting $\varphi = \pi \circ \Phi$, where $\Phi : \mathbb{D} \to \mathcal{R}$ is a covering map.

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A sheaf theoretic model for analytic Hilbert modules

Gadadhar Misra

(joint work with Shibananda Biswas)

Let Ω be a bounded open connected set in $\mathbb{C}^m$ and $\mathcal{M}$ be a Hilbert module over the function algebra $\mathcal{A}(\Omega)$ (see [6]). The study of the natural class $\mathrm{B}_n(\Omega)$, discussed below, was initiated in [1, 2]. A different approach was given in [3]. Let $D_{\mathbf{T}} : \mathcal{M} \to \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the operator $f \mapsto \big(T_1 f, \ldots, T_m f\big)$, where T_i is the operator determined by the adjoint of the module action $(z_i, f) \mapsto z_i \cdot f$, $1 \le i \le m$, $f \in \mathcal{M}$. Let $\mathrm{B}_n(\Omega)$ be the set of those Hilbert modules $\mathcal{M}$ for which $\operatorname{ran} D_{\mathbf{T}-w}$ is closed, $\operatorname{span}_{w\in\Omega} \ker D_{\mathbf{T}-w}$ is dense and $\dim\ker D_{\mathbf{T}-w} = n$ for all $w \in \Omega$. A Hilbert module $\mathcal{M}$ in $\mathrm{B}_n(\Omega)$ determines a holomorphic Hermitian vector bundle on Ω. It is then proved that isomorphic Hilbert modules correspond to equivalent vector bundles and vice-versa (see [1, 2]). Also, these papers provide a model for the

Hilbert modules in $\mathrm{B}_n(\Omega)$ by showing that they can be realized as a Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel. The module action is then simply the pointwise multiplication. Examples are Hardy and the Bergman modules over the ball and the poly-disc in $\mathbb{C}^m$. However, many natural examples of Hilbert modules fail to be in the class $\mathrm{B}_n(\Omega)$. For instance, $H_0^2(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ is not in $\mathrm{B}_n(\mathbb{D}^2)$. The problem is that the dimension of the joint kernel $\mathbb{K}(w) := \ker D_{\mathbf{T}-w}$ is no longer a constant (cf. [4]):

$$\dim H_0^2(\mathbb{D}^2) \otimes_{\mathcal{A}(\mathbb{D}^2)} \mathbb{C}_w = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

Here $\mathbb{C}_w$ is the one dimensional module over the algebra $\mathcal{A}(\mathbb{D}^2)$, where the module action is given by the map $(f, w) \mapsto f(w)$ for $f \in \mathcal{A}(\mathbb{D}^2)$ and $w \in \mathbb{C}$. We outline an attempt to systematically study examples like the one given above using methods of complex analytic geometry.

For a Hilbert module $\mathcal{M}$ over a function algebra $\mathcal{A}(\Omega)$, not necessarily in the class $\mathrm{B}_1(\Omega)$, motivated by the correspondence of vector bundles with locally free sheaf, we construct a sheaf of modules $\mathcal{S}^{\mathcal{M}}(\Omega)$ over $\mathcal{O}(\Omega)$ corresponding to $\mathcal{M}$. We assume that $\mathcal{M}$ possesses all the properties for it to be in the class $\mathrm{B}_1(\Omega)$ except that the dimension of the joint kernel $\mathbb{K}(w)$ need not be constant. We note that sheaf models have occured, as a very useful tool, in the study of analytic Hilbert modules (cf. [7]). Although, the model we describe below is somewhat different.

Let $\mathcal{S}^{\mathcal{M}}(\Omega)$ be the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk at $w \in \Omega$ is $\big\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\big\}$, or equivalently, $\mathcal{S}^{\mathcal{M}}(U) = \Big\{ \sum_{i=1}^n \big(f_{i|U}\big)g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U)\Big\}$ for U open in Ω.

Proposition 1. The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is coherent.

Proof. The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is generated by the set of functions $\{f : f \in \mathcal{M}\}$. Let $\mathcal{S}_J^{\mathcal{M}}(\Omega)$ be the subsheaf generated by the set of functions $J = \{f_1, \ldots, f_\ell\} \subseteq \mathcal{M} \subseteq \mathcal{O}(\Omega)$. Thus $\mathcal{S}_J^{\mathcal{M}}(\Omega)$ is coherent. An application of Noether's Lemma [8] then guarantees that $\mathcal{S}^{\mathcal{M}}(\Omega) = \cup_{J\,\text{finite}} \mathcal{S}_J^{\mathcal{M}}(\Omega)$ is coherent. □

We note that the coherence of the sheaf implies, in particular, that the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$ is generated by a finite number of elements $g_1, \ldots, g_n$ from $\mathcal{O}(\Omega)$.

If K is the reproducing kernel for $\mathcal{M}$ and $w_0 \in \Omega$ is a fixed but arbitrary point, then for w in a small neighborhood U of w_0, we obtain the following decomposition theorem.

Theorem 1. Suppose g_i^0, $1 \leq i \leq n$, be a minimal set of generators for the stalk $(\mathcal{S}^{\mathcal{M}})_0 := (\mathcal{S}^{\mathcal{M}})_{w_0}$. Then we have

$$K(\cdot, w) := K_w = g_1^0(w)K_w^{(1)} + \cdots + g_n^0(w)K_w^{(n)},$$

where $K^{(p)} : U \to \mathcal{M}$, $1 \leq k \leq n$, is anti-holomorphic. Moreover, the elements $K_{w_0}^{(p)}$, $1 \leq p \leq n$ are linearly independent in $\mathcal{M}$, they are eigenvectors for the adjoint of the action of $\mathcal{A}(\Omega)$ on the Hilbert module $\mathcal{M}$ at w_0 and are uniquely determined by these generators.

We also point out that the Grammian $G(w) = (\!(\langle K_w^{(p)}, K_w^{(q)}\rangle)\!)_{p,q=1}^n$ is invertible in a small neighborhood of w_0 and is independent of the generators $g_1, \ldots, g_n$. Thus $t : w \mapsto (K_w^{(1)}, \ldots, K_w^{(n)})$ defines a holomorphic map into the Grassmannian $\mathrm{G}(\mathcal{H}, n)$ on the open set U. The pull-back E_0 of the canonical bundle on $\mathrm{G}(\mathcal{H}, n)$ under this map then define a holomorphic Hermitian bundle on U. Clearly, the decomposition of K given in our Theorem is not canonical in anyway. So, we can't expect the corresponding vector bundle E_0 to reflect the properties of the Hilbert module $\mathcal{M}$. However, it is possible to obtain a canonical decomposition following the construction in [3]. It then turns out that the equivalence class of the corresponding vector bundle E_0 obtained from this canonical decomposition is an invariant for the isomorphism class of the Hilbert module $\mathcal{M}$. These invariants are by no means easy to compute. At the end of this note, we indicate, how to construct invariants which are more easily computable.

For now, the following Corollary to the decomposition theorem is immediate.

Corollary 1. The dimension of the joint kernel $\mathbb{K}(w)$ is greater or equal to the number of minimal generators of the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$.

Now is the appropriate time to raise a basic question. Let $\mathfrak{m}_w \subseteq \mathcal{A}(\Omega)$ be the maximal ideal of functions vanishing at w. Since we have assumed $\mathfrak{m}_w\mathcal{M}$ is closed, it follows that the dimension of the joint kernel $\mathbb{K}(w)$ equals the dimension of the quotient module $\mathcal{M}/(\mathfrak{m}_w\mathcal{M})$. However it is not clear if one may impose natural hypothesis on $\mathcal{M}$ to ensure

$$\mathcal{M}/(\mathfrak{m}_w\mathcal{M}) = \dim \mathbb{K}(w) = (\mathcal{S}^{\mathcal{M}})_w/(\mathfrak{m}(\mathcal{O}_w)(\mathcal{S}^{\mathcal{M}})_w),$$

where $\mathfrak{m}(\mathcal{O}_w)$ is the maximal ideal in $\mathcal{O}_w$, as well.

More generally, suppose $p_1, \ldots, p_n$ generate $\mathcal{M}$. Then $\dim \mathbb{K}(w) \leq n$ for all $w \in \Omega$. If the common zero set V of these is $\{0\}$ then $(p_1)_0, \ldots, (p_n)_0$ need not be a minimal set of generators for $(\mathcal{S}^{\mathcal{M}})_0$. However, we show that they do if we assume $p_1, \ldots, p_n$ are homogeneous of degree k, say. Further more, basis for $\mathbb{K}(0)$ is the set of vectors:

$$\big\{p_1(\bar{\boldsymbol\partial})\}K(\cdot, w)_{|\,w=0}, \ldots, p_n(\bar{\boldsymbol\partial})\}K(\cdot, w)_{|\,w=0}\big\},$$

where $\bar{\boldsymbol\partial} = (\bar\partial_1, \ldots, \bar\partial_m)$.

Going back to the example of $H_0^2(\mathbb{D}^2)$, we see that it has two generators, namely z_1 and z_2. Clearly, the joint kernel $\mathbb{K}(w) := \ker D_{(M_1^* - \bar w_1, M_2^* - \bar w_2)}$ at $w = (w_1, w_2)$ is spanned by $\{z_1 \otimes_{\mathcal{A}(\mathbb{D}^2)} 1_w, z_2 \otimes_{\mathcal{A}(\mathbb{D}^2)} 1_w\} = \{w_1 K_{H_0^2(\mathbb{D}^2)}(z, w), w_2 K_{H_0^2(\mathbb{D}^2)}(z, w)\}$ which consists of two vectors that are linearly dependent except when $w = (0, 0)$. We also easily verify that

$$\big(\mathcal{S}^{H_0^2(\mathbb{D}^2)}\big)_w \cong \begin{cases} \mathcal{O}_w & w \neq (0,0) \\ \mathfrak{m}\,(\mathcal{O}_0) & w = (0,0). \end{cases}$$

Since the reproducing kernel

$$K_{H_0^2(\mathbb{D}^2)}(z, w) \quad = \quad K_{H^2(\mathbb{D}^2)}(z, w) - 1 = \frac{z_1\bar w_1 + z_2\bar w_2 - z_1 z_2 \bar w_1 \bar w_2}{(1 - z_1\bar w_1)(1 - z_2\bar w_2)},$$

we find there are several choices for $K_w^{(1)}$ and $K_w^{(2)}$, $w \in U$. However, all of these choices disappear if we set $\bar{w}_1\theta_1 = \bar{w}_2$ for $w_1 \neq 0$, and take the limit:

$$\lim_{(w_1,w_2)\to 0} \frac{K_{H_0^2(\mathbb{D}^2)}(z,w)}{\bar{w}_1} = K_0^{(1)}(z) + \theta_1 K_0^{(2)}(z) = z_1 + \theta_1 z_2$$

because $K_0^{(1)}$ and $K_0^{(2)}$ are uniquely deteremined by Theorem 1. Similarly, for $\bar{w}_2\theta_2 = \bar{w}_1$ for $w_2 \neq 0$, we have

$$\lim_{(w_1,w_2)\to 0} \frac{K_{H_0^2(\mathbb{D}^2)}(z,w)}{\bar{w}_2} = K_0^{(2)}(z) + \theta_2 K_0^{(1)}(z) = z_2 + \theta_2 z_1.$$

Thus we have a Hermitian line bundle on the complex projective space $\mathbb{P}^1$ given by the frame $\theta_1 \mapsto z_1 + \theta_1 z_2$ and $\theta_2 \mapsto z_2 + \theta_2 z_1$. The curvature of this line bundle is then an invariant for the Hilbert module $H_0^2(\mathbb{D}^2)$ as shown in [5]. This curvature is easily calculated and is given by the formula $\mathcal{K}(\theta) = (1 + |\theta|^2)^{-2}$.

The decomposition theorem yields similar results in many other examples.

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A classification of homogeneous operators in the Cowen-Douglas class

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(joint work with Gadadhar Misra)

A bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ is said to be *homogeneous* if its spectrum is contained in the closed unit disc and for every Möbius transformation g of the unit disc $\mathbb{D}$, the operator $g(T)$ defined via the usual holomorphic functional calculus, is unitarily equivalent to T. This class of operators originally appeared in the work of G. Misra and was studied in the articles [1],[3], [4] and [5], among others.

A Fredholm operator T on a Hilbert space $\mathcal{H}$ is said to be in the Cowen - Douglas class of the domain $\Omega \subseteq \mathbb{C}$ if its eigenspaces E_w, $w \in \Omega$ are of constant finite dimension. In the paper [2], Cowen and Douglas show that

(a) $E \subseteq \Omega \times \mathcal{H}$ with fiber E_w at $w \in \Omega$ is a holomorphic Hermitian vector bundle over Ω, where the Hermitian structure is given by

$$\|s_w\|_w = \|\iota_w s_w\|_{\mathcal{H}},\ s_w \in E_w,$$

and $\iota_w : E_w \to \mathcal{H}$ is the inclusion map;

(b) isomorphism classes of E correspond to unitary equivalence classes of T;

(c) the holomorphic Hermitian vector bundle E is irreducible if and only if the operator T is irreducible.

Our first non-trivial result is that a Cowen-Douglas operator is homogeneous if and only if the corresponding bundle is homogeneous under $\tilde{G}$, the universal covering group of the Möbius group. We describe below all irreducible homogeneous holomorphic Hermitian vector bundles over the unit disc and determine which ones of these correspond to homogeneous operators (necessarily irreducible) in the Cowen-Douglas class.

Let $\mathfrak{t} \subseteq \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ be the Lie algebra $\mathbb{C}h + \mathbb{C}y$, where

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By holomorphic induction, linear representations (ϱ, V) of the algebra $\mathfrak{t} \subseteq \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$, that is, pairs $\varrho(h), \varrho(y)$ of linear transformations satisfying $[\varrho(h), \varrho(y)] = -\varrho(y)$ provide a para-metrization of the homogeneous holomorphic vector bundles.

The $\tilde{G}$ - invariant Hermitian structures on the homogeneous holomorphic vector bundle E (making it into a homogeneous holomorphic Hermitian vector bundle), if they exist, are given by $\varrho(\tilde{\mathbb{K}})$ - invariant inner products on the representation space. Here $\tilde{\mathbb{K}}$ is the stabilizer of 0 in $\tilde{G}$.

An inner product can be $\varrho(\tilde{\mathbb{K}})$ - invariant if and only if $\varrho(h)$ is diagonal with real diagonal elements in an appropriate basis. We are interested only in Hermitizable bundles, that is, those that admit a Hermitian structure. So, we will assume without restricting generality, that the representation space of ϱ is $\mathbb{C}^n$ and that $\varrho(h)$ is a real diagonal matrix.

Since $[\varrho(h), \varrho(y)] = -\varrho(y)$, we have $\varrho(y)V_\lambda \subseteq V_{\lambda-1}$, where $V_\lambda = \{\xi \in \mathbb{C}^n : \varrho(h)\xi = \lambda\xi\}$. Hence $(\varrho, \mathbb{C}^n)$ is a direct sum, orthogonal for every $\varrho(\tilde{K})$ - invariant inner product of "elementary" representations, that is, such that

$$\varrho(h) = \begin{pmatrix} -\eta I_0 & & \\ & \ddots & \\ & & -(\eta+m)I_m \end{pmatrix} \text{ with } I_j = I \text{ on } V_{-(\eta+j)} = \mathbb{C}^{d_j}$$

and

$$Y := \varrho(y) = \begin{pmatrix} 0 & & & & \\ Y_1 & 0 & & & \\ & Y_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & Y_m & 0 \end{pmatrix}, \; Y_j : V_{-(\eta+j-1)} \to V_{-(\eta+j)}.$$

We denote the corresponding elementary Hermitizable bundle by $E^{(\eta,Y)}$.

The invariant Hermitian structures on it are given by positive definite block diagonal matrices H. $E^{(\eta,Y)}$ is irreducible if and only if Y is not the H - orthogonal direct sum of two operators with the same sub-diagonal structure as Y.

We note that $(E^{(\eta,Y)}, H) \cong (E^{(\eta,AYA^{-1})}, A^{*-1}HA)$ for any block diagonal invertible A. Therefore every homogeneous holomorphic Hermitian vector bundle is isomorphic with one of the form $(E^{(\eta,Y)}, I)$.

Now we proceed to a construction. Let $\eta > 0$ and let Y be as above. For $\lambda > 0$, let $\mathbb{A}^{(\lambda)}$ be the Hilbert space of holomorphic functions on the unit disc with reproducing kernel $(1 - z\bar{w})^{-2\lambda}$. $\tilde{G}$ acts on it with the multiplier $g^\lambda(z)$. Let $\mathbf{A}^{(\eta)} = \oplus_{j=0}^m \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$. The elements of $\mathbf{A}^{(\eta)}$ are just the sections of $E^{(\eta,0)}$ in a natural trivialization. For f in $\mathbf{A}^{(\eta)}$, we denote by f_j, the part of f in $\mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$. We define $\Gamma^{(\eta,Y)}f$ as the $\mathbb{C}^n$ - valued holomorphic function whose part in $\mathbb{C}^{d_\ell}$ is given by

$$\left(\Gamma^{(\eta,Y)} f\right)_\ell = \sum_{j=0}^{\ell} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} f_j^{(\ell-j)}$$

for $\ell \geq j$. For invertible block diagonal N on $\mathbb{C}^n$, we also define $\Gamma_N^{(\eta,Y)} := \Gamma^{(\eta,Y)} \circ N$. It can be verified that $\Gamma_N^{(\eta,Y)}$ is a $\tilde{G}$ - equivariant isomorphism of $\mathbf{A}^{(\eta,0)}$ as a homogeneous holomorphic vector bundle onto $E^{(\eta,Y)}$. The image $K_N^{(\eta,Y)}$ of the reproducing kernel of $\mathbf{A}^{(\eta)}$ is then a reproducing kernel for $E^{(\eta,Y)}$. A computation gives that $K_N^{(\eta,Y)}(0,0)$ is a block diagonal matrix such that its ℓ'th block is

$$K_N^{(\eta,Y)}(0,0)_{\ell,\ell} = \sum_{j=0}^{\ell} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} N_j N_j^* Y_{j+1}^* \cdots Y_\ell^*.$$

We set $H_N^{(\eta,Y)} = K_N^{(\eta,Y)}(0,0)^{-1}$. We have now constructed a family $(E^{(\eta,Y)}, H_N^{(\eta,Y)})$ of elementary homogeneous holomorphic Hermitian vector bundles with a reproducing kernel ($\eta > 0$, Y as before, N invertible block diagonal).

Our main result is the following.

Theorem 2. Every elementary homogeneous holomorphic vector bundle E with a reproducing kernel arises from the construction given above. These vector bundles are exactly the ones that correspond to irreducible homogeneous Cowen-Douglas operators.

By one of our previous remarks, every homogeneous holomorphic Hermitian vector bundle can be put in the form $(E^{(\eta,Y)}, I)$. We can then write down a system of inequalties that characterize the pairs (η, Y) such that $(E^{(\eta,Y)}, I)$ corresponds to a Cowen-Douglas operator.

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