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Multiplier Ideal Sheaves in Algebraic and Complex Geometry

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April 12th – April 18th, 2009

Abstract. The workshop *Multiplier Ideal Sheaves in Algebraic and Complex Geometry*, organised by Stefan Kebekus (Freiburg), Mihai Paun (Nancy), Georg Schumacher (Marburg) and Yum-Tong Siu (Cambridge MA) was held April 12th – April 18th, 2009. Since the previous Oberwolfach conference in 2004, there have been important new developments and results, both in the analytic and algebraic area, e.g. in the field of the extension of L^2-holomorphic functions, the solution of the ACC conjecture, log-canonical rings, the Kähler-Ricci flow, Seshadri constants and the analogues of multiplier ideals in positive characteristic.

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Introduction by the Organisers

The workshop *Multiplier Ideal Sheaves in Algebraic and Complex Geometry*, organised by Stefan Kebekus (Freiburg), Mihai Paun (Nancy), Georg Schumacher (Marburg) and Yum-Tong Siu (Cambridge MA) was held April 12th – April 18th, 2009. Since the previous Oberwolfach conference in 2004, there have been important new developments and results, both in the analytic and algebraic area. This meeting included several leaders in the field as well as many young researchers.

The title of the workshop stands for phenomena and methods, closely related to both the analytic and the algebraic area. The aim of the workshop was to present recent important results with particular emphasis on topics linking different areas, as well as to discuss open problems.

The original approach involving the theory of partial differential equations and subelliptic estimates was addressed in several contributions, including existence theorems for L^2-holomorphic functions and applications of multiplier ideal sheaves to solutions of the Ricci-flow and the Monge-Ampère equation. Further areas included the study of Seshadri numbers, canonical models, as well as log canonical varieties and their canonical rings. The solution of the ACC conjecture for log canonical thresholds was presented. Furthermore, the analogues of multiplier ideals in positive characteristic were discussed.

Workshop: Multiplier Ideal Sheaves in Algebraic and Complex Geometry

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Abstracts

Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties

Lawrence Ein
(joint work with Tommaso de Fernex, Mircea Mustaţă)

Let k be an algebraically closed field of characteristic zero. Log canonical varieties are varieties with mild singularities that provide the most general context for the Minimal Model Program. More generally, one considers the log canonicity condition on pairs $(X, \mathfrak{a}^t)$, where $\mathfrak{a}$ is a proper ideal sheaf on X (most of the times, it is the ideal of an effective Cartier divisor), and t is a nonnegative real number. Given a log canonical variety X over k, and a proper nonzero ideal sheaf $\mathfrak{a}$ on X, one defines the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$ of the pair $(X, \mathfrak{a})$. This is the largest number t such that the pair $(X, \mathfrak{a}^t)$ is log canonical. One makes the convention $\mathrm{lct}(0) = 0$ and $\mathrm{lct}(\mathcal{O}_X) = \infty$. One also defines a local version of the log canonical threshold at a point $p \in X$, which we denote by $\mathrm{lct}_p(\mathfrak{a})$. The log canonical threshold is a fundamental invariant in birational geometry, see for example [9], [7], or Chapter 9 in [12].

Shokurov's ACC Conjecture [13] says that the set of all log canonical thresholds on varieties of any fixed dimension satisfies the ascending chain condition, that is, it contains no infinite strictly increasing sequences. This conjecture attracted considerable interest due to its implications to the Termination of Flips Conjecture (see [2] for a result in this direction). The first unconditional results on sequences of log canonical thresholds on smooth varieties of arbitrary dimension have been obtained in [5], and they were subsequently reproved and strengthened in [10].

Theorem 1. *For every n, the set*

$$\mathcal{T}_n^{\mathrm{sm}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \textit{ is smooth}, \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

of log canonical thresholds on smooth varieties of dimension n satisfies the ascending chain condition.

As we will see, every log canonical threshold on a variety with quotient singularities can be written as a log canonical threshold on a smooth variety of the same dimension. Therefore for every n the set

$$\mathcal{T}_n^{\mathrm{quot}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \text{ has quotient singularities}, \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

is equal to $\mathcal{T}_n^{\mathrm{sm}}$, and thus the ascending chain condition also holds for log canonical thresholds on varieties with quotient singularities.

In order to extend the result to log canonical thresholds on locally complete intersection varieties, we consider a more general version of log canonical thresholds. Given a variety X and an ideal sheaf $\mathfrak{b}$ on X such that the pair $(X, \mathfrak{b})$ is log canonical, for every nonzero ideal sheaf $\mathfrak{a} \subsetneq \mathcal{O}_X$ we define the *mixed log canonical threshold* $\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a})$ to be the largest number c such that the pair $(X, \mathfrak{b} \cdot \mathfrak{a}^c)$ is log canonical. Note that when $\mathfrak{b} = \mathcal{O}_X$, this is nothing but $\mathrm{lct}(\mathfrak{a})$. Again, one sets $\mathrm{lct}_{(X,\mathfrak{b})}(0) = 0$ and $\mathrm{lct}_{(X,\mathfrak{b})}(\mathcal{O}_X) = \infty$. The following is our main result.

Theorem 2. *For every n, the set*

$$\mathcal{M}_n^{\text{l.c.i.}} := \{\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a}) \mid X \textit{ l.c.i.}, \dim X = n, \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X, \mathfrak{a} \neq \mathcal{O}_X, (X, \mathfrak{b}) \textit{ log can.}\}$$

of mixed log canonical thresholds on l.c.i. varieties of dimension n satisfies the ascending chain condition.

Corollary 3. *For every n, the set*

$$\mathcal{T}_n^{\text{l.c.i.}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \textit{ l.c.i.}, \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

of log canonical thresholds on l.c.i. varieties of dimension n satisfies the ascending chain condition.

We will use Inversion of Adjunction (in the form treated in [6]) to reduce Theorem 2 to the analogous statement in which X ranges over smooth varieties. More precisely, we show that all sets

$$\mathcal{M}_n^{\text{sm}} := \{\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a}) \mid X \text{ smooth}, \dim X = n, \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X, \mathfrak{a} \neq \mathcal{O}_X, (X, \mathfrak{b}) \text{ log can.}\}$$

satisfy the ascending chain condition. It follows by Inversion of Adjunction that every mixed log canonical threshold of the form $\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a})$, with $\mathfrak{a}$ and $\mathfrak{b}$ ideal sheaves on an l.c.i. variety X, can be expressed as a mixed log canonical threshold on a (typically higher dimensional) smooth variety. This is the step that requires us to work with mixed log canonical thresholds. The key observation that makes this approach work is that if X is an l.c.i. variety with log canonical singularities, then $\dim_k T_x X \leq 2 \dim X$ for every $x \in X$. This implies that the above reduction to the smooth case keeps the dimension of the ambient variety bounded.

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Remarks on a theorem by H. Tsuji

Bo Berndtsson

Let X be a compact complex manifold of dimension n with ample canonical bundle, K_X. If ϕ is a metric on K_X, then e^ϕ is a volume form on X. If ϕ has positive curvature so that $i\partial\bar\partial\phi$ is a positive form, we can also equip the manifold X with the Kahler metric $\omega^\phi := i\partial\bar\partial\phi$, with the associated volume form

$$\omega_n^\phi := (\omega^\phi)^n/n!.$$

Let V be the volume of the manifold for this latter volume form; it is independent of the choice of ϕ. The metric ω^ϕ satisfies the *Kähler-Einstein equation* if

$$\omega_n^\phi/V = ce^\phi,$$

where c is a constant. We shall always normalize our Kahler potentials so that $c = 1$.

By a theorem of Aubin and Yau (see eg [3]), a Kähler-Einstein metric always exists if K_X is ample, and its (normalized) Kahler potential $\phi := \phi_{KE}$ is uniquely determined.

In a very interesting paper, H. Tsuji [4] has recently shown that the Kähler-Einstein potential can be obtained as a limit of a sequence of metrics computed iteratively by Bergman kernel constructions. (He has also continued this study in [5] in a much more general setting, but this note concerns only the case of ample canonical bundle.) The aim of the talk is to give an alternate proof of Tsuji's theorem, using Bergman kernel asymptotics by Bouche-Tian-Catlin- Zelditch (see e g [6]) . This method also allows us to get uniform convergence (Tsuji proved convergence in L^1) at an explicit rate, and to obtain existence and a similar convergence result for balanced metrics (see below).

Given any (smooth) metric ϕ on K_X, and any natural number k we get an associated metric on the space $H^0(X, (k+1)K_X)$ of global holomorphic sections of $(k+1)K_X$ by

$$\|u\|^2_{k\phi} = \int_X |u|^2 e^{-k\phi}.$$

Let $B_{k\phi}$ be its associated Bergman kernel, defined as

$$B_{k\phi} = \sum |u_j|^2,$$

where u_j is an orthonormal basis for $H^0(X,(k+1)K_X)$. Obviously

$$\int_X B_{k\phi}e^{-k\phi} = d_k,$$

the dimension of $H^0(X,(k+1)K_X)$. Let

$$\beta_k(\phi) = \frac{1}{k+1}(\log B_{k\phi} - \log d_k);$$

it is again a metric on K_X, and it satisfies

$$\int_X e^{(k+1)\beta_k(\phi)-k\phi} = 1.$$

From this it follows easily from Hölder's inequality that

$$\int_X e^{\beta_k(\phi)} \leq 1,$$

if the integral of e^ϕ is at most one. (We also see that equality holds exactly when ϕ is a fixed point of the map β_k.)

The operator 'ϕ maps to $\beta(\phi)$' therefore maps the set of metrics on K_X with this property to itself.

Theorem 1. *Let ϕ be an arbitrary continuous metric on K_X. Let $k >> 0$ and define iteratively a sequence of metrics ϕ_m by $\phi_0 = \phi$ and*

$$\phi_{m+1} = \beta_{k+m}(\phi_m).$$

Then

$$\sup_X |\phi_m - \phi_{KE}|$$

tends to zero at the rate $1/m^2$.

This theorem is due to Tsuji, with slightly weaker convergence. (Tsuji writes his conclusion as $\limsup \phi_m = \phi_{KE}$. This seems to be not quite enough for applications, but his method gives L^1-convergence.)

One interesting consequence of Tsuji's theorem is the plurisubharmonic variation of Kahler-Einstein metrics in the relative case. If we let Z be an $n+l$ dimensional manifold smoothly and properly fibered over an l-dimensional base we can apply Tsuji's theorem to each fiber. Since we can choose the initial metric ϕ to have plurisubharmonic variation when the fiber varies, it follows from the results of [1] that all the fiberwise Bergman kernels will have plurisubharmonic variation as well. Taking limits, we conclude that the fiberwise Kähler-Einstein metrics ϕ_{KE} will have plurisubharmonic variation too. A stronger result in this direction has been proved recently by Schumacher [2], who gets an explicit lower bound for the curvature of the variation, giving in particular strict plurisubharmonicity where the Kodaira-Spencer class does not vanish.

Let us say that a metric ϕ is *k-balanced* if it is a fixed point for the map β_k, so that

$$B_{k\phi}/d_k = e^{(k+1)\phi}.$$

Theorem 2. *If $m >> 0$ there is a unique m-balanced metric, $\phi_{(m)}$. Moreover*

$$\sup_X |\phi_{(m)} - \phi_{KE}|$$

tends to zero at the rate $1/m^2$.

This result can be proved using similar methods as in the proof of Theorem 1. It should be noted, that whereas the convergence results in Theorems 1 and 2 use the existence of Kahler-Einstein metrics, the existence result for balanced metrics does not.

The main tool in our proof is the follwing lemma that compares Bergman kernels to sub- (or super-) solutions to the Kahler-Einstein equation.

Lemma 3. *Let ϕ_0 be a smooth metric on K_X with positive curvature form satisfying*

$$\omega_n^{\phi_0}/V \geq e^{\phi_0 + a},$$

where a is constant. Let ϕ be an arbitrary continuous metric on K_X such that

$$e^{\phi} \geq Ce^{\phi_0},$$

where C is constant. Then

$$e^{\beta_k(\phi)} \geq C^{k/(k+1)} e^{a/(k+1)} (1 - \epsilon_k)^{1/(k+1)} e^{\phi_0}.$$

Here ϵ_k is a sequence of positive numbers, depending only on ϕ_0, tending to zero at rate at most $1/k$, and in the special case $\phi_0 = \phi_{KE}$, ϵ_k tends to zero at the rate $1/k^2$. A similar estimate from above of $e^{\beta_k(\phi)}$ holds if we revers the two inequalities in the assumption.

This lemma follows easily from the Bergman kernels asymptotics of ϕ_0 if we compare the Bergman kernel for ϕ with the Bergman kernel for ϕ_0. If we take $\phi_0 = \phi_{KE}$, we can take $a = 0$ and apply the lemma iteratively starting with some constant C. We then get better and better estimates as we go along (since $k/(k+1) < 1$) , and Theorem 1 follows.

To show the existence of balanced metrics, without using the existence of Kähler-Einstein metrics, we take ϕ_0 arbitrary with positive curvature form, so that ϕ_0 satisfies the first hypothesis of the lemma for some a. The lemma then shows that the operator β_k maps the set of metrics ϕ with $|\phi - \phi_0| \leq A$ into itself if A is large enough, and from there we can deduce the existence of a fixed point as a point maximizing the integral of $e^{\beta_k \phi}$.

As a last comment we can note that we may change Tsuji's construction by defining

$$\tilde{\phi}_j = b_{k_j}(\tilde{\phi}_{j-1}),$$

where k_j is some sequence tending to infinity. Then we still get convergence to ϕ_{KE} as long as $\sum 1/k_j$ diverges.

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Fake projective planes and complex exotic quadrics

Sai-Kee Yeung

The first theme of the talk is to update the results on classification of fake projective planes. The following main result is the consequence of results of Prasad-Yeung [3] and Cartwright-Steger [2]. A fake projective plane is a smooth complex surface which has the same Betti numbers as $P_{\mathbb{C}}^2$ but which is not biholomorphic to $P_{\mathbb{C}}^2$. Here is the conclusion.

There are altogether one hundred fake projective planes up to biholomorphism in twenty-six non-empty classes of fake projective planes. This consists of fifty pairs of non-isometric Riemannian fourfolds, each of which consists of two biholomorphically distinct fake projective planes with conjugate complex structures. There can at most be one more class of fake projective planes, corresponding to very specific number fields.

The second theme of the talk is to introduce an approach to study the complex exotic quadrics. We say that a complex surface is a complex exotic quadric if it is homeomorphic but not biholomorphic to the complex quadric in $P_{\mathbb{C}}^3$. It is well-known that Hirzebruch surfaces of even type are complex exotic quadrics. A folklore conjecture is that these are the only complex exotic quadrics (cf. [1], [4]). In particular, there should not be complex exotic quadrics of general type.

Suppose M is an complex exotic quadric of general type. Our approach consists of two steps. The first step is algebraic, stating that the nef cone of the M consists of convex linear combinations of the two line bundles L_1, L_2 satisfying $L_1 \cdot L_1 = 0 = L_2 \cdot L_2$ and $L_1 \cdot L_2 = 1$. The second step is analytic, studying an appropriate limit of the Bergman kernels generated by sections of $nL_i + aK_M$ for some fixed $a > 0$ as $n \to \infty$.

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On the extension of L^2 holomorphic functions from analytic sets with singularities

Takeo Ohsawa

Let M be a complex manifold of dimension n, let E be a holomorphic vector bundle over M, let S be a closed complex analytic subset of M, and let K be the canonical line bundle of M. Given a C^∞ volume form dV on M, a C^∞ fiber metric h of E and a measure $d\mu$ on S, we denote by $A^2(S, E\otimes K, h\otimes(dV)^{-1}, d\mu)$ the space of L^2 holomoprhic sections of $E\otimes K$ over S with respect to $h\otimes(dV)^{-1}$ and $d\mu$. $A^2(S, E\otimes K, h\otimes(dV)^{-1}, d\mu)$ is independent of dV, so that we shall denote it by $A^2(M, E\otimes K, h)$.

For any locally integrable function $\psi : M \longrightarrow [-\infty, \infty)$, the spaces $A^2(S, E\otimes K, e^{-\psi}h\otimes(dV)^{-1}, d\mu)$ and $A^2(M, E\otimes K, e^{-\psi}h)$ are defined similarly. We shall call $e^{-\psi}h$ a singular fiber metric of E. Given a singular fiber metric $\tilde{h}$ of E, a bounded linear operator $I : A^2(S, E\otimes K, \tilde{h}\otimes(dV)^{-1}, d\mu) \longrightarrow A^2(M, E\otimes K, \tilde{h})$ satisfying $I(f)_{|S} = f$ for any f will be called an extension operator for $(E\otimes K, \tilde{h}\otimes(dV)^{-1})$ from $(S, d\mu)$ to (M, dV).

Let $\phi : M \longrightarrow [-\infty, 0)$ be a continuous function. We shall say that ϕ has a pole along S if the following are satisfied.

(1) $\phi^{-1}(-\infty) = S$,
(2) $\phi_{|M\setminus S}$ is C^∞,
(3) $e^{-\phi}$ is not integrable on an open set $U \subset M$ whenever $U\cap S \neq \varnothing$.

Given a function ϕ which has a pole along S, we say that $d\mu$ is a residual majorant of (dV, ϕ) if the inequality

$$\limsup_{r\longrightarrow\infty} \int_{-r<\phi<-r+1} \rho e^{-\phi} dV \leq \int_S \rho d\mu \tag{4}$$

holds for any nonnegative continuous function ρ with compact support on M.

We say that (E, h) is ϕ-positive if there exists a positive number τ_0 such that $(E, he^{-(1+\tau)\phi})$ are Nakano semipositive on $M\setminus S$ if $\tau \in [0, \tau_0]$. We shall denote the supremum of such τ_0 by $\nu(h, \phi)$.

Let X be a closed subset of M. We say that X is L^2-negligible if, for any point $p \in X$ and for any neighbourhood $V \ni p$, every L^2 holomorphic n-form on $V\setminus X$ is holomorphically extendible to V.

The main result of [1] was

Theorem 1. *Let M be a complex manifold with a C^∞ volume form dV, let E be a holomorphic vector bundle over M with a C^∞ fiber metric h, let S be a closed complex analytic subset of M equipped with a measure $d\mu$, and let $\phi : M \longrightarrow [-\infty, 0)$ be a continuous function with poles S. Suppose that $d\mu$ is a residual majorant of (dV, ϕ), h is ϕ-positive, and that there exists an L^2-negligible set $X \subset$*

M such that $M\backslash X$ is Stein and $S \cap X$ is nowhere dense in S. Then for any plurisubharmonic function ψ on M, there exists an extension operator

$$I : A^2(S, E \otimes K, e^{-\psi} h \otimes (dV)^{-1}, d\mu) \longrightarrow A^2(M, E \otimes K, e^{-\psi} h) \tag{5}$$

whose norm is bounded by a constant depending only on $\nu(h, \phi)$.

Applicability questions aside, this has been all what can be said by extending [2], Any small positive step from here should require a completely new method. The following was announced (essentially), but there turned out to be a gap in the proof, so that I would like to leave it as an open question.

Conjecture 2. *In the situation fo theorem 1, let $x \in S\backslash X$. Assume that $n \geq 3$ and that $\psi = 0$ on a neighbourhood U of x. Then there exists a neighbourhood V of x with $V \Subset U$, such that, for any holomorphic section f of $E \otimes K$ over S satisfying*

$$\int_{S\backslash V} e^{-\psi} |f|^2 d\mu < \infty, \tag{6}$$

there exists a holomorphic extension $\tilde{f}$ of f to M satisfying

$$\int_M e^{-\psi} |\tilde{f}|^2 dV \leq C \int_{S\backslash V} e^{-\psi} |f|^2 d\mu. \tag{7}$$

Here C is a constant depending von h and V, but not on the choices of ψ of f.

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Morse Inequalities and Deformations of Compact Complex Manifolds

Dan Popovici

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Fix a Hermitian metric, that is a C^∞ positive-definite $(1,1)$-form ω, on X. According to [3], a *Kähler current* is a d-closed current T of bidegree $(1,1)$ such that $T \geq \varepsilon\, \omega$ on X for some constant $\varepsilon > 0$. Kähler currents need not exist on an arbitrary X. The problem of characterising compact complex manifolds X that carry a Kähler current was settled in [1] in the following form. Recall that X is said to be a *Fujiki class $\mathcal{C}$ manifold* if there exists a proper holomorphic bimeromorphic map (i.e. a holomorphic modification) $\mu : \tilde{X} \to X$ from a compact *Kähler* manifold $\tilde{X}$ to X. Similarly, X is said to be a *Moishezon manifold* if there exists a holomorphic modification $\mu : \tilde{X} \to X$ such that $\tilde{X}$ is a projective manifold.

Theorem 1. *(a) (***Demailly-Paun** [1]*) A compact complex manifold X is a Fujiki class $\mathcal{C}$ manifold if and only if there exists a Kähler current T on X.*
*(b) (***Ji-Shiffman** [3]*) A compact complex manifold X is Moishezon if and only if there exists a Kähler current T on X whose De Rham cohomology class $\{T\}$ is integral.*

By Kodaira's Embedding Theorem, a projective manifold is a compact Kähler manifold carrying a Kähler form of *integral* De Rham cohomology class. The weaker, bimeromorphically equivalent, versions stand in a similar relation: Moishezon manifolds are the integral class version of Fujiki class $\mathcal{C}$ manifolds as far as Kähler currents are concerned.

The object of our work is to study the behaviour of the projective and Kähler properties of compact complex manifolds in the limit under holomorphic deformations. According to [4], a *complex analytic family of compact complex manifolds* is a proper holomorphic submersion $\pi : \mathcal{X} \to \Delta$ between complex manifolds $\mathcal{X}$ and Δ. Thus the fibres $X_t := \pi^{-1}(t)$ are (smooth) compact complex manifolds of equal dimensions n. Here the base Δ will be a ball about the origin in some $\mathbb{C}^m$. It suffices to take $m = 1$. Our motivation comes from the following.

Conjecture 2. *(***folklore***) Suppose that the fibre X_t is Kähler for every $t \in \Delta^\star := \Delta \setminus \{0\}$. Then the limit fibre X_0 is expected to be a Fujiki class $\mathcal{C}$ manifold.*

Pending a solution of this conjecture, we announce a proof of the following integral class version of it.

Theorem 3. *(*[6]*) Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is projective for every $t \in \Delta^\star := \Delta \setminus \{0\}$. Then the limit fibre X_0 is Moishezon.*

We shall now outline the main steps in the proof of Theorem 3 and a strategy to tackle Conjecture 2. The overarching idea goes back to the work of J. -P. Demailly:

• *Step 1*: choose Kähler forms ω_t on the fibres X_t with $t \neq 0$ in such a way that for a sequence $t_k \to 0$ the forms ω_{t_k} converge weakly to a current T of type $(1,1)$ on the limit fibre X_0;

• *Step 2*: prove Morse inequalities to ensure the existence of a Kähler current S on X_0 in the same De Rham cohomology class as T.

Step 1 has been settled in [6] and is common to the approaches to Theorem 3 and Conjecture 2. Let X denote the differentiable manifold underlying all fibres X_t and let $(J_t)_{t\in\Delta}$ be the holomorphic family of complex structures on $(X_t)_{t\in\Delta}$. Choose now a family $(\gamma_t)_{t\in\Delta}$, varying in a C^∞ way with $t \in \Delta$, of Gauduchon metrics on the X_t's (i.e. each γ_t is a positive-definite $(1,1)$-form for J_t satisfying $\partial_t\bar\partial_t\gamma_t^{n-1} = 0$). Such a family always exists by the proof of Gauduchon's Vanishing Excentricity Theorem [2].

If X_t is projective for all $t \in \Delta^\star$ (as in Theorem 3), there exists a non-zero *integral* De Rham cohomology class $\alpha \in H^2(X, \mathbb{Z})$ such that, for every $t \in \Delta^\star$,

α can be represented by a 2-form which is of J_t-type $(1,1)$. Furthermore, α can be chosen in such a way that α is an ample class on X_t for all $t \neq 0$ outside a countable union Σ' of analytic subsets of $\Delta^\star$. Let $\Sigma := \{0\} \cup \Sigma' \subset \Delta$. The main difficulty in settling *Step 1* is proving uniform boundedness for the masses of Kähler forms ω_t w.r.t. the relevant powers of the Gauduchon metrics γ_t.

Proposition 4. *([6]) The C^∞ family $(\gamma_t)_{t\in\Delta}$ of Gauduchon metrics on the fibres $(X_t)_{t\in\Delta}$ can be chosen with the following property. For every $t \in \Delta \setminus \Sigma$ and every J_t-Kähler form ω_t belonging to the class α, there exists a constant $C > 0$ independent of t such that*

$$0 < \int\limits_{X_t} \omega_t \wedge \gamma_t^{n-1} \leq C, \quad \textit{for all} \quad t \in \Delta \setminus \Sigma,$$

after possibly shrinking Δ about 0.

The proof of this result proceeds comparatively painlessly under the extra assumption that the Hodge number $h^{0,1}(t) := \dim H^{0,1}(X_t, \mathbb{C})$ does not jump at $t = 0$. This will be the case *a posteriori* for all Hodge numbers $h^{p,q}(t)$, but ruling out the jumping of $h^{0,1}(t)$ at $t = 0$ on *a priori* grounds seems to be a daunting task. Our approach in [Pop09] is to prove the uniform mass boundedness above even allowing for the mythical possibility of jumping. To this end, we introduce a new type of metric that we call a *strongly Gauduchon metric*: $\partial_t\gamma_t^{n-1}$ is required to be $\bar\partial_t$-exact rather than just $\bar\partial_t$-closed as the Gauduchon condition required. Unlike Gauduchon metrics, *strongly Gauduchon metrics* need not exist on an arbitrary compact complex manifold. We prove that their existence is equivalent to the non-existence of a non-zero $(1,1)$-current that is both ≥ 0 and d-exact. Moreover, since the $\partial\bar\partial$-lemma holds on every X_t with $t \neq 0$, by the Kähler assumption, every Gauduchon metric on the generic fibre is *strongly Gauduchon*. We prove as well that if the limit fibre X_0 carries a *strongly Gauduchon metric* γ_0, then Proposition 4 follows. Finally, by a positivity argument and a study of eigenvalues, we prove that any C^∞ family $(\gamma_t)_{t\in\Delta}$ of Gauduchon metrics on the fibres X_t can be modified to a C^∞ family of *strongly Gauduchon metrics* (i.e. γ_0 becomes *strongly Gauduchon*) and Proposition 4 follows.

Proposition 4 implies the weak compactness of the family $(\omega_t)_{t\in\Delta\setminus\Sigma}$. One can therefore extract a subsequence (ω_{t_k}), $t_k \to 0$, converging weakly to a current $T \geq 0$ on X_0 which lies in the same *integral* cohomology class α and is of type $(1,1)$ for J_0. If the Kähler forms ω_t have been chosen to have prescribed volume forms by means of the Aubin-Yau theorem, the limit current T is easily seen to satisfy the extra positivity condition $\int\limits_{X_0} T_{ac}^n > 0$, where T_{ac} is the absolutely continuous part of T in the Lebesgue decomposition.

Step 2 consists in an application on X_0 of the *singular Morse inequalities* that were the main result in [5].

Theorem 5. *(reformulation of Theorem 1.3. in [5]) Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Suppose there exists a d-closed $(1,1)$-current T on X whose De Rham cohomology class is integral and which satisfies*

$$(i)\ T \geq 0 \ \ on\ X\,; \quad (ii) \int\limits_X T_{ac}^n > 0.$$

Then the cohomology class of T contains a Kähler current S. Implicitly, X is Moishezon.

In the situation of Conjecture 2, the class α cannot be chosen constant in general. However, it is sufficient for our purposes to choose a family of Kähler classes α_t, $t \neq 0$, such that the volume of α_t is uniformly bounded below by a strictly positive constant. This can be arranged using the Gauss-Manin connection. Thus, *Step 1* can be run identically to the case of Theorem 3 producing a limit current on X_0 satisfying the positivity assumptions (i) and (ii) in Theorem 5. The only difference to the projective case is that the De Rham class of T need not be integral. Conjecture 2 would follow if the integral class assumption on T could be removed from the statement of Theorem 5. This is Demailly's conjecture on transcendental Morse inequalities.

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Estimates for the Kähler-Ricci flow

Duong Hong Phong

(joint work with Jacob Sturm)

Let $(X, \omega_0 = \frac{i}{2} g^0_{\bar{k}j} dz^j \wedge d\bar{x}^k)$ be a compact Kähler manifold of dimension n. The Kähler-Ricci flow is the following parabolic non-linear heat equation

$$\dot{g}_{\bar{k}j}(t) = -(R_{\bar{k}j} - \mu g_{\bar{k}j}), \qquad g_{\bar{k}j}(0) = g^0_{\bar{k}j}. \tag{1}$$

We shall restrict to the case where $c_1(X) > 0$, $\omega_0 \in \pi c_1(X)$, and $\mu = 1$, where the fixed points of the flow are precisely Kähler-Einstein metrics, the characterization of whose existence is still an open question at this time. For a discussion of conjectures of Yau [20] and notions of K-stability due to Tian [18] and Donaldson [5] regarding the existence of constant scalar curvature metrics in a given Kähler polarization, see e.g. [15]. Since the flow preserves the Kähler class, we may set $g_{\bar{k}j} = g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \phi$, $\omega_\phi = \omega_0 + \frac{i}{2}\partial\bar{\partial}\phi$, and rewrite the Kähler-Ricci flow

$$\dot{\phi} = \log \frac{\omega_\phi^n}{\omega_0^n} + \phi - u(z,0), \qquad \phi(0) = c_0 \tag{2}$$

as a flow of potentials, where c_0 is a constant, and for each $g_{\bar{k}j}(t)$, we have defined the Ricci potential $u(z,t)$ by

$$R_{\bar{k}j} - g_{\bar{k}j} = \partial_j \partial_{\bar{k}} u, \qquad \int e^u \omega_\phi^n = \int_X \omega_\phi^n \equiv V. \tag{3}$$

It is well-known that the flow exists for $t \in [0, \infty)$ [1, 19], so the main issue is its convergence.

1. Multiplier ideal sheaves

The convergence of the Kähler-Ricci flow admits the following characterization in terms of multiplier ideal sheaves:

Theorem 1. [10] *Let the constant c_0 in the equation (2) be chosen to be*

$$c_0 = \int_0^\infty e^{-t} \|\nabla \dot{\phi}\|^2_{L^2} dt + \frac{1}{V} \int_X u(z,0) \omega_0^n. \tag{4}$$

(which is a well-defined, finite quantity, independent of the choice of initial conditions for the flow). Then the Kähler-Ricci flow converges if and only if there exists $p > 1$ so that

$$\sup_{t \geq 0} \int_X e^{-p\phi} \omega_0^n < \infty. \tag{5}$$

Besides the (unpublished) works of Perelman, the original proof of this theorem relied heavily on the works of Kolodziej [7]. We note that the characterization given in Theorem 1 of convergence for the Kähler-Ricci flow is very close to Nadel's criterion for the non-existence of Kähler-Einstein metrics [9, 4]. However, it is more complicated here due the facts that the ϕ's are *not* normalized to satisfy $\sup_X \phi = 0$, and that the characterization is in terms of a *dynamical* rather than a *static* multiplier ideal sheaf, in Siu's terminology [17]. A criterion for the Kähler-Ricci flow idential to Nadel's can be found in [16]. For an application of Theorem 1 to the convergence of the Kähler-Ricci flow on Del Pezzo surfaces, see Heier [6] (and also the abstract of his talk at this workshop).

2. The $\bar{\partial}$ operator on vector fields

Next, we discuss some estimates which hold in all generality for the Kähler-Ricci flow. They are formulated in terms of the Ricci potential u, the L^2 norm of its gradient

$$Y(t) = \int_X |\nabla u|^2 \omega_\phi^n \tag{6}$$

and the lowest strictly positive eigenvalue $\lambda(t)$ for the $\bar\partial$ operator on vector fields

$$\lambda(t) = \inf_{W\perp H^0(X,T^{1,0})} \frac{\|\bar\partial W\|^2}{\|W\|^2}. \tag{7}$$

Theorem 2. [11] *The following estimates always hold for the Kähler-Ricci flow on a compact Kähler manifold* (X,ω_0), $\omega_0\in\pi c_1(X)$*:*

(a) $\|R(t)-n\|_{C^0}\leq C\,Y(t-1)^{\frac{1}{2(n+1)}}$

(b) There exists N, $\rho_0,\cdots,\rho_N>0$*, depending on* n *with* $\frac12(\rho_0+\cdots+\rho_N)>1$*, so that*

$$\dot Y(t)\leq -2\lambda(t)Y(t)+2\lambda(t)\mathrm{Fut}(\pi(\nabla u))+C\prod_{j=0}^{N}Y(t-j)^{\frac12\rho_j}. \tag{8}$$

where π *is the orthogonal projection onto the space of holomorphic vector fields, and* Fut *is the Futaki invariant, defined on holomorphic vector fields* W *by*

$$\mathrm{Fut}(W)=\int_X (Wu)\,\omega_\phi^n. \tag{9}$$

Observe that (a) is a strong smoothing statement, since the left hand side can be identified with $\|\Delta u\|_{C^0}$, and (a) asserts that this quantity can be controlled by $\|\nabla u\|_{L^2}$, at the cost of moving a unit back in time. The statement (b) provides a way to establish the exponential decay of $Y(t)$, when the Futaki invariant vanishes and the eigenvalues λ_t are uniformly bounded from below by a positive constant [11]. This last condition is intimately related to a stability condition introduced in [14] with respect to the group $Diff(X)$ of diffeomorphism of X, namely that the closure of the orbit of the complex structure of (X,ω_0) under $Diff(X)$ does not contain any complex structure $\tilde J$ with a strictly higher number of independent holomorphic vector fields.

For applications of Theorem 2 to the Kähler-Ricci flow on manifolds of positive bisectional curvature, see [12, 2]. A different approach is proposed in [3].

For a generalization of Theorem 2 to the modified Kähler-Ricci flow and solitons, see [13].

For a very recent different version of (b) of Theorem 2 with a simpler proof, see [8].

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Gonality, Seshadri numbers and singular potentials

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(joint work with Jun-Muk Hwang)

Let X be a compact Riemann surface of genus $g \geq 2$. A basic invariant associated to the hyperbolic geometry of X is its injectivity radius ρ_X with respect to the hyperbolic metric ds^2_X. The assignment $X \to \rho_X$ defines a function $\rho : \mathcal{M}_g \longrightarrow (0, +\infty)$ on the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. It is known that ρ is a topological Morse function on $\mathcal{M}_g$ (see e.g. [S] where 2ρ is denoted by *syst*). On the other hand, $\mathcal{M}_g$, as the moduli space of algebraic curves of genus g, has been studied extensively in the context of algebraic geometry. In particular, there are many special algebraic subvarieties and algebraic stratifications of $\mathcal{M}_g$ arising from the theory of algebraic curves, which provide rich geometry of $\mathcal{M}_g$ as an algebraic variety.

The present work [5] reported in this talk is an outgrowth of our attempt to study the interplay between the hyperbolic geometry and the algebraic geometry of Riemann surfaces. More specifically, we will investigate the gonality stratification of $\mathcal{M}_g$, which is one of the most fundamental stratifications of $\mathcal{M}_g$. Recall that the gonality $\delta = \delta(X)$ of X is the minimum of the degrees of non-constant surjective holomorphic maps from X onto the Riemann sphere $\mathbb{P}^1$, and it takes values between 2 and $\lceil \frac{g}{2} \rceil + 1$ (with the lower bound being attained precisely when X is a hyperelliptic curve). Here $\lceil x \rceil$ denotes the smallest integer $\geq x$. Thus the values of this algebro-geometric invariant define a stratification of $\mathcal{M}_g$, and it is well known that for each integer δ satisfying $2 \leq \delta \leq \lceil \frac{g}{2} \rceil + 1$, the stratum with gonality δ is a non-empty algebraic subvariety in $\mathcal{M}_g$.

To state our result, we denote the geodesic distance function on X with respect to ds^2_X by $d_X(\cdot,\cdot)$. For each $x \in X$ and $R > 0$, we denote by $B(x,R) := \{y \in X \,|\, d_X(x,y) < R\}$ the geodesic ball of X centered at x and of radius R. It is easy to see that for each $x \in X$, there exists a unique local holomorphic isometry $\sigma_x : B(x,\rho_X) \longrightarrow B(x,\rho_X)$ such that $\sigma_x(x) = x$ and $d\sigma_x\big|_{T_xX} = -\mathrm{Id}_x$, where Id_x is the identity map on T_xX. Note that the individual σ_x's may be defined on a possibly bigger domain than $B(x,\rho_X)$. Next we consider the Cartesian product $X \times X$, and denote its diagonal by $D := \{(x,y) \in X \times X \,|\, x = y\}$. For any given $r > 0$, we consider the *geodesic tubular neighborhood of D in $X \times X$* given by $W_r := \{(x,y) \in X \times X \,|\, d_X(x,y) < r\} \supset D$. For each $x \in X$, we denote the graph of σ_x by $\mathrm{Graph}(\sigma_x) := \{(y,\sigma_x(y)) \,|\, y \in B(x,\rho_X)\} \subset X \times X$. It is easy to see that $\widehat{D}_{(x,x)} := \mathrm{Graph}(\sigma_x) \cap W_{\rho_X}$ is a 1-dimensional complex submanifold of W_{ρ_X} intersecting D only at (x,x), and we call $\widehat{D}_{(x,x)}$ the *anti-diagonal* of W_{ρ_X} at (x,x). For a complex analytic subvariety V of an open set in $X \times X$, we simply denote by $\mathrm{Vol}\,(V)$ its volume with respect to the product metric on $X \times X$ induced by ds^2_X. It is easy to see that for each $0 < r \leq \rho_X$, the value of $\mathrm{Vol}\,(\widehat{D}_{(x,x)} \cap W_r)$ is the same for all $x \in D$, and we simply denote the common value by $\mathrm{Vol}\,(\widehat{D}_{(\cdot,\cdot)} \cap W_r)$. We denote by $K_{X\times X}$ the canonical line bundle on $X \times X$. The Seshadri number $\epsilon(K_{X\times X}, D)$ of $K_{X\times X}$ along D is simply given by

$$\epsilon(K_{X\times X}, D) := \sup\{\, \epsilon \in \mathbb{R} \,\big|\, K_{X\times X} - \epsilon D \text{ is nef on } X \times X\}$$

(see e.g. [8], Remark 5.4.3). Our main result is the following

Theorem 1. *([5]) Let X be a compact Riemann surface of genus ≥ 2.*
(i) Then for any real number r satisfying $0 < r \leq \rho_X$ and any purely 1-dimensional complex analytic subvariety V of the geodesic tubular neighborhood W_r of the diagonal D in $X \times X$, one has

$$Vol(V) \geq 8\pi \sinh^2(\frac{r}{4}) \cdot (V \cdot D) = Vol(\widehat{D}_{(\cdot,\cdot)} \cap W_r) \cdot (V \cdot D).$$

In particular, for each $0 < r \leq \rho_X$ and each value s of $V \cdot D$, the above lower bound is attained by the volume of some (and hence any) V consisting of the intersection of W_r with the union of s copies of anti-diagonals counting multiplicity.
(ii) As a consequence, for all $\alpha \in \mathbb{R}$ satisfying $0 \leq \alpha \leq 4\sinh^2(\frac{\rho_X}{4})$, the $\mathbb{R}$-divisor class $K_{X\times X} - \alpha D$ is nef on $X \times X$; and equivalently, one has

$$\epsilon(K_{X\times X}, D) \geq 4\sinh^2(\frac{\rho_X}{4}).$$

Our approach for obtaining the lower bound for $\epsilon(K_{X\times X},\, D)$ (or more generally, that for the volumes of complex analytic varieties in $X \times X$) is adapted from the authors' earlier works ([2], [3]) on an analogous problem for the Seshadri number of the canonical line bundle at a given point for a smooth compact quotient

Y of a bounded symmetric domain Ω (see also [4]). An essentially new difficulty in our present situation is that the product metric on $X \times X$ does not have a Kähler potential near D, whereas in [2] and [3], the Bergman kernel function of Ω descends readily to a local potential for the Kähler metric (induced from the Bergman metric on Ω) near each point in Y. To overcome this, we introduce an auxiliary semi-Kähler form which does admit a potential function in geodesic tubular neighborhoods of D (the choice of the auxiliary semi-Kähler form is essentially unique if optimal result is to be achieved). Thereafter, an analogue of the argument in [2] and [3] is carried out to modify its potential function into a singular one with desired pole-order along D. From Theorem 1, one obtains the following

Corollary 2. *([5]) Let X be a compact Riemann surface of genus $g \geq 2$ and of gonality δ. Then we have* $\rho_X \leq 2\cosh^{-1}\left(\frac{g\delta}{g+\delta-1}\right) \leq 2\cosh^{-1}\delta.$

The special case of Corollary 2 for hyperelliptic Riemann surfaces (i.e. when $\delta = 2$) also follows from the argument (with minor adaptation) of Katz-Sabourau in ([7], Proof of Proposition 3.6). However, the argument of Katz-Sabourau depends on the existence of hyperelliptic involutions for such Riemann surfaces, and it does not appear to generalize readily to the cases when $\delta > 2$. From Corollary 1, one deduces readily the the following

Corollary 3. *([5]) Let $\{X_i\}_{i=1}^{\infty}$ be a tower of compact (hyperbolic) Riemann surfaces. Then we have* $\delta(X_i) \longrightarrow \infty$ *as* $i \longrightarrow \infty$.

We remark that with $\{X_i\}_{i=1}^{\infty}$ as in Corollary 2, a result of Yeung [10] implies that K_{X_i} is very ample, i.e., $\delta(X_i) \geq 3$, for all sufficiently large i.

Next we recall from ([8], p. 76) the invariant associated to a compact Riemann surface X of genus $g \geq 2$ given by

$$t(X) := \inf\{t > 0 \,|\, \frac{t+1}{2g-2}K_{X\times X} - D \text{ is nef on } X \times X\} = \frac{2g-2}{\epsilon(K_{X\times X}, D)} - 1,$$

which is useful in the study of the ample cone over the symmetric product S^2X of X (see e.g. [8], p. 78). It is also known that one always has $t(X) \leq g$ and that $t(X) \leq g-1$ if and only if X is non-hyperelliptic (cf. e.g. [8], p. 77]). Moreover, it follows from a result of Kouvidakis [6] that $t(X) \leq \frac{g}{[\sqrt{g}]}$ for very general X (see [8], p. 76]), where $[x]$ denotes the greatest integer $\leq x$. This upper bound of $t(X)$ is not known to hold over an open subset of $\mathcal{M}_g$. Nonetheless, by combining Theorem 1 and the explicit examples of Riemann surfaces constructed by Buser-Sarnak [1], we have

Corollary 4. *([5]) There exist a constant $c > 0$, a strictly increasing sequence of positive integers $\{g_i\}$ and a corresponding sequence of non-empty open subsets $U_i \subset \mathcal{M}_{g_i}$ (in the classical topology) such that* $t(X) \leq c \cdot g_i^{\frac{2}{3}}$ *for all* $X \in U_i$, $i = 1, 2, 3, \cdots$.

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Seshadri constants and generation of jets

Thomas Bauer

(joint work with Christoph Schulz and Tomasz Szemberg)

Seshadri constants. Consider a smooth projective variety X of dimension n over $\mathbb{C}$ and an ample line bundle L on X. The *Seshadri constant* of L at a point $x \in X$ is the number

$$\varepsilon(L,x) \stackrel{\text{def}}{=} \max\{\,\varepsilon \geq 0 \mid f^*L - \varepsilon E \text{ nef}\,\} \in \mathbb{R} \ ,$$

where f is the blow-up of X at x with exceptional divisor E over x. This invariant, introduced by Demailly in [6], may be thought of as measuring the local positivity of L at the point x. One has the two basic estimates

$$0 < \varepsilon(L,x) \leq \sqrt[n]{L^n} \ ,$$

where the first one comes from (the easy part of) Seshadri's criterion for ampleness, and the second one from Kleiman's theorem.

While Seshadri constants were originally introduced in the context of the Fujita conjecture, they are today viewed as interesting invariants in their own right. The guiding questions are:

- What geometric information do Seshadri constants encode?
- What upper and lower bounds – or even explicit values – can one find for surfaces and higher-dimensional varieties?

Local positivity on abelian varieties. Consider a polarized abelian variety (X, L) of dimension n. The two basic facts on Seshadri constants, which both come from homogeneity, are:

- The number $\varepsilon(L, x) = \varepsilon(L)$ does not depend on the point $x \in X$, and
- one has the lower bound $\varepsilon(L) \geq 1$.

The author has shown in [1] the lower bound

$$\varepsilon(L) \geq \frac{1}{4}\sqrt[n]{2L^n}$$

for the very general polarized abelian variety (X, L) of fixed type. For abelian surfaces of Picard number one, the Seshadri constants can even be explicitly computed by means of a diophantine equation (see [2, Sect. 6]): If L is ample of type $(1, d)$, then

$$\varepsilon(L) = 2d \cdot \frac{k_0}{\ell_0} = \frac{2d}{\sqrt{2d + \frac{1}{k_0^2}}} \ ,$$

where (k_0, ℓ_0) is the minimal solution of Pell's equation

$$\ell^2 - 2dk^2 = 1 \ .$$

Recently, we studied in joint work with Ch. Schulz [3] the case, where X is the self-product $E \times E$ of an elliptic curve. While the product $E \times E$ might seem to be an easy case at first glance, the challenge here is to determine $\varepsilon(L)$ explicitly for *all* ample line bundles L on X. Note that the ample cone is 4-dimensional or 3-dimensional, depending on whether E has complex multiplication or not. In [3] we determine these Seshadri constants, which in effect means that we can explicitly describe the *Seshadri function*

$$\mathrm{Nef}(X) \longrightarrow \mathbb{R}, \quad L \mapsto \varepsilon(L) \ .$$

on the nef cone of X. Interesting features in this situation are: (1) All Seshadri constants (of integral line bundles) are integers, as they are computed by elliptic curves, and (2) every elliptic curve "matters", i.e., for each of the countably many elliptic curves $F \subset X$ there is an ample line bundle L on X such that F computes the Seshadri constant $\varepsilon(L)$. Our methods use some nice geometry of numbers: To show the existence of the elliptic curves in question amounts to finding lattice points in suitably small balls.

Generation of jets. One knows from [6, Theorem 6.4] that the Seshadri constant $\varepsilon(L, x)$ of an ample line bundle L at a point x on a smooth projective variety can be characterized as the rate of growth of the number of jets that the linear series $|kL|$ generate at x for $k \gg 0$: One has

$$\varepsilon(L, x) = \lim_{k \longrightarrow \infty} \frac{s(kL, x)}{k} \ ,$$

where $s(kL, x)$ is the number of jets that $|kL|$ generates at x, i.e. the maximal integer s such that the evaluation map $H^0(X, kL) \longrightarrow H^0(X, kL \otimes \mathcal{O}_X/m_x^{s+1})$ is surjective. So, if we knew the sequence of integers $(s(kL, x))_k$ (which is a rather theoretical possibility), then we could just compute $\varepsilon(L,x)$ as the limit of a sequence. In joint work with T. Szemberg [4] we study the converse question:

Once $\varepsilon(L,x)$ is known, what can we say about the sequence of numbers $(s(kL,x))_k$?

We consider first Fano varieties, where our results yield a nice characterization of projective space in terms of Seshadri constants, which also follows from work of Bonavero, Campana, and Wiśniewski [5]:

If X is a Fano variety of dimension n such that $\varepsilon(-K_X,x) = n+1$ for some $x \in X$, then $X \simeq \mathbb{P}^n$.

On Fano varieties different from $\mathbb{P}^n$ one has $\varepsilon(-K_X,x) \le n$ for all points x (see [4, Theorem 2]).

Secondly, we study varieties with trivial canonical bundle. If X is such a variety, $n = \dim(X)$, and L ample, then an immediate application of vanishing for big and nef divisors yields the following estimates:

(a) If $\varepsilon(L,x) = \sqrt[n]{L^n}$ and $\varepsilon(L,x) \in \mathbb{Z}$, then

$$k \cdot \varepsilon(L,x) - (n+1) \le s(kL,x) \le k \cdot \varepsilon(L,x) \ ,$$

(b) else

$$\lfloor k \cdot \varepsilon(L,x) \rfloor - n \le s(kL,x) \le \lfloor k \cdot \varepsilon(L,x) \rfloor \ .$$

(Here $\lfloor \cdot \rfloor$ denotes the round-down.) So there are for every $k \ge 1$ only $n+2$ resp. $n+1$ possible values for the number $s(kL,x)$. In concrete situations, we can provide further restrictions on the possible values: For instance, on smooth quartic surfaces $X \subset \mathbb{P}^3$, we have for $k \gg 0$ only the two possibilities

$$s(\mathcal{O}_X(k),x) = 2k-3 \quad \text{or} \quad s(\mathcal{O}_X(k),x) = 2k-2 \ .$$

A particularly intriguing example is given by theta functions: Consider a principally polarized abelian surface (X,Θ). One knows from [7] that $\varepsilon(\Theta) = \frac{4}{3}$, so that there are by (b) for every k only three possible values for the number $s(k\Theta,x)$ of jets that pluri-theta divisors generate. While $\varepsilon(\Theta,x) = \varepsilon(\Theta)$ is independent of the point x, the numbers $s(k\Theta,x)$ do depend on x, at least for $k=1$ and $k=2$. Here are two open questions, whose answer would be very interesting to know:

- Is $s(k\Theta,x)$ independent of x when $k \gg 0$?
- Is the sequence of numbers $s(k\Theta,x)$ (for large k) the same for every principally polarized abelian surface (X,Θ) or does it depend on the moduli?

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Seshadri constants and geometry of surfaces

Tomasz Szemberg
(joint work with Thomas Bauer, Andreas Knutsen, Wioletta Syzdek)

Let X be a smooth projective variety and L a nef line bundle on X. Recall that the number

$$\varepsilon(X, L; x) := \inf \frac{L \cdot C}{\operatorname{mult}_x C}$$

is the *Seshadri constant of* L *at the point* $x \in X$ (the infimum being taken over all irreducible curves C passing through x). The Seshadri constant $\varepsilon(X)$ of the variety X is defined taking the infimum over all points $x \in X$ and all ample line bundles L. It is not known if the Seshadri constant $\varepsilon(X)$ is positive in general (this question was posed by Demailly already in his fundamental paper [3, Question 6.9]). If the Picard number of the variety is 1, then $\varepsilon(X) > 0$ by the Seshadri criterion for ampleness and one may ask for some effective estimates. We recall the following result from [7, Theorem 7].

Theorem 1. *Let S be a smooth projective surface with $\rho(S) = 1$ and let L be an ample line bundle on S. Then for any point $x \in S$*

(S) $\varepsilon(S, L; x) \geq 1$ *if S is not of general type and*
(G) $\varepsilon(S, L; x) \geq \frac{1}{1+\sqrt[4]{K_S^2}}$ *if S is of general type.*

Moreover both bounds are sharp.

For surfaces with arbitrary Picard number we pose the following conjecture.

Conjecture 2. *Let S be a minimal surface, L an ample line bundle on S and $x \in S$ an arbitrary point, then*

$$\varepsilon(S, L; x) \geq \frac{1}{2 + \sqrt[4]{|K_S^2|}}.$$

The situation becomes much nicer if one studies very general points instead of arbitrary. This is due to the fact that Seshadri constants considered as a function of x are lower semi-continuous in the topology which closed sets are at most countable unions of Zariski closed sets. In particular there is an open subset in the topology on which the Seshadri numbers are constant and maximal among values attained for a fixed L at all points x of X. This maximal value will be denoted by $\varepsilon(X, L; 1)$. For surfaces, we have the following result of Ein and Lazarsfeld [4].

Theorem 3 (Ein-Lazarsfeld)**.** *Let X be a smooth projective surface and L a nef and big line bundle on X. Then*

$$\varepsilon(X, L; 1) \geq 1.$$

There are easy examples showing that one cannot get a better lower bound in general. However, following some ideas of Nakamaye [6] we show that examples of this kind can be classified.

Theorem 4. *Let X be a smooth surface and L an ample line bundle on X. If*

$$\varepsilon(X, L; 1) < \sqrt{\frac{7}{9}} \cdot \sqrt{L^2} \ ,$$

then

(a) *either X is fibred by curves computing Seshadri constant of L at their general points, or*
(b) *X is a rational surface.*

The key ingredient for the proof is the following Lemma, which was proved in [5] and independently by Bastianelli in [1].

Lemma 5. *Let S be a smooth projective surface. Suppose that $\mathcal{C}_U = \{(C_u, x_u) \mid u \in U\}$ is a family of pointed curves as above parametrized by a 2-dimensional subset $U \subset \mathrm{Hilb}(S)$ and C is a general member of this family. Let $\widetilde{C}$ be its normalization. Then*

$$C^2 \geq m(m-1) + \mathrm{gon}(\widetilde{C}),$$

where gon(D) denotes the gonality of a curve D.

As a consequence we get the following strengthening of results in [2, Thms. 2 and 3] characterizing genus 2 fibrations on surfaces of general type.

Corollary 6. *Let X be a smooth projective surface such that K_X is big and nef. If $K_X^2 \geq 5$, then either*

(a) *$\varepsilon(K_X, 1) > 2$, or*
(b) *$\varepsilon(K_X, 1) = 2$, and there exists a pencil of curves of genus 2 computing $\varepsilon(K_X, x)$ for x very general.*

In the opposite direction, if X is a smooth minimal surface of general type such that there is a genus 2 fibration $f : X \longrightarrow B$ over a smooth curve B, then

$$\varepsilon(K_X, 1) \leq 2 \ ,$$

and if $K_X^2 \geq 4$, then actually

$$\varepsilon(K_X, 1) = 2 \ .$$

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Lower bounds for Seshadri constants on $\mathbb{CP}^2$ blown up in 10 points

Thomas Eckl

Conjecture 1 (Nagata, [7]). *Let $p_1, \ldots, p_n$ be $n \geq 10$ points on $\mathbb{CP}^2$ in general position, and let $\pi : X \to \mathbb{CP}^2$ be the blow up of these n points. Furthermore, call H the divisor class of a line on $\mathbb{CP}^2$, and denote the exceptional divisor over p_i with E_i. Then the $\mathbb{R}$-divisor*

$$\pi^* H - \epsilon \sum_{i=1}^{n} E_i$$

is nef iff $0 \leq \epsilon \leq \frac{1}{\sqrt{n}}$.

Hence $\frac{1}{\sqrt{n}}$ is the multi-point Seshadri constant of $p_1, \ldots, p_n \in \mathbb{CP}^2$, for the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$.
It is well known that Nagata's conjecture can be deduced from another conjecture on the dimension of linear systems on $\mathbb{CP}^2$ (see e.g. [1]):

Conjecture 2 (Harbourne-Gimigliano-Hirschowitz [5, 4, 6]). *Let $p_1, \ldots, p_n$ be n points on $\mathbb{CP}^2$ in general position, and let $\pi : X \to \mathbb{CP}^2$ be the blow up of these n points. Furthermore, call H the divisor class of a line on $\mathbb{CP}^2$, and denote the exceptional divisor over p_i with E_i. Given a degree d and n multiplicities $m_1, \ldots, m_n$, the linear system $|d\pi^* H - \sum_{i=1}^{n} m_i E_i|$ has the expected dimension*

$$\max(-1, \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2})$$

iff there exists no (-1)-curve C on X such that

$$C.(d\pi^* H - \sum_{i=1}^{n} m_i E_i) \leq -2.$$

An effective way of this deduction can be given, by restricting the linear systems, for which the Harbourne-Hirschowitz conjecture must be checked. This uses a new characterization of multi-point Seshadri constants:

Theorem 3 (Eckl [3]). *Let L be an ample on a smooth projective complex surface X, and $p_1, \ldots, p_n \in X$ arbitrary points. Then the multi-point Seshadri constant of $p_1, \ldots, p_n$ for L equals*

$$\sup_{k; D_1, D_2} \frac{\min_j \min_{i=1,2}(\mathrm{mult}_{x_j} D_i)}{k},$$

where the supremum is taken over all pairs of divisors $D_1, D_2 \in |kL|$ *such that* $p_1, \ldots, p_n$ *are isolated points in* $D_1 \cap D_2$.

With this characterization we can show

Theorem 4 (Eckl [3]). *Let* $p_1, \ldots, p_n$ *be* $n \geq 10$ *points on* $\mathbb{CP}^2$ *in general position, and let* $\pi : X \to \mathbb{CP}^2$ *be the blow up of these* n *points. If* (d_i, m_i), $i \in \mathbb{N}$, *is a sequence of integer pairs, such that the linear system* $|d_i \pi^* H - m_i \sum_{j=1}^{n} E_j|$ *is non-empty of expected dimension, and* $\frac{d_i}{m_i} \stackrel{i\to\infty}{\longrightarrow} \frac{a}{\sqrt{n}}$ *then the* $\mathbb{R}$*-divisor*

$$\pi^* H - \frac{a}{\sqrt{n}} \sum_{j=1}^{n} E_j$$

is nef on X.

An often used method to study linear systems of the form $|d\pi^* H - \sum_{i=1}^{n} m_i E_i|$ on $\mathbb{CP}^2$ blown up in n points is the degeneration method:

> Degenerate $\mathbb{CP}^2$ blown up in n points into a union of varieties being the blow up of $\mathbb{CP}^2$ in less points (or points in special position), degenerate the linear system, prove that the degenerate linear system restricted to each of the components has expected dimension, try to glue along the intersections, and conclude by using semi-continuity.

In a recent preprint [2], Ciliberto and Miranda set up an iterative procedure which improves in each step the bounds obtained from such a degeneration. They start by blowing up and flopping a (-1)-curve destroying the non-specialty in one of the degeneration components. Such a curve must exist if we believe in the Harbourne-Hirschowitz conjecture. Then they twist the line bundle of the complete linear system in question appropriately, and continue to work on the flopped degeneration. The flopping improves the situation on the component with the bad (-1)-curve, without worsening it on other components in the cases considered by Ciliberto and Miranda. In this way they obtain the (up to now) best known lower bound $\frac{55}{174}$ for the Seshadri constant of 10 points on $\mathbb{CP}^2$.
As expected some streamlining of this machinery is achieved by using Theorem 4 from above, because with its help it is not necessary to study all linear systems $|d\pi^* H - m \sum_{j=1}^{10} E_j|$ with $\frac{d}{m}$ above a certain bound. Furthermore it may be more convenient not to work on the complete flops but on the intermediate blow ups: The flops introduce non-normal components, and to argue on linear systems restricted to them anyway requires to go back to their normalizations. Finally, one can use the special position of the points blown up on the components and uniformly apply the following generalization of a criterion of Harbourne [5], or variants of it, to obtain non-specialty of linear systems:

Proposition 5. *Let* $p_1, \ldots, p_n$ *be* n *points on* $\mathbb{CP}^2$ *in general position, and let* $\pi : X \to \mathbb{CP}^2$ *be the blow up of these* n *points. Furthermore, let* F *be a line*

bundle on X such that F is not he pull back of a line bundle on $\mathbb{CP}^2$ blown up in $p_1, \dots, p_{n-1}$.
Suppose that $C \subset X$ is a reduced curve with irreducible components $C_1, \dots, C_k$ such that

$$(K_X + C).C_i < F.C_i, \quad i = 1, \dots, k.$$

Then the linear system of global sections of F is non-special if the linear system of global sections of $F \otimes \mathcal{O}(-C)$ is non-special.

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Invariant multiplier ideal sheaves on Fano surfaces

Gordon Heier

In the seminal paper [3], Nadel introduced the notion of a multiplier ideal subsheaf of the sheaf of holomorphic functions on a complex manifold. He proved that the non-existence of certain invariant multiplier ideal sheaves is a sufficient criterion for the existence of a Kähler-Einstein metric on a given Fano manifold.

In a much more recent development, the paper [4] gave an analogous, but slightly weaker, sufficient criterion for the convergence of the Kähler-Ricci flow on a given Fano manifold.

This talk will be about the contents of the papers [1] and [2], whose goal is to clarify the question to what extent the above-mentioned sufficient criteria can be used to prove the existence of Kähler-Einstein metrics or the convergence of the Kähler-Ricci flow on Fano surfaces. The main results are as follows.

Theorem 1. *Let X be a Fano surface obtained by blowing up $\mathbb{P}^2$ in $3, 4,$ or 5 points in general position. Then the conditions in Nadel's sufficient criterion are satisfied.*

Theorem 2. *Let X be one of the following Fano surfaces.*

(1) *$\mathbb{P}^2$ blown up in 4 points in general position,*
(2) *$\mathbb{P}^2$ blown up in 5 points in general position with* $\mathrm{Aut}(X) = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4, \mathbb{Z}_2^4 \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2)$, *or* $\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_5 \rtimes \mathbb{Z}_2)$.

Then the conditions in Phong-Sesum-Sturm's sufficient criterion are satisfied.

The proofs draw on a variety of techniques from algebraic geometry and group theory, including the classification of Fano surfaces, cohomology vanishing theorems and the representation theory of finite groups. A sketch of the proofs for the case of 4 points will be given in the talk. The interested reader can find the exact nature of the sufficient criteria, all details of the proofs as well as some further results in [1] and [2].

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Finite generation, non-vanishing etc.

Vladimir Lazić

It has been a main goal of Algebraic Geometry since the work of the Italian school on surfaces at the turn of the twentieth century to find a meaningful classification of algebraic varieties, or at least of smooth ones. It has become clear in the 1980s that the category of non-singular varieties is not large enough, and that certain mild singularities must be allowed. Even if a variety X is smooth, the notion of *singularities* is represented by a presence of a divisor Δ on X such that the *adjoint bundle* $K_X + \Delta$ satisfies certain conditions. We call (X, Δ) a *log pair.* Pairs appear naturally even on surfaces, as a consequence of Kodaira's bundle formula. The corresponding classification theory is now known as the Minimal Model Program, or Mori Theory.

If (X, Δ) is a pair, we can associate to it a *log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0\big(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor)\big),$$

where $\lfloor\cdot\rfloor$ denotes the component-wise round-down of a Weil divisor. It is a long standing conjecture, present in various forms since [8] appeared, that the log canonical ring is finitely generated when singularities are mild. The conjecture has many important structural consequences for the geometry of X.

The most widely used class is that of *kawamata log terminal*, or klt, singularities. Namely, we say that a pair (X, Δ) is klt if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$, where $\mathcal{J}(X, \Delta)$ is the corresponding multiplier ideal. Then it was proved in the remarkable paper [1] that a log canonical ring of a klt pair is finitely generated. The proof therein uses heavily techniques of the Minimal Model Program, and derives the finite generation as a standard consequence of the general scheme of the theory.

In my talk I present a new approach to the problem, which avoids completely Mori theory, and is by induction on the dimension. The natural idea is to pick a smooth divisor S on X and to restrict the algebra to it. If the restricted algebra is finitely generated we might hope that the generators lift to generators of $R(X, K_X + \Delta)$. In order to obtain something meaningful on S, by adjunction formula S should appear with coefficient 1 in Δ, which needs to be arranged.

However, even if the restricted algebra were finitely generated, the same might not be obvious for the kernel of the restriction map. So far this seems to have been the greatest conceptual issue in attempts to prove the finite generation by the plan just outlined. The idea to resolve the kernel issue is to view $R(X, K_X + \Delta)$ as a subalgebra of a larger algebra, which would a priori contain generators of the kernel. In practice this means that the new algebra will have higher rank grading.

The main result is thus [4], [5]:

Theorem. *Let X be a projective variety, and for $i = 1, \ldots, \ell$ let $D_i = k_i(K_X + \Delta_i + A)$ be an integral divisor on X, where A is an ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a klt pair. Then the graded ring*

$$R(X; D_1, \ldots, D_\ell) = \bigoplus_{(m_1,\ldots,m_\ell)\in\mathbb{N}^\ell} H^0\big(X, \mathcal{O}_X(m_1D_1 + \cdots + m_\ell D_\ell)\big)$$

is finitely generated.

The basic theory of higher rank finite generation is developed in [3] and related to so-called *b-divisors*. The technical core of the paper rests on deep extension theorems developed in [2] which are based on [7], and on techniques from [3] which are used to prove that certain superlinear functions are piecewise linear.

Parts of the proof involve showing that certain convex sets in the set of divisors on X are in fact rational polytopes. One of these sets contains divisors Δ such that, for a fixed ample divisor A, $K_X + \Delta + A$ is pseudo-effective. Therefore it is natural to use a characterisation of pseudo-effectivity for such adjoint bundles from [6], also derived without Mori theory. The result states that if such an adjoint divisor is pseudo-effective, then in fact there is an effective divisor numerically equivalent to it. I discuss this and related results also in the talk.

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Generic semi-positivity of cotangent bundles

Thomas Peternell

We fix a projective manifold X of dimension n. The aim of the lecture was to present and discuss semi-positivity properties of the cotangent bundle of X, the prototype of these results being Miyaoka's famous theorem:

The cotangent bundle of a projective manifold is "generically nef" unless the manifold is uniruled.

Our first result is the following sharpening of Miyaoka's uniruledness criterion:

Theorem 1. *Let X be a projective manifold, $(\Omega^1_X)^{\otimes m} \longrightarrow \mathcal{S}$ a torsion free coherent quotient for some $m \in \mathbb{N}$. Then* $\det \mathcal{S}$ *is pseudo-effective if X is not uniruled.*

A vector bundle E is *generically nef* if $E|C$ is nef on the general curve cut out by very ample linear systems of sufficiently high degree.
A line bundle L is *pseudo-effective* if $c_1(L)$ lies in the closure of the Kähler cone. To sharpen generic nefness to pseudo-effectivity in the theorem, we use the characterization [1] of pseudo-effective line bundles by moving curves which are images of very ample curves above by birational morphisms. Our proof here is not entirely algebro-geometric (Mehta-Ramanathan no longer applies), and rests on analytic methods. In fact we use the following result due to M. Toma, which in turn is the key to Theorem 2, a main ingredient in the proof of Theorem 1.

Theorem. *Let α be a class in the interior of $\overline{ME}(X)$ and $\mathcal{E}$ and $\mathcal{F}$ two α-polystable locally free sheaves. Then $\mathcal{E} \otimes \mathcal{F}$ is again α-polystable.*

Recall that $\overline{ME}(X)$ denotes the movable cone of the $n-$dimensional projective manifold X. We say that $\alpha \in \overline{ME}(X)$ is *geometric*, if there exists a modification $\pi : \tilde{X} \longrightarrow X$ from the projective manifold $\tilde{X}$ and ample line bundles H_i such that

$$\alpha = \lambda \pi_*(H_1 \cap \ldots \cap H_{n-1})$$

with a positive multiple λ. By definition, $\overline{ME}(X)$ is the closed cone generated by the geometric classes.
If $\lambda \in \mathbb{Q}_+$, we say that α is *rational geometric*.
If $\mathcal{E}$ and $\mathcal{F}$ are torsion free sheaves, then we put

$$\mathcal{E} \hat{\otimes} \mathcal{F} = (\mathcal{E} \otimes \mathcal{F})/\mathrm{tor}.$$

Toma's result is used to show the following - well-known in case of an ample polarization.

Theorem 2. Let $\alpha \in \overline{ME}(X)$ be a rational geometric class and let $\mathcal{E}$ and $\mathcal{F}$ be $\alpha-$semi-stable torsion free sheaves on X. Then $\mathcal{E} \hat{\otimes} \mathcal{F}$ is again $\alpha-$semi-stable.

An important consequence of Theorem 1 is:

Theorem 3. *Let X be a projective manifold. Suppose that Ω^p_X contains for some p a subsheaf whose determinant is big (i.e. has Kodaira dimension $n = \dim X$). Then K_X is big, i.e. $\kappa(X) = n$.*

The uniruledness criterion of Theorem 1 has also other applications, e.g. one can prove that a variety admitting a section in a tensor power of the tangent bundle with a zero, must be uniruled.

Theorem 1 is actually a piece in a larger framework. To explain this, we consider subsheaves $\mathcal{F} \subset \Omega^p_X$ for some $p > 0$. Then one can form $\kappa(\det \mathcal{F})$ and take the supremum over all $\mathcal{F}$. This gives a refined Kodaira dimension $\kappa^+(X)$, introduced in [2]. Conjecturally

$$\kappa^+(X) = \kappa(X) \qquad (*)$$

unless X is uniruled. Theorem 3 is nothing but this conjecture in case $\kappa^+(X) = \dim X$.
We prove in [3] the conjecture (*) in several other cases. It is actually a consequence of the following more general conjecture, which moreover deals only with line bundles:

Conjecture: *Suppose X is a projective manifold, and suppose a decomposition*

$$NK_X = A + B$$

with some positive integer N, an effective divisor A (one may assume A spanned) and a pseudo-effective line bundle B. Then

$$\kappa(X) \geq \kappa(A).$$

The special case $A = \mathcal{O}_X$ implies that $\kappa(X) \geq 0$ if X is not uniruled, using the preceeding result, and the pseudo-effectiveness of K_X when X is not uniruled ([1]).
In another direction we establish the special case in which B is numerically trivial:

Theorem 4. *Let X be a projective complex manifold, and $L \in \mathrm{Pic}(X)$ be numerically trivial. Then:*

(1) $\kappa(X, K_X + L) \leq \kappa(X)$.
(2) *If $\kappa(X) = 0$, and if $\kappa(X, K_X + L) = \kappa(X)$, then L is a torsion element in the group $\mathrm{Pic}^0(X)$.*

In particular, if mK_X is numerically equivalent to an effective divisor, then $\kappa(X) \geq 0$.

This result permits, in particular, to handle numerically trivial line bundles in the study of the conjecture $C_{n,m}$ on irregular manifolds.

Another application of Theorem 1 concerns the study of universal covers $\tilde{X}$ of complex projective n-dimensional manifolds X. The Shafarevich conjecture asserts that $\tilde{X}$ is holomorphically convex, i.e. admits a proper holomorphic map onto a Stein space. There are two extremal cases:

- either $\tilde{X}$ is compact and so $\pi_1(X)$ is finite or
- $\tilde{X}$ is a modification of a Stein space, hence through the general point of $\tilde{X}$ there is no positive-dimensional compact subvariety.

This latter case happens in particular for X a modification of an Abelian variety or a quotient of a bounded domain. It is conjectured (see [5], and [4] for the Kähler case) that X should then admit a holomorphic submersion onto a variety of general type with Abelian varieties as fibres, after a suitable finite étale cover and birational modification. This follows up to dimension 3 from the solutions of the conjectures of the Minimal Model Program. We prove here a special case and a weaker statement in every dimension:

Theorem 5. *Let X be a normal n-dimensional projective variety with at most rational singularities.*
(1) Suppose that the universal cover of X is not covered by its positive-dimensional compact subvarieties. Then X is of general type if $\chi(\mathcal{O}_X) \neq 0$.
(2) If X has at most terminal singularities and $\tilde{X}$ does not contain any compact subvariety of positive dimension (eg. X is Stein), then either K_X is ample, or K_X is nef, $K_X^n = 0$, and $\chi(\mathcal{O}_X) = 0$.

This theorem is deduced from Theorem 1 via the comparison theorem [2], which relates the geometric positivity of subsheaves in the cotangent bundle to the geometry of $\tilde{X}$.

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From Multiplier ideals to equicharacteristic étale sheaves

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(joint work with Mircea Mustaţă)

Building upon the work of the school of tight closure – founded by Hochster and Huneke in the early nineties – Hara, Yoshida, and Takagi [9, 8] defined generalized test ideals as a positive characteristic, resolution free variant of multiplier ideals. A question – trivial for multiplier ideals – whether the jumps (called F-thresholds) in the resulting filtration are rational and discrete was not answered in the original treatment. Our goal in this talk is to show how techniques which are at the heart of proving discreteness [4, 3] are related to the cohomology of p-torsion étale sheaves. We will illustrate this with an example of Miller [10] who gives a bound on the Euler characteristic of p-torsion étale sheaves on smooth curves.

When measuring the severity of singularities the standard approach (in characteristic zero) is to use resolution of singularities. As this is not (yet) available in positive characteristic, one has to resort to alternative methods. The key there is the classical result of Kunz which states that a local ring is regular if and only if the pth power map, i.e. the Frobenius, is flat. This suggests that one could use the failure of the flatness of the Frobenius in the study of singularities.

We let $X = \operatorname{Spec} R$ be a regular and F-finite scheme over a field of positive characteristic. Then the Frobenius acts on the set of ideals of R by sending an ideal $\mathfrak{a}$ of R to $F(\mathfrak{a}) = \mathfrak{a}^{[p]} = R\langle r^p | r \in \mathfrak{a}\rangle$, the ideal generated by the pth powers of the elements of $\mathfrak{a}$. A key step in the construction of the generalized test ideals is an inverse of this operation on ideals $\mathfrak{a} \mapsto \mathfrak{a}^{[p]}$. This can be described as follows. Define

$$\mathfrak{a}^{[\frac{1}{p}]} = \text{smallest ideal } \mathfrak{b} \text{ such that } \mathfrak{b}^{[p]} \supseteq \mathfrak{a}$$

There are some other equivalent definitions of $\mathfrak{a}^{[\frac{1}{p}]}$ and one that is useful for seeing the relation to multiplier ideals is as follows: Let $C : F_*\omega_X \longrightarrow \omega_X$ the Cartier operator, i.e. the trace of the Frobenius under duality of finite morphisms, then $\mathfrak{a}^{[\frac{1}{p}]}\omega_X = C(F_*\mathfrak{a}\omega_X)$. In any case we define the generalized test ideal

$$\tau(X, \mathfrak{a}^t) = (\mathfrak{a}^{\lceil tp^e \rceil})^{[\frac{1}{p^e}]} \text{ for } e \gg 0$$

For this definition to make sense we observe that the expressions on the right hand side for increasing e form an increasing sequence of ideals of R, which by noetherian-ness of R stabilizes.

Suppose a log resolution is available also in positive characteristic, then one may also use that same definition as in characteristic zero to define a multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$. The argument in [9] shows in this situation that one always has an inclusion

$$\tau(X, \mathfrak{a}^t) \subseteq \mathcal{J}(X, \mathfrak{a}^t).$$

Example 1. *For monomial ideals in a polynomial ring* [7]*, or more generally torus fixed ideals in a toric variety* [1] *one always has equality. There are some partial results for bi-nomial ideals by Shibuta and Takagi (cf. Takagi's talk in this* *workshop). If $\mathfrak{a}$ is the ideal of the cone over an elliptic curve E then one has equality for all t if and only if E is ordinary, i.e.* $\dim_{\mathbb{F}_p} H^1_{et}(E, \mathbb{F}_p) = 1$.

In particular it would be of great interest to find similar conditions as the last one for a more general class of varieties.

Definition 2. *We call a number $\lambda \in \mathbb{R}$ an F-threshold for $(R, \mathfrak{a})$ if for all $\epsilon > 0$ the test ideal $\tau(R, \mathfrak{a}^\lambda) \neq \tau(R, \mathfrak{a}^{\lambda-\epsilon})$.*

The question we want to address is the rationality and discreteness of these F-thresholds. Note that in the case of the jumping numbers for multiplier ideals their rationality and discreteness was trivial since everything is determined from one fixed resolution. However in the definition of the F-thresholds, one has a priori infinitely many conditions to check.

Theorem 3 ([4],[3])**.** *Let R be a regular and F-finite and $\mathfrak{a}$ an ideal of R. In each of the following situations*

(1) *R is essentially of finite type over a field*
(2) *$\mathfrak{a} = (f)$ is a principal ideal*

are the F-thresholds a discrete set of rational numbers.

The key point in proving this statement is to show that a certain descending chain stabilizes (this leads to discreteness which then easily implies rationality). If we consider the case of a principal ideal the descending chain which needs to stabilize in order that the number $\frac{a}{q-1}$ for $q = p^e$ is not an accumulation point of F-thresholds is obtained as follows: Consider the map $\gamma : M_0 = R \xrightarrow{r \mapsto f^a \otimes r^q} F^{e*}R = F^{e*}M_0$. Observe that the smallest R submodule M_1 of M_0 such that $\gamma(M_0) \subseteq F^{e*}M_1$ is equal to $\tau(R, f^{a/q}) = (f^a)^{[\frac{1}{q}]}$. Iterating this construction we observe that $M_i = \tau(R, f^{a(1+q+\ldots+q^{i-1})/q^i}) = (f^{a(1+q+\ldots+q^{i-1})})^{[\frac{1}{q^i}]}$. Since the exponents in this sequence of test ideals converges to $a/(q-1)$, the stabilization of this sequence is equivalent to $a/(q-1)$ not being an accumulation point of F-thresholds. There are several different ways one can proof the stabilization of this descending chain. In [2] this appears as a by-product of a much more general result on the existence of so-called minimal γ-modules.

To see how this relates to the cohomology of p-torsion étale sheaves we consider the case of a smooth, projective curve C. In characteristic zero (or for coefficients with torsion not divisible by p) the Grothendieck-Ogg-Shafarevich formula expresses the Euler characteristic of a constructible étale sheaf N in terms of the genus g of the curve, the generic rank of N, and some purely local terms. In the case that N is a constructible sheaf of $\mathbb{F}_p$ vector-spaces one cannot hope for an exact expression like this (because of non-ordinaryness as in the example of elliptic curves). However, suppose that $N = M^F = \{m \in M | F(m) = m\}$ arises as the fixed points of a coherent $\mathcal{C}_C$-module M with a Frobenius action, Pink [11] observed that one obtains always a lower bound (this comes from the Artin-Schreier sequence):

$$\chi(C, N) \geq \chi(C, M) = (1-g)\,\mathrm{rankN} + \deg M$$

By work of Emerton and Kisin [6] or Böckle and Pink [5] such M always exists. Miller [10] observed that (the dual of) the minimal γ-module alluded to above is of maximal degree among such, and hence yields the best possible bound. Furthermore, it is *canonically* attached to N and thus can be used to obtain new *coherent* invariants for the constructible sheaf N.

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Computations of log canonical thresholds

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(joint work with Takafumi Shibuta)

Several approaches to log canonical thresholds are known to exist. In this note, we will explain how to compute log canonical thresholds using characteristic p techniques.

Definition 1. *Let X be a nonsingular algebraic variety over a field of characteristic zero, $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf of X and $x \in X$ be a point lying in the zero locus of $\mathfrak{a}$. Fix a log resolution $\pi : Y \longrightarrow X$ of $\mathfrak{a}$ where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Write*

$$F = \sum_{i=1}^{r} a_i E_i, \quad K_{Y/X} = \sum_{i=1}^{r} k_i E_i.$$

The log canonical threshold of $\mathfrak{a}$ at $x \in X$ is defined to be

$$\mathrm{lct}_x(\mathfrak{a}) = \min\{(k_i + 1)/a_i \mid x \in \pi(E_i)\}$$

(when x is not contained in the zero locus of $\mathfrak{a}$, we put $\mathrm{lct}_x(\mathfrak{a}) = \infty$). The definition of $\mathrm{lct}_x(\mathfrak{a})$ is independent of the choice of the log resolution π.

Since the log canonical threshold is defined via a log resolution, it is difficult to compute it. A notable exception is the case of monomial ideals.

Proposition 2 ([2, Example 5]). *Let $\mathfrak{a} = (x^{\mathbf{c}_1}, \dots, x^{\mathbf{c}_s})$ be a monomial ideal of the polynomial ring $k[x_1, \dots, x_n]$ over a field k and $P(\mathfrak{a}) \subseteq \mathbb{R}^d$ be the Newton polytope of $\mathfrak{a}$. Then*

$$\begin{aligned}\mathrm{lct}_0(\mathfrak{a}) &= \sup\{t \in \mathbb{R}_+ \mid \mathbf{1} \in t \cdot P(\mathfrak{a})\} \\ &= \max\left\{ \sum_{i=1}^s \lambda_i \,\middle|\, \sum_{i=1}^s c_{ij}\lambda_i \le 1 \text{ for all } 1 \le j \le n,\ \lambda_i \in \mathbb{Q}_{\ge 0} \right\},\end{aligned}$$

where $\mathbf{c}_i = (c_{i1}, \dots, c_{in})$ for all $i = 1, \dots, s$.

Watanabe and the author introduced in [4] a characteristic $p > 0$ analogue of log canonical thresholds.

Definition 3 ([4, Definition 2.1]). *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$, then for each $e \in \mathbb{N}$, we set $\nu_{\mathfrak{a}}(p^e)$ to be the largest nonnegative integer r such that $\mathfrak{a}^r$ is not contained in $\mathfrak{m}^{[p^e]} := (a^{p^e} | a \in \mathfrak{m})$. Then the F-pure threshold of $\mathfrak{a}$ is defined to be*

$$\mathrm{fpt}(\mathfrak{a}) = \lim_{e \longrightarrow \infty} \frac{\nu_{\mathfrak{a}}(p^e)}{p^e}.$$

Let A be the localization of $\mathbb{Z}$ at some nonzero integer a. We fix a nonzero ideal $\mathfrak{a}$ of the polynomial ring $A[x_1, \dots, x_n]$ such that $\mathfrak{a} \subseteq (x_1, \dots, x_n)$. Let $\mathfrak{a}_{\mathbb{Q}} := \mathfrak{a} \cdot \mathbb{Q}[x_1, \dots, x_n]$ and $\mathfrak{a}_p := \mathfrak{a} \cdot \mathbb{F}_p[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$, where p is a prime number which does not divide a and $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. We call the pair $(\mathbb{F}_p[x_1, \dots, x_n]_{(x_1, \dots, x_n)}, \mathfrak{a}_p)$ the reduction of $(\mathbb{Q}[x_1, \dots, x_n], \mathfrak{a}_{\mathbb{Q}})$ to characteristic p.

Hara and Yoshida discovered a connection between log canonical thresholds and F-pure thresholds.

Theorem 4 ([1]). *Let the notation be as above.*

(1) *If $p \gg 0$, then $\mathrm{fpt}(\mathfrak{a}_p) \le \mathrm{lct}_0(\mathfrak{a}_{\mathbb{Q}})$.*
(2) $\mathrm{lct}_0(\mathfrak{a}_{\mathbb{Q}}) = \lim_{p \longrightarrow \infty} \mathrm{fpt}(\mathfrak{a}_p)$.

Using this theorem, we obtain the following main technical result.

Proposition 5 ([3, Proposition 2.1]). *Let $S := k[x_1, \dots, x_n]$ be the n-dimensional polynomial ring over a field k of characteristic zero. Let $\mathfrak{a} = (f_1, \dots, f_r)$ be an ideal of S generated by binomials $f_i = x^{\mathbf{a}_i} - \gamma_i x^{\mathbf{b}_i}$, where $\mathbf{a}_i = (a_{i1}, \dots, a_{in}), \mathbf{b}_i = (b_{i1}, \dots, b_{in}) \in \mathbb{Z}^n_{\ge 0} \setminus \{\mathbf{0}\}$ and $\gamma_i \in k^*$ for all $i = 1, \dots, r$. Put*

$$A := \begin{pmatrix} a_{11} & \cdots & a_{r1} & b_{11} & \cdots & b_{r1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{rn} & b_{1n} & \cdots & b_{rn} \\ 1 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 1 \end{pmatrix} \in M_{\mathbb{Z}}(n+r, 2r),$$

and consider the following linear programming problem:

$$\text{Maximize: } \sum_{i=1}^{r}(\mu_i+\nu_i),$$
$$\text{Subject to: } A\ (\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_r)^{\mathrm{T}} \le \mathbf{1},\ \mu_i,\nu_i \in \mathbb{Q}_{\ge 0}.$$

Suppose that there exists an optimal solution $(\mu,\nu)=(\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_r)$ *such that* $A\ (\mu,\nu)^{\mathrm{T}} \ne A\ (\mu',\nu')^{\mathrm{T}}$ *for all other optimal solutions* $(\mu',\nu')\ne(\mu,\nu)$. *Then the log canonical threshold* $\mathrm{lct}_0(\mathfrak{a})$ *is equal to its optimal value* $\sum_{i=1}^{r}(\mu_i+\nu_i)$.

We use the above proposition to generalize Howald's result to the case of binomial ideals.

Theorem 6 ([3, Theorem 0.1])**.** *Let* k *be a field of characteristic zero and let* $\mathfrak{a}=(f_1,\dots,f_r)\subseteq(x_1,\dots,x_n)$ *be an ideal of* $k[x_1,\dots,x_n]$ *generated by binomials* $f_i=x_1^{a_{i1}}\cdots x_n^{a_{in}}-\gamma_i x_1^{b_{i1}}\cdots x_n^{b_{in}}$, *where* $a_{ij},b_{ij}\in\mathbb{Z}_{\ge0}$ *and* $\gamma_i\in k$ *for all* $i=1,\dots,r$ *and* $j=1,\dots,n$. *Suppose that* $\mathfrak{a}$ *contains no monomials and, in addition, that one of the following conditions is satisfied:*

1. $f_1,\dots,f_r$ *form a regular sequence for* $k[x_1,\dots,x_n]$,
2. $f_1,\dots,f_r$ *form the canonical system of generators of the defining ideal of a monomial curve in* $\mathbb{A}_k^3$ *(in this case,* $r\le 3$*).*

Then the log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ *of* $\mathfrak{a}$ *at the origin is equal to*

$$\max\left\{\sum_{i=1}^{r}(\mu_i+\nu_i)\ \middle|\ \sum_{i=1}^{r}(a_{ij}\mu_i+b_{ij}\nu_i)\le 1 \text{ for all } 1\le j\le n,\ \mu_i+\nu_i\le 1,\ \mu_i,\nu_i\in\mathbb{Q}_{\ge0}\right\}.$$

Example 7. (1) *Let* $\mathfrak{a}=(x_1^4-x_2x_3^2, x_2^4-x_1^3x_3, x_3^3-x_1x_2^3)$ *be the defining ideal of the monomial curve* $\operatorname{Spec} k[t^9,t^{10},t^{13}]$ *in* $\mathbb{A}_k^3$. *Then*

$$(\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3)=(5/24,0,0,1/2,0,1/6)$$

is an optimal solution of the linear programming problem in Theorem 6. Thus, $\mathrm{lct}_0(\mathfrak{a})=5/24+0+0+1/2+0+1/6=7/8$.

(2) *Let* $\mathfrak{a}=(x_1^3-x_4^2, x_2^2-x_1x_4, x_3^2-x_2x_4)$ *be the defining ideal of the monomial curve* $\operatorname{Spec} k[t^8,t^{10},t^{11},t^{12}]$ *in* $\mathbb{A}_k^4$. *Then*

$$(\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3)=(1/9,1/3,1/2,0,2/3,1/3)$$

is an optimal solution of the linear programming problem in Theorem 6. Thus, $\mathrm{lct}_0(\mathfrak{a})=1/9+1/3+1/2+0+2/3+1/3=35/18$.

(3) *Let* $\mathfrak{a}=(x_1^5-x_2x_4^2, x_2^7-x_3^4x_4, x_3^3-x_1x_4^2, x_4^7-x_1^3x_2^6x_3^2, x_1^4x_2^6-x_3x_4^5)$ *be the defining ideal of the monomial curve* $\operatorname{Spec} k[t^{53},t^{63},t^{85},t^{101}]$ *in* $\mathbb{A}_k^4$. *Then* $\mathfrak{a}$ *does not satisfy the assumption in Theorem 6, but we can still apply Proposition 5 to this situation. It is easy to check that*

$$(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)=(1/5,1/14,1/3,0,0,1/2,0,0,0,0)$$

is an optimal solution of the linear programming problem in Proposition 5 and, in addition, this solution satisfies the assumption in Proposition 5. Thus, $\mathrm{lct}_0(\mathfrak{a}) = 1/5 + 1/14 + 1/3 + 0 + 0 + 1/2 + 0 + 0 + 0 + 0 = 116/105.$

Question 8. *Let* $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ *be a binomial ideal which contains no monomials and let* $f_1, \ldots, f_r$ *be a system of minimal binomial generators for* $\mathfrak{a}$. *Then do* $f_1, \ldots, f_r$ *satisfy the assumption in Proposition 5? We don't know any counterexample for the moment.*

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Okounkov bodies on low-dimensional varieties

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(joint work with Catriona Maclean)

In his influential articles [6] and [7], Okounkov explains how to associate a convex body $\Delta(D) \subseteq \mathbb{R}^n$ to an ample divisor D on an n-dimensional smooth variety X equipped with a complete flag of subvarieties $Y_{n-1} \supset Y_{n-2} \ldots \supset Y_0$. The convex body $\Delta(D)$ then encodes a lot of information on the asymptotic behaviour of the complete linear system $|D|$. In their excellent survey article [5] Lazarsfeld and Mustaţă extend Okounkov's construction to big divisors, which we now summarize, and prove various properties of these convex bodies.

For simplicity, start with a projective variety of dimension n defined over an uncountable algebraically closed field of arbitrary characteristic. Certainly no harm is done if we assume that we work over the complex numbers. Fix a complete flag

$$X = Y_0 \supset Y_1 \supset \ldots \supset Y_{n-1} \supset Y_n = \mathrm{pt}$$

with Y_i being a smooth irreducible subvariety of codimension i in X. For a given big divisor D the choice of the flag determines a valuation-like function

$$\begin{aligned} \nu_{Y_\bullet, D} \colon H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \quad &\longrightarrow \quad \mathbb{Z}^n \\ s \quad &\mapsto \quad \nu(s) \stackrel{\text{def}}{=} (\nu_1(s), \ldots, \nu_n(s)) \end{aligned}$$

where the values of the $\nu_i(s)$'s are defined in the following manner. We set

$$\nu_1(s) \stackrel{\text{def}}{=} \mathrm{ord}_{Y_1}(s) \ .$$

Dividing s by a local equation of Y_1, we obtain a section $\tilde{s_1} \in H^0(X, D - \nu_1(s)Y_1)$ not vanishing identically along Y_1. This way, upon restricting to Y_1, we arrive at a non-zero section

$$s_1 \in H^0(Y_1, (D - \nu_1(s)Y_1)|_{Y_1}) .$$

Then we set

$$\nu_2(s) \stackrel{\text{def}}{=} \text{ord}_{Y_2}(s_1) .$$

Continuing in this fashion, we can define all the integers $\nu_i(s)$. The image of the function $\nu_{Y_\bullet, D}$ in $\mathbb{Z}^n$ is denoted by $v(D)$. With this in hand, we define the *Okounkov body of D with respect to the flag $Y_\bullet$* to be

$$\Delta_{Y_\bullet}(D) \stackrel{\text{def}}{=} \text{ the convex hull of } \bigcup_{m=1}^{\infty} \frac{1}{m} \cdot v(mD) \subseteq \mathbb{R}^n .$$

Remark. The construction is most likely familiar to some extent from the theory of toric varieties. This is no coincidence, since when taking a torus-invariant complete flag, the rational polytope P_D commonly associated to a torus-invariant big divisor D will be a translate of $\Delta(D)$.

To see a very simple example of this phenomenon, take $\mathbb{P}^n$ with a complete flag of linear subspaces. Then the function ν on the sections of $\mathcal{O}(1)$ gives the lexicographic order; the Okounkov body of a divisor in $\mathcal{O}(1)$ is an n-dimensional simplex.

One of the illustrations of the new theory in [5] is the case of surfaces, where they give an explicit description $\Delta(D)$. Here Zariski decomposition of divisors provides a tool sufficiently strong for making such a concrete description possible. A complete flag on the surface S consists of a smooth curve C on S, and a point x on C.

Given a surface S equipped with a flag (C, x) and a $\mathbb{Q}$-divisor D, Lazarsfeld and Mustaţă define real numbers ν and μ by setting

$$\nu = \text{ the coefficient of } C \text{ in the negative part of the Zariski decomposition of } D$$

$$\mu = \sup\{t | D - tC \text{ is big } \} .$$

Equivalently, ν is the minimal real number for which C is *not* in the support of the negative part of the Zariski decomposition of $D - \nu C$. As it turns out, the Okounkov body of D 'lives' over the interval $[\nu, \mu]$ described by two functions, $\alpha(t)$ and $\beta(t)$ as follows. Let N_t be the negative part of the Zariski decomposition of $D - tC$ and set $P_t = D - tC - N_t$. Note that although P_t is nef, it is not necessarily effective, though it is linearly equivalent to an effective divisor for all t with $D - tC$ effective. By setting

$$\alpha(t) = \text{ord}_x(N_t|_C), \quad \beta(t) = \text{ord}_x(N_t|_C) + P_t \cdot C .$$

Lazarsfeld and Mustaţă prove the following theorem: the Okounkov body $\Delta(D)$ is given by the inequalities

$$\Delta(D) = \{(t, y) \in \mathbb{R}^2 | \nu \le t \le \mu, \alpha(t) \le y \le \beta(t)\} .$$

As a consequence of [1], it is observed in particular that α and β are both piecewise linear and rational on any interval $[\nu, \mu']$ where $\mu' < \mu$.

The first result we intend to present is a sharpening of the Lazarsfeld–Mustaţă statement for surfaces. We also give a complete characterization of the rational convex polygons which can appear as Okounkov bodies of surfaces. More precisely, we prove the following.

Theorem. *The Okounkov body of a big divisor D on a smooth projective surface S is a convex polygon with rational slopes.*

A rational polygon $\Delta \subseteq \mathbb{R}^2$ is up to translation the Okounkov body $\Delta(D)$ of a divisor D on some smooth projective surface S equipped with a complete flag (C, x) if and only if the following set of conditions is met.

There exists a rational number $\mu > 0$, and α, β piecewise linear functions on $[0, \mu]$ such that

(1) $\alpha \leq \beta$,
(2) *β is a convex function,*
(3) *α is increasing, concave and $\alpha(0) = 0$;*

moreover

$$\Delta = \left\{(t, y) \in \mathbb{R}^2 \,|\, 0 \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\right\} .$$

It is in general quite difficult to say anything specific about asymptotic invariants of divisors on higher-dimensional varieties. In particular, there is little in the way of regularity that we can expect. It is established in [5] building on a classical example of Cutkosky [3], that there exist big divisors on higher-dimensional varieties with non-polyhedral Okounkov bodies.

Fano varieties, however, enjoy many favourable properties, which guarantee that all previously known asymptotic invariants behave in a 'rational polyhedral way' on them. Hence, one could hope that Okounkov bodies associated to divisors on Fano varieties turn out to be rational polytopes. This however is not the case, as the following example will show.

Theorem. *There exists a Fano threefold X equipped with a flag $X = Y_0 \supset Y_1 \supset Y_1 \supset Y_2 \supset Y_3$ such that for almost any ample divisor D on X, the Okounkov body of D with respect to the flag $Y_\bullet$ is not a polyhedron.*

Our example is heavily based on a work of Cutkosky [2], where he produces a quartic surface $S \subseteq \mathbb{P}^3$ such that the nef and effective cones of S coincide and are round. The Néron-Severi space $N^1(S)$ is isomorphic to $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ and the intersection form $q(x, y, z) = 4x^2 - 4y^2 - 4z^2$. In particular he shows that the divisor class $(1, 0, 0)$ on S corresponds to a very ample divisor class $[L]$ and the projective embedding corresponding to L realizes S as a quartic surface in $\mathbb{P}^3$. We obtain our Fano threefold by blowing up $\mathbb{P}^3$ along an irreducible curve of class $(1, 0, 0)$.

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Automorphism groups of invariant domains in the complexification of isotropy irreducible homogeneous spaces

Xiang-Yu Zhou

Let K be a connected compact Lie group and L be a closed subgroup of K, $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ be (universal) complexifications of K and L, then $X = K/L$ is a compact homogenous space and $X_{\mathbb{C}} = K_{\mathbb{C}}/L_{\mathbb{C}}$ is a homogeneous Stein manifold which is a complexification of X. Without loss of generality, we may assume that K acts effectively on $X = K/L$. There is a natural holomorphic action of $K_{\mathbb{C}}$ on $X_{\mathbb{C}}$ given by the left translation. Let $D \subset X_{\mathbb{C}}$ be a K-invariant domain, where a domain means a connected open set.

In the present talk, we present some background and an outline proof of the following theorem.

Theorem. Let K/L be an isotropy irreducible homogeneous spaces (except a couple of cases). Then $Aut(D)_0 = K$.

Corollary. When (K, L) is a symmetric pair, then $Aut(D)_0 = K$.

Definition. A homogeneous space K/L is said to be isotropy irreducible if the isotropy (which is just adjoint) representation of L is irreducible on the vector space k/l, where k and l are Lie algebras of K and L; K/L is said to be strongly isotropy irreducible if the isotropy (adjoint) representation of the identity component L_0 of L is irreducible on the vector space k/l.

Example. Irreducible Riemannian symmetric spaces are strongly isotropy irreducible.

Isotropy irreducible homogeneous spaces are classified, and the isometry groups of the spaces are explicitly given and just equal to K for effective action of K except a couple of cases, see [10, 9]. Isotropy irreducible homogeneous space is an important class of homogeneous Einstein manifold. It's also close to representation theory.

The above theorem extends the well-known result on Reinhardt domain. Let $D \subset\subset (\mathbb{C}^*)^n$ be a Reinhardt domain. It's well-known that $Aut(D)$ is compact and the identity component $Aut(D)_0$ of $Aut(D)$ is exactly T^n, see [2], [1], [5], [7].

In [13], Zhou proved the following result.

Theorem ([13]). Let $D \subset\subset \ K_{\mathbb{C}}/L_{\mathbb{C}}$ be a K-invariant domain, then $Aut(D)$ is compact.

Under a more assumption that (K, L) is a symmetric pair, the result is due to Fels and Geati [4].

The proof of the above theorem is to relate the automorphism groups to the isometric groups of some K-orbit via several reduction steps. In the steps, a result of Zhou's about the univalence of the envelope of holomorphy of invariant domains plays a key role.

Theorem ([12]). Let M be a Stein manifold, $K_{\mathbb{C}}$ holomorphically act on M. Let $D \subset M$ be a K-invariant orbit connected domain. Then the envelope of holomorphy $E(D)$ of D is schlicht and orbit convex if and only if the envelope of holomorphy $E(K_{\mathbb{C}} \cdot D)$ of $K_{\mathbb{C}} \cdot D$ is schlicht. Furthermore, in this case, $E(K_{\mathbb{C}} \cdot D) = K_{\mathbb{C}} \cdot E(D)$.

This result unifies and extends many known results. In particular, we have the following theorem.

Theorem ([12]). Let K be a connected compact Lie group and L be a closed subgroup of K. If L is connected, then any K-invariant domain D in $X_{\mathbb{C}} = K_{\mathbb{C}}/L_{\mathbb{C}}$ has schlicht envelope of holomorphy.

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Vanishing periods in complex geometry

Daniel Barlet

We begin to introduce the subject by considering a proper holomorphic function $f : X \longrightarrow D$ on a complex manifold of dimension $n+1$ such that

$$\{df = 0\} \subset \{f = 0\}.$$

For ω a smooth $(p+1)$-differential form on X such that $d\omega = 0$ and $df \wedge \omega = 0$ and for an horizontal family of p-cycles in the fibers of f, $(\gamma_s)_{s\in\widetilde{D}^*}$, where $\widetilde{D}^*$ is the universal cover of the punctured disc $D^* := D\setminus\{0\}$, we define

$$\varphi(s) := \int_{\gamma_s} \omega/df$$

which is a multivalued holomorphic function on D^*. It admits a finite expansion near $s = 0$

$$\varphi(s) = \sum_{\alpha\in\mathbb{Q}, j\in[0,n]} c_{\alpha,j}(s).s^{\alpha-1}.(\log s)^j$$

where $c_{\alpha,j} \in \mathcal{O}(D)$. This type of function describes the way a "period" vanishes in such a degenerating family of compact comex manifolds $(X_s)_{s\in D}$, where we set $X_s := f^{-1}(s)$.

We introduce in general (without assuming that f is proper), the complex $(\widetilde{\ker\, df^\bullet}, d^\bullet)$ on X and we show that there exists natural operations $a := \times f$ and $b := df \wedge d^{-1}$ on the cohomology sheaves of this complex satisfying the commutation relation

$$ab - ba = b^2. \qquad (@)$$

We define the (non commutative) $\mathbb{C}$-algebra

$$\widetilde{A} := \left\{ \sum_{\nu\geq 0} b^\nu.P_\nu(a) \right\}$$

where $P_\nu \in \mathbb{C}[x]$ for each $\nu \geq 0$, with the commutation relation (@).

Theorem 1. *(see* [1]*) For any non constant holomorphic function $f : X \longrightarrow D$ on a complex manifold X, there exists a natural complex of sheaves of left $\widetilde{A}$-modules which is canonically quasi-isomorphic to the complex $(\widetilde{\ker}\ df^\bullet, d^\bullet)$, such that the left $\widetilde{A}$-module structure on its cohomology sheaves coincides with the operations a and b defined above.*

Of course this result impies that all natural operations on the complex of sheaves $(\widetilde{\ker}\ df^\bullet, d^\bullet)$ will be compatible with the left $\widetilde{A}$-structure on cohomology.

Theorem 2. *(see* [2]*) Assuimg f proper, for each $i \geq 0$ the hypercohomology $\mathbb{H}^i(K, (\widetilde{\ker}\ df^\bullet, d^\bullet))$ is a left $\widetilde{A}$-module with the following properties:*

(1) *The b-torsion is a finite dimensional $\mathbb{C}$-vector space.*
(2) *The quotient $\mathbb{H}^i/(b\text{-torsion})$ is a geometric (a, b)-module.*

Recall that an (a, b)**-module** is a left $\widetilde{A}$-module E which is free of finite type on $\mathbb{C}[[b]] \subset \widetilde{A}$. It is called **regular** when its saturation

$$E^\# := \sum_{j \geq 0} (b^{-1}.a)^j.E \subset E[b^{-1}]$$

is again of finite type on $\mathbb{C}[[b]]$.
It is called **geometric** when moreover any root of its Bernstein polynomial B_E which is the minimal polynomial of the action of $b^{-1}.a$ on $E^\#/b.E^\#$ is in $\mathbb{Q}^{-*}$.

The notion of geometric (a, b)-module is a "generalisation" of the classical notion of "Brieskorn module" for a holomorphic function with an isolated singularity. The geometric condition encodes the regularity theorem for the Gauss-Manin connection, the Monodromy theorem and the "positivity theorem" for characteristic exponents of B. Malgrange.

In a second part we explain how the study of the "monogenetic" (a, b)-module $\widetilde{A}.x$, for x a given element of a regular (a, b)-module, gives much more precise information on the "asymptotics" of x (think that $x = \varphi(s) := \int_{\gamma_s} \omega/df$ for $[\omega] \in \mathbb{H}^{p+1}$ as above). We give two structure theorems for such "monogenetic" (a, b)-modules and we show that the Bernstein polynomial of such a regular (a, b)-module is rather simple to compute ion such a case. In fact, it coincides with the characteristic plynomial of $b^{-1}.a$ action on $E^\#/b.E^\#$ in this case, so it has nice functorial behaviour. We conclude with an explicit computation of an example making obvious that this approach gives much precise results than the classical computations.
For detail see [3].

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Non-algebraicity criterion for foliations of general type

Jun-Muk Hwang
(joint work with Eckart Viehweg)

This is a report on my joint-work [1] with Eckart Viehweg. Let X be a complex projective manifold. By a rank-1 foliation $\mathcal{F}$ on X, we mean a line subbundle $\mathcal{F} \subset T_X$ of the tangent bundle of X. Throughout, we will consider only rank-1 foliation and omit the term 'rank 1' from now on. The dual bundle $\mathcal{F}^*$, equipped with the natural quotient map $\Omega^1_X \longrightarrow \mathcal{F}^*$ will be called the *canonical bundle* of the foliation and will be denoted by $\mathcal{K}$. The foliation is said to be *of general type* if its canonical bundle $\mathcal{K} = \mathcal{F}^*$ is big. The kernel of the quotient map $\Omega^1_X \longrightarrow \mathcal{K}$ is called the *conormal* bundle of the foliation and will be denoted by $\mathcal{Q}$, i.e.,

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{K} \longrightarrow 0.$$

Given a foliation $\mathcal{F}$, one fundamental question is whether the leaves are algebraic curves. If all leaves are algebraic curves, we will say that the foliation is *algebraic*. Our main result is the following non-algebraicity criterion.

Non-algebraicity Theorem Let X be a projective manifold and $\mathcal{F} \subset T_X$ be a foliation of general type. If $\mathcal{F}$ is algebraic, then the Iitaka dimension $\kappa(\det \mathcal{Q})$ of the determinant of the conormal bundle is equal to $\dim X - 1$.

The essence of the proof is the following structure theorem for algebraic foliations. Denote by $\mathfrak{f} : \mathcal{C} \longrightarrow \mathcal{M}_g^{[N]}$ be the universal family of curves of genus g with level N-structure for some integer $N \geq 3$ and denote by $\omega_{\mathfrak{f}}$ the relative dualizing sheaf.

Structure Theorem For an algebraic foliation

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{K} \longrightarrow 0$$

with leaves of genus > 1 on a projective manifold X and given $N \geq 3$, there is a diagram

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{\eta} & V & \xrightarrow{\sigma} & X \\ \mathfrak{f}\downarrow & \spadesuit & f\downarrow & & \downarrow \\ \mathcal{M}_g^{[N]} & \xleftarrow{\varphi} & W & \longrightarrow & \mathrm{Chow}_X \end{array}$$

where

(1) V, W are projective manifolds,
(2) σ is surjective and generically finite,
(3) $\spadesuit$ is a fiber product,
(4) $\sigma^*\mathcal{K} = \eta^*\omega_{\mathfrak{f}}$ and $f^*\Omega^1_W \subset \sigma^*\mathcal{Q}$.

In fact, the bigness of $\sigma^*\mathcal{K}$ shows that η is generically finite over its image. Hence φ is generically finite over its image and $\varphi^*\Omega^1_{\mathcal{M}_g^{[N]}} \longrightarrow \Omega^1_W$ is generically surjective. Now it is well-known that $\Omega^1_{\mathcal{M}_g^{[N]}}$ is ample (see e.g. [3]). Thus $\det \Omega^1_W$

is big on W. It follows that $\kappa(\det \mathcal{Q}) = \dim W = \dim X - 1$. Thus Structure Theorem implies Non-algebraicity Theorem.

The proof of Structure Theorem uses the classical Reeb Stability Theorem for foliations (see e.g. [2]).

One application of Non-algebraicity Theorem is the following. Let (M, ω) be a projective symplectic manifold. Given a non-singular hypersurface $X \subset M$, the restriction of the symplectic form ω on the tangent space of X at each point $x \in X$ has 1-dimensional kernel, defining a foliation on X, which we will call the *characteristic foliation of* X induced by ω.

Theorem on Characteristic Foliations Let M be a non-singular projective variety of dimension ≥ 4 with a symplectic form ω. Let $X \subset M$ be a non-singular hypersurface of general type. Then the characteristic foliation on X induced by ω cannot be algebraic.

In fact, for the characteristic foliation on X, $\mathcal{K}$ is exactly the canonical bundle of X. Thus the characteristic foliation is of general type. Moreover, the symplectic form ω induces a symplectic form on the conormal bundle $\mathcal{Q}$, which implies that $\det \mathcal{Q}$ is trivial. Thus $\kappa(\det \mathcal{Q}) \neq \dim X - 1$ and the foliation cannot be algebraic by Non-algebraicity Theorem.

One particular case of Theorem on Characteristic Foliations is worth noting:

Corollary Let $A = \mathbb{C}^{2n}/\Lambda$ be an even-dimensional principally polarized abelian variety with smooth theta divisor. Fix any linear coordinate $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on $\mathbb{C}^{2n}$ and let $\theta(p_1, \ldots, p_n, q_1, \ldots, q_n)$ be the Riemann theta function associated to the period Λ. For a very general (i.e. outside a countable union of proper subvarieties) point $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ on the theta divisor, the solution $(p_i(t), q_i(t))$ of the Hamiltonian flow

$$\frac{dp_i}{dt} = -\frac{\partial \theta}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial \theta}{\partial p_i}, \quad i = 1, \ldots, n$$

with initial value $p_i(0) = a_i, q_i(0) = b_i, i = 1, \ldots, n$, cannot have the locus of an algebraic curve on A.

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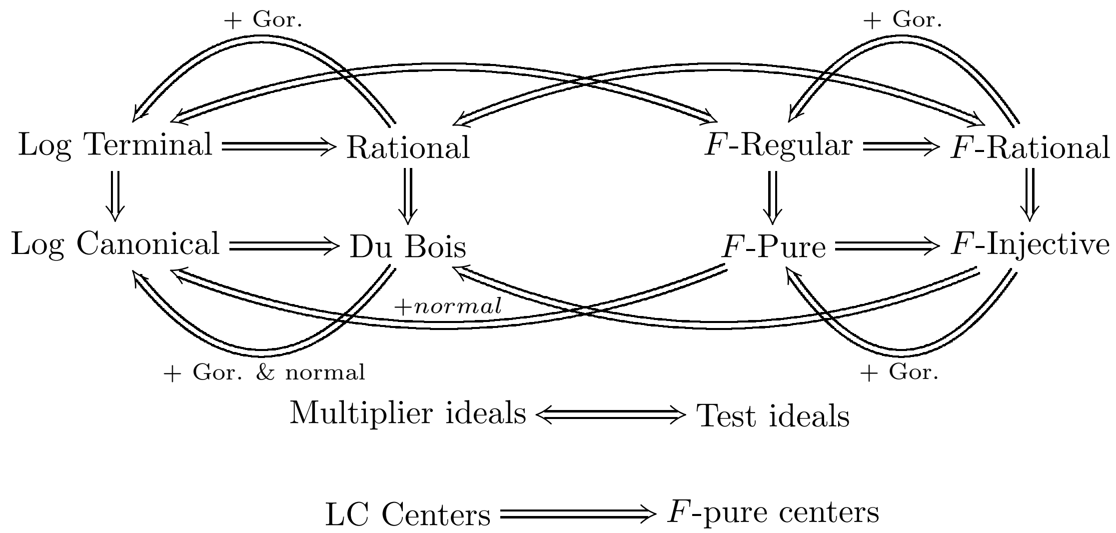
A restriction theorem for positive characteristic analogues of multiplier ideals in arbitrary codimension

Karl Schwede

For the past 30 years, people have been aware of connections between singularities defined by the Frobenius action in characteristic $p > 0$ and singularities defined by a resolution of singularities in characteristic zero. For example, the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ after reduction to characteristic $p \gg 0$, is equal to the test ideal $\tau(X_p, \Delta_p, \mathfrak{a}_p^t)$, see [10], [17], [6], [8] and [18]. These relationships are summarized in the diagram below.



The relations between the diagram's left-hand-side (characteristic zero) and its right-hand-side (characteristic $p > 0$) were worked on by many people; for example, see [3], [4], [16], [7], [5], [11], [17], [8] [6], [1], [20], [19], [13], [12], [14]. One should note that the arrow linking F-pure and log canonical singularities (respectively, F-injective and Du Bois singularities) only goes one way. In these cases, the converse direction (ie, a log canonical singularity is F-pure after reduction to characteristic $p > 0$ for infinitely many p) is open.

We discuss now log canonical centers and their characteristic $p > 0$ analogues. Consider a pair (X, Δ) where X is normal and quasi-projective over $\mathbb{C}$ and Δ is an effective $\mathbb{Q}$-divisor. A subvariety $W \subset X$ is called a *log canonical center of* (X, Δ) if there exists some proper birational map $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ normal and a divisor $E_i \subset \widetilde{X}$ such that $\pi(E_i) = W$ and $a_i \leq -1$, here a_i is the discrepancy of E_i for the pair (X, Δ). An alternate characterization of a log canonical center is the following:

- for every $\epsilon > 0$ and every effective Cartier divisor G passing through η_W, the generic point of W, the pair $(X, \Delta + \epsilon G)$ does *NOT* have log canonical singularities at η_W.

One can use this description of a log canonical center to create a characteristic $p > 0$ analogue (simply replace the words "log canonical" with "F-pure", wherever they appear).

There is another (very useful) way to think about these *F-pure centers* however. First we discuss a positive characteristic interpretation of pairs (X, Δ) such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. In particular, there is a bijection of sets:

$$\left\{ \begin{array}{c} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ such} \\ \text{that } (p^e - 1)(K_X + \Delta) \text{ is Cartier} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Line bundles } \mathscr{L} \text{ and non-zero} \\ \text{elements of } \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathscr{L}, \mathcal{O}_X) \end{array} \right\} \Big/ \sim$$

Where the equivalence relation on the right identifies two maps if they agree up to pre-multiplication by a unit of $H^0(X, \mathcal{O}_X)$. Also note that if Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p, then there exists some $e > 0$ such that $(p^e - 1)(K_X + \Delta)$ is Cartier. In the above correspondence, $\mathscr{L}$ is $\mathcal{O}_X((1 - p^e)(K_X + \Delta))$. The correspondence then follows from the isomorphism $F^e_* \mathcal{O}_X((p^e - 1)\Delta) \cong \mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((1 - p^e)(K_X + \Delta)), \mathcal{O}_X)$ (due to Grothendieck duality for a finite morphism, see [9]). See [15] for more details.

One can then show the following:

Proposition 1. [14], [15] *Suppose that Δ is a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$. Set $\phi_\Delta : F^e_* \mathscr{L} \to \mathcal{O}_X$ to be a morphism corresponding to Δ (via the above correspondence). Then $W \subset X$ is an F-pure center of (X, Δ) if and only if $\phi_\Delta(F^e_*(I_W \mathscr{L})) \subseteq I_W$ (here I_W is the ideal sheaf corresponding to W).*

Using this characterization, one can prove the following result.

Theorem. [15] *Suppose that X is an integral normal F-finite noetherian scheme essentially of finite type over a field of characteristic $p > 0$. Further suppose that Δ is an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p. Let $W \subseteq X$ be an closed subscheme that satisfies the following properties:*

(a) *W is integral and normal.*
(b) *(X, Δ) is F-pure at the generic point of W.*
(c) *W is a center of F-purity for (X, Δ).*

Then there exists a canonically determined effective divisor Δ_W on W satisfying the following properties:

(i) *$(K_W + \Delta_W) \sim_{\mathbb{Q}} (K_X + \Delta)|_W$, furthermore if $(p^e - 1)(K_X + \Delta)$ is Cartier then $(p^e - 1)(K_W + \Delta_W)$ is also Cartier.*
(ii) *(X, Δ) is sharply F-pure near W if and only if (W, Δ_W) is sharply F-pure.*
(iii) *W is minimal among centers of sharp F-purity for (X, Δ), with respect to containment of topological spaces if and only if (W, Δ_W) is strongly F-regular.*
(iv) *For any ideal sheaf $\mathfrak{a}$ which doesn't vanish on W, there is a naturally defined ideal sheaf $\tau_b(X, \not\subseteq W; \Delta, \mathfrak{a}^t)$, which philosophically corresponds to an analogue of an adjoint ideal in arbitrary codimension, such that $\tau(X, \not\subseteq W; \Delta, \mathfrak{a}^t)|_W = \tau(W; \Delta_W, \overline{\mathfrak{a}}^t)$, "the test ideal of $(R, \Delta, \overline{\mathfrak{a}}^t)$".*

Using (iv) combined with Fedder's criterion, [3], and using the technique of [2], one can then prove the following.

Theorem. [Blickle, –, Takagi, Zhang] *Suppose that* X *is a normal variety and* Δ *is a divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier with index not divisible by* $p > 0$*. Suppose further that* $\mathfrak{a}$ *is an ideal sheaf on* X*. Then the* F*-jumping numbers of* $\tau(X, \Delta, \mathfrak{a}^t)$ *are a discrete set of rational numbers.*

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Mild singularities

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Classification of algebraic varieties is one of the most fundamental questions in algebraic geometry. It is far from being completed, but we have a relatively detailed plan on how to proceed.

First, one obtains a canonical model in order to find a natural polarization on a birational model of the given variety. This is usually done via the Minimal Model Program, first producing a minimal model and then its canonical model using base point freeness. There have been spectacular advances in this theory recently, the main example being [1]. There are other approaches to constructing the canonical model, e.g., [3], but in any case the purpose of this note is to discuss something else, so I will leave it to the reader to explore the details.

Once the canonical model is found, it may be embedded into a projective space via some power of the canonical bundle. The necessary power only depends on the Hilbert polynomial due to Matsusaka's Big Theorem [4, 5, 6]. Then the moduli space is constructed by taking a quotient of an appropriate subscheme of the corresponding Hilbert scheme.

One may consider the produced moduli space and the procedure to determine the moduli point of any given variety the answer to the classification problem. The moduli space is, in some sense, a (non-discrete) "list" of preferred models in a class of varieties one aims to classify.

A major technical difficulty arises from the fact that the canonical model of a smooth projective variety is usually singular. But even if one restricted to the study of smooth models, a meaningful theory should include information on degenerations. In other words, one would like to obtain a *compact* moduli space. In this case there is no way out; one must work with singular spaces.

Fortunately, the singularities that are necessary to consider can be controlled and stay relatively "mild". Nevertheless, it makes the treatment technical and to some extent perhaps even threatening for a newcomer.

The main purpose of this talk was to discuss some of the singularities that occur in this program, their relationships and significance. One of the main applications discussed was the following joint result with János Kollár proved in [2]:

Theorem 1. *Let $\phi : X \longrightarrow B$ be a flat projective morphism such that all fibers are log canonical. Then the cohomology sheaves $h^i(\omega_\phi^\bullet)$ are flat over B, where $\omega_\phi^\bullet$ denotes the relative dualizing complex of ϕ.*

Corollary 2. *Under the same hypethesis, assume that one of the fibers of ϕ is Cohen-Macaulay. Then so are all the fibers.*

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