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Topological and Variational Methods for Partial Differential Equations

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Introduction by the Organisers

This was a very successful and enjoyable workshop which showed the diversity and vitality of the area. The meeting was attended by 48 participants from 16 countries representing all continents (except Antarctica). In the 26 talks given during the course of the week, both leading experts and promising young mathematicians were invited to present recent trends and new developments in the field. Most of the talks dealt with nonlinear elliptic and parabolic equations, while special emphasis was laid on

- singularities and concentrating solutions
- the interaction between PDE and geometry
- Liouville type theorems
- symmetry and symmetry breaking.

A number of important talks were concerned with solutions of nonlinear elliptic equations on all of Euclidean space. For example, in some talks surprisingly complicated solutions tending to a constant were discussed together with connections with differential geometry. Another talk presented new solutions for elliptic systems modeling quadruple junction structures. The importance and future potential of this work was illustrated when several participants were invited to give lectures at the International Congress of Mathematicians in 2010. The atmosphere of the workshop was fruitful and stimulating, which was most visible outside the scheduled lecture time when ideas were exchanged in numerous scientific discussions in small groups. Here some promising joint research projects from experts with different methodological background were initiated. The feedback of the participants was very positive; there was a clear consensus that due to the strong dynamics of the field there is a need for regular meetings within the unique atmosphere of the MFO.

Workshop: Topological and Variational Methods for Partial Differential Equations

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Abstracts

Global Dynamics of Spike Solutions to the Allen-Cahn Equation and Invariant Manifolds from Approximations

Peter W. Bates
(joint work with Kening Lu, Chongchun Zeng)

We view particles as peak-like approximate minimizers for the energy functional

$$E(u) \equiv \int_\Omega (\frac{\varepsilon^2}{2}|\nabla u|^2 - F(u)),$$

where $F(u)$ may be like $\frac{u^{p+1}}{p+1} - \frac{u^2}{2}$, $p > 1$, or some more general nonlinearity having a similar shape for $u > 0$ and Ω is a smoothly bounded domain in $\mathbb{R}^n$. We speak of 'approximate' minimizers since we discuss non-equilbrium states, which have 'condensed' to localized states and which evolve by the gradient flow of E:

$$(*) \qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u) & (t,x) \in [0,\infty) \times \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \varepsilon << 1$. It is assumed that $f = F'$ is such that there is a non-degenerate positive radially symmetric ground state (e.g. $u^p - u$, $p > 1$, subcritical, or $u(u-a)(1-u)$ with $0 < a < \frac{1}{2}$).

We note that stationary states with peaks have been discussed in detail by many authors, starting with papers by Ni and Takagi (see, e.g. [2, 3, 4]) where typically variational methods are used. The idea is that a rescaled version of the ground state, centered at a point on $\partial\Omega$ is close to being a critical point of the energy functional with the first order error being proportional to the mean curvature of $\partial\Omega$.

In fact, various authors have used three techniques: Variational, namely, constrained minimization or saddle point reduction, reducing the equation to a family, Γ, of states parameterized by $\partial\Omega$ in the vicinity of a critical point of the mean curvature; A nonlinear Lyaponov-Schmidt reduction to Γ; Dynamical systems, looking for a quasi-invariant manifold as a graph over Γ.

These methods are all based on the same idea: Peak states are **strongly unstable** in certain directions,**strongly stable** in a finite co-dimensional submanifold, **very weakly stable or unstable in "translational" directions (along** Γ). This concept is known as "**Normal Hyperbolicity**, which has been shown to be necessary and sufficient for the persistence of invariant manifolds under perturbation.

Our manifold Γ is not invariant and, strictly speaking, normal hyperbolicity is only defined for invariant manifolds. Here we develop a theory for *approximately* invariant, *approximately* normally hyperbolic manifolds for semiflows in Banach space, proving the existence of truly invariant normally invariant manifolds in the vicinity of manifolds that are approximately so. We expect that this abstract result will be useful in a variety of settings when one can construct approximate solutions to PDEs, as in the case of the Allen-Cahn equation above.

To demonstrate, we build an approximately invariant normally hyperbolic manifold by taking the rescaled radially symmetric ground state: w satisfying

$$\begin{cases} \Delta w + f(w) = 0, & y \in \mathbb{R}^n, \\ w(0) = \max w(y), & w > 0, \\ w(y) \to 0, & |y| \to \infty. \end{cases}$$

With $L_0 \equiv \Delta + f'(w) : W^{2,q}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $\sigma(L_0) \cap (-b, \infty) = \{\lambda_1, 0\}$, for some $b > 0$; $\lambda_1 > 0$ is the principle eigenvalue, and the eigenspace of 0 is spanned by $\{\frac{\partial w}{\partial y_j} : j = 1, 2, \ldots, n\}$.
Define

$$|u|^q_{k,\varepsilon} = \Sigma^k_{i=0} \varepsilon^{qi-n} \Sigma_{|\alpha|=i} |\partial^\alpha u|^q_{L^q(\Omega)}.$$

The phase space will be taken as $X = (W^{2,q}(\Omega), |\cdot|_{0,\varepsilon})$.

For any $p \in \partial\Omega$, let

$$\tilde{w}_{\varepsilon,p}(x) = w(\frac{x-p}{\varepsilon}).$$

Since $\tilde{w}_{\varepsilon,p}$ does not satisfy the boundary condition, it will be modified: Given any $v : \partial\Omega \to R$, let h be the solution of

$$\begin{cases} \varepsilon^2 \Delta h + f'(0) h = 0, & x \in \Omega, \\ \frac{\partial h}{\partial n} = v, & x \in \partial\Omega. \end{cases}$$

Define a linear operator Bc by $Bc(v) = h$. For $p \in \partial\Omega$, let $W_{\varepsilon,p} = \tilde{w}_{\varepsilon,p} - Bc(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial n})$. Define the smooth imbedding $\psi_\varepsilon : \partial\Omega \to L^2(\Omega)$ by

$$\psi_\varepsilon(p) \equiv W_{\varepsilon,p}$$

and the approximate invariant manifold

$$M_\varepsilon = \psi_\varepsilon(\partial\Omega).$$

The boundary correction $Bc(\frac{\partial w_{\varepsilon,p}}{\partial n})$ is (better than) order $O(\varepsilon)$ in terms of $|\cdot|_{k,\varepsilon}$ for any $k \geq 0$. Let $v(x) > 0$ be the first eigenfunction, corresponding to the eigenvalue λ_1, of the linearized operator L_0.
For any $p \in \partial\Omega$, define $\tilde{v}_{\varepsilon,p}(x) = v(\frac{x-p}{\varepsilon})$, $\quad V_\varepsilon(p) = \tilde{v}_{\varepsilon,p} - Bc(\frac{\partial}{\partial n} \tilde{v}_{\varepsilon,p})$, and $X^u_{\varepsilon,p} = \text{span}\{V_\varepsilon\}$, $\ X^c_{\varepsilon,p} = T_{\psi_\varepsilon(p)} M_\varepsilon$, $\quad X^s_{k,\varepsilon,p} = (X^c_{\varepsilon,p} \oplus X^u_{\varepsilon,p})^\perp$. Then M_ε is approximately invariant and normally hyperbolic in the sense of the abstract results, provided ε is sufficiently small.

Obtain an inflowing center-stable invariant manifold W^{cs} and an overflowing center-unstable invariant manifold W^{cu}. These are C^j sections of the vector bundles $(M_\varepsilon, X^s_{\varepsilon,k,p})$ and $(M_\varepsilon, X^u_{\varepsilon,p})$, respectively.

By taking their intersection, we obtain an invariant manifold $\tilde{M}_\varepsilon$ in a small $W^{k,q}$ neighborhood of M_ε, which therefore consists of spike-like functions.

Finally, one can compute the vector field on $\tilde{M}_\varepsilon$ induced by the equation, obtaining a dynamical system on $\partial\Omega$ for the location of the peak of the spike.

The results can be summarized as

Theorem(BLZ[1])

Under the assumptions mentioned above, for any sufficiently small $\varepsilon > 0$, there exists a smooth mapping $\Psi_\varepsilon : \partial\Omega \to W^{2,2}_\varepsilon(\Omega)$ such that

(1) For any $q \in (n, \infty)$, there exists $C > 0$ independent of $p \in \partial\Omega$ and $\varepsilon > 0$ such that

$$|\Psi_\varepsilon(p) - w(\frac{\cdot - p}{\varepsilon})|_{C^0((\partial\Omega, \frac{1}{\varepsilon^2} < \cdot, \cdot >), W^{2,2}_\varepsilon(\Omega) \cap W^{2,q}_\varepsilon(\Omega))} \le C\varepsilon$$

$$|\Psi_\varepsilon(p) - w(\frac{\cdot - p}{\varepsilon})|_{C^1((\partial\Omega, \frac{1}{\varepsilon^2} < \cdot, \cdot >), W^{2,2}_\varepsilon(\Omega) \cap W^{2,q}_\varepsilon(\Omega))} \to 0.$$

(2) There exists a unique $\tilde{p} \in \partial\Omega$ such that $\max_{x \in \bar{\Omega}} \Psi_\varepsilon(p)(x) = \Psi_\varepsilon(p)(\tilde{p})$. Moreover $|p - \tilde{p}| < C\varepsilon^2$ for some $C > 0$ independent of $0 < \varepsilon << 1$.

(3) $M^*_\varepsilon \equiv \Psi_\varepsilon(\partial\Omega)$ is a normally hyperbolic invariant manifold of the flow generated by the PDE (*).

(4) Equation (*) induces a vector field $Y_\varepsilon(p)$ on $\partial\Omega$ that satisfies

$$|Y_\varepsilon(p) - \gamma\varepsilon^3 \nabla\kappa(p)| \le C\varepsilon^4$$

for some $C > 0$ independent of $p \in T_p\partial\Omega$ where
$\kappa(p) = H(p) \cdot N(p)$ and $H(p)$ is the mean curvature vector of $\partial\Omega$ and

$$\gamma = \frac{1}{3} \int_{\partial\mathbb{R}^n_+} \left[\frac{w'(|y|)}{|y|}\right]^2 y_j^4 dy > 0.$$

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Asymptotics of Grow-Up Solutions and Global Attractors of Non-Dissipative PDEs

Nitsan Ben-Gal

There has been much study in recent decades regarding the asymptotics of solutions and the decomposition of attracting sets for the dissipative and fast non-dissipative ($f(u)$ superlinear) forms of the scalar reaction-diffusion equation

(1) $$u_t = u_{xx} + f(u), \quad x \in [0, L]$$

with boundary conditions. While the dissipative form of (1) guarantees the existence of a compact global attractor, the fast non-dissipative form of (1) leads to finite time blow-up of solutions.

This talk introduces recent results which determine the asymptotics of solutions and the decomposition of the attracting set for the slowly non-dissipative scalar parabolic PDE

$$\begin{gathered} u_t = u_{xx} + \underbrace{bu + g(u)}_{f(u)} \\ x \in [0,\pi], \quad t \geq 0, \quad u_x(0) = u_x(\pi) = 0 \\ b > 0;\ g(u)\ \textit{bounded},\ C^2,\ \textit{Lipschitz} \end{gathered} \tag{2}$$

The equation with linearly growing nonlinearity $f(u)$ as in (2) induces a dynamical system wherein there are no blow-up solutions, but for any positive b there exists a subset of solutions to (2) wherein solutions grow to infinity in infinite time. These solutions are termed "grow-up solutions" of the dynamical system.

Due to the existence of grow-up solutions we cannot construct a global attractor in the classical sense. We introduce the concept of a non-compact global attractor and discuss how such an object may be decomposed, which leads to the following theorem:

Theorem 1. *The non-compact global attractor of Equation (2) is the union of the bounded equilibria of (2), their connecting heteroclinics, the equilibria at infinity, transfinite heteroclinics connecting bounded equilibria with equilibria at infinity, and the infinite heteroclinics which connect equilibria within infinity.*

In the classic dissipative case the primary tools of this decomposition are the time map, lap number, and y-map. The "Connection Problem", which seeks to obtain the complete decomposition of the global attractor via the determination of which connecting heteroclinics exist and which are blocked, was solved by Brunovský and Fiedler through the use of the first two tools and the invention of the y-map [1, 2] for dissipative systems.

We address the implications of the time map and lap number for the global bifurcation diagram and non-compact attractor for equations of the form (2), especially the ability of the time map to determine all bounded equilibria of (2). We extend the y-map to slowly non-dissipative systems as well as to a broader range of boundary conditions than originally pursued. The extended y-map, using the properties of the lap number which were derived by Matano [5], allows us to determine all possible asymptotic behavior of solutions to (2) as well as providing the necessary tools for determining uniquely all bounded heteroclinics originating at any given bounded equilibria.

Recent results from the thesis of Hell [4] provide us with a Conley index at infinity. This tool allows for the study of the equilibria at infinity and the construction of heteroclinics which connect such equilibria to each other. Even with this new tool, the techniques used to solve the Connection Problem in the dissipative case are insufficient to determine which equilibria at infinity a bounded equilibria connects to.

To address this issue we adapt the concept of inertial manifolds first introduced by Foias, Sell, and Temam [3]. Inertial manifolds are a valuable tool for describing the large-time behavior of a dissipative dynamical system via the reduction of the infinite-dimensional case to the finite-dimensional. Via adaptation of the properties necessary to the construction of inertial manifolds and alteration of the construction process, we were able to prove the existence of inertial manifolds for equations of the form (2).

This provides us with the necessary convergence results with which to uniquely determine to which equilibria at infinity any transfinite heteroclinic will connect. Combining this with our results on bounded heteroclinics and the work of Hell within infinity allows us to uniquely determine all elements of each type in Theorem 1. This leads to our main result:

Theorem 2. *Given a stationary solution v of (2), all stationary solutions, both finite and infinite, to which v connects are uniquely determined. Thus, the non-compact global attractor of (2) may be decomposed explicitly.*

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Periodic and Bloch solutions to a magnetic nonlinear Schrödinger equation

Mónica Clapp

(joint work with Renato Iturriaga and Andrzej Szulkin)

The behavior of a charged particle in the presence of an external magnetic field B and an electric field is described by the magnetic Schödinger operator

$$L_{A,V} = (-i\nabla + A)^2 + V,$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is an electric potential and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential associated to B, that is, $\mathrm{curl}A = B$. In the language of differential forms, A is a 1-form $A = A_1 dx_1 + \cdots + A_N dx_N$ and $\mathrm{curl}A := dA = \sum_{j<k} b_{jk} dx_j \wedge dx_k$, where $b_{jk} = (\mathrm{curl}A)_{jk} = \partial_j A_k - \partial_k A_j$.

We consider the nonlinear problem

$$(\wp_A) \quad \begin{cases} (-i\nabla + A)^2 u + Vu = |u|^{p-2}u, \\ u \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}), \quad \nabla u + iAu \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

where $A \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in C^{0,\alpha}(\mathbb{R}^N)$ are 2π-periodic in each variable $x_1, ..., x_N$, $V > 0$, and $p \in (2, 2^*)$ with $2^* := \infty$ if $N = 2$ and $2^* := \frac{2N}{N-2}$ if $N \geq 3$.

Existence of solutions to the equation in $(\wp_A)$ with periodic and nonperiodic data whose absolute value vanishes at infinity has been shown for example in [7, 9, 1, 10, 6, 4, 3, 5], both in the classical and semiclassical regime. There is an extensive literature on this subject in the nonmagnetic case $A = 0$.

We address two questions: First, the gauge-dependence problem if one considers only 2π-periodic solutions and second, the multiplicity question for solutions whose absolute value is 2π-periodic.

Recall that every closed 2-form B on $\mathbb{R}^N$ is exact, that is, there exists a 1-form A such that $\mathrm{curl}A = B$. Moreover, if $\tilde{A}$ is another 1-form with $\mathrm{curl}\tilde{A} = B$, then $A - \tilde{A}$ is the gradient of a function φ. A straightforward computation shows that

$$u \text{ solves } (\wp_A) \Longleftrightarrow e^{-i\varphi}u \text{ solves } (\wp_{\tilde{A}}). \tag{1}$$

This is called the gauge invariance. It says that the choice of A with fixed $\mathrm{curl}A = B$ does not affect the solutions of $(\wp_A)$ in any essential way, as long as we allow arbitrary solutions.

Since our data are periodic, it is natural to consider periodic solutions. Now, if we are interested only in 2π-periodic solutions the situation changes drastically. In this case, problem $(\wp_A)$ can be interpreted as a problem on the N-dimensional flat torus $\mathbb{T}^N := \mathbb{R}^N/2\pi\mathbb{Z}^N$ which has nontrivial topology. This has the effect that a 2π-periodic closed 2-form B might not be the curl of a 2π-periodic 1-form [2]. A necessary and sufficient condition for this to happen is that the mean value of B over $[0, 2\pi]^N$ is 0. Moreover, two 2π-periodic 1-forms A and $\tilde{A}$ do not necessarily differ by the gradient of a 2π-periodic function, so there is no obvious one-to-one correspondence between the 2π-periodic solutions of $(\wp_A)$ and those of $(\wp_{\tilde{A}})$ as given by (1). However, $\tilde{A}$ differs from $A + z$ by the gradient of a 2π-periodic function φ for some $z \in \mathbb{R}^N$. This leaves us with comparing problems $(\wp_{A+z})$ for different choices of $z \in \mathbb{R}^N$. So the question is, does the choice of $z \in \mathbb{R}^N$ affect the solutions of $(\wp_{A+z})$ in some essential way? Now, in the context of quantum physics, a relevant quantity is the absolute value of the solution: $|u(x)|^2$ can be interpreted as the (unnormalized) probability density of finding a particle at x. Note that (1) establishes a one-to-one correspondence which preserves the absolute value of the solutions. So one may ask whether there is a one-to-one correspondence associating to each 2π-periodic solution u_0 of $(\wp_A)$ a 2π-periodic solution u_z of $(\wp_{A+z})$ with the same absolute value, i.e. $|u_z| = |u_0|$. We address this question and prove the following.

Theorem 1. *Assume that problem $(\wp_A)$ has a nowhere vanishing 2π-periodic solution u_0. Then there exists a quadric $\mathcal{Q}$ of codimension at least one containing* *the origin with the following property: If* $(\wp_{A+z})$ *has a* 2π*-periodic solution* u_z *such that* $|u_z| = |u_0|$, *then* $z \in \mathcal{Q} + \mathbb{Z}^N$.

By a quadric we mean, as usual, the set of zeros of a quadratic polynomial. So Theorem 1 implies, in particular, that $(\wp_{A+z})$ does not have a 2π-periodic solution u_z with $|u_z| = |u_0|$ for a.e. $z \in \mathbb{R}^N$.

It is also natural to look for solutions to $(\wp_A)$ which are not necessarily periodic but whose absolute value is periodic. Moreover, from the point of view of physics it is desirable that $(\wp_A)$ is invariant with respect to the gauge choice for the vector potential A. This suggests looking at solutions of the form $\psi(x) = e^{iz\cdot x}u(x)$ with $z \in \mathbb{R}^N$ and u a 2π-periodic function. Following the usual terminology, we call them *Bloch solutions.* More precisely, they are Bloch solutions with *real* quasimomentum z (in general, $z \in \mathbb{C}^N$ is admitted, see e.g. [8]). They have 2π-periodic absolute value and render problem $(\wp_A)$ gauge invariant, more precisely,

$$\psi \text{ is a Bloch solution of } (\wp_{A+z}) \iff e^{iz\cdot x}\psi \text{ is a Bloch solution of } (\wp_A).$$

In quantum mechanical models two solutions ψ and $e^{i\gamma}\psi$, $\gamma \in \mathbb{R}$, represent the same state. So the state space is, in fact, the space of $\mathbb{S}^1$-orbits $[\psi] := \{e^{i\gamma}\psi : \gamma \in \mathbb{R}\}$ of the Sobolev space of complex-valued functions ψ where the solutions are seaked. The group of translations $\mathbb{Z}^N$ acts also on the $\mathbb{S}^1$-orbit space in the obvious way. Basically, Bloch solutions are solutions whose $\mathbb{S}^1$-orbit is 2π-periodic.

A 2π-periodic solution u_z of $(\wp_{A+z})$ gives rise to a Bloch solution $\psi_z := e^{iz\cdot x}u_z$ of $(\wp_A)$ with the same absolute value. Hence, the gauge-dependence question for 2π-periodic solutions is related to the multiplicity question for Bloch solutions. As a consequence of Theorem 1 we obtain the following.

Theorem 2. *Assume there exists* $\varepsilon_0 > 0$ *such that for every* $|z| < \varepsilon_0$ *problem* $(\wp_{A+z})$ *has a nonwhere vanishing* 2π*-periodic solution* u_z. *Then problem* $(\wp_A)$ *has an uncountable family of Bloch solutions* $(\psi_\alpha)_{\alpha\in\mathcal{I}}$ *such that*

$$|\psi_\alpha| \neq |\psi_\beta| \quad \textit{if } \alpha \neq \beta.$$

Moreover, if u_z *is a ground state, then for each* $\delta > 0$ *there is an uncountable set* $\mathcal{J} \subset \mathcal{I}$ *such that*

$$\left| \int_{[0,2\pi]^N} |\psi_\alpha|^p - \int_{[0,2\pi]^N} |u_0|^p \right| < \delta \quad \forall \alpha \in \mathcal{J}.$$

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On a conjecture by De Giorgi in large dimensions

Manuel del Pino
(joint work with Michal Kowalczyk, Juncheng Wei)

We consider the Allen-Cahn equation

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{1}$$

E. De Giorgi [3] formulated in 1978 the following celebrated conjecture:

(DG) *Let u be a bounded solution of equation* (1) *such that $\partial_{x_N} u > 0$. Then the level sets $[u=\lambda]$ are hyperplanes, at least for dimension $N \le 8$.*

Equivalently, under the above conditions the statement asserts the existence of $a \in \mathbb{R}^N$, $b \in \mathbb{R}$, $|a| = 1$ such that u has the form

$$u(x) = w(a \cdot x - b)$$

where $w(t)$ is the unique solution of

$$w'' + (1-w^2)w = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

namely $w(t) = \tanh(t/\sqrt{2})$. De Giorgi conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [5] and for $N = 3$ by Ambrosio and Cabré [1]. Savin [6] proved its validity for $4 \le N \le 8$ under a mild additional assumption. (DG) is a statement parallel to *Bernstein's theorem* for minimal graphs which in its most general form, due to Simons [8], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N-1$ variables, must be a hyperplane if $N \le 8$. Bombieri, De Giorgi and Giusti [2] proved that this fact is false in dimension $N \ge 9$, by constructing a nontrivial entire solution to the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8. \tag{2}$$

by means of the super-subsolution method. Let us write

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u,v)$ with the symmetry property $F(u,v) = -F(v,u)$ if $u \geq v$. In addition we can show that F becomes asymptotic to a function homogeneous of degree 3 that vanishes on the cone $u = v$. Let $\Gamma = \{x_9 = F(x')\}$ be the minimal BDG graph so predicted, and let us consider for $\alpha > 0$ its dilation $\Gamma_\alpha = \alpha^{-1}\Gamma$, which is also a minimal graph. Our result, which disproves statement (DG) in dimensions 9 or higher is the following.

Theorem 1. [4] *Let $N = 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of equation* (1) *such that*

$$\partial_{x_9} u_\alpha(x) > 0 \quad \textit{for all } x \in \mathbb{R}^9,$$

and such that for $x = y + t\nu(\alpha y)$, where $y \in \Gamma_\alpha$ and ν is a choice of normal to Γ we have

$$u(x) = w(t) + o(1),$$

where $|t| < \frac{\delta}{\alpha}$ and $o(1) \to 0$ uniformly as $\alpha \to 0$.

Let us consider coordinates to describe points in $\mathbb{R}^9$ near Γ_α, $x = y + t\nu(\alpha y)$, $y \in \Gamma_\alpha$, $|t| < \frac{\delta}{\alpha}$. Then we choose as a first approximation $\mathtt{w}(x) := w(t + h(\alpha y))$ where h is a smooth, small function on Γ, to be determined. Looking for a solution of the form $\mathtt{w} + \phi$, the problem becomes essentially reduced to

$$\Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0 \quad \text{in } \Gamma_\alpha \times \mathbb{R}$$

where $S(\mathtt{w}) = \Delta\mathtt{w} + f(\mathtt{w})$, $E = \chi_{|z|<\alpha^{-1}\delta}\, S(\mathtt{w})$, $N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi)$, $f(\mathtt{w}) = \mathtt{w}(1-\mathtt{w}^2)$, and $B(\phi)$ is a second order linear operator with small coefficients, also cut-off for $|z| > \delta\alpha^{-1}$. Rather than solving the above problem directly we consider a projected version of it:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{in } \Gamma_\alpha \times \mathbb{R} \tag{3}$$

$$\int \phi(y,z)w'(z)\,dz \;=\; 0 \quad \text{for all } y \in \Gamma_\alpha \tag{4}$$

A solution to this problem can be found in such a way that it respects the size and decay rate of the error E, which is roughly of the order $\sim r(\alpha y)^{-3}e^{-|z|}$, this is made precise with the use of a linear theory for the projected problem in weighted Sobolev norms and an application of contraction mapping principle. Finally h s found so that $c(y) \equiv 0$. We have $c(y)\int w'^2 dz = \int (E + N(\phi))w'\,dz$ and thus we get reduced to a (nonlocal) nonlinear PDE in Γ of the form

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{in } \Gamma, \quad h = 0 \quad \text{on } \Gamma \cap [u = v], \tag{5}$$

where $M(h)$ is a small operator which includes nonlocal terms. A solvability theory for the Jacobi operator in weighted Sobolev norms is then devised, with the crucial ingredient of the presence of explicit barriers for inequalities involving the linear operator above, and asymptotic curvature estimates by Simon [7]. Using this theory, problem (5) is finally solved by means of contraction mapping principle. The monotonicity property follows from maximum principle applied to the linear equation satisfied by $\partial_{x_9} u$.

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Concentration phenomena in 2D for exponential-type nonlinearities

Pierpaolo Esposito
(joint work with Juncheng Wei)

The basic problem we are interested in is the equation

$$\begin{cases} -\Delta u = \lambda^+ \frac{V^+ e^u}{\int_\Omega V^+ e^u} - \lambda^- \frac{V^- e^{-u}}{\int_\Omega V^- e^{-u}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (MF)$$

where $\lambda^+, \lambda^- \geq 0$, $V^+, V^- : \Omega \to [0, +\infty)$ are smooth potentials and $\Omega \subset \mathbb{R}^2$ is a smooth domain.

Problem (MF) is of interest in fluid mechanics, self-dual Gauge theories, conformal geometry, etc. For example, in fluid mechanics the stationary Euler equations for a planar, incompressible and homogeneous fluid can be rewritten in terms of the vorticity function ω, and an "ansatz" on the relation among the vorticity function and the stream function ψ reduces the Euler equations to a single equation on ψ of the form (MF) with a general R.H.S.
Many choices of the "ansatz" (Stuart, Joyce-Montgomery and Tur-Yanovski ansatz) are physical meaningful and lead to vortex-type configurations. They are all of exponential type and lead to equations in the general form (MF) with potentials V^+, V^- possibly vanishing like $|x - p_0|^{2\alpha}$, $\alpha \in \mathbb{N}$.

In the regular exponential case ($\lambda^- = 0$ and $\inf_\Omega V^+ > 0$) the asymptotic analyis of non-compact sequences of solutions is deeply understood as well as their existence by a perturbative construction. In the non-compact case the R.H.S. in (MF) concentrates to a sum of Dirac deltas supported at the so-called blow-up points.
In the singular exponential case ($\lambda^- = 0$ and $V \sim |x - p_0|^{2\alpha}$ as $x \to p_0$) the asymptotic analysis is harder and has received significant contributions just recently. The construction of non-compact sequences of solutions is open so far and has recently got a positive answer just on simply connected domains (see [1]).

The asymptotic analysis of the sinh −Gordon case ($V^+ = V^-$, $\lambda^+, \lambda^- > 0$ and $\inf_\Omega V^+ > 0$) has been achieved in [3] as a quantization result on the masses of the limiting Dirac deltas. The authors leave the question of the simple/multiple character of the blow-up points as an open problem. Note that in the corresponding pure exponential situation the blow-up points are always simple. In collaboration with J. Wei, in [2] we give a partial negative answer. By perturbative methods (the so-called nonlinear Lyapunov-Schimdt reduction) we construct a sequence of solutions of (MF) on the unit ball with $V^+ = V^- = 1$ and Neumann boundary condition as a super-position of a positive peak at the origin and three negative peaks on the vertices of a small equilateral triangle. In this way, the origin is a multiple blow-up point due to the collapse of four peaks (a positive and three negative ones). In the talk I'll describe this construction and exhibit the expansion of the so-called reduced energy to give an idea on how such solutions can be obtained.

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Symmetric quadruple phase transitions

Changfeng Gui

(joint work with Michelle Schatzman)

In this talk, we discuss a model which models the quadruple junction via generalized Allen-Cahn equation as in [3]. We introduce a quadruple well potential W with each well (global minimum point) representing a phase of grain. In material science, crystalline materials with a quadruple junction structure is studied in [7], [5], etc.

Assuming that all phases of the grain interior are of equal status, the potential W can be chosen so that it has the symmetry of a regular tetrahedron. The physical state of a crystalline material may be represented by an order parameter V which is a $\mathbb{R}^3$ -vector valued function. The order parameter V has a constant value $\mathbf{u}_0$ in each grain, where $\mathbf{u}_0$ corresponds to one of the wells of the potential. The following energy functional is used to gauge the physical state

$$E_{\Omega,\epsilon}(V) = \int_\Omega \epsilon|\nabla V|^2 + \frac{1}{\epsilon}Q(V)dx \tag{1}$$

where $\epsilon > 0$ is a small physical constant which is related to the difussion rate and therefore the thickness of the grain boundary. The dynamics of the physical state can be modeled by the gradient flow of the energy functional, i.e.

$$V_t = \epsilon \Delta V - \frac{1}{\epsilon} \nabla_V Q(V), \quad x \in \Omega,\, t > 0. \tag{2}$$

A very interesting problem in this model is the formation of quadruple junction where all four grains meet. The finer structure of quadruple junction can be expressed as a scaling of a solution of the Euler -Lagrange equation

$$-\Delta U + \nabla^2 Q(U) = 0, \quad U : \mathbb{R}^3 \to \mathbb{R}^3 \tag{3}$$

with suitable asymptotic behavior at infinity which shows a quadruple structure. We construct rigorously such a quadruple junction solution with proper symmetry, in the same spirit as done in [4] for triple junction solutions. However, the technical details of constructing quadruple junction solution turns out to be much more complicated, due to both the complexity of the three dimensional geometries and the new additional structure of the solution. Indeed, more interesting phenomena arise in this model compared to the two dimensional problem of triple junctions. We have to construct first one dimensional transition profile between two phases (heteroclinic solution) with target space $\mathbb{R}^3$ and study its special properties; then reproduce a triple junction solution with target space $\mathbb{R}^3$, which is not just the trivial generalization of [4] due to the extra dimension of the target space and the fourth well of the potential. Moreover, the structure of the quadruple junction solution has a subtle two dimensional transition layer connecting different triple junctions in different faces, which requires delicate analysis.

To state our main result, the following notation and assumptions are introduced.

(Q1): *The potential Q is a nonnegative function of class C^3 from $\mathbb{R}^3$ to $\mathbb{R}$ which is invariant under Γ, i.e.*

$$\forall \gamma \in \Gamma : \quad Q \circ \gamma = Q.$$

(Q2): *W vanishes only at the points in $\mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ in $\mathbb{R}^3$, and the Hessian $\nabla^2 Q(\mathbf{x})$ is nondegenerate for all $\mathbf{x} \in \mathbf{X}$. The eigenvalues of $\nabla^2 Q(\mathbf{x})$ are the strictly positive numbers $\lambda_1 \le \lambda_2 \le \lambda_3$ for $\mathbf{x} \in \mathbf{X}$.*

(Q3): *There is a constant $R_0 > 0$ such that*

$$\nabla Q(u) \cdot u := (u, \nabla Q(u)) \ge 0, \quad \forall u \in \mathbb{R}^3 \text{ with } |u| \ge R_0 \tag{4}$$

Choose any two wells $\mathbf{x}, \mathbf{y} \in \mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, and consider the minimization problem

$$\mathbf{e_{xy}} = \min \left\{ \int_{\mathbb{R}} \frac{1}{2}|v'|^2 + Q(v)\, dt : v \in H^1_{\text{loc}}(\mathbb{R})^2, \quad v(-\infty) = \mathbf{x}, v(\infty) = \mathbf{y} \right\}. \tag{5}$$

It can be shown that $\mathbf{e_{xy}} > 0$ and the minimizer exists, as in [8], [1]. The minimizer represents the transition profile between phases x and y.

Let z be a minimizer connecting $\mathbf{a}$ to $\mathbf{b}$ and be called heteroclinic connection; it satisfies the Euler-Lagrange equation

$$-z'' + \nabla_z Q(z) = 0. \tag{6}$$

The operator A defined by

$$D(A) = H^2(\mathbb{R})^3, \quad Av = -v'' + \nabla^2 Q(z)v \tag{7}$$

is a self-adjoint nonnegative operator; the lower bound of its essential spectrum is governed by its behavior at infinity: it is equal to the lower bound of the spectrum of $\nabla^2 Q(\mathbf{a})$, i.e. λ_1. It is easy to see that z' belongs to the kernel of A.

The main assumption is the *non degeneracy* of the heteroclinic connection z, namely

(Q4): the kernel of A is spanned by z'.

The main result reported in this talk is the following existence theorem of symmetric quadruple junction solution.

Theorem: Under the assumptions **(Q1)-(Q3)** for the potential Q and **(Q4)** for the linearized operator A, there exists a solution U in $C^2(\mathbb{R}^3, \mathbb{R}^3)$ to the Euler-Lagrange equation (3) with a symmetric quadruple structure. Namely, U satisfies $U \circ \gamma = \gamma \circ U$ for $\gamma \in \Gamma$ and, for any vector $\mathbf{x} \in \mathbb{R}^3$ with $\mathbf{x} \cdot (\mathbf{a} - \mathbf{b}) > 0$, $\mathbf{x} \cdot (\mathbf{a} - \mathbf{c}) > 0$, $\mathbf{x} \cdot (\mathbf{a} - \mathbf{d}) > 0$,

$$\lim_{t \to \infty} |U(t\mathbf{x}) - \mathbf{a}| = 0. \tag{8}$$

The symmetry of U also yields the corresponding limits of U relative to other wells $\mathbf{b}, \mathbf{c}, \mathbf{d}$.

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Symmetry and monotonicity of least energy solutions

Louis Jeanjean
(joint work with J. Byeon, M. Maris, M. Squassina)

In [3] we give a simple proof of the fact that for a large class of autonomous quasilinear elliptic equations and systems the solutions that minimize the corresponding energy in the set of all solutions are radially symmetric. We require just continuous non linearities and no cooperative conditions for systems. Thus, in particular, our results cannot be obtained by using the moving planes method.

In the case of scalar equations, we also prove that any least energy solution has a constant sign and is monotone with respect to the radial variable.

Our results of symmetry are based on a general approach, just developed by M. Maris [6] to study the symmetry of the minimizers of a functional under one (or several) constraint. Basically his result says that if the problem admits a minimizer and if any minimizer is at least a C^1 function then any minimizer is a radial function.

In [3] we show that, under some general "abstracts" conditions any least energy solution can be viewed as a minimizer of a functional on a constraint and we then apply the results of [6]. These "abstracts" conditions are known to hold in any situation (within our framework) where it has been previously proved the existence of least energy solutions. The result on the constant sign is obtained by a simple scaling argument.

As a special case of our results, we show that under the same conditions that guarantee the existence of solutions (see [1]) the least energy solutions of

$$-\Delta u = g(u), \quad u \in H^1(R^N) \tag{1}$$

are all radially symmetric, of given sign and monotone. This answer a conjecture of P.L. Lions [5] and for the corresponding system's version of (1) a conjecture of Brézis and Lieb [2].

Motivated by [3] we derive in [4] the existence of least energy solutions for a general class of scalar quasilinear equations set on R^N. Checking that the "abstract" conditions hold we extend our results of symmetry and monotonicity to this class.

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Entire Solutions of the Allen-Cahn equation and Complete Embedded Minimal Surfaces of Finite Total Curvature in $\mathbb{R}^3$

Michał Kowalczyk

(joint work with M. del Pino, J. Wei)

We establish a correspondence between minimal surfaces M which are complete, embedded and have finite total curvature in $\mathbb{R}^3$, and finite Morse index bounded, entire solutions of the Allen-Cahn equation $\Delta u + f(u) = 0$ in $\mathbb{R}^3$, where $f = -F'$ with F bistable and balanced, for instance $F(u) = \frac{1}{4}(1-u^2)^2$. We assume additionally that M is non-degenerate, in the sense that its bounded Jacobi fields are all originated from rigid motions (this is known for instance for a Catenoid and for the Costa-Hoffman-Meeks surface of any genus) We prove that for any small $\alpha > 0$ the Allen-Cahn equation has a bounded solution u_α whose 0-level set lies close to the blown-up surface $M_\alpha := \alpha^{-1}M$. We prove that u_α is *non-degenerate* and find that its Morse index coincides with the index of the minimal surface. A continuum of solutions of this type, with ends eventually diverging logarithmically from M_α is also found. Our construction suggests parallels of De Giorgi conjecture for general bounded solutions of finite Morse index.

Equivariant Yamabe problem and Hebey–Vaugon conjecture

Farid Madani

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Denote by $I(M, g)$, $C(M, g)$ and R_g the isometry group, the conformal transformations group and the scalar curvature respectively. Let G be a subgroup of the isometry group $I(M, g)$. E. Hebey and M. Vaugon [2] considered the following problem:

Is there some $G-$invariant metric g_0 which minimizes the functional

$$J(g') = \frac{\int_M R_{g'} \mathrm{d}v_{g'}}{(\int_M \mathrm{d}v_{g'})^{\frac{n-2}{n}}}$$

where g' belongs to the $G-$invariant conformal class of metrics g defined by:

$$[g]^G := \{\tilde{g} = e^f g / f \in C^\infty(M),\ \sigma^* \tilde{g} = \tilde{g} \quad \forall \sigma \in G\}$$

The positive answer would have two consequences. The first is that there exists a $I(M,g)-$invariant metric g_0 conformal to g such that the scalar curvature R_{g_0} is constant. The second is that the A. Lichnerowicz's conjecture [3], stated below, is true.

Lichnerowicz conjecture. *For every compact Riemannian manifold (M,g) which is not conformal to the unit sphere S_n endowed with its standard metric, there exists a metric $\tilde{g}$ conformal to g for which $I(M,\tilde{g}) = C(M,g)$, and the scalar curvature $R_{\tilde{g}}$ is constant.*

To such metrics correspond functions which are necessarily solutions of the Yamabe equation. In other words, if $\tilde{g} = \psi^{\frac{4}{n-2}} g$, ψ is a $G-$invariant smooth positive function then ψ satisfies

$$\frac{4(n-1)}{n-2}\Delta_g \psi + R_g \psi = R_{\tilde{g}} \psi^{\frac{n+2}{n-2}}.$$

The classical Yamabe problem, which consists to find a conformal metric with constant scalar curvature on a compact riemannian manifold, is the particular case of the problem above when $G = \{\mathrm{id}\}$. Denote by $O_G(P)$ the orbit of $P \in M$ under G, W_g the Weyl tensor associated to the manifold (M,g) and ω_n the volume of the unit sphere S_n. We define the integer ω at the point P as

$$\omega(P) = \inf\{i \in \mathbb{N} / \|\nabla^i W_g(P)\| \neq 0\}$$

Hebey–Vaugon conjecture. *If (M,g) is not conformal to the sphere S_n endowed with its standard metric g_{can}, or if the action of G has no fixed point, then the following inequality holds*

$$\inf_{g' \in [g]^G} J(g') < n(n-1)\omega_n^{2/n} (\inf_{Q \in M} \mathrm{card} O_G(Q))^{2/n} \tag{1}$$

This conjecture is the generalization of T. Aubin's conjecture [1] for the Yamabe problem corresponding to $G = \{\mathrm{id}\}$, where the constant in the right side of the inequality is equal to $\inf_{g' \in [g_{can}]} J(g')$ for S_n. In this case, the conjecture is completely proved.

Using the test function of T. Aubin [1] and R. Schoen [4], E. Hebey and M. Vaugon [2] proved the conjecture when the action of G is free over M, when the dimension of M is between 3 and 11 or when there exists $P \in M$ with minimal orbit (finite) such that $\omega(P) > (n-6)/2$ or $\omega(P) \leq 2$.

In this talk we will show how the Hebey–Vaugon conjecture solves the equivariant Yamabe problem, using Sobolev embedding in presence of symmetries. We give some ideas about the proof of the following new result :

Hebey–Vaugon conjecture is valid if there exists a point P with minimal orbit (finite) under G such that $\omega(P) < 15$.

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Minimal surfaces in CR manifolds

Andrea Malchiodi

(joint work with Jih-Hsin Cheng, Jenn-Fang Hwang and Paul Yang)

Let us consider a three dimensional manifold M. A *contact structure* ξ on is a completely non-integrable two-dimensional distribution on M, while a *contact form* Θ is a non-zero 1-form on M which annihilates ξ. We will always assume Θ to be *oriented*, namely that $d\Theta(u,v) > 0$ if (u,v) is an oriented basis of ξ. The *Reeb vector field* is the unique vector field T such that $\Theta(T) = 1$ and such that $d\Theta(T,\cdot) = 0$. A *CR structure* compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. We assume that also J is oriented, namely that for every non-zero vector field X, the couple (X, JX) is an oriented basis of ξ. A *CR manifold* (or pseudo-hermitian) is a manifold endowed with a CR structure and with a global contact form Θ. We have a natural volume form

$$V(\Omega) = \int_\Omega \Theta \wedge d\Theta$$

and a metric defined on ξ called *Levi form*

$$L_\Theta(v,w) = d\Theta(v, Jw).$$

For the Heisenberg group H^1, we have the standard choices

$$\hat{e}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \hat{T} = \frac{\partial}{\partial z}, \quad \hat{\Theta} = x\,dy - y\,dx + dz,$$

while the CR structure J is defined as $J\hat{e}_1 = \hat{e}_2$.

Consider next a regular surface Σ embedded in M^3. If $p \in \Sigma$ and if $T_p\Sigma \neq \xi(p)$ (as it happens generically), we define $e_1(p)$ as the unique (up to the sign) unit vector belonging to $T_p\Sigma \cap \xi(p)$, and $e_2(p) = J(p)e_1(p)$. Assuming that Σ is the boundary of an open set Ω, if we take a variation of this set in the direction fe_2 we have

$$\delta_{fe_2} V(\Omega) = \int_\Sigma f\omega; \qquad \delta_{fe_2} \int_\Sigma \omega = -\int_\Sigma fH\omega,$$

for some explicit 2-form ω (depending on Σ and on the CR structure) and for some function H. We call ω the *(pseudo)-area form* of Σ and H the *(pseudo)-mean curvature.*

For the Heisenberg group, our definition coincide with those in [CDG], [DGN], [Pau]. Moreover, the area element ω coincides with the three dimensional Hausdorff measure of Σ, considered in [B] and in [FSS].

Graphs in the Heisenberg group. Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let Σ be the graph of u

$$\Sigma = \left\{(x, y, u(x, y)) \mid (x, y) \in \mathbb{R}^2\right\}.$$

It turns out that, at regular points, $e_1 \in T\Sigma \cap \xi$ is given by

$$e_1 = \frac{1}{D}\left[-(u_y + x)\begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} + (u_x - y)\begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}\right],$$

where

$$D = \left[(u_x - y)^2 + (u_y + x)^2\right]^{\frac{1}{2}}.$$

One also finds that

$$H = \frac{1}{D^3}\{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\},$$

so the equation $H \equiv 0$ is

$$(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0 \tag{$*$}$$

In [CHMY], see also [CH] and [GP] for some extensions, the following classification result was proved.

Theorem A. *The only entire C^2 smooth solutions to $(*)$ are of the form*

(1.1) $u = ax + by + c$ *(a plane with a,b,c being real constants);*

(1.2) $u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay)$

(a, b being real constants such that $a^2 + b^2 = 1$ and $g \in C^2$).

The main ingredient for proving Theorem A is the analysis of the *singular points* of Σ (or of u), which are given by

$$S(u) = \left\{(x, y) \in \mathbb{R}^2 \;:\; u_x - y = u_y + x = 0\right\}.$$

For a minimal graph we have the following characterization of the singular points.

Proposition *Let Ω be a domain in the $xy-$plane. Let $u \in C^2(\Omega)$ be a solution of $(*)$. Let p_0 be a singular point of u. Then either p_0 is isolated in $S(u)$ or there exists a small neighborhood of p_0 which intersects with $S(u)$ in exactly a C^1 smooth curve through p_0.*

The analysis of the singular points can also be employed to study surfaces with bounded (p)-mean curvature in general 3-dimensional CR manifolds. We have indeed the following result.

Theorem B. *Let M be a pseudohermitian 3-manifold. Let Σ be a closed, connected surface, C^2 smoothly immersed in M with bounded p-mean curvature. Then the genus of Σ is less than or equal to 1. In particular, there are no constant p-mean curvature or p-minimal surfaces Σ of genus greater than one in M.*

The above proposition has been applies in [RR] to classify C^2 isoperimetric sets in H^1, according to a well known conjecture by P.Pansu. Other results concerning Pansu's conjecture can be found in [MR].

According to the results in [CHY] and [R] however, weak solution of $(*)$ might have lower regularity than C^2, and in particular in [CHY] a condition is given to characterize minimizers in terms of the angles formed by singular curves and characteristic curves (projections onto the $x-y$ plane of the integral curves of e_1). By this reason, in [CHMY2], the structure of the singular set for C^1 solutions is being studied. The following result has been proved, deriving an ordinary differential equation for the above quantity D along characteristic curves.

Theorem C. *Consider a C^1 smooth p-minimal graph over a plane domain Ω. Let p be a singular point in Ω. Then either p is an isolated singular point, i.e., there exists a neighborhood $V \subset \Omega$ of p such that V contains no other singular points except p, or there exists at least one C^0 singular curve $\gamma : [0,1] \to \Omega$ (i.e., γ is continuous and $\gamma(s)$ is a singular point for each $s \in [0,1]$) such that $\gamma(0) = p$ and $\gamma(1) \neq p$.*

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Existence of solutions for singular problems by perturbation methods

Marcelo Montenegro

The purpose of this note is to describe how to approximate some classes of singular equations by nonsingular equations. We obtain a solution to each nonsingular problem and estimates guaranteeing that the limiting function is a solution of the original problem.

The following problem was studied in [4]

$$
\begin{cases} -\Delta u = \chi_{\{u>0\}}\big(- u^{-\beta} + \lambda u^p\big) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}
$$

$0 < \beta < 1$ and $0 < p < 1$.

Theorem 1. *There exists a maximal solution for every $\lambda > 0$. There is constant $\lambda^* > 0$ such that for $\lambda > \lambda^*$ the maximal solution is positive. And for $\lambda < \lambda^*$, the maximal solution vanishes on a set of positive measure.*

We solve problem (1) by perturbing the equation as $-\Delta u + \frac{u}{(u+\varepsilon)^{1+\beta}} = u^p$. The solutions $u_\varepsilon \searrow u$ pointwisely and

$$
\int_\Omega u(-\Delta\varphi) + \int_{\{u>0\}} \frac{1}{u^\beta}\varphi \le \lambda \int_\Omega u^p \varphi, \tag{2}
$$

$\forall \varphi \in C^2(\overline{\Omega})$, $\varphi \ge 0$, $\varphi = 0$ on $\partial\Omega$.

There are two approaches to show that u is indeed a solution of (1). Relation (2) tells us that u is a maximal subsolution. We then regularize it and show that $u \in C^{1,\frac{1-\beta}{1+\beta}}$ and indeed solve the problem (1). In doing this, we need to obtain a local estimate $|\nabla u| \le C u^{\frac{1-\beta}{2}}$ in $\Omega' \subset\subset \Omega$. One of the main ingredients is the Harnack type lemma below.

Lemma 1.1. *For every ball $B_r(p) \subset \Omega$ there are constants $c_0, \tau > 0$ depending only on n and β such that if*

$$
\fint_{\partial B_r(p)} u \ge c_0 r^{\frac{2}{1+\beta}}, \ \text{then } u(x) \ge \tau \fint_{\partial B_r(p)} u \quad \text{a.e. in } B_{r/2}(p)
$$

The second approach relies on an estimate for u_ε by the maximum principle, namely $|\nabla u_\varepsilon| \le C u_\varepsilon^{\frac{1-\beta}{2}}$ in $\Omega' \subset\subset \Omega$. The idea to obtain such an estimate is to define $v = \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^{1-\beta}} \varphi_1^2$, where φ_1 is the first eigenfunction of the Laplacean with zero boundary condition. The function v has a maximum at $x_0 \in \Omega$, and then $\Delta v(x_0) \le 0$. If the estimate is not true, it is possible to take a constant $C > 0$ independently of ε such that $\sup v > C$ and by computation $\Delta v(x_0) > 0$, a contradiction. Using the estimate and multiplying the equation by an adequate test function, we let $\varepsilon \to 0$ in the equation to get a weak solution.

The next problem was studied in [6]

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}\big(\log u + \lambda u^p\big) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

Both approaches described above work in this case and an analogous result to theorem (1) holds true. The estimate obtained for the maximal subsolution (which is a solution) is $|\nabla u| \le Cu$ in $\Omega' \subset\subset \Omega$ and $u \in C^{1,1}$, a better regularity than the one for (1). This is roughly explained since $logu$ is less singular than $-1/u^\beta$. The estimate by maximum principle is $|\nabla u_\varepsilon| \le Cu_\varepsilon$ in $\Omega' \subset\subset \Omega$.

The fully nonlinear problem is addressed in a work in progress with E. Teixeira. We consider

$$\begin{cases} F(D^2u) &= G\left(x, u, |\nabla u|^2\right) & \text{in} & \Omega \\ \qquad u &= f & \text{on} & \partial\Omega \end{cases}$$

with $f \in C^{1,\alpha}(\partial\Omega)$ and $G\colon \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ a C^1 function. Following [2], $F\colon \mathrm{Sym}(d\times d) \to \mathbb{R}$ and $F(0) = 0$. The uniform ellipticity reads as follows: $\exists\lambda, \Lambda$, $0 < \lambda \le \Lambda$ such that

$$F(\mathcal{M} + \mathcal{N}) \le F(\mathcal{M}) + \Lambda\|\mathcal{N}^+\| - \lambda\|\mathcal{N}^-\|, \ \forall \mathcal{M}, \mathcal{N} \in \mathrm{Sym}(d \times d).$$

In order to state our Lipschitz estimate, let $\phi\colon (0, \infty) \to \mathbb{R}$ such that $\liminf_{s\to\infty} \phi(s) \ge 0$. We define the asymptotic behavior of ϕ passing 0, $\kappa\colon (0, 1) \to (0, \infty)$ by $\kappa(\varepsilon) := \inf\{s : \phi(s) > -\varepsilon\}$.

Theorem 2. *Let $u \in C^3(\Omega)$ be a solution. Define*

$$\sigma(|p|) := \inf_{(x,u)} \frac{D_uG(x, u, |p|^2)|p|^2 - \big|D_xG(x, u, |p|^2)\big|\,|p|}{G^2(x, u, |p|^2)}$$

assume $S := \liminf_{|p|\to\infty} \sigma(|p|) \ge 0$. Then $\max_{\overline{\Omega}} |\nabla u| \le C$, where C depends only on d, λ, Λ, $\|f\|_{C^{1,\alpha}}$ and the asymptotic behavior of σ passing 0.

The proof runs by defining $v = |\nabla u|^2$. We compute $D_{i,j}v$ and use the equation. Since v has a maximum at $x_0 \in \Omega$, we use the asymptotic behavior to conclude the estimate. It is not a proof by contradiction.

Specializing the function G we study the problem

$$\begin{cases} F(D^2u) &= \beta(u)\Gamma\left(|\nabla u|^2\right) & \text{in} & \Omega \\ \qquad u &= f & \text{on} & \partial\Omega, \end{cases} \tag{3}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ and $\Gamma\colon [0, \infty) \to \mathbb{R}$ are $C^{1,\alpha}$ functions. We have two consequences of Theorem 2.

Corollary 2.1. *If $\inf_u \frac{\beta'(u)}{\beta(u)^2} > -\infty$ and $\frac{\Gamma(\tau)}{\tau} \to +\infty$ as $\tau \to +\infty$, then $\max_{\overline{\Omega}} |\nabla u| \le C$.*

Corollary 2.2. *If β is nondecreasing, $|\beta| + |\beta'| > 0$ and $\liminf_{\tau\to\infty} \Gamma(\tau) > 0$, then $\max_{\overline{\Omega}} |\nabla u| \le C$.*

The approach to solve (3) is by considering again a perturbed problem

$$\text{(4)} \qquad \begin{cases} F\left(D^2u_\epsilon\right) &= \beta_\epsilon(u_\epsilon)\Gamma\left(|\nabla u_\epsilon|^2\right) & \text{in} & \Omega \\ u_\epsilon &= f & \text{on} & \partial\Omega. \end{cases}$$

Using Corollary 2.1 we derive existence of a Lipschitz viscosity solution for

$$\text{(5)} \qquad \begin{cases} F(D^2u) &= \dfrac{1}{u^q}\chi_{\{u>0\}}\Gamma\left(|\nabla u|^2\right) & \text{in} & \Omega \\ u &= f & \text{on} & \partial\Omega, \end{cases}$$

with $q \geq 1$, $\Gamma \geq 0$, Γ superlinear and F concave. In this case $\beta_\epsilon(u) = 1/u^q$ for $u > \varepsilon$ and $\beta_\epsilon(u) = \varepsilon$ for $u < -\varepsilon$. Between $-\varepsilon$ and ε, $\beta_\epsilon(u)$ is a fourth order polinomial. Since $\beta_\epsilon(u)$ is not monotone, Perron's method should be adapted by adding a term ku in both sides of the equation. This gives a solution u_ε to (4). The estimate of Theorem 2 permits us to let $u_\varepsilon \to u$, thus obtaining a viscosity solution of (5).

Another existence of viscosity solution result can be obtained using Corollary 2.2 for the problem

$$\text{(6)} \qquad \begin{cases} F(D^2u) &= \chi_{\{u>0\}}\Gamma\left(|\nabla u|^2\right) & \text{in} & \Omega \\ u &= f & \text{on} & \partial\Omega. \end{cases}$$

In this case β_ε is defined as follows. Let ρ be a smooth function supported in $[0,1]$, $\rho > 0$ in $(0,1)$ and normalized as to $\int_{\mathbb{R}} \rho = 1$. We define

$$\beta_\epsilon(s) := \frac{1}{2}\int_0^{s/\epsilon} \rho(\tau)d\tau - \frac{1}{2}\int_0^{-s/\epsilon} \rho(\tau)d\tau + \frac{1}{2} + \epsilon,$$

which satisfy the assumptions of Corollary 2.2.

Equations similar to (5) and (6) have been treated in [3, 5]. The solutions of the equations my exhibit a free boundary, which regularity can be studied with techniques from [1].

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Nonlinear Liouville theorems and applications

Pavol Quittner

(joint work with Thomas Bartsch, Peter Poláčik, Philippe Souplet)

Consider classical solutions of the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1}$$

where $p > 1$ is subcritical in the Sobolev sense:

$$p < p_S := \frac{N+2}{(N-2)_+}.$$

Liouville theorems state that if u is an entire solution of (1) (a solution defined for all times, positive and negative) and u is contained in an admissible class of functions then $u \equiv 0$. For example, if $p < p_B := N(N+2)/(N-1)^2$ then (1) does not possess nontrivial solutions in the class of nonnegative functions (see [1]). It is not known whether the exponent p_B can be replaced with the critical Sobolev exponent p_S. The following theorem due to [2, 4, 5] supports this conjecture.

Theorem 1 *Let $1 < p < p_S$. Equation (1) does not possess nontrivial classical solutions in the class of radial functions with bounded zero number.*

Here "radial function" means a function $u = u(x,t)$ which is radially symmetric with respect to the spatial variable x, hence $u(x,t) = U(|x|,t)$. The notion "function with bounded zero number" is defined as follows: Let $I \subset [0,\infty)$ and $J \subset \mathbb{R}$ be intervals and $U = U(r,t) : I \times J \to \mathbb{R}$ be a continuous function. We say that U is a function with bounded zero number if the number

$$\begin{aligned} z_I(U(\cdot,t)) := \sup\{k : \exists r_1, \dots, r_{k+1} \in I,\ & r_1 < r_2 < \cdots < r_{k+1}, \\ & U(r_i,t) \cdot U(r_{i+1},t) < 0 \text{ for } i = 1,2,\dots,k\} \end{aligned}$$

is finite and bounded uniformly in $t \in J$. Of course, one takes $I = [0,\infty)$ and $J = \mathbb{R}$ in Theorem 1.

It is known that Liouville-type theorems and scaling arguments can be used in order to obtain precise estimates for solutions of related problems in bounded and unbounded domains (see [3, 4]). As an application of such estimates based on Theorem 1, the following result has been proved in [5]: Consider the problem

$$\left\{ \begin{array}{ll} u_t - \Delta u = m(t)f(u), & |x| < R,\ t \in (0,T), \\ u = 0, & |x| = R,\ t \in (0,T), \\ u(x,0) = u(x,T), & |x| < R, \end{array} \right. \tag{2}$$

where $R, T \in (0,\infty)$, $x \in \mathbb{R}^N$,

$$m \in W^{1,\infty}([0,T]) \text{ is positive, } m(0) = m(T), \tag{3}$$

$$\left\{ \begin{array}{l} f \in C^1(\mathbb{R}),\ f(0) = 0,\ f'(0) \le 0, \\ |f'(u)| \le C(1 + |u|^{r-1}),\ r < p_S, \end{array} \right. \tag{4}$$

and

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|^{p-1}u} = 1 \quad \text{for some } p \in (1, p_S). \tag{5}$$

Theorem 2 *Assume* (3)*,* (4)*,* (5)*. Fix* $Z \in \{0, 1, 2, \dots\}$*. Then there exists a radial solution* $u(x,t) = U(|x|,t)$ *of* (2) *satisfying* $z_{(0,R)}(U(\cdot,t)) = Z$ *for all* t*.*

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Symmetry and nonexistence results for low Morse index solutions to semilinear elliptic equations

Filomena Pacella

(joint work with Francesca Gladiali and Tobias Weth)

Consider the class of semilinear elliptic equations

$$-\Delta u = f(|x|, u) \quad \text{in } \Sigma \tag{1}$$

where Σ is a radial (but not necessarily bounded) subdomain of $\mathbb{R}^N$, $N \geq 2$ and $f : \overline{\Sigma} \times \mathbb{R} \to \mathbb{R}$ is locally a $C^{1,\alpha}$-function. In the case where $\Sigma \neq \mathbb{R}^N$, we consider (1) together with Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Sigma. \tag{2}$$

We are interested in the question whether certain classes of solutions inherit - at least partially - the symmetry of the underlying domain. When $\Sigma = \mathbb{R}^N$, the moving plane method can be used to show radial symmetry of positive solutions of (1) under further assumptions on f and on the decay of the solutions at infinity. The first results in this direction were derived in the famous paper of Gidas, Ni and Nirenberg [9] and were extended in various directions in several papers (see e.g. [4, 5, 11] and the references therein). Other symmetry results, using a different method based on symmetrization techniques, were obtained in [2].
Very few results are available in the case when the solution changes sign or the underlying domain is an annulus or the exterior of a ball in $\mathbb{R}^N$. In part this is due to the fact that the study of nodal solutions in unbounded domains presents

several technical difficulties, and in many cases (see e.g. [7, 1]) these solutions form a set with a more complicated structure than the set of positive solutions. In the following, we present some symmetry results based on Morse index information which we have obtained in [14] and [10]. For this we need to recall a few definitions.

Definition 1. *A function $u \in C(\overline{\Sigma})$ is said to be foliated Schwarz symmetric if there is a unit vector $p \in \mathbb{R}^N$, $|p| = 1$ such that $u(x)$ only depends on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$ and u is nonicreasing in θ.*

Now let us denote by Q_u the quadratic form corresponding to a solution u of (1) and (2), i.e.

$$Q_u(\phi, \psi) = \int_\Sigma [\nabla\phi\nabla\psi - V_u(x)\phi\psi]\, dx, \quad \psi, \phi \in C_0^1(\Sigma) \tag{3}$$

where $V_u : \mathbb{R}^N \to \mathbb{R}$ is given by $V_u(x) = f'(|x|, u(x)) = \frac{\partial f}{\partial u}(|x|, u(x))$ and $C_0^1(\Sigma)$ denotes the space of all C^1-functions $\Sigma \to \mathbb{R}$ with compact support strictly contained in Σ.

Definition 2. *We say that a C^2-solution of (1), (2)*

- *is stable if $Q_u(\psi, \psi) \geq 0$ for all $\psi \in C_0^1(\Sigma)$;*
- *has Morse index equal to $K \geq 1$ if K is the maximal dimension of a subspace X of $C_0^1(\Sigma)$ such that*

$$Q_u(\psi, \psi) < 0 \quad \text{for all } \psi \in X \setminus \{0\}.$$

We note that, if the underlying domain Σ is bounded, than the Morse index of a solution u of (1), (2) coincides with the number of negative Dirichlet eigenvalues of $-\Delta + V_u$ on Σ. Our main symmetry results now read as follows (see [14] and [10])).

Theorem 3. *Suppose that $f(|x|, s)$ has a convex derivative $f'(|x|, s) = \frac{\partial f}{\partial s}(|x|, s)$ for every $x \in \Sigma$. Then every solution u of (1) and (2) with $|\nabla u| \in L^2(\Sigma)$ and Morse index $j \leq N$ is foliated Schwarz symmetric.*

Theorem 4. *Suppose that $f(|x|, s)$ is convex in the s-variable for every $x \in \Sigma$. Then every solution u of (1) and (2) with $|\nabla u| \in L^2(\Sigma)$ and Morse index $j \leq N$ is foliated Schwarz symmetric.*

We note that the assumption $|\nabla u| \in L^2(\Sigma)$ is automatically satisfied if Σ is bounded, i.e. if Σ is a ball or an annulus in $\mathbb{R}^N$ centered at the origin. By passing from bounded domains (which were considered in [14] and [13]) to unbounded domains, we not only encounter technical difficulties but also new phenomena since the shape of solutions depends in a subtle way both on decay assumptions and the class of nonlinearities. Theorem 4 in particular improves the main result in [12], where only solutions with Morse index less than or equal to one were considered.

We remark that under the assumptions of Theorem 3 or 4 there are only two possibilities, namely either the solution is radially symmetric or is strictly monotone in the polar angle, and both cases occur in particular examples. In contrast, stable solutions u are radially symmetric even without any convexity assumption on the nonlinearity f. Indeed we have:

Theorem 5. *Every stable solution u of (1) and (2) such that $|\nabla u| \in L^2(\Sigma)$ is radial. If in addition $\Sigma = \mathbb{R}^N$ and f does not depend on $|x|$ then u is constant.*

This Theorem generalizes and complements results on stable solutions obtained in [3] and [6]. The main difference here is that we also consider the domain $\mathbb{R}^N \setminus B$ and - as in the other results - allow $|x|-$dependence of the nonlinearity. On the other hand, in case f does not depend on $|x|$ and the underlying domain Σ is unbounded, we can deduce the following nonexistence results from Theorems 3 and 4.

Theorem 6. *Assume that $\Sigma = R^N$ and $f = f(s)$, i.e., f does not depend on x and that either f is convex or f' is convex. Then there are no sign changing solutions u of (1) with*

$$|\nabla u| \in L^2(\mathbb{R}^N), \qquad u(x) \to 0 \quad \text{as } |x| \to \infty$$

and Morse index $j \leq N$.

Theorem 7. *Assume that $\Sigma = \mathbb{R}^N \setminus B$ and $f = f(s)$, i.e., f does not depend on x and that either f is convex or f' is convex. Then there are no solutions u – neither positive nor sign changing – of (1) and (2) with*

$$|\nabla u| \in L^2(\Sigma), \qquad u(x) \to 0 \quad \text{as } |x| \to \infty$$

and Morse index $j \leq N$.

These nonexistence results are rather easy corollaries of Theorems 3 and 4 *except* in the case where nodal solutions in $\mathbb{R}^N \setminus B$ have to be excluded. They apply in particular to semilinear elliptic equations of the type

$$-\Delta u = |u|^{\frac{4}{N-2}} u \qquad \text{in } \mathbb{R}^N, \tag{4}$$

where $3 \leq N \leq 6$, and

$$-\Delta u + u = |u|^\sigma u, \qquad u \in H^1(\mathbb{R}^N) \text{ or } u \in H_0^1(\mathbb{R}^N \setminus B), \tag{5}$$

where $\sigma \geq 1$ and $\sigma < \frac{4}{N-2}$ if $N > 3$. In the case of (4). the assumptions

$$|\nabla u| \in L^2(\Sigma), \qquad u(x) \to 0 \quad \text{as } |x| \to \infty \tag{6}$$

are automatically satisfied since it is proved in [8] that every classical solution u with finite Morse index belongs to the space $D^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|v\|_{D^{1,2}(\mathbb{R}^N)} = \|\nabla v\|_{L^2(\mathbb{R}^N)}$. We believe that the properties (6) hold for finite Morse index solutions corresponding to a more general class of nonlinearities, although they are certainly not satisfied in the case of the Allen-Cahn nonlinearity $f(u) = u - u^3$.
Since we allow the nonlinearity to depend on $|x|$, Theorem 3 applies to nonautonomous problems of the type

$$-\Delta u + V(|x|)u = |u|^\sigma u, \qquad u \in H^1(\mathbb{R}^N) \text{ or } u \in H_0^1(\mathbb{R}^N \setminus B) \tag{7}$$

where $\sigma \geq 1$ and $\sigma < \frac{4}{N-2}$ if $N > 3$. As a consequence, all solutions of (7) with $|\nabla u| \in L^2$ and Morse index less than or equal to N are foliated Schwarz symmetric. If we restrict our attention to positive solutions, the same statement is true also for $0 < \sigma < 1$, since Theorem 4 applies in this case.

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On the Lane-Emden problem for the p-Laplacian

Enea Parini

(joint work with Christopher Grumiau)

In [1] we consider the Lane-Emden equation for the p-Laplacian

$$
\text{(1)} \qquad \begin{cases} -\Delta_p u &= \lambda |u|^{q-2} u \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega \end{cases}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda > 0$ and $1 < p < q < p^*$ (with $p^* = \frac{np}{n-p}$ if $p < n$ and $p^* = \infty$ otherwise). Weak solutions of the equation are critical points of the functional

$$\varphi_q(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q.$$

defined on the Sobolev space $W_0^{1,p}(\Omega)$. It can be easily seen that φ_q is not bounded from below; to overcome this difficulty, one defines the *Nehari manifold*

$$\mathcal{N}_q = \Big\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} \,|\, d\varphi_q(u)(u) = 0 \Big\}$$

and the *nodal Nehari set*

$$\mathcal{M}_q = \{ u \in W_0^{1,p}(\Omega) \,|\, u^+, u^- \in \mathcal{N}_q \}$$

where

$$d\varphi_q(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega |u|^{q-2}uv.$$

All the critical points of φ_q are contained in $\mathcal{N}_q$, and all the sign-changing critical points belong to $\mathcal{M}_q$. It can be proved that there exists a minimizer of φ_q on $\mathcal{N}_q$ (resp. $\mathcal{M}_q$), which is a positive (resp. sign-changing) critical point of φ_q and which therefore is called *ground state solution* (resp. *least energy nodal solution*) of (1).

The behaviour of families of ground state and least energy nodal solutions as $q \to p$ for a fixed $\lambda > 0$ is also investigated. Let us denote by $\lambda_1(p;\Omega)$ and $\lambda_2(p;\Omega)$ the first and the second eigenvalues of the p-Laplacian respectively. The following theorems hold.

Theorem 1. *Let $\{u_q\}_{q>p}$ be a family of ground state solutions of the Lane-Emden problem.*
(i) If $\lambda < \lambda_1(p;\Omega)$, then u_q diverge to infinity as $q \to p$.
(ii) If $\lambda = \lambda_1(p;\Omega)$, then u_q converge to a first eigenfunction of $-\Delta_p$ as $q \to p$.
(iii) If $\lambda > \lambda_1(p;\Omega)$, then u_q converge to zero as $q \to p$.

Theorem 2. *Let $\{u_q\}_{q>p}$ be a family of least energy nodal solutions of the Lane-Emden problem.*
(i) If $\lambda < \lambda_2(p;\Omega)$, then u_q diverge to infinity as $q \to p$.
(ii) If $\lambda = \lambda_2(p;\Omega)$, then u_q converge to a second eigenfunction of $-\Delta_p$ as $q \to p$.
(iii) If $\lambda > \lambda_2(p;\Omega)$, then u_q converge to zero as $q \to p$.

A natural question is whether the above mentioned results hold also when $1 < q < p$. In this case the functional φ_q has a different structure, and the Nehari manifold is no longer a closed subset of $W_0^{1,p}(\Omega)$, since there exists a sequence $\{u_n\}$ of solutions of (1) such that $u_n \to 0$ as $n \to \infty$ (see [2]). However, it is possible to prove that the equation admits a unique positive solution (see [3]). The investigation of this problem represents surely an interesting research topic.

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Tower of bubbles in almost critical problems

Angela Pistoia
(joint work with Monica Musso)

In this paper we are interested in the construction of solutions to the slightly super critical problem

$$\Delta u + \lambda\, \epsilon^{\frac{N-4}{N-2}} u + u^{\frac{N+2}{N-2}+\epsilon} = 0 \text{ in } \Omega,\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 5$, λ is a positive parameter and ϵ is supposed to be small and positive.

We are also interested in the construction of sign changing solutions to the slightly sub critical problem problem

$$\Delta u + |u|^{\frac{4}{N-2}-\epsilon} u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and ϵ is supposed to be small and positive.

In a celebrated paper, Brezis and Nirenberg [1] established that the problem

$$\Delta u + \mu u + u^{\frac{N+2}{N-2}+\epsilon} = 0 \text{ in } \Omega,\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

for $\epsilon = 0$, in a general bounded smooth domain Ω, is solvable for $0 < \mu < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions. This result is optimal, since the condition $\mu < \lambda_1$ is also necessary for existence. On the other hand, Pohozaev's identity, gives nonexistence for $\mu \leq 0$ and for any $\epsilon \geq 0$, in star-shaped domains.

In [2] the authors study the problem of existence for solutions to (1) in the case Ω is the unit ball. In this context, they construct families of radially symmetric solutions u_ϵ to (1) such that $u_\epsilon(0) = \|u_\epsilon\|_\infty \to \infty$ as $\epsilon \to 0$. To be more precise, they find that, given any number $k \geq 1$, there exist solutions of the form

$$u_\epsilon(y) = \alpha_N \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^{\frac{4}{N-2}}\, |y|^2} \right)^{\frac{N-2}{2}} M_j \,(\, 1 \ + \ o(1)\,) \quad \text{as } y \to 0\,,$$

where $M_j \to +\infty$ and $M_j = o(M_{j+1})$, as $\epsilon \to 0$, for all j and $\alpha_N = (N(N-2))^{\frac{4}{N-2}}$. The shape of the solutions described above is the one of a super position of bubbles, namely sum of functions of the form

$$U_{\delta,\zeta}(x) = \alpha_N \left(\frac{\delta}{\delta^2 + |x-\zeta|^2} \right)^{\frac{N-2}{2}},$$

with $M_j = \delta^{-\frac{N-2}{2}}$ and $\zeta = 0$. The family of functions $U_{\delta,\zeta}$, for any parameter $\delta > 0$ and any point $\zeta \in \mathbb{R}^N$, is known to be the only bounded solutions to the *limit problem* $\Delta u + u^{\frac{N+2}{N-2}} = 0$ in $\mathbb{R}^N$.

The phenomena discovered in [2], namely the existence of solutions to a certain semilinear elliptic problem involving the critical Sobolev exponent with the shape of a superposition of several bubbles centered at the same point but with different scaling parameters, was new and somewhat surprising, since for $\epsilon = 0$ and $\mu \to 0^+$ only a single bubble is present.

The method used to prove the result in [2] strongly relies on the symmetry of the problem. In [3], the authors could extend the construction described above to a more general class of domains, namely domains whose associated Robin's function has a non degenerate critical point. More precisely, if the domain Ω is such that the corresponding Robin's function has a non degenerate critical point ζ, one of the results contained in [3] states that solutions to (1) of the form of a tower of bubbles centered at ζ do exist. We point out that such a nondegeneracy condition is satisfied, for example, if Ω is a perturbation of a convex and axially symmetric domain.

In [4] we further extend the result in [2] and [3] to any bounded smooth domain. Indeed we prove that in any bounded domain Ω problem (1) does admit solutions with the shape of a tower of bubbles and we remove the assumption on the non-degeneracy of the critical point of the Robin's function.

Theorem 1. Assume $N \geq 5$. Then, given an integer $k \geq 1$, there exists a number $\bar{\lambda}_k > 0$ such that if $\lambda > \bar{\lambda}_k$ there exists $\epsilon_k > 0$ such that for any $\epsilon \in (0, \epsilon_k)$ there exist points ${\xi_j}_\epsilon^{(i)} \in \Omega$, positive numbers ${d_j}_\epsilon^{(1)} < {d_j}_\epsilon^{(2)}$, $j = 1, \ldots, k$ and two solutions $u_\epsilon^{(i)}$, $i = 1, 2$, of Problem (1) of the form

$$u_\epsilon(y) = \alpha_N \sum_{j=1}^{k} \left(\frac{{d_j}_\epsilon^{(i)} \epsilon^{\frac{2j-1}{N-2}}}{\left({d_j}_\epsilon^{(i)} \epsilon^{\frac{2j-1}{N-2}}\right)^2 + |y - {\xi_j}_\epsilon^{(i)}|^2} \right)^{\frac{N-2}{2}} + \Theta_\epsilon(y), \tag{3}$$

where, as ϵ goes to 0, $\|\Theta_\epsilon\|_{\mathrm{H}_0^1(\Omega)} \to 0$, $\tau\left({\xi_j}_\epsilon^{(i)}\right) \to \min_{x\in\Omega} \tau(x)$ and ${d_j}_\epsilon^{(i)} \to {d_j}^{(i)} > 0$, for $i = 1, 2$ and $j = 1, \ldots, k$.

As far as it concerns problem (2), in [5], the authors prove that if Ω is axially symmetric, problem (2) has a sign-changing solution with the shape of a tower of bubble with alternate signs, centered at the center of symmetry of the domain. In [4] we extend such a result to an arbitrary domain.

Theorem 2. Assume $N \geq 3$. Then, given an integer $k \geq 1$, there exists $\epsilon_k > 0$ such that for any $\epsilon \in (0, \epsilon_k)$ there exist points ${\xi_j}_\epsilon \in \Omega$, positive numbers

d_{j_ϵ} $j = 1, \ldots, k$ and a solutions u_ϵ of Problem (2) of the form

$$u_\epsilon(y) = \alpha_N \sum_{j=1}^{k} (-1)^j \left(\frac{d_{j_\epsilon} \epsilon^{\frac{2j-1}{N-2}}}{\left(d_{j_\epsilon} \epsilon^{\frac{2j-1}{N-2}}\right)^2 + |y - \xi_{j_\epsilon}|^2} \right)^{\frac{N-2}{2}} + \Theta_\epsilon(y), \tag{4}$$

where, as ϵ goes to 0, $\|\Theta_\epsilon\|_{\mathrm{H}^1_0(\Omega)} \to 0$, $\tau\left(\xi_{j_\epsilon}\right) \to \min_{x\in\Omega} \tau(x)$ and $d_{j_\epsilon} \to d_j > 0$, for $j = 1, \ldots, k$.

The proof of our results is based on a nonlinear Liapunov-Schmidt reduction.

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Singularities of complex solutions of the Burgers equation

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(joint work with Vladimír Šverák)

We consider the viscous 1D Burgers equation

$$u_t + uu_x = u_{xx} \tag{1}$$

in $\mathbb{R} \times (0, \infty)$ with initial condition $u(x, 0) = u_0(x)$, where we allow u_0 to be complex-valued. A well-known fact about equation (1) is that the transformation $u = -2v_x/v$, called the Cole-Hopf transformation, leads to standard heat equation $v_t = v_{xx}$ for v. The singularities of u correspond to the zeros of v. For real valued functions, v cannot have zeros if they are not present in $v(x, 0)$ and one sees immediately that for u_0 real and "sufficiently regular" the initial value problem for equation (1) has a unique smooth global solution. We show that complex valued solutions do develop singularities. One can then ask about the nature of the singular set. We show that, roughly speaking, if there are no singularities present in the initial data, then the singularities of u are isolated in $\mathbb{R} \times (0, \infty)$. In a "typical situation" the number of singularities of such a solution u will be finite. However, we have examples of solutions with infinitely many (isolated) singularities.

We now formulate our results more precisely. For a complex-valued $u \in L^1(\mathbb{R})$ we define $U(x) = \int_{-\infty}^{x} u(\xi)\, d\xi$ and $v = \exp(-U/2)$. Vice-versa, given a complex-valued $v \in W^{1,1}_0(\mathbb{R})$ (the space of all absolutely continuous functions that have the derivative in $L^1(\mathbb{R})$) with $v(x) \neq 0$ in $\mathbb{R}$ and $v(x) \to 1$ as $x \to -\infty$, we let $u = -2v_x/v$. For time-dependent functions on $\mathbb{R}$ we apply the above transformations at each time level.

A well known simple calculation shows that u satisfies equation (1) if and only if v satisfies the standard heat equation $v_t = v_{xx}$.

We can now solve the initial value problem for equation (1) with a complex valued $u_0 \in L^1(\mathbb{R})$ as follows. Set $v_0(x) = \exp\{-\frac{1}{2}\int_{-\infty}^{x} u_0(\xi)\,d\xi\}$ and let v be the bounded solution of the heat equation with initial data v_0. It is easy to check that there is $T > 0$ such that $|v(x,t)| \geq \varepsilon > 0$ in $\mathbb{R} \times (0,T)$ and hence $u = -2v_x/v$ is a well-defined local-in-time solution of equation (1) with $u(x,0) = u_0(x)$.

The following result gives a sufficient condition for the global existence of such a solution.

Proposition 1. *In the notation above, assume that $u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} |\operatorname{Im} u_0| \leq 2\pi$. Then equation (1) has a global smooth solution u with $u(x,0) = u_0(x)$. If in addition $|\int_{\mathbb{R}} \operatorname{Im} u_0| < 2\pi$, then $\sup_x |u(x,t)| \to 0$ as $t \to \infty$.*

On the other hand, there are solutions which develop singularities.

Proposition 2. *For each $\delta > 0$ there exists a smooth, compactly supported (complex-valued) u_0 with $\int_{\mathbb{R}} \operatorname{Im} |u_0| < 2\pi + \delta$ such that the solution of equation (1) with initial condition u_0 blows up in finite time.*

The above results show that solutions with initial conditions satisfying $|\int_{\mathbb{R}} \operatorname{Im} u_0| = \int_{\mathbb{R}} |\operatorname{Im} u_0| = 2\pi$ lie on the "threshold between blow up and decay." We next look at the asymptotic behavior of such threshold solutions.

Proposition 3. *Assume $u_0 \in L^1(\mathbb{R})$ is compactly supported, with $|\int_{\mathbb{R}} \operatorname{Im} u_0| = \int_{\mathbb{R}} |\operatorname{Im} u_0| = 2\pi$. Then there exist a real γ and a complex β with $\operatorname{Im} \beta \neq 0$ such that the solution u of equation (1) with $u(x,0) = u_0(x)$ satisfies*

$$u(x,t) = \frac{-2}{(x - \gamma\sqrt{2t}) + \beta} + O\left(\frac{1}{\sqrt{t}}\right), \quad (t \to \infty) \tag{2}$$

uniformly in $x \in \mathbb{R}$. The constants γ and β are determined by u_0. In particular, $\gamma = 0$ if and only if $\int_{\mathbb{R}} \operatorname{Re} u_0 = 0$.

Note that $-2/(x+b)$, $b \in \mathbb{C} \setminus \mathbb{R}$, is a family of steady states of (1). The solution in the previous result approaches the family as $t \to \infty$, converging to one of them if and only if $\gamma = 0$.

We next examine the structure of the singular set of u which corresponds to the nodal set of v.

Theorem 4. *Let v be a bounded complex-valued solution of the heat equation in $\mathbb{R} \times (0,\infty)$. Assume v has no zeros in some neighborhood of $\mathbb{R} \times \{0\}$. Then all zeros of v in $\mathbb{R} \times (0,\infty)$ are isolated.*

We remark that typically v has only finitely many zeros, but there are initial conditions for which v has infinitely many zeros:

Proposition 5. *There exists a smooth (complex-valued) function* $v_0 \in W_0^{1,1}(\mathbb{R})$ *such that* $v_0(-\infty) = 1$, $|v_0(x)| \geq \varepsilon_0 > 0$ *for any* $x \in \mathbb{R}$, *and the solution* v *of the heat equation with* $v(\cdot, 0) = v_0$ *vanishes at infinitely many points* $(0, \tau_k)$, *with* $\tau_k \to \infty$.

The proofs of the above results are given in [3]. This research was inspired by [2], where complex-valued solutions of 3D incompressible Navier-Stokes equations which develop singularities were found. Finally, we mention that in a recent paper [1], Lu Li has carried out a detailed analysis of the behavior of complex solutions of equation (1) near their singularities. In particular, she has given all possible blow up rates (they are all of type II) and rescaled blow up profiles of such solutions.

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On existence and behavior of minimizers for the Schrödinger-Poisson-Slater problem

David Ruiz

Our starting point is the system of Hartree-Fock equations:

(1)
$$-\Delta\psi_k + (V(x) - E_k)\psi_k + \psi_k(x)\int_{\mathbb{R}^3}\frac{|\rho(y)|^2}{|x-y|}\,dy - \sum_{j=1}^{N}\psi_j(x)\int_{\mathbb{R}^3}\frac{\overline{\psi_j(y)}\psi_k(y)}{|x-y|}\,dy = 0,$$

where $\psi_k : \mathbb{R}^3 \to \mathbb{C}$ form an orthogonal set in H^1, $\rho = \frac{1}{N}\sum_{j=1}^{N}|\psi_j|^2$, $V(x)$ is an exterior potential and $E_k \in \mathbb{R}$. This system appears in Quantum Mechanics in the study of a system of N particles.

In (1), the last term is usually called the exchange term, and is the most difficult term to be treated. A very simple approximation of this term was given by Slater [12] in the form:

$$\sum_{j=1}^{N}\psi_j\int_{\mathbb{R}^3}\frac{\overline{\psi_j(y)}\psi_k(y)}{|x-y|}\,dy \sim C_s\rho^{1/3}\psi_k,$$

where C_s is a positive constant.

By a mean field approximation, the local density ρ can be estimated as $\rho = |u|^2$, where u is a solution of the problem:

$$-\Delta u(x) + V(x)u(x) + Bu(x)\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}\,dy = C|u(x)|^{2/3}u(x).$$

This system receives the name of Schrödinger-Poisson-Slater system (see for instance [2, 4]). In recent years problem (2) has been object of intensive research, see [1, 7, 5, 6, 9, 10] and the references therein.

In this paper we are interested in the following version of the Schrödinger-Poisson-Slater problem:

$$-\Delta u + u + \lambda \left(u^2 \star \frac{1}{|x|} \right) u = |u|^{p-2} u, \tag{2}$$

where $\lambda > 0$. Our approach is variational, that is, we will look for solutions of (2) as critical points of the associated energy functional $I_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$,

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.$$

In [10] the following existence results were given:

	λ small	$\lambda \geq 1/4$
$2 < p < 3$	Two solutions $\inf I_\lambda\vert_{H^1_r} > -\infty$	No solutions $\inf I_\lambda = 0$
$p = 3$	One solution $\inf I_\lambda\vert_{H^1_r} = -\infty$	No solutions $\inf I_\lambda = 0$
$3 < p < 6$	One solution $\inf I_\lambda\vert_{H^1_r} = -\infty$	One solution $\inf I_\lambda\vert_{H^1_r} = -\infty$

In the above table, "Two solutions" (respectively, "One solution") means that there exist at least two (respectively, one) positive radial solutions. On the other hand, "No solution" means that there are no nontrivial solutions.

Let us consider the case $p \in (2,3)$. With λ small, there exist two solutions, one of them corresponding to a minimum of $I_\lambda|_{H^1_r}$. This solution must blow up as $\lambda \to 0$, since $I_0|_{H^1_r}$ is unbounded below. The aim of the talk is to describe the asymptotic behavior of those radial minimizers.

A partial answer is given in [6, 9]. In those papers, by using a perturbation technique, solutions of (2) with a certain behavior are found (for λ small). Moreover, those solutions correspond to local minima of $I_\lambda|_{H^1_r}$ and their energy tend to $-\infty$ as $\lambda \to 0$, so it is quite reasonable to think that those solutions correspond to global minima. However, those solutions are provided only if $p < 18/7$.

At this point, some natural questions arise: what is the meaning of the value $p = 18/7$? How do minimizers behave if $p \in (18/7, 3)$? Here we intend to answer both questions.

By making the change of variables $v(x) = \varepsilon^{\frac{2}{p-2}} u(\varepsilon x)$, $\varepsilon = \lambda^{\frac{p-2}{4(3-p)}}$, we obtain:

$$-\Delta v + \varepsilon^2 v + \left(v^2 \star \frac{1}{|x|} \right) v = |v|^{p-2} v. \tag{3}$$

This motivates the study of the limit problem:

$$-\Delta v + \left(v^2 \star \frac{1}{|x|} \right) v = |v|^{p-2} v. \tag{4}$$

Problem (4) can be thought of as a zero mass problem (see [3]), but under the action of a nonlocal term. To start with, $H^1(\mathbb{R}^3)$ is not the right space to study it. It seems quite clear that the right space should be:

$$E = E(\mathbb{R}^3) = \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy < +\infty\}.$$

The double integral expression is the so-called Coulomb energy of the wave, and has been very studied, see for instance [8]. We also denote $E_r = E(\mathbb{R}^3)$ the subspace of radial functions.

In [11] we give the following general inequality on the Coulomb energy term:

Theorem 1. *Given $\alpha > 1/2$, there exists $c = c(\alpha) > 0$ such that for any $u : \mathbb{R}^N \to \mathbb{R}$ measurable function, we have:*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-2}}\,dx\,dy \geq c\left(\int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{\frac{N-2}{2}}(1+|\log|x||)^{\alpha}}\,dx\right)^2. \tag{5}$$

In particular, $E \subset L^2(\mathbb{R}^3,\ |x|^{-\frac{1}{2}}(1+|\log|x||)^{-\alpha}\,dx)$ continuously.

We are not aware of any lower bound for the Coulomb energy in this fashion. We think that this inequality can be very useful in other frameworks, such as the Hartree equation or the Thomas-Fermi-Von Weizsäcker model.

Making use of (5) we obtain the following embedding result:

Theorem 2. *$E_r(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ continuously for $p \in (\frac{18}{7}, 6]$, and the inclusion is compact for $p \in (\frac{18}{7}, 6)$. Moreover, $E_r(\mathbb{R}^3)$ is not included in $L^p(\mathbb{R}^3)$ for $p < \frac{18}{7}$.*

Thanks to Theorem 2, for $p \in (\frac{18}{7}, 6]$, the functional $J : E_r \to \mathbb{R}$,

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3} |v|^p\,dx,$$

is well-defined, C^1, and its critical points correspond to solutions of (4). Moreover:

Theorem 3. *For any $p \in (18/7, 3)$, J is coercive and weak lower semicontinuous. Therefore, it attains its infimum, which is negative. As a consequence, (4) has a positive solution in E.*

Finally, Theorem 3 can be used to describe the asymptotic behavior of the radial minimizers of I_λ (see [11]):

Theorem 4. *Suppose that $p \in (18/7, 3)$ and let u_λ be a minimizer of $I_\lambda|_{H^1_r}$. Then, as $\lambda \to 0$,*

$$u_\lambda = \varepsilon^{-\frac{2}{p-2}} v_\varepsilon\left(\frac{x}{\varepsilon}\right),$$

where $\varepsilon = \lambda^{\frac{p-2}{4(3-p)}}$ and $d(v_\varepsilon, K) \to 0$. Here $K \subset E$ is the set of minimizers:

$$K = \{v \in E :\ J(v) = \min J\},$$

and $d(v, K) = \inf\{\|v - w\|_E :\ w \in K\}$. In particular, given $\lambda_n \to 0$, we have that $\varepsilon_n \to 0$ and $v_{\varepsilon_n} \to v$ in E (up to a subsequence) where v is a minimizer of J.

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Eigenvalue problem for the 1-Laplace operator

Friedemann Schuricht

(joint work with Bernd Kawohl, Zoja Milbers)

For $\Omega \subset \mathbb{R}^n$ open bounded with Lipschitz boundary the variational problem

$$\int_\Omega |Du|^p\,dx \to \text{Min!} \quad \text{subject to} \quad \int_\Omega |u|^p\,dx = 1 \tag{1}$$

in the Sobolev space $\mathcal{W}_0^{1,p}(\Omega)$ is related to the intensively studied eigenvalue problem for the p-Laplace operator

$$-\operatorname{div}|Du|^{p-2}Du = \lambda |u|^{p-2}u\,.$$

In the limit case $p=1$ minimizers of problem (1) do not belong to $\mathcal{W}_0^{1,1}(\Omega)$ in general. Therefore we have to consider

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1} \to \text{Min!} \tag{2}$$

subject to

$$G(u) := \int_\Omega |u|\,dx = 1 \tag{3}$$

in the space $BV(\Omega)$ of functions of bounded variation (here the surface integral replaces homogeneous boundary conditions). This problem formally leads to the equation

$$-\operatorname{div}\frac{Du}{|Du|} = \lambda\frac{u}{|u|}\,.$$

Since typical solutions of (2), (3) are characteristic functions of a subset of Ω, this equation is highly degenerate and needs a suitable interpretation.

In Kawohl & Schuricht [3] it is shown that problem (2), (3) has always a solution $u \in BV(\Omega)$ which is called *eigenfunction* of the 1-Laplace operator and satifies an equation

$$-\operatorname{div} z = \lambda s \quad \text{a.e. on } \Omega, \qquad \lambda = E(u) \tag{4}$$

where $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ and $s \in \mathcal{L}^\infty(\Omega)$ are related to u by the conditions

$$\|z\|_{\mathcal{L}^\infty} = 1\,, \quad \operatorname{div} z \in \mathcal{L}^{p'}(\Omega)\,, \quad E(u) = -\int_\Omega u \operatorname{div} z\, dx \tag{5}$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$) and

$$s(x) \in \operatorname{Sgn}(u(x)) \ \text{ a.e. on } \Omega \quad \text{ where } \quad \operatorname{Sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha > 0, \\ [-1, 1] & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases} \tag{6}$$

In [3] it is even shown that for any s satisfying (6) there is some vector field z with (5) such that equation (4) is satisfied, i.e. a minimizer u has to satisfy infinitely many Euler-Lagrange equations in general. We call u a solution of the *single* eigenvalue equation if it satisfies (4) for one selection s satisfying (6) and a corresponding z with (5) and we call u a solution of the *multiple* eigenvalue equation if it satisfies (4) for any selection s satisfying (6) with corresponding vector fields z.

Now a natural question is that for higher eigensolutions of the 1-Laplace operator. But it is not immediately clear how to define them. The multiple eigenvalue equation might be satisfied merely by a minimizer and the single equation has "too many" solutions (there is always a continuum of eigenvalues where "most" of them have a continuum of normalized eigenfunctions). Thus we claim to define eigenfunctions as critical points of E with respect to the constraint $G(u) = 1$. But what are critical points if E and G are convex but nondifferentiable functions. They can be defined by means of the weak slope $|dF|(u)$ replacing $\|F'(u)\|$ for a merely continuous or even lower semicontinuous function F (cf. Degiovanni & Marzocchi [2]). More precisely, we call $u \in BV(\Omega)$ *eigenfunction* of the 1-Laplace operator if the weak slope $|d(E + I_K)|(u) = 0$ where I_K denotes the indicator function of the set $K := \{v \mid G(v) = 1\}$. The weak slope depends on the norm chosen in the underlying space and it seems to be reasonable to study critical points also with respect to the $\mathcal{L}^q$-norm instead of the BV-norm for $1 \le q \le \frac{n}{n-1}$. Correspondingly, we call $u \in BV(\Omega)$ a BV-eigenfunction or $\mathcal{L}^q$-eigenfunction. It turns out that

$$\text{any } \mathcal{L}^q\text{-eigenfunction is also a } BV\text{-eigenfunction} \tag{7}$$

and that

(8) any BV-eigenfunction satisfies the single eigenvalue equation.

In Milbers & Schuricht [4] the existence of a sequence of pairs $\{\pm u_k\} \subset BV(\Omega)$ of $\mathcal{L}^q$-eigenfunctions with corresponding eigenvalues $\lambda_k = E(u_k) \to \infty$ is verified (partial results can also be found in Chang [1]). By (7) and (8) each eigenfunction u_k satisfies the single eigenvalue equation. Unfortunately these eigenfunctions cannot be identified by the single eigenvalue equation, since it has many solutions that do not seem to be critical points of the corresponding variational problem. Therefore we are looking for further necessary conditions for critical points that might at least reduce the set of candidates for eigenfunctions.

In the calculus of variations it is known that the evaluation of so called inner variations does not provide an additional condition to the usual Euler-Lagrange equation for smooth problems, but it might give an additional information in the nonsmooth case as present here. Thus, for an eigenfunction u we consider perturbations of the form

$$v(x,t) := u(x + t\xi(x)) \quad \text{for } \xi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^n)\,, \quad |t| \text{ small.}$$

This way, for any $\mathcal{L}^1$-eigenfunction $u \in BV(\Omega)$ of the 1-Laplace operator we can derive the following additional necessary condition that

$$\int_\Omega \langle z, D\xi\, z\rangle - \operatorname{div}\xi\, d|Du| = -\lambda \int_\Omega |u| \operatorname{div}\xi\, dx \quad \text{for all } \xi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^n) \tag{9}$$

where $\lambda = E(u)$ is the corresponding eigenvalue and $z \in \mathcal{L}^\infty(\Omega, \mathbb{R}^n)$ is given by the polar decomposition $Du = z|Du|$ (notice that this z is somehow related to the z in (4) but it is not necessarily the same). It turns out that this new condition easily rules out the solutions of the single eigenvalue equation that had been thought not to be eigenfunctions. Moreover this condition allows to derive further consequences for $\mathcal{L}^1$-eigenfunctions. It is still open in general how far the solutions of the single eigenvalue equation *and* of (9) are also eigenfunctions.

The situation is much clearer in $\mathbb{R}^1$. Using the results of Chang [1], we readily obtain that there is a unique correspondence between the solutions of the single eigenvalue equation (4) combined with (9) and the $\mathcal{L}^1$-eigenfunctions. Interestingly, the set of BV-eigenfunctions is strictly larger than that of $\mathcal{L}^1$-eigenfunctions.

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The proof of the Lane-Emden conjecture in four space dimensions

Philippe Souplet

The so-called Lane-Emden conjecture asserts that the elliptic system

$$\begin{cases} -\Delta u = v^p, & x \in \mathbb{R}^n, \\ -\Delta v = u^q, & x \in \mathbb{R}^n, \end{cases}$$

$(p, q > 0)$ has no positive classical solution if and only if the pair (p, q) lies below the Sobolev critical hyperbola, i.e.

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}.$$

This statement is the analogue of the celebrated Gidas-Spruck [3] Liouville-type theorem for the scalar case. Up to now, the conjecture had been proved for radial solutions [5, 8], in $n \leq 3$ space dimensions [9, 6], and in certain subregions below the critical hyperbola for $n \geq 4$ [2, 11, 5, 9, 4, 7, 1].

We shall report on our recent work [10], where we establish the conjecture in 4 space dimensions and obtain a new region of nonexistence for $n \geq 5$. Our proof is based on a delicate combination involving Rellich-Pohozaev type identities, a comparison property between components via the maximum principle, Sobolev and interpolation inequalities on S^{n-1}, and feedback and measure arguments.

Such Liouville-type nonexistence results have many applications in the study of nonvariational elliptic systems, namely for a priori estimates, existence and singularity analysis.

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Large time behavior in a viscous Hamilton-Jacobi equation with degenerate diffusion

Christian Stinner

We study weak solutions of the one-dimensional viscous Hamilton-Jacobi equation

$$u_t = (|u_x|^{p-2}u_x)_x + |u_x|^q \quad \text{in } (-R,R)\times(0,\infty) \tag{1}$$

with homogeneous Dirichlet boundary conditions, where $R > 0$, $p > 2$ and $1 < q < p-1$. For these solutions we investigate the convergence to steady states via a Lyapunov functional, which is constructed with a technique developed by Zelenyak (see [12]).

The more general problem

$$u_t = \Delta_p u + a\,|\nabla u|^q \quad \text{in } \Omega\times(0,\infty), \qquad \text{where } \Omega\subset\mathbb{R}^n,\ p\geq 2,\ q>0,\ a\in\{-1,1\},$$

$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known p-Laplacian operator and $n\in\mathbb{N}$ is arbitrary, possesses many different qualitative behaviors.

In particular, the semilinear equation, which corresponds to the case $p = 2$, has been widely studied by several authors. Concerning the large time behavior of nonnegative solutions to the semilinear equation in a bounded domain with homogeneous Dirichlet boundary conditions, it has been shown that for $q \geq 1$ and $a \in \{-1, 1\}$ any global classical solution converges to zero (see [3], [8]), whereas gradient blow-up occurs in case of $q > 2$ and $a = 1$ for large initial data (see [7]). Moreover, in case of $q \in (0, 1)$ the phenomenon of extinction in finite time has been shown for $a = -1$ (see [3]), while for $a = 1$ there is a one parameter family of nonnegative steady states and any solution evolving from sufficiently regular initial data converges uniformly to one of these stationary solutions (see [4]).

Concerning the quasilinear equation with $p > 2$ and $q > 1$, the large time behavior of solutions to the Cauchy problem has been studied recently (see [1], [2], [5], [6]). In particular, the existence of the critical exponents $q_1 = p-1$ and $q_* = p - \frac{n}{n+1}$, which separate different kinds of behavior, was established. But to the best of our knowledge, no result implying the convergence to a nonzero state is known for the quasilinear equation in a bounded domain with homogeneous Dirichlet boundary conditions.

In this talk, we show the existence of a one parameter family of nonnegative stationary solutions of (1). Moreover, we prove the existence of a global weak solution of (1) which converges to one nonnegative steady state w_ϑ as $t\to\infty$. This behavior is observed for all initial data $u_0 \in C^1([-R,R])$ satisfying the zero boundary condition. In particular, the limit w_ϑ fulfills $w_\vartheta \not\equiv 0$ in case of $u_0 \geq 0$ and $u_0 \not\equiv 0$, while the solution of (1) tends to zero for nonpositive initial data.

Although this behavior corresponds to the result observed for the semilinear problem (see [4]), the proofs differ significantly. We also establish the existence of a Lyapunov functional using a method from [12]. But as the weak solution of

(1) is not regular, we can only use a Lyapunov functional for classical solutions of certain approximative and parabolic problems. For these approximate functions u_ε, we obtain two estimates involving the derivatives $(u_\varepsilon)_t$ and $(u_\varepsilon)_{xx}$, respectively. Applying now a version of the Aubin-Lions lemma (see [10]) and adapting a method which has been used for other degenerate parabolic equations (see [11]), we can prove the convergence to the steady states.

The results presented in this talk will be published in [9].

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Symmetry properties and existence of solutions to nonlinear Schrödinger equations with singular electro-magnetic potentials

Susanna Terracini

(joint work with Laura Abatangelo)

We are concerned with differential operators of the form

$$\left(i\nabla - \frac{A(\theta)}{|x|}\right)^2 - \frac{a(\theta)}{|x|^2} \tag{1}$$

where $\frac{A(\theta)}{|x|}$ is the vector magnetic potential associated to the magnetic field $B = \mathrm{d}\frac{A(\theta)}{|x|}$ with $A(\theta) \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^N)$, $\frac{a(\theta)}{|x|^2}$ is the electric potential and both $A(\theta)$ and

$a(\theta)$ are in $\in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$. A quadratic form can be associated to this differential operator, that is

$$\int_{\mathbb{R}^N} \left| \left(i\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2. \tag{2}$$

We shall always assume the quadratic form to be stricly positive definite.

As a natural domain to study the properties of the quadratic form one can consider the closure of compactly supported functions $C_C^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to the quadratic form: this turns out to be $D^{1,2}(\mathbb{R}^N) \cap L^2(|x|^{-2}\mathrm{d}x)$. In this space we are interested in weak solutions to

$$\left(i\nabla - \frac{A(\theta)}{|x|} \right)^2 u - \frac{a(\theta)}{|x|^2} u = u^{2^*-1} \qquad \text{in } \mathbb{R}^N \setminus \{0\} \tag{3}$$

and in particular in their symmetry properties.

To investigate these questions, we refer to solutions which minimize the Rayleigh quotient over suitable spaces of symmetric elements of $D^{1,2}(\mathbb{R}^N) \cap L^2(|x|^{-2}\mathrm{d}x)$

$$\frac{\displaystyle\int_{\mathbb{R}^N} \left| \left(i\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} . \tag{4}$$

We are concerned also with Aharonov-Bohm type potentials. In $\mathbb{R}^2$ a vector potential associated to the Aharonov-Bohm magnetic field has the form

$$\mathcal{A}(x_1, x_2) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

where $\alpha \in \mathbb{R}$ stands for the circulation of $\mathcal{A}$ around the thin solenoid (see [3] for further details). In this paper we consider the analogous of these potentials in $\mathbb{R}^N$ for $N \geq 3$, that is

$$\alpha(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0 \right) \qquad (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{R}^{N-2} .$$

Our main result can be stated as follows:

Theorem 1. *Assume $N \geq 4$ and $a(\theta_0) > 0$. There exist $\epsilon > 0$ such that, when $\|B\|_{L^N(B(\theta_0,r))} < \epsilon$, the equation (3) admits at least one entire solutions in $D^{1,2}(\mathbb{R}^N) \cap L^2(|x|^{-2}dx)$. The same conclusion holds in dimension $N = 3$ provided $\int_{\mathbb{S}(N-1)} a(\theta) d\theta \geq 0$*

When $a(\theta) \leq 0$ the minimization of the Rayleigh quotient fails, cause of the diamagnetic inequality. However, one can still obtain existence and multiplicity of solutions in symmetric cases.

Theorem 2. *Assume $a(\theta) \equiv a \in \mathbb{R}^-$ and assume that A commutes with the subgroup of rotations on a fixed plane and on its orthogonal. There exist $a^* < 0$ such that, when $a < a^*$, the equation (3) admits at least two distinct entire* *solutions in $D^{1,2}(\mathbb{R}^N) \cap L^2(|x|^{-2}\,dx)$: one is radially symmetric while the second one is only invariant under a discrete group of rotations on the first two variables.*

In order to distinguish the two solution, we make use of the following result that, we believe, can be of independent interest.

Theorem 3. *Suppose $u \in D^{1,2}(\mathbb{R}^N) \cap L^2(|x|^{-2}\,dx)$ has a double cylindrical summetry, i.e. $u = u(r_1, r_2)$ (where $r_1 = \sqrt{x_1^2 + x_2^2}$ and $r_2 = \sqrt{x_3^2 + \cdots + x_N^2}$) is a solution to*

$$-\Delta u - \frac{a}{|x|^2} u = f(x, u) \tag{5}$$

with $a \in \mathbb{R}^-$ and $f : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ being a Carathéodory function, C^1 with respect to z, such that it satisfies the growth restriction

$$|f_z'(x, z)| \leq C(1 + |z|^{2^*-2})$$

for a.e. $x \in \mathbb{R}^N$ and for all $z \in \mathbb{C}$.

If the solution u has Morse index $m(u) \leq 1$, then u is a radial solution, that is $u = u(r)$ where $r = \sqrt{x_1^2 + \cdots + x_N^2}$.

These are joint works with Laura Abatangelo.

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Liouville type theorems for a class of non-cooperative elliptic systems

Tobias Weth

(joint work with E.N. Dancer, J.C. Wei)

We study the set of solutions of the nonlinear elliptic system

$$(P) \qquad \begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\ \quad u, v > 0 \quad \text{in } \Omega, \qquad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

in a smooth domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$ with coupling parameter $\beta \in \mathbb{R}$. This system arises in the study of Bose-Einstein double condensates, see e.g. [5]. Here we report results obtained in [2] concerning the impact of the parameters on Liouville type theorems, a priori bounds and the existence of multiple solutions of (P). Our first result is the following.

Theorem 1. *If $N \leq 3, \beta > -\sqrt{\mu_1\mu_2}$, there exists a constant $C = C(\beta, \mu_1, \mu_2, \Omega) > 0$ such that for any solution (u, v) of (P) we have*

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq C$$

The assumption on β in this Theorem is optimal. More precisely, consider the fully symmetric case $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$ and $V_1 = V_2 \equiv 0$. Then, by a rescaling, (P) becomes

$$(1) \qquad \begin{cases} -\Delta u + u = u^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v + v = v^3 + \beta u^2 v & \text{in } \Omega, \\ \quad u, v > 0 \quad \text{in } \Omega, \quad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The critical value $-\sqrt{\mu_1\mu_2}$ corresponds to $\beta = -1$ in (1), which is now invariant under the reflection $(u, v) \to \sigma(u, v) = (v, u)$. This invariance is essential for the following result.

Theorem 2. *Let $N \leq 3$.*
(a) If $\beta \leq -1$, then system (1) admits a sequence $(u_k, v_k)_k$ of solutions with

$$\|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)} \to \infty.$$

(b) For any positive integer k there exists a number $\beta_k > -1$ such that, for $\beta < \beta_k$, system (1) has at least k pairs $(u, v), (v, u)$ of solutions.

We briefly add some comments.
1. For $\beta > -1$, every positive solution of the Dirichlet problem for the scalar equation $-\Delta u + u = u^3$ in Ω gives rise to a *diagonal* solution $\frac{1}{\sqrt{1+\beta}}(u, u)$ of (1). In contrast, it will be evident from our construction that the solutions obtained in Theorem 2 have different components u, v. Moreover, for $\beta \neq 1$, system (1) does not admit nontrivial solutions (u, v) with $u \neq v$ and $u \leq v$ or $v \leq u$ (as is easily seen by multiplying the first equation of (1) with v, the second equation with u and integrating). Consequently, all solutions obtained in Theorem 2 have intersecting components.
2. The proof of Theorem 2 relies on a variant of Liusternik-Schnirelman theory on a submanifold M (depending on β) of the underlying energy space $H_0^1(\Omega) \times H_0^1(\Omega)$. The importance of this manifold is given by the following properties; it contains all solutions of (1), it is invariant under the reflection σ, and σ has no fixed points in M if $\beta \leq -1$.
3. The multiplicity statements in Theorem 2 carry over to the corresponding problem in the full space $\mathbb{R}^N$ if compactness is restored by restricting to radial functions. More precisely, with essentially the same proof we can show that, for $\beta \leq -1$, system (1) admits infinitely many radial bound state solutions if $\Omega = \mathbb{R}^N$, and the number of radial bound states tends to infinity as $\beta \searrow -1$, $\beta > -1$.
4. If $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, a different approach based on a corresponding parabolic problem shows the existence of radial solutions of (1) with a prescribed number of intersections of u and v, see [14].

A priori bounds for systems like (P) have been studied extensively in recent years, see [11, 3, 4, 12, 15] and the references therein. With the exception of [11], in all these papers the a priori bounds are derived from Liouville type theorems for the corresponding limiting elliptic system by means of a well known rescaling argument of Gidas and Spruck [7]. In our case the limiting system is

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega, \\ -\Delta v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega. \end{cases} \tag{2}$$

If β is nonnegative, this system is cooperative, so techniques based on the maximum principle allow to prove the nonexistence of nontrivial nonnegative solutions of (2) in $\Omega = \mathbb{R}^N$ and $\Omega = \mathbb{R}^N_+ := \{x \in \mathbb{R}^N \,:\, x_N > 0\}$ together with Dirichlet boundary conditions, see e.g. [4] and [12]. In the following we improve these results by relaxing the condition on β. More precisely, we have:

Theorem 3. *If $N \leq 3$, $\beta > -\sqrt{\mu_1\mu_2}$, and (u,v) is a classical solution of the system 2 either in $\Omega = \mathbb{R}^N$ or in $\Omega = \mathbb{R}^N_+ := \{x \in \mathbb{R}^N \,:\, x_N > 0\}$ and satisfying Dirichlet boundary conditions $u = v = 0$ on $\partial\mathbb{R}^N_+$, then $(u,v) = 0$.*

For $N = 1, 2$, these Liouville theorems are rather simple consequences of nonexistence results for solutions of the differential inequality $-\Delta w \geq w^3$ obtained in [1, 6, 8, 9]. The case $N = 3$ is essential more involved, since $-\Delta w \geq w^3$ admits solutions if the underlying domain is a half space in $\mathbb{R}^3$, see [9]. We combine a doubling lemma of Poláčik, Quittner and Souplet [10] with a uniform Hopf type estimate on boundary derivatives and a variant of Pohozaev's identity to deal with this case. This procedure is new and might be useful for other non-cooperative elliptic systems.

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Remarks on some quasilinear critical problems

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(joint work with Sébastien de Valeriola)

Abstract

In this paper we prove the almost everywhere convergence of the gradient of Palais-Smale sequences, allowing us to apply the Brezis-Lieb lemma. This leads us to show that infima are attained, and thus to prove the existence of optimal solutions for some critical problems. Our method does not use the concentration-compactness principle.

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In the framework of the classical paper by Brezis and Nirenberg ([2]), the analysis of a critical minimization problem involves

(1) a strict inequality,
(2) a convergence theorem.

The first step is always difficult and technical. On the other hand, the second step uses only the Hilbert space structure of $W_0^{1,2}(\Omega)$ and the Brezis-Lieb lemma ([1]).

This method is applicable to many other problems, like the existence of optimal functions for the critical Poincaré-Sobolev inequality in $W^{1,2}(\Omega)$ (see [3]).

For quasilinear critical problems, the spaces $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ are replaced by $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$. Since there is no Hilbert space structure, various sophisticated tools were used.

(1) The concentration-compactness principle for the critical Dirichlet problem, for the critical Neumann problem, for the critical Poincaré-Sobolev inequality and for the critical trace inequality.
(2) Approximation by subcritical problems and regularity theorems, for the critical Dirichlet problem and for the critical trace inequality.

Our aim is to deduce the convergence theorems for quasilinear critical problems from an elementary result. We prove the almost everywhere convergence of the gradients of Palais-Smale sequences. It is then possible to use the Brezis-Lieb lemma for the sequence of the gradients instead of the Hilbert space structure.

Our main result on the almost everywhere convergence of the gradients is the following.

We define

$$T(s) = \begin{cases} s & \text{if } |s| \leqslant 1, \\ \frac{s}{|s|} & \text{if } |s| > 1, \end{cases}.$$

Theorem 1. *Let $p > 1$, let Ω be an open bounded subset of $\mathbb{R}^N$ and let $(u_n) \subset W^{1,p}(\Omega)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\right) \cdot \nabla T(u_n - u)dx \;\to\; 0, \quad n \to \infty. \tag{1}$$

Then

(1) *there exists a subsequence (u_{n_k}) such that*

$$\nabla u_{n_k} \to \nabla u \quad a.e. \text{ on } \Omega,$$

(2)

$$\lim_{n\to\infty} \left(\int_\Omega |\nabla u_n|^p \, dx - \int_\Omega |\nabla (u_n - u)|^p \, dx\right) = \int_\Omega |\nabla u|^p \, dx,$$

(3) *for any $1 \leqslant q < p$, $u_n \to u$ in $W^{1,q}(\Omega)$.*

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Infinitely many positive solutions for elliptic problems with critical growth

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(joint work with L.P.Wang and J.Wei)

We will present a new technique to construct infinitely many positive solutions for some elliptic problems with critical growth.

Firstly, we consider the following prescribed scalar curvature problem:

$$\begin{cases} -\Delta u = K(y) u^{\frac{N+2}{N-2}}, \; u > 0 \quad \text{in } R^N \\ u \in D^{1,2}(R^N), \end{cases} \tag{1}$$

where $K(y) \geq K_0 > 0$.

We assume that $K(r)$ satisfies the following condition:

(K): There is a constant $r_0 > 0$, such that

$$K(r) = K(r_0) - c_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad r \in (r_0 - \delta, r_0 + \delta),$$

where $c_0 > 0$, $\theta > 0$ are some constants, and the constant m satisfies $m \in [2, N-2)$.

Let

$$x_j = (r_k \cos\frac{2\pi(j-1)}{k}, r_k \sin\frac{2\pi(j-1)}{k}, 0) \in R^N, \quad j = 1, \cdots, k,$$

and

$$U_{a,\lambda} = \frac{\alpha_N \lambda^{\frac{N-2}{2}}}{(1 + \lambda^2|y-a|^2)^{\frac{N-2}{2}}},$$

where α_N is the constant, such that $-\Delta U_{a,\lambda} = U_{a,\lambda}^{\frac{N+2}{N-2}}$. Then we have the following result:

Theorem 1 (with J.Wei): Suppose that $N \geq 5$ and $K(r)$ satisfies (K). Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, problem (1) has a solution u_k of the form

$$u_k = \sum_{j=1}^{k} U_{x_{j,k}, \mu_k} + \omega_k,$$

where as $k \to +\infty$, $\|\omega_k\|_{L^\infty} = o\big(k^{\frac{N-2}{N-2-m}}\big)$, $\mu_k = \Lambda_k k^{\frac{N-2}{N-2-m}}$

$$r_k \in (r_0 - \frac{1}{\mu_k^{1+\bar\theta}}, r_0 + \frac{1}{\mu_k^{1+\bar\theta}}), \quad \Lambda_k \in (\Lambda_1, \Lambda_2).$$

In particular, (1) has infinitely many solutions.

Since (1) does not satisfy the Palais-Smale condition, it is hard to use the variational techniques to obtain a multiplicity result for (1). In Theorem 1, we construct solutions with large number of bubbles near a local maximum set $|y| = r_0$ of the function $K(|y|)$, and thus obtain the existence of infinitely many solutions for (1). Note that these solutions have very large energy and are non-radial. To obtain such solutions, condition (K) is nearly necessary. See [6, 9].

Next, we consider

$$\begin{cases} -\Delta u + \lambda u = u^{\frac{N+2}{N-2}}, \ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{2}$$

where Ω is a bounded domain in R^N, ν is the outward unit normal of $\partial\Omega$ at y.

If the nonlinear term in (2) is replaced by u^p and p is sub-critical, then C.S.Lin, W.M. Ni and Takagi [8] proved the u must be a constant if $\lambda > 0$ is small. From this result, Lin and Ni made the following conjecture [7]:

Lin-Ni conjecture: If $\lambda > 0$ is small, then (2) only has a constant solution.

Assume that Ω is the unit ball. Then Adimurthi-Yadava [1, 2, 3], Budd, Knapp and Peletier [4] proved that

- if $N = 3$ or $N \geq 7$, and u is radially symmetric, then u is a constant if $\lambda > 0$ is small;
- if $N = 4, 5$ or 6, problem (2) admits a non-constant radial solution.

So we see that the Lin-Ni's conjecture is not always true if $N = 4, 5, 6$.

If $N = 3$ and Ω is convex, then Lin-Ni's conjecture is true. See [11, 10]. On the other hand, Druet, Robert and Wei proved [5] that assuming $N \geq 7$ and the mean curvature $H(x) \neq 0$ for all $x \in \partial\Omega$, if the solution u_λ satisfies

$$\int_\Omega u_\lambda^{\frac{2N}{N-2}} \leq C, \tag{3}$$

for some constant $C > 0$, independent of λ, then for λ small, $u_\lambda \equiv$ *constant*.

By the results mentioned above, we see that if $N \geq 7$, the solutions are either radially symmetric (in the case of the a ball), or have bounded energy, then the solutions must be a constant if λ is small. But these results can not cover the case that the solutions are non-radial with large energy. Recall that the solutions we constructed for the prescribed scalar curvature problem are non-radial with large energy. Using the same technique, we can now construct counter examples for the Lin-Ni's conjecture for all dimensions $N \geq 3$.

Theorem 2 (with L.P.Wang and J.Wei): Suppose that $N \geq 3$. Then, there are non-convex domains Ω, such that problem (2) has infinitely many solutions for any fixed $\lambda > 0$. Thus, Lin-Ni's conjecture is not always true in non-convex domain.

Remark: By [11, 10], the assumption that Ω is non-convex is necessary when $N = 3$.

Theorem 3 (with L.P.Wang and J.Wei): Suppose that $N \geq 4$. Then, there are convex domains Ω, such that problem (2) has infinitely many solutions for any fixed $\lambda > 0$. Thus, Lin-Ni's conjecture is not always true in convex domain.

Remark: By [11, 10], the assumption $N \geq 4$ is necessary.

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Solutions with moving singularities for a semilinear parabolic equation

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(joint work with Shota Sato)

We study the semilinear parabolic equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \tag{1}$$

where $p > 1$ is a parameter. It is known that for $N \geq 3$ and $p > p_{sg} := N/(N-2)$, (1) has a singular steady state $\psi_\infty(x) \in C^\infty(\mathbb{R}^N \setminus \{\xi_0\})$ with a singular point $\xi_0 \in \mathbb{R}^N$ that is explicitly expressed as

$$\psi_\infty(x) = L|x-\xi_0|^{-m}, \quad m = \frac{2}{p-1}, \quad L = \{m(N-m-2)\}^{1/(p-1)}.$$

Clearly, the spatial singularity of $u = \psi_\infty$ persists for all $t > 0$, but the singular point does not move in time.

In [1], we studied the existence of a solution of (1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

Definition. The function $u(x,t)$ is said to be a solution of (1) with a moving singularity at $\xi(t) \in \mathbb{R}^N$ for $t \in (0,T)$, where $0 < T \leq \infty$, if the following conditions hold:

(i) $u,\ u^p \in C([0,T); L^1_{loc}(\mathbb{R}^N))$ satisfy (1) in the distribution sense.

(ii) $u(x,t)$ is defined on $\{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0,T)\}$, and is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

(iii) $u(x,t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0,T)$.

Concerning the solutions with singularities, it turns out that

$$p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}, \quad N > 2,$$

plays a crucial role. We note that p_* is larger than p_{sg} and is smaller than the Sobolev critical exponent $p_S := (N+2)/(N-2)$. In our previous paper [1], for $p_{sg} < p < p_*$, we established the time-local existence, uniqueness and comparison principle for a solution with a moving singularity of the Cauchy problem (1) with the initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $u_0 \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative function. Given the motion $\xi(t)$ of a singularity and the initial data $u_0(x)$ satisfying some conditions, it can be shown that for some $T > 0$, there exists a solution of (1) and (2) with a moving singularity at $\xi(t)$.

Our aim here is to find a time-global solution with a moving singularity. To this aim, we first consider a forward self-similar solution of the form

$$u = (t+1)^{-1/(p-1)} \varphi((t+1)^{-1/2} x - a), \tag{3}$$

where $a \in \mathbb{R}^N$ is a given point. If $\varphi(z)$ satisfies

$$\Delta_z \varphi + \frac{z+a}{2} \cdot \nabla_z \varphi + \frac{1}{p-1} \varphi + \varphi^p = 0, \qquad z \in \mathbb{R}^N, \tag{4}$$

in the distribution sense, then u defined by (3) satisfies (1) in the distribution sense. Moreover, if

(A1) $\varphi(z)$ is defined on $\mathbb{R}^N \setminus \{0\}$ and is twice continuously differentiable,

(A2) $\varphi(z) \to \infty$ as $z \to 0$, then u defined by (3) may become a time-global solution with a singularity at $\xi(t) = (t+1)^{1/2} a$.

In order to state our result, we define Λ to be a set of $p > p_{sg}$ such that the equality

$$(-m + i)(N - m + i - 2) + pm(N - m - 2) = j(N + j - 2)$$

holds for some

$$i \in \{1, 2, \ldots, [m]\} \text{ and } j \in \{0, 1, 2, \ldots, i\},$$

where $[m]$ denotes the largest integer not greater than m. Clearly Λ is a finite set.

Concerning the existence of a forward self-similar solution with a moving singularity, we have the following result.

Theorem 1. *Let $N \geq 3$. Suppose that $p \notin \Lambda$ and*

$$p_{sg} < p < \begin{cases} p_* & \text{if} \quad N \leq 10, \\ \frac{N+2}{N-1} & \text{if} \quad N > 10. \end{cases}$$

Then there exists $\delta > 0$ such that if $|a| < \delta$, (4) has a solution $\varphi(z)$ satisfying (A1), (A2) *with the property*

$$0 < \varphi(z) < C|z|^{-m} \quad \text{on } \mathbb{R}^N \setminus \{0\}$$

for some $C > 0$.

Our next result is concerning the existence of a more general time-global solution of (1) with a moving singularity.

Theorem 2. *Assume that N and p satisfy the same conditions as in Theorem 1. Then there exists $\delta > 0$ such that if $\xi(t)$ satisfies $\xi(t) \in C^{i+1+\alpha}([0,\infty);\mathbb{R}^N)$ for $i = [\frac{[m-\lambda_1]+1}{2}]$ and some $\alpha > 0$, and*

$$\left\| \exp\left(\frac{\cdot}{2}\right) \xi_t(\exp(\cdot) - 1) \right\|_{C^i([0,\infty);\mathbb{R}^N)} < \delta,$$

(1) has a time-global solution with a moving singularity at $\xi(t)$. Here λ_1 is a positive constant defined by

$$\lambda_1 := \frac{N - 2 - \sqrt{(N-2)^2 - 4pm(N-m-2)}}{2}.$$

We remark that if $\xi(t) \in C^{i+1+\alpha}([0,\infty);\mathbb{R}^N)$ satisfies

$$|\xi^{(j)}(t)| \leq \beta(t+1)^{-j+\frac{1}{2}}, \quad j = 1, 2, \ldots, i+1$$

for sufficiently small $\beta > 0$ and $i = [\frac{[m-\lambda_1]+1}{2}]$, then $\xi(t)$ satisfies the assumption of Theorem 2. This means that the singular point $\xi(t)$ can connect any two points in $\mathbb{R}^N$. More details can be found in [2].

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