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Noncommutative Geometry

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Abstract. Many of the different aspects of Noncommutative Geometry were represented in the talks. The list of topics that were covered includes in particular new insight into the geometry of a noncommutative torus, local index formulae in various situations, C*-algebras and dynamical systems associated with number theoretic structures, new methods in K-theory for noncommutative algebras as well as new progress in quantum field theory using concepts from noncommutative geometry.

Mathematics Subject Classification (2000): 46Lxx, 19xx, 81Txx.

Introduction by the Organisers

Noncommutative geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly interdisciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and theoretical quantum physics. On the basis of ideas and methods from algebraic and differential topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. This includes K-theory, cyclic homology and the theory of spectral triples. Areas of intense research in recent years are related to topics such as index theory, quantum groups and Hopf algebras, the Novikov and Baum-Connes conjectures as well as to the study of specific questions in other fields such as number theory, modular forms, topological dynamical systems, renormalization theory, theoretical high-energy physics and string theory. Many results elucidate important properties of fascinating specific classes of examples that arise in many applications.

The talks at this meeting covered substantial new results and insights in several of the different areas in Noncommutative Geometry. The workshop was attended by 51 participants.

Workshop: Noncommutative Geometry

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Abstracts

Large scale geometry and topological quantum computing (after Freedman and Kitaev)

John Roe

This talk reported on some aspects of my interaction with the topological quantum computing group at Microsoft's "Station Q" in Santa Barbara, which is headed by Mike Freedman. I was not reporting on specific new results so much as trying to explain a group of questions. The initial focus of the discussion is Freedman's MSRI talk [2] referenced below.

This talk begins in a way that is very congenial for students of noncommutative geometry. Freedman considers infinite unitary matrices (a_{ij}), $i, j \in \mathbb{Z}$, which satisfy a control condition of the form $|a_{ij}| \leq C|i-j|^{-r}$. Define the 'flow' f_{ij} from i to j to be $|u_{ij}|^2 - |u_{ji}|^2$. The first result in the talk is that the "total flow" along the axis—the sum $\sum_{i<c\leq j} f_{ij}$ —does not depend on the constant c and is *quantized*: the total flow is an integer. There is a similar result for projection matrices parameterized over $\mathbb{Z}^2$.

For noncommutative geometers there is a simple explanation. Let $\mathcal{A}$ denote the $*$-algebra of controlled matrices over $\mathbb{Z}$. The pair $(\ell^2(\mathbb{Z}), F)$, where $Fe_i = \pm e_i$ (the sign being $+$ if $i \geq c$ and $-$ otherwise) is a 1-summable Fredholm module over $\mathcal{A}$, and Freedman's flow formula is its cyclic character $\tau_F(u, u^*)$. In fact, the K-theory of the C^*-algebra completion of $\mathcal{A}$ (the "translation algebra") can be computed [6] and one can see that it is $\mathbb{Z}$, exactly detected by the Fredholm module we have described.

The challenge posed by Freedman in his talk is to construct a "multiplicative" version of a similar theory. To explain the word "multiplicative" imagine a quantum spin system [1, Section 6.2] having a copy of $\mathbb{C}^2$ located at each point of $\mathbb{Z}$. The relevant Hilbert space is a (restricted) infinite tenor product of copies of these $\mathbb{C}^2$, and $\ell^2(\mathbb{Z})$ appears in this infinite tensor product as a subspace—the "one electron sector"—spanned by those tensor products of basis elements only one of which is not the vacuum vector. It is proposed to extend the "coarse" ideas above to multi-electron states and to classify associated projections that are "topologically protected". One says that a projection p is "topologically protected" if for some appropriately large class of perturbations x, $pxp = \lambda_x p$, where λ_x is a scalar. This condition is trivially satisfied if p has rank 1, but the object is to find topologically protected projections of high rank.

It has been proposed [3] that topologically protected projections could allow the construction of quantum computing elements (qbits) which are "protected" against decoherence.

An example (not a coarse-geometric one) of such a construction is provided by the *toric code* of A. Kitaev [4]. This operates as follows. Let X be an oriented 2-manifold provided with a cellular structure—Kitaev works with the torus divided into N^2 squares in the standard way, but we could consider any triangulation. Let the Hilbert space V be the tensor product of one copy of $\mathbb{C}^2$ for each edge (so V has dimension 2^{2N^2} in the example above). Consider operators on V as follows: For each vertex s, the operator A_s is the tensor product of copies of the matrix $\sigma' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on the edges adjacent to s and the identity operator on all other edges; and for each face p the operator B_p is the tensor product of copies of the matrix $\sigma'' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the edges adjacent to p and the identity operator on all other edges. Because each (vertex, face) pair have either 0 or 2 edges in common, while σ' and σ'' anticommute, it follows that all the operators A_s and B_p commute and can be simultaneously diagonalized. Look at the common $+1$-eigenspace of all these operators, which is the "ground state" of the "Hamiltonian" $H = -\sum A_s - \sum B_p$. What is this space?

The basis vectors for the tensor product space V are in $1:1$ correspondence with subsets S of the set of edges of X, These standard basis vectors e_S are eigenvectors for all the operators A, and in order that $A_s e_S = +e_S$ for all vertices s it is necessary and sufficient that an *even* number of the edges of S be adjacent to each vertex. In other words, the common $+1$ eigenspace of all the A operators is the group algebra $W = \mathbb{C}Z_1(X;\mathbb{Z}_2)$ of the group of mod 2 1-cycles of X.

Consider now the B operators. For a face p, the operator B_p acts on W as an involution which exchanges the basis elements of all the $\mathbb{C}^2$ factors for edges adjacent to p (and leaves the others alone). If we identify $W = \mathbb{C}Z_1(X;\mathbb{Z}_2)$ as above this action is induced by the additive action of the *boundary* of the (mod 2) 2-chain $[p]$ on the group of 1-cycles $Z_1(X;\mathbb{Z}_2)$. Consequently the common $+1$ eigenspace of all the A and B operators is identified with the fixed point set of the group of 1-boundaries, $B_1(X;\mathbb{Z}/2)$, acting on W; that is

$$(\mathbb{C}Z_1(X;\mathbb{Z}_2))^{B_1(X;\mathbb{Z}_2)} = \mathbb{C}(Z_1(X;\mathbb{Z}_2)/B_1(X;\mathbb{Z}_2)) = \mathbb{C}H_1(X;\mathbb{Z}_2)$$

the group algebra of the first homology group. In the torus case this is a 4-dimensional vector space.

It can be shown using a Schur's Lemma argument that this subspace is "protected" in the sense that any perturbation supported in a set of diameter smaller than the injectivity radius acts as a scalar on the range of the projection.

I am grateful to M. Freedman, A. Kitaev and S. Morrison for discussions.

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Twisted spectral triples and local index formula

Henri Moscovici

The basic template for a space in noncommutative geometry is encoded in the notion of a spectral triple. This was adapted in [4], by means of twisting by an automorphism of the algebra of coordinates, to include certain type III spaces, such as the spaces of leaves of codimension 1 foliations or of conformal foliations of arbitrary codimension. Given that the primary effect of the twisting automorphism is the replacement of the bimodule of noncommutative differential forms of a spectral triple with a bimodule of twisted differential forms, one could have reasonably expected the characteristic classes of a twisted spectral triple to be captured by a twisted version of the Connes-Chern character. Somewhat surprisingly, it turned out that no cohomological twisting is actually occurring, and that Connes' original construction of the Chern character in K-homology [1], landing in cyclic cohomology, remains in fact operative in the twisted case as well. One question that naturally arises is whether the Connes-Chern character of a twisted spectral triple can be expressed in local terms as in [3], by means of residue integrals that eliminate all quantum infinitesimal perturbations of order strictly larger than 1.

After discussing several geometric examples of spectral triples of type III, including an adelic version based on the Shimura variety for GL_2, we presented a local index formula for a special class of a twisted spectral triples where the twisting is implemented by scaling automorphisms. Compared with the proofs in the untwisted case (*cf.* [3], [5]), the novel feature resides in the fact that the twisted analogues of the JLO entire cocycle and of its retraction [2] are no longer cocycles in their respective cyclic cohomology bicomplexes. We showed however (see [6] for details) that the passage to the infinite temperature limit, respectively the integration along the full temperature range against the Haar measure of the positive half-line (which uses the finite part at one end, as in [2]), has the remarkable effect of curing in both cases the deviation from the cocycle, resp. transgression, identity. The end result is a local cocycle which computes the total characteristic class of the given twisted spectral triple in terms of residues of zeta-type functions resembling those corresponding to the untwisted case, but involving iterated twisted commutators rather than ordinary commutators.

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Boundary maps and Residues

Denis Perrot

Some years ago Cuntz and Quillen were able to show that excision holds in complete generality for periodic cyclic (co)homology of associative algebras [2]. That is, given any extension (short exact sequence) of algebras over $\mathbb{C}$,

$$(E) \;:\; 0 \to \mathscr{B} \to \mathscr{E} \to \mathscr{A} \to 0 \tag{1}$$

there exists an associated six-term exact sequence relating the periodic cyclic homology of $\mathscr{B}$, $\mathscr{E}$, $\mathscr{A}$, and similarly for cohomology. Using the abstract properties of the theory, Nistor [3] then proved that the connecting morphism $HP_1(\mathscr{A}) \to HP_0(\mathscr{B})$ of the cyclic homology exact sequence is compatible, via the Chern-Connes character, with the *index map* induced by the extension (E) on algebraic K-theory in low degrees:

$$\mathrm{Ind}_E : \; K_1(\mathscr{A}) \to K_0(\mathscr{B}) \; . \tag{2}$$

In principle this allows to state a general "higher index theorem", in the sense that the pairing of any periodic cyclic cohomology class $[\tau] \in HP^0(\mathscr{B})$ with the image of (2) can be computed as the pairing of its boundary $E^*([\tau])$ with $K_1(\mathscr{A})$. Here $E^* : HP^0(\mathscr{B}) \to HP^1(\mathscr{A})$ denotes the connecting morphism in cohomology. However, although explicit formulas for $E^*([\tau])$ do exist a priori, they turn out to be extremely complicated in general, and moreover they do not give rise to *local* formulas in contrast with, for instance, the residue index theorem of Connes and Moscovici [1].

In this talk we present an explicit construction of the connecting morphism E^* which avoids as much as possible the use of excision. One knows from the work of Cuntz and Quillen that any cyclic cohomology class $[\tau] \in HP^0(\mathscr{B})$ can be represented by a trace over an adequate extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ of $\mathscr{B}$, or equivalently by a trace over some power of this extension (think for example about the operator trace on a Schatten ideal). Our basic observation is the following: if the extensions $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ and $0 \to \mathscr{B} \to \mathscr{E} \to \mathscr{A} \to 0$ fit together in a commutative diagram with three rows and columns, then $E^*([\tau])$ is explicitly given by a fairly simple formula based on a "renormalization" procedure. The proof that actually *any* cyclic cohomology class over $\mathscr{B}$ can be represented in this way requires the knowledge of excision. Fortunately many cyclic cohomology classes appear naturally equipped with the required diagram, so we are able to circumvent excision completely in this situation.
Let us mention that the term "renormalization" is inspired by our previous work on the bivariant Chern character for quasihomomorphisms [5, 6, 7], where it was

argued that this procedure yields local index formulas automatically. This is related to the well-known anomalies of quantum field theory [4]. In fact when the extension (E) is *invertible*, the map E^* coincides with the bivariant Chern character of the odd quasihomomorphism associated to the extension. This allows to give an alternative proof of Nistor's index theorem: for any $[\tau] \in HP^0(\mathscr{B})$ and $[g] \in K_1(\mathscr{A})$, one has the equality of pairings

$$\langle [\tau], \mathrm{Ind}_E([g]) \rangle = \langle E^*([\tau]), [g] \rangle \tag{3}$$

that is, the index map is adjoint to the connecting morphism in periodic cyclic cohomology.

Then we show on the example of the family index theorem that our construction of $E^*([\tau])$ effectively leads to local formulas. We consider a proper submersion of smooth manifolds without boundary $M \to B$. A canonical extension (E) is obtained by taking $\mathscr{E}$ as the algebra of smooth families of (fiberwise) classical pseudodifferential operators of order zero, $\mathscr{B}$ as the ideal of order -1 pseudodifferential operators, and $\mathscr{A}$ as the commutative algebra of smooth functions over the cotangent sphere bundle of the fibers. The projection $\mathscr{E} \to \mathscr{A}$ thus carries a family of pseudodifferential operators to its family of leading symbols. Then any de Rham cycle in the base manifold B gives rise to a cyclic cocycle τ over the algebra $\mathscr{B}$; notice however that this requires to choose a connection on the submersion. Using zeta-function renormalization, we find that $E^*([\tau])$ is given explicitly in terms of a fiberwise Wodzicki residue [9] applied to some families of pseudodifferential operators, involving the connection and its curvature. Interestingly, the formula is a higher analogue of the famous Radul cocycle [8]. Hence if Q is a family of elliptic pseudodifferential operators with symbol class $[g] \in K_1(\mathscr{A})$, the pairing between $[\tau]$ and the "index bundle" $\mathrm{Ind}_E([g]) \in K_0(\mathscr{B})$ is the evaluation of this higher Radul cocycle on certain polynomials built from Q, its parametrix P, and the connection.

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Relative Connes-Chern character for manifolds with boundary

Matthias Lesch
(joint work with Henri Moscovici and Markus J. Pflaum)

Let M be a compact smooth m-dimensional manifold with boundary $\partial M \neq 0$. Assuming that M possesses a Spin^c structure, the fundamental class in the relative K-homology group $K_m(M, \partial M)$ can be realized analytically in terms of the Dirac operator D associated to the given Spin^c structure and to a Riemannian metric on M. Indeed, according to [1, Prop. 3.1], if D_B is a realization of an appropriate boundary value problem for D then the bounded operator $F = \mathsf{D}_B(\mathsf{D}_B^* \mathsf{D}_B + 1)^{-1/2}$ defines a relative Fredholm module over the pair of C^*-algebras $\big(\mathcal{C}(M), \mathcal{C}(\partial M)\big)$, hence an element $[\mathsf{D}] \in K_m(M, \partial M)$.

The index map $\operatorname{Index}_{[\mathsf{D}]} : K^\bullet(M, \partial M) \to \mathbb{Z}$, defined by the pairing of $[\mathsf{D}]$ with the K-theory, can be expressed in cyclic cohomological terms by means of Connes' Chern character in K-homology [6]. Since

$$K_\bullet(M \setminus \partial M) \simeq KK_\bullet(\mathcal{J}^\infty(M, \partial M); \mathbb{C}),$$

one can compute the Connes-Chern character of $[\mathsf{D}]$ by restricting F, or directly D, to the dense (and closed under holomorphic functional calculus) subalgebra $\mathcal{J}^\infty(M, \partial M) = \big\{ f \in \mathcal{C}^\infty(M) \;\big|\; f \text{ vanishes to infinite order at } \partial M \big\}$ of $\mathcal{C}_0(M \setminus \partial M) = \mathcal{C}\big(\{ f \in \mathcal{C}(M) \;|\; f_{|\partial M} = 0 \}\big)$. The resulting periodic cyclic cocycle, which can be computed as in [6, Part I, §6], corresponds via the canonical isomorphism $HP^\bullet\big(\mathcal{J}^\infty(M, \partial M)\big) \simeq H_\bullet^{\mathrm{dR}}(M \setminus \partial M; \mathbb{C})$ [3] to the relative de Rham class of the $\hat{A}$-current associated to the Riemannian metric. In fact, one can recover the $\hat{A}$-form itself out of local representatives for the Connes-Chern character (*cf.* [5, Remark 4, p. 119]). However, the boundary ∂M is not reflected in any way in these calculations.

It is the purpose of the present project to find explicit representations for the Connes-Chern character of the fundamental relative K-homology class $[\mathsf{D}] \in K_\bullet(M, \partial M)$ that allow to retain geometric information about the boundary. A significant step in this direction has already been taken by Getzler [7], who used the setting of Melrose's b-calculus [9] to construct an entire version of the relative Connes-Chern character. Devised for the treatment of infinite-dimensional geometries, entire cyclic cohomology is less effective than periodic cyclic cohomology in the finite-dimensional case. To remedy this drawback, we give explicit cocycle realizations for the Connes-Chern character in the relative cyclic cohomology bicomplex associated to the pair of algebras $\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)\big)$. This is achieved by adapting to the relative context the retraction procedure of [5], which converts the entire Connes-Chern character into the periodic one.

More concretely, we fix an exact b-metric g on M, and denote by D the corresponding b-Dirac operator. We define for each $t > 0$ and any $k \geq m = \dim M$,

$n \equiv m \pmod 2$, a pair of cochains $\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k+1}(\mathsf{D}_\partial)\big)$ over $\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)\big)$, given by the following formulæ

$$
\begin{aligned}
{}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}) &:= \sum_{j\geq 0} {}^{\mathrm{b}}\mathrm{Ch}^{k-2j}(t\mathsf{D}) + B\, {}^{\mathrm{b}}\mathrm{T\not{c}h}_t^{k+1}(\mathsf{D}) \\
\mathrm{ch}_t^{k+1}(\mathsf{D}_\partial) &:= \sum_{j\geq 0} \mathrm{Ch}^{k-2j+1}(t\mathsf{D}_\partial) + B\, \mathrm{T\not{c}h}_t^{k+2}(\mathsf{D}_\partial);
\end{aligned} \tag{1}
$$

here $\mathrm{Ch}^\bullet(\mathsf{D}_\partial)$ denote the components of the Jaffe-Lesniewski-Osterwalder realization [8] of the Connes-Chern character in entire cyclic cohomology, ${}^{\mathrm{b}}\mathrm{Ch}^\bullet(\mathsf{D})$ stand for their b-counterparts, and the components $\mathrm{T\not{c}h}_t^\bullet(\mathsf{D}_\partial)$, resp. ${}^{\mathrm{b}}\mathrm{T\not{c}h}_t^\bullet(\mathsf{D})$, are manufactured out of the canonical transgression formula as in [5]. $i : \partial M \to M$ denotes the inclusion. On has

$$
(b+B)\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D})\big) \;=\; \mathrm{ch}_t^{k+1}(\mathsf{D}_\partial)\circ i^* \tag{2}
$$

which shows that $\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k+1}(\mathsf{D}_\partial)\big)$ is a cocycle in the relative total (b,B)-complex of the pair of algebras $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$.

The main result about these cocycles reads as follows:

Theorem 1. *The pair of retracted cochains* $\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k+1}(\mathsf{D}_\partial)\big)$, $t>0$, $k \geq m = \dim M, k-m \in 2\mathbb{Z}$ *is a cocycle in the relative total complex*

$$\mathrm{Tot}_\oplus^\bullet \mathcal{B}C^{\bullet,\bullet}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)).$$

Its class in $HC^n(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ *is independent of* $t>0$ *and its class in* $HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ *is independent of* k.

Denote by $\omega_{\mathsf{D}}, \omega_{\mathsf{D}_\partial}$ *the local index forms of* D *resp.* D_∂ [2, Thm. 4.1]. *Then one has a pointwise limit*

$$\lim_{t\to 0+}\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k-1}(\mathsf{D}_\partial)\big) = \Big(\int_{{}^{\mathrm{b}}M}^{\mathrm{b}} \omega_{\mathsf{D}}\wedge\,.\,, \int_{\partial M}\omega_{\mathsf{D}_\partial}\wedge\,.\Big),$$

in particular $\big({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k-1}(\mathsf{D}_\partial)\big)$ *represents the Chern character of* $[\mathsf{D}] \in KK_m(C_0(M);\mathbb{C}) = K_m(M,\partial M)$.

For proving this Theorem one has to deal with the asymptotic behavior of expressions of the form $\int_{\Delta_n} {}^{\mathrm{b}}\mathrm{Tr}\big(a_0 e^{-\sigma_0 t\mathsf{D}^2}[a_1,\mathsf{D}]e^{-\sigma_1 t\mathsf{D}^2}\ldots[a_k,D]e^{-\sigma_k t\mathsf{D}^2}\big)d\sigma$, where Δ_n denotes the standard simplex $\{\sigma_0+\ldots+\sigma_k=1, \sigma_j\geq 0\}$. The difficulty here is twofold: firstly the b–trace is a regularization of the trace to b–pseudodifferential operators on the *non-compact* b-manifold. Secondly, the expression inside the b-trace is more complicated than just a single heat operator. To explain our main technical result let $A_0,\ldots,A_k$ b–differential operators of order $a_0,\ldots,a_k$. Then the Schwartz kernel of $A_0e^{-\sigma_0 t\mathsf{D}^2}A_1e^{-\sigma_1 t\mathsf{D}^2}\cdot\ldots\cdot A_ke^{-\sigma_k t\mathsf{D}^2}$ is well–known to have a *pointwise* asymptotic expansion [4]:

$$
\begin{aligned}
&\Big(A_0e^{-\sigma_0 t\mathsf{D}^2}A_1e^{-\sigma_1 t\mathsf{D}^2}\cdot\ldots\cdot A_ke^{-\sigma_k t\mathsf{D}^2}\Big)(p,p) \\
&\quad =: \sum_{j=0}^n a_j(A_0,\ldots,A_k,\mathsf{D})(p)\; t^{\frac{j-\dim M - a}{2}} + O_p(t^{(n+1-a-\dim M)/2}),
\end{aligned} \tag{3}
$$

where $a = \sum_{j=0}^{k} a_j$. The asymptotic expansion is only *locally* uniform in p. It is not *globally* uniform on the non-compact manifold M. A further subtlety arises from the fact that the function $a_j(A_0, \dots, A_k, \mathsf{D})$ is not necessarily integrable. However, a partie finie type regularized integral, which we denote by $\int_{{}^{\mathrm{b}}M} a_j(A_0, \dots, A_k, \mathsf{D}) d\,\mathrm{vol}$, exists.

Theorem 2. *For the* b*–trace we then have an asymptotic expansion*

$$
\begin{aligned}
&{}^{\mathrm{b}}\mathrm{Tr}\Big(A_0 e^{-\sigma_0 t\mathsf{D}^2} A_1 e^{-\sigma_1 t\mathsf{D}^2} \dots A_k e^{-\sigma_k t\mathsf{D}^2}\Big) \\
&\quad = \sum_{j=0}^{n} \int_{{}^{\mathrm{b}}M} a_j(A_0, \dots, A_k, \mathsf{D}) d\,\mathrm{vol}\; t^{\frac{j-\dim M - a}{2}} + \\
&\qquad + O\Big(\big(\prod_{j=1}^{k} \sigma_j^{-a_j/2}\big) t^{(n+1-a-\dim M)/2} \Big).
\end{aligned}
\tag{4}
$$

Under a technical assumption we can also calculate the limit as $t \to \infty$ of $({}^{\mathrm{b}}\mathrm{ch}_t^k(\mathsf{D}), \mathrm{ch}_t^{k+1}(\mathsf{D}_\partial))$. This will give a formal argument to identify the relative cohomology class of the pair $({}^{\mathrm{b}}\mathrm{ch}_\infty^k(\mathsf{D}), \mathrm{ch}_\infty^{k+1}(\mathsf{D}_\partial))$ with the b–analogue of the Connes' "unlocal" Chern character constructed from the phase $F = \mathsf{D}/|\mathsf{D}|$.

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Noncommutative geometric correspondences and an algebraic formula for the KK-product

Bram Mesland

Spectral triples [5] are a central notion in noncommutative geometry. The data for a spectral triple consist of a $\mathbb{Z}/2$-graded C^*-algebra A, acting on a likewise graded Hilbert space $\mathcal{H}$, and a selfadjoint unbounded odd operator D in $\mathcal{H}$, with compact resolvent, such that the subalgebra

$$\mathcal{A} := \{a \in A : [D, a] \in B(\mathcal{H})\},$$

is dense in A. The above commutator is understood to be graded. The motivating example is the Dirac operator acting on the Hilbert space of L^2-sections of a compact spin manifold M.

In [10, 11], a categorical framework for noncommutative geometry is developed. This is done by introducing a notion of morphism for spectral triples. Spectral triples are the unbounded cycles for K-homology [4], and their bivariant generalization are the cycles for Kasparov's KK-theory [8]. The central feature of KK-theory is the Kasparov product

$$KK_i(A, B) \otimes KK_j(B, C) \to KK_{i+j}(A, C).$$

Here A, B and C are C^*-algebras, and the product allows one to view KK as a category. The unbounded picture of this theory was introduced by Baaj and Julg [1]. In this picture the external product

$$KK_i(A, B) \otimes KK_j(A', B') \to KK_{i+j}(A\overline{\otimes}B, A'\overline{\otimes}B'),$$

is given by an algebraic formula, as opposed to Kasparov's original approach, which is more analytic in nature.

When dealing with unbounded operators, one is forced to consider dense subalgebras of C^*-algebras. They inherit a natural operator space structure by the representation

$$\pi : \mathcal{A} \to \mathrm{End}^*_B(\mathcal{E} \oplus \mathcal{E})$$
$$a \mapsto \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix}.$$

Blecher [2, 3] showed that such operator algebras admit a notion of C^*-module, and the Haagerup tensor product is well behaved on such modules. The resulting theory coincides with the usual one in the C^*-case. His work plays a crucial role in our construction, and leads to a natural definition of smooth C^*-algebra and smooth modules over them [10].

In order to describe the internal Kasparov product of unbounded KK-cycles, we introduce a notion of connection for unbounded cycles $(\mathcal{E}, D)$. This is a universal connection

$$\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B),$$

in the sense of Cuntz and Quillen [7], such that $[\nabla, D]$ extends to a completely bounded operator. The topological tensor product used here is again the Haagerup tensor product for operator spaces. A smoothness condition on connections has to be taken into account as well.

It should be noted that these connections are different from the ones considered in [9]. There, in the spirit of [6], a sufficient condition is given for a cycle to be the product of two given cycles. In the above setting, the product of two cycles with connection is given by an algebraic formula :

$$(\mathcal{E}, S, \nabla) \circ (\mathcal{F}, T, \nabla') = (\mathcal{E}\tilde{\otimes}_B \mathcal{F}, S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T, 1\tilde{\otimes}_\nabla \nabla').$$

Here

$$1 \otimes_\nabla T(e \otimes f) := (-1)^{\partial e}(e \otimes Tf + \nabla_T(e)f),$$

where ∇_T is the connection induced by ∇ from the derivation $b \mapsto [T, b]$. The connection $1\tilde{\otimes}_\nabla \nabla'$ is defined analogously. Thus, cycles with connection form a category, and the bounded transform

$$(\mathcal{E}, D, \nabla) \mapsto (\mathcal{E}, D(1 + D^2)^{-\frac{1}{2}}),$$

defines a functor from this category to the category KK. This framework can be used to obtain the noncommutative torus as a smooth quotient of the irrational rotation action on the circle [11]. More examples that fit into this framework are of course desirable.

There are several interesting questions to be asked about the newly constructed category of spectral triples. The notion of isomorphism, a refinement of Morita equivalence, is to be investigated, as well as the relation with Connes-Skandalis correspondences in the commutative case [6]. Another important problem is the existence of connections, up to KK-equivalence, which might lead to a description of KK as a quotient of the above category, and possibly to a smooth KK-theory for spectral triples.

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On the local index theorem of Connes and Moscovici

Uuye Otgonbayar

The notion of a spectral triple, due to Connes [3], encodes the data for an index problem: Associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we have an index map

$$\mathrm{Index}_{\mathcal{D}} : K_0(\mathcal{A}) \to \mathbb{Z}.$$

If the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable, that is, $e^{-t\mathcal{D}^2}$ is of trace class for all $t > 0$, then the Jaffe-Lesniewski-Osterwalder (JLO) character $\mathrm{Ch}_{\mathcal{D}}$ of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in entire cyclic cohomology computes the index as

$$\mathrm{Index}_{\mathcal{D}}(e) = \langle \mathrm{Ch}_{\mathcal{D}}, e \rangle, \quad e \in K_0(\mathcal{A}).$$

See [7, 6] for more details. This is a vast generalization of the McKean-Singer formula and the constant term of the asymptotic expansion of the character $\mathrm{Ch}_{\sqrt{t}\mathcal{D}}$ as $t \to 0+$ gives the local index formula [2].

We study the multiplicative property of the JLO character and show that it is compatible with the A_∞-exterior product structure of Getzler and Jones, using ideas from [5, 1]. More precisely, we have the following. See [8] for more details.

For unital algebras $\mathcal{A}_i$, let $\times : C_\bullet(\mathcal{A}_1) \otimes C_\bullet(\mathcal{A}_2) \to C_\bullet(\mathcal{A}_1 \otimes \mathcal{A}_2)$ denote the Hochschild shuffle product and let $B_r : C_\bullet(\mathcal{A}_1) \otimes \cdots \otimes C_\bullet(\mathcal{A}_r) \to C_{r+\bullet}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r)$ denote the Getzler-Jones operations [4].

Theorem 1. (1) *Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be θ-summable spectral triples. Then*

$$\mathrm{Ch}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha_1 \times \alpha_2) = \mathrm{Ch}_{\mathcal{D}_1}(\alpha_1)\mathrm{Ch}_{\mathcal{D}_2}(\alpha_2),$$

for $\alpha_1 \in C_\bullet(\mathcal{A}_1)$ and $\alpha_2 \in C_\bullet(\mathcal{A}_2)$.

(2) *Let $(\mathcal{A}_i, \mathcal{H}_i, \mathcal{D}_i)$, $1 \le i \le r$, be θ-summable spectral triples. Then*

$$\mathrm{Ch}_{\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_r} B_r(\alpha_1, \ldots, \alpha_r) = \frac{1}{r!} B\mathrm{Ch}_{\mathcal{D}_1}(\alpha_1) \ldots B\mathrm{Ch}_{\mathcal{D}_r}(\alpha_r),$$

for $\alpha_i \in C_\bullet(\mathcal{A}_i)$, $1 \le i \le r$. In particular, for $r = 2$,

$$\mathrm{Ch}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha_1 \times' \alpha_2) = \frac{1}{2} B\mathrm{Ch}_{\mathcal{D}_1}(\alpha) B\mathrm{Ch}_{\mathcal{D}_2}(\alpha_2),$$

where $\times' = B_2$ is the cyclic shuffle product.

As a corollary, we get a multiplicative character. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a θ-summable spectral triple. We define the *perturbed JLO character* as

$$\mathrm{Ch}^{\mathrm{pert}}_\bullet = \mathrm{Ch}_\bullet + \frac{1}{\sqrt{2}} B\mathrm{Ch}_{\bullet-1}.$$

Corollary 2. *Let* $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ *and* $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ *be* θ*-summable spectral triples. Then for* $\alpha \in C_\bullet(\mathcal{A}_1)$ *and* $\beta \in C_\bullet(\mathcal{A}_2)$

$$\mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha \times \beta + \alpha \times' \beta) = \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1}(\alpha) \cdot \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_2}(\beta).$$

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The signature package on Witt spaces

Paolo Piazza

(joint work with Pierre Albin, Eric Leichtnam, Rafe Mazzeo)

Let X be an orientable closed compact Riemannian manifold with fundamental group Γ. Let X' be a Galois Γ-covering and $r : X \to B\Gamma$ a classifying map for X'. What we call 'the signature package' for the pair $(X, r : X \to B\Gamma)$ refers to the following collection of results:

(1) the signature operator with values in the Mischenko bundle $r^*E\Gamma \times_\Gamma C^*_r\Gamma$ defines a signature index class $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$, $* \equiv \dim X \bmod 2$;
(2) the signature index class is a bordism invariant; more precisely it defines a group homomorphism $\Omega^{\mathrm{SO}}_*(B\Gamma) \to K_*(C^*_r\Gamma)$;
(3) the signature index class is a homotopy invariant;
(4) there is a K-homology signature class $[\eth_{\mathrm{sign}}] \in K_*(X)$ whose Chern character is, rationally, the Poincaré dual of the L-Class;
(5) the assembly map $\beta : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ sends the class $r_*[\eth_{\mathrm{sign}}]$ into $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$;
(6) if the assembly map is rationally injective one can deduce from the above results the homotopy invariance of Novikov higher signatures.

For history and background see [4] and for a survey we refer to [10].
The signature package is sometime decorated by the following item:

- there is a (C^*-algebraic) symmetric signature $\sigma_{C^*_r\Gamma}(X, r) \in K_*(C^*_r\Gamma)$, which is topologically defined, a bordism invariant $\sigma_{C^*_r\Gamma} : \Omega^{\mathrm{SO}}_*(B\Gamma) \to K_*(C^*_r\Gamma)$ and equal to the signature index class.

Item 3) can then proved by establishing the homotopy invariance of the symmetric signature.

The main goal of my talk was to report on recent joint work with Albin, Leichtnam and Mazzeo, where the signature package is stated and proved for a wide class of stratified pseudomanifolds known as Witt spaces. Most of our results are available on arXiv (June 2009).

A stratified pseudomanifold is a Witt space if the even dimensional links have vanishing middle degree lower middle perversity intersection homology. See [1], [6], [7]. We shall assume that our Witt spaces are endowed with a Thom-Mather stratification. Witt spaces can be endowed with iterated conic metrics and doing analysis on them means doing analysis on the regular part endowed with such an incomplete metric.

For Witt spaces the lower and upper middle perversity intersection homology groups are isomorphic and satisfy Poincaré duality. Moreover, for each topological space X, for example $X = B\Gamma$, there is a well defined Witt bordism group $\Omega_*^{\mathrm{Witt}}(X)$. This was thoroughly studied by Siegel, see [14]. We can thus expect the existence of a Witt-symmetric signature and we can freely talk about Witt-bordism invariance. Notice that intersection homology is not a homotopy invariant theory; however it is stratified homotopy invariant [5] and this is the notion that we need to employ in the analogue of item 3) and item 6)

In trying to extend the signature package to Witt spaces one is confronted with several challenging questions. The first one is the essential self-adjointness of the signature operator associated to an iterated conic metric, as well as its Fredholm property (on its domain endowed with the graph norm). Using methods inspired by the edge calculus of Mazzeo [11], see also [12], we first prove the following

Theorem 1. *Let $\widehat{X}$ be any smoothly stratified pseudomanifold which satisfies the Witt hypothesis. Let g be any iterated conic metric on the regular part of $\widehat{X}$. Denote by $\eth$ either the Hodge-de Rham operator $d+\delta$ or the signature operator $\eth_{\mathrm{sign}}$ associated to g. Then the following is true:*

- *Let u be in the maximal domain of $\eth$ as an operator on $L^2_{\mathrm{iie}}(X; {}^{\mathrm{iie}}\Lambda^* X)$. Then*
$$u \in \bigcap_{0<\epsilon<1} \rho^{\epsilon} H^1_{\mathrm{iie}}(X; {}^{\mathrm{iie}}\Lambda^* X).$$
- *The maximal domain $\mathcal{D}_{\max}(\eth)$ is compactly embedded in L^2_{iie}.*

As a consequence, the minimal and maximal domains of $\eth$ are equal, the de Rham operator and the signature operator are essentially self-adjoint and have only discrete spectrum of finite multiplicity. Moreover, there is a well defined signature class $[\eth_{\mathrm{sign}}] \in K_*(\widehat{X})$, with $* = \dim \widehat{X} \mod 2$, which is independent of the choice of the iterated conic metric on the regular part of $\widehat{X}$. In particular, in the even dimensional case, the index of the signature operator is well-defined.

If $\widehat{X}' \to \widehat{X}$ is a Galois covering with group Γ and $r : \widehat{X} \to B\Gamma$ is the classifying map, then the signature operator $\widetilde{\eth}_{\mathrm{sign}}$ with coefficients in the Mishchenko bundle, *together with the $C^*_r\Gamma$-Hilbert module $L^2_{\mathrm{iie},\Gamma}(X; {}^{\mathrm{iie}}\Lambda^*_\Gamma X)$ define an unbounded Kasparov $(\mathbb{C}, C^*_r\Gamma)$-bimodule and hence a class in $KK_*(\mathbb{C}, C^*_r\Gamma) = K_*(C^*_r\Gamma)$, which we call the index class associated to $\widetilde{\eth}_{\mathrm{sign}}$ and denote by $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$. If $[[\eth_{\mathrm{sign}}]] \in KK_*(C(\widehat{X}) \otimes C^*_r\Gamma, C^*_r\Gamma)$ is the class obtained from $[\eth_{\mathrm{sign}}] \in KK_*(C(\widehat{X}), \mathbb{C})$ by tensoring with $C^*_r\Gamma$, then $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$ is equal to the Kasparov product of the class defined by the Mishchenko bundle $[\widetilde{C^*_r\Gamma}] \in KK_0(\mathbb{C}, C(\widehat{X}) \otimes C^*_r\Gamma)$ with $[[\eth_{\mathrm{sign}}]]$:*

$$\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) = [\widetilde{C^*_r\Gamma})] \otimes [[\eth_{\mathrm{sign}}]] \tag{1}$$

In particular, the index class $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$ does not depend on the choice of the adapted metric on the regular part of $\widehat{X}$. Finally, if $\beta : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ denotes the assembly map in K-theory, then

$$\beta(r_*[\eth_{\mathrm{sign}}]) = \mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \text{ in } K_*(C^*_r\Gamma) \tag{2}$$

In this statement ρ is the product of the radial functions along the Thom-Mather tubular neighbourhoods.

The part of our theorem stating the essential self-adjointness of $\eth$ as well as the discreteness of the spectrum had been established by Cheeger in [3], using heat kernel techniques.

Notice that the above theorem establishes, in particular, item 1), 5) and (part of) 4) of the signature package on Witt spaces.

As far as the other items are concerned, we observe first of all that the higher signatures in the Witt context are defined using the homology L-class of Goresky-MacPherson; thus if $\widehat{X}$ is a Witt space then there is a well defined homology class $L_*(\widehat{X}) \in H_*(\widehat{X}, \mathbb{Q})$ whose definition employs the intersection pairing in intersection homology. If $r : \widehat{X} \to B\Gamma$ is a classifying map, then the higher signatures are defined as the collection of numbers:

$$\{\langle \alpha, r_* L_*(\widehat{X})\rangle\,, \quad \alpha \in H^*(B\Gamma, \mathbb{Q})\}\,.$$

It has been proved by Moscovici and Wu, see [13], that the Chern character of the signature K-homology class of a Witt space is precisely equal to the homology L-class $L_*(\widehat{X})$ rationally. This completes item 4). The (rational) Witt-bordism invariance of the signature index class can be proved more or less as in the closed case, see [9]; this establishes item 2). More delicate is the proof of the stratified homotopy invariance of the signature index class; this is achieved by generalizing to the Witt context the analytic proof of Hilsum and Skandalis, see [8]. This establishes item 3) in the Witt context. Thus, all items but the last one have been established in the Witt context. However, the stratified homotopy invariance of the numbers $\{\langle \alpha, r_* L_*(\widehat{X})\rangle\,, \quad \alpha \in H^*(B\Gamma, \mathbb{Q})\}$ follows from the rational injectivity of the assembly map exactly as in the closed case. Finally, for the notion of symmetric signature we employ recent work of Banagl, see [2], where, using techniques from surgery theory, the notion of symmetric signature for a Witt space with fundamental group Γ is rigorously introduced as an element in $L^*(\mathbb{Q}\Gamma)$. Using the surjectivity of the natural map $\Omega^{\mathrm{SO}}_*(B\Gamma) \to \Omega^{\mathrm{Witt}}_*(B\Gamma)$, due to Sullivan, we can prove that our signature index class is equal, rationally, to the C^*-symmetric signature (the latter being, by definition, the image of the symmetric signature defined by Banagl under the composite $L^*(\mathbb{Q}\Gamma) \to L^*(C_r^*\Gamma) \to K_*(C_r^*\Gamma)$). This completes the signature package in the Witt context.

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Harmonic maps between noncommutative tori

Jonathan Rosenberg

(joint work with Varghese Mathai [5])

1. Motivation

In classical sigma-models in string theory, the fields are maps $g\colon \Sigma \to X$, where Σ is closed and 2-dimensional, representing a *string worldsheet*, and the target space X is 10-dimensional space-time. The leading term in the action is

$$S(g) = \int_\Sigma \|\nabla g(x)\|^2 d\sigma(x), \tag{1}$$

where σ is a volume measure on Σ, and critical points of the action are just harmonic maps $\Sigma \to X$. There is now increasing evidence that in some cases, space-time should be a *noncommutative* space. What should replace maps $g\colon \Sigma \to X$ and the action (1) when X becomes noncommutative? It's natural to start with the simplest interesting case, where X is a noncommutative 2-torus (or rotation algebra) $A = A_\Theta$. We are primarily interested in the case where Θ is irrational.

Naively, since a map $g\colon \Sigma \to X$ is equivalent to a C^*-algebra morphism $C_0(X) \to C(\Sigma)$, one's first guess would be to consider $*$-homomorphisms $A \to C(\Sigma)$, where Σ is still an ordinary 2-manifold. But if A is simple, there are no nontrivial such maps. Hence we are led to consider a sigma-model based on $*$-homomorphisms between noncommutative tori. **Note:** The case where X is commutative (but Σ noncommutative) was studied in [1].

2. Maps between irrational rotation algebras

We begin by classifying maps between irrational rotation algebras, using what is known about their ordered K-theory (see, e.g., [6]).

Theorem 1. *Fix Θ and θ in $(0, 1)$, both irrational, and $n \in \mathbb{N}$, $n \geq 1$. There is a unital $*$-homomorphism $\varphi\colon A_\Theta \to M_n(A_\theta)$ if and only if $n\Theta = c\theta + d$ for some $c, d \in \mathbb{Z}$, $c \neq 0$. Such a $*$-homomorphism φ can be chosen to be an isomorphism onto its image if and only if $n = 1$ and $c = \pm 1$.*

This can be reformulated in the following more algebraic language. In what follows, Tr denotes the normalized trace on A_θ, extended as usual to matrices. The monoid M also appears in the theory of Hecke operators.

Lemma 2. *Let M be the submonoid (**not** a subgroup) of $GL(2, \mathbb{Q})$ consisting of matrices in $M_2(\mathbb{Z})$ with non-zero determinant, i.e., of integral matrices having inverses that are not necessarily integral. Then M is generated by $GL(2, \mathbb{Z})$ and by the matrices of the form $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, $r \in \mathbb{Z} \setminus \{0\}$.*

Theorem 3. *Fix Θ and θ in $(0, 1)$, both irrational. Then there is a non-zero $*$-homomorphism $\varphi\colon A_\Theta \to M_n(A_\theta)$ for some n, not necessarily unital, if and only if Θ lies in the orbit of θ under the action of the monoid M of Lemma 2 on $\mathbb{R}$ by linear fractional transformations. The possibilities for $\mathrm{Tr}(\varphi(1_{A_\Theta}))$ are precisely the numbers $t = c\theta + d > 0$, $c, d \in \mathbb{Z}$ such that $t\Theta \in \mathbb{Z} + \theta\mathbb{Z}$. Once t is chosen, n can be taken to be any integer $\geq t$.*

The maps in Theorems 1 and 3 can always be chosen to be smooth (i.e., to send the canonical smooth subalgebra A_Θ^∞ to $M_n(A_\theta^\infty)$).

One might want a finer classification to include the induced map on K_1 as well. Because of [2, Theorem 7.3], which applies because of [3], for any given possible map on K_0, there is a $*$-homomorphism inducing any desired group homomorphism $\mathbb{Z}^2 \cong K_1(A_\Theta) \to K_1(A_\theta) \cong \mathbb{Z}^2$, including the 0-map. In particular, A_θ always has proper (i.e., non-invertible) unital $*$-endomorphisms. There is one case where these are especially well-behaved. The following theorem is a slight improvement on a result of Kodaka [4].

Theorem 4. *Suppose θ is irrational. Then there is a (necessarily injective) unital $*$-endomorphism $\Phi\colon A_\theta \to A_\theta$, with image $B \subsetneq A_\theta$ having nontrivial relative commutant and with a conditional expectation of index-finite type from A_θ onto B, if and only if θ is a quadratic irrational number.*

The maps Φ in Theorem 4 can be chosen to be smooth and to induce an arbitrary group endomorphism of $K_1(A_\theta)$. But when θ is not a quadratic irrational, we do not know if A_θ has any *smooth* proper $*$-endomorphisms.

3. The harmonic map equation

Now that we understand maps between irrational rotation algebras, it is time to study the analogue of the action functional (1).

Definition 5. Let φ denote a unital $*$-homomorphism $A_\Theta \to A_\theta$. As before, denote the canonical generators of A_Θ and A_θ by U and V, u and v, respectively. The natural analogue of $S(g)$ in our situation is

$$\begin{aligned} S(\varphi) = \mathrm{Tr}\Big(&\delta_1(\varphi(U))^*\delta_1(\varphi(U)) + \delta_2(\varphi(U))^*\delta_2(\varphi(U)) \\ &+ \delta_1(\varphi(V))^*\delta_1(\varphi(V)) + \delta_2(\varphi(V))^*\delta_2(\varphi(V))\Big). \end{aligned} \tag{2}$$

Critical points for this action are called *harmonic maps.* Here δ_1 and δ_2 are the infinitesimal generators for the "gauge action" of the group $\mathbb{T}^2$ on A_θ. More precisely, δ_1 and δ_2 are defined on the smooth subalgebra A_θ^∞ by the formulas

$$\delta_1(u) = 2\pi i u, \quad \delta_2(u) = 0, \quad \delta_1(v) = 0, \quad \delta_2(v) = 2\pi i v.$$

Note that $S(\varphi)$ in (2) is just the sum $E(\varphi(U)) + E(\varphi(V))$, where for a unitary $W \in A_\theta^\infty$,

$$E(W) = \mathrm{Tr}\Big(\delta_1(W)^*\delta_1(W) + \delta_2(W)^*\delta_2(W)\Big). \tag{3}$$

It was conjectured in [7] that the "special" unitaries $u^n v^m$ minimize the energy E in the various connected components of $U(A_\theta^\infty)$. This has now been proved by Hanfeng Li:

Theorem 6 (Hanfeng Li)**.** *For W in the connected component of $U(A_\theta^\infty)$ containing $u^n v^m$, $E(W) \geq E(u^n v^m) = 4\pi^2(m^2 + n^2)$, with equality if and only if $W = \lambda u^n v^m$ for some $\lambda \in \mathbb{T}$.*

This implies something we had proved in a number of special cases:

Corollary 7 ("Minimal Energy Conjecture")**.** *Suppose $\varphi\colon A_\theta^\infty \circlearrowleft$ is a $*$-endomorphism inducing the map on K_1 given by $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$. Then $S(\varphi) \geq 4\pi^2(p^2+q^2+r^2+s^2)$, with equality if and only if $\varphi(u) = \lambda u^p v^q$, $\varphi(v) = \mu u^r v^s$, $\lambda, \mu \in \mathbb{T}$.*

In general, we would like to understand the nature of all critical points of (2), not just the minima. The most interesting test case seems to be related to the nontrivial Morita equivalences between A_θ and $A_{1/\theta}$. While we don't yet know how to treat this when θ is irrational, the case of $\theta = n \in \mathbb{N} \smallsetminus \{0\}$ is already nontrivial. In this case, $A_n \cong C(\mathbb{T}^2)$ and $A_{1/n}$ is the algebra of sections of the endomorphism bundle of a rank-n vector bundle over $\mathbb{T}^2$ of Chern class 1 mod n.

Theorem 8. *Let e be a self-adjoint rank-1 smooth projection in $A_{1/n}$ and let $\varphi_e \colon C(\mathbb{T}^2) \xrightarrow{\cong} eA_{1/n}e$ be defined by $\varphi_e(U) = eu^n$, $\varphi_e(V) = ev^n$, where u and v are the canonical unitaries in $A_{1/n}$. Then φ_e is harmonic exactly when e is harmonic in the sense of* [1], *and solutions do exist.*

When $n = 2$, we have constructed explicit solutions to the harmonic map equation, and they turn out to be related to solutions of the equation governing a nonlinear pendulum.

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Fibrations with noncommutative fibres

Siegfried Echterhoff
(joint work with Ryszard Nest and Hervé Oyono-Oyono)

We study certain noncommutative analogues of the classical Serre-fibrations in topology. Our theory is relative to a given (co-)homology theory on the category of C*-algebras (or a suitable sub-category). The basic examples are K-theory or K-homology and we shall basically restrict to the case of K-theory in this report. The content of this report is based on the papers [2, 3] but some of the ideas (with interesting applications to the computation of certain K-theory groups) also appeared in [1]. We refer the reader to these papers for more details.

Recall that a C*-algebra bundle (or $C_0(X)$-algebra) over a Hausdorff locally compact base space X is a C*-algebra A together with a nondegenerate $*$-homomorphism $\Phi : C_0(X) \to ZM(A)$, where $ZM(A)$ denotes the center of the multiplier algebra of A. We shall often write $A(X)$ to indicate that A is equipped with the structure of a C*-algebra bundle over X. If $x \in X$, then $I_x = \Phi(C_0(X \setminus \{x\}))A$ is a closed ideal in A and $A_x := A/I_x$ is called the fibre of $A(X)$ at x. Then each $a \in A(X)$ induces a map $x \mapsto a_x := a + I_x \in A_x$ and in this way we may regard a as a section of a bundle of C*-algebras over X with fibre A_x at x.

An interesting example of a C*-algebra bundle is given by the group C*-algebra $C^*(H)$ of the discrete Heisenberg group H, which can be described as the universal C*-algebra generated by three unitaries u, v, w subject to the relation $uv = wvu$. It is a C*-algebra bundle over the circle $\mathbb{T}$ with structure map $\Phi(C(\mathbb{T})) \to Z(C^*(H))$ given by the functional calculus of the central unitary w, i.e., $\Phi(f) = f(w)$ for all $f \in C(\mathbb{T})$. The fibre $C^*(H)_z$ over $z = e^{2\pi i\theta} \in \mathbb{T}$ is just the irrational rotation algebra A_θ. Since A_θ is simple for irrational values of θ and Morita equivalent to $C(\mathbb{T}^2)$ for rational values of θ, we see that this bundle is highly non-trivial. But it turns out that this bundle becomes locally trivial in K-theory, so that it will give a prominent example for the noncommutative Serre-fibrations as defined below.

If $f : Y \to X$ is any continuous map, we define the pull-back $f^*A(Y)$ of $A(X)$ along f as the balanced tensor product

$$f^*A(Y) := C_0(Y) \otimes_{C_0(X)} A(X),$$

where the $C_0(X)$ acts on $C_0(Y)$ via $\varphi \cdot g(y) = \varphi(f(y))g(y)$ for $\varphi \in C_0(X)$ and $g \in C_0(Y)$. The action of $C_0(Y)$ on the first factor gives $f^*A(Y)$ the structure of a C*-algebra bundle over Y with fibre $A_{f(y)}$ at each $y \in Y$.

Definition 1. We say that $A(X)$ is a *K-fibration*, if for every p-simplex Δ^p and any continuous map $f : \Delta^p \to X$ the evaluation map $\mathrm{ev}_v : f^*A(\Delta^p) \to A_{f(v)}$ induces an isomorphism $K_*(f^*A(\Delta^p)) \cong K_*(A_{f(v)})$ for all $v \in \Delta^p$. We say that $A(X)$ is a *KK-fibration*, if $\mathrm{ev}_v : f^*A(\Delta^p) \to A_{f(v)}$ is a KK-equivalence for all $v \in \Delta^p$.

It is clear that every KK-fibration is a K-fibration and the converse holds if $A(X)$ and all fibres A_x satisfy the UCT. Here is a list of examples:

(1) Every locally trivial C*-algebra bundle $A(X)$ is a KK-fibration. In particular, every continuous trace algebra $A(X)$ is a KK-fibration (up to Morita equivalence, these are just the locally trivial bundles with fiber $\mathcal{K}(H)$).
(2) If $A(X)$ is a K-fibration (resp. KK-fibration) and if $\mathbb{Z}^n$ acts fiber-wise on $A(X)$, then the crossed product $A(X) \rtimes \mathbb{Z}^n$ is a K-fibration (resp KK-fibrations) over X with fibres $A_x \rtimes \mathbb{Z}^n$. A similar result holds for fiber-wise actions of $\mathbb{R}^n$.
(3) Using the Baum-Connes conjecture, the statement in the previous item can be extended to fiber-wise crossed products $A(X) \rtimes G$ by any amenable (or a-T-menable) group G, provided the crossed products $A(X) \rtimes K$ of all *compact* subgroups K of G are K-fibrations (resp. KK-fibrations). This is always true, if $A(X)$ is a continuous trace algebra, of if G has no compact subgroup.

An interesting class of examples of KK-fibrations are given by the noncommutative principal torus bundles: using a result of Phil Green, we know that if Y

is a *commutative* principal $\mathbb{T}^n$-bundle over the base space X, then $C_0(Y) \rtimes \mathbb{T}^n \cong C_0(X, \mathcal{K})$. This observation gave the motivation for

Definition 2. A C*-bundle $A(X)$ is called a noncommutative principal $\mathbb{T}^n$-bundle (or NCP $\mathbb{T}^n$-bundle), if there exists a fiber-wise action of $\mathbb{T}^n$ on $A(X)$ such that $A(X) \rtimes \mathbb{T}^n \cong C_0(X, \mathcal{K})$.

If $A(X)$ is a NCP $\mathbb{T}^n$-bundle, then it follows from Takesaki-Takai duality that

$$A(X) \otimes \mathcal{K} \cong (A(X) \rtimes \mathbb{T}^n) \rtimes \mathbb{Z}^n \cong C_0(X, \mathcal{K}) \rtimes \mathbb{Z}^n,$$

where all crossed products are by fiber-wise actions. Thus it follows from the list of examples mentioned above, that all NCP $\mathbb{T}^n$-bundles are KK-fibrations. A particular example is given by the Heisenberg bundle $C^*(H)(\mathbb{T})$: the action of $(z_1, z_2) \in \mathbb{T}^2$ on $C^*(H)$ given on the generators u, v, w by

$$(z_1, z_2) \cdot u = z_1 u, \quad (z_1, z_2) \cdot v = z_2 v, \quad \text{and} \quad (z_1, z_2) \cdot w = w$$

satisfies the requirement that $C^*(H) \rtimes \mathbb{T}^2 \cong C(\mathbb{T}, \mathcal{K})$. In particular, it follows that the Heisenberg bundle, although highly non-trivial in any standard way, is locally trivial in K-theory.

If $A(X)$ is any K-fibration, the associated K-theory group bundle $\mathcal{K}_*(A) := \{K_*(A_x) : x \in X\}$ comes with some additional structure: If $\gamma : [0,1] \to X$ is any path in X which joins the points x and y, then γ induces an isomorphism $c_\gamma : K_*(A_x) \overset{\cong}{\to} K_*(A_y)$ via the composition

$$K_*(A_x) \overset{\mathrm{ev}_{0,*}^{-1}}{\to} K_*(\gamma^* A[0,1]) \overset{\mathrm{ev}_{1,*}}{\to} K_*(A_y).$$

The isomorphism c_γ only depends on the homotopy class of γ and preserves composition of paths. Thus we may consider the (simplicial) cohomology of the base X with coefficients in the K-theory group bundle $\mathcal{K}_*(A)$ over X. This then allows to formulate and prove a complete analogue of the Leray-Serre spectral theorem for K-fibrations (see [3] for the precise formulation). In principle, it computes the K-theory of the total algebra $A(X)$ in terms of the K-theory of the fibers A_x and the cohomology of the base X.

An interesting problem when studying C*-algebra bundles is to decide whether two such bundles $A(X)$ and $B(X)$ are $\mathcal{R}KK$-equivalent in the sense of Kasparov, i.e., whether there exists an invertible class in $\mathcal{R}KK(X; A(X), B(X))$, which is Kasparov's bundle version of his bivariant K-theory. It is not difficult to check that if $A(X)$ and $B(X)$ are K-fibrations which are $\mathcal{R}KK$-equivalent, then their K-theory group bundles are isomorphic and their spectral sequences coincide. Thus the K-theory group bundle and the spectral sequence serve as new invariants for $\mathcal{R}KK$-equivalence. A test case to see the usefulness of these invariants is given by the NCP $\mathbb{T}^n$-bundles defined above, since their structure is quite well understood (see [2, §2]). Indeed, in this case we have the following results (see [2, 3]):

Theorem. *Suppose that $A(X)$ is an NCP $\mathbb{T}^n$-bundle over the path connected space X. Then $A(X)$ is $\mathcal{R}KK$-equivalent to a commutative principal $\mathbb{T}^n$-bundle if and only if the associated K-theory group bundle is trivial.*

Moreover, if X is a finite simplicial complex, then $A(X)$ is $\mathcal{R}KK$-equivalent to the trivial bundle $C_0(X \times \mathbb{T}^n)$ if and only its K-theory group bundle is trivial and all d_2-maps in the Leray-Serre spectral sequence vanish.

Since the K-theory group bundle of the Heisenberg bundle is nontrivial (the isomorphism of the fibre $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ at $z = 1$ by the path $\gamma : [0,1] \to \mathbb{T}; \gamma(t) = e^{2\pi i t}$ is given by multiplication with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) we see that the Heisenberg-group bundle is not $\mathcal{R}KK$-equivalent to any commutative principal $\mathbb{T}^2$-bundle. On the other hand, the pull-back of the Heisenberg-group bundle to any simply connected space X via any continuous map $f : X \to \mathbb{T}$ is $\mathcal{R}KK$-equivalent to the trivial bundle!

Unfortunately, so far we are not able to decide in general whether two given non-trivial NCP $\mathbb{T}^n$-bundles $A(X)$ and $B(X)$ are $\mathcal{R}KK$-equivalent if both have non-trivial isomorphic K-theory group bundles. For a discussion of this problem we refer to [3, §5].

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The Baum-Connes conjecture for free orthogonal quantum groups

Christian Voigt

The aim of this talk is to discuss K-theoretic properties of the following C^*-algebras introduced by Wang [8].

Definition 1. Let $n \in \mathbb{N}$. The free orthogonal quantum group $A_o(n)$ is the universal C^*-algebra with self-adjoint generators $u_{ij}, 1 \leq i, j \leq n$ and relations

$$\sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{ij}.$$

If we write $u = (u_{ij})$, then the above relations are equivalent to saying that u is an orthogonal matrix. The abelianization of $A_o(n)$ is isomorphic to the algebra $C(O(n))$ of functions on the orthogonal group $O(n)$.
On $A_o(n)$ there exists a comultiplication $\Delta : A_o(n) \to A_o(n) \otimes A_o(n)$ given by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Together with this comultiplication, $A_o(n)$ is a compact quantum group in the sense of Woronowicz. However, in the sequel we shall rather consider it as the full group-C^*-algebra of a discrete quantum group instead. In this picture, $A_o(n)$ may be viewed as a quantum analogue of a free group.
The reduced C^*-algebra $A_o(n)_{\mathsf{red}}$ is the image of $A_o(n)$ in the GNS-representation of its Haar integral. It is known [2] that $A_o(n)$ is not amenable for $n > 2$, that is, the canonical map

$$\lambda : A_o(n) \to A_o(n)_{\mathsf{red}}$$

is not an isomorphism.
From the work of Meyer and Nest [6], [5] arises the formulation of an analogue of the Baum-Connes conjecture for the discrete quantum group corresponding to $A_o(n)$, and our main result is the following theorem.

Theorem 2. *Let $n > 2$. Then the discrete quantum group $A_o(n)$ satisfies the strong Baum-Connes conjecture.*

This may be formulated equivalently by saying that $A_o(n)$ has a γ-element and that $\gamma = 1$. As a consequence one obtains the following result.

Theorem 3. *Let $n > 2$. Then the free orthogonal quantum group $A_o(n)$ is K-amenable. In particular, the map*

$$K_*(A_o(n)) \to K_*(A_o(n)_{\mathsf{red}})$$

is an isomorphism.
The K-theory of $A_o(n)$ is

$$K_0(A_o(n)) = \mathbb{Z}, \qquad K_1(A_o(n)) = \mathbb{Z}.$$

These groups are generated by the class of 1 *in the even case and the class of the fundamental matrix u in the odd case.*

We remark that the notion of K-amenability, which was introduced by Cuntz for discrete groups in [4], carries over to the setting of quantum groups in a natural way.
In the proof of theorem 2 we use the theory of monoidal equivalence introduced by Bichon, de Rijdt and Vaes [3] to transfer the Baum-Connes problem for $A_o(n)$ into a problem concerning $SU_q(2)$ and the Podleś sphere. This step relies on fundamental work of Banica [1]. The crucial part of the argument is a detailed analysis of the equivariant KK-theory of the standard Podleś sphere. Our constructions in connection with the Podleś sphere are based on [7]. Finally, the K-theory computation for $A_o(n)$ involves some homological algebra for triangulated categories worked out in [5].
As a consequence of theorem 3 we obtain the following result, in the same way as in the classical case of free groups.

Theorem 4. *For $n > 2$ the reduced C^*-algebra $A_o(n)_{\mathsf{red}}$ does not contain nontrivial projections.*

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Phase transition on the Toeplitz algebra of the affine semigroup over $\mathbb{N}$

Marcelo Laca

(joint work with Iain Raeburn)

Recall that a quasi-lattice ordered group is a pair (G, P) in which G is a group and P is a submonoid with no inverses satisfying

$$xP \cap yP = \begin{cases} zP & \text{when } xP \cap yP \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

When it exists, z is the (unique) *smallest upper bound* of x and y, denoted $x \vee y = z$. When $xP \cap yP = \emptyset$ we use the notation $x \vee y = \infty$ for convenience.

The Toeplitz (or left regular representation) of P on $\ell^2(P)$ is given by $T_x \varepsilon_y = \varepsilon_{xy}$ where $\{\varepsilon_y : y \in P\}$ is the canonical orthonormal basis; it satisfies the covariance condition introduced by A. Nica:

$$(T_x T_x^*)(T_y T_y^*) = \begin{cases} T_{x\vee y} T_{x\vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

This allows for a "Wick-ordering" of products:

$$T_x^* T_y = T_x^*(T_x T_x^* T_y T_y^*) T_y = T_x^*(T_{x\vee y} T_{x\vee y}^*) T_y = T_{x^{-1}(x\vee y)} T_{y^{-1}(x\vee y)}^*.$$

Hence the linear span of the products of the form $T_a T_b^*$ is dense in the Toeplitz algebra $\mathcal{T}(P)$, which can be viewed as a semigroup crossed product, of the commutative C*-algebra generated by the projections $T_x T_x^*$ by the natural action of P [3]. As such, this Toeplitz algebra carries a canonical dual action of $\widehat{G}$ when G is abelian or, in general, a canonical dual coaction of G itself.

In our work we show that the group $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ of orientation-preserving affine transformations of the rational numbers is quasi-lattice ordered by its submonoid

$\mathbb{N} \rtimes \mathbb{N}^{\times}$. In the resulting order, which is given by

$$(r, x) \leq (s, y) \iff x^{-1}(s - r) \in \mathbb{N} \text{ and } x|y,$$

the additive order is scaled by the multiplicative component x, and thus

$$(m, a) \vee (n, b) = \begin{cases} \infty & \text{if } \gcd(a, b) \nmid (m - n), \\ (l, \operatorname{lcm}(a, b)) & \text{if } \gcd(a, b) \mid (m - n), \end{cases}$$

where l is the smallest element of $(m + a\mathbb{N}) \cap (n + b\mathbb{N})$, a set that is nonempty only in the second case.

We study the associated Toeplitz C^*-algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ and show that it is universal for isometric representations that are covariant in the sense of Nica. Specifically, $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ is generated by isometries s and $\{v_p : p \in \mathcal{P}\}$ (where $\mathcal{P}$ is the set of prime numbers) satisfying the relations

(T1) $v_p s = s^p v_p$,

(T2) $v_p v_q = v_q v_p$,

(T3) $v_p^* v_q = v_q v_p^*$ when $p \neq q$,

(T4) $s^* v_p = s^{p-1} v_p s^*$, and

(T5) $v_p^* s^k v_p = 0$ for $1 \leq k < p$,

and, in fact, this is a presentation of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$. We use this presentation to show that the C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ recently introduced by Cuntz in [2] is the boundary quotient of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ in the sense of [1], which involves imposing a maximal set of extra relations to those defining the Toeplitz algebra. We then extend to $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ the natural dynamics considered by Cuntz on the quotient $\mathcal{Q}_{\mathbb{N}}$ and study the KMS states of this C*-dynamical system.

Recall that if (A, σ) is a C*-dynamical system, an element $a \in A$ is σ-analytic if the map $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$. The σ-analytic elements form a dense *-subalgebra. A state φ on A satisfies the *KMS condition with respect to σ at inverse temperature* $\beta \in \mathbb{R}$ (φ is a σ-KMS$_\beta$ state), if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a)) \qquad \forall a, b \in A, \text{ analytic}.$$

For $\beta = 0$ one specifically requires φ to be σ-invariant, but this is automatic for $\beta \neq 0$. We shall also distinguish between *KMS$_\infty$ states*, which are the limits of KMS$_\beta$ states as $\beta \to \infty$ and *ground states*, which are those for which the function $z \mapsto \varphi(b\sigma_z(a))$ is bounded on the upper half plane.

Our main result is a phase transition theorem giving the KMS$_\beta$ (equilibrium) states of the C*-dynamical system $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times}), \sigma)$ for all values of the inverse temperature β. For $\beta < 1$ there are no KMS$_\beta$ states; for $\beta \in [1, 2]$ there is a unique KMS$_\beta$ state ψ_β given by

$$\psi_\beta(s^m v_a v_b^* s^{*n}) = \begin{cases} 0 & \text{if } a \neq b \text{ or } m \neq n \\ a^{-\beta} & \text{if } a = b \text{ and } m = n; \end{cases}$$

this state obviously factors through the conditional expectation of the dual coaction of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$. For $\beta \in (2, \infty]$ the KMS$_\beta$ states are indexed by probability measures on the circle, with the extremal states $\psi_{\beta,z}$ for $z \in \mathbb{T}$ given by

$$\psi_{\beta,z}(s^m v_a v_b^* s^{*n}) = \\ = \begin{cases} 0 & \text{if } a \neq b \text{ or } m \not\equiv n \pmod a, \\ \dfrac{1}{a\zeta(\beta-1)} \displaystyle\sum_{\{x\,:\,a|x|(m-n)\}} x^{1-\beta} z^{(m-n)/x} & \text{if } a = b \text{ and } m \equiv n \pmod a. \end{cases}$$

There is a further phase transition at $\beta = \infty$, with the KMS_∞ states indexed by the probability measures on the circle as above, but in which the ground states are indexed by all the states on the classical Toeplitz algebra $\mathcal{T}(\mathbb{N})$ generated by the unilateral shift s.

As β passes from 2^- to 2^+ the symmetry of equilibrium changes from the above mentioned coaction of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ (for $\beta < 2$) to the quotient coaction of $\mathbb{Q}_+^*$ (for $\beta > 2$), which can be viewed more classically as an action of the infinite torus $\widehat{\mathbb{Q}_+^*}$. Somewhat mysteriously, this results in the circular symmetry-breaking observed at the level of KMS_β states: $\phi_{\beta,z} \mapsto \phi_{\beta,\lambda z}$ (for $\lambda \in \mathbb{T}$). However, in contrast to what happens at the level of states, this broken symmetry does not ostensibly come from an action of $\mathbb{T}$ on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, for a symmetry group θ_λ such that $\theta_\lambda(S) = \lambda S$ for $\lambda \in T$ would be incompatible with the first relation $v_p s = s^p v_p$ in the presentation of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$.

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Quantum complex projective spaces: Fredholm modules, K-theory, spectral triples

Francesco D'Andrea

In this talk I presented some results about the geometry of quantum complex projective spaces. It is a joint work with L. Dąbrowski and G. Landi [6, 7].

Let me start with some basic considerations about Fredholm modules for commutative spaces. A non-trivial 1-summable even Fredholm module $(\mathcal{A}_0, \mathcal{H}_0, F_0, \gamma_0)$ over the algebra $\mathcal{A}_0 := \mathbb{C}$ (say, of continuous functions on the space with one point) is given by the representation $a \mapsto a \oplus 0$ of $a \in \mathbb{C}$ on $\mathcal{H}_0 := \mathbb{C} \oplus \mathbb{C}$, with grading $\gamma_0 = 1 \oplus -1$ and with F_0 the operator that interchanges the two components, $F_0(x \oplus y) = y \oplus x \;\forall\, x, y \in \mathbb{C}$. The Chern map in K-theory $\mathrm{ch}_0 : K_0(\mathbb{C}) \to \mathbb{Z}$ in this case is given by

$$\mathrm{ch}_0[p] = \tfrac{1}{2}\,\mathrm{Tr}(\gamma_0 F_0 [F_0, p]) \equiv \mathrm{Trace}(p) \tag{1}$$

(where p is a matrix projection, Trace(p) is the matrix trace of p and Tr is the composition of Trace with the trace over $\mathcal{H}_0$). This Fredholm module is a generator of the K-homology of $\mathbb{C}$, and matrix projections are classified by their rank.

Given a compact Hausdorff space X, we can use an irreducible representation — i.e. a map $C(X) \to \mathbb{C}$, $f \mapsto f(x)$, with $x \in X$ — to obtain a Fredholm module over $C(X)$ as a pullback of the Fredholm module over $\mathbb{C}$: the corresponding map $K_0(C(X)) \to \mathbb{Z}$ evaluated on a projection gives the dimension of the fiber at x of the corresponding vector bundle. If X is connected, this number is independent of x and gives the rank of the vector bundle (for a given $[p] \in K_0(C(X))$ the map $X \to \mathbb{Z}$, $x \mapsto \mathrm{Trace}(p(x))$, is locally constant since p is continuous; it is then constant if X is connected). In the commutative case, the rank is all the information we can get from Fredholm modules constructed with irreducible representations. This is because irreducible representations of $C(X)$ are highly degenerate.

A nice thing about complex quantum projective spaces $\mathbb{C}\mathrm{P}^n_q$ (here $n \geq 1$ and $0 < q < 1$) is that we can construct generators of the K-homology group $K^0(\mathcal{A}_n)$ (where $\mathcal{A}_n = \mathcal{A}(\mathbb{C}\mathrm{P}^n_q)$ is the algebra of "polynomial functions" on $\mathbb{C}\mathrm{P}^n_q$) by simply working with irreducible representations. These are constructed by induction in [7] using a surjective $*$-algebra morphism $\mathcal{A}_n \to \mathcal{A}_{n-1}$. At each step n Fredholm modules are obtained by pullback, and an additional one $(\mathcal{A}_n, \mathcal{H}_n, F_n, \gamma_n)$ is constructed using the unique (modulo equivalences) faithful irreducible representation of $\mathcal{A}_n$ (the only one that is not the pullback of a representation of $\mathcal{A}_{n-1}$). These $n+1$ Fredholm modules are 1-summable, and generate the group $K^0(\mathcal{A}_n) \simeq \mathbb{Z}^{n+1}$, as it is proved by pairing them with generators of the K-theory group $K_0(\mathcal{A}_n)$ using a formula similar to (1):

$$\mathrm{ch}_n([p]) = \tfrac{1}{2}\,\mathrm{Tr}(\gamma_n F_n [F_n, p]) \ .$$

The geometric picture is clear. For $n = 1$, we are computing the dimension of the fiber of the restriction of noncommutative vector bundles over $\mathbb{C}\mathrm{P}^1_q$ to the 'classical point' $\{*\} \in \mathbb{C}\mathrm{P}^1_q$ (given by the unique character of the algebra): this gives the 'rank'. A second independent Fredholm module is obtained by using the unique (modulo equivalences) faithful irreducible representation of $\mathcal{A}_1$, and the corresponding map ch_1 computes the 'monopole charge' of the bundle. For $n = 2$, the restriction of a vector bundle over $\mathbb{C}\mathrm{P}^2_q$ to $\{*\} \in \mathbb{C}\mathrm{P}^2_q$ gives the rank, and the restriction to $\mathbb{C}\mathrm{P}^1_q \subset \mathbb{C}\mathrm{P}^2_q$ gives the 'monopole charge'. A third Fredholm module coming from the unique (modulo equivalences) faithful irreducible representation of $\mathcal{A}_2$ gives the analogue of the 'instanton number'. And so on for higher n.

Another important aspect of the geometry of $\mathbb{C}\mathrm{P}^n_q$ is the one concerning spectral triples. Recall that *spectral triples* (also called "unbounded Fredholm modules") provide a non-commutative generalization of the notion of smooth manifold (cf. [1]). Now, as conformal structures are conformal classes of (pseudo-)Riemannian metrics, similarly Fredholm modules are 'conformal classes' of spectral triples: given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, a Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ can be obtained by replacing D with the bounded operator $F := D(1 + D^2)^{-\frac{1}{2}}$ (one can use

$F := D|D|^{-1}$ if D is invertible), and viceversa any K-homology class has a representative that arise from a spectral triple through this exact construction. Passing from bounded to unbounded Fredholm modules is convenient since it allows to use powerful tools such as local index formulas.

In the case of $\mathbb{C}\mathrm{P}^n_q$, in [7] we studied even regular spectral triples of any metric dimension $d \in \mathbb{R}^+$ whose "conformal class" is the Fredholm module $(\mathcal{A}_n, \mathcal{H}_n, F_n, \gamma_n)$ discussed above. They are constructed by giving explicitly the spectral decomposition of the Dirac operator. These spectral triples are not equivariant, and have no $q \to 1$ analogue. Furthermore, even regular spectral triples on q-spaces don't give very interesting local index formulas (for example, the unique term surviving in Connes-Moscovici local cocycle [2] is the non-local one, cf. [4, 5, 3]).

On $\mathbb{C}\mathrm{P}^1_q$ (also known as *standard Podleś sphere*) a more interesting local index formula is given in [11], and is obtained using the *non-regular* and *equivariant* spectral triple constructed in [8]. In [12] it is explained the geometrical nature of the spectral triple in [8] and implicitly suggested how to generalize this construction to $\mathbb{C}\mathrm{P}^n_q$: using the action of the Hopf algebra $\mathcal{U}_q(\mathfrak{su}(n+1))$, in particular the action of quasi-primitive elements, that are external derivations on $\mathbb{C}\mathrm{P}^n_q$. This idea has been used in [10] to construct — on any quantum irreducible flag manifold, including $\mathbb{C}\mathrm{P}^n_q$ — a Dirac operator D realizing the unique finite-dimensional covariant real differential calculus on the space. It is not clear whether D gives a spectral triple or not, as the compact resolvent condition is not proven.

Equivariant spectral triples on $\mathbb{C}\mathrm{P}^n_q$ are constructed in [6], in complete analogy with the $q = 1$ case, by using the fact that complex projective spaces are Kähler manifolds: in particular they admit a homogeneous Kähler metric (for the action of $SU(n+1)$), called *Fubini-Study metric*. The result is a family of (equivariant, even) spectral triples $(\mathcal{A}_n, \mathcal{H}_N, D_N)$ labelled by $N \in \mathbb{Z}$: $\mathcal{H}_0$ are the noncommutative analogue of antiholomorphic forms (in fact, they give a finite-dimensional covariant differential calculus on $\mathbb{C}\mathrm{P}^n_q$), D_0 is the analogue of the Dolbeault-Dirac operator, $\mathcal{H}_N$ is the tensor product of $\mathcal{H}_0$ with 'sections of line bundles' with monopole charge N over $\mathbb{C}\mathrm{P}^n_q$, D_N is the twist of D_0 with the Grassmannian connection of the line bundle. If n is odd (in this case $\mathbb{C}\mathrm{P}^n$ is a spin manifold) for $N = \frac{1}{2}(n+1)$ we have a *real* spectral triple whose Dirac operator is a deformation of the Dirac operator for the Fubini-Study metric.

The spectrum of D_N is computed by relating D_N^2 to the Casimir of $\mathcal{U}_q(\mathfrak{su}(n+1))$: in this way we prove that for $q < 1$ eigenvalues of D_N grow exponentially, hence the spectral triple is 0^+-summable. For $q = 1$ we find (as expected) the same spectrum computed in [9].

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Enhancing T-duality

Calder Daenzer

If a torus is expressed as a vector space modulo a full rank lattice, $T = V/\Lambda$, the associated dual torus is the Pontryagin dual $\widehat{\Lambda} := \mathrm{Hom}(\Lambda; U(1))$ of the lattice. Generalizing this, a principal torus bundle with a gerbe on it may have a T-dual, which is a principal dual-torus bundle with a dual gerbe. There is a transform taking an object to its T-dual, and the transform is topological in the sense that it preserves twisted K-theory which is a topological invariant.

Topological T-duality has sparked interest in noncommutative geometry because some principal bundles with gerbes which were thought to not have T-duals, were discovered by Mathai-Rosenberg [4] to have T-duals that are families of noncommutative tori. On the other hand, according to the SYZ-conjecture [5], T-duality is the mechanism which implements mirror symmetry between Calabi-Yau manifolds. This is no longer topological T-duality though, it involves an understanding of the behavior of complex and symplectic structures under the T-dualization transform. These aspects of T-duality are not at all well understood. In fact the first place where T-duality was realized as an interesting symmetry was in theoretical physics, where it was used to relate the many different forms of string theory. In physics T-duality has grown into an industry with many varieties, and several recent constructions involve forming T-duals of orbifold quotients of tori. T-duality for orbifold quotients is understood for a single torus quotient, but the generalization to families of such objects and to families with gerbes over them has not yet been made[1].

[1]It is natural to wonder why gerbes are an important feature of T-duality. The mathematical answer to that question is that the T-dual of a nontrivial torus bundle without gerbe is a dual-torus bundle with a $U(1)$-gerbe, so these objects are unavoidable. The physics answer is that a $U(1)$-gerbe carries information about the "B-field" which couples to a string worldsheet; this is

Today I will sketch solutions to two of these problems. The first, which was joint work with Jonathan Block in [1], is understanding the behavior of complex structures under T-duality. The second, which comprises a forthcoming paper [3], is to describe topological T-duality for families of orbifold quotients of tori.

Complex structure and T-duality. The goal here is to describe an analogue of the derived category of coherent sheaves for gerbes over DM-stacks and for families of noncommutative tori, then to describe T-duality as functor between two such derived categories. A smooth DM-stack can be presented by a proper étale complex groupoid. A gerbe on one of these is given by a groupoid 2-cocycle, and this data is encoded in the associated twisted groupoid algebra. A family of noncommutative tori can be described similarly.

Generalizing the Dolbeault dga of a complex manifold, there is a graded algebra $\mathcal{A}^\bullet$ of anti-holomorphic differential forms on the groupoid, whose degree zero component is the twisted groupoid algebra of the gerbe. Choosing a connection on the gerbe determines a $\overline{\partial}$-operator analogous to the Dolbeault differential.

We construct a DG-category $\mathcal{P}_\mathcal{A}$ of "$\mathcal{A}^\bullet$-modules," whose associated homotopy category defines the derived category of coherent sheaves on a gerbe over a DM-stack. The objects of $\mathcal{P}_\mathcal{A}$ are meant to be finite complexes of finitely generated projective $\mathcal{A}^0$-modules with a so-called $\overline{\partial}$-superconnection whose curving is the curvature of the gerbe. If the gerbe is trivial and the DM-stack is just a manifold, such complexes correspond to resolutions of coherent sheaves by complexes of vector bundles; this makes the connection with the classical derived category.

However, $\mathcal{A}^\bullet$ is nonunital when the gerbe is nontrivial. A nonunital algebra admits in general no nondegenerate finitely generated projective modules, so the objects of $\mathcal{P}_\mathcal{A}$ mentioned above cannot work. We thus consider instead complexes of nondegenerate projective modules with a certain nuclearity-up-to-homotopy condition which replaces finite generation. But forming tensor products of such infinitely generated modules (which is necessary for describing Morita morphisms) then becomes an analytical question. Following the work of Ralf Meyer, we found that the most flexible method for treating this issue is to work in the context of bornological vector spaces. Thus $\mathcal{A}^\bullet$ and objects of $\mathcal{P}_\mathcal{A}$ are equipped with bornological structures, and a core theorem in this paper states that $\mathcal{A}^\bullet$ is a multiplicatively convex, complete, quasi-unital bornological algebra, and that for such algebras, categories of homotopy-nuclear complexes behave well with respect to Morita equivalence.

Finally, T-duality is expressed in this framework as a DG-quasi-equivalence between the associated DG-categories mentioned above. This DG-quasi-equivalence is strictly weaker than Morita equivalence, and expresses T-duality as a transform in derived noncommutative complex geometry.

T-duality for orbifold torsors. The goal here is to describe the analogue of T-duality for certain torsors over orbifold groups. I will give a precise definition of what I mean by orbifold groups below, but the class is meant to include the

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analogous to the $U(1)$-bundle which carries information about the electromagnetic field potential that couples to a particle worldline.

quotient $S^1/\mathbb{Z}_2$ of the circle group by the automorphism $e^{i\theta} \mapsto e^{-i\theta}$, a quotient of a torus by a finite group of rotations, and certain orbifolds which occur at boundary points of the moduli space of Kähler metrics on a Kummer surface.

Consider a two term complex of locally compact abelian groups $G \xrightarrow{d} H$. From this complex one can form the transformation groupoid

$$\mathrm{tr}(G \to H) := (G \times H \rightrightarrows H)$$

whose source, range and composition maps are

$$s(g,h) := h, \ t(g,h) := d(g)h, \ (g_1, d(g_2)h) \circ (g_2, h) := (g_1 g_2, h).$$

Groupoids of the form $tr(H \to G)$ are group objects in the category of topological groupoids, so two such groupoids are "equivalent" when there is Morita equivalence between them that is compatible with the extra group laws.

Note that $\mathrm{tr}(0 \to G)$ refers to a unit groupoid, while $\mathrm{tr}(G \to 0)$ refers to a group viewed as a groupoid with one object. Also, if $\mathbb{Z}/n\mathbb{Z} \hookrightarrow S^1$ denotes the inclusion of the n-th roots of unity into the circle, then $\mathrm{tr}(\mathbb{Z}/n\mathbb{Z} \hookrightarrow S^1)$ is a quotient of a circle by those rotations, so these groupoids present quotients by rotations. An example of equivalent groupoids is when $G \to H$ is injective and has closed image, in which case $tr(G \to H)$ is equivalent to $\mathrm{tr}(0 \to H/G)$. The latter object might seem simpler, but the flexibility to use different presentations is crucial because in describing nontrivial families of these spaces or gerbes over them, some presentations work while others may not. For example there are nontrivial gerbes on the torus which can only be presented on $\mathrm{tr}(\mathbb{Z}^n \to \mathbb{R}^n)$.

The orbifold $S^1/\mathbb{Z}_2$ cannot be described by these groupoids because it is a quotient by group automorphisms rather than translations. Quotients by automorphisms are obtained when a group F acts by automorphisms of G and H and intertwines the homomorphism $d : G \to H$. In this case the semidirect product groupoid $F \ltimes \mathrm{tr}(G \to H)$ will present the quotient by these automorphisms. Such an object is what I mean by an orbifold group.

Finally, torsors whose fibers are of the form $F \ltimes \mathrm{tr}(G \to H)$ are given by cocycle data in a way analogous to how principal bundles are given by Čech 1-cocycles. In our situation, it is a 1-cocycle in groupoid hypercohomology $\mathbb{H}^1(\mathcal{H}; G \to H)$, where $G \to H$ is viewed as a complex supported in degrees -1 and 0 and $\mathcal{H}$ is any groupoid which acts on $G \to H$ analogously to F.

Examples of torsors which can be described in this framework: Suppose $\mathcal{H} = F \times \mathcal{M}$ is the product of a group F with a Čech groupoid $\mathcal{M}$ for some space M. Then when $F = G = 0$, the 1-cocycle is data for a principal H-bundle over M. When $F = H = 0$ the 1-cocycle is actually a 2-cocycle (since we are computing hypercohomology and G is in degree -1) and corresponds to a G-gerbe over M. When $F \neq 0$ we are considering actions of F that vary along the base M.

Here is the mechanism for T-dualizing an orbifold quotient of a torus, presented as $F \ltimes \mathrm{tr}(\Lambda \to V)$:

$$F \ltimes \mathrm{tr}(\Lambda \to V) \overset{\text{Mod out V}}{\rightsquigarrow} F \ltimes \mathrm{tr}(\Lambda \to 0) \overset{\text{Pont. dualize}}{\rightsquigarrow} F \ltimes \mathrm{tr}(0 \to \widehat{\Lambda})$$

where on the right side F acts via the dual action.

To construct T-duals for orbifold torsors rather than just a single orbifold, I extended the notion of Pontryagin duality to orbifold torsors and applied the analogous transform. Gerbes over orbifold torsors are treated similarly.

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The Connes-Marcolli GL_2-system and adelic mixing

Sergey Neshveyev

In order to define an analogue of the Bost-Connes system for general number fields Connes and Marcolli introduced a universal system for imaginary quadratic fields [2]. The underlying C^*-algebra A can be described as a corner of the crossed product of $C_0(\mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)))$ by the Hecke algebra $\mathcal{H}(\mathrm{GL}_2^+(\mathbb{Q}),\mathrm{SL}_2(\mathbb{Z}))$. The action of $\mathbb{R}$ on the Hecke algebra given by $\sigma_t([g]) = \det(g)^{it}[g]$ defines the dynamics of the system. Connes and Marcolli classified KMS$_\beta$-states for $\beta > 2$ and showed that they are of type I. In [4] we proved that for every $\beta \in (1,2]$ there exists a unique KMS$_\beta$-state φ_β. These states are defined by certain product-measures $\mu_{\mathbb{H}} \times \mu_{\beta,f}$ on $\mathbb{H} \times \mathrm{Mat}_2(\mathbb{A}_f)$. For $\beta = 2$ the measure $\mu_{\beta,f}$ is the Haar measure on $\mathrm{Mat}_2(\mathbb{A}_f)$ normalized so that $\mathrm{Mat}_2(\hat{\mathbb{Z}}) = 1$. A natural question is what is the type of the states φ_β, or equivalently, what is the type of the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathbb{H} \times \mathrm{Mat}_2(\mathbb{A}_f), \mu_{\mathbb{H}} \times \mu_{\beta,f})$. It is natural to consider a slightly more general problem by replacing $\mathbb{H}$ by $\mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{GL}_2^+(\mathbb{Q})$ by $\mathrm{GL}_2(\mathbb{Q})$. Let μ_∞ be a Haar measure on $\mathrm{PGL}_2(\mathbb{R})$.

Theorem 1. *For every $\beta \in (1,2]$ the action of $\mathrm{GL}_2(\mathbb{Q})$ on*

$$(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f), \mu_\infty \times \mu_{\beta,f})$$

is amenable, ergodic, of type III$_1$. In particular, $\pi_{\varphi_\beta}(A)''$ is the injective factor of type III$_1$.

It is easy to compute the modular flows of the above actions. For $\beta = 2$ the theorem then takes the following essentially equivalent form.

Corollary 2. *The action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathrm{Mat}_2(\mathbb{A})$ with its Haar measure is ergodic and amenable.*

The computation of the ratio set in the proof of the theorem is based on distribution of prime numbers and the following form of equidistribution of Hecke points, which is a consequence of a form of adelic mixing for GL_2 [1, 3].

Proposition 3. *Let* Γ *be a group of the form* $\mathrm{GL}_2(\mathbb{Z}) \cap r\mathrm{GL}_2(\mathbb{Z})r^{-1}$*, where* $r \in \mathrm{GL}_2(\mathbb{Q})$*, and* f *a compactly supported continuous function on* $\Gamma\backslash\mathrm{PGL}_2(\mathbb{R})$*. Then*

$$\frac{1}{R_\Gamma(g)} \sum_{h \in \Gamma\backslash\Gamma g \Gamma} f(hx) \to \int f \, d\nu_\infty$$

as $R_\Gamma(g) \to \infty$ *for every* $x \in \Gamma\backslash\mathrm{PGL}_2(\mathbb{R})$*, where* $R_\Gamma(g) = |\Gamma\backslash\Gamma g\Gamma|$ *and* ν_∞ *is the unique right* $\mathrm{PGL}_2(\mathbb{R})$*-invariant probability measure on* $\Gamma\backslash\mathrm{PGL}_2(\mathbb{R})$*. Furthermore, the convergence is uniform on compact sets.*

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New holomorphivcally closed subalgebras of C^*-algebras of hyperbolic groups

Michael Puschnigg

The search for smooth subalgebras of C^*-algebras is an important problem in Noncommutative Geometry. Recall that a dense subalgebra of a Banach algebra is smooth (or closed under holomorphic functional calculus) if the spectra of its elements in both algebras coincide. Certain invariants of Banach algebras like K-groups or local cyclic cohomology groups do not change under passage to a smooth subalgebra. Finding a particularly small and well behaved smooth subalgebra is therefore often an important first step towards the calculation of such invariants.

In this talk we presented the construction of a new class of smooth subalgebras for any unconditional Banach algebra over the group ring of a word-hyperbolic group Γ. Recall that a Banach algebra over $\mathbb{C}\Gamma$ is unconditional if the norm of an element $\sum_g a_g \, u_g \in \mathbb{C}\Gamma$ depends only on the absolute values $(|a_g|)_{g\in\Gamma}$ of its Fourier coefficients.

In particular, we find a smooth unconditional Banach subalgebra $B(\Gamma) \subset C^*_r(\Gamma)$, such that every trace on the group ring $\mathbb{C}\Gamma$ which is supported in finitely many conjugacy classes of Γ extends to a bounded trace on $B(\Gamma)$.

This result has applications to delocalized L^2-invariants of negatively curved manifolds and to the local cyclic cohomology of the reduced group C^*-algebra.

Our construction is based on two things.

The first is a universal unconditional cross-norm $\otimes_{uc}$ on the algebraic tensor product of unconditionally normed spaces. It differs in general from the projective cross-norm.

The second is the linear operator $\Delta_S : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$

$$\Delta_S(u_g) = \sum_{\substack{g'g''=g \\ l_S(g')+l_S(g'')=l_S(g)}} u_{g'} \otimes u_{g''},$$

where l denotes the word length with respect to the finite symmetric set S of generators of Γ. If $\mathcal{A}\Gamma$ is an unconditional Banach algebra over the group ring of a word-hyperbolic group Γ, then

$$\| \Delta_S(\alpha\beta) \|_{uc} \leq C \left(\| \Delta_S(\alpha) \|_{uc} \| \beta \| + \| \alpha \| \| \Delta_S(\beta) \|_{uc} \right), \quad \forall \alpha, \beta \in \mathbb{C}\Gamma.$$

In particular, the closure of the group ring in the graph norm of the operator is a dense and holomorphically closed subalgebra of $\mathcal{A}\Gamma$. Iteration of the procedure leads to the desired algebras.

It should be noted that the proof depends strongly on hyperbolicity and that there is no indication that a similar result holds for other classes of groups.

Algebraic K-theory and algebraic approximation

Andreas Thom
(joint work with Guillermo Cortiñas)

In this talk I presented joint work with Guillermo Cortiñas. The talk was centered around a conjecture which arose from the work of Johnathan Rosenberg (see [1]) about negative algebraic K-theory of algebras of continuous functions on compact Hausdorff spaces. Negative algebraic K-theory is a natural invariant of a ring, which is defined in terms of projective modules over the ring itself and certain polynomial Laurent extensions over it. Rosenberg conjectured that the assignment

$$X \mapsto K_{-n}(C(X))$$

is homotopy invariant as a functor on the category of compact Hausdorff topological spaces. Here, $C(X)$ denotes the ring of complex-valued continuous functions on X. I presented a proof of this conjecture and put forward the following conjecture, which contains Rosenberg's conjecture as the special case $\Gamma = \mathbb{Z}^n$.

Conjecture 1. *Let Γ be a group. The functor*

$$X \mapsto K_0^{\text{alg}}(C(X)[\Gamma])$$

is homotopy invariant on the category of compact Hausdorff topological spaces.

Here, $C(X)[\Gamma]$ denotes the group ring of Γ with coefficients in the ring $C(X)$. The relation to Rosenberg's original question comes from the natural isomorphism

$$K_0(C(X)[\mathbb{Z}^n]) \cong \bigoplus_{k=0}^{n} K_{-k}(C(X))^{\oplus \binom{k}{n}},$$

which was proved by Rosenberg. Building on work Eric Friedlander and Marc Walker [2], we managed to prove that this conjecture for torsionfree groups is a consequence of the Farrell-Jones Isomorphism Conjecture with coefficients. Indeed, the Farrell-Jones Isomorphism Conjecture reduces the whole problem to a study of $\mathbb{Z}^n$ and this case can to be done by a sophisticated direct argument. The polyhedral case was implicitly contained in the work of Friedlander and Walker, see [2].

As a key tool we use a technique of *algebraic approximation* which considers the algebra $C(X)$ as a union of its finitely generated subalgebras. More precisely, for every finite set $F \subset C(X)$, we consider the algebra $A_F = \mathbb{C}[f \in F] \subset C(X)$. Since these subalgebras are reduced, they are algebras of regular functions on affine algebraic varieties V_F. This allows to introduce algebraic techniques such as desingularization. Hironaka desingularization gives a (more of less) canonical smooth quasi-projective variety $\tilde{V}_F$ and a proper algebraic morphism $f_F \colon \tilde{V}_F \to V_F$. Considering the special case $C(\beta\mathbb{N}) = \ell^\infty(\mathbb{N})$, the inclusion $A_F \subset \ell^\infty(\mathbb{N})$ is dual (in the sense of Gel'fand) to a map $\mathbb{N} \to V_F$ with pre-compact image. Since f_F is proper, such maps can be lifted through the desingularization $\tilde{V}_F$. This finally leads to the conclusion that the negative algebraic K-theory of $\ell^\infty(\mathbb{N})$ vanishes since this is the case for smooth quasi-projective varieties. Similar conclusions hold for group rings $\ell^\infty(\mathbb{N})[\Gamma]$. The argument reveals that the Farrell-Jones Isomorphism Conjecture with coefficients in rings like $C(X)$ or $\ell^\infty(\mathbb{N})$ can give interesting results about the algebraic structure of complex group rings themselves. Indeed, non-existence of exotic classes in $K_0(\ell^\infty(\mathbb{N})[\Gamma])$ can be interpreted as an *algebraic compactness* statement.

An intriguing and simple question which remains untouched in the general case is the following: Are homotopic projections in matrices over a complex group ring similar? Hopefully, the technique of algebraic approximation will allow to treat this question even in the absence of general results about the validity of the Farrell-Jones Isomorphism Conjecture. This is a topic of current research.

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Non-Hausdorff Symmetries of C*-Algebras

Ralf Meyer

(joint work with Alcides Buss and Chenchang Zhu)

This lecture introduces the results of the eprints [1] and [2], which are the beginning of a program to study actions of higher categories on C*-algebras. Here "higher categories" means "2-groupoids." My original motivation for doing this was the desire to establish the Baum–Connes Conjecture for some non-Hausdorff, locally Hausdorff groupoids. In the beginning of my lecture, I explain why this requires viewing such groupoids as 2-groupoids. Then I describe some examples of 2-categories. Finally, I introduce the notion of a weak group action by correspondences on a C^*-algebra. This notion is a special case of general constructions from category theory. It turns out to be equivalent to the notion of a saturated Fell bundle familiar to operator algebraists.

The main tool to establish bijectivity or split injectivity of the *Baum–Connes assembly map* for a particular groupoid is the *Dirac dual Dirac method*. We first recall this method under some simplifying assumptions.

Let G be, say, a Lie groupoid, and let $\mathcal{E}G$ be a universal proper G-space. To simplify, we assume that $\mathcal{E}G$ is a bundle of smooth manifolds over the object space G^0 of G. We may also assume that the projection map $p\colon \mathcal{E}G \to G^0$ is G-equivariantly K-oriented. Then the Dirac operators along the fibres of p define a class D in $\mathrm{KK}^G(\mathrm{C}_0(\mathcal{E}G), \mathrm{C}_0(G^0))$, called the *Dirac morphism* for G (see [4]).

The Baum–Connes assembly map turns out to be equivalent to the map on reduced crossed products with G induced by D (see [3]). Hence the Baum–Connes assembly map is invertible if D is invertible. A one-sided inverse for D is called a *dual Dirac morphism*, its existence implies split injectivity of the assembly map. Most positive results about the Baum–Connes Conjecture use such a dual Dirac morphism. For instance, Jean-Louis Tu established the Baum–Connes Conjecture for Hausdorff amenable groupoids using such techniques, extending a result by Gennadi Kasparov and Nigel Higson for groups.

Why does Tu's result require the groupoid G to be Hausdorff? If G is a non-Hausdorff locally Hausdorff groupoid, then we should expect the universal proper G-space to be non-Hausdorff locally Hausdorff as well. Then the algebra $\mathrm{C}_0(\mathcal{E}G)$ does not really capture the topology of $\mathcal{E}G$, so that the Dirac morphism above cannot be defined on $\mathrm{C}_0(\mathcal{E}G)$.

There is an obvious solution to this problem: a non-Hausdorff manifold should be described by a groupoid. More precisely, we take a covering by Hausdorff open subsets and form the resulting Čech groupoid. Its groupoid C^*-algebra A should be the correct substitute for $\mathrm{C}_0(\mathcal{E}G)$ in the case where $\mathcal{E}G$ is a non-Hausdorff, locally Hausdorff manifold bundle over G^0. Unfortunately, this construction is not natural on the level of *-homomorphisms: the action of G on $\mathcal{E}G$ does not induce an action by *-homomorphisms on A. The action is only well-defined up to inner automorphisms. In what sense does G act on A?

Once this question is settled, the next step will be to extend the definition of (reduced) crossed products and equivariant bivariant KK-theory to such actions and to establish that the Dirac morphism exists and is invertible for some class of non-Hausdorff groupoids. But so far our program has only taken the very first steps of defining the right notion of action.

The crucial point for this is to view G not as a groupoid but as a *2-groupoid*, that is, a 2-category in which all arrows are equivalences. This means that we replace the non-Hausdorff space of arrows of G by a Čech groupoid, introducing a second layer of arrows between arrows, which are called *2-arrows*. This three-layered structure carries various composition operations that reflect the category structure on G. Instead of listing the data and conditions for a 2-category, we illustrate this notion by examples.

The first example of a 2-category is the *2-category of categories*. Its objects are categories, its arrows are functors, and its 2-arrows are natural transformations between functors. We may compose arrows and natural transformations in an obvious way. Furthermore, natural transformations $F_1 \Rightarrow G_1$ and $F_2 \Rightarrow G_2$ between composable functors induce a unique natural transformation $F_1F_2 \Rightarrow G_1G_2$. More precisely, there are two obvious formulas for such a natural transformation, which agree. Finally, there are identity functors and identity natural transformations. This is all the data needed for a 2-category.

A second classical example is the *homotopy 2-category of spaces*. Its objects are topological spaces, its arrows are continuous maps, and its 2-arrows are homotopy classes of homotopies between these maps. Once again, we may compose arrows and 2-arrows, and there is a second composition of 2-arrows that produces homotopies between composites of homotopic maps.

Our third example is more relevant to our problem: it is the correspondence 2-category of C*-algebras. Its objects are C*-algebras, arrows are correspondences, and 2-arrows are isomorphisms of correspondences. Here a correspondence from A to B is a Hilbert B-module $\mathcal{H}$ with an essential *-representation of A by adjointable operators on $\mathcal{H}$. Isomorphisms are, by convention, required to be unitary. The composition of correspondences is a balanced tensor product operation familiar from Kasparov theory or from the composition of imprimitivity bimodules in the theory of Morita equivalence. This tensor product operation is clearly natural with respect to isomorphisms of correspondences. Unfortunately, this example also exhibits a technical problem: the composition of correspondences is only well-defined (or at least well-behaved) up to isomorphism. This isomorphism is canonical enough not to cause trouble, but this means that the correspondence 2-category is only a *weak 2-category* (also called a *bicategory*).

Why is it useful to consider isomorphic correspondences as different? One obvious reason is the construction of correspondences from ordinary *-homomorphisms. Any essential *-homomorphism f from A to the multiplier algebra of B gives a correspondence $[f]$ with underlying Hilbert module B and A acting on the left by f. It is easy to see that isomorphisms $[f] \cong [f']$ are in bijection with unitary intertwiners between f and f', that is, unitary multipliers u of B that satisfy $uf(a)u^* = f'(a)$ for all $a \in A$. In particular, the isomorphism class of $[f]$ does not distinguish inner automorphisms from the identity. It is usually not such a good idea to identify inner automorphisms with the identity automorphism. In a sense, they are equivalent to the identity automorphism, but the correct way to make this precise is 2-categorical.

Several notions from higher category theory specialise to interesting concepts for the 2-category of C*-algebras. Here we only consider one non-trivial example: weak group actions by correspondences are equivalent to saturated Fell bundles. More examples are described in [2].

The notion of a weak group action is a special case of a *functor* between 2-categories. It is important here to replace equalities between arrows by isomorphisms of arrows. These isomorphisms of arrows become part of the data. They are required to satisfy certain coherence laws, which ensure that different ways of constructing 2-arrows from equalities always produce the same result. We now illustrate this in the case of a group action.

A group action of a group G on a C^*-algebra A is given by automorphisms α_g for $g \in G$ that satisfy $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$ for all $g_1, g_2 \in G$ and $\alpha_1 = \mathrm{Id}_A$.

A weak group action on A by correspondences requires the following data: correspondences α_g from A to itself for all $g \in G$, isomorphisms of correspondences $\omega(g_1, g_2)$ between $\alpha_{g_1}\alpha_{g_2}$ and $\alpha_{g_1 g_2}$ for all $g_1, g_2 \in G$ and an isomorphism u between α_1 and Id_A. The isomorphisms $\omega(g_1, g_2)$ and u are part of the data. If $g_1, g_2, g_3 \in G$, then there are two ways of proving the equation $\alpha_{g_1}\alpha_{g_2}\alpha_{g_3} = \alpha_{g_1 g_2 g_3}$ for a classical group action, via $\alpha_{g_1 g_2}\alpha_{g_3}$ or via $\alpha_{g_1}\alpha_{g_2 g_3}$. This generates a coherence law, which requires that the two corresponding isomorphisms of correspondences between $\alpha_{g_1}\alpha_{g_2}\alpha_{g_3}$ and $\alpha_{g_1 g_2 g_3}$ are equal. Similarly, there are two ways of proving $\alpha_g \alpha_1 \cong \alpha_g$, using u or $\omega(g, 1)$, and there are two ways of proving $\alpha_1 \alpha_g \cong \alpha_g$. This gives two coherence laws involving u. A weak group action is given by the data α_g, u, and $\omega(g_1, g_2)$, satisfying these three coherence laws

The assumptions imply that α_g and $\alpha_{g^{-1}}$ are inverse to each other up to isomorphism. This implies that α_g must be an imprimitivity bimodule. Thus the correspondences in a weak group action are imprimitivity bimodules. It then follows that the spaces $\alpha_{g^{-1}}$ for $g \in G$ form a *saturated Fell bundle.* Conversely, any saturated Fell bundle comes from a weak group action. Thus isomorphism classes of saturated Fell bundles correspond bijectively to isomorphism classes of weak actions of G on C^*-algebras. Here A is the unit fibre of the Fell bundle.

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The Gauss-Bonnet theorem for the noncommutative two torus

Alain Connes

In this paper we shall show that the value at the origin, $\zeta(0)$, of the zeta function of the Laplacian on the non-commutative two torus, endowed with its canonical conformal structure, is independent of the choice of the volume element (Weyl factor) given by a (non-unimodular) state. We had obtained, in the late eighties, in an unpublished computation, a general formula for $\zeta(0)$ involving modified logarithms of the modular operator of the state. We give here the detailed computation and prove that the result is independent of the Weyl factor as in the classical case, thus proving the analogue of the Gauss-Bonnet theorem for the noncommutative two torus.

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Emergent Geometry

Denjoe O'Connor

Spacetime geometry may not have always existed, it may in fact be an emergent low energy concept. This possibility is illustrated by a simple, three matrix, model in which a geometrical phase emerges as the system is cooled. The characteristic non-analyticity of the thermodynamic functions are very similar to those of the dimer and six-vertex models. The dimer model is shown to also have an emergent geometrical phase and one is conjectured for the six-vertex model. This suggests that phase transitions, in which the underlying geometry emerges, form new universality classes of topological phase transitions.

We will be interested in the 3-matrix model [1, 2] with global $SO(3)$ symmetry and energy

$$E = \frac{Tr}{N}\left(-\frac{1}{4}[D_j, D_k]^2 + \frac{2\mathrm{i}}{3}\epsilon_{jkl} D_j D_k D_l\right) . \tag{1}$$

The partition function is given by $Z(\beta, N) = \int [dD_j] \mathrm{e}^{-S(D)}$ where $S(D) = -\beta E(D)$ and $\beta = 1/T$ with T the temperature. The critical points of the energy are given by $[D_k, ([D_j, D_k] - \mathrm{i}\epsilon_{jkl} D_l)] = 0$. So representations of the Lie algebra of $SU(2)$ are critical points with energy $E_{saddle} = -\frac{1}{6}\frac{Tr}{N}(D_j^2)$. The minimum energy configuration is $D_j = L_j$ with $E_0 = -\frac{N^2-1}{24}$, and L_j satisfy $[L_j, L_j] = \mathrm{i}\epsilon_{jkl} L_l$ and $L_j L_j = \frac{N^2-1}{4}\mathbf{1}$, i.e. it is the irreducible representation of $su(2)$ of dimension N.

Geometry. Let $N_j = \frac{2}{\sqrt{N^2-1}} L_j$, so that $N_1^2 + N_2^2 + N_3^2 = \mathbf{1}$. This looks like a sphere but with the non-commutative "coordinates": $[N_1, N_2] = \frac{2\mathrm{i}}{\sqrt{N^2-1}} N_3$, which become those of a commutative sphere for $N \to \infty$. In the physics literature, this system of matrix equations is said to describe the round fuzzy sphere.

Alternatively, consider the Dirac operator $\mathbb{D} = \sigma_j [D_j, \cdot] + 1$ with σ_j the Pauli matrices. There is a natural geometry associated with the ground state via the spectral triple $(\mathbb{H}, \mathrm{Mat}_N, \mathbb{D}_0)$, where the algebra is Mat_N with trace norm and $\mathbb{D}_0 = \sigma_i [L_i, \cdot] + 1$. This Dirac operator has the same spectrum as that of the standard $SO(3)$ invariant Dirac operator on the commutative two sphere, except that at high energies the spectrum is cutoff. The spectral triple then specifies the ground state geometry to be that of a fuzzy sphere.

The fuzzy sphere, like other geometries, can be used as a background for field theory. One can describe a scalar field theory by the functional

(2) $$S_N(\Phi, a, b, c) = Tr(-a[L_j, \Phi]^2 + b\Phi^2 + c\Phi^4)$$

with Φ an $N \times N$ matrix and L_j described above. In the large N limit the action $S_N(\Phi, a, b, c)$ converges to $S(\phi, r, \lambda) = \int_{\mathbf{S}^2} \left(\frac{1}{2}\phi\Delta\phi + \frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right)$ with Δ the round Laplacian on the commutative sphere of unit area, $\mathbf{S}^2$. More precisely, if ϕ is a scalar field on $\mathbf{S}^2$ with the expansion $\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_{lm}$ in spherical harmonics, Y_{lm}, then the matrix $\Phi = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} c_{lm}\hat{Y}_{lm}$ expanded in polarization tensors [4], $\hat{Y}_{lm}$, which form a basis for Mat_N and satisfying $[L_3, \hat{Y}_{lm}] = m\hat{Y}_{lm}$ and $[L_j, [L_j, \hat{Y}_{lm}] = l(l+1)\hat{Y}_{lm}$. The coefficients, c_{lm}, then specify both ϕ and Φ and

(3) $$\lim_{N\to\infty} \left| S(\phi, r, \lambda) - S_N(\Phi, \frac{1}{2N}, \frac{r}{2N}, \frac{\lambda}{4!N}) \right| \to 0 \,.$$

I.e. in the large N limit the matrix functional for Φ converges to the standard functional for a scalar field on the two sphere.

Similarly, if we consider small fluctuations around the minimum of (1) with $D_j = L_j + A_j$ then E yields a field theory on the fuzzy sphere for noncommutative Yang-Mills gauge fields, A_i, with field strength $F_{jk} = i[L_j, A_k] - i[L_j, A_k] + \epsilon_{jkl}A_l + i[A_j, A_k]$. The gauge fields, A_i, include a scalar field, $\Phi = \frac{1}{\sqrt{N^2-1}}(D_j - L_j)^2$. In the large N limit Φ becomes a scalar field normal to $\mathbf{S}^2$ imbedded in $\mathbf{R}^3$.

Increasing the temperature and studying the system by both Monte Carlo simulations and perturbative techniques [1, 2] we find the system has a phase transition, where the partition function is non-analytic. The quantity $\mathbb{S} =< S >$ approaches the value $\mathbb{S}_- = \frac{5}{12}$ as the transition is approached from the fuzzy sphere phase and jumps to $\mathbb{S}_+ = \frac{3}{4}$ in the high temperature phase. The transition has a jump in the entropy which indicates that the system undergoes a change in order at a microscopic level. The specific heat $C = C_v/N^2$, where $C_v =< S^2 > - < S >^2$, diverges as this critical temperature is approached from below, with $C \approx A_-(T_c - T)^{-\alpha}$. Our analysis gives the critical point $\beta_c = (\frac{8}{3})^3$ and the critical exponent $\alpha = \frac{1}{2}$. We find that *as the temperature is increased the fuzzy sphere expands and evaporates.*

A closer look at the transition in the high temperature phase, suggests that $X_i = (\frac{\beta}{N^2})^{1/4} D_i$ becomes largely independent of both β and N. In fact fluctuations in the high temperature phase appear to be around three mutually commuting matrices. Initial results indicate that the joint distribution of their eigenvalues form a solid ball of radius R. The distribution of eigenvalues of X_3 is then: $\rho(x) = \frac{3}{4R^3}(R^2 - x^2)$. Numerically, we find $R \approx 2$.

The Dimer Model. Curiously a very classical model called the dimer model has very similar thermodynamic properties [3]. Mathematicians study it as a model to count tilings of graphs. For a physicist it has many faces but it can be thought of as a lattice model for a two dimensional Fermion. The simplest example of the model is when the lattice is taken to be hexagonal, with the three edges associated with each vertex having activities $a = \mathrm{e}^{-\beta\epsilon_a}$, $b = \mathrm{e}^{-\beta\epsilon_b}$ and $c = \mathrm{e}^{-\beta\epsilon_c}$. These determine the probability of a bond being active and the orientation of a lozenge in the tiling of the dual triangular lattice.

The partition function $Z(N, M, a, b, c) = \sum_{tilings} a^{N_a} b^{N_b} c^{N_c}$, where N_i is the number of active bonds of type i and $N_a + N_b + N_c = NM$, since the lattice must be completely covered. When the activities are set to one, Z counts the number of lozenge tilings of the dual triangular lattice.

By assigning signs judiciously to the adjacency matrix, Kasteleyn, Fisher and Temperley observed that, one can convert $Z(a, b, c)$ into a Pfaffian of, what is now called, a Kasteleyn matrix.

A Kasteleyn matrix, K, is a signed weighted adjacency matrix for the lattice. On any simply connected planar domain, the modulus of the Pfaffian of K gives the partition function of the dimer model. For a toroidal lattice $Z = \frac{1}{2}\left(-Z_{00} + Z_{\frac{1}{2}0} + Z_{0\frac{1}{2}} + Z_{\frac{1}{2}\frac{1}{2}}\right)$, where the subscripts determine the periodicity of K around the cycles of the torus.

In the thermodynamic limit the logarithm of the bulk partition function per dimer, $W = \frac{\ln Z}{NM}$, is found to be $W = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \ln(c - a\mathrm{e}^{-i\theta} - b\mathrm{e}^{i\phi})$.

If one of the weights is larger than the sum of the other two, then W is the logarithm of that weight. For $a = b = e^{-\beta}$ and $c = 1$ the low temperature phase has frozen order in the c direction, with a phase transition at $\beta = \ln 2$. Expanding around $\beta_c = \ln 2$, we have $W = 0$ for $\beta > \ln 2$ while for $\ln 2 \geq \beta$ we have $W(\beta) \simeq \frac{4\sqrt{2}}{3\pi}(\ln 2 - \beta)^{\frac{3}{2}}$. We see that the specific heat, $C = \beta^2 \frac{\partial^2 W}{\partial \beta^2}$, is zero in the low temperature frozen phase and diverges with critical exponent $\alpha = \frac{1}{2}$ as the transition is approached from the high temperature side.

The spectrum of K is given by $p^{Hex}(z, w) = 1 - \frac{1}{z} - w$ with $z = e^{x+i\theta}$ and $w = e^{y+i\phi}$ where $x = \ln(a/c)$ and $y = \ln(b/c)$. In the high temperature phase there are exactly two pairs of angles (Θ, Φ) and $(-\Theta, -\Phi)$ at which $p^{Hex} = 0$. More generally, the zero locus of p^{Hex} is a domain in the x, y plane bounded by curves on which these angles become either 0 or π. Expanding near the zero

$$p^{Hex} \approx -\frac{2\pi i a}{N} e^{-i\Theta} \left(n + u + \tau(m + v)\right) \quad \text{where} \quad \tau = \frac{Nb}{Ma} e^{i(\Theta+\Phi)}$$

and we see the spectrum of K becomes that of a massless Dirac fermion on a two dimensional torus whose modular parameter, $\tau = \tau_0 + i\tau_1$, is determined by the dimer weights. As the system is cooled and the transition is approached the angle of the torus collapses, $\tau \to \tau_0$. The system reverts to a frozen lattice system in a transition where the continuum geometry collapses. A careful analysis shows that, in the high temperature phase, the finite size effects are precisely those of a Dirac Fermion propagating on the continuum torus with modular parameter τ in the presence of a gauge potential with zero field strength but with flat connection, and holonomies round the cycles of the torus. The holonomies are determined by the precise nature of the limit.

We conjecture [3] that the six-vertex model, which has similar thermodynamic properties, when defined on a torus, has a transition where the geometry collapses again, but the transition occurs at a critical τ with $\tau_1 \neq 0$.

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Group field theory

Razvan Gurau

Group field theory is the higher-dimensional generalization of random matrix models. As it has built-in scales and automatically sums over metrics and discretizations, it provides a combinatoric origin for space time. Its graphs facilitate a new approach to algebraic topology. I exemplify this approach by introducing a graph's cellular structure and associated homology.

Noncommutative Geometry and Particle Physics

Walter D. van Suijlekom
(joint work with Jord Boeijink, Thijs van den Broek)

The following is based on two projects in progress with Jord Boeijink and Thijs van den Broek, respectively. Both are centered around the spectral action principle [3, 4]. This principle allows for a derivation of physical Lagrangians –or, rather, action functionals– from a spectral triple. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we can consider so-called inner fluctuations $D \to D + A$ of the Dirac operator by self-adjoint elements of the form $A = \sum_j a_j[D, b_j]$; these are the gauge fields. The *spectral action* is given by

$$\text{Trace } f(D + A)$$

with f a cut-off function on the real line. The *fermionic action* is given by $\langle \psi, (D + A)\psi\rangle$ for $\psi \in \mathcal{H}$. Both are invariant under the action of the group of unitaries in $\mathcal{A}$; this is the gauge group.

The first project [1] generalizes the noncommutative geometrical description of Einstein–Yang–Mills theory to topologically non-trivial $(SU(n))$ Yang–Mills fields. Recall that Chamseddine and Connes considered a spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), L^2(S) \otimes M_n(\mathbb{C}), \partial\!\!\!/ \otimes 1)$ on a compact Riemannian spin manifold M, deriving from it along the above lines (effective) $SU(n)$ Yang–Mills theory, coupled to gravity [4] (cf. also [6]). However, the topological structure of the Yang–Mills fields (i.e. connections on vector bundles) is trivial in this case, which reflects the tensor product structure of the above spectral triple.

We construct a spectral triple on a module algebra consisting of sections of an algebra bundle. The inner fluctuations for this spectral triple can be interpreted as a family of connections on the algebra bundle (tensored with the spinor bundle). Moreover, the spectral action principle gives the Yang–Mills action for a connection one-form on a $PSU(n)$ principal bundle. The above algebra bundle is naturally associated to this principal bundle and we identify the unitaries of the algebra with the sections of its adjoint bundle, that is, the gauge group in the usual sense.

The second project [2] explores another generalization of the noncommutative Einstein–Yang–Mills system, starting by studying its supersymmetric properties. It was already noted in [5] (cf. also [4]) that with the fermions in the adjoint representation of $SU(n)$ this theory possesses supersymmetry. In fact, the spectral action plus the fermionic action is symmetric under an exchange of the gauge field (gluon) and the fermion (gluino). The next step we take is to include quarks as well as their superpartners, the so-called squarks. It turns out that this can be done whilst leaving the algebra unchanged. Actually, this last point is crucial, since the group of unitaries in the algebra should remain the same gauge group $SU(n)$. Naturally, quarks are added to the Hilbert space and the squarks are realized as inner fluctuations, after introducing a non-trivial Dirac operator on the finite-dimensional part. Now, the spectral action and the fermionic action give the Lagrangian of supersymmetric QCD, which is one of the building blocks of the full (minimally) supersymmetric Standard Model of high-energy physics.

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Spectral Triples of Holonomy Loops

Jesper Møller Grimstrup

(joint work with Johannes Aastrup, Ryszard Nest, Mario Paschke)

In this talk I will present recent results concerning an intersection of quantum gravity with noncommutative geometry. The overall theme of this research is to apply the ideas and tools of noncommutative geometry directly to the setup of canonical quantum gravity. The aim is a new approach to quantum gravity which combines mathematical rigor with elements of unification.

So far, we have successfully constructed a semi-finite spectral triple which encodes the kinematical part of quantum gravity. Further, we have identified certain semi-classical states for which the spectral triple construction renders the Dirac Hamiltonian in 3+1 dimensions.

In more detail, we have constructed a semi-finite spectral triple over an algebra of holonomy loops. This construction is related to a configuration space of connections which, in turn, can be related to a formulation of gravity, due to Ashtekar, in terms of connections. In this spectral triple construction, the Dirac type operator has a natural interpretation as a global functional derivation operator. In terms of canonical gravity, it is an infinite sum of certain flux operators, conjugate to the holonomy loop operators. The interaction between the Dirac type operator and the algebra reproduces the structure of the Poisson bracket of general relativity. Thus, the spectral triple contains, a priori, the kinematical part of general relativity, in the sense that it involves information tantamount to a representation of the Poisson structure of general relativity.

The construction of the semi-finite spectral triple is based on an inductive system of embedded graphs. Although the construction works for a large class of ordered graphs we find that a system of 3-dimensional nested, cubic lattices has a clear physical interpretation. In fact, we find certain semi-classical states for which the Dirac type operator descents to the Dirac Hamiltonian in 3+1 dimensions. This semi-classical limit only works for cubic lattices and provides an interpretation of the lattices as a choice of a coordinate system. Here, the lapse and shift fields, which encode the choice of time-variable, can be understood in terms of certain degrees of freedom found in the Dirac type operator.

The Hilbert space in the spectral triple construction can be interpreted as a kinematical hilbert space of quantum gravity. Thus, a natural next step in the construction is to formulate and implement the Hamiltonian constraint. Here, the concreteness of the appearance of the lapse and shift fields raises the hope that this construction might hint towards a natural formulation of this constraint, which should exactly implement invariance under changes in the choice of the time-coordinate. A priori it is certainly possible to formulate a quantized Hamiltonian constraint – just by interchanging classical variables with flux and loop operators. The goal, however, is to find some principle and quantity which appears natural.

The spectral triple construction raises many additional question. For instance, we do not yet know how to extract the classical algebra of functions on the (spatial) manifold. Here, the key question is whether the function algebra, should it emerge from the construction in a semi-classical limit, will be commutative, or whether it will pick up a noncommutative factor stemming from the noncommutativity of the holonomy loops. If the latter should be the case it would make contact to the work of Alain Connes on the standard model of particle physics, which is based on an almost commutative algebra.

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Progress in solving a noncommutative QFT in four dimensions

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(joint work with Harald Grosse)

This report is based on [1].

In previous work [2] we have proven that the following action functional for a ϕ^4-model on four-dimensional Moyal space gives rise to a renormalisable quantum field theory:

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{1}$$

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation and becomes self-dual at $\Omega = 1$. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [3, 4]. The vanishing of the β-function at $\Omega = 1$ was next proven in [5] at three-loop order and finally by Disertori, Gurau, Magnen and Rivasseau [6] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible. The Landau ghost problem is solved.

The action functional (1) is most conveniently expressed in the matrix base of the Moyal algebra [2]. For $\Omega = 1$ it simplifies to

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\;, \tag{2}$$

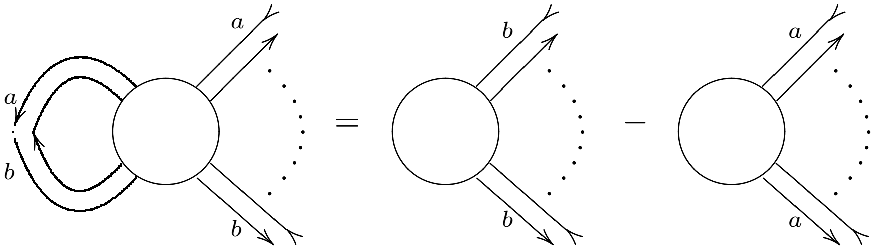
$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\;, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;. \tag{3}$$

The model only needs wavefunction renormalisation $\phi \mapsto \sqrt{Z}\phi$ and mass renormalisation $\mu_{bare} \to \mu$, but no renormalisation of the coupling constant [6] or of $\Omega = 1$. All summation indices $m, n, \dots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$, and $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size.

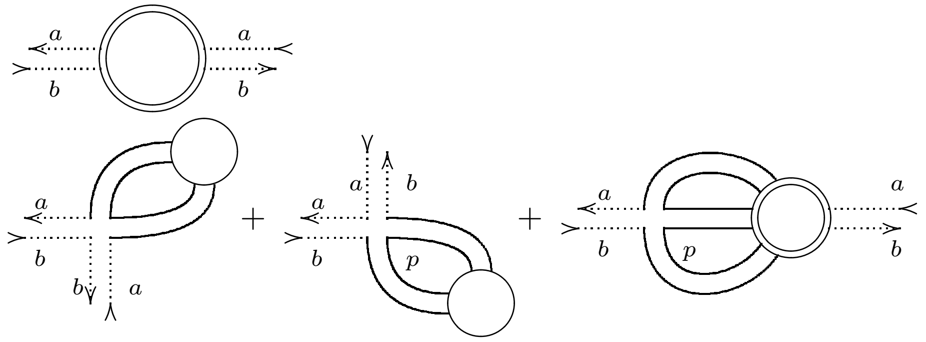
The key step in the proof [6] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a| - |b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$:

(5) $Z(|a| - |b|)$ 

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(6) $\Gamma_{ab} =$ 

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\begin{aligned} \Gamma_{ab} &= Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b}\Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)}\Big) \\ &= Z^2\lambda \sum_p \Big(\frac{1}{H_{ap} - \Gamma_{ap}} + \frac{1}{H_{bp} - \Gamma_{bp}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)}\Big) \,. \end{aligned} \tag{7}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} in H (3). Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices $a, b, \dots$ in $\mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (7) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2 \frac{\alpha}{1-\alpha}$, $|p| =: \mu^2 \frac{\rho}{1-\rho}$ and

$$\Gamma^{ren}_{ab} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1 - \frac{1}{G_{\alpha\beta}}\Big) , \tag{8}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [1]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$\begin{aligned} G_{\alpha\beta} = 1 + \lambda\bigg(&\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\ &+ \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\ &- \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y}\bigg) , \end{aligned} \tag{9}$$

where $\alpha, \beta \in [0, 1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \quad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho} , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} ,$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function [1].

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$\begin{aligned} G_{\alpha\beta} = 1 &+ \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\ &+ \lambda^2\Big\{AB\big((I_{\overset{\alpha}{\bullet}} - \alpha) + (I_{\overset{\beta}{\bullet}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1)\big) \\ &\quad + A\big(\beta I_{\overset{\beta}{\bullet}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\ &\quad + B\big(\alpha I_{\overset{\alpha}{\bullet}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big)\Big\} + \mathcal{O}(\lambda^3) , \end{aligned} \tag{10}$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

$$\begin{aligned} I_\alpha &:= \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha) , \\ I_{\overset{\alpha}{\bullet}} &:= \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2 . \end{aligned} \tag{11}$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

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Real multiplication and quantum statistical mechanics

Jorge Plazas

The explicit class field theory problem for a field k asks for explicit generators of k^{ab}, the maximal abelian extension of k, and the corresponding Galois action of $G_k^{ab} = Gal(\overline{k}/k)^{ab}$ on these generators. In the case of number fields a complete solution of this problem is known only for $\mathbb{Q}$ (Kronecker-Weber theorem) and imaginary quadratic extensions of $\mathbb{Q}$ (complex multiplication theory). In both of these cases generators of $k^{ab} \subset \mathbb{C}$ can be given as special values of explicit holomorphic functions. These algebraic numbers can as well be recovered as special values of L-functions associated to ideal classes in k. For the general number field case the values obtained from the corresponding L-functions are conjectured to provide generators for the maximal abelian extension, this is the content of Stark's conjectures (cf. [8]). The next simplest case for which there is not yet a complete solution to the explicit class field theory problem is the case of real quadratic fields. Based on deep analogies with the theory of elliptic curves Y. Manin proposed in [6] the use of noncommutative tori as geometric objects associated to real quadratic fields. Elliptic curves admitting nontrivial self isogenies correspond to lattices generated by imaginary quadratic irrationalities and play a central role in the solution of the explicit class field theory problem for the corresponding imaginary quadratic fields. Noncommutative tori admitting nontrivial self Morita equivalences correspond to pseudo-lattices generated by real quadratic irrationalities and might provide the right framework to develop in this context real multiplication theory for real quadratic fields.

In the last years quantum statistical mechanical systems of arithmetic relevance have been constructed. For a C^*-dynamical system (A, σ_t) encoding arithmetic data a rational subalgebra $\mathcal{A}_0 \subset A$ is given and special values arise as $\varphi(a)$ for $a \in \mathcal{A}_0$ and φ a extremal KMS_∞ state. Explicit class field theory of $\mathbb{Q}$ and imaginary quadratic fields can be recovered in this context (see [1] and [4, 5]). These cases fall into the recently developed framework of endomotives which starting with the relevant arithmetic-geometric data provide a construction of the algebra $\mathcal{A}_0$

together with a time evolution on its natural C^*-completion (see [3]). It is clear in view of these results that the study of endomotives associated to real quadratic fields could provide relevant information for the explicit class field theory problem in this case. In particular taking into account Manin's real multiplication program it is natural to ask the question of whether it is possible to associate endomotives to real multiplication noncommutative tori and their rings of Morita endomorphisms. We briefly discus how some examples can be constructed.

By definition an Artin endomotive over a field k is determined by a projective system $\{X_i\}_{i\in I}$ of zero-dimensional varieties over k and an abelian semigroup S of endomorphisms of the pro-variety $\varprojlim_{i\in I} X_i$. The system $(A_{\mathbb{Q}}, \sigma_t)$ studied in [1] corresponds to the endomotive $\mathcal{E}_{\mathbb{Q}}$ associated to projective limit $\{\mathbb{Z}/n\mathbb{Z}\}_{n\mathbb{N}^*}$ where the groups $\mathbb{Z}/n\mathbb{Z}$ are viewed as finite algebraic group schemes over $\mathbb{Q}$; the semigroup of endomorphisms in this case being $S = \mathbb{N}^*$.

Let $\theta \in \mathbb{R}$ be a real quadratic irrationality and let $k = \mathbb{Q}(\theta)$ be the real quadratic field generated by θ. Let $g \in SL_2(\mathbb{Z})$ be a hyperbolic matrix such that $g\theta = \theta$ and write

$$g^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \qquad n = 1, 2, \ldots$$

It can be assumed that $c_1 \geq a_1 + d_1 + 1$. Under these conditions we have that $c_n | c_m$ if and only if $n|m$ and we obtain therefore a projective system $\{\mathbb{Z}/c_n\mathbb{Z}\}_{n\in\mathbb{N}^*}$ which defines a quotient endomotive $\mathcal{E}^1_{\theta,g}$ of $\mathcal{E}_{\mathbb{Q}}$. For any matrix g as above and every $n = 1, 2, \ldots$ a bimodule realizing a nontrivial Morita self equivalence of the noncommutative torus $\mathbb{T}_\theta$ can be constructed by considering the natural representation of the finite Heisenberg group $\mathrm{Heis}(\mathbb{Z}/c_n\mathbb{Z})$ (see [2]). In particular the corresponding direct limit algebra in $\mathcal{E}^1_{\theta,g}$ gives the "profinite" part of the Fock space of the bimodule associated to g.

Using the continued fraction expansion of elements in k and the fact that the corresponding convergents give rise to matrices of the above form we can construct endomotives $\mathcal{E}^2_{\Gamma,\theta}$ where the projective system is given in terms of double products of the groups defining $\mathcal{E}^1_{\theta,g}$ and for which the semigroup of endomorphisms is given by the positive elements of an order Γ in $\mathcal{O}_k$. The analysis of these endomotives is yet to be carried out in full but it is expected that formulas analogous to those introduced by Zagier in [9] will play a role making therefore contact with Stark's conjectures for the real quadratic field case (see [7]).

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A duality result for C*-algebras associated with global fields

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(joint work with Joachim Cuntz)

Ring C*-algebras were first introduced by Cuntz in [1]. Motivated by the Bost-Connes system, Cuntz associated C*-algebras to the integers $\mathbb{Z}$ or to the natural numbers $\mathbb{N}$, respectively. Let me explain his construction.

1. What is a ring C*-algebra?

This question has four answers, in the sense that there are four equivalent ways of describing the ring C*-algebra of the integers. Here is the first one:

Consider unitaries $\{u^a\colon a \in \mathbb{Z}\}$ and isometries $\{s_b\colon b \in \mathbb{Z}^\times\}$ ($\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$) on $\ell^2(\mathbb{Z})$ given by

$$u^a(\xi_r) = \xi_{a+r} \text{ and } s_b(\xi_r) = \xi_{b\cdot r}$$

where $\{\xi_r\colon r \in \mathbb{Z}\}$ is the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. The ring C*-algebra of $\mathbb{Z}$ is given by

$$\mathfrak{A}[\mathbb{Z}] := C^*(\{u^a, s_b\colon a \in \mathbb{Z}, b \in \mathbb{Z}^\times\}) \subseteq \mathcal{L}(\ell^2(\mathbb{Z})).$$

This can be viewed as the reduced ring C*-algebra of $\mathbb{Z}$ given by the regular representation of the ring $\mathbb{Z}$. Alternatively, we can look for a full version:

Consider the universal C*-algebra generated by

$$\text{unitaries } \{u^a\colon a \in \mathbb{Z}\} \text{ and isometries } \{s_b\colon b \in \mathbb{Z}^\times\}$$

satisfying the relations

$$I. \qquad u^a s_b u^c s_d = u^{a+bc} s_{bd} \text{ for all } a, c \in \mathbb{Z}; b, d \in \mathbb{Z}^\times$$

$$II. \qquad \sum_{a+(b)\in\mathbb{Z}/b\mathbb{Z}} u^a e_b u^{-a} = 1 \text{ for all } b \in \mathbb{Z}^\times (\text{where } e_b := s_b s_b^*).$$

To justify II., note that $u^a e_b u^{-a}$ is given by $\xi_r \mapsto \mathbb{1}_{[a+b\mathbb{Z}]}(r)\xi_r$ on $\ell^2(\mathbb{Z})$, where $\mathbb{1}_{[a+b\mathbb{Z}]}$ is the characteristic function attached to $a + b\mathbb{Z} \subseteq \mathbb{Z}$. In other words, $u^a e_b u^{-a}$ is the orthogonal projection onto $\ell^2(a + b\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$. Thus, II. reflects the fact that $\mathbb{Z}$ is the disjoint union $\dot\cup_{a+b\mathbb{Z}\in\mathbb{Z}/b\mathbb{Z}}(a + b\mathbb{Z})$.

Now, it turns out that this universal C*-algebra coincides with $\mathfrak{A}[\mathbb{Z}]$. One way to prove this result is to show that the universal C*-algebra is purely infinite and simple, and that relations I. and II. are satisfied in $\mathfrak{A}[\mathbb{Z}]$. So, we have already seen a second answer to our question, a second way of characterizing $\mathfrak{A}[\mathbb{Z}]$.

To obtain a third description of our ring C*-algebra, let us consider $\mathfrak{D}[\mathbb{Z}] = C^*(\{u^a e_b u^{-a} \colon a \in \mathbb{Z}, b \in \mathbb{Z}^\times\})$, the C*-subalgebra of $\mathfrak{A}[\mathbb{Z}]$ generated by the projections $u^a e_b u^{-a}$. $\mathbb{Z}$ acts on $\mathfrak{D}[\mathbb{Z}]$ via conjugation with the unitaries u^a, and $\mathbb{Z}^\times$ acts by conjugation with the isometries s_b (the second action is an action of a semigroup via endomorphisms). Therefore, we can consider the associated crossed product, and again, we get a description of our ring C*-algebra:

$$\mathfrak{A}[\mathbb{Z}] \cong \mathfrak{D}[\mathbb{Z}] \rtimes \mathbb{Z} \overset{e}{\rtimes} \mathbb{Z}^\times .$$

Up to now we only used two properties of the integers: Namely, that $\mathbb{Z}$ is an integral domain and that $\mathbb{Z}$ has finite quotients. Therefore, it is straightforward to generalize the construction to integral domains (commutative, unital rings without zero-divisors) with finite quotients (R has finite quotients if for all nontrivial $b \in R$, $\#\,[R/(b)] < \infty$). This is done in [2], where the connection to the Bost-Connes system and its higher dimensional analogues is studied as well.

Now, the question arises how to generalize the notion of ring C*-algebras to arbitrary rings. First of all, it turns out that commutativity of the ring is not needed in the construction. There are two problems left: zero-divisors and infinite quotients. For a zero-divisor b, the formula $s_b(\xi_r) = \xi_{b \cdot r}$ would not define a bounded operator in general. One easy way out is to consider regular elements only (i.e. to exclude zero-divisors). This is done in [4], and in special cases, this already leads to reasonable results. Still, in the general case, one would probably lose too much information. An alternative way would be to replace isometries by partial isometries in our definitions. But there does not seem to be a reasonable way to organize the support projections of these partial isometries implementing multiplication of the ring. So, this is still an open problem. As against that, the second problem concerning infinite quotients has been solved in [4]. If the ring has infinite quotients, relation II. will no longer make sense. To avoid summing up infinitely many projections, the idea is to consider additional generators, namely projections, together with relations which substitute relation II. This can be formalized using the notion of a finitely additive, projection-valued spectral measure. A similar idea can be used to construct semigroup C*-algebras and semigroup crossed products. For the details, see [4].

2. K-theory

To compute the K-theory of $\mathfrak{A}[\mathbb{Z}]$, it is helpful to refine the crossed product description of $\mathfrak{A}[\mathbb{Z}]$. We observe that $\mathfrak{D}[\mathbb{Z}]$ is a commutative C*-algebra with Spec $(\mathfrak{D}[\mathbb{Z}]) \cong \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Thus,

$$\mathfrak{A}[\mathbb{Z}] \cong C(\widehat{\mathbb{Z}}) \rtimes \mathbb{Z} \overset{e}{\rtimes} \mathbb{Z}^\times \sim_M C_0(\mathbb{A}_f) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times .$$

We have $\mathbb{Q}^\times = \{\pm 1\} \times \mathbb{Q}_{>0}$, and $\mathbb{Q}_{>0}$ is a free abelian group generated by the primes. Now, the idea is to compute the K-theory of $C_0(\mathbb{A}_f) \rtimes \mathbb{Q} \rtimes \{\pm 1\}$ using the description of $\mathfrak{A}[\mathbb{Z}]$ as a universal C*-algebra and then to use the Pimsner-Voiculescu sequence to determine $K_*(C_0(\mathbb{A}_f) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times)$. But it turns out that it is only possible to compute the K-groups in the Pimsner-Voiculescu sequences when there is additional information about the multiplicative action of $\mathbb{Q}_{>0}$. This is exactly provided by the duality result, which states for $\mathbb{Q}$:

$$C_0(\mathbb{A}_\infty) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times \sim_M C_0(\mathbb{A}_f) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times.$$

Here, $\mathbb{A}_\infty$ is the infinite adele space, which is $\mathbb{R}$ in the case of $\mathbb{Q}$. $\mathbb{A}_f$ is the finite adele ring. $\mathbb{Q}$ acts additively and $\mathbb{Q}^\times$ acts multiplicatively. By the way, this is the fourth description of the ring C*-algebra $\mathfrak{A}[\mathbb{Z}]$.

This duality theorem tells us that the multiplicative action is equivariantly homotopic to the trivial action, once one can show that it is enough to consider the multiplicative action only, that one can leave out addition. This homotopy argument does not apply in the totally disconnected space $\mathbb{A}_f$, so it is really a crucial step to pass over to $\mathbb{A}_\infty \cong \mathbb{R}$. The final result is:

$$K_*(\mathfrak{A}[\mathbb{Z}]) \cong \Lambda^* (\mathbb{Q}_{>0})$$

where K_0 corresponds to products of even and K_1 corresponds to products of odd numbers of pairwise distinct primes. Note that K_* is the direct sum of K_0 and K_1, viewed as a graded abelian group.

The same strategy should apply to rings of integers in arbitrary number fields. Namely, it is possible to prove the duality result in full generality (i.e. for every global field), even in an equivariant way:

Theorem. *Let K be a global field and Γ be a subgroup of $K^\times$. Then,*

$$C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma \sim_M C_0(\Gamma \cdot \widehat{\mathcal{O}}) \rtimes (\Gamma \cdot \mathcal{O}) \rtimes \Gamma,$$

where the groups act via (inverse) affine transformations.

Here, $\mathcal{O}$ is the ring of integers if K is a number field, and the integral closure of $\mathbb{F}_p[T]$ for function fields. $\widehat{\mathcal{O}}$ is the profinite completion of $\mathcal{O}$.

Up to now, the remaining steps of our K-theoretic computations can only be carried out if the roots of unity in K are ± 1. The final result is the following:

Theorem. *Let K be a number field and $\mathcal{O}$ the ring of integers in K. We assume that the set of roots of unity in K is given by $\mu = \{\pm 1\}$. Choose a free abelian subgroup Γ of $K^\times$ with $K^\times = \mu \times \Gamma$. Then:*

$$K_*(\mathfrak{A}[\mathcal{O}]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \# \{v_{\mathbb{R}}\} = 0 \\ \Lambda^*(\Gamma) & \text{if } \# \{v_{\mathbb{R}}\} \text{ is odd} \\ \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)) & \text{if } \# \{v_{\mathbb{R}}\} \text{ is even and } \geq 2. \end{cases}$$

Here we consider graded tensor products where $K_0(C^*(\mu))$ and $\mathbb{Z}/2\mathbb{Z}$ are trivially graded. We take the diagonal grading on the direct sum. $\# \{v_{\mathbb{R}}\}$ is the number of real places of K.

The reader may consult [3] for the details.

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Holomorphic structures on the quantum 2-sphere

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In the present talk we present a natural complex structure on the Podleś quantum 2-sphere – which, with additional structure, we identify with the quantum projective line $\mathbb{CP}^1_q$ – that resembles in many aspects the analogous structure on the classical Riemann sphere. We shall concentrate on both algebraic and analytic aspects. While at the algebraic level the complex structure we are using on the quantum projective line was already present in literature, we move from this to the analytic level of holomorphic functions and sections. Indeed, it is well known that there are finitely generated projective modules over the quantum sphere that correspond to the canonical line bundles on the Riemann sphere in the $q \to 1$ limit. In this paper we study a holomorphic structure on these projective modules and give explicit bases for the space of corresponding holomorphic sections. Since these projective modules are in fact bimodules we can define, in terms of their tensor products, a quantum homogeneous coordinate ring for $\mathbb{CP}^1_q$. We are able to identify this ring with the coordinate ring of the quantum plane.

We start with the notion of a complex structure on an involutive algebra as a natural and minimal algebraic requirement on structures that ought to be present in any holomorphic structure on a noncommutative space; we also give several examples, some of which are already present in the literature. We then define holomorphic structures on modules and bimodules and indicate, in special cases, a tensor product for bimodules. Then we look at the quantum projective line $\mathbb{CP}^1_q$ and its holomorphic structure defined via a differential calculus. This differential calculus, induced from the canonical left covariant differential calculus on the quantum group $\mathrm{SU}_q(2)$, is the unique left covariant one on $\mathbb{CP}^1_q$. We compute explicit bases for the space of holomorphic sections of the canonical line bundles $\mathcal{L}_n$ (labelled by the 'degree' $n \in \mathbb{Z}$) on $\mathbb{CP}^1_q$ and we notice that they follow a pattern similar to the classical commutative case. This allows us to compute the quantum homogeneous coordinate ring of $\mathbb{CP}^1_q$, and to show that it coincides with the coordinate ring of the quantum plane. Finally, we look for a possible positive

Hochschild cocycle on the quantum sphere. Given that there are no non-trivial 2-dimensional cyclic cocycles on the quantum 2-sphere we formulate a notion of twisted positivity for twisted Hochschild and cyclic cocycles and show that a natural twisted Hochschild cocycle φ is positive. The twist here is induced on $\mathbb{C}\mathrm{P}^1_q$ by the modular automorphism of the quantum $\mathrm{SU}_q(2)$.

Among the many open questions we mention: 1) uniqueness of the holomorphic structure; 2) extremality of the twisted Hochschild 2-cocycle φ; 3) abstract perturbations of conformal structures by Beltrami differentials; 4) a Gauss-Bonnet theorem along the line of the results on the noncommutative torus presented by A.Connes at this meeting.

This talk is based on [1].

References

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Quantization

Eli Hawkins

A Poisson structure on a manifold M is a bilinear bracket on the commutative algebra of smooth functions $\mathcal{C}^\infty(M)$ that is antisymmetric, is a derivation on both arguments, and satisfies the Jacobi identity. The motivation for considering such a structure is that these are the properties of the commutator bracket in a noncommutative algebra.

Strict Deformation Quantization was first defined by Rieffel [6] in terms of a fixed vector space with variable product, involution, and norm, each depending on a parameter $\hbar$ that varies over an interval. This vector space is identified with a space of functions on M, and for $\hbar = 0$, the product is required to be the (commutative) pointwise product. For $\hbar \approx 0$, the commutator is required to be approximately $i\hbar$ times the Poisson bracket. Other axioms imply that the norm completions of these algebras form a continuous field of C*-algebras.

Intuitively, this can be thought of as a continuous path through the space of all possible algebras. The path begins at a commutative algebra of functions on M, and the Poisson structure defines a tangent to this path.

Examples show that it is too restrictive to require that $\hbar$ vary over an interval, or that there be a one to one correspondence between functions and algebra elements. A more flexible way to formulate strict deformation quantization is in terms of a subset $I \subset \mathbb{R}$ (of $\hbar$ values), a continuous field of C*-algebras $A \to I$, and a linear "quantization map" $Q : \mathcal{A}_0 \to \Gamma(I, A)$ from a dense subalgebra $\mathcal{A}_0 \subset \mathcal{C}_0(M)$ to the sections of A. The most important axioms require that at $\hbar = 0 \in I$, $Q_0 : \mathcal{A}_0 \to A_0$ is a homomorphism, and for any $f, g \in \mathcal{A}_0$, the commutator is approximated by the Poisson bracket:

$$[Q_\hbar(f), Q_\hbar(g)] \approx i\hbar Q_\hbar(\{f, g\}) \mod o(\hbar).$$

This notation means that the norm of the difference between the left and right sides goes to 0 faster than $\hbar$.

Formal deformation quantization [1, 7] was actually defined first. Instead of a variable product, there is a *formally* variable product. This can be expressed most simply as a product on the space $\mathcal{C}^\infty(M)[[\hbar]]$ of formal power series in $\hbar$ with coefficients in the space of smooth functions. This is required to reduce to the ordinary pointwise product modulo $\hbar$, and the commutator is required to equal $i\hbar\{\ \cdot\ ,\ \cdot\ \}$ modulo $\hbar^2$. Intuitively, this is a formal path through the space of algebras, or at best it is the jet of a path.

It is also possible to consider formal deformation quantization to some finite order $\hbar^n$. This means an associative product on $\mathcal{C}^\infty(M)[\hbar]/\hbar^{n+1}$. If $n \geq 2$, then the properties of the Poisson bracket are implied by the associativity of the formal product. The derivation property of the Poisson bracket is the order $\hbar$ part of the equivalent identity for the commutator. The Jacobi identity for the Poisson bracket is the order $\hbar^2$ part of that identity for the commutator. So, if $n \leq 1$, then the Jacobi identity is an unnecessary condition; we could just as well consider first order formal deformation quantizations for "almost" Poisson structures which violate the Jacobi identity.

This same issue arises in strict deformation quantization. Hanfeng Li [5] has given a general construction of strict deformation quantization that works for any almost Poisson manifold. The problem is that the definition of strict deformation quantization does not require sufficiently good behavior for small $\hbar$. A solution is to require that the strict deformation induces a formal deformation quantization to order $n \geq 2$. This can be thought of as the order n jet of that path through the space of algebras.

So, we may require an "order n" axiom [3]:

$$\mathbb{A} := \{a \in \Gamma(I, A) \mid \exists f_0, \ldots, f_n \in \mathcal{A}_0 : a_\hbar \approx Q_h(f_0 + \cdots + f_n\hbar^n) \mod o(\hbar^n)\}$$

is a *-subalgebra of $\Gamma(I, A)$.

The polynomial $f_0 + f_1\hbar + \cdots + f_n\hbar^n$ is essentially a Taylor expansion of the section a about $\hbar = 0$. So, the space $\mathbb{A}$ should be thought of as the continuous sections of A that are n times differentiable at $\hbar = 0$. Intuitively, an order n strict deformation quantization is a continuous path through the space of algebras that is n times differentiable at $\hbar = 0$.

A quantization map serves to identify functions with operators, but the specific identification is not really important. These axioms of quantization serve to restrict the quantization map in the limit $\hbar \to 0$, but for finite $\hbar$, it is really quite arbitrary. Two quantization maps should be viewed as equivalent if they give the same notion of differentiability at $\hbar = 0$. In other words, the subalgebra $\mathbb{A}$ — rather than the map Q — expresses the important structure of a strict deformation quantization.

Can the axioms for an order n quantization be reexpressed without reference to a quantization map? Firstly, everything is done modulo the ideal of sections whose norms approach 0 faster than $\hbar^n$; I shall denote this as $o(\hbar^n)$. The quantization map Q serves to identify the quotient algebra $\mathbb{A}/o(\hbar^n)$ with $\mathcal{A}_0[\hbar]/\hbar^{n+1}$ as a $\mathbb{C}[\hbar]/\hbar^{n+1}$-module. This implies that the quotient is a free module. Conversely, if we simply require that $\mathbb{A}/o(\hbar^n)$ be a free module, then there must exist some identification, and this lifts to a map satisfying all the axioms for a quantization map.

So, the definition of an order n quantization can be reexpressed in terms of a continuous field of C*-algebras and a subalgebra of "nice" sections which are thought of as being n-times differentiable at $\hbar = 0$. However, this definition is unsymmetrical. Why should differentiability only be imposed at one point? It would be more natural to require differentiability all along the path.

First, define the algebra of continuously n times differentiable functions on $I \subset \mathbb{R}$ as

$$\mathcal{C}^n(I) := \{F \in \mathcal{C}(I) \mid \forall \hbar \in I \; \exists U \subseteq \mathbb{R} \text{ a neighborhood of } \hbar \\ \text{such that } \exists G \in \mathcal{C}^n(U) \text{ such that } F|_{I\cap U} = G|_{I\cap U}\}.$$

Now, we can define a *$\mathcal{C}^n$-field of* C*-*algebras* over $I \subset \mathbb{R}$ to be a continuous field of C*-algebras $A \to I$ with a dense subspace of sections $\mathbb{A} \subset \Gamma(I, A)$ such that:

- $\mathbb{A}$ is a $\mathcal{C}^n(I)$-*-algebra.
- For any $\hbar_0 \in I$, the quotient
$$\mathbb{A}/o(\hbar - \hbar_0)^n$$
is a free module of the quotient algebra
$$\mathcal{C}^n(I)/o(\hbar - \hbar_0)^n.$$
- $\mathbb{A}$ is not a proper subspace of anything satisfying the preceding axioms.

Kontsevich's results on formal deformation quantization [4] showed that there is a one to one correspondence between isomorphism classes of formal deformations of Poisson structures and isomorphism classes of formal deformation quantizations. In particular, there is a unique (isomorphism class of a) formal deformation quantization associated to not deforming the Poisson structure at all. If we think of a formal deformation quantization as a formal path through the space of algebras, then this is analogous to a geodesic, determined uniquely by its starting point ($\mathcal{C}^\infty(M)$) and a tangent vector (the Poisson structure).

But these are just formal paths. Imagine that there are "geodesic" strict deformation quantizations. If this were true, then a Poisson manifold would determine a unique preferred strict deformation quantization, and at $\hbar = 1$ this would give a specific C*-algebra corresponding to the Poisson manifold.

This is not true; for example, a symplectic 2-torus can be quantized as a family of matrix algebras or of the much more interesting noncommutative torus algebras. However, it may not be far from true. Many examples of Poisson manifolds do have a very natural quantization, once some small choices are made in the construction.

It is my hope that there does exist a natural "quantization" construction of a C*-algebra from a Poisson manifold with some additional structure. My proposed outline of such a constriction is described in [2].

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