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Mini-Workshop: Geometry of Quantum Entanglement

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Abstract. The workshop aimed at developing interactions between researchers from quantum information theory and from asymptotic geometric analysis. A central notion discussed was the phenomenon of quantum entanglement, which naturally leads to geometric considerations in high-dimensional vector spaces. In these spaces, phenomena such as concentration of measure become prominent and may invalidate our low-dimensional intuition.

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Introduction by the Organisers

The goal of the workshop was to bring together researchers working in the areas of quantum information theory and asymptotic geometric analysis to intensify interaction between the two groups. A number of results obtained over the last few years has shown that potential for such interaction existed. On the other hand, while there had been, for example, workshops at the Banff International Research Center and at the Fields Institute in Toronto focused on the interaction between Quantum computation/information and the theory of operator algebras, and of course numerous programs/conferences on the subject of Quantum computation/information as such, no events centered precisely on the interface addressed by the present workshop took place to date.

Let us give a high level overview of the nature of links between high dimensional convex geometric analysis and quantum information theory. The main objects of study in quantum information theory are *states* and *channels*. For every particular physical system, the corresponding states and channels form convex sets. It is also of importance to consider various subsets of these sets (such as the set of separable states), often also convex. A systematic analysis of these sets via conventional geometric, analytic and numerical methods is generally feasible only for very small systems: if one works with more than just a few qubits or qudits, our sets "live" in a space of a rather high dimension. Therefore one gets into the realm of *asymptotic geometric analysis*, which deals exactly with quantitative study of such high-dimensional objects and phenomena by identifying and exploiting "approximate" symmetries of various problems that escaped the earlier "too qualitative" or "too rigid" methods. While classically analyzing high-dimensional phenomena often suffers from the *curse of dimensionality* (the complexity of the problem exploding with the increase in dimension so that the question quickly ceases to be tractable), we may say that asymptotic geometric analysis exploits the *blessing of dimensionality*, with the symmetries mentioned above becoming apparent only when the dimension is large. Moreover it is well-known in asymptotic geometric analysis, that many low-dimensional intuitions are wrong in high dimensions, and so one may expect various surprising discoveries as a result of applying the techniques of that field in the present context. The idea behind the workshop was to publicize, and ultimately to systematically exploit this unique perspective, which up to now appeared in quantum information theory only on an ad hoc basis.

Mini-Workshop: Geometry of Quantum Entanglement

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Abstracts

On randomizing channels with short Kraus decompositions

Guillaume Aubrun

The goal of this educational talk was to present in details some applications of ε-net arguments to problems in quantum information theory. We first consider the problem of *universal entanglers*, which fits especially well to this setting. We then focus on the problem of finding *almost randomizing channels* with short Kraus decompositions.

1. Universal entanglers

The following definition appeared in [3]: a unitary operator U on $\mathbf{C}^d \otimes \mathbf{C}^d$ is called a universal entangler if for every separable pure state ψ on $\mathbf{C}^d \otimes \mathbf{C}^d$, the state $U\psi$ is entangled.

The question of existence of universal entanglers is answered in [3] using algebraic geometry. Consider the projective space $P = \mathbf{P}(\mathbf{C}^d \otimes \mathbf{C}^d)$, and let $Sep \subset P$ be the subset of separable (=product) states. Then Sep is a projective variety (often called the Segré variety) of dimension $2d - 2$. The same holds for $U(Sep)$. We are now in position to apply the projective intersection theorem :

Let A, B be two projective varieties in $\mathbf{P}(\mathbf{C}^n) = \mathbf{P}^{n-1}$. Then

- If $\dim A + \dim B \geq n - 1$, then $A \cap B \neq \emptyset$,
- Conversely, if $\dim A + \dim B < n - 1$, then generically $A \cap B = \emptyset$. This means for example the following: if U is a random Haar-distributed unitary matrix, then $A \cap U(B)$ is almost surely empty.[1]

As an application, universal entanglers exist (and are generic) in dimension $d \geq 4$ [3]. Now suppose that we want a quantitative version: can a single gate U map every separable state to a very entangled state ? At this point, probabilistic techniques come naturally into the picture.

Proposition [2]. Let $U \in \mathcal{U}(\mathbf{C}^d \otimes \mathbf{C}^d)$ be a random Haar-distributed unitary matrix. Then with probability larger than $1 - \exp(-cd)$, for every separable state ϕ, all the Schmidt coefficients of $U\phi$ are bounded by C/d (here c and C denote absolute constants).

To prove the proposition we need to show that with large probability,

$$A := \max_{x_i} \Re \langle x_1 \otimes x_2 | U | x_3 \otimes x_4 \rangle \leq \sqrt{\frac{C}{d}},$$

where the maximum is taken over 4-tuples of unit vectors x_i in $\mathbf{C}^d$. To this end we use an ε-net argument which decomposes into three steps.

[1] The first part of the statement is very classical, see for example [4]. This second part of the statement does not seem to appear in algebraic geometry textbooks. We thank Cédric Bonnafé, Daniel Perrin and Olivier Wittenberg for proofs and discussions.

(1) There exists a ε-net $\mathcal{N}$ in the unit sphere of $\mathbf{C}^n$ with cardinality less than $(1+2/\varepsilon)^{2n}$. This is proved using a standard volumetric argument [6].
(2) For fixed unit vectors $(x_i)_{1\leq i\leq 4}$, we have
$$\Re\langle x_1 \otimes x_2 | U | x_3 \otimes x_4\rangle \leq C'/\sqrt{d}$$
with probability larger than $1-\exp(-c'd)$ for absolute constants c', C'. This step just amounts to computing the volume of a spherical cap in a unit sphere.
(3) We need to pass from the net to the whole sphere. Define B to be
$$B := \max_{\bar{x}_i \in \mathcal{N}^4} \Re\langle \bar{x}_1 \otimes \bar{x}_2 | U | \bar{x}_3 \otimes \bar{x}_4\rangle,$$
By the union bound, $B \leq C'/\sqrt{d}$ with positive probability. Obviously, $B \leq A$. We check than $A \leq B + 4\varepsilon A$ using the fact that the quantity we maximize is *linear* in its arguments. We now choose ε to be $1/8$, so that $A \leq 2B$. This completes the proof.

A possibly surprising feature of the above proof is that, whereas the Schmidt numbers become smaller and smaller when the dimension increases, the resolution of the net can be chosen to remain constant.

We also ask the question of finding explicitly such a "quantitatively performant" universal entangler.

2. Almost randomizing channels

Let $\Phi : \mathcal{M}(\mathbf{C}^d) \to \mathcal{M}(\mathbf{C}^d)$ be a quantum channel. The Kraus rank of Φ is the minimal N so that Φ admits a decomposition of the form
$$\Phi(X) = \sum_{i=1}^{N} A_i X A_i^\dagger.$$
The Kraus rank of Φ is always bounded by d^2. We are interested in the following class of channels, introduced in [5]. A channel Φ is called ε-randomizing if $||\Phi(\rho) - \rho_*||_\infty \leq \varepsilon/d$ for every state ρ (here $\rho_* = Id/d$ denotes the maximally mixed state). As in the previous section, the intersection theorem for projective varieties implies that a channel mapping every state to a full rank state has Kraus rank at least $2d-1$. As before, this can be made quantitative using probabilistic arguments.

Theorem [5, 1]. Let $(U_i)_{1\leq i\leq N}$ be N independent random Haar-distributed random unitary $d\times d$ matrices. If $N \geq Cd/\varepsilon^2$, then the random channel
$$\Phi(X) = \frac{1}{N}\sum_{i=1}^{N} U_i X U_i^\dagger$$
is ε-randomizing with large probability.

The proof mimics the previous one, except than one needs the Bernstein inequalities to estimate individual probabilities (see [5, 1]).

This theorem is optimal for such random channels. A possibly hard question is to derandomize the construction: how to find deterministically a single channel using $O(d)$ Kraus operators with the (generic) property of being ε-randomizing ? As a step in this direction, we give a construction using less random bits, using Pauli matrices. A random Pauli matrix on $(\mathbf{C}^2)^{\otimes k}$ is by definition, the tensor product of k independent random Pauli matrices.

Theorem [1]. Let $(P_i)_{1\leq i\leq N}$ be N independent random Pauli matrices on $(\mathbf{C}^2)^{\otimes k}$. Let $d = 2^k$. If $N \geq C(\varepsilon) d \log^6 d$, then the random channel

$$\Phi(X) = \frac{1}{N}\sum_{i=1}^{N} P_i X P_i^\dagger$$

is ε-randomizing with positive probability.

Here the proof requires more advanced tools, such that Dudley's entropic integral and fine properties of covering numbers. Note the presence of (needed) extra logarithmic factors. On the other hand, this gives examples of ε-randomizing channels with *local* Kraus operators, a property which is important for some applications.

References

[1] G. Aubrun, *On almost randomizing channels with a short Kraus decomposition*, Comm. Math. Phys. **288**, 1103-1116 (2009).

[2] G. Aubrun and I. Nechita, unpublished.

[3] J. Chen, R. Duan, Z. Ji, M. Ying and J. Yu, *Existence of universal entangler*, J. Math. Phys. **49**, 012103 (2008).

[4] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52** (1977).

[5] P. Hayden, D. Leung, P. W. Shor and A. Winter, *Randomizing quantum states: constructions and applications*, Comm. Math. Phys. **250**, 371–391 (2004).

[6] G. Pisier, *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics **94** (1989).

Entanglement Catalysis

Guillaume Aubrun

We review some results about *entanglement catalysis*, due to Turgut and Aubrun-Nechita.

Let ψ be a pure state on a bipartite quantum system $H_A \otimes H_B$. We write λ_ψ for the vector of Schmidt coefficients of ψ. Note that λ_ψ is a probability vector (i.e. a vector with positive coordinates summing up to 1). We write P_d for the set of probability vectors of dimension d. If $x, y \in P_d$, we write $x \prec y$ if y majorizes x. This means that there exists a bistochastic matrix B so that $x = By$. There are other equivalent definitions, cf [5, 3].

Majorization is very relevant to quantum information theory (QIT), as the following theorem due to Nielsen [4] shows. If ψ is a quantum state shared by Alice and Bob, it is possible for them to LOCC-transform it into another quantum state ϕ if and only if $\lambda_\psi \prec \lambda_\phi$.

The interplay of majorization and tensor product leads to phenomena which are surprising from the QIT point of view. Consider for example the following definitions:

- If $x, y \in P_d$, one writes $x \prec_C y$ if there exists $z \in P_k$ for some $k \in \mathbf{N}$ so that $x \otimes z \prec y \otimes z$. Such a vector z is called a *catalyst*.
- If $x, y \in P_d$, one writes $x \prec_M y$ if there exists a integer $n \geq 1$ so that $x^{\otimes n} \prec y^{\otimes n}$.

One can find examples of vectors x, y so that $x \prec_C y$ (resp. $x \prec_M y$) while $x \not\prec y$. Physically, this means that some LOCC transformations are possible only in presence of a certain catalyst in the environnement (resp. only when we simultaneous perform n such transformations). This raises a question: given x and y, how can we check that $x \prec_C y$ or $x \prec_M y$? It is known that $x \prec_M y$ implies $x \prec_C y$.

For $p \in \mathbf{R}$ and $x \in P_d$, we define $N_p(x)$ as $\sum_{i=1}^d x_i^p$. Note that $N_p(x \otimes y) = N_p(x)N_p(y)$. Now consider the following conditions

(A) For every $p \geq 1$, $N_p(x) \leq N_p(y)$.
(B) For every $p \in [0, 1]$, $N_p(x) \geq N_p(y)$.
(C) For every $p \leq 0$, $N_p(x) \leq N_p(y)$.

It is easy to check that $x \prec_C y$ or $x \prec_M y$ implies that conditions (A), (B), (C) hold. The converse cannot be true since it is known that the relation $\prec_C$ is not a closed relation. We however have the following results.

Theorem 1 [6]. Let $x, y \in P_d$ with nonzero coordinates. The following are equivalent

(1) Conditions (A), (B), (C) hold.
(2) There is a sequence (x_n) in P_d tending to x so that $x_n \prec_C y$ for every n.

Actually Turgut's result is more precise, see [6].

Theorem 2 [2]. Let $x, y \in P_d$ with nonzero coordinates. The following are equivalent

(1) Conditions (A), (B) hold.
(2) There is a sequence (x_n) in P_{d+1} tending to x so that $x_n \prec_C y$ for every n.
(3) There is a sequence (x_n) in P_{d+1} tending to x so that $x_n \prec_M y$ for every n.

Theorem 3 [1, 2]. Let $x, y \in P_d$ with nonzero coordinates. The following are equivalent

(1) Condition (A) holds.
(2) There is a sequence (x_n) in P_{m_n} tending to x so that $x_n \prec_C y$ for every n (the dimension m_n is allows to go to infinity with n).
(3) There is a sequence (x_n) in P_{m_n} tending to x so that $x_n \prec_M y$ for every n (the dimension m_n is allows to go to infinity with n).

The convergence is understood with respect to the ℓ_1 norm.

The proof of Theorem 1 is based on a discretization argument. Amazingly, the key lemma is the following fact about polynomials: if a polynomial $P \in \mathbf{R}[X]$ satisfies $P(x) > 0$ for any $x \geq 0$, then P can be written as a quotient of polynomials with non-negative coefficients.

On the other hand, the proof of Theorems 2 and 3 is derived from a similar statement about probability measures (where stochastic domination replaces majorization and convolution replaces tensor product), which is a consequence of standard large deviations theory.

It is still an open question whether Theorem 1 holds for $\prec_M$ instead of $\prec_C$.

References

[1] G. Aubrun and I. Nechita, *Catalytic majorization and ℓ_p norms*, Comm. Math. Phys. **278** (2008), no. 1, 133–144.

[2] G. Aubrun and I. Nechita, *Stochastic ordering for iterated convolutions and catalytic majorization*, Ann. Inst. Henri Poincar **45**, 611-625 (2009).

[3] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997.

[4] M. Nielsen, *Conditions for a class of entanglement transformations*, Phys. Rev. Lett. **83**, 436 (1999).

[5] M. Nielsen, *An introduction to majorization and its applications to quantum mechanics*, preprint.

[6] S. Turgut, *Catalytic transformations for bipartite pure states*, J. Phys. A: Math. Theor. **40** 12185-12212, (2007).

Bell inequalities with and without entanglement

Ingemar Bengtsson

In entanglement theory we consider a system whose Hilbert space has dimension N^2, with the understanding that only a subgroup of its unitary group—$SU(N) \times SU(N)$—is under the control of the experimenters. Described in this way, this is a special case of a situation that has been much discussed in the context of coherent states. Coherent states form a very special orbit of some compact Lie group G corresponding to transformations under experimental control [1]. The question is to what extent entanglement can be illuminated if viewed in this more general context; interesting attempts in this direction have been provided by Klyachko and his coworkers [2], by Barnum et al. [3], and by Marek Kuś in his talk at this workshop.

The very special orbit that goes under the name of coherent state is, in itself, a Kähler space, and as such it admits a symplectic form and can serve as a classical phase space. In the case of entanglement it consists of the separable states. We refer to these states as the 'classical' states" [4]. Typically other very special orbits arise. The maximally entangled orbit on which the symplectic form vanishes is an example.

To prevent the canvas from being too large I confine myself to two instances of the general setup: spin coherent states, for which the dynamical group $G = SU(2)$, and bipartite entanglement. Among many similarities, let me mention that the convex hull of the spin coherent states shares with the convex hull of the separable states the property that the minimal faces are one dimensional line segments in both cases. It is easy to see why this is so, because the convex hull must be a subset of the convex hull of all pure states. In the latter, the minimal faces are Bloch balls. The minimal faces of the 'classical' states must be subsets of these Bloch balls, so the question whether they can be larger than just line segments comes down to the question whether three 'classical' states can be linearly dependent. In projective Hilbert space this is the question whether three 'classical' states can lie on a single projective line. For the spin coherent states a small calculation confirms that this never happens [5]. In the spin 1 case this can be seen directly, since the coherent states form a conic section in a complex projective plane. This conic section intersects any projective line in exactly two points (counting multiplicities). In the case of two entangled qubits the separable states form the Segre hyperboloid in projective 3-space, and a projective line will intersect the hyperboloid in more than two points only in the exceptional case that the whole line lies within the Segre hyperboloid. The case of arbitrary dimension behaves similarly [6]. Hence the two 'classical' convex hulls share two important properties: they have full dimension, and they have an "edgy" classical feel to them, much like a classical simplex.

At the other end of the spectrum we find the orbit on which the symplectic form vanishes. In the two simplest cases, dimension 3 for the spin coherent states and dimension 2×2 for entangled states, they form real projective spaces—and very obviously they share the property that the dimension of their convex hulls is smaller than that of the set of all mixed states, while their minimal faces are those of the set of real density matrices.

These analogies provide a beginning for a justification of the epithet 'classical'. But in entanglement theory the really important criterion for whether a pure state should be regarded as entangled or 'classical' is whether it violates a Bell inequality or not [7]. Does this criterion have an analogue when the dynamical group is $G = SU(2)$? Perhaps surprisingly, it has [8, 2]. The simplest case, that of a three dimensional Hilbert space, has been studied in detail. It is possible to use five Hilbert space vectors with an orthogonality graph in the form of a pentagram, and to argue using the logic of the Kochen-Specker theorem [9] that any realistic and non-contextual theory containing such observables must give rise to an inequality that, as a matter of fact, is violated by quantum mechanics. By varying the five vectors used to derive the inequality one can show that every state will violate some version of this inequality—with the exception of the spin coherent states, which in this way earn their epithet 'classical'.

The set of all pentagram inequalities, and the various uses one can make of them, were described in my talk, and in much more detail in a paper that I coauthored

[10]. Experimental realisation of the corresponding "Kochen-Specker paradox" is a challenging but perhaps not impossible task.

References

[1] A. M. Perelomov: *Generalized Coherent States and Their Applications*, Springer, Berlin 1986.

[2] A. Klyachko, *Dynamic symmetry approach to entanglement*, in J.-P. Gazeau et al. (eds): Proc. NATO Advanced Study Inst. on Physics and Theoretical Computer Science, IOS, Amsterdam 2007.

[3] H. Barnum, *Generalizations of entanglement based on coherent states and convex sets*, Phys. Rev. **A68** (2003), 032308.

[4] M. Kuś and I. Bengtsson, *'Classical' quantum states*, Phys. Rev. **A80** (2009), 022319.

[5] M. Sadiq, *The convex hull of the spin coherent states*, M.Sc. thesis, Stockholm University (2009).

[6] E. Alfsen and F. Shultz, *Unique decompositions, faces, and automorphisms of separable states*, arXiv:0906.1761.

[7] N. Gisin, *Bell's inequality holds for all non-product states*, Phys. Lett. **A154** (1991), 201.

[8] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, *A simple test for hidden variables in the spin-1 system*, Phys. Rev. Lett. **101** (2008), 020403.

[9] S. Kochen and E. Specker, *The problem of hidden variables in quantum mechanics*, J. Math. Mech. **17** (1967), 59.

[10] P. Badziąg, I. Bengtsson, A. Cabello, H. Granström, and J.-Å. Larsson, *Pentagrams and paradoxes*, arXiv:0909.4713.

Min- and Max- Relative Entropies of Entanglement

Nilanjana Datta

One of the fundamental quantities in Quantum Information Theory is the relative entropy between two states. Other entropic quantities, such as the von Neumann entropy of a state, the conditional entropy and the mutual information for a bipartite state, are obtainable from the relative entropy. Many basic properties of these entropic quantities can be derived from those of the relative entropy. The strong subadditivity of the von Neumann entropy, which is one of the most powerful results in Quantum Information Theory, follows easily from the monotonicity of the relative entropy. Other than acting as a parent quantity for other entropic quantities, the relative entropy itself has an operational meaning. It serves as a measure of distinguishability between different states.

A fundamental problem of Quantum Information Theory is the determination of optimal rates of information-processing tasks such as storage and transmission of information, or manipulation of entanglement, Traditionally, these were obtained under the assumptions that the underlying resources, e.g. information sources, communication channels and entanglement resources, are available for a large number of independent uses. In other words, it was assumed that resources were (1) memoryless and that (2) they were used an asymptotically large number of times. The optimal rates evaluated under these assumptions are known to be given in terms of entropic quantities which are all obtainable from the quantum relative entropy.

In reality, however, the above assumptions cannot be justified, since resources are used a finite number of times and there are unavoidable correlations between their successive uses. Hence, it is important to lift the above assumptions and evaluate optimal rates of information-processing tasks for a finite number (or even a single use) of the underlying resources. Such a rate is referred to as a one-shot rate. In this seminar we introduce two new relative entropy quantities, namely the *min- and max- relative entropies* which are seen to act as parent quantities for optimal rates of quantum information-processing tasks in the one-shot scenario.

The *max- relative entropy* of two operators ρ and σ, such that $\rho \geq 0$, $\mathrm{Tr}\rho \leq 1$ and $\sigma \geq 0$, is defined by

$$D_{\max}(\rho||\sigma) := \log \min\{\lambda : \rho \leq \lambda\sigma\} \tag{1}$$

The *min- relative entropy* of two operators ρ and σ, such that $\rho \geq 0$, $\mathrm{Tr}\rho \leq 1$ and $\sigma \geq 0$, is defined by

$$D_{\min}(\rho||\sigma) := -\log \mathrm{Tr}\big(\pi_\rho \sigma\big) \ , \tag{2}$$

where π_ρ denotes the projector onto $\mathrm{supp}\,\rho$, the support of ρ. It is well-defined if $\mathrm{supp}\,\rho$ has non-zero intersection with $\mathrm{supp}\,\sigma$. Note that

$$D_{\min}(\rho||\sigma) = \lim_{\alpha\to 0^+} S_\alpha(\rho||\sigma), \tag{3}$$

where $S_\alpha(\rho||\sigma)$ denotes the *quantum relative Rényi entropy* of order α, with $0 < \alpha < 1$, defined by:

$$S_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\rho^\alpha \sigma^{1-\alpha}. \tag{4}$$

These relative entropies, satisfy interesting mathematical properties which will be presented in the seminar. In particular, they are both non-zero when ρ and σ are states (i.e., positive operators of unit trace), and are monotonous under quantum operations (i.e., completely positive trace-preserving (CPTP) maps). The min- and max- relative entropies of two states ρ and σ are related to the quantum relative entropy $S(\rho||\sigma) := \mathrm{Tr}\big[\rho \log \rho - \rho \log \sigma\big]$ as follows:

$$D_{\min}(\rho||\sigma) \leq S(\rho||\sigma) \leq D_{\max}(\rho||\sigma). \tag{5}$$

The consideration of the min- and max- relative entropies lead naturally to the definition of two new entanglement monotones, defined below. Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are isomorphic Hilbert spaces of dimension d. For given orthonormal bases $\{|i^A\rangle\}_{i=1}^d$ and $\{|i^B\rangle\}_{i=1}^d$ a maximally entangled state of rank $M \leq d$ is given by

$$|\Psi_M^{AB}\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i^A\rangle|i^B\rangle.$$

Let $\mathcal{D}(\mathcal{H})$ denote the set of states and let $\mathcal{S}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})$ denote the set of separable states.

Definition 1. *The max-relative entropy of entanglement of $\rho \in \mathcal{D}(\mathcal{H})$ is given by*

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\max}(\rho||\sigma), \tag{6}$$

and its min-relative entropy of entanglement by

$$E_{\min}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\min}(\rho||\sigma), \tag{7}$$

We also define smoothed versions of the quantities we consider as follows

Definition 2. *For any $\varepsilon > 0$, the smooth max-relative entropy of entanglement of $\rho \in \mathcal{D}(\mathcal{H})$ is given by*

$$E^{\varepsilon}_{\max}(\rho) := \min_{\bar{\rho} \in B^{\varepsilon}(\rho)} E_{\max}(\bar{\rho}), \tag{8}$$

where $B^{\varepsilon}(\rho) := \{\bar{\rho} \in \mathcal{D}(\mathcal{H}) : F(\bar{\rho}, \rho) \geq 1 - \varepsilon\}$ and $F(\rho, \rho') := \mathrm{Tr}\sqrt{\rho^{\frac{1}{2}}\rho'\rho^{\frac{1}{2}}}$ denotes the fidelity of two states. .

The smooth min-relative entropy of entanglement of $\rho \in \mathcal{D}(\mathcal{H})$ is defined as

$$E^{\varepsilon}_{\min}(\rho) := \max_{\substack{0 \leq A \leq I \\ \mathrm{Tr}(A\rho) \geq 1-\varepsilon}} \min_{\sigma \in \mathcal{S}} \left(-\log \mathrm{Tr}(A\sigma)\right). \tag{9}$$

These smoothed max- and min- relative entropies of entanglement have interesting operational significances in one-shot entanglement manipulation under the so-called *non-entangling* or *separability-preserving* maps defined below.

Definition 3. *A completely positive trace-preserving (CPTP) map Λ is said to be a non-entangling (or separability preserving) map if $\Lambda(\sigma)$ is separable for any separable state σ. We denote the class of such maps by SEPP*[1]*.*

Such maps can be further generalized to yield the so-called δ-non-entangling maps, for any given $\delta > 0$.

Definition 4. *For any given $\delta > 0$ we say a map Λ is a δ-non-entangling map if $R_G(\Lambda(\sigma)) \leq \delta$ for every separable state σ. We denote the class of such maps by δ-SEPP.*

The smoothed min-relative entropy of entanglement is related to the one-shot distillable entanglement under non-entangling maps.

Theorem 1. *For any state ρ and any $\varepsilon \geq 0$,*

$$\lfloor E^{\varepsilon}_{\min}(\rho) \rfloor \leq E^{(1),\varepsilon}_{D,SEPP}(\rho) \leq E^{\varepsilon}_{\min}(\rho), \tag{10}$$

where $E^{(1),\varepsilon}_{D,SEPP}(\rho)$ denotes the one-shot distillable entanglement defined as follows:

$$E^{(1),\varepsilon}_{D,SEPP}(\rho) := \max_{M,\Lambda} \{\log M : F(\Lambda(\rho), \Psi_M) \geq 1 - \varepsilon, \Lambda \in SEPP, M \in \mathbb{Z}^+\}.$$

The max-relative entropy of entanglement is related to the one-shot catalytic entanglement cost under δ-non-entangling maps.

[1]The acronym comes from the name *separability preserving.*

Theorem 2. *For any $\delta > 0$ there exists a positive integer K, such that for any state ρ*

$$E^{\varepsilon}_{\max}(\rho\otimes\Psi_K)-\log K-\log(1+\delta) \le \widetilde{E}^{(1),\varepsilon}_{C,\delta-SEPP}(\rho) \le E^{\varepsilon}_{\max}(\rho\otimes\Psi_K)-\log K, \tag{11}$$

where $\widetilde{E}^{(1),\varepsilon}_{C,\delta-SEPP}(\rho)$ denotes the one-shot catalytic entanglement cost, defined as follows:

$$\tilde{E}^{(1),\varepsilon}_{C,SEPP}(\rho) := \min_{M,K,\Lambda} \quad \{\log M : \Lambda(\Psi_M \otimes \Psi_K) = \rho' \otimes \Psi_K, \\ F(\rho,\rho') \ge 1-\varepsilon, \Lambda \in SEPP, M, K \in \mathbb{Z}^+\}.$$

References

[1] N. Datta, "Min- and Max- Relative Entropies and a New Entanglement Monotone," IEEE, Trans. Inf. Theory, vol. 55, 2816, 2009.

[2] N. Datta, "Max- Relative Entropy of Entanglement, alias Log Robustness," Int. Jour. Q. Info, **7**, 475, 2009.

[3] F.G.S.L. Brandão and N. Datta, "One-shot entanglement manipulation under non-entangling maps," accepted for publication in IEEE Trans. Inf. Thy.

On additivity of minimal output entropy for quantum channels

Michał Horodecki

(joint work with Fernando Brandao, Andrzej Grudka and Łukasz Pankowski)

The question whether the minimal output entropy of quantum channels is additive had been open for quite a long time. While the most interesting case is when the entropy is the von Neumann one, the more general Rényi p-entropies (or equivalently p-norms) were also studied. Several additivity results had been first obtained for particular classes of channels, including unital qubit channels [7] and entanglement breaking channels [8] for the von Neumann entropy (cf. [1] for a more complete list). The first counterexample was obtained for $p \ge 4.79$ [9], for the so called Werner-Holevo channel. Subsequently Winter [10] proved nonadditivity for all $p > 2$ which was pushed by Hayden [6] until all $p > 1$ (See also [5]). Finally Hastings has shown nonadditivity for $p = 1$ which is the von Neumann entropy case [4].

However the counterexamples to additivity, apart from that of [9] are nonconstructive: they hold for randomly picked channels. Moreover the proof of nonadditivity by Hastings is quite complicated. In the paper [2] all the details are worked out explicitly, however the structure of the proof is the same as that of Hasting's. In particular, exact probability distribution of singular values of random bipartite state is used.

The purpose of the talk is two-fold. First I will quickly present a constructive counterexample for $p > 2$ given in [3]. It is based on antisymmetric subspace. For p tending to 2, the needed dimension tends to infinity. Second, I will present a simpler proof of Hastings' result given in [1]. The main difference is that we use large deviation bounds rather than exact probability distribution. Moreover, we consider a slightly different class of channels, extending thereby the range channels which violate additivity.

References

[1] F. G. S. L. Brandão and M. Horodecki. *On Hastings' counterexamples to the minimum output entropy additivity conjecture.* (2009) arXiv:0907.3210.
[2] M. Fukuda, C. King, and D. Moser. *Comments on Hastings' additivity counterexamples.* (2009), arXiv:0905.3697.
[3] A. Grudka, M. Horodecki, and Ł. Pankowski. *Constructive counterexamples to additivity of minimum output Rényi entropy of quantum channels for all* $p > 2$ (2009) arXiv:0911.2515.
[4] M. B. Hastings. *Superadditivity of communication capacity using entangled inputs.* Nature Phys., **5** (2009) 255–257, arXiv:0809.3972.
[5] P. Hayden and A. Winter. *Counterexamples to the maximal* p*-norm multiplicativity conjecture for all* $p > 1$. Comm. Math. Phys., **284** (2008) 263–280, arXiv:0807.4753.
[6] P. Hayden. *The maximal* p*-norm multiplicativity conjecture is false.* (2007) arXiv:0707.3291.
[7] Ch. King. *Additivity for a class of unital qubit channels.* (2001) quant-ph/0103156.
[8] P. W. Shor. *Additivity of the classical capacity of entanglement-breaking quantum channels* J. Math. Phys., **43** (2002) 4334–4340, quant-ph/0201149.
[9] R.F. Werner and A.S. Holevo. *Counterexample to an additivity conjecture for output purity of quantum channels.* (2002) quant-ph/0203003.
[10] Andreas Winter. *The maximum output* p*-norm of quantum channels is not multiplicative for any* $p > 2$. (2007) arXiv:0707.0402.

Geometry of the set of 'classical states'

Marek Kuś
(joint work with Ingemar Bengtsson)

We show that several classes of mixed quantum states in finite-dimensional Hilbert spaces which can be characterized as being, in some respect, 'most classical' can be described and analyzed in a unified way. Among the states we consider are separable states of distinguishable particles and unentangled states of indistinguishable fermions and bosons,

In the situation that we will be concerned with here, there is a Lie group G of preferred symmetries of the system. It may be the group $SU(N) \times SU(M)$ of local unitaries acting on a bipartite composite system or the single $SU(N)$ group in the case of indistinguishable particles; in both cases entanglement properties of the system are left unchanged. The group action in an appropriate Hilbert space $\mathcal{H}$ of the quantum system in question divides the set of states i.e. points in the projective space $\mathbb{P}(\mathcal{H})$ into disjoint orbits. There is be a special orbit which is in itself a symplectic and indeed a Kähler manifold with respect to the natural symplectic and complex structures on $\mathbb{P}(\mathcal{H})$ – the orbit through the highest weight vector of an irreducible representation of G on $\mathcal{H}$ ('coherent states' [1]). In all considered cases these are exactly nonentangled states of a composite system.

The nonentangled mixed states are defined by first identifying pure states, i.e. points in the projective space with one dimensional orthogonal projectors

$$\mathbb{P}(\mathcal{H}) \ni [v] \mapsto P_v \in End(\mathcal{H}), \quad P_v \cdot v = v, \quad P_v \cdot v^{\perp} = 0, \tag{1}$$

for $v \in \mathcal{H}$ corresponding to $[v] \in \mathbb{P}(\mathcal{H})$ and $v^\perp$ orthogonal to v, and taking the convex hull of the pure unentangled states.

The vectors v on the orbit through the highest weight vector can be characterized in terms of a positive semidefinite linear operator L acting on $\mathcal{H} \otimes \mathcal{H}$ [2]. To this end we consider the root-space decomposition of $\mathfrak{g}_{\mathbb{C}}$ – the Lie algebra of the complexification of the group G,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha} \tag{2}$$

take a normalized basis in $\mathfrak{g}_{\mathbb{C}}$ fulfilling $B(X_\alpha, X_{-\alpha}) = 1 = B(H_i, H_i)$, $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, $H_i \in \mathfrak{t}$, where B is the Killing-Cartan form, and finally define

$$\mathbf{L} = C_2 \otimes I + I \otimes C_2 + 2\sum_{\alpha>0} \Big(X_\alpha \otimes X_{-\alpha} + X_{-\alpha} \otimes X_\alpha \Big) + 2\sum_{i=1}^{r} H_i \otimes H_i, \tag{3}$$

where C_2 is the quadratic Casimir operator

$$C_2 = \sum_{\alpha>0} \Big(X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha \Big) + \sum_{i=1}^{r} H_i H_i. \tag{4}$$

A vector $v \in \mathcal{H}$ belongs to the orbit through the highest weight vector if and only if

$$\mathbf{L}(v \otimes v) = \langle 2\boldsymbol{\lambda} + 2\boldsymbol{\delta}, 2\boldsymbol{\lambda} \rangle (v \otimes v), \tag{5}$$

where $\boldsymbol{\lambda}$ is the highest weight, $\boldsymbol{\delta}$ – the half-sum of positive roots and $\langle \cdot, \cdot \rangle$ – the corresponding Killing-Cartan form in the dual space $\mathfrak{g}^*_{\mathbb{C}}$.

$\mathbf{L}$ is a positive semidefinite operator on $\mathcal{H} \times \mathcal{H}$ and $\langle 2\boldsymbol{\lambda} + 2\boldsymbol{\delta}, 2\boldsymbol{\lambda} \rangle$ is its largest eigenvalue. Thus the operator $A = \langle 2\boldsymbol{\lambda} + 2\boldsymbol{\delta}, 2\boldsymbol{\lambda} \rangle \mathbb{I} - \mathbf{L}$ is positive semidefinite and can be used to characterize pure nonentangles states *via*

$$\mathrm{Tr} A P_v = 0, \tag{6}$$

i.e. by expectation values of a physical observable.

To the positive semidefinite $A \in End(\mathcal{H} \otimes \mathcal{H})$ there corresponds *via* the Jamiołkowski-Choi isomorphism [3] a completely positive operator $\Lambda \in End\big(End(\mathcal{H})\big)$. Its complete positivity is equivalent to the existence of the Kraus decomposition [4]

$$\Lambda(\rho) = \sum_{\mu=1} T_\mu \rho T_\mu^\dagger, \quad T_\mu \in End(\mathcal{H}), \tag{7}$$

for an arbitrary (mixed) state $\rho \in End(\mathcal{H})$. The expectation value $C(P_v) := \mathrm{Tr} A P_v$ for a pure state P_v can be expressed as

$$C(P_v) = \sum_{\mu=1} \frac{|\langle v | T_\mu \cdot \overline{v} \rangle|^2}{\langle v | v \rangle}, \tag{8}$$

where $\langle \cdot | \cdot \rangle$ is the scalar product in $\mathcal{H}$.

In order to characterize mixed unentangled states we invoke the following simple theorem [5]

Theorem 1. *Let E be the set of all extreme points of a compact convex set K in a finite dimensional real vector space V. For every non-negative function $f : E \to \mathbb{R}_+$ we may define its extension $f_K : K \to \mathbb{R}_+$ by*

$$f_K(x) = \inf_{x=\sum p_i x_i} \sum p_i f(x_i) \tag{9}$$

where the infimum is taken with respect to all expressions of x in the form of convex combinations of points x_i from E. Let now E_0 be a compact subset of E with the convex hull $K_0 = conv(E_0) \subset K$. If f is continuous and vanishes exactly on E_0, then the function f_K is convex on K and vanishes exactly on K_0.

In our cases K is the set of all states, E – the set of pure states, and E_0 – the set of pure 'classical' states. Consequently we may characterize the mixed unentangled states as those with vanishing

$$C_A(\rho) = \inf \sum_{k=1}^{K} p_k C_A(P_k), \tag{10}$$

where the infimum is taken over all their convex decompositions into rank one projectors

$$\rho = \sum_{k=1}^{K} p_k P_k, \quad p_k > 0, \quad \sum_{k=1}^{K} p_k = 1. \tag{11}$$

Using (10) together with (8) we (a) recovered the measures of entanglement for distinguishable and indistinguishable particles [6], known in the literature in cases of low-dimensional (c.f. $\mathcal{H}$ [7],[8],[9]), (b) constructed appropriate measures for arbitrary dimensions and arbitrary number of subsystems [10], (c) gave effective estimates for $C_A(\rho)$ allowing for discrimination between entangled and unentangled states.

References

[1] A. Perelomov, Generalized coherent states and their applications, Springer, Heidelberg (1986).

[2] W. Lichtenstein, *A system of quadrics describing the orbit of the highest weight vector*, Proc. Am. Math. Soc. **84** (1982), 605–608.

[3] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefinite operators*, Rep. Math. Phys. **3** (1972), 275–278.

[4] M. D. Choi, *Completely positive linear maps on complex matrices*, Linear Alg. Appl. **10** (1975), 285–290.

[5] J. Grabowski, M. Kuś, and G. Marmo, *Geometry of quantum systems: density states and entanglement* J. Phys. A: Math. Gen. **38** (2005), 10217–1024.

[6] M. Kuś and I. Bengtsson *"Classical" quantum states*, Phys. Rev. A **80** (2009), 022319.

[7] W. K. Wootters, *Entanglement of formation of an arbitrary state of two qubits*, Phys. Rev. Lett. **80** (1998), 2245–2248.

[8] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein, and D. Loss, *Quantum correlations in two-fermion systems*, Phys. Rev. A **64** (2001), 022303.

[9] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein, *Quantum correlations in systems of identical particles*, Ann. Phys. **299** (2002), 88–127.

[10] M. Kotowski, M. Kotowski, and M. Kuś, *Universal nonlinear entanglement witnesses*, to be published.

Properties of extremal positive maps on matrix algebras

Marcin Marciniak

Let K and H denote complex Hilbert spaces, and let $B(K)$ and $B(H)$ be C^*-algebras of bounded linear operators on K and H respectively. By $B(K)^+$ and $B(H)^+$ we denote the convex cones of positive definite operators from $B(K)$ and $B(H)$ respectively. Consider a bounded linear map $\phi : B(K) \to B(H)$. We say that ϕ is a *positive map* if $\phi(B(K)^+) \subset \phi(B(H)^+)$. For any $k \in \mathbb{N}$ we denote by $M_k(B(K))$ (respectively $M_k(B(H))$) the C^*-algebra of square $k \times k$-matrices with coefficients from $B(K)$ (respectively $B(H)$). We say that the map ϕ is *k-positive* if $M_k(B(K)) \to M_k(B(H)) : [X_{ij}] \mapsto [\phi(X_{ij})]$ is a positive map. Analogously, ϕ is said to be *k-copositive* if the map $M_k(B(K)) \to M_k(B(H)) : [X_{ij}] \mapsto [\phi(X_{ji})]$ positive. The map ϕ is called *completely positive* (respectively *completely copositive*) if it is k-positive (respectively k-copositive) for any $k \in \mathbb{N}$.

Let $\mathcal{P}(K, H)$ denote the convex cone of positive maps acting from $B(K)$ into $B(H)$. The main purpose of our talk is describe some properties of extremal elements of this cone. Let us remind that only few examples of extremal positive maps are known. The most familiar examples are of the form $\phi(X) = AXA^*$ and $\phi(X) = AX^TA^*$, where $A : K \to H$ is some bounded linear operator while T stands for the transposition. The convex hull of maps of the first form is precisely the cone of completely positive (CP) maps while maps of the second form generate the cone completely copositive (coCP) maps. Our first result says that extremal positive maps which are not of the above two forms have are highly nonregular: they are not 2-positive.

Further, we analyse rank properties of extremal positive maps. It turns out the CP and coCP extremal maps can be characterized as those positive maps which have the property that $\operatorname{rank} \phi(P) \leq 1$ for any 1-dimensional projection P from $B(K)$, i.e ϕ is rank-1 nonincreasing. It leads to the conclusion that nondecomposable extremal positive maps can be described as those extremal maps with $\operatorname{rank} \phi(P) \geq 2$ for at least one 1-dimensional projection P.

Next we prove that under the additional assumption that ϕ is locally completely positive (LCP) if ϕ is extremal then either ϕ is rank-1 nonincreasing or $\operatorname{rank} \phi(P) \neq 1$ for any 1-dimensional projection P. As a consequence we get a partial solution of Robertson's prolem: we show that if $\operatorname{rank} \phi(P) = 1$ for some 1-dimensional projection P and ϕ is extremal and LCP then it is automatically CP.

References

[1] M. Marciniak, *On extremal positive maps between type I factors*, Banach Center Publ. (to appear).

[2] A. G. Robertson, *Schwarz inequalities and the decomposition of positive maps on C^*-algebras*, Math. Proc. Camb. Phil. Soc. 94 (1983), 291–296.

[3] E. Størmer, *Decomposable positive maps on C^*-algebras*, Proc. Amer. Math. Soc. 86 (1980), 402–404.

A unified treatment of the convexity of relative entropy and certain trace functionals, with conditions for equality

Mary Beth Ruskai
(joint work with Anna Jenčová)

We introduce a generalization of relative entropy derived from the Wigner-Yanase-Dyson entropy for $p \in (0, 2)$; for $p = 1$ it reduces to the usual relative entropy. For positive definite A we introduce the operations of left and right multiplication denoted by L_A and R_A respectively. We then give a simple, self-contained proof that the map $(X, A, B) \mapsto \text{Tr}X^*(L_A + tR_B)^{-1}X$ is convex for any $t \in \mathbf{R}$. Combining this with easily verified integral representations, yields a proof of the joint convexity of our function. Moreover, special cases yield the joint convexity of relative entropy, and for the map $(A, B) \mapsto \text{Tr}K^*A^pKB^{1-p}$, Lieb's [4] joint concavity for $0 < p < 1$ and Ando's [1] joint convexity for $1 < p < 2$.

This approach allows us to obtain conditions for equality, and easily demonstrate that the conditions are independent of p. We also obtain conditions for equality in a number of inequalities which follow from our convexity results. These include monotonicity under partial traces, and some Minkowski type matrix inequalities proved by Lieb and Carlen [2] for mixed (p, q) norms. For extensions to three spaces the equality conditions are identical to the conditions for equality [3] in the strong subadditivity of quantum entropy.

References

[1] T. Ando, "Concavity of certain maps on positive definite matrices and applications to Hadamard products", *Lin, Alg. Appl.* **26**, 203-241 (1979)
[2] Eric A. Carlen, Elliott H. Lieb "A Minkowski Type Trace Inequality and Strong Subadditivity of Quantum Entropy II: Convexity and Concavity" *Lett. Math. Phys.* **83**, 107–126 (2008). arXiv:0710.4167
[3] P. Hayden, R. Jozsa, D. Petz, A. Winter "Structure of states which satisfy strong subadditivity of quantum entropy with equality" *Commun. Math. Phys.* **246**, 359–374, (2004)4 arXiv:quant-ph/0304007
[4] E. H. Lieb, "Convex trace functions and the Wigner-Yanase-Dyson conjecture", *Adv. Math.* **11**, 267–288 (1973).

The role of permutational symmetry in violations of additivity

Mary Beth Ruskai

This talk describes some open questions arising from attempts to extend the construction in [1] of channels described at the end of M. Horodecki's talk. Channels of this type can be found which violate additivity of minimal Renyi entropy for any $p > 2$. Although this has an obvious extension to multiple tensor products, the analysis becomes more complex when the output space is $\mathbf{C}_d^{\otimes m}$, the environment is $\mathbf{C}_d^{\otimes(N-m)}$ and the input remains the anti-symmetric subspace of $\mathbf{C}_d^{\otimes N}$. For channels with such Stinespring representations, the output is equivalent to the convex set of density matrices called N-representable by quantum chemists. Characterizing this set in the case $m = 2$ is a long-standing open question which was recently shown to be QMA-complete.

Roughly speaking, one expects the anti-symmetry associated with fermions to restrict the purity of the output. However, it is well-known that for $m = 2$, there are states for which one eigenvalue is close to $1/N$ associated with a type of pairing called "boson condensation". This suggests that it might be more useful to consider the case $m = 3$, about which very little is known. Although the N-representability problem is extremely challenging, only the extreme points need to be characterized to study optimal output purity of channels of the type described above. Even if these channels do not violate additivity for $p \leq 2$, there is much to be learned by studying them.

References

[1] Andrzej Grudka, Michael Horodecki, Lukasz Pankowski "Constructive counterexamples to additivity of minimum output Ryi entropy of quantum channels for all $p > 2$" arXiv:0911.2515

Dualities and positivity in the study of quantum entanglement

Łukasz Skowronek

(joint work with Erling Størmer and Karol Życzkowski)

The aim of the talk was to discuss the important role played by convex cone dualities and positivity conditions in the theory of quantum information.

Let $\mathcal{H}$ be a finite-dimensional ($\dim V = d$) vector space over $\mathbb{C}$ equipped with a Hermitian inner product. Denote with $|\psi_+\rangle$ the maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|\alpha\rangle|\alpha\rangle$ in $\mathcal{H}\otimes\mathcal{H}$. Convex cone dualities and some specific properties of the cones of k-superpositive and k-positive maps are crucial for the following theorem [1],

Theorem 3. *Let $\Phi \in J^{-1}(H(V\otimes V))$ and $k \in \mathbb{N}$. The following conditions are equivalent:*

(1) $\Phi \in \mathcal{SP}_k(\mathcal{H})$,

(2) $\Psi\circ\Phi \in \mathcal{SP}_k(\mathcal{H})\ \forall_{\Psi\in\mathcal{P}_k(\mathcal{H})}$,

(3) $\Psi \circ \Phi \in \mathcal{CP}(\mathcal{H})\ \forall_{\Psi \in \mathcal{P}_k(\mathcal{H})}$,
(4) $\mathrm{Tr}\left(|\psi_+\rangle\langle\psi_+|\left(\not{\Vdash} \otimes \Psi \circ \Phi\right)\left(|\psi_+\rangle\langle\psi_+|\right)\right) \geqslant 0\ \forall_{\Psi \in \mathcal{P}_k(\mathcal{H})}$.

The symbol J denotes the Jamiołkowski isomorphism,

$$J : L\left(L(V), L(V)\right) \ni \Phi \mapsto (\Phi \otimes \not{\Vdash})\, |\psi_+\rangle \langle\psi_+| \in L(V \otimes V), \tag{1}$$

whereas $\mathcal{SP}_k(\mathcal{H})$, $\mathcal{CP}(\mathcal{H})$ and $\mathcal{P}_k(\mathcal{H})$ stand for k-superpositive, completely positive and k-positive maps, resp. [1]. Theorem 3 can easily be used to prove a generalization of the positive maps criterion by Horodeccy [2] to k-positive maps,

Proposition 1 (k-positive maps criterion)**.** *Let $\rho \in H(V \otimes V)$. The operator ρ is k-entangled if and only if it satisfies*

$$(\Phi \otimes \not{\Vdash})\, \rho \geqslant 0\, \forall_{\Psi \in \mathcal{P}_k(V)}. \tag{2}$$

An even further generalization of the positive maps criterion has recently been found by Størmer, who uses a special class of so-called *mapping cones* [3] to prove results similar to 3. Mapping cones are cones of maps s.t. $\Phi \circ \Psi \circ \Upsilon$ is an element of the cone whenever Ψ is in the cone and Φ, Υ are completely positive. In particular, in [4] it was proved that an analogue of point 3) in Theorem 3 holds for all symmetric mapping cones, i.e. cones that are invariant w.r.t. the adjoint $^* : \Phi \mapsto \Phi^*$ and transpose $t : \Phi \mapsto t \circ \Phi \circ t$. It should be noticed that both the sets $\mathcal{P}_1(\mathcal{H})$ and $\mathcal{P}_2(\mathcal{H})$ play an important role in the theory of quantum information. The set of positive maps, $\mathcal{P}_1(\mathcal{H})$ is very closely related to the set of entanglement witnesses (see e.g.[6]), whereas the set of 2-positive maps is in correspondence with the set of one-copy undistillable quantum states [5].

It was explained how the set of positive maps or entanglement witnesses can be described using a system of polynomial inequalities. In the example of a three-parameter family of operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with matrices

$$[A(a,b,c)] = \begin{bmatrix} A_{00,00} & A_{00,01} & A_{00,10} & A_{00,11} \\ A_{01,00} & A_{01,01} & A_{01,10} & A_{01,11} \\ A_{10,00} & A_{10,01} & A_{10,10} & A_{10,11} \\ A_{11,00} & A_{11,01} & A_{11,10} & A_{11,11} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & a & 0 & 0 \\ \bar{a} & \frac{1}{2} & b & 0 \\ 0 & \bar{b} & \frac{1}{2} & c \\ 0 & 0 & \bar{c} & \frac{1}{2} \end{bmatrix}, \tag{3}$$

it was demonstrated that the polynomial inequalities can sometimes be solved explicitly [7]. In this way, explicit conditions for matrices (3) to correspond to an entanglement witness have been obtained in terms of a, b, c.

References

[1] Ł. Skowronek, E. Størmer, K. Życzkowski, *Cones of positive maps and their duality relations*, J. Math. Phys. 50 (2009), 062106

[2] M. Horodecki, P. Horodecki, R. Horodecki, *Separability of mixed states: Necessary and sufficient conditions*, Phys. Lett. A 223 (1996), 1

[3] E. Størmer, *Duality of cones of positive maps*, Münster J. of Math. 2 (2009), 299-310

[4] E. Størmer, *Mapping cones of positive maps*, preprint arXiv:0906.0472

[5] L. Clarisse, *Characterization of distillability of entanglement in terms of positive maps*, Phys. Rev. A 71 (2005), 032332

[6] I. Bengtsson, K. Życzkowski, *Geometry of Quantum States. An introduction to Quantum Entanglement*, Cambridge University Press 2006

[7] Ł. Skowronek, K. Życzkowski, *Positive maps, positive polynomials and entanglement witnesses*, J. Phys. A: Math. Theor. 42 (2009), 325302

Contribution to discussion: On conditions for block positivity

Hans-Juergen Sommers

One is interested in conditions for positive maps or equivalently for block positive matrices M with K^2 blocks M_{ij} of size $N \times N$ or N^2 blocks M^{mn} of size $K \times K$. Thus the matrix M with $M_{ij} = M_{ji}^\dagger$ or $M^{mn} = M^{nm\dagger}$ is a Hermitian matrix of size $KN \times KN$. In his talk on this miniworkshop Stanisław Szarek has put forward the conjecture that block positivity, i.e.

(1) $$\sum_{i,j=1}^{N} \bar{t}_i t_j M_{ij} \geq 0 \quad \text{in matrix sense} \quad \forall t_i \in C$$

implies

(2) $$(\mathrm{Tr} M)^2 \geq \mathrm{Tr} M^2$$

i.e. positivity of the second coefficient of the characteristic polynomial of M. Here I will give a proof for that and generalize it to the statement

(3) $$M_{ii} \geq 0, \quad \mathrm{Tr} M_{ii} \mathrm{Tr} M_{jj} \geq \mathrm{Tr} M_{ij} M_{ij}^\dagger \quad \text{for } i,j = 1,2,\ldots,N\ .$$

For $i = j$ this follows from the positivity of M_{jj}, which is a simple consequence of (1), but for $i \neq j$ (3) is not trivial. It will be enough to prove (3) for $i = 1$ and $j = 2$.

Let me write proposition (1) in a slightly different form

(4) $$\langle \xi | M_{ij} | \xi \rangle \geq 0 \quad \text{in matrix sense} \ \ \forall \xi \in C^K\ .$$

This implies

$$\langle \xi | M_{11} | \xi \rangle \langle \xi | M_{22} | \xi \rangle - |\langle \xi | M_{12} | \xi \rangle|^2 \geq 0 \quad \forall \xi \in C^K\ .$$

M_{12} is a complex matrix and thus can be written in polar decomposition as $M_{12} = \sqrt{M_{12} M_{12}^\dagger} U$ with some unitary matrix U. To get rid of phases we choose a special basis $U|n\rangle = \mathrm{e}^{i\phi_n}|n\rangle$. Then we find $|\langle m | M_{12} | n \rangle|^2 = |\langle m | \sqrt{M_{12} M_{12}^\dagger} | n \rangle|^2 \leq \langle m | \sqrt{M_{12} M_{12}^\dagger} | m \rangle \langle n | \sqrt{M_{12} M_{12}^\dagger} | n \rangle = |\langle m | M_{12} | m \rangle| |\langle n | M_{12} | n \rangle$ where we have used Cauchy-Schwartz inequality. Now we may use proposition (4) (for $i = 1$ and $j = 2$ and $|\xi\rangle = |n\rangle$ or $|m\rangle$) to find
$|\langle m | M_{12} | n \rangle|^2 \leq \sqrt{\langle m | M_{11} | m \rangle \langle m | M_{22} | m \rangle \langle n | M_{11} | n \rangle \langle n | M_{22} | n \rangle} \leq \frac{1}{2}(\langle m | M_{11} | m \rangle \langle n | M_{22} | n \rangle + \langle m | M_{22} | m \rangle \langle n | M_{11} | n \rangle)$.
This is valid for all $|m\rangle, |n\rangle$ including $|m\rangle = |n\rangle$. Summing over all m and n we obtain (3) for $i = 1$ and $j = 2$. This completes the proof of (3). Summing (3) over all i and j one finds Szarek's conjecture (2). One should mention, that by switching indices one proves in the same way

(5) $$M^{mm} \geq 0, \quad \mathrm{Tr} M^{mm} \mathrm{Tr} M^{nn} \geq \mathrm{Tr} M^{mn} M^{mn\dagger} \quad \text{for } m,n = 1,2,\ldots,K.$$

So we have found a set of necessary conditions (3) and (5) for block positivity. One may ask, wether it is possible to extend them to sufficient conditions. As reported by Łucasz Skowronek in his talk on this workshop this might be difficult since it is not possible to derive sufficient conditions using only a finite number of conditions of type $\sum_{ij} \bar{t}_i t_j \langle\xi|M_{ij}|\xi\rangle \geq 0$ (this means a finite number of vectors t and ξ). If (3) and (5) are not already sufficient one might try as the next candidate

$$\begin{aligned}(6) \qquad & \mathrm{Tr}M_{ii}\mathrm{Tr}M_{jj}\mathrm{Tr}M_{kk} + \mathrm{Tr}M_{ij}M_{jk}M_{ki} \\ & +\mathrm{Tr}M_{ik}M_{kj}M_{ji} - \mathrm{Tr}M_{ij}M_{ji}\mathrm{Tr}M_{kk} \\ & -\mathrm{Tr}M_{ik}M_{ki}\mathrm{Tr}M_{jj} - \mathrm{Tr}M_{jk}M_{kj}\mathrm{Tr}M_{ii} \geq 0\end{aligned}$$

which may diagrammatically be obtained as disconnected closed Fermion 1-loops with directed line $-M_{ij}$ from i to j and for each closed cycle (eg. (i) or (ij) or (ijk)) one writes a trace Tr and a -1 sign for a Fermion loop. Note that (6) is locally unitarily invariant and at least valid in some extremal cases. If inequalities (6) are valid for all i, j, k including coinciding indices, (6) may be summed up to get positivity of the third coefficient of the characteristic polynomial of M:

$$(\mathrm{Tr}M)^3 + 2(\mathrm{Tr}M^3) - 3\mathrm{Tr}M^2\mathrm{Tr}M \geq 0\ .$$

It is obvious how to generalize (6) to higher powers of M, but one would expect them to be true only for powers of M not larger than K, or the corresponding inequalities with switched indices l, m, n for powers of M not larger than N. Otherwise one could possibly derive positivity for all coefficients of the characteristic polynomial for M, which means full positivity.

Geometry in multidimensional spaces and the set of entangled states

Stanislaw J. Szarek

We present a selections of notions and results of Geometric Functional Analysis (a.k.a. Asymptotic Geometric Analysis, or AGA) that are relevant to Quantum Information Theory (QIT), in particular those that allow to calculate various geometric parameters (or reasonably estimate them) for various sets of quantum states and quantum channels, giving insights as to their shape/size and the like.This includes volume radii, mean widths, ellipsoids associated in canonical ways with convex bodies, the concentration of measure phenomenon, and the role of duality. Crucial aspects are the high dimensionality of the spaces in which our objects live, or on which our functions act , and the usually present convexity hypotheses. [Further applications of these and similar connections between AGA and QIT will be exemplified in talks by G. Aubrun and E. Werner.]

Finally, we will state a few questions. (One of the questions was answered during the workshop by Hans-Juergen Sommers, who gave a lecture presenting the solution and further problems suggested by the solution.)

References

[1] G. Aubrun and S. Szarek, *Tensor products of convex sets and the volume of separablestates on N qudits.* Phys. Rev. A. **73**, 022109 (2006); eprint arXiv.org:quant-ph/0503221

[2] L. Gurvits and H. Barnum, *Largest separable balls around the maximally mixed bipartite quantum state.* Phys. Rev. A 66, 062311 (2002).

[3] L. Gurvits and H. Barnum, *Better bound on the exponent of the radius of the multipartite separable ball.* Phys. Rev. A 72, 032322 (2005).

[4] R. Hildebrand, *Entangled states close to the maximally mixed state.* Phys. Rev. A 75, 062330 (2007); eprint arXiv:quant-ph/0702040

[5] S. Szarek, *The volume of separable states is super-doubly-exponentially small in the number of qubits.* Phys. Rev. A **72**, 032304 (2005); ; eprint arXiv.org:quant-ph/0310061

[6] S. Szarek, E. Werner and K. Życzkowski, *Geometry of sets of quantum maps: a generic positive map acting on a high-dimensional system is not completely positive.* J. Math. Phys. 49, 032113 (2008); eprint arXiv.org:0710.1571

Entanglement of bosons and fermions - two qubits and larger systems

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(joint work with Markus Tiersch, Fernando de Melo, Marek Kuś, Florian Mintert and Andreas Buchleitner)

Entanglement manifests itself most counterintuitively in measurements at remote detectors which exhibit correlations stronger than allowed by realistic local theories [1]. The entanglement between particles of a different type, like electrons and protons, i.e. non-identical particles, can be defined rigorously [2] since every particle and therewith every subsystem is assigned one unique Hilbert space. The wave function of identical particles, however, has to be (anti)symmetrized for (fermions) bosons. Hence this necessary identification of individual subsystems with single particle Hilbert spaces breaks down. The formal labels corresponding to the Hilbert spaces have no physical discriminating power anymore. Instead, one must find distinctive properties that allow us to distinguish subsystems in order to assign an identity to the particles and therewith to be able to apply an entanglement measure.

As we argue here, a natural subsystem structure for two identical particles is the one induced by the measurement process: The particles measured by a pair of detectors are the entities between which entanglement can be defined. We assume that the detectors are described by projection operators $\hat{O}_L$ and $\hat{O}_R$. The local single-particle observables measured by this pair of detectors then correspond to the expectation values of the following operator:

$$\hat{O}_d(\hat{\alpha},\hat{\beta}) = \underbrace{\hat{O}_L \otimes \hat{\alpha}}_{1} \otimes \underbrace{\hat{O}_R \otimes \hat{\beta}}_{2} + \underbrace{\hat{O}_R \otimes \hat{\beta}}_{1} \otimes \underbrace{\hat{O}_L \otimes \hat{\alpha}}_{2} , \tag{1}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are observables on the internal degrees of freedom of the particles, such as the spin. The formal particle labels (1) and (2) in Eq. (1) do not allow to distinguish the particles physically. Due to the symmetrization postulate, the observable has to be symmetric in the particle labels.

The information on possible measurement outcomes at the two detectors is contained in the detector-level density matrix ρ_d which can be reconstructed as follows:

$$\hat{\rho}_d = N \sum_{i,j} \hat{\chi}_i \otimes \hat{\chi}_j \mathrm{Tr}\left(\hat{O}_d(\hat{\chi}_i, \hat{\chi}_j)\hat{\rho}_a\right), \tag{2}$$

where the sum is taken over the complete set of observables $\hat{\chi}_i, \hat{\chi}_j$, and the normalization N ensures that $\mathrm{Tr}(\hat{\rho}_d) = 1$. Thus Eq. (2) describes the density matrix of the internal degrees of freedom as reconstructed by the detection procedure. It has the appropriate subsystem structure set by the measurement setup. The *physical entanglement* of any state $\hat{\rho}_a$, with respect to a given set of detectors, can thus be inferred by application of any entanglement measure on the detector-level density matrix $\hat{\rho}_d$.

Due to the inclusion of the measurement process, the resulting values of entanglement measures differ considerably to previous works [3]. A state of the form

$$\frac{1}{\sqrt{2}} \left(|A \uparrow, B \downarrow\rangle + \delta \cdot |B \downarrow, A \uparrow\rangle\right) \tag{3}$$

where δ indicates whether the particles are bosons ($\delta = 1$) or fermions ($\delta = -1$), is considered unentangled by previous approaches [3], because it can be written as a single Slater determinant ($\delta = -1$) or permanent ($\delta = 1$). For a situation with $|A\rangle\langle A| = \hat{O}_L$ and $|B\rangle\langle B| = \hat{O}_R$, *physical entanglement* corresponds to previous approaches. However, by imposing this setting, one explicitly assumes that experimentalists are always willing and able to choose such detectors corresponding to the external states of the particles as above. Often, one instead encounters a situation in which the detector level state is entangled [4]. Most prominently, this can be found in experiments with photons scattered by a beamsplitter in a Hong-Ou-Mandel setup [5, 6]: Two photons of opposite polarization enter one input port of the beam splitter each and have a 50% probability to exit at any of the two output ports. Due to the absence of interaction, the structure of the state of Eq. (3) remains the same throughout the process. At the input ports, the photons occupy one local mode corresponding to the arms of the beam splitter, a measurement at these input arms would hence yield a separable state. At the output ports, however, the photons are in a superposition of modes, their wave-functions do not correspond unambiguously to one detector each, instead the detectors project on linear combinations of the particles' states:

$$\hat{O}_L = \frac{1}{2}\left(|A\rangle + |B\rangle\right)\left(\langle A| + \langle B|\right) \tag{4}$$

$$\hat{O}_R = \frac{1}{2}\left(|A\rangle - |B\rangle\right)\left(\langle A| - \langle B|\right). \tag{5}$$

Consequently, at detector level, the state of Eq. (3) is indeed entangled. This cannot be inferred from the structure of the state alone, but it can only be understood incorporating the measurement process. On the other hand, a state which cannot be written as a single slater determinant or permanent, and is consequently entangled according to [3], does not necessarily exhibit quantum correlations at detector level.

The identicalness of particles can thus enhance or decrease quantum correlations at detector level with respect to the correlations expected from a study of the state alone. A full understanding of the entanglement of identical particles needs therefore an incorporation of the measurement setup in the formalism. Examples and applications for a series of states and detector settings is discussed in [4].

References

[1] J. Bell, Physics **1**, 195 (1964); J. F. Clauser *et al.*, Phys. Rev. Lett. **23**, 880 (1969).
[2] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[3] K. Eckert*et al.*, Ann. Phys. **299**, 88 (2002); J. Schliemann *et al.*, Phys. Rev. A **64**, 022303 (2001).
[4] M. C. Tichy, F. de Melo, M. Kuś, F. Mintert, and A. Buchleitner, Entanglement of identical particles and the detection process, arXiv:0902.1684, 2009.
[5] Z. Y. Ou and L. Mandel, Phys. Rev. Lett. **61**, 50 (1988).
[6] S. Bose and D. Home, Phys. Rev. Lett. **88**, 050401 (2002); N. Paunkovic *et al.*, Phys. Rev. Lett. **88**, 187903 (2002).

Concentration of entanglement dynamics in high dimensions

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(joint work with Fernando de Melo and Andreas Buchleitner)

In most contexts of physics it is not only the state of a physical system that is of interest but also its dynamics. The time evolution of a state can be described by a linear, completely positive, trace-preserving map [7] that maps the initial state at time 0 to the final state $\rho(t) = \Lambda_t[\rho(0)]$ at time t. For pure initial states, i.e. if $\rho(0)$ is a projector with trace 1, these maps constitute all possible, physical dynamics of the quantum system. In order to not only detect, but also quantify entanglement of a quantum state, one employs an entanglement measure E, that is a map from quantum states to the nonnegative reals with $E(\rho) = 0$ if and only if ρ is separable [7]. Under the dynamics on the set of states as described by Λ_t, we ask, how does the amount of entanglement of the quantum system change, i.e. what the map is that relates $E[\rho(0)]$ and $E[\rho(t)]$.

For the smallest physical system that can exhibit entanglement, a system of two two-level atoms A and B with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$, it is possible to derive an exact evolution equation for entanglement. This relation requires a specific entanglement measure, concurrence [8], which is defined for pure states $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$C(|\psi\rangle) = \sqrt{2(1 - \operatorname{Tr} \rho_A^2)}, \tag{1}$$

where $\rho_A = \operatorname{Tr}_B |\psi\rangle\langle\psi|$ denotes the partial trace over one of the subsystems. This definition for pure states can be extended to mixed states by means of the convex roof construction [12],

$$C(\rho) = \inf_{\{p_i,|\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle), \tag{2}$$

with the infimum over all possible convex decompositions into pure states, i.e. $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$ and $\sum_i p_i = 1$ with $p_i > 0$. If the system starts in a pure state $|\chi\rangle$ and only on of the subsystems undergoes arbitrary dynamics Λ_t, the final state is $\rho(t) = \mathbb{I} \otimes \Lambda_t|\chi\rangle\langle\chi|$, and the entanglement as quantified by concurrence evolves according to [4]:

$$C[\rho(t)] = C(|\chi\rangle)C(\mathbb{I} \otimes \Lambda_t|\Phi^+\rangle\langle\Phi^+|). \tag{3}$$

This means that, given a Λ_t, the entanglement of all initially pure states evolves similarly, and is merely rescaled by the initial amount of entanglement. The second factor determines the evolution, and is the concurrence of a the isomorphic state of the map Λ_t according to the Choi-Jamiołkowski-isomorphism [10, 11]. This state is obtained by applying the map to one subsystem of a maximally entangled state $|\Phi^+\rangle = \sum_{i=1}^2 |e_i\rangle \otimes |f_i\rangle/\sqrt{2}$, where the $|e_i\rangle$ and $|f_i\rangle$ form a basis in the respective Hilbert spaces. Therefore, operationally, a maximally entangled state suffices to benchmark the entanglement dynamics for a given dynamics.

Factorization relations like (3) appear also in more general cases for mixed initial states and two dynamics that act locally on the respective subsystems. In general, however, they provide only upper bounds, e.g. for $\rho(t) = \Lambda_A \otimes \Lambda_B[\rho(0)]$ the bound is

$$C[\rho(t)] \leq C[\rho(0)]C(\mathbb{I} \otimes \Lambda_A|\Phi^+\rangle\langle\Phi^+|)C(\mathbb{I} \otimes \Lambda_B|\Phi^+\rangle\langle\Phi^+|), \tag{4}$$

which is exact for suitable combinations of dynamics and states, and a good indicator in the regime of weakly mixing Λ [3].

An extension to systems of higher dimension is possible, and the above evolution equations and bounds persist if the entanglement measure concurrence is replaced with the entanglement monotone G-concurrence [9] for Hilbert spaces $\mathbb{C}^n \otimes \mathbb{C}^n$ [5].

If one employs results of high dimensional geometry in the form of Levy's lemma [13], and the monotonicity and convexity properties of entanglement measures and dynamical maps in general, rather than the algebraic properties of a specific entanglement measure, it is possible to identify a typical evolution of entanglement among all initial pure states [2]. Thereby we can extend the treatment of entanglement for pure states [6] to its dynamics [1]. After identifying pure states with points on a high dimensional sphere, Levy's lemma states that Lipschitz functions, here $E(\Lambda_t(\cdot))$ for suitable entanglement measures E like the trace-distance from the set of separable states, are almost constant on the sphere. In particular, a deviation from the mean for a randomly uniformly chosen initial state is exponentially suppressed in the size of the deviation squared, and the dimension of the underlying Hilbert space. Since the Hilbert space dimension of composite quantum systems scales exponentially in the number of constituents, we find a double exponentially suppression of the deviation from the mean in the number of constituents, which is easily observed for examples of few constituents [1]. Similar to the above evolution equations, we find the operational interpretation, that a single generic pure state suffices to determine the entanglement evolution of all other initially pure states with great probability.

References

[1] M. Tiersch, F. de Melo, and A. Buchleitner, *Universality in open system entanglement dynamics*, preprint, arXiv:0810.2506 (2008).

[2] M. Tiersch, *Bechmarks and Statistics of Entanglement Dynamics*, Ph.D. thesis, University of Freiburg, Germany, 2009. (http://www.freidok.uni-freiburg.de/volltexte/6878/pdf/diss_tiersch_2009.pdf)

[3] M. Tiersch, F. de Melo, T. Konrad, and A. Buchleitner, *Equation of motion for entanglement*, Quantum Information Processing **8**, 523–534 (2009).

[4] T. Konrad, F. de Melo, M. Tiersch, C. Kasztelan, A. Aragão, and A. Buchleitner, *Evolution equation of quantum entanglement*, Nature Physics **4**, 99–102 (2008).

[5] M. Tiersch, F. de Melo, and A. Buchleitner, *Entanglement evolution in finite dimensions*, Physical Review Letters **101**, 170502 (2008).

[6] P. Hayden, D. W. Leung, and A. Winter, *Aspects of generic entanglement*, Communications in Mathematical Physics **265**, 95–117 (2006).

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.

[8] W. K. Wootters, *Entanglement of formation of an arbitrary state of two qubits*, Physical Review Letters **80**, 2245–2248 (1998).

[9] G. Gour, *Family of concurrence monotones and its applications*, Physical Review A **71**, 012318 (2005).

[10] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and its Applications **10**, 285–290 (1975).

[11] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefiniteness of operators*, Reports on Mathematical Physics **3**, 275–278 (1972).

[12] A. Uhlmann, *Entropy and optimal decompositions of states relative to a maximal commutative subalgebra*, Open Systems and Information Dynamics **5**, 209–227 (1998).

[13] M. Ledoux, *The Concentration of Measure Phenomenon*, Number 89 in Mathematical Surveys and Monographs, American Mathematical Society, 2001.

Non-additivity of Rényi entropy and Dvoretzky's Theorem

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(joint work with Guillaume Aubrun and Stanisław Szarek)

Many major questions in quantum information theory can be formulated as additivity problems. These questions have received considerable attention in recent years, culminating in Hastings' work showing that the minimal output von Neumann entropy of a quantum channel is not additive. He used a random construction inspired by previous examples due to Hayden and Winter, who proved non-additivity of the minimal output p-Rényi entropy for any $p > 1$. In this short note, we show that the Hayden–Winter analysis can be simplified (at least conceptually) by appealing to Dvoretzky's theorem. Dvoretzky's theorem is a fundamental result of asymptotic geometric analysis, which studies the behaviour of geometric parameters associated to norms in $\mathbb{R}^n$ (or equivalently, to convex bodies) when n becomes large. Such connections between quantum information theory and high-dimensional convex geometry promise to be very fruitful.

If $\mathcal{H}$ is a Hilbert space, we will denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$, and by $\mathcal{D}(\mathcal{H})$ the set of *density matrices* on $\mathcal{H}$, i.e., positive semi-definite trace one operators on $\mathcal{H}$ (or *states* on $\mathcal{H}$, or – more properly – states on $\mathcal{B}(\mathcal{H})$). Most often we will have $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$, and we will then write $\mathcal{M}_n$ for $\mathcal{B}(\mathbb{C}^n)$.

For $p \geq 1$, the *p-Rényi entropy* of a state ρ is defined as

$$S_p(\rho) = \frac{1}{1-p} \log(\mathrm{tr}\rho^p).$$

A linear map $\Phi : \mathcal{M}_m \to \mathcal{M}_d$ is called a *quantum channel* if it is completely positive and trace-preserving. The *minimal output p-Rényi entropy* of Φ is then defined as

$$S_p^{\min}(\Phi) = \min_{\rho \in \mathcal{D}(\mathbb{C}^m)} S_p(\Phi(\rho)).$$

The *Additivity Conjecture* [1] asserted that the following equality held for every pair Φ, Ψ of quantum channels

$$S_p^{\min}(\Phi \otimes \Psi) \stackrel{?}{=} S_p^{\min}(\Phi) + S_p^{\min}(\Psi). \tag{1}$$

The most important case, $p = 1$, has been shown to be equivalent to a number of central questions in quantum information theory [7]. Of course, had the conjecture been true for every $p > 1$, it would have held also for $p = 1$ by continuity.

The conjecture has been recently disproved for *all* values of $p \geq 1$. Early (explicit) counterexamples for $p > 4.79$ were due to Holevo and R. F. Werner [8]. Subsequently, the case $p > 1$ was settled by Hayden and Winter in [5], and finally Hastings found a counterexample to the additivity conjecture for $p = 1$ [4].

We show here that a large part of the analysis by Hayden and Winter is actually a fallout of Dvoretzky's theorem, a classical result in high-dimensional convex geometry dating to the 1960s [2]. We note that this approach, at least in its present form, does not cover Hastings' construction.

It will be more convenient to study a multiplicative version of the conjecture, already considered in [1]. Instead of the Rényi entropy, we will work with the Schatten p-norm $\|\sigma\|_p = \big(\mathrm{tr}(\sigma^\dagger\sigma)^{p/2}\big)^{1/p}$. If $p > 1$ and ρ is a state, then $S_p(\rho) = \frac{p}{1-p} \log \|\rho\|_p$, and so the study of $S_p^{\min}(\Phi)$ is replaced by that of $\max_{\rho \in \mathcal{D}(\mathbb{C}^m)} \|\Phi(\rho)\|_p$, or the *maximum output p-norm.* The latter quantity has a nice functional-analytic interpretation: it equals $\|\Phi\|_{1\to p}$, i.e., the norm of Φ as an operator from $(\mathcal{M}_m, \|\cdot\|_1)$ to $(\mathcal{M}_d, \|\cdot\|_p)$. This allows to rewrite conjecture (1) in a multiplicative form

$$\|\Phi \otimes \Psi\|_{1\to p} \stackrel{?}{=} \|\Phi\|_{1\to p}\|\Psi\|_{1\to p}. \tag{2}$$

The inequality " $\geq$ " is trivial, so the conjecture asked if " $\leq$ " was always true.

The Hayden–Winter construction can be described as follows. Let $V : \mathbb{C}^m \to \mathbb{C}^d \otimes \mathbb{C}^d$ be a random isometry (chosen with respect to the Haar measure) and $\Phi : \rho \mapsto \mathrm{tr}_2(V\rho V^\dagger)$ be the corresponding quantum channel from $\mathcal{M}_m$ into $\mathcal{M}_d$. We show that Dvoretzky's theorem implies that for $m \sim d^{1+1/p}$, such random quantum channel typically satisfies

$$\|\Phi\|_{1\to p} \sim d^{1/p-1}. \tag{3}$$

Here, and throughout the remainder of the paper, $\sim$ means "equivalent up to a universal multiplicative constant."

Take as the second channel the (complex) conjugate channel $\bar{\Phi}$ and let $|\psi\rangle$ be the maximally entangled state in $\mathbb{C}^m \otimes \mathbb{C}^m$. It is shown in [5] that $(\Phi \otimes \bar{\Phi})(|\psi\rangle\langle\psi|)$ has an eigenvalue $\geq m/d^2$, which implies that with the above choice of m,

$$\|\Phi \otimes \bar{\Phi}\|_{1\to p} \geq \|\Phi \otimes \bar{\Phi}\|_{1\to\infty} \geq m/d^2 \sim d^{1/p-1}.$$

On the other hand, again with the same choice of m, by (3)

$$\|\Phi\|_{1\to p} = \|\bar{\Phi}\|_{1\to p} \sim d^{1/p-1},$$

and thus

$$\|\Phi\|_{1\to p}\, \|\bar{\Phi}\|_{1\to p} \sim \left(d^{1/p-1}\right)^2 \ll d^{1/p-1}, \tag{4}$$

so that we obtain a violation of the multiplicativity provided that $d^{1/p-1} \leq 1/C$, i.e., $d \geq C^{p/(p-1)}$, where C is the absolute constant hidden behind the $\sim$ symbol.

It can be shown that for channels Φ as above, $\|\Phi\|_{1\to p} = \max_{x\in\mathcal{W}} \left(\|x\|_{2p}/\|x\|_2\right)^2$, where $\mathcal{W} \subset \mathcal{M}_d$ is an m-dimensional subspace. The behavior of the ratio between the Euclidean norm and some other norm on subspaces of given dimension is a quantity that has been extensively studied in geometry of Banach spaces. The most classical result in this direction is Dvoretzky's theorem. The version of Dvoretzky's theorem that is relevant here is due to Milman [6].

Dvoretzky's theorem *Consider the n-dimensional Euclidean space (real or complex) endowed with the Euclidean norm $|\cdot|$ and some other norm $\|\cdot\|$ such that, for some $b > 0$, $\|\cdot\| \leq b|\cdot|$. Denote $M = \mathbb{E}\|X\|$, where X is a random variable uniformly distributed on the unit Euclidean sphere. Let $\varepsilon > 0$ and let $m \leq c\varepsilon^2(M/b)^2 n$, where $c > 0$ is an appropriate (computable) universal constant. Then, for most m-dimensional subspaces E (in the sense of the invariant measure on the corresponding Grassmannian) we have*

$$\forall x \in E, \quad (1-\varepsilon)M|x| \leq \|x\| \leq (1+\varepsilon)M|x|.$$

In the Hayden–Winter construction, $\mathcal{W} \subset \mathcal{M}_d$ is a random m-dimensional subspace distributed according to the Haar measure on the Grassmann manifold and we want to control the ratio $\|x\|_{2p}/\|x\|_2$ uniformly on $\mathcal{W}$, where $2p =: q > 2$. Thus the context in which one needs to apply Dvoretzky's theorem is the Schatten q-norm on the complex space $\mathcal{M}_d$ for $q > 2$, in particular $n = d^2$, $\|\cdot\| = \|\cdot\|_q$

and $|\cdot| = \|\cdot\|_2$, the Hilbert–Schmidt norm. This has been done, e.g., in the 1977 paper [3]. The conclusion is that if $m \sim d^{1+2/q} = d^{1+1/p}$, then the inequality

$$d^{1/q-1/2}\|x\|_2 \leq \|x\|_q \leq Cd^{1/q-1/2}\|x\|_2 \tag{5}$$

holds (for some constant $C \geq 1$ that does not depend on d for all x in a typical m-dimensional subspace of $\mathcal{M}_d$.

For completeness, let us comment on the details of the derivation of (5) from Dvoretzky's theorem. What we need is to find (or estimate) the quantities b, M appearing in the theorem. Clearly, for all $x \in \mathcal{M}_d$,

$$d^{1/q-1/2}\|x\|_2 \leq \|x\|_q \leq \|x\|_2, \tag{6}$$

which yields the value of the parameter $b = 1$.

As we mentioned above, the fact that $M \sim d^{1/q-1/2}$ is implicit in the argument from [3]. A simple argument to get an upper bound for M goes as follows. Let X be a random variable uniformly distributed on the Hilbert–Schmidt sphere in $\mathcal{M}_d$. It is easy to check, using an elementary ε-net argument, that the expectation of $\|X\|_\infty$ is bounded by $C_0 d^{-1/2}$ for some absolute constant C_0. Using the (pointwise) inequality $\|X\|_q \leq \|X\|_2^{2/q}\|X\|_\infty^{1-2/q}$ and Hölder's inequality, we get

$$M = \mathbb{E}\|X\|_q \leq \big(\mathbb{E}\|X\|_\infty\big)^{1-2/q} \leq \big(C_0 d^{-1/2}\big)^{1-2/q} = C_0^{1-2/q} d^{1/q-1/2}.$$

References

[1] G. G. Amosov, A. S. Holevo and R. F. Werner, *On some additivity problems in quantum information theory.* Probl. Inform. Transm. 36 (4) (2000), 25–34; arXiv:math-ph/0003002

[2] A. Dvoretzky, *Some Results on Convex Bodies and Banach Spaces.* In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, Jerusalem Academic Press, Jerusalem; Pergamon, Oxford 1961, pp. 123–160.

[3] T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimension of almost spherical sections of convex bodies.* Acta Math. 139 (1977), no. 1-2, 53–94.

[4] M. B. Hastings, *Superadditivity of communication capacity using entangled inputs.* Nature Physics **5**, 255 (2009)

[5] P. Hayden and A. Winter, *Counterexamples to the maximal p-norm multiplicativity conjecture for all $p > 1$.* Comm. Math. Phys. 284(1):263-280, 2008; arXiv:0807.4753

[6] V. Milman, *A new proof of the theorem of A. Dvoretzky on sections of convex bodies.* Funct. Anal. Appl. 5 (1971), 28–37 (English translation)

[7] P. W. Shor, *Equivalence of additivity questions in quantum information theory.* Comm. Math. Phys. 246 (2004), no. 3, 453–472.

[8] R. F. Werner and A. S. Holevo, *Counterexample to an additivity conjecture for output purity of quantum channels.* J. Math. Phys. 43(9):4353-4357 (2002); arXiv:quant-ph/0203003.

On geometry of quantum entanglement

Karol Życzkowski

(joint work with I. Bengtsson, M.Kuś and S. Szarek)

Entanglement is one of the most mysterious features of quantum theory. Many years ago it was called by Schrödinger "the characteristic trait of quantum mechanics", since it does not have a direct classical analogue.

In this talk we advocate a geometric approach to investigate various features of quantum entanglement. This phenomenon is easiest to characterize for pure states describing quantum systems which consist of two subsystems, labeled by A and B. Any pure state $|\psi\rangle$ belongs then to a Hilbert space with the tensor product structure, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. In the physics literature, a pure state from $\mathcal{H}$ is called *separable* if it has a product form,

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle, \tag{1}$$

where $|\phi_A\rangle \in \mathcal{H}_A$ and $|\phi_B\rangle \in \mathcal{H}_B$. Any state $|\psi\rangle \in \mathcal{H}$ which is not of the form (1) is called *entangled*.

Note that the entanglement is defined *with respect to* a given tensor product structure, which corresponds in physics to splitting of the entire system into two concrete subsystems A and B. It is then clear that the definition of entanglement is not invariant with respect to a global unitary operation U acting on $\mathcal{H}$. However it is invariant with respect to any *local* unitary operation with a tensor product structure $U = U_A \otimes U_B$, which act locally on each of subsystems.

To introduce a measure of entanglement (with respect to a prescribed partition of the system into subsystems A and B) of any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ one defines its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$. The *entanglement entropy* of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := \ -\mathrm{Tr}\ \sigma \ln \sigma, \tag{2}$$

and varies form zero (for any separable state) to $\ln N$ for any maximally entangled state of an $N \times N$ system. In general, the more mixed the partial trace σ, the more entangled the initial pure state $|\psi\rangle$.

The simplest composite quantum systems consists of two *qubits* (quantum bits) - subsystems described in a two–dimensional complex Hilbert space each. Since the states differing by an overall phase are identified, the set of all (normalized) pure states forms a complex projective space $\mathbb{C}P^3$. The set of separable (product) pure states is equivalent to the Cartesian product of two spheres, $\mathbb{C}P^1 \times \mathbb{C}P^1$. As schematically shown below, the entire space $\mathbb{C}P^3$ can be thus stratified into strata of quantum states of the same entanglement.

Analyzing the space of pure quantum states of any bipartite system we demonstrate that the degree of entanglement of a given state can be related to its distance to the closest separable state. Figure 2 presents the entropy of entanglement (2) for a 3-d cross-section of the 6-d set of all pure states of the two–qubit system.

The above analysis can be extended for bipartite systems of larger dimension. For instance, the orbit of states locally equivalent to the maximally entangled state of a $N \times N$ system has $N^2 - 1$ dimensions and the topology of the coset space $U(N)/U(1)$. However, it is not so simple to describe in a similar way the entanglement of multi–partite systems.

Even in the case of three qubits, described in the $2^3 = 8$ dimensional Hilbert space with the three–fold tensor product structure, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, several

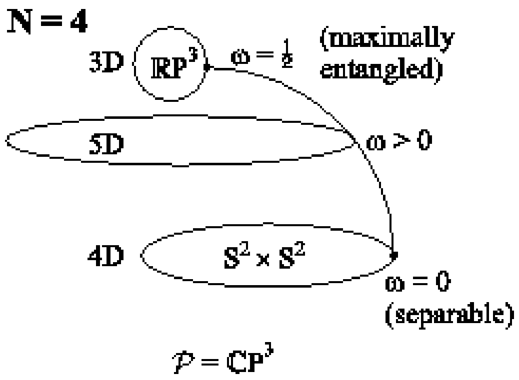

FIGURE 1. Stratification of the space $\mathcal{P}_4 = \mathbb{C}P^3$ of pure quantum states of a two-qubit system with respect to the degree of entanglement ω, determined by the Fubini–Study distance to the set of product states. This quantity is equal to zero for separable states and equals 1/2 for any maximally entangled state. The manifold of these states forms a three dimensional real projective space $\mathbb{R}P^3$. A generic 5–D stratum of an intermediate entanglement $0 < \omega < 1/2$ has the local structure of $U(2)/[U(1)]^2 \times \mathbb{R}P^3$.

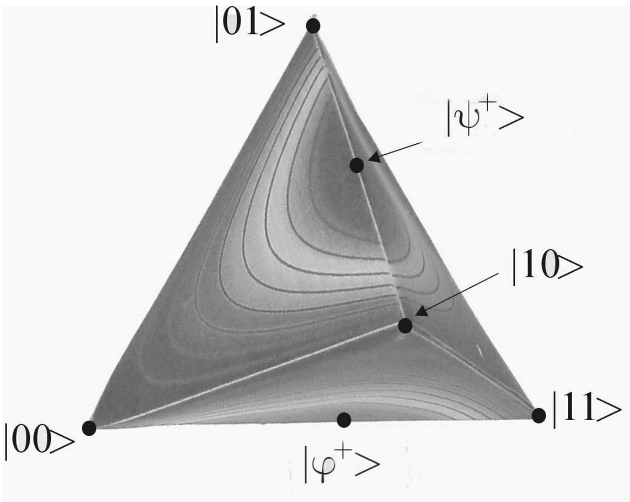

FIGURE 2. Curves of an equal entropy plotted on the tetrahedron representing the three dimensional set of real pure states of two qubits. Four corners of the tetrahedron represent separable product states. The maximally entangled *Bell state* $|\psi^+\rangle$ is formed by a symmetric superposition of the states $|01\rangle$ and $|10\rangle$.

questions remain open. Up to my knowledge the topology of the orbit containing states locally equivalent to one of the two distinguished states

a) GHZ state, $|\psi_{\mathrm{GHZ}}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$,

b) W–state, $|\psi_{\rm W}\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
is still not very well understood. Similarly, it would be instructive to stratify the space $\mathcal{P}_8 = \mathbb{C}P^7$ containing three-qubit pure states into the strata of equal entanglement and investigate their structure.

Studying the geometry of the convex body of mixed quantum states acting on an N–dimensional Hilbert space and demonstrate that it belongs to the class of sets of a constant height. The same property characterizes the set of all separable states of a two–qubit system. These results contribute to our understanding of quantum entanglement and its dynamics.

References

[1] M. Kuś and K. Życzkowski, *Geometry of entangled states*, Phys. Rev. **A 63** (2001), 032307-13

[2] I. Bengtsson, J. Brännlund and K. Życzkowski, *CP^n, or, entanglement illustrated*, Int. J. Mod. Phys. **A17** (2002), 4675-4696.

[3] M. Sinołęcka, K. Życzkowski and M. Kuś, *Manifolds of equal entanglement for composite quantum systems*, Acta Phys. Pol. **B 33** (2002), 2081-2095.

[4] K. Życzkowski and H.–J. Sommers, *Hilbert–Schmidt volume of the set of mixed quantum states*, J. Phys. **A 36** (2003), 10115-10130.

[5] S. Szarek, I. Bengtsson and K. Życzkowski, *On the structure of the body of states with positive partial transpose*, J. Phys. **A** 39 (2006), L119–L126.

[6] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States. An Introduction to Quantum Entanglement*, Cambridge University Press, Cambridge, (2006).

Generalized numerical range as a versatile tool to study quantum entanglement

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(joint work with M.D. Choi, C. Dunkl, J. Holbrook, P. Gawron, J. Miszczak, Z. Puchała, and Ł. Skowronek)

Let X be an operator acting on an N-dimensional complex Hilbert space $\mathcal{H}_N$. Let $W(X)$ denote its *numerical range* [1], i.e. the set of all λ such that there exists a normalized state $|\psi\rangle \in \mathcal{H}_N$, $||\psi|| = 1$, which satisfies $\langle\psi|X|\psi\rangle = \lambda$. We are going to analyze various generalizations of this definition in view of their possible applications in the theory of quantum information.

Take any integer number k such that $1 \le k \le N$ and define a subset of the complex plane given by

$$W_k(X) = \big\{\lambda \in \mathbb{C} : P_k X P_k = \lambda P_k\big\}, \qquad (1)$$

where P_k is an arbitrary k–dimensional projection operator. Note that this definition reduces the to standard numerical range for $k = 1$. For $k > 1$ the sets $W_k(X)$ are called *higher-rank numerical ranges* [2, 3] and they satisfy the following inclusion relation $W_1(X) \supseteq W_2(X) \supseteq \ldots \supseteq W_N(X)$.

It was recently shown that for any normal operator X its higher rank numerical range forms a convex set [4, 5]. This generalization of the standard numerical range is interesting from the mathematical perspective [6] and also in relation to quantum error correction codes [7, 8, 9].

Let us now take an arbitrary composite number, $N = KM$, and consider the Hilbert space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ with a tensor product structure. Following [10, 11] we define the *product numerical range* $W_\otimes$ of X, with respect to this tensor product structure,

$$(2) \qquad W_\otimes(X) \ := \ \{\langle\psi_A \otimes \psi_B|X|\psi_A \otimes \psi_B\rangle : |\psi_A\rangle \in \mathcal{H}_K, |\psi_B\rangle \in \mathcal{H}_M\} ,$$

where the states $|\psi_A\rangle \in \mathcal{H}_K$ and $|\psi_B\rangle \in \mathcal{H}_M$ are normalized.

We analyze operators acting on a tensor product Hilbert space and investigate their product numerical range, product numerical radius and product C–numerical radius. Concrete bounds for the product numerical range for Hermitian operators are derived. Product numerical range of a non-Hermitian operator forms a subset of the standard numerical range. While the latter set is convex, the product range need not be convex nor simply connected [12].

The product numerical range of a tensor product is equal to the Minkowski product of numerical ranges of individual factors. As an exemplary application of these algebraic tools in the theory of quantum information, we study block positive matrices, entanglement witnesses and consider the problem of finding minimal output entropy of a quantum channel. Furthermore, we apply product numerical range to solve the problem of local distinguishability for a family of two unitary gates.

For an arbitrary operator A which acts on an N dimensional complex Hilbert space $\mathcal{H}_N$ we introduce its *numerical shadow* as a probability distribution P_A defined on the complex plane

$$(3) \qquad P_A(z) \ := \ \int_{\Omega_N} \mathrm{d}\mu(\psi) \ \delta\Big(z - \langle\psi|A|\psi\rangle\Big) \ ,$$

where $\mu(\psi)$ denotes the unique unitarily invariant (Fubini-Study) measure on the set Ω_N of N–dimensional pure quantum states.

We show that for any normal operator A acting on $\mathcal{H}_N$, such that $AA^* = A^*A$, its shadow covers its numerical range with the probability corresponding to a projection of a *regular* N–simplex embedded in $\mathbb{R}^{N-1}$ into the plane. As the numerical range of a generic non–normal matrix is not a polygon, the corresponding numerical shape occurs to be a more complicated probability distribution. Numerical shadow of such an exemplary matrix of order three, $A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & i & 1 \\ 0 & 0 & -1 \end{pmatrix}$ with respect to real states shown in Fig. 1a resembles an artist's image of $\mathbb{R}P^3$.

This notion may also be generalized to give a *restricted* shadow of an operator,

$$(4) \qquad P_A^R(z) \ := \ \int_{\Omega_R} \mathrm{d}\mu_R(\psi) \ \delta\big(z - \langle\psi|A|\psi\rangle\big) \ ,$$

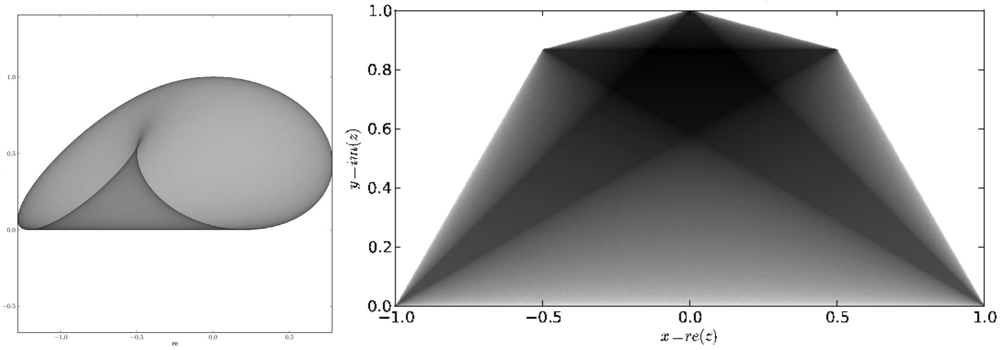

FIGURE 1. Numerical shadow restricted to real states for the operators A_3 and $A_5 = \mathrm{diag}[1, \exp(i\pi/3), i, \exp(i2\pi/3), -1]$. Observe that the inner dark pentagon in the right panel allows one to identify the numerical range of rank $k = 2$ of A_5.

where $\mu_R(\psi)$ denotes the Fubini-Study measure restricted to the set Ω_R and normalized, $\int_{\Omega_R} \mathrm{d}\mu_R(\psi) = 1$. For instance, one can consider the set of real states and analyze the shadow restricted to real states.

Assume now that the dimension N is composite, so one can define the sets of separable pure states and maximally entangled states. In analogy to the notion of product numerical range one can thus analyze numerical shadow restricted to separable (maximally entangled) states only. In the simplest case of $N = 2 \times 2$ the numerical shadow of a unitary matrix of size 4 is presented in Fig. 2

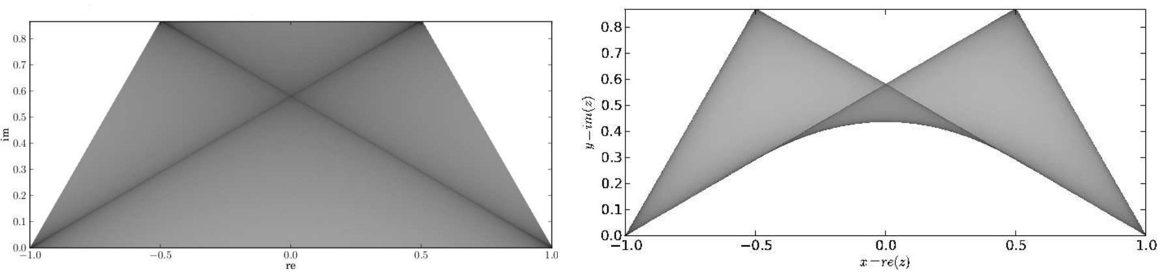

FIGURE 2. Numerical shadow for the operator

$$A_4 = \mathrm{diag}[1, \exp(i\pi/3), \exp(i2\pi/3), -1]$$

restricted to a) real states and b) real separable states.

Investigating numerical shadows of several operators of a given composite dimension with respect to the set of separable (maximally entangled) states one gains information about the structure of these multi-dimensional sets. On the other hand, knowing the numerical shadow of an unknown observable it is possible to identify this observable. We believe that the advocated approach based on geometrization of the algebraic notions provides a further contribution to our understanding of the geometry of quantum entanglement [13].

References

[1] A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, (1994)

[2] M.-D. Choi, D. W. Kribs, and K. Życzkowski, *Higher-Rank Numerical Ranges and Compression Problems*, Lin. Alg. Appl. **418** (2006), 828-839.

[3] M.-D. Choi, J. A. Holbrook, D. W. Kribs, and K. Życzkowski, *Higher-rank numerical ranges of unitary and normal matrices*, Operators and Matrices **1** (2007), 409-426.

[4] C.-K. Li, and N.-S. Sze, *Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations*, *Proc. Amer. Math. Soc.* **136** (2008), 3013-3023.

[5] H. Woerdeman, *The higher rank numerical range is convex*, Lin. & Multilin. Algebra **56** (2008), 65-67.

[6] M.-D. Choi, M. Giesinger J.A. Holbrook, D.W. Kribs, *Geometry of higher-rank numerical ranges*, Lin. & Multilin. Algebra **56** (2008), 53-64.

[7] M.-D. Choi, D. W. Kribs, and K. Życzkowski, *Quantum error correcting codes from the compression formalism*, Rep. Math. Phys. **58** (2006), 77-91.

[8] D.W. Kribs, A. Pasieka, K. Życzkowski, *Entropy of a quantum error correction code*, Open Syst. Inf. Dyn. **15** (2008), 329-343.

[9] K. Majgier, H. Maassen, and K. Życzkowski, *Protected Subspaces in Quantum Information* preprint arXiv:0902.4126 and Quantum Information Processing (2010), *to appear*

[10] G. Dirr, U. Helmke, M. Kleinsteuber, T. Schulte-Herbrüggen, *Relative-numerical ranges for applications in quantum control and quantum information*, Linear and Multilinear Algebra **56** (2008), 2

[11] T. Schulte-Herbrüggen, G. Dirr, U. Helmke., S. J. Glaser, *The significance of the C-numerical range and the local-numerical range in quantum control and quantum information*, Linear and Multilinear Algebra **56** (2008)

[12] P. Gawron, Z. Puchała, J. A. Miszczak, Ł. Skowronek, M.-D. Choi, K. Życzkowski *Product numerical range: a versatile tool in the theory of quantum information*, preprint arXiv: 0905.3646 (2009).

[13] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States. An Introduction to Quantum Entanglement*, Cambridge University Press, Cambridge, (2006).

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