Mathematisches Forschungsinstitut Oberwolfach

Report No. 55/2009

DOI: 10.4171/OWR/2009/55

Mini-Workshop: Modeling and Understanding Random Hamiltonians: Beyond Monotonicity, Linearity and Independence

Organised by
Günter Stolz (Birmingham)
Ivan Veselić (Chemnitz)

December 6th – December 12th, 2009

Abstract. The mini-workshop was devoted to the spectral analysis of random Schrödinger-type operators. While this topic has been intensively studied by physicists and mathematicians for several decades, more recently there has been particular attention devoted to models where the random parameters enter the model in a non-monotone or non-linear way. Most of the established methods applied for random operators, in fact, hinge on the presence of monotonicity w. r. t. randomness. Thus the treatment of non-monotone models forces a deeper analysis of the structure of random Hamiltonians and, in particular, the interplay of the kinetic and the potential energy parts.

Mathematics Subject Classification (2000): 82B44, 81Q10, 47B80.

Introduction by the Organisers

The mini-workshop was organised by G. Stolz and I. Veselić and brought together sixteen participants from six countries. An introductory talk by one of the organizers was followed by 15 lectures of participants, including survey talks as well as lectures on results of recent research. Nevertheless, the mini-workshop format left time for intense collaborative work which was pursued in small group discussions throughout the week.

The workshop activities focused on open problems in the theory of localization of random Schrödinger operators. A central theme was to discuss difficulties which arise due to the lack of monotonicity properties in some models of random Schrödinger operators. Such monotonicity properties have been heavily exploited in many of the rigorous results obtained for the Anderson model over the last three decades. While the standard discrete and continuum Anderson models depend monotonically on the random parameters, this is not the case for other important quantum mechanical models of disordered media. Examples of such models are Anderson-type models with sign-indefinite single site potentials, models for structural disorder such as the Poisson and random displacement models or random wave guides, Schrödinger operators including random magnetic fields or random spin matrices, certain ergodic Hamiltonians with "pseudo-random" properties (induced e.g. by the skew-shift or doubling map), or random Schrödinger operators with discretely distributed random parameters such as the Bernoulli-Anderson model or Laplacians on random graphs.

A number of talks focussed on some of the central tools in localization theory where monotonicity properties (or their lack) play a significant role in proofs: Wegner estimates, fractional moment bounds, and Lifshits tail asymptotics of the integrated density of states. Some mechanisms were identified which allow to recover monotonicity properties in some of the models or replace them with other tools such as arguments involving convexity or analyticity. One main reason for the interest in these results is that they generally require a better understanding of the underlying physics, in particular, the interaction of kinetic and potential energy in the form of uncertainty principle relations. In fact, many existing proofs which use the monotone dependence of the potential on the random parameters disregard completely the properties of the kinetic energy part of the Schrödinger operator.

Most of the participants have been working in the field of random operators before, which enabled intense and flexible discussions during and after the lectures. A few experts from other fields (namely asymptotic analysis and probability theory) have been invited with the intent to provide new tools which may be used to tackle the challenges not approachable by current methods. For instance, the lecture by W. König gave insight into how probabilistic methods are used to yield a detailed analysis of the intermittency phenomenon for the parabolic Anderson model. Similar ideas may lead to a proof of localization which does not hinge on the regularity of the individual random variables. On the final day of the workshop a lecture (by W. Kirsch) was presented in a joint session with participants of the Mini-Workshop on "Geometry of Quantum Entanglement".

Summarizing, one can say that the discussions at the workshop led to a better understanding of common themes in various non-monotone models, which had previously been investigated for specific random Hamiltonians. A clearer picture arose of the difficulties due to non-monotonicity as well as how (and if) they can be remedied. In addition, a number of open challenges were identified, examples being:

- Find a localization proof for the discrete multi-dimensional Bernoulli-Anderson model. In particular: How can recent work by Bourgain and Kenig in the continuum be carried over to discrete models?
- Identify an analogue or replacement of the usual unique continuation property for solutions of *discrete* Schrödinger operators.

- Show localization at the spectral edges of Laplacians on percolation graphs in the supercritical regime.
- Find ways to understand that infinite volume quantities (such as the IDS) are "smoother" than their finite volume counterparts.
- Use multiple averaging to show that the expectation of the eigenvalue counting function has regularity beyond the one of the distribution of a single random variable.

Mini-Workshop: Modeling and Understanding Random Hamiltonians: Beyond Monotonicity, Linearity and Independence

Table of Contents

Abstracts

Wegner-type bounds for discrete alloy-type models

Ivan Veselić

A discrete alloy type model is a family of operators $H_\omega = H_0 + V_\omega$ on $\ell^2(\mathbb{Z}^d)$. Here H_0 denotes an arbitrary symmetric operator. In most applications H_0 is the discrete Laplacian on $\mathbb{Z}^d$. The random part V_ω is a multiplication operator

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k) \tag{1}$$

defined in terms of an i. i. d. sequence $\omega_k\colon \Omega \to \mathbb{R}, k \in \mathbb{Z}^d$ of random variables each having a density f, and a single site potential $u \in \ell^1(\mathbb{Z}^d;\mathbb{R})$. It follows that the mean value $\bar u := \sum_{k\in\mathbb{Z}^d} u(k)$ is well defined. We will assume throughout the paper that u does not vanish identically and that $f \in BV$. Here BV denotes the space of functions with finite total bounded variation and $\|\cdot\|_{BV}$ denotes the corresponding norm. The mathematical expectation w.r.t. the product measure associated with the random variables $\omega_k, k \in \mathbb{Z}^d$ will be denoted by $\mathbb{E}$.

The estimates we want to prove do not concern the operator $H_\omega, \omega \in \Omega$, but rather its finite box restrictions. For $L \in \mathbb{N}$ we denote the subset $[0,L]^d \cap \mathbb{Z}^d$ by Λ_L, its characteristic function by χ_{Λ_L}, the canonical inclusion $\ell^2(\Lambda_L) \to \ell^2(\mathbb{Z}^d)$ by ι_L and the adjoint restriction $\ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda_L)$ by π_L. The finite cube restriction of H_ω is then defined as $H_{\omega,L} := \pi_L H_0 \iota_L + V_\omega \chi_{\Lambda_L} : \ell^2(\Lambda_L) \to \ell^2(\Lambda_L)$. For any $\omega \in \Omega$ and $L \in \mathbb{N}$, the restriction $H_{\omega,L}$ is a selfadjoint finite rank operator. In particular its spectrum consists entirely of real eigenvalues $E(\omega,L,1) \le E(\omega,L,n) \le \cdots \le E(\omega,L,\sharp\Lambda_L)$ counted including multiplicities. Note that if u has compact support, then there exist an $n \in \mathbb{N}$ and an $x \in \mathbb{Z}^d$ such that supp $u \subset \Lambda_{-n} + x$, where $\Lambda_{-n} := \{-k \mid k \in \Lambda_n\}$. We may assume without loss of generality $x = 0$ without restricting the model (1). The number of points in the support of u is denoted by $\operatorname{rank} u$. Now we are in the position to state our bounds on the expected number of eigenvalues of finite box Hamiltonians $H_{\omega,L}$ in a compact energy interval $[E-\epsilon, E+\epsilon]$.

Theorem 1. *Assume that the single site potential u has support in Λ_{-n}. Then there exists a constant c_u depending only on u such that for any $L \in \mathbb{N}$, $E \in \mathbb{R}$ and $\epsilon > 0$ we have,*

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E+\epsilon]}(H_{\omega,L})\right]\right\} \le c_u \,\|f\|_{BV}\, \operatorname{rank} u\ \epsilon\,(L+n)^{d\cdot(n+1)}$$

The next Theorem applies to single site potentials $u \in \ell^1(\mathbb{Z}^d)$ with non vanishing mean $\bar u \neq 0$. Let $m \in \mathbb{N}$ be such that $\sum_{\|k\|\ge m} |u(k)| \le |\bar u/2|$. Here $\|k\| = \|k\|_\infty$ denotes the sup-norm.

Theorem 2. *Assume $\bar u \neq 0$ and that f has compact support. Then we have for any $L \in \mathbb{N}$, $E \in \mathbb{R}$ and $\epsilon > 0$*

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E+\epsilon]}(H_{\omega,L})\right]\right\} \le \frac{8}{\bar u}\,\|f\|_{BV}\, \min\left(L^d, \operatorname{rank} u\right)\, \epsilon\,(L+m)^d$$

If, in addition, supp $u \subset \Lambda_{-n}$*, we have for any* $L \in \mathbb{N}$*,* $E \in \mathbb{R}$ *and* $\epsilon > 0$*,*

$$\mathbb{E}\left\{\mathrm{Tr}\left[\chi_{[E-\epsilon,E+\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{4}{\bar{u}}\,\|f\|_{BV}\ \mathrm{rank}\,u\ \epsilon\,(L+n)^d$$

If the operator H_ω has a well defined integrated density of states $N\colon \mathbb{R} \to \mathbb{R}$, meaning that

$$\lim_{L\to\infty} \frac{1}{L^d}\mathbb{E}\left\{\mathrm{Tr}\left[\chi_{(-\infty,E]}(H_{\omega,L})\right]\right\} = N(E)$$

at all continuity points of N, then the second statement of Theorem 2 implies that the integrated density of states is Lipschitz continuous. Consequently its derivative, the density of states, exists for almost all $E \in \mathbb{R}$.

Theorem 3. *Assume that* $d = 1$*,* f *has compact support and that there exist* $s \in (0,1)$ *and* $C \in (0,\infty)$ *such that* $|u(k)| \leq Cs^{|k|}$ *for all* $k \in \mathbb{Z}$*. Then there exist* $c_u \in (0,\infty)$ *and* $D \in \mathbb{N}_0$ *depending only on* u *such that for each* $\beta > D/|\log s|$ *there exists a constant* $K_\beta \in (0,\infty)$ *such that for all* $L \in \mathbb{N}$*,* $E \in \mathbb{R}$ *and* $\epsilon > 0$*,*

$$\mathbb{E}\left\{\mathrm{Tr}\left[\chi_{[E-\epsilon,E+\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{8}{c_u}\,\|f\|_{BV}\ \epsilon\,L\,(L+\beta\log L+K_\beta)^{D+1}$$

One can use the above results to prove spectral localisation with the aid of multiscale analysis. The necessary initial scale estimate can be obtained from the Wegner bound and a sufficiently large choice of disorder.

Theorem 4. *Assume that* H_0 *is the discrete Laplacian on* $\mathbb{Z}^d$*, and* f *and* u *have compact support. Then there exists an* $\epsilon \in (0,\infty)$ *depending only on* u*, such that if* $\|f\|_{BV} \leq \epsilon$*, then* H_ω *has almost surely no continuous spectral component and all its eigenfunctions decay at infinity with an exponential rate.*

References

[1] H. von Dreifus and A. Klein, *Localization for random Schrödinger operators with correlated potentials*, Comm. Math. Phys., 140(1):133–147, 1991

[2] W. Kirsch, *An Invitation to Random Schrödinger operators*, 2007, `http://www.arXiv.org/abs/0709.3707`

[3] F. Klopp, *Localization for some continuous Random Schrödinger Operators*, Commun. Math. Phys., 167:553–569, 1995.

[4] I. Veselić, *Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators*, Lett. Math. Phys., 59(3): 199–214, 2002.

[5] I. Veselić, *Wegner estimate for discrete alloy-type models*, `http://www.ma.utexas.edu/mp_arc/c/09/09-100.pdf`

Regularity of the Green function for the Anderson model with spin

Alexander Elgart
(joint work with Gian Michele Graf)

We study the behavior of the system governed by the Hamiltonian

$$H_\omega := -\Delta \otimes I_2 + \lambda \sum_{n\in\mathbb{Z}} I \otimes V_\omega(n)\,, \tag{1}$$

acting on the Hilbert space $\mathcal{H} = L_2(\mathbb{Z}\otimes\mathbb{C}^2)$. Here Δ is the discrete Laplacian, accounting for a hopping in the electronic degrees of freedom, and the matrix $V_\omega(n) \in M_{2,2}$ is a randomly chosen Pauli matrix, describing the spin:

$$V_\omega(n) := \sigma(k_\omega(n)) := k_\omega(n)\cdot\sigma\,, \quad k_\omega(n) \in S^2\,. \tag{2}$$

We assume that k_ω are independent, identically distributed random variables, with the uniform probability density function on the unit sphere.

Let P_n denote a (two dimensional) projection to the site $n\in\mathbb{Z}$, let H_Λ be the natural restriction of H_ω to the set $\Lambda\subset\mathbb{Z}$, and let $\Gamma(\Lambda)$ stand for a collection of bonds connecting Λ with its complement in $\mathbb{Z}$. By $|S|$ we will denote the cardinality of the set $S\in\mathbb{Z}$. The quantity of the interest is the typical asymptotic behavior of the (matrix valued in our case) Green function

$$G_{nk}(E) := P_n(H_\omega - E + i0)^{-1}P_k\,; \quad G^\Lambda_{nk}(E) := P_n(H_\Lambda - E + i0)^{-1}P_k\,. \tag{3}$$

The main result we establish is

Theorem 1. *For every finite set Λ, any $x,y\in\Lambda$, and all $E\in\mathbb{R}$ satisfying $|E-E_j| > \sqrt{\delta}$ for a set of 14 values of j, with $\delta>0$, we have uniform bounds*

$$\mathbb{E}\,\|G^\Lambda_{xx}(E)\|^{\frac{s}{40}} \quad < \quad \frac{C_s(\delta)}{\lambda^s}\,; \tag{4}$$

$$\mathbb{E}\,\|G^\Lambda_{xy}(E)\|^{\frac{s}{400}} \quad < \quad \frac{C_s(\delta)}{\lambda^s} \tag{5}$$

for all $0<s<1$.

This estimate is sufficient as an input for the fractional moment method, and can be used to establish Anderson localization in the perturbative regimes. As oppose to the monotone Anderson model [1], we have to average over not only the randomness associated with the sites x,y, but also over their local environment to obtain the desired regularity.

To obtain this result, we exploit the continuous fraction structure of the Schrödinger operator. We were also forced to obtain a quantitative version of the Schur complement formula [3] that controls the number of small (but necessary zero) singular values of the Hermitian matrix. Another important tool which is used in the analysis is a refined version of Woodbury matrix identity. The method is stable in higher dimension as well, but the control over the Hölder exponent becomes ridiculously weak in this case.

References

[1] M. Aizenman, J. H. Schenker, R. M. Friedrich, and D. Hundertmark, *Finite volume fractional moment criteria for Anderson localization*, Commun. Math. Phys. **224** (2001), 219–253.

[2] R. Horn and C. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge (1990).

[3] F. Z. Zhang, editor *The Schur complement and its applications*, Numerical Methods and Algorithms **4**, Springer-Verlag, New York (2005).

Quasi-one-dimensional random operators: random phase property, Lyapunov spectrum and delocalization

Hermann Schulz-Baldes

(joint work with Rudo Römer and Christian Sadel)

A random phase property establishing a link between quasi-one-dimensional random Schrödinger operators and full random matrix theory is advocated. Briefly summarized it states that the random transfer matrices placed into a normal system of coordinates act on the isotropic frames and lead to a Markov process with a unique invariant measure which is of geometric nature. On the elliptic part of the transfer matrices, this measure is invariant under the full hermitian symplectic group of the universality class under study. While the random phase property can up to now only be proved in special models or in a restricted sense, we provide strong numerical evidence that it holds in the Anderson model of localization. A main outcome of the random phase property is a perturbative calculation of the Lyapunov exponents which shows that the Lyapunov spectrum is equidistant and that the localization lengths for large systems in the unitary, orthogonal and symplectic ensemble differ by a factor 2 each. In an Anderson-Ando model on a tubular geometry with magnetic field and spin-orbit coupling, the normal system of coordinates is calculated and this is used to derive explicit energy dependent formulas for the Lyapunov spectrum.

The second topic of the talk concerns the Altland-Zirnbauer symmetry classification for disordered systems. It is reviewed for quasi-one-dimensional systems with particular focus on the associated isotropic frames. It shows that the isotropic frames can be identified with the maximal compact subgroup of the transfer matrix group except in the cases where there is a non-trivial $\mathbb{Z}_2$ invariant. For those systems as well as systems with different number of left and right movers it can be proved that the spectral measure of associated model operators are absolutely continuous.

Fractional moment method for discrete alloy-type models

Martin Tautenhahn
(joint work with Alexander Elgart, Ivan Veselić)

The discrete alloy-type model is the discrete random Schrödinger operator $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, where Δ denotes the discrete Laplace operator and V_ω denotes the multiplication with the function $V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k u(x-k)$. Here, $\omega = \{\omega_k\}_{k\in\mathbb{Z}^d}$ is a sequence of independent and identically distributed (i. i. d.) random variables each distributed with the density $\rho \in W^{1,1}(\mathbb{R})$, and $u : \mathbb{Z}^d \to \mathbb{R}$ is assumed to have compact support $\Theta = \operatorname{supp} u = \{k \in \mathbb{Z}^d : u(k) \neq 0\}$. For $\Gamma \subset \mathbb{Z}^d$ we denote by $H_\Gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$ the natural restriction of H_ω to the set Γ. For $x, y \in \mathbb{Z}^d$ (resp. Γ) and $z \in \mathbb{C} \setminus \mathbb{R}$, we set $G_\omega(z; x, y) = \langle \delta_x, (H_\omega - z)^{-1}\delta_y\rangle$ (resp. $G_\Gamma(z; x, y) = \langle \delta_x, (H_\Gamma - z)^{-1}\delta_y\rangle$). The symbol $\mathbb{E}$ denotes the average with respect to the randomness.

One of the fundamental results in the theory of random Schrödinger operators is the physical phenomenon of localization, i. e. the almost sure spectrum of H_ω has only pure point spectrum with probability one. In the higher dimensional case there are two methods to prove localization in the case of large disorder or at extreme energies: the multiscale analysis [8, 7, 5] and the fractional moment method [3, 1, 9, 2]. We are interested in the fractional moment method for the discrete alloy-type model. Our results are

Theorem 1 ([6])**.** *Let $d = 1$, $n = \max\Theta - \min\Theta + 1$, $s \in (0, 1)$ and assume that $\|\rho'\|_{L^1}$ is sufficiently small. Then there are constants $C, m \in (0,\infty)$ such that*

$$\mathbb{E}\big\{|G_\omega(z; x, y)|^{s/2n}\big\} \le C\mathrm{e}^{-m|x-y|}$$

for all $x, y \in \mathbb{Z}$ with $|x - y| \ge 4n$ and all $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 2 ([10])**.** *Assume that the function $\hat{u}\colon [0, 2\pi)^d \to \mathbb{C}$, defined by*

$$\hat{u}(\theta) = \sum_{k\in\mathbb{Z}^d} u(k)\mathrm{e}^{\mathrm{i}k\cdot\theta},$$

does not vanish. Let $s \in (0, 1)$ and $\Lambda \subset \mathbb{Z}^d$ be finite. Then there exists a constant C_u depending only on u, such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all $x, y \in \Lambda$,

$$\mathbb{E}\big\{|G_\Lambda(z; x, y)|^s\big\} \le \big(C_u\|\rho'\|_{L^1}\big)^s \frac{2^{1+s}s^{-s}}{1-s}.$$

There are several relatives to our results. In the case where $u(0) = 1$ and $u(k) = 0$ for $k \in \mathbb{Z}^d \setminus \{0\}$, i. e. in the i. i. d. Anderson model, exponential decay of averaged fractional moments has been shown, e. g., in [3, 4]. They also allow the potential values at different lattice sites to be correlated random variables by imposing some regularity conditions. However, these regularity conditions are in general not satisfied for the discrete alloy type model, see [10]. In the case where $u(k) \ge 0$ for all $k \in \mathbb{Z}^d$ the method of [2] applies. Indeed, in [2] the fractional moment method is developed for the continuum alloy-type model with sign fixed single-site potential, and these results carry over immediately to the discrete case. The fact that $u(k) \geq 0$ plays an important role in the proofs.

We allow the single site potential u to change its sign. As a consequence, the dependence of H_ω on the random coupling constants ω_k is no longer monotone. Notice, that the proof of Theorem 1 does not use monotonicity at all, while the proof of Theorem 2 is based on a transformation of random variables to recover monotonicity.

References

[1] M. Aizenman, *Localization at weak disorder: some elementary bounds*, Commun. Math. Phys. **6** (1994), 1163–1182.

[2] M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker and G. Stolz, *Moment analysis for localization in random Schr¨odinger operators*, Invent. Math. **163** (2006), 343–413.

[3] M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: an elemantary derivation*, Commun. Math. Phys. **157** (1993), 245–278.

[4] M. Aizenman, J. H. Schenker, R. M. Friedrich and D. Hundertmark, *Finite-volume fractional-moment criteria for Anderson localization*, Commun. Math. Phys. **224** (2001), 219–253.

[5] H. von Dreifus and A. Klein, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124** (1989), 285–299.

[6] A. Elgart, M. Tautenhahn and I. Veselić, *Exponential decay of Green's function for Anderson models on $\mathbb{Z}$ with single-site potentials of finite support*, Preprint (2009), arXiv:0903.0492v1 [math-ph] .

[7] J. Fröhlich, F. Martinelli, E. Scoppola and T. Spencer, *Constructive proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **101** (1985), 21–46.

[8] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), 151–184.

[9] G. M. Graf, *Anderson localization and the space-time characteristic of continuum states*, J. Stat. Phys. **75** (1994), 337–346.

[10] M. Tautenhahn and I. Veselić, Spectral properties of discrete alloy-type models, submitted to the Proceedings of 16th International Congress of Mathematical Physics (2009), Prague.

Lifshitz tails for Schrödinger operators with non-sign definite random potentials

Shu Nakamura
(joint work with Frédéric Klopp)

We discuss recent results on Lifshitz tails for alloy-type (or generalized alloy-type) Schrödinger operators with the local potential which does not have fixed sign ([1, 2]). Usual proof of Lifshitz tail relies on the monotonicity of the random perturbation with respect to random variables, and thus we cannot apply these method directly to show Lifshitz singularities for our model.

Here we consider

$$H_\omega = -\triangle + V_p + V_\omega \quad \text{on } L^2(\mathbb{R}^d)$$

where V_p is a $\mathbb{Z}^d$-periodic background potential, and

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma v(x-\gamma).$$

Here $\{\omega_\gamma\}$ are i.i.d. random variables with the common distribution μ, and $v \in C_c^0(\Lambda_1(0))$, where $\Lambda_1(0)$ is the unit cube with the center at the origin. We suppose $\operatorname{Supp}\mu \subset [a,b]$ with $\{a,b\} \subset \operatorname{Supp}\mu$, and V_p and v are symmetric with respect to reflections about $\{x \mid x_j = 0\}$, $j = 1,\dots,d$.

Let H_λ^N be the operator $-\triangle + V_p + \lambda v$ on $L^2(\Lambda_1(0))$ with Neumann boundary conditions. We denote $E(\lambda) = \inf\sigma(H_\lambda^N)$. We note that $E(\lambda)$ is a concave function in λ, and hence $E_- := \min\{E(\lambda) \mid \lambda \in [a,b]\}$ is attained either at $\lambda = a$ or b.

Theorem 1 ([1]) If $E(a) \neq E(b)$, then the Lifshitz tail holds at the bottom of the spectrum, i.e., $\inf\sigma(H_\omega) = E_-$ almost surely, and

$$\limsup_{E\to E_-} \frac{\log|\log N(E)|}{\log(E-E_-)} \le -\frac{d}{2},$$

where $N(E)$ is the integrated density of states for H_ω.

The main step of the proof employs a simple operator inequality:

$$H_\omega \ge c\big[-\triangle + V_p + \sum_{\gamma\in\mathbb{Z}^d}(\omega_\gamma - a)\big] \quad \text{on } L^2(\mathbb{R}^d)$$

with some $c > 0$, where we suppose $E_- = E(a)$. This in turn follows from another operator inequality:

$$H_\lambda^N \ge c\big[-\triangle + V_p + (\lambda - a)\big]^N \quad \text{on } L^2(\Lambda_1(0)),$$

where both sides are Neumann operators. This (simple but rather surprising) argument relies on the fact:

$$H^1(\mathbb{R}^d) \subset \bigoplus_{\gamma\in\mathbb{Z}^d} H^1(\Lambda_1(\gamma))$$

and that the form domain of the Neumann operator on $\Omega \subset \mathbb{R}^d$ is $H^1(\Omega)$.

If $E(a) = E(b)$ then we have the following somewhat weaker result:

Theorem 2 ([2]) If $E(a) = E(b)$ and if μ is not Bernoulli, then the Lifshitz tail holds at the bottom of the spectrum, i.e., $\inf\sigma(H_\omega) = E_-$ almost surely, and

$$\limsup_{E\to E_-} \frac{\log|\log N(E)|}{\log(E-E_-)} \le -\frac{1}{2},$$

where $N(E)$ is the integrated density of states for H_ω.

In this case, the above comparison theorem does not hold, and we need to use completely different method. In particular, we cannot use the argument involving the Temple inequality. We use, instead:

- Neumann decomposition to long pseudo 1D domains.

- Poincaré type inequality for long pseudo 1D domains with periodic background potential.
- The positivity of the Dirichlet-to-Neumann operator for positive Schrödinger operators on small domains.

Combining these, we can obtain the necessary lower bound of lowest eigenvalues for H_ω restricted to large boxes to show the Lifshitz singularities.

Note that in [2] we consider Schrödinger operators with generalized alloy-type random potentials, which takes finitely many forms randomly at each $\gamma \in \mathbb{Z}^d$. We then combine the result with concavity argument to show Theorem 2. Our general result applies also to some random displacement models discussed in a talk by Günter Stolz.

References

[1] F. Klopp and S. Nakamura: Spectral extrema and Lifshitz tails for non monotonous alloy type models. Commun. Math. Phys. **287**, 1133-1143 (2009).
[2] F. Klopp and S. Nakamura: Lifshitz tails for generalized alloy type random Schrödinger operators. Preprint, March 2009 (`http://arxiv.org/abs/0903.2105`)

About currents, magnetic perturbations, magnetic barriers and magnetic guides in quantum Hall systems

François Germinet
(joint work with Nicolas Dombrowski and Georgi Raikov)

Since a seminal paper of Halperin [H], the physics of the Quantum Hall effect can be studied from two points of view: bulk and edge. They both give rise to quantized currents measured through, respectively, the Hall conductance and the edge conductance. These two points of view coincide since these two conductances are simultaneously quantized.

BULK:

In quantum Hall systems, namely 2DEG submitted to a transverse constant magnetic field, localized states are responsible for the celebrated plateaux of the quantum Hall effect. Where the Hall conductance is discontinuous, non trivial transport has been proved to take place in [GKS] for electric disorder. We provide a similar picture but with magnetic disorder. The random magnetic potential is shown to create both strongly localized states at the edges of the spectrum and dynamical delocalization near the center of the band in the sense that wave packets travel at least at a given minimum speed. We thus consider 2D-random magnetic perturbations of the Landau Hamiltonian and prove a transition between dynamical localization and dynamical delocalization inside an arbitrary number of bands.

The proof of localization exploits the Wegner estimate of Hislop and Klopp [HK], revisited by Ghribi, Hislop and Klopp [GrHK], together with a simple weak disorder argument to start the multiscale analysis, provided some information on the location of the spectrum that we address in a separate argument; then dynamical localization follows from [GK]. Delocalization is proved along the lines of [GKS]; in particular the Hall conductance is quantized, constant in the region of localization and jumps by one as a Landau level is crossed.

We further exhibit an explicit family of small periodic magnetic perturbations for which the splitting gives rise to a full interval of spectrum. This is achieved by direct computation using translation invariance of our potential in one direction. Such examples are then good enough to be randomized and used as random magnetic fields.

ONE EDGE:

The wall is designed by an Iwatsuka magnetic field [Iw], a y-independent magnetic field with a decaying profile in the x-axis. As a matter of fact the particle is subjected to, say, a strong magnetic field on the left half plane, and to a weaker one on the right half plane, creating currents along such an interface. The edge conductance for these currents is explicitly computed and is quantized.

Perturbations are also of magnetic nature. As a preliminary but essential result, we prove that magnetic perturbations carried by magnetic fields compactly supported in the x-axis do not affect the edge conductance. Next, we consider non compactly supported perturbations that do not vanish at infinity, and provide a sum rule similar to that obtained in [CG]. Namely, the edge conductance of the perturbed system is the sum of the edge conductance of the magnetic confining potential and of the edge conductance of the system without magnetic wall defined by a reference Landau Hamiltonian perturbed by the magnetic potential. This enables us to compute the edge conductance of the perturbed Hamiltonian when energies fall inside a gap of some reference Landau Hamiltonian perturbed by the magnetic potential. To consider energies corresponding to localized states, one has to go one step further and regularize the trace that defines the edge conductance.

TWO EDGES:

If we now consider a magnetic strip created by two large positive magnetic fields and a (not too big) magnetic field in the middle, the net current flowing along these axes is zero, like in the electric case. An interesting phenomenon appears when the two walls are generated by magnetic fields of opposite signs. Existence of quantized current is proved, with quantization equal to two times the value provided by the quantum Hall effect in the particular case of opposite value of the magnetic strengths. Such currents are sometimes called “snake currents” in the physics literature.

References

[CG] J.-M. Combes and F. Germinet: Edge and impurity effects on quantization of Hall currents, Comm. Math. Phys. **256**, 159–180 (2005)

[GK] F. Germinet and A. Klein: Explicit finite volume criteria for localization in continuous random media and applications. Geom. Funct. Anal. **13** 1201-1238 (2003)

[GKS] F. Germinet, A. Klein and J. Schenker: Dynamical delocalization in random Landau Hamiltonians. Annals of Math. **166**, 215-244 (2007)

[GrHK] F. Ghribi, P.D. Hislop and F. Klopp, Localization for Schrödinger operators with random vector potentials Contemporary Mathematics, vol. 447, “Adventures in mathematical physics”, Eds Germinet, Hislop, 123-138 (2007)

[H] B.I. Halperin: Quantized Hall conductance, current carrying edge states and the existence of extended states in a two-dimensional disordered potential. Phys. Rev. B **25**, 2185-2190 (1982)
[HK] P.D. Hislop and F. Klopp: The integrated density of states for some random operators with nonsign definite potentials, J. Funct. Anal. **195** 12–47 (2002).
[Iw] A. Iwatsuka: Examples of absolutely continuous Schrödinger operators in magnetic fields, Publ. Res. Inst. Math. Sci. **21**, 385–401 (1985)

Spectral properties of weak-disorder quantum waveguides

Denis Borisov
(joint work with Ivan Veselić)

We consider random quantum waveguides in $\mathbb{R}^2$, determined by the following data. Let $(\omega_k)_{k\in\mathbb{Z}}$ be a sequence of independent, identically distributed, non-trivial random variables taking values in the interval $[0,1]$, $\kappa > 0$ a global coupling constant, $l \geq 1$ the length of one (periodicity) cell of the waveguide, and $g \in C_0^2(0,l)$ a single bump function. The following function determines the shape of the waveguide

$$G(x_1,\omega) := \sum_{k\in\mathbb{Z}} \omega_k\, g(x_1 - kl),$$

which is defined as the set

$$D_{\kappa,\omega} := \{x \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, \kappa G(x_1,\omega) < x_2 < \kappa G(x_1,\omega) + \pi\}.$$

The operator we consider is the Dirichlet Laplacian in $L_2(D_{\kappa,\omega})$. The main aim is to obtain an initial length scale estimate for such operator.

For the formulation of our results we need to consider also finite segments of the infinite waveguide. For $N \in \mathbb{N}$ we let

$$D_{\kappa,\omega}(N) := \{x \in \mathbb{R}^2 \mid 0 < x_1 < Nl, \kappa G(x_1,\omega) < x_2 < \kappa G(x_1,\omega) + \pi\}.$$

Denote by $\Gamma_{\kappa,\omega}(N)$ the upper and lower part of the boundary of $D_{\kappa,\omega}(N)$, i.e.,

$$\begin{aligned}\Gamma_{\kappa,\omega}(N) :=&\{x \in \mathbb{R}^2 \mid 0 < x_1 < Nl, x_2 = \kappa G(x_1,\omega)\}\\ &\cup\{x \in \mathbb{R}^2 \mid 0 < x_1 < Nl, x_2 = \kappa G(x_1,\omega) + \pi\}.\end{aligned}$$

The remaining part of the boundary $\partial D_{\kappa,\omega}(N) \setminus \Gamma_{\kappa,\omega}(N)$ is denoted by $\gamma_{\kappa,\omega}(N)$. Let $\mathcal{H}_{\kappa,\omega}(N)$ be the negative Laplace operator on $D_{\kappa,\omega}(N)$ with Dirichlet boundary conditions on $\Gamma_{\kappa,\omega}(N)$ and Neumann b.c. on $\gamma_{\kappa,\omega}(N)$. The lowest eigenvalue of $\mathcal{H}_{\kappa,\omega}(N)$ is indicated by $\lambda_{\kappa,\omega}(N)$.

Since the global coupling constant $\kappa > 0$ is arbitrary we may assume without loss of generality

$$\max\{\|g\|_{C[0,l]}, \|\|g'\|_{C[0,l]}, \|g''\|_{C[0,l]}\} = 1. \tag{1}$$

Denote the distribution measure of ω_k by μ. Then $\mathbb{P} = \bigotimes_{k\in\mathbb{Z}} \mu$ denotes the product measure on the configuration space $\Omega = \times_{k\in\mathbb{Z}}[0,1]$ whose elements we denote by ω. Set

$$\widetilde{g} := g - \frac{1}{l}\int_0^l g(t)\,\mathrm{d}\,t, \quad c_2 = \frac{9}{10}\|\widetilde{g}\|_{L_2(0,l)}, \quad c_3 = \frac{3}{5000}\|\widetilde{g}\|^2_{L_2(0,l)}.$$

Our first result gives the probabilistic estimate for $\lambda_{\kappa,\omega}(N)$.

Theorem 1. *Let g and μ as above be given, and $\gamma > 17$. Then there exists an initial scale N_1 such that if $N \geq N_1$ then the interval*

$$I_N := \left[\frac{2N^{\frac{1}{\gamma}-\frac{1}{2}}}{\mathbb{E}\{\omega_k\}\sqrt{c_2}}, c_3 N^{-\frac{15}{2\gamma}}\right]$$

is non empty. There exists a positive constant $c_4 > 0$ such that if $N \geq N_1$ and $\kappa \in I_N$ then

$$\mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 \leq N^{-1}\right) \leq N^{1-\frac{1}{\gamma}}\ \mathrm{e}^{-c_4 N^{1/\gamma}}.$$

By Combes-Thomas estimate this theorem implies an initial length scale estimate.

Theorem 2. *Under the assumptions of the previous theorem there exist constants C_1 and C_2 independent on N so that for all sets*

$$A, B \subset D_{\kappa,\omega}, \quad \mathrm{dist}\,(A,B) = \delta > 0,$$

the estimate

$$\mathbb{P}\left\{\|\chi_A\big(\mathcal{H}_{\kappa,\omega}(N)\big)^{-1}\chi_B\| \leq C_1 N \mathrm{e}^{-C_2 N^{-1/2}\delta}\right\} \geq 1 - N^{1-\frac{1}{\gamma}}\ \mathrm{e}^{-c_4 N^{1/\gamma}}$$

holds true for $N \geq N_1$, where χ_A and χ_B are the characteristic functions of the sets A and B.

D. Borisov was partially supported by RFBR (grant No. 08-01-97016) and the grant of the President of Russia for young scientists (MD-453.2010.1) and for Leading Scientific Schools.

Correlated Random Potentials and Dominated Schrödinger Cocycles

David Damanik

(joint work with Artur Avila)

We consider ergodic Schrödinger operators

$$[H_\omega\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n)$$

where $V_\omega(n) = f(T^n\omega)$ with Ω a compact metric space, $T : \Omega \to \Omega$ a homeomorphism, μ a T-ergodic Borel probability measure, and $f : \Omega \to \mathbb{R}$ continuous.

Let us discuss two examples. The Bernoulli-Anderson Model is obtained as follows: $\Omega = \{0,1\}^{\mathbb{Z}}$, $(T\omega)_n = \omega_{n+1}$, $\mu = \bigotimes_{n\in\mathbb{Z}}\nu$, where $\nu(0) = p$ and $\nu(1) = 1-p$, and $f(\omega) = \lambda\omega_0$. It is known that this model is spectrally and dynamically localized for every $p \in (0,1)$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

The Doubling Map Model is obtained as follows: $\Omega = \mathbb{R}/\mathbb{Z}$, $T\omega = 2\omega$, and $\mu =$ Leb. It is widely expected that this model is spectrally and dynamically localized for most non-constant continuous sampling functions $f : \Omega \to \mathbb{R}$. Partial results for small f are due to Chulaevsky-Spencer, Bourgain-Schlag, Sadel-Schulz-Baldes, and Avila-Damanik. Note that this model may be considered as the half-line restriction of a canonical whole-line model.

Motivated by these examples, we consider potentials generated by a uniformly hyperbolic homeomorphism T and a general (Hölder) continuous f. Using a recent generalization of Fürstenberg's Theorem, due to various authors including Avila, Bonatti, Gómez-Mont, and Viana, we describe a way to prove that the Lyapunov exponent associated with these Schrödinger operators is positive for most energies. This procedure is based on mutual accessibility of period points of T and inverse spectral theory for periodic Schrödinger operators.

Crucial to this approach is the notion of a dominated cocycle and the resulting holonomy maps that allow one to formulate a sufficient criterion for the positivity of the Lyapunov exponent in terms of the absence of a family of probability measures $\{\nu_\omega\}_{\omega\in\Omega}$ on $\mathbb{RP}^1$ that are invariant under the holonomies and the projective action of the cocycle. Indeed, the existence of such families for the energy-indexed family of cocycles associated with the given Schrödinger operators for a too large set of energies leads (via analyticity arguments) to the conclusion that the spectra corresponding to periodic points of T coincide, either locally or globally, which is a situation that cannot occur if there are sufficiently many periodic points.

An example of continuous matrix-valued Anderson model

Hakim Boumaza

We study localization properties of the following operator :

$$(1) \qquad H_\ell(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}} + V + \sum_{n\in\mathbb{Z}} \begin{pmatrix} c_1\omega_1^{(n)}\mathbf{1}_{[0,\ell]}(x-\ell n) & & 0 \\ & \ddots & \\ 0 & & c_N\omega_N^{(n)}\mathbf{1}_{[0,\ell]}(x-\ell n) \end{pmatrix},$$

acting on $L^2(\mathbb{R})\otimes\mathbb{C}^N$. The constants $c_1,\dots,c_N$ are non-zero real numbers, $\ell>0$ and V is a real symmetric matrix. The $(\omega_i^{(n)})_{n\in\mathbb{Z}}$ are sequences of independent and identically distributed (*i.i.d.*) random variables of common law ν such that $\{0,1\}\subset \operatorname{supp}\nu$ and $\operatorname{supp}\nu$ is bounded. Let Σ denote the almost-sure spectrum of $H_\ell(\omega)$.

This operator is an example of a larger class of quasi-one dimensional random Schrödinger operators of the form :

$$(2) \qquad H(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}} + \sum_{n\in\mathbb{Z}} V_\omega^{(n)}(x-\ell n),$$

acting on $L^2(\mathbb{R})\otimes\mathbb{C}^N$, where $N\geq 1$ is an integer and $\ell>0$ is a real number. For every $n\in\mathbb{Z}$, the functions $x\mapsto V_\omega^{(n)}(x)$ are symmetric matrix-valued functions,

supported on $[0, \ell]$ and bounded uniformely on x, n and ω. The sequence $(V_\omega^{(n)})_{n\in\mathbb{Z}}$ is a sequence of *i.i.d.* random variables.
In [1], we have already proven that, under suitable assumptions on the Fürstenberg group of these operators (*i.e.* the group generated by the transfer matrices), valid on an interval $I \subset \mathbb{R}$, they exhibit localization properties on I, both in the spectral and dynamical sense. After looking at the regularity properties of the Lyapunov exponents and of the integrated density of states, we had to prove a Wegner estimate and apply a multiscale analysis scheme to prove localization for these operators.
For the operator $H_\ell(\omega)$, we prove that for almost every background potential V, and for small values of the parameter ℓ, away from a finite set of critical energies, there will be localization in a certain compact interval depending only on ℓ and N.

Theorem 1. *For Lebesgue-almost every real symmetric matrix V, there exist a finite set $\mathcal{S}_{\mathrm{V}} \subset \mathbb{R}$ and a real number $\ell_C = \ell_C(N) > 0$ such that, for every $\ell \in (0, \ell_C)$, there exists a compact interval $I(N, \ell) \subset \mathbb{R}$ with the property :*

on every open interval $I \subset I(N, \ell) \setminus \mathcal{S}_{\mathrm{V}}$, with $I \cap \Sigma \neq \emptyset$, $H_\ell(\omega)$ exhibits exponential and dynamical localization on I.

According to the general result of [1], we only have to prove that the Fürstenberg group of $H_\ell(\omega)$ is Zariski-dense in the symplectic group for all energies in an interval except a finite set of critical energies. We actually prove a stronger statement which will be that the Fürstenberg group of $H_\ell(\omega)$ is equal to the symplectic group. The proof of this result is based upon a denseness criterion in semisimple Lie groups due to Breuillard and Gelander ([2]). All we have to do to apply this criterion is to construct elements of the Fürstenberg group close to the identity (that is why we need ℓ small), then to compute their logarithms and finally to show that these logarithms generate the Lie algebra of the symplectic group. The first two steps can be done for any background potential V, the third one is true only for a generic V. We actually show that it is true for a particular V_0 defined as the matrix having a null diagonal and coefficients on the upper and lower diagonals all equal to 1 and then we use an argument of analytic continuation to obtain the result for almost every real symmetric matrix V.
The result we obtain here for $H_\ell(\omega)$ could be improved in several ways. First we should be able to get rid of the assumption of $\ell > 0$ to be small and prove the same localization result for any ℓ. Simultaneously we should obtain that we have localization for all energies except a discrete set and not only in a bounded interval. Finally, we would like to replace, in the random potential, the characterisitic functions of $[0, \ell]$ by any function in $L^1(\mathbb{R})$ compactly supported in $[0, \ell]$.

References

[1] H. Boumaza, *Localization for a matrix-valued Anderson model*, Math. Phys. Anal. Geom. **12**(3), 255–286 (2009), DOI:10.1007/s11040-009-9061-3

[2] E. Breuillard and T. Gelander, On dense free subgroups of Lie groups, J. Algebra **261**(2), 448–467 (2003)

Low energy properties of the random displacement model

Günter Stolz
(joint work with Jeff Baker and Michael Loss)

We consider the random displacement model, a random Schrödinger operator given by $H_\omega = -\Delta + V_\omega(x)$ in $L^2(\mathbb{R}^d)$, where

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q(x - i - \omega_i).$$

The *single site potential* q is real-valued, bounded, supported in $[-r, r]^d$ for some $r \in (0, 1/2)$ and reflection symmetric in each variable. The displacements $\omega = (\omega_i)_{i\in\mathbb{Z}^d}$ are i.i.d. random vectors in $\mathbb{R}^d$ with distribution μ supported in the cube $[-d_{max}, d_{max}]^d$, where $d_{max} = \frac{1}{2} - r$, giving non-overlapping sites in V_ω.

In [1] a spectrally minimizing periodic configuration ω^* was identified, i.e. a configuration with the property that $\inf \sigma(H_{\omega^*}) = \inf \Sigma =: E_0$, where Σ is the almost sure spectrum of H_ω. This configuration is characterized through clusters of 2^d neighboring single-sites located in adjacent corners of their supporting unit cell.

One may find examples of single-site potentials q such that $\inf \sigma(H_\omega) = \inf \Sigma$ for *all* configurations ω, thus providing situations where $\inf \Sigma$ is not a fluctuation boundary of the spectrum. In all other cases, for example if q is nontrivial and sign-definite, and under the additional assumption $r < 1/4$, it was shown in [2] that, up to translations, ω^* is the unique minimizing periodic configuration if $d \geq 2$. In dimension $d = 1$ there are many other minimizing periodic configurations, which are characterized by the requirement that equally many ω_i take values d_{max} and $-d_{max}$, respectively, and none of them lies in $(-d_{max}, d_{max})$.

This difference between the one and multi-dimensional cases also leads to different low energy asymptotics of the integrated density of states of H_ω. An extreme case is given by the one-dimensional *Bernoulli displacement model*, where the support of μ is $\{\pm d_{max}\}$ with $\mathbb{P}(\omega_i = d_{max}) = \mathbb{P}(\omega_i = -d_{max}) = 1/2$. In this case it was shown in [2] that

$$N(E) \geq \frac{C}{\ln^2(E - E_0)}$$

for some $C > 0$ and E near E_0. In particular, the IDS is not Hölder-continuous at E_0.

On the other hand, it was shown in [5] that the uniqueness result from [2] implies a weak form of Lifshits tails for the IDS if $d \geq 2$. More precisely, it was shown in [5] that if $\operatorname{supp} \mu$ is finite and all 2^d corners $(\pm d_{max}, \ldots, \pm d_{max})$ are contained in $\operatorname{supp} \mu$, then

$$\limsup_{E \downarrow E_0} \frac{\log |\log N(E)|}{\log(E - E_0)} \leq -\frac{1}{2}.$$

The above results have led to a better understanding of the low-energy properties of the random displacement model. In fact, in an ongoing collaboration with F. Klopp, M. Loss and S. Nakamura we expect to find a proof of spectral and dynamical localization for the multi-dimensional random displacement model near the bottom of the spectrum. So far, results on localization for this model are only known in a semi-classical regime [3] and in certain situations with small displacements [4].

References

[1] J. Baker, M. Loss and G. Stolz, *Minimizing the ground state energy in a randomly deformed lattice*, Comm. Math. Phys. **283** (2008), 397-415

[2] J. Baker, M. Loss and G. Stolz, *Low energy properties of the random displacement model*, J. Funct. Anal. **256** (2009), 2725-2740

[3] F. Klopp, *Localization for semiclassical continuous random Schrödinger operators. II. The random displacement model*, Helv. Phys. Acta **66** (1993), 810-841

[4] F. Ghribi and F. Klopp, *Localization for the random displacement model at weak disorder*, Preprint `http://arxiv.org/abs/0904.2890`

[5] F. Klopp and S. Nakamura, *Lifshitz tails for generalized alloy type random Schrödinger operators*, Preprint `http://arxiv.org/abs/0903.2105`

Some Spectral Properties of Discrete Displacement Operators

Roger Nichols

(joint work with Günter Stolz)

In this talk we consider the spectral properties of discrete displacement operators on $\ell^2(\mathbb{Z}^d)$. A displacement operator is specified by $L = (L_i)_{i=1}^d, (b_i)_{i=1}^d \in \mathbb{N}^d$ with $b_i \leq L_i$ for $i = 1, \ldots, d$ and a function $q : \mathbb{Z}^d \to \mathbb{R}$ which is supported in $\times_{i=1}^d [1, b_i]$ and reflection symmetric about each of the d axes. For a "displacement configuration" $\omega \in (\times_{i=1}^d [0, L_i - b_i])^{\mathbb{Z}^d}$, a Hamiltonian is given by $h_\omega = h_0 + V_\omega$, where h_0 is the negative discrete Laplacian and V_ω is multiplication by the function

$$V_\omega(n) = \sum_{k \in \mathbb{Z}^d} q(n - kL - \omega_k).$$

As h_0 is a bounded operator, the family $\{h_\omega\}_\omega$ is a uniformly bounded family of self-adjoint operators and one is led to consider the existence of minimizing and maximizing configurations, ν^{min} and ν^{max}, for which

$$\begin{aligned} \min \sigma(h_{\nu^{min}}) &= \inf_\omega \min \sigma(h_\omega) \\ \max \sigma(h_{\nu^{max}}) &= \sup_\omega \max \sigma(h_\omega). \end{aligned}$$

We show that an extremal configuration is given by the cluster configuration, $\nu^* = \nu^{min} = \nu^{max}$, where the components of ν^* are given by $\nu_k^* = \langle \delta_1((-1)^{k_i})(L_i - b_i) \rangle_{i=1}^d$. This is the discrete analogue of the extremal configuration given by Baker, Loss, and Stolz for the continuum displacement model on $L^2(\mathbb{R}^d)$, [1].

As an application, we consider the random Bernoulli displacement model, which is a simple one-dimensional discrete displacement model with $L_1 = 2$, and $q = \lambda\delta_1$ for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$. For $\omega \in \{0, 1\}^{\mathbb{Z}}$, we assume $\omega_k = 0$ (resp. $\omega_k = 1$) with probability $p \neq 0$ (resp. $1 - p$), so that $\{h_{\omega,\lambda}\}_\omega$, with $h_{\omega,\lambda} = h_0 + V_\omega$, is an ergodic Schrödinger operator. Standard results from ergodic operator theory provide that

$\{h_{\omega,\lambda}\}_\omega$ has a deterministic spectrum, Σ_λ, in the sense that $\sigma(h_{\omega,\lambda}) = \Sigma_\lambda$ for almost every ω. Using the extremal configuration ν^* with $\nu^*_k = \delta_1((-1)^k)$, and positive solutions to eigenvalue equations, we calculate Σ_λ and we show that Σ_λ contains at least one gap for every $\lambda \neq 0$. We rely heavily on the characterization of the almost sure spectrum as the closure of the union of the spectra over all periodic configurations:

$$\Sigma_\lambda = \overline{\cup_{\omega \text{ periodic}} \sigma(h_{\omega,\lambda})}$$

and the fact that the extremal configuration ν^* is periodic. For $0 < |\lambda| \leq 2$, the almost sure spectrum Σ_λ is calculated explicitly by showing $\Sigma_\lambda = \sigma(h_{\nu^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$ where ω^1 is the periodic configuration with components $\omega^1_k = 1$. We are not able to handle the case $|\lambda| > 2$, but conjecture $\Sigma_\lambda = \sigma(h_{\nu^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$ in this case as well; we provide numerics to back the conjecture.

References

[1] J. Baker, M. Loss, and G. Stolz: *Minimizing the ground state energy in a randomly deformed lattice.* Commun. Math. Phys. **283**, No. 2, 397-415 (2008).

[2] J. Baker: *Spectral Properties of Displacement Models.* Ph.D. Thesis, University of Alabama at Birmingham (2007).

[3] W. Kirsch: An Invitation to Random Schroedinger Operators. arXiv:0709.3707v1.

[4] W. Kirsch, P. Müller: *Spectral properties of the Laplacian on bond-percolation graphs.* Math. Z. 252 (2006) 899 - 916.

[5] G. Teschl: *Jacobi Operators and Completely Integrable Nonlinear Lattices.* Matematical Surveys and Monographs **72**, American Mathematical Society, Providence (2000).

The universality classes in the parabolic Anderson model

Wolfgang König

We consider the parabolic Anderson model (PAM), the Cauchy problem for the heat equation with random potential

$$\begin{aligned} \partial_t u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z), \qquad t>0, z \in \mathbb{Z}^d, \\ u(t,0) &= \delta_0(z), \end{aligned}$$

where $\xi = (\xi(z))_{z\in\mathbb{Z}^d}$ is an i.i.d. random potential, and Δ is the discrete Laplace operator. This model describes a random mass flow through a random field of sinks and sources; in a branching process with random rates and migration it decribes the expected number of particles at time t in the site z.

The main task is the description of the random function $u(t,\cdot)$ and of its total mass, $U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z)$, asymptotically for large t. In particular, we aim at the understanding of those small regions from which the overwhelming part of the total mass stems from. Using a Fourier expansion, it is suggested that these regions are determined by those regions with (close to) maximal local Dirichlet eigenvalues of the Anderson Hamiltonian $\Delta + \xi$, and these are determined by the extraordinarily high potential peaks. This gives an intuitive interpretation of the spectrum of the Anderson Hamiltonian close to its top.

It turns out in a series of papers by various authors, that, under some mild regularity assumption on the potential distribution in the upper tail, there are basically only four different universality classes of asymptotic behaviors, determined by the size of the relevant islands and by a characteristic variational problem [1]. In the talk, we explain the characteristics of these classes and the general picture.

The main starting point of all proofs is a probabilistic representation of the solution in terms of the well-known Feynman-Kac formula,

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\delta_z(X_t)\Big],$$

where $(X_s)_{s\in[0,\infty)}$ is a continuous-time random walk in $\mathbb{Z}^d$ with generator Δ. We explain the large-t behavior in terms of this formula, i.e., in terms of the optimal behavior of the underlying random walk in the random field ξ.

References

[1] R. van der Hofstad, W. König and P. Mörters, *The universality classes in the parabolic Anderson model*, Commun. Math. Phys. **267:2** (2006), 307-353.

Some Random Hamiltonians modeling Superconductors

Werner Kirsch
(joint work with Bernd Metzger, Peter Müller)

We consider random operators that arise in models of superconductors (BCS theory). The operators are block matrices of the form:

$$M \;=\; \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

where A and B are selfadjoint operators in the Hilbert space $\mathcal{H}_0 = \ell^2(\mathbb{Z}^d)$.

The operator M acts in the natural way on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$. The first component in that space describes a particle, say an electron, the second component describes a hole. The electron can convert into a hole which mimics the creation of a Cooper pair and a hole can convert into an electron modeling the decay of a Cooper pair.

In our investigation the operator A is an Anderson Hamiltonian of the form

$$H_\omega \;=\; H_0 + V_\omega$$

where H_0 is the discrete Laplacian and $V_\omega(n)$ is a sequence of independent and identically distributed random variables. The operator B is a diagonal matrix with entries $b_\omega(n)$.

If $b_\omega(n) \geq b_0 > 0$ then M has a spectral gap between $-b_0$ and b_0. This is typical for superconductors.

Under reasonable assumptions on b we can prove a Wegner estimate for M showing that the density of states exists.

We also investigate the behavior of the density of states near the spectral edges $\pm b_0$. If $b = b_0$ is constant, then the density of states n has a square root singularity at the points $\pm b_0$, which is independent of the dimension d.

For certain cases we can prove Lifshitz behavior of the density of states near $\pm b_0$ for random b.

2×2 matrices, quadratic forms, and representation theorems

Vadim Kostrykin

(joint work with Luka Grubišić, Konstantin A. Makarov, Krešimir Veselić)

Consider a 2×2 matrix

$$B = \begin{pmatrix} a_+ & v \\ v^* & a_- \end{pmatrix}$$

with $a_\pm \in \mathbb{R}$ and $v \in \mathbb{C}$. Without loss of generality we can assume that $a_+ \geq 0$ and $a_- \leq 0$. The eigenvalues and the corresponding eigenvectors of the matrix B can be calculated explicitly in the form

$$\begin{aligned} \lambda_+ &= a_+ + |v| \tan\theta = vw \tan\theta, \\ \lambda_- &= a_- - |v| \tan\theta = v^* w^* \tan\theta \end{aligned} \tag{1}$$

and

$$x_+ = \begin{pmatrix} 1 \\ w \tan\theta \end{pmatrix}, \quad x_- = \begin{pmatrix} -w^* \tan\theta \\ 1 \end{pmatrix}, \tag{2}$$

where $w := \dfrac{v^*}{|v|}$ and $\theta := \dfrac{1}{2} \arctan \dfrac{2|v|}{a_+ - a_-} \leq \dfrac{\pi}{4}$ is the angle between the vectors x_+ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Let now $\mathcal{H}_0 \oplus \mathcal{H}_1$ be an orthogonal decomposition of the Hilbert space $\mathcal{H}$. Assume that B is a bounded self-adjoint operator in $\mathcal{H}$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ it can be represented as a block operator matrix

$$B = \begin{pmatrix} A_+ & V \\ V^* & A_- \end{pmatrix},$$

where the operators $A_\pm$ act in the Hilbert spaces $\mathcal{H}_\pm$, respectively, and the operator V maps $\mathcal{H}_-$ in $\mathcal{H}_+$ continuously.

There is a big amount of literature addressing the spectral theory of block operator matrices (see, e.g., [9] and references quoted therein). Surprisingly, the spectrum and the spectral subspaces of the operator B under the assumption that both A_+ and $-A_-$ are positive definite, can be described by formulae completely analogous to (1) and (2). In particular, the spectrum of the operator B is a union of two disjoint sets belonging to $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively,

$$\begin{aligned} &\operatorname{spec}(B) = \operatorname{spec}(A_+ + VW \tan\Theta) \cap \operatorname{spec}(A_- - V^* W_* \tan\Theta_*), \\ &\operatorname{spec}(A_+ + VW \tan\Theta) \subset \mathbb{R}_+, \quad \operatorname{spec}(A_- - V^* W_* \tan\Theta_*) \subset \mathbb{R}_-, \end{aligned}$$

The operators $A_+ + VW\tan\Theta$ and $A_- - V^*W_*\tan\Theta_*$ are in general not self-adjoint but similar to self-adjoint ones. The spectral subspaces of the operator B corresponding to the sets $\mathbb{R}_+$ and $\mathbb{R}_-$ possess representations as graph subspaces,

$$\text{Ran } E_B(\mathbb{R}_+) = \left\{ \begin{pmatrix} x \\ W\tan\Theta x \end{pmatrix} \,\middle|\, x \in \mathcal{H}_+ \right\},$$
$$\text{Ran } E_B(\mathbb{R}_-) = \left\{ \begin{pmatrix} -W_*\tan\Theta_* x \\ x \end{pmatrix} \,\middle|\, x \in \mathcal{H}_- \right\}.$$

Here W and W_* are partial isometries defined by the polar decompositions $X = W|X|$ and $X^* = W_*|X^*|$ of the operators X and X^*, respectively, where $X : \mathcal{H}_0 \to \mathcal{H}_1$ is a unique contractive solution to the Riccati equation

$$A_-X - XA_+ - XVX + V^* = 0.$$

The operators Θ and Θ_* are self-adjoint operator satisfying $0 \le \Theta, \Theta_* \le \pi/4$ and $\tan\Theta = |X|$, $\tan\Theta_* = |X^*|$. Geometrically they have the meaning of angles between the subspaces $\mathcal{H}_+$ and $\text{Ran}E_B(\mathbb{R}_+)$ (see, e.g., [1]).

A generalization of the results described above to the case of unbounded operators defined by indefinite (that is, not necessarily semibounded) quadratic forms will appear in [3], [4], [5]. In particular, we provide new straightforward proofs of the first and the second representation theorems for indefinite quadratic forms. Alternative approaches to the representation theorems for indefinite quadratic forms have been developed by A. McIntosh [6], [7], by Nenciu [8], and by Fleige, Hassi, and de Snoo (see [2] and references quoted therein).

References

[1] C. Davis and W. M. Kahan, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal. **7** (1970), 1 – 46.

[2] A. Fleige, S. Hassi, and H. de Snoo, *A Krein space approach to representation theorems and generalized Friedrichs extensions*, Act Sci. Math. (Szeged) **66** (2000), 633 – 650.

[3] L. Grubišić, V. Kostrykin, K. A. Makarov, K. Veselić, *Representation theorems for indefinite quadratic forms revisited*, in preparation.

[4] L. Grubišić, V. Kostrykin, K. A. Makarov, K. Veselić, *The tan* 2Θ *theorem for indefinite quadratic forms*, in preparation.

[5] L. Grubišić, V. Kostrykin, K. A. Makarov, K. Veselić, *Indefinite quadratic forms and weak solutions to the Riccati equation*, in preparation.

[6] A. McIntosh, *Bilinear forms in Hilbert space*, J. Math. Mech. **19** (1970), 1027 – 1045.

[7] A. McIntosh, *Hermitian bilinear forms which are not semibounded*, Bull. Am. Math. Soc. **76** (1970), 732 – 737.

[8] G. Nenciu, *Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms*, Commun. Math. Phys. **48** (1976), 235 – 247.

[9] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, 2008.

Reporter: Hakim Boumaza

Participants

Prof. Dr. Denis Borisov
Bashkir State Pedagogical
University
October Rev.St.3a
450000 Ufa
RUSSIA

Prof. Dr. Hakim Boumaza
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. David Damanik
Dept. of Mathematical Sciences
Rice University
P. O. Box 1892
Houston , TX 77251
USA

Prof. Dr. Alexander Elgart
Department of Mathematics
Virginia Polytechnic Institute and
State University
Blacksburg , VA 24061-0123
USA

Prof. Dr. Francois Germinet
Departement de Mathematiques
Universite de Cergy-Pontoise
33, boulevard du Port
F-95011 Cergy-Pontoise Cedex

Prof. Dr. Gian Michele Graf
Institut für Theoretische Physik
ETH Zürich
Hönggerberg
CH-8093 Zürich

Prof. Dr. Werner Kirsch
Mathematisches Institut der
Fernuniversität Hagen
Lützowstr. 125
58095 Hagen

Prof. Dr. Wolfgang D. König
Weierstraß-Institut für
Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.
Mohrenstr. 39
10117 Berlin

Prof. Dr. Vadim Kostrykin
Fachbereich Mathematik
Johannes Gutenberg Universität
Staudingerweg 9
55128 Mainz

Prof. Dr. Daniel Lenz
Mathematisches Institut
Friedrich-Schiller-Universität Jena
07737 Jena

Prof. Dr. Shu Nakamura
Graduate School of
Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Roger Nichols
Department of Mathematics, CH 452
University of Alabama at Birmingham
Birmingham , AL 35294-1170
USA

Prof. Dr. Hermann Schulz-Baldes
Department Mathematik
Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
91054 Erlangen

Prof. Dr. Günter Stolz
Department of Mathematics, CH 452
University of Alabama at Birmingham
Birmingham , AL 35294-1170
USA

Martin Tautenhahn
Fakultät für Mathematik
TU Chemnitz
09107 Chemnitz

PD Dr. Ivan Veselic
Fakultät für Mathematik
TU Chemnitz
09107 Chemnitz