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Combinatorial Representation Theory

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Abstract. The workshop brought together researchers from different fields in representation theory and algebraic combinatorics for a fruitful interaction. New results, methods and developments ranging from classical and modular representation theory, the theory of symmetric functions and Lie theory to cluster algebras and connections to physics and geometry were discussed.

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Introduction by the Organisers

The workshop on *Combinatorial Representation Theory*, was organised by Christine Bessenrodt (Hannover), Francesco Brenti (Roma), Alexander Kleshchev (Eugene) and Arun Ram (Melbourne). It was attended by 54 participants coming from Europe, North America, Japan and Australia. In the 23 long and 8 short talks – many given by young participants – fascinating new developments and significant progress on deep conjectures were presented. The schedule still left ample time for many discussions; in the ideal environment of the institute there was a lively exchange of ideas, also with researchers in the "Research in Pairs" program. Indeed, there were exciting questions and discoveries every evening, continuing cooperations as well as starting new joint research.

The scope of the meeting embraced representations coming from many directions: finite and algebraic groups and different types of algebras and Lie algebras. The fruitful focus point was the common interest in combinatorial aspects, dealt with by very different methods; in many cases, the representation theory of the symmetric groups and related groups or related algebras plays an important rôle.

A series of talks was dedicated to the recently discovered Khovanov-Lauda-Rouquier (KLR) algebras. These graded algebras defined for any Lie type have many important connections to Lie theory, categorification, and representation theory of symmetric groups and Hecke algebras. D. Hill spoke on his work with Melvin and Mondragon on the classification of irreducible modules over KLR algebras of finite type along the program initiated by Kleshchev and Ram. M. Vazirani presented her work with Lauda which yields another classification of irreducible modules over KLR algebras of arbitrary type via defining and identifying a crystal structure on them. A. Mathas talked on his work with Hu on a cellular structure on cyclotomic KLR algebras of finite and affine type A. In view of the work of Brundan and Kleshchev this transports into a completely new *graded* cellular structure on cyclotomic Hecke algebras, which promises to be very important.

Modular representation theory was also a focus point. S. Ariki spoke about modular branching rules for affine Hecke algebras of type A. This is related to the identification of various classifications of irreducible modules over affine Hecke algebras. N. Jacon spoke on canonical bases in higher level Fock spaces in connection with Ariki's categorification theorem and modular representation theory of cyclotomic Hecke algebras. M. Fayers reported on the recent progress concerning the classification of irreducible Specht modules over cyclotomic Hecke algebras.

His work with S. Khoroshkin on the representation theory of usual and twisted Yangians was described by M. Nazarov, while S. Goodwin spoke on his work with J. Brown on representations of finite W-algebras of classical types corresponding to special nilpotent orbits, generalizing work of Brundan and Kleshchev in type A.

Classical representation theory continues to be a central topic. O. Yacobi's talk had classical invariant theory as its core, and I. Gordon discussed problems from the invariant theory of complex reflection groups, whose solutions involved the representation theory of rational Cherednik algebras. C. Caselli introduced projective reflection groups and showed that Gelfand models and bases for their coinvariant algebras can be obtained via the combinatorics of a dual projective reflection group. Some of these results are new even for the Weyl groups of type D. V. Miemietz explained deep results with W. Turner extending the applications of the Dlab-Ringel construction from finite dimensional to infinite dimensional algebras. G. Malle talked on work with G. Navarro, explaining combinatorics underlying the confirmation of a recent conjectured character criterion of nilpotent blocks of finite groups for many quasi-simple groups, J. Comes spoke on the representation theory of Deligne's tensor category $\underline{\mathrm{Re}}\mathrm{p}(S_t)$ (for each $t \in \mathbb{C}$) related to the symmetric group S_t, when t is an integer, and G. Han explained techniques for discovering surprising new hook length formulas. A. Henderson spoke on enumerative results for nilpotent orbits in classical groups of type B or C which refine results by Lusztig.

In an important talk which stood on its own, P. Fiebig spoke on his work with Arakawa on blocks of representations of affine Kac-Moody algebras at the critical level. He explained how to relate such blocks to the blocks of the category of modules over an associated small quantum group.

Symmetric function theory continues to be a vibrant area of research. M. Yip explained how the alcove walk techniques from the proof of the Ram-Yip formula for expanding Macdonald polynomials in terms of monomial symmetric functions apply to provide a Littlewood-Richardson rule for Macdonald polynomials; this generalizes the classical rule for multiplying Weyl characters and recent results on the product of Hall-Littlewood symmetric functions. J. Haglund and C. Lenart covered several aspects of the relation between the Ram-Yip formula and the Haglund-Haiman-Loehr formula; their relations to expansions in terms of Demazure bases and quasisymmetric functions were discussed by Haglund and Yip. The theory of quasisymmetric functions has seen increasing importance in the last ten years, including applications to Macdonald polynomials and Kazhdan-Lusztig theory. As explained by S. van Willigenburg, the quasisymmetric Schur functions satisfy natural analogues of many results that hold for the classical Schur functions, providing a good justification for their name. This opens natural major lines of research. A. Schilling presented a Murnaghan-Nakayama rule for noncommutative k-Schur functions, a result which is new even in the commutative case.

Connections to mathematical physics and representations of affine Lie algebras appeared in several talks and blossomed in the talk of J. de Gier explaining how certain parabolic Kazhdan-Lusztig polynomials and Macdonald polynomial theories enter in the solutions of eigenvalue problems in statistical physics: specifically, fully packed loop models. M. Marietti presented a closed combinatorial formula for the parabolic Kazhdan-Lusztig polynomials of the tight quotients of the symmetric groups; this implies the known formula for the maximal quotients and relies on a new class of superpartitions with fermionic number one. The talk of C. Stroppel focused on the interplay between symmetric functions, geometry, mathematical physics (quantum cohomology, Bethe Ansatz and Verlinde formulas). J. Kujawa explained the notion of categorical dimension in ribbon categories and a suitable graphical calculus; these categories include categories of finite dimensional representations of groups, Lie algebras, superalgebras and quantum groups.

Deep connections to geometry appeared also in other talks. S. Gaussent explained some of the "compression" by using walks on the one-skeleton of an affine Tits building. T. Lam gave detailed information about the structure of the totally positive part of a loop group, and a mysterious similarity between his product constructions and the cluster algebra constructions of regular functions in the coordinate ring of the unipotent part of a Kac-Moody group which appeared in J. Schröer's talk was noticeable. Indeed, the cluster algebras introduced by Fomin and Zelevinsky are a very important and active research area. L. Williams, J. Schröer and D. Hernandez impressively used cluster algebra technology in the study of preprojective algebras, canonical bases, total positivity, moduli spaces of surfaces, and the finite dimensional representations of affine Lie algebras. A highlight was L. Williams' talk on a proof of the celebrated nonnegativity conjecture for cluster algebras arising from punctured surfaces. Her result gives a combinatorial interpretation of the coefficients of the Laurent monomials in the cluster variables, and includes the nonnegativity result for cluster algebras of finite type.

Workshop: Combinatorial Representation Theory

Table of Contents

Abstracts

Catalan Numbers for complex reflection groups

Iain G. Gordon
(joint work with Stephen Griffeth)

This is a report on joint work with S. Griffeth which will be published as [3]. The motivation for the talk was also the beautiful construction by Y. Berest and O. Chalykh of quasi-invariants in [1]. Underlying both of these works is the appearance of the so-called KZ-twists, which first appeared in the work of Opdam, [5], [6].

1.1. Let $\mathfrak{h}$ be a complex vector space of dimension n and W a finite complex reflection group with reflection representation $\mathfrak{h}$. We assume that $\mathfrak{h}$ is irreducible as a representation of W. Denote by $\mathcal{A}$ the set of reflecting hyperplanes, and $\mathcal{R}$ the set of pseudo-reflections. We let $N := |\mathcal{A}|$ and $N^* := |\mathcal{R}|$. Define the *generalised Coxeter number* h to be the integer

$$h = \frac{N + N^*}{n}. \tag{1}$$

1.2. Let P denote ring of polynomial functions on $\mathfrak{h}$. The coinvariant algebra $P^{\mathsf{co}W} := P/(P_+^W)$ carries the regular representation of W. Given $U \in \mathsf{Irrep}(W)$, the *exponents* of U

$$e_1(U) \leq \ldots \leq e_{\dim U}(U)$$

are the homogeneous embedding degrees of U in $P^{\mathsf{co}W}$. Set $d_i = e_i(V) + 1$ for $i = 1, \ldots, n$, the degrees of a minimal set of homogeneous elements generating P^W.

1.3. Let q be an indeterminate, and for any positive integer i, set $[i]_q := 1 + q + \cdots + q^{i-1}$. For a non-negative integer m, we define the mth *q-Fuss-Catalan number* of $(\mathfrak{h}, W)$ to be

$$C_W^{(m)}(q) = \prod_{i=1}^{n} \frac{[mh + 1 + e_i(\Psi^m(\mathfrak{h}^*)^*)]_q}{[d_i]_q}, \tag{2}$$

where $\Psi \in \mathrm{Perm}(\mathsf{Irrep}(W))$ is to be explained below.

Theorem 1. *The rational function $C_W^{(m)}(q)$ belongs to $\mathbb{N}[q]$. Two reasons [1] for this are:*

(1) $C_W^{(m)}(q)$ is the Hilbert series of $(P/(\Theta))^W$ where Θ is a homogeneous system of parameters of degree $mh+1$ carrying the W-representation $\Psi^m(\mathfrak{h}^*)$;

(2) $C_W^{(m)}(q)$ is the graded character of a finite dimensional irreducible representation of the spherical rational Cherednik algebra $U_{m+1/h}(W)$.

[1] There is actually a restriction on the validity of this theorem at the moment: one must assume that the Iwahori-Hecke algebras $\mathcal{H}_q(W)$'s form a flat family as q varies. This is known in all but finitely many cases, and conjectured in general.

1.4. The proof of the theorem proceeds like many others in this field. One confirms the simplest case (here $m = 0$) and then uses some kind of shift functor to pass from m to $m+1$. Iteration proves the theorem. In our case the shift functor comes from passing from category $\mathcal{O}_c(W)$ for the rational Cherednik algebra $H_c(W)$ (an algebra that depends on the choice of a parameter $c \in \mathbb{C}[\mathcal{R}]^{\mathrm{ad}W}$) to $\mathcal{O}_{c+\mathbf{1}}(W)$. Category $\mathcal{O}_c(W)$ has a set of standard representations $\Delta_c(U)$ labelled by $U \in \mathsf{Irrep}(W)$ and it admits an analytically constructed functor $\mathsf{KZ_c} : \mathcal{O}_c(W) \longrightarrow \mathcal{H}_{q(c)}(W)$-mod to the Iwahori-Hecke algebra $\mathcal{H}_{q(c)}(W)$ where $q(c) = \exp(2\pi i c)$. Now KZ_c is an equivalence for generic choices of c. So observing that if $s \in \mathbb{Z}[\mathcal{R}]^{\mathrm{ad}W}$ then $\mathcal{H}_{q(c)}(W) = \mathcal{H}_{q(c+s)}(W)$, one can construct $\Psi_s \in \mathrm{Perm}(\mathsf{Irrep}(W))$ by the rule $\mathsf{KZ}_{c+s}^{-1} \circ \mathsf{KZ}_c(\Delta_c(U)) = \Delta_{c+s}(\Psi_s(U))$ for generic c. The permutation Ψ in the statement of the theorem is then $\Psi_{\mathbf{1}}$.

1.5. The permutation Ψ_s is called a KZ-twist. It appeared first in the work of Opdam, [5] and [6], where it was used to explain certain symmetries among the exponents of irreducible representations of W. In the recent work of Berest and Chalykh, [1], a new construction is given (based directly on rational Cherednik algebra representation theory) which links to the study of quasi-invariants and hence integrable systems, and which removes a great deal of the dependence on flatness properties of Hecke algebras. It seems that the KZ-twists should play a fundamental role in the representation theory of rational Cherednik algebras.

1.6. The description of $C_W^{(1)}(q)$ in the second part of the theorem allows us to define a (q,t)-Catalan number for all W. This agrees with the definition of Garsia-Haiman in the symmetric group case.

1.7. The q-Fuss Catalan numbers display a shadow of the cyclic sieving phenomenon. Let d be a *regular* number for (V, W) and let $\zeta = \exp(2\pi\sqrt{-1}/d)$. Then $C_W^{(m)}(\zeta^t)$ is a positive integer for all m and all t. In general, this is not understood combinatorially, but do see [2].

1.8. The pair $(\mathfrak{h}, W)$ is *well-generated* if W can be generated by n pseudo-reflections. In this case the formula for the q-Fuss-Catalan number simplifies,

$$C_W^{(m)}(q) = \prod_{i=1}^{n} \frac{[mh + d_i]_q}{[d_i]_q}.$$

This is the standard definition of q-Fuss-Catalan numbers for well-generated groups which is used throughout the literature. Furthermore, a case-by-case observation made by Malle [4] in studying Galois automorphisms of $\mathcal{H}_q(W)$ shows that $\Psi(\mathfrak{h}^*) = \mathfrak{h}^*$ if and only if $(\mathfrak{h}, W)$ is well-generated. Thus, the first part of the theorem above confirms a conjecture of [2]

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Cluster algebras arising in Lie theory

Jan Schröer
(joint work with Christof Geiß, Bernard Leclerc)

Let Q be a finite quiver, and let $\mathfrak{g}$ be the associated symmetric Kac-Moody Lie algebra. The Kac-Moody group attached to $\mathfrak{g}$ as in [4, Chapter VI] comes with a pair of subgroups N and N_- (denoted by $\mathcal{U}$ and $\mathcal{U}^-$ in [4]).

For an element w in the Weyl group W of $\mathfrak{g}$ let $N(w) := N \cap (w^{-1}N_-w)$ and $N'(w) := N \cap (w^{-1}Nw)$. These are pro-unipotent pro-groups. The group $N(w)$ is finite-dimensional of dimension length(w), and $N'(w)$ is infinite dimensional if $\mathfrak{g}$ is infinite dimensional. Multiplication in N yields a bijective map $N(w) \times N'(w) \to N$. Thus the algebra $\mathbb{C}[N]^{N'(w)}$ of $N'(w)$-invariant functions on N is isomorphic to the coordinate ring $\mathbb{C}[N(w)]$ of $N(w)$.

Next, let Λ be the preprojective algebra associated to Q, and let $S_1,\dots,S_n$ be the 1-dimensional simple Λ-modules corresponding to the vertices of Q. For a nilpotent Λ-module X and $\mathbf{a} = (a_r,\dots,a_1) \in \mathbb{N}^r$ let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of flags

$$X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$$

of submodules of X such that $X_{k-1}/X_k \cong S_{i_k}^{a_k}$ for all $1 \le k \le r$. The varieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ were first introduced by Lusztig for his Lagrangian construction of the enveloping algebra $U(\mathfrak{n})$, where $\mathfrak{n}$ denotes the positive part of $\mathfrak{g}$. Let $x_i(t)$ be the one-parameter subgroup of N associated to the simple root α_i of $\mathfrak{g}$. For each reduced expression $\mathrm{i} = (i_r,\dots,i_1)$ of w, we get an injective map

$$\underline{x}_{\mathrm{i}} : (t_r,\dots,t_2,t_1) \mapsto x_{i_r}(t_r)\cdots x_{i_2}(t_2)x_{i_1}(t_1)$$

from $(\mathbb{C}^*)^r$ to N. Dualizing Lusztig's construction, we can associate with X a regular function $\varphi_X \in \mathbb{C}[N]$ satisfying

$$\varphi_X(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \sum_{\mathbf{a}\in\mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X})\mathbf{t}^{\mathbf{a}}.$$

Here $\mathbf{t} = (t_r,\dots,t_1) \in (\mathbb{C}^*)^r$, $\mathbf{t}^{\mathbf{a}} := t_r^{a_r}\cdots t_2^{a_2}t_1^{a_1}$, and χ denotes the topological Euler characteristic. The functions in $\mathbb{C}[N]^{N'(w)}$ are already determined by their values on $\{\underline{x}_{\mathbf{i}}(\mathbf{t}) \mid \mathbf{t} \in (\mathbb{C}^*)^r\}$. We want to determine which of the φ_X are contained in $\mathbb{C}[N]^{N'(w)}$.

The functions φ_X generate $\mathbb{C}[N]$ as a vector space, and they capture many aspects of the representation theory of the Kac-Moody algebra $\mathfrak{g}$. For example, one can use φ-functions to construct all integrable highest weight $\mathfrak{g}$-modules. Also, the dual of Lusztig's semicanonical basis of $U(\mathfrak{n})$ consists of a subset of all φ-functions. (One can identify the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$ with $\mathbb{C}[N]$.) Conjecturally, the specialized dual canonical basis also consists of φ-functions.

Buan, Iyama, Reiten and Scott [1] have attached to each element w of the Weyl group W of $\mathfrak{g}$ a subcategory $\mathcal{C}_w$ of $\mathrm{mod}(\Lambda)$. They show that the categories $\mathcal{C}_w$ are Frobenius categories and the corresponding stable categories $\underline{\mathcal{C}}_w$ are Calabi-Yau categories of dimension two. (These results were also discovered and proved independently in [2] in the special case when w is an *adaptable* element of W.)

For a Λ-module X and a simple module S_j let $\mathrm{socle}_{(j)}(X) := \mathrm{socle}_{S_j}(X)$ be the sum of all submodules U of X with $U \cong S_j$. For a sequence $(j_t, \ldots, j_1)$ of indices with $1 \le j_p \le n$ for all p, there is a unique chain

$$0 = X_t \subseteq \cdots \subseteq X_1 \subseteq X_0 \subseteq X$$

of submodules of X such that $X_{p-1}/X_p = \mathrm{socle}_{(j_p)}(X/X_p)$. Define

$$\mathrm{socle}_{(j_t,\ldots,j_1)}(X) := X_0.$$

Again, let $\mathbf{i} = (i_r, \ldots, i_1)$ be a reduced expression of an element w of W. By $I_1, \ldots, I_n$ we denote the injective envelopes of the simple Λ-modules $S_1, \ldots, S_n$, respectively. For $1 \le k \le r$ let

$$V_k := \mathrm{socle}_{(i_k\ldots,i_1)}(I_{i_k}),$$

and set $V_{\mathbf{i}} := V_1 \oplus \cdots \oplus V_r$. (The module $V_{\mathbf{i}}$ is dual to the cluster-tilting object constructed in [1, Section III.2].) Let $\mathcal{C}_w$ be the subcategory of $\mathrm{mod}(\Lambda)$ generated by $V_{\mathbf{i}}$. The category $\mathcal{C}_w$ depends only on w, and not on the chosen reduced expression $\mathbf{i}$ of w. (If Q is a Dynkin quiver, and $w = w_0$ is the longest Weyl group element, then $\mathcal{C}_w = \mathrm{mod}(\Lambda)$.)

Theorem 1. *For any nilpotent Λ-module X the following are equivalent:*
(i) $\varphi_X \in \mathbb{C}[N]^{N'(w)}$,
(ii) $X \in \mathcal{C}_w$.

For each $1 \le k \le r$ there is a canonical embedding

$$\iota_k \colon V_{k^-} \to V_k$$

where $k^- := \max\{0 \le s \le k-1 \mid i_s = i_k\}$ and $V_0 := 0$. Let M_k be the cokernel of ι_k.

Theorem 2. *The following hold:*
(i) $\mathbb{C}[N]^{N'(w)} = \mathbb{C}[\varphi_{M_1}, \ldots, \varphi_{M_r}]$.
(ii) *The subset $\mathcal{S}^*_w := \mathcal{S}^* \cap \mathbb{C}[N]^{N'(w)}$ of the dual semicanonical basis $\mathcal{S}^*$ of $\mathbb{C}[N]$ is a $\mathbb{C}$-basis of $\mathbb{C}[N]^{N'(w)}$.*

From the definitions it looks as if the functions φ_X are very difficult to compute. However, the functions φ_{V_k} can be interpreted as *generalized minors*, and they can be described very explicitly by a simple recursion. The tuple $(\varphi_{V_1}, \ldots, \varphi_{V_r})$ serves now as an *initial cluster* of a cluster algebra structure on $\mathbb{C}[N]^{N'(w)}$. Starting with $(\varphi_{V_1}, \ldots, \varphi_{V_r})$ we can apply (combinatorially defined) Fomin-Zelevinsky cluster mutations. This yields again tuples of φ-functions $(\varphi_{T_1},\dots,\varphi_{T_r})$, called *clusters*. Expressions of the form

$$\varphi_{T_1}^{m_1}\varphi_{T_2}^{m_2}\cdots\varphi_{T_r}^{m_r}$$

with $m_k \geq 0$ for all k are called *cluster monomials*. The functions φ_{T_k} can be described explicitly. In particular, the Euler characteristics $\chi(\mathcal{F}_{\mathbf{i},\mathbf{a},T_k})$ can be obtained inductively from the initial cluster $(\varphi_{V_1},\dots,\varphi_{V_r})$.

Theorem 3. *The algebra $\mathbb{C}[N]^{N'(w)}$ is a cluster algebra in a natural way, and the set of cluster monomials is a subset of $\mathcal{S}_w^*$.*

The proofs of Theorems 1, 2 and 3 can be found in [3].

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Positivity and positive bases for cluster algebras from surfaces

Lauren Williams

(joint work with Gregg Musiker, Ralf Schiffler)

Since their introduction by Fomin and Zelevinsky [FZ1], cluster algebras have been related to diverse areas of mathematics such as total positivity, quiver representations, Teichmüller theory, tropical geometry, Lie theory, and Poisson geometry. A main outstanding conjecture about cluster algebras is the *positivity conjecture*, which says that if one fixes a cluster algebra $\mathcal{A}$ and an *arbitrary* seed $(\mathbf{x},\mathbf{y},B)$, one can express each cluster variable $x \in \mathcal{A}$ as a Laurent polynomial with *positive coefficients* in the variables of $\mathbf{x}$.

There is a class of cluster algebras arising from *surfaces with marked points*, introduced by Fomin, Shapiro, and Thurston in [FST] (generalizing work of Fock and Goncharov [FG1, FG2] and Gekhtman, Shapiro, and Vainshtein [GSV]), and further developed in [FT]. This class is quite large: (assuming rank at least three) it has been shown [FeShTu] that all but finitely many skew-symmetric cluster algebras of *finite mutation type* come from this construction. Note that the class of cluster algebras of finite mutation type in particular contains those of finite type.

In this paper we give a combinatorial expression for the Laurent polynomial which expresses any cluster variable in terms of any seed, for any cluster algebra arising from a surface. As a corollary, we prove the positivity conjecture for all such cluster algebras.

A *cluster algebra* $\mathcal{A}$ of rank n is a subalgebra of an *ambient field* $\mathcal{F}$ isomorphic to a field of rational functions in n variables. Each cluster algebra has a distinguished set of generators called *cluster variables*; this set is a union of overlapping algebraically independent n-subsets of $\mathcal{F}$ called *clusters*, which together have the structure of a simplicial complex called the *cluster complex*. The clusters are related to each other by birational transformations of the following kind: for every cluster $\mathbf{x}$ and every cluster variable $x \in \mathbf{x}$, there is another cluster $\mathbf{x}' = \mathbf{x} - \{x\} \cup \{x'\}$, with the new cluster variable x' determined by an *exchange relation* of the form

$$xx' = y^+ M^+ + y^- M^-.$$

Here y^+ and y^- lie in a *coefficient semifield* $\mathbb{P}$, while M^+ and M^- are monomials in the elements of $\mathbf{x} - \{x\}$. There are two dynamics at play in the exchange relations: that of the monomials, which is encoded in the exchange matrix, and that of the coefficients.

A classification of finite type cluster algebras – those with finitely many clusters – was given by Fomin and Zelevinksy in [FZ2]. They showed that this classification is parallel to the famous Cartan-Killing classification of complex simple Lie algebras, i.e. finite type cluster algebras either fall into one of the infinite families A_n, B_n, C_n, D_n, or are of one of the exceptional types E_6, E_7, E_8, F_4, or G_2. Furthermore, the *type* of a finite type cluster algebra depends only on the dynamics of the corresponding exchange matrices, and not on the coefficients.

Our main results of [MSW] are combinatorial formulas for cluster expansions of cluster variables with respect to any seed, in any cluster algebra coming from a surface. Our formulas are manifestly positive, so as a consequence we obtain the following result.

Theorem 1. *Let $\mathcal{A}$ be any cluster algebra arising from a surface, where the coefficient system is of geometric type, and let Σ be any initial seed. Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.*

Our results generalize those in [S2], where cluster algebras from the (much more restrictive) case of surfaces without punctures were considered. This work in turn generalized [ST], which treated cluster algebras from unpunctured surfaces with a very limited coefficient system that was associated to the boundary of the surface. The very special case where the surface is a polygon and coefficients arise from the boundary was covered in [S], and also in unpublished work [CP, FZ3]. See also [Pr2]. Recently [MS] gave an alternative formulation of the results of [S2], using perfect matchings as opposed to *T-paths*.

Many others have worked on finding Laurent expansions of cluster variables, and on the positivity conjecture. However, most of the results so far obtained have strong restrictions on the cluster algebra, the choice of initial seed or on the system of coefficients. For finite type cluster algebras, the positivity conjecture with respect to a bipartite seed follows from [FZ4, Corollary 11.7]. Other work [M] gave cluster expansions for coefficient-free cluster algebras of finite classical types with respect to a bipartite seed.

A recent tool in understanding Laurent expansions of cluster variables is the connection to quiver representations and the introduction of the cluster category [BMRRT] (see also [CCS1] in type A). The work [CR] used this approach to make progress towards the positivity conjecture for coefficient-free acyclic cluster algebras, with respect to an acyclic seed.[1] Building on [HL] and [CK2], Nakajima recently used quiver varieties to prove the positivity conjecture for cluster algebras that have at least one bipartite seed, with respect to any cluster [N]. This is a very strong result, but it does not overlap very much with our Theorem 1. Note that a bipartite seed is in particular acyclic, but not every acyclic type has a bipartite seed; e.g. the affine type $\tilde{A}_2$ does not. Further, the only surfaces that give rise to acyclic cluster algebras are the polygon with 0, 1, or 2 punctures, and the annulus (corresponding to the finite types A and D, and the affine types $\tilde{D}$ and $\tilde{A}$, respectively). All other surfaces yield non-acyclic cluster algebras, see [FST, Corollary 12.4].

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Combinatorics Associated to Type A Nonsymmetric Macdonald Polynomials

Jim Haglund

In 1988 Macdonald [8],[9] introduced symmetric functions $P_\lambda(X;q,t)$ which contain most of the previously studied families of symmetric functions as special cases. The $P_\lambda(X;q,t)$ are multivariate orthogonal polynomials which have become increasingly important in recent years. In 1995 Macdonald [10] introduced a refinement of this theory involving polynomials $E_\alpha(X;q,t)$, now called nonsymmetric Mcadonald polynomials, which also satisfy an orthogonality relation, and which are a basis for the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n](q,t)$ whose coefficients are rational functions in q,t. Here λ is a partition and α a weak composition. There are versions of the P_λ and E_α for arbitrary affine root systems, and Cherednik showed many of the properties of Macdonald polynomials have an interpretation in terms of the representation theory of his double affine Hecke algebra.

The P_λ and E_α have "integral forms" J_λ and $\mathcal{E}_\alpha$ associated to them, which are just scalar multiples of them which clear all denominators, resulting in a polynomial (i.e. an element of $\mathbb{Q}[x_1,\ldots,x_n,q,t]$). A few years ago Haiman, Loehr and the speaker [2] proved a combinatorial formula for the J_λ, and in subsequent work [3] obtained a corresponding combinatorial expression for the $\mathcal{E}_\alpha$. We will mostly use the notational conventions occurring in the discussion of the E_α formula from Appendix C of [1]. It involves nonattacking fillings, which are fillings of the diagram α' whose ith column has height α_i, with positive integers so that no two entries in the same row are equal, and no two entries in successive rows, with the entry in the upper row strictly to the right of the lower entry, are equal. Then

(1)
$$\mathcal{E}_\alpha(X;q,t)=\sum_\sigma x^\sigma q^{\mathrm{maj}}t^{\mathrm{coinv}}\prod_{\substack{s\in\alpha'\\ \sigma(s)\neq\sigma(South(s))}}(1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1})\prod_{\substack{s\in\alpha'\\ \sigma(s)=\sigma(South(s))}}(1-t),$$

where South(s) is the square right below s. The statistic maj is just the sum of the major index of the columns, while the more intricate statistic coinv is a sum, over pairs of squares in the same row, of a generalized concept of coinversion. Arm and leg lengths for composition diagrams are the same as in work of Knop and Sahi on Jack polynomials [6].

In (1) there is also a "basement" consisting of a row of squares below the diagram, which are filled with the numbers $(n, n-1, \ldots, 1)$, and which are used in the computation of maj, coinv, and the description of nonattacking. To get the $\mathcal{E}_\alpha$ we need to use the diagram with column heights $(\alpha_n, \ldots, \alpha_1)$. A corresponding formula for J_λ, where λ is the partition rearrangement of α, can be obtained by simply changing the basement to $(2n, 2n-1, \ldots, n+1)$. Also, by changing the basement to $(1, 2, \ldots, n)$ and letting the ith column have height α_i, we get the version of the nonsymmetric Macdonald polynomial studied by Marshall [11], which we denote $\mathcal{E}'_\alpha$, which are essentially related to the $\mathcal{E}_\alpha$ by reversing the order of the variables, reversing the order of the parts of α, and sending $q \to 1/q$, $t \to 1/t$.

Note that the J_λ version of (1) implies that for $k \in \mathbb{N}$,

$$J_\lambda(X; q, q^k)/(1-q)^n|_{m_\lambda} \mathbb{N}[q], \tag{2}$$

i.e. the coefficient of a monomial symmetric function in (2) is a positive polynomial in q, since when $t = q^k$, each of the factors $(1 - q^{\text{leg}+1} t^{\text{arm}+1})$ or $(1-t)$ becomes $(1-q^m)$ for some m. There are n of these factors, and combining them with the n powers of $1-q$ in the denominator of (2) we get a product of q-integers. Maple calculations indicate a stronger condition holds, namely Schur positivity.

Conjecture 1 For $k \in \mathbb{N}$,

$$J_\lambda(X; q, q^k)/(1-q)^n|_{s_\lambda} \in \mathbb{N}[q]. \tag{3}$$

During the talk Arun Ram suggested that Conjecture 1 can be embedded in a family of conjectures, where you expand $J_\lambda(X; q, q^m)$ in terms of the basis $J_\mu(X; q, q^{m-1})$, with a positivity condition for each m. Since $P_\mu(X; q, q) = s_\mu$, Ram's conjecture for $m = 2, 3, ..., k$ implies Conjecture 1. (Since the P_μ are not quite the J_μ, some slight modification in the statement of Ram's conjecture is needed.) After the talk Ram described some geometric heuristics involving Macdonald polynomials and quotients of determinants to the speaker which led Ram to his conjecture. These heuristics suggest some version of this phenomenon should hold for the $E_\alpha(X; q, t)$.

There is a lot of interesting combinatorics associated to the case $q = t = 0$ of (1). It is known that the Demazure character, or key polynomial, $\mathcal{K}_\alpha(x_1, \ldots, x_n)$ equals $\mathcal{E}_\alpha(x_1, \ldots, x_n; 0, 0)$, and furthermore the Demazure atom, or standard bases, $\mathcal{A}_\alpha(x_1, \ldots, x_n)$ equals $\mathcal{E}'_\alpha(x_1, \ldots, x_n; 0, 0)$. Standard bases were introduced by Lascoux and Schützenberger [7] in their study of Schubert varieties. They showed that the Schubert polynomial is a positive sum of key polynomials, and that the key polynomial is a positive sum of Demazure atoms. Further results on key polynomials were obtained by Reiner and Shimozono [14]. Now if you start with an identity of Macdonald which expresses P_λ as a sum, over compositions α whose rearrangement α^+ into partition order is λ, of $\mathcal{E}'_\alpha(x_1,\dots,x_n;q,t)$, and then set $q=t=0$, we get

$$s_\lambda = \sum_{\substack{\alpha \\ \alpha^+=\lambda}} \mathcal{A}_\alpha(x_1,\dots,x_n). \tag{4}$$

S. Mason [12], [13] has given a combinatorial proof of this identity by introducing a generalization of the RSK algorithm.

Recently K. Luoto, S. Mason, S. van Willigenburg and the speaker [4], [5] have introduced a new basis for the ring of quasisymmetric functions called quasisymmetric Schur functions, denoted $\mathrm{QS}_\beta(x_1,\dots,x_n)$, where β is a (strong) composition of n. It is defined as the sum, over all (weak) compositions α which are shuffles of the parts of β and $n-\ell(\beta)$ zeros, of $\mathcal{A}_\alpha$. Properties of Mason's RSK algorithm are used to show these functions are quasisymmetric, and also to give a decomposition of them into Gessel's fundamental basis F_β. (QS_β equals the sum, over all standard tableaux T of β^+ which get mapped under Mason's RSK to one of the shapes α occurring in the decomposition of QS_β into atoms, of $F_{\mathrm{des}(T)}$.) The QS functions satisfy a refinement of the Littlewood-Richardson rule, as well as many other well-known properties of Schur functions. In a paper to be presented at the FPSAC 2010 conference this summer, A. Lauve and S. Mason have used this refined Littlewood-Richardson rule and other properties of QS functions to obtain an explicit basis of the quotient ring $\mathrm{QSYM}_n/\mathrm{SYM}_n$, where QSYM_n and SYM_n are the rings of quasisymmetric functions and symmetric functions in n variables, thus resolving a conjecture of F. Bergeron and C. Reutenauer.

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A Littlewood-Richardson rule for Macdonald polynomials

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In [M88], Macdonald introduced a remarkable family of orthogonal polynomials $P_\lambda(q,t)$ associated with root systems. For special values of q and t, they specialize to various well-known functions, including Weyl characters and spherical functions for p-adic groups. These polynomials are a basis for symmetric functions, and are a common generalization of Schur functions s_λ, monomial symmetric functions, Hall-Littlewood polynomials, and the symmetric Jack polynomials. The symmetric Macdonald polynomials are indexed by dominant weights of the weight lattice P.

Classically, the Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$ are the structure constants of the ring of symmetric functions with respect to the Schur basis:

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

In the representation theory of the general linear group $\mathrm{GL}_n\mathbb{C}$, the Littlewood-Richardson coefficients also give the multiplicity of the irreducible highest weight module $V(\nu)$ in $V(\lambda)\otimes V(\mu)$. The coefficient $c^\nu_{\lambda\mu}$ is given combinatorially as the number of Young tableaux of shape $\nu\backslash\lambda$ admitting a Littlewood-Richardson filling of type μ.

Littelmann introduced the path model in [Li94] as a tool for calculating formulas for characters of complex symmetrizable Kac-Moody algebras, and showed that it can also be used to compute Littlewood-Richardson coefficients. Instead of a sum over tableaux, his formula for $c^\nu_{\lambda\mu}$ is a sum over certain paths in the vector space $P\otimes_{\mathbb{Z}}\mathbb{R}$, where the endpoint (weight) of a path takes the place of the filling of a tableau. Several variations of the Littelmann path model were introduced to obtain character formulas, including the gallery model of Gaussent-Littelmann [GL02], and the model of Lenart-Postnikov [LP08] based on λ-chains. In [R06], Ram developed the alcove walk model for working in the affine Hecke algebra, and the paper [RY] showed that alcove walks are a useful tool for expanding products of intertwining operators of the double affine Hecke algebra.

Cherednik developed the theory of double affine Hecke algebras, using it to solve Macdonald's constant term conjectures [C95a], and in [C95b], he showed that products of intertwining operators of the double affine Hecke algebra generate the nonsymmetric Macdonald polynomials $E_\lambda(q,t)$, which are a family of orthogonal polynomials indexed by points of the weight lattice. These polynomials were first introduced by Opdam [O95] in the case $q \to 1$ (see [M03, p.147]). By applying a symmetrizing operator $\mathbf{1}_0$ to E_λ, one can obtain the symmetric polynomials P_λ.

In this talk, we show that the alcove walk model can be used to calculate products of monomials and intertwining operators of the double affine Hecke algebra, and give a product formula for two symmetric Macdonald polynomials

$$P_\mu P_\lambda = \sum_p c_{\mathtt{wt}(p)}(q,t) P_{-w_0 \mathtt{wt}(p)}, \tag{1}$$

where $\mathtt{wt}(p)$ is the weight of the path p, and w_0 is the longest element of the Weyl group. The sum is over alcove walks of type determined by μ and contained in the dominant chamber, and the coefficients $c_{\mathtt{wt}(p)}(q,t)$ are rational functions in q and t.

This is a generalization of the classical formula for products of Weyl characters. When $q = 0$, Equation (1) reduces to Schwer's product formula [Sc06, Theorem 1.3] for Hall-Littlewood polynomials in terms of positively folded galleries.

In the Type A case, Lenart [Le09] compressed the alcove walk formula in [RY] to obtain a tableau formula for Macdonald polynomials similar to the Haglund-Haiman-Loehr formula [HHL05], so it is natural to ask, is there a tableau version of (1) that generalizes the Pieri rule in Macdonald [M88, (6.24)]?

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On restricted representations of affine Kac–Moody algebras at the critical level

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(joint work with Tomoyuki Arakawa)

I want to give an overview on a recent research project on the structure of the restricted category $\mathcal{O}$ over an affine Kac–Moody algebra at the critical level. In contrast to the case of any other level, the representation theory at the critical level is not yet very well understood in general.

There is an approach, due to Frenkel and Gaitsgory, that links D-modules on an affine Grassmannian to representations at the critical level via a Beilinson–Bernstein type localization functor (cf. [4] for an outline of this approach). So far, this approach culminated in the determination of the I^0-equivariant category of critical representations with central characters corresponding to regular opers. These categories can be thought of as versions of category $\mathcal{O}$ with fixed central character, but generalized action of the Cartan, cf. [5].

We propose an essentially different, rather Koszul-dual approach towards the critical level representation theory via a Jantzen-type deformation theory. In particular, we aim to relate critical level representations to the topology of the Langlands-dual affine Grassmannian.

1.1. **Affine Kac–Moody algebras.** Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, and let $\widehat{\mathfrak{g}}$ be the associated affine Kac–Moody algebra. Recall that $\widehat{\mathfrak{g}}$ is constructed starting from the loop algebra $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$, which turns out to admit a unique (up to isomorphism) non-split central extension $\widetilde{\mathfrak{g}}=\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$, and $\widehat{\mathfrak{g}}$ is obtained from $\widetilde{\mathfrak{g}}$ by adding a grading operator D. If we denote by $k(\cdot,\cdot)\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ the Killing-form, then Lie bracket on $\widehat{\mathfrak{g}}=\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}D$ is given by

$$\begin{aligned}[K,\widehat{\mathfrak{g}}]&=0,\\ [D,x\otimes t^n]&=nx\otimes t^n\\ [x\otimes t^n,y\otimes t^m]&=[x,y]\otimes t^{m+n}+\delta_{n,-m}nk(x,y)K\end{aligned}$$

for $x,y\in\mathfrak{g}$, $m,n\in\mathbb{Z}$.

Let us fix a Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{b}$. The corresponding Borel and Cartan subalgebras of $\widehat{\mathfrak{g}}$ are

$$\begin{aligned}\widehat{\mathfrak{b}}&:=\mathfrak{g}\otimes t\mathbb{C}[t]\oplus\mathfrak{b}\oplus\mathbb{C}K\oplus\mathbb{C}D,\\ \widehat{\mathfrak{h}}&:=\mathfrak{h}\oplus\mathbb{C}K\oplus\mathbb{C}D.\end{aligned}$$

Then $\widehat{\mathfrak{h}}$ acts semisimply on $\widehat{\mathfrak{g}}$ with finite dimensional weight spaces (via the adjoint action).

Let us denote by $\mathfrak{h}^\star$ and $\widehat{\mathfrak{h}}^\star$ the linear duals of $\mathfrak{h}$ and $\widehat{\mathfrak{h}}$, resp. Then the dual of the projection $\widehat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}K\oplus\mathbb{C}D\to\mathfrak{h}$ along the decomposition is an inclusion $\mathfrak{h}^\star\to\widehat{\mathfrak{h}}^\star$, which allows us to think of $\mathfrak{h}^\star$ as a subspace in $\widehat{\mathfrak{h}}^\star$. Let $\Delta\subset\mathfrak{h}^\star$ be the set of roots of $\mathfrak{g}$ and denote by $\delta \in \widehat{\mathfrak{h}}^\star$ the element that is dual to D with respect to the decomposition $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$. Explicitly, δ is the element given by

$$\delta(\mathfrak{h} \oplus \mathbb{C}K) = 0,$$
$$\delta(D) = 1.$$

Then the set $\widehat{\Delta} \subset \widehat{\mathfrak{h}}^\star$ of the roots of $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ is

$$\widehat{\Delta} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} \cup \{n\delta \mid n \neq 0\}.$$

The first set on the right hand side is called the set of *real* roots, the second is called the set of *imaginary* roots.

1.2. **Affine category $\mathcal{O}$.** We denote by $\mathcal{O}$ the full subcategory of the category of $\widehat{\mathfrak{g}}$-modules that contains all objects on which $\widehat{\mathfrak{h}}$ acts semisimply and $\widehat{\mathfrak{b}}$ locally finitely. The simple objects on $\mathcal{O}$ are easy to parametrize: if we denote for $\lambda \in \widehat{\mathfrak{h}}^\star$ by $L(\lambda)$ the simple $\widehat{\mathfrak{g}}$-module with highest weight λ, then $\{L(\lambda)\}_{\lambda \in \widehat{\mathfrak{h}}^\star}$ is a full set of representatives for the simple isomorphism classes in $\mathcal{O}$.

The simple highest weight module $L(\delta)$ is invertible, i.e. it is one-dimensional and there is an isomorphism $L(\delta) \otimes_{\mathbb{C}} L(-\delta) \cong L(0) = \mathbb{C}_{triv}$. Hence there is a *shift equivalence* $T = \cdot \otimes L(\delta)$ on the category of $\widehat{\mathfrak{g}}$-modules. Then T, as well as T^{-1}, preserve the subcategory $\mathcal{O}$, and we have $T(L(\lambda)) \cong L(\lambda + \delta)$.

Let $\mathcal{O} = \prod_\Lambda \mathcal{O}_\Lambda$ be a full block decomposition. Then we can identify each index Λ with the subset $\{\lambda \in \widehat{\mathfrak{h}}^\star \mid L(\lambda) \in \mathcal{O}_\Lambda\}$. Here is a categorical characterization of the critical level:

Definition 1. *We say that $\mathcal{O}_\Lambda$ is* critical *if it is preserved by T, i.e. if $T(\mathcal{O}_\Lambda) = \mathcal{O}_\Lambda$.*

The Kac–Kazhdan theorem [6], together with BGG-reciprocity [3, 7], yield the following characterization of the critical blocks: $\mathcal{O}_\Lambda$ is critical iff for some (or, equivalently, all) $\lambda \in \Lambda$ we have $\lambda + \delta \in \Lambda$ iff $\lambda(K) = -\rho(K)$ for some (or all) $\lambda \in \Lambda$. Here ρ is an element in $\widehat{\mathfrak{h}}^\star$ that takes the value 1 on each simple coroot (note that even though ρ is not well-defined, $\rho(K)$ is). The value $\lambda(K)$ is called the *level* of $L(\lambda)$, and we call $-\rho(K)$ the *critical* level.

1.3. **The graded center and restricted representations.** Let us fix a critical block $\mathcal{O}_\Lambda$. For $n \in \mathbb{Z}$ we let $\mathcal{A}_n = \mathcal{A}_{\Lambda,n} := \mathrm{Hom}'(\mathrm{id}_{\mathcal{O}_\Lambda}, T^n)$ be the vector space of natural transformations $z\colon \mathrm{id} \to T^n$ (considered as functors $\mathcal{O}_\Lambda \to \mathcal{O}_\Lambda$) that have the property that for any $M \in \mathcal{O}_A$ and any $m \in \mathbb{Z}$ we have $T^m z^M = z^{T^m M}\colon T^m M \to T^{m+n} M$. Then the direct sum

$$\mathcal{A} = \mathcal{A}_\Lambda = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$$

carries a canonical structure of a commutative, associative, graded algebra. It is called the *graded center* of $\mathcal{O}_\Lambda$.

Definition 2. *We call an object M of $\mathcal{O}_\Lambda$* restricted *if for any $n \in \mathbb{Z}$, $n \neq 0$ and any $z \in \mathcal{A}_n$ the homomorphism $z^M\colon M \to T^n M$ is zero.*

We denote by $\overline{\mathcal{O}_\Lambda} \subset \mathcal{O}_\Lambda$ the full subcategory of restricted objects. Note that the inclusion functor $\overline{\mathcal{O}_\Lambda} \to \mathcal{O}_\Lambda$ admits a left and a right adjoint.

1.4. **The restricted linkage principle.** Denote by $\widehat{\mathcal{W}} \subset \mathrm{GL}(\widehat{\mathfrak{h}}^\star)$ the affine Weyl group. It is generated by the affine reflections $s_\alpha \colon \widehat{\mathfrak{h}}^\star \to \widehat{\mathfrak{h}}^\star$ that are associated to the *real* roots α. The *dot-action* of $\widehat{\mathcal{W}}$ on $\widehat{\mathfrak{h}}^\star$ is obtained by shifting the linear action in such a way that $-\rho$ becomes a fixed point, i.e. for $w \in \widehat{\mathcal{W}}$ and $\lambda \in \widehat{\mathfrak{h}}^\star$ it is given by

$$w.\lambda = w(\lambda + \rho) - \rho.$$

Again, this action does not depend on the choice of ρ.

Let $\mathcal{O}_\Lambda$ be a critical block, and denote by $\widehat{\mathcal{W}}(\Lambda) \subset \widehat{\mathcal{W}}$ the subgroup generated by the reflections s_α associated to those real roots α that satisfy $2(\lambda+\rho, \alpha) \in \mathbb{Z}(\alpha, \alpha)$, where $(\cdot,\cdot)\colon \widehat{\mathfrak{h}}^\star \times \widehat{\mathfrak{h}}^\star \to \mathbb{C}$ denotes any non-degenerate, invariant bilinear form. Again, this definition does not depend on the choice of ρ. Now, by the Kac-Kazhdan theorem and the BGG-reciprocity, the set Λ is stable under the dot-action of $\widehat{\mathcal{W}}(\Lambda)$.

Let $\overline{\mathcal{O}_\Lambda} = \prod_\Gamma \overline{\mathcal{O}_\Gamma}$ be the full block decomposition. Again we identify Γ with the set $\{\lambda \in \widehat{\mathfrak{h}}^\star \mid L(\lambda) \in \overline{\mathcal{O}_\Gamma}\}$. The main result of [2] is that this decomposition corresponds to the orbit decomposition of Λ:

Theorem 3. *Any $\Gamma \subset \Lambda$ is a $\widehat{\mathcal{W}}(\Lambda)$-orbit.*

The main ingredients in the proof of the above result are a deformation theory for representations at the critical level, and an explicit result on the structure of the *subgeneric* blocks that is obtained in [1]. Using the above linkage principle we hope that we are able to relate a critical restricted block to the principal block of the category of modules over an associated small quantum group. This should then yield character formulas for the simple critical highest weight modules and confirm a conjecture of Feigin and Frenkel (cf. [1]).

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Simple tensor products of representations of quantum affine algebras

David Hernandez

Let $q \in \mathbb{C}^*$ which is not a root of unity and let $\mathcal{U}_q(\mathfrak{g})$ be a quantum affine algebra (not necessarily simply-laced or untwisted). Let $\mathcal{F}$ be the tensor category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$.
In my talk at the Oberwolfach Workshop, I presented the main result of [7], expected in various papers of the vast literature about $\mathcal{F}$.

Theorem 1 [7] *Let $S_1, \cdots, S_N$ be objects of $\mathcal{F}$. The tensor product $S_1 \otimes \cdots \otimes S_N$ is simple if and only if $S_i \otimes S_j$ is simple for any $i < j$.*

The "only if" part of the statement is known: it is an immediate consequence of the commutativity of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of $\mathcal{F}$ proved in [5] (see [6] for the twisted types). The "if" part of the statement is proved in [7].

The following is an extended version of the introduction of [7].

If the reader is not familiar with the representation theory of quantum affine algebras, he may wonder why such a result is non trivial. Indeed, in tensor categories associated to "classical" representation theory, there are "few" non trivial tensor products of representations which are simple. For instance, let V, V' be non-zero simple finite-dimensional modules of a simple algebraic group in characteristic 0. Then, it is well-known that $V \otimes V'$ is simple if and only if V or V' is of dimension 1. But in positive characteristic there are examples of non trivial simple tensor products given by the Steinberg theorem. And in $\mathcal{F}$ there are "many" simple tensor products of non trivial simple representations. For instance, it is proved in [3] that for $\mathfrak{g} = \hat{sl}_2$ an arbitrary simple object V of $\mathcal{F}$ is *real*, i.e. $V \otimes V$ is simple. Although it is known [10] that there are non real simple objects in $\mathcal{F}$ in general, many other examples of non trivial simple tensor products can be found in [8].

The statement of Theorem 1 has been conjectured and proved by several authors in various special cases. The result is proved for $\mathfrak{g} = \hat{sl}_2$ in [3], for a special class of modules of the Yangian of gl_n attached to skew Young diagrams in [12], for tensor products of fundamental representations in [1, 4], for a special class of tensor products satisfying an irreducibility criterion in [2], for a certain "small" subtensor category $\mathcal{C}_1$ of $\mathcal{F}$ when $\mathfrak{g}$ is simply-laced in [8].
So, even in the case $\mathfrak{g} = \hat{sl}_3$, Theorem 1 had not been established. Our complete proof is valid for arbitrary simple objects of $\mathcal{F}$ and for arbitrary $\mathfrak{g}$.

Let us give a few first comments. Theorem 1 allows to produce simple tensor products $V \otimes V'$ where $V = S_1 \otimes \cdots \otimes S_k$ and $V' = S_{k+1} \otimes \cdots \otimes S_N$. Besides it implies that $S_1 \otimes \cdots \otimes S_N$ is real if we assume that the S_i are real in addition to the assumptions of Theorem 1.

The main ingredients of the proof are the following : the parametrization of simple objects of $\mathcal{F}$ [3], a cyclicity property of tensor product of fundamental representations [2, 9, 14], the theory of Frenkel-Reshetikhin q-characters [5, 4], a "filtration" of $\mathcal{F}$ by tensor subcategories [8], the notion of truncated q-characters [8], a certain property of tensor products of l-weight vectors (analogs of weight vectors for q-characters) that we establish [7], a compatibility property of intertwining operators with a decomposition of q-characters that we establish [7].

Our result is stated in terms of the tensor structure of $\mathcal{F}$. Thus, it is purely representation theoretical. But we have three additional motivations, related respectively to physics, topology, combinatorics, and also to other structures of $\mathcal{F}$.

First, although the category $\mathcal{F}$ is not braided (in general $V \otimes V'$ is not isomorphic to $V' \otimes V$), $\mathcal{U}_q(\mathfrak{g})$ has a *universal R-matrix* in a completion of the tensor product $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$. In general the universal R-matrix can not be specialized to finite-dimensional representations, but it gives rise to $V(z) \otimes V' \to V' \otimes V(z)$ which depend meromorphically on a formal parameter z (here the representation $V(z)$ is obtained by homothety of spectral parameter). From the physical point of view, it is an important question to localize the zeros and poles of these operators. The reducibility of tensor products of objects in $\mathcal{F}$ is known to have strong relations with this question. This is the first motivation to study irreducibility of tensor products in terms of irreducibility of tensor products of pairs of constituents [1].

Secondly, if $V \otimes V'$ is simple the universal R-matrix can be specialized and we get a well-defined intertwining operator $V \otimes V' \to V' \otimes V$. In general the action of the R-matrix is not trivial. As the R-matrix satisfies the Yang-Baxter equation, when V is real we can define an action of the braid group $\mathcal{B}_N$ on $V^{\otimes N}$ (as for representations of quantum groups of finite type). It is known [13] that such situations are important to construct *topological invariants*.

Finally, in a tensor category, there are natural important questions such as the parametrization of simple objects or the decomposition of tensor products of simple objects in the Grothendieck ring. But another problem of the same importance is the factorization of simple objects V in *prime objects*, i.e. the decomposition $V = V_1 \otimes \cdots \otimes V_N$ where the V_i can not be written as a tensor product of non trivial simple objects. This problem for $\mathcal{F}$ is one of the main motivation in [8]. When we have established that the tensor products of some pairs of prime representations are simple, Theorem 1 gives the factorization of arbitrary tensor products of these representations.

This factorization problem is related to the program of realization of *cluster algebras* in $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ initiated in [8] when $\mathfrak{g}$ is simply-laced. A cluster algebra has a distinguished set of generators called *cluster variables*. A notion of *compatibility* of cluster variables comes with the definition of cluster algebras (cluster variables are compatible if they occur in the same seed). A product of compatible cluster variables is called a *cluster monomial*. Let us recall the notion of *monoidal* *categorification* due to Leclerc [8]. A tensor category $\mathcal{C}$ is said to be a monoidal categorification of a cluster algebra $\mathcal{A}$, if there is a ring isomorphism $\phi : K_0(\mathcal{C}) \to \mathcal{A}$, where $K_0(\mathcal{C})$ is the Grothendieck ring of $\mathcal{C}$, such that ϕ induces bijections

$$\{\text{Classes of real simple objects of } \mathcal{C}\} \leftrightarrow \{\text{Cluster monomials of } \mathcal{A}\},$$

$$\{\text{Classes of prime real simple objects of } \mathcal{C}\} \leftrightarrow \{\text{Cluster variables of } \mathcal{A}\}.$$

If one can establish a monoidal categorification, we get results about $\mathcal{A}$ (positivity, linear independence of cluster monomials) and $\mathcal{C}$ (Clebsch-Gordan coefficients, factorization in prime modules). A cluster algebra $\mathcal{A}$ of finite type (ADE) has a monoidal categorification $\mathcal{C}_1$ which is a tensor subcategory of $\mathcal{F}$ for $\mathcal{U}_q(\hat{\mathfrak{g}})$ where $\mathfrak{g}$ has the type of $\mathcal{A}$. This was proved in [8] for types A, D_4 and in [11] for the other types. In the proof of [8], the statement of Theorem 1 for $\mathcal{C}_1$ is a crucial step (the proof of Theorem 1 in this case is drastically simplified; several new technical ingredients are used in the general case). It reduces the proof of the irreducibility of tensor products of representations corresponding to compatible cluster variables to the proof of the irreducibility of the tensor products of pairs of simple representations corresponding to compatible cluster variables. We plan to use Theorem 1 in the future to establish monoidal categorifications associated to non necessarily simply-laced quantum affine algebras, involving categories different than the small subcategories $\mathcal{C}_1$ considered in [8, 11].

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The modular branching rule for the affine Hecke algebra of type A

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(joint work with Nicolas Jacon, Cédric Lecouvey)

Let F be an algebraically closed field, $q \in F^\times$, and H_n the extended affine Hecke algebra of type A. We denote the generators by $T_1, \dots, T_{n-1}$ and $X_1^{\pm 1}, \dots X_n^{\pm 1}$ as usual. It is well-known that the classification of simple H_n-modules is reduced to the case when the eigenvalues of $X_1, \dots, X_n$ on the modules are powers of q. Hence, we consider simple modules with this property hereafter. We assume that $q \neq 1$ and its multiplicative order is e. Then, by work of Ginzburg and Lusztig, we know that if $F = \mathbb{C}$ then such simple modules are parametrized by aperiodic multisegments. Further, the cellular algebra technique in [4] tells us that the classification depends only on e and does not depend on the characteristic of F. Hence, as long as the classification of simple modules is concerned, we may assume $F = \mathbb{C}$ and $q = \sqrt[e]{1}$.

Recall that the geometric method says that such simple H_n-modules are labelled by the canonical basis of the quantized enveloping algebra of type $A^{(1)}_{e-1}$ when we vary n through $\mathbb{N}$, and that the aperiodic multisegments label the canonical basis elements. In particular, the set of aperiodic multisegments carries a crystal structure: we note that the identification of H_n with the convolution algebra of the Steinberg variety is given by hand, and there are at least two ways to do so. The choice of the crystal structure is different for the two identifications, but both are isomorphic to $B(\infty)$. We denote by L_ψ the simple H_n-module labelled by an aperiodic multisegment ψ of size n. It is difficult to obtain explicit matrix representations of the generators on L_ψ, so that purely algebraic construction of L_ψ is desirable. One of our main results is to provide this.

Let Λ be a nonnegative integral linear combination of fundamental weights Λ_i, for $i \in \mathbb{Z}/e\mathbb{Z}$. Then we have the cyclotomic quotient H_n^Λ of H_n. Write

$$\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_\ell}.$$

Dipper, James and Mathas showed that the sequence $(\gamma_1, \dots, \gamma_\ell)$ defines a cellular algebra structure on H_n^Λ, and simple H_n^Λ-modules are parametrized by Kleshchev multipartitions by [1]. Recall that an ℓ-partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ is *Kleshchev* if

$$\lambda^{(\ell)} \otimes \cdots \otimes \lambda^{(1)} \in B(\Lambda) \subseteq B(\lambda_{\gamma_\ell}) \otimes \cdots \otimes B(\Lambda_{\gamma_1}).$$

Here, $B(\Lambda)$ is identified with the connected component of the tensor product crystal that contains the empty ℓ-partition. In particular, if we vary n through $\mathbb{N}$, the set of Kleshchev multipartitions carries a crystal structure. We denote by D^λ the simple H_n^Λ-module labelled by a Kleshchev multipartition λ of size n.

As any simple H_n-module is a simple H_n^Λ-module, for some Λ, we already know how to construct all simple H_n-modules in the purely algebraic method of cellular algebras. Thus, to construct L_ψ in a purely algebraic manner, it suffices to know the module correspondence between the geometric construction and the algebraic construction. This may be done by proving the modular branching conjecture for

H_n. Note that it is not proved in [7]. The modular branching conjecture for H_n is the statement that $\tilde{e}_i L_\psi = L_{\tilde{e}_i \psi}$, where $\tilde{e}_i$ on the left hand side is the module theoretic Kashiwara operator, and $\tilde{e}_i$ on the right hand side is the Kashiwara operator from the crystal structure on the set of aperiodic multisegments. We note that the modular branching conjecture for H_n^Λ is already established in [2].

Theorem 1.
(1) *The modular branching conjecture for H_n holds.*
(2) *The module correspondence $D^\lambda \mapsto L_\psi$ is given by the crystal embedding*

$$B(\Lambda) \hookrightarrow B(\infty) \otimes T_\Lambda : \lambda \mapsto \psi \otimes t_\Lambda,$$

where the crystal $T_\Lambda = \{t_\Lambda\}$ is such that

$$\mathrm{wt}(t_\Lambda) = \Lambda, \;\; \epsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty, \;\; \tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = 0.$$

The proof of (1) uses the multiplicity one result from Vazirani's thesis [8], but we do not need [7]. We also remark that in the case when q is not a root of unity, it was proved in [9] if we assume facts relating the algebraic construction and the geometric construction in this case.

We briefly explain main ideas of the proof. The first point is how to understand the modular branching geometrically. Following Ginzburg, let $a = (s, q)$ and

$$\mathcal{N}_n^a = \{X \in Mat(n, n, \mathbb{C}) \mid X^n = O, \; Ad(s)X = qX\},$$

where s is a diagonal matrix whose nonzero entries are powers of q. Then we define

$$\tilde{\mathcal{N}}_n^a = \{(X, F = (F_k))\} \xrightarrow{\pi_n^a} \mathcal{N}_n^a$$

to be the first projection, where the complete flag F satisfies the conditions

(i) F_k is spanned by k eigenvectors of s, (ii) $XF_k \subseteq F_{k-1}$,

for $1 \le k \le n$. If we further assume that F_n is obtained from F_{n-1} by adding an eigenvector of s with eigenvalue q^i, for $i \in \mathbb{Z}/e\mathbb{Z}$, we obtain a union of connected components of $\tilde{\mathcal{N}}_n^a$. We denote it by $p_i \tilde{\mathcal{N}}_n^a$.

On the other hand, $a = (s, q)$ defines a central character of H_n and we may consider the specialized Hecke algebra H_n^a which is obtained by specializing the center $Z(H_n)$ to $\mathbb{C}$. Then, $H_n^a \simeq \mathrm{Ext}^*(\pi_{n!}^a \mathbb{C}, \pi_{n!}^a \mathbb{C})$ by [5]. Let $H_{n-1,n}^a$ be the $\mathbb{C}$-subalgebra of H_n^a generated by $T_1, \ldots, T_{n-2}$ and $X_1, \ldots, X_n$. Then X_n is central in $H_{n-1,n}^a$ and its eigenvalue q^i defines a central idempotent p_i of $H_{n-1,n}^a$. Let V_j be the eigenspace of s for the eigenvalue q^j, for $j \in \mathbb{Z}/e\mathbb{Z}$. $\mathcal{N}_n^a$ is nothing but the space of nilpotent representations of the cyclic quiver of length e having the dimension vector $(\dim V_j)_{j \in \mathbb{Z}/e\mathbb{Z}}$. We denote $\dim V_i = m + 1$ and consider

$$\pi_{n-1,n}^a : p_i \tilde{\mathcal{N}}_n^a \to \mathcal{N}_n^a \times \mathbb{P}^m$$

defined by $(X, F) \mapsto (X, F_{n-1} \cap V_i)$. Note that $F_{n-1} \cap V_i$ is a hyperplane in V_i. We set $A = H^b_{n-1}$, $B = p_i H^a_{n-1,n} p_i$, $C = H^a_n$ and

$$A' = \mathrm{Ext}^*(\pi^b_{n-1!}\mathbb{C}, \pi^b_{n-1!}\mathbb{C}),$$
$$B' = \mathrm{Ext}^*(\pi^a_{n-1,n!}\mathbb{C}, \pi^a_{n-1,n!}\mathbb{C}),$$
$$C' = \mathrm{Ext}^*(\pi^a_{n!}\mathbb{C}, \pi^a_{n!}\mathbb{C}),$$

where $b = (s', q)$ is obtained from $a = (s, q)$ by ignoring one eigenvalue q^i of s. The modular branching is to consider $\tilde{e}_i L_\psi = \mathrm{Top}(p_i L_\psi)$, for a simple C-module L_ψ. Then $p_i L_\psi$ is a B-module and its action on $\mathrm{Top}(p_i L_\psi)$ factors through A. We may identify $A/\mathrm{Rad}\, A \longleftarrow B \longrightarrow C$ with $A'/\mathrm{Rad}\, A' \leftarrow B' \to C'$. In the latter setting, we may show that $L_{\tilde{e}_i\psi}$ appears in $\mathrm{Top}(p_i L_\psi)$ by using the following lemma by Kashiwara.

Lemma 2. *Let φ be a multisegment of size $n-1$. Then*

$$\mathrm{Ind}_i(IC_\varphi) \simeq \left(\oplus_{j=0}^{\epsilon_i(\varphi)} IC_{\tilde{f}_i\varphi}[\epsilon_i(\varphi) - 2j]\right) \bigoplus \left(\oplus_{j\in\mathbb{Z}} R_{\varphi,j}[j]\right),$$

where $R_{\varphi,j}$ are certain perverse sheaves and

$$-\epsilon_i(\varphi) + 2 \leq j \leq \epsilon_i(\varphi) - 2.$$

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Symmetric quasi-hereditary algebras

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(joint work with Volodymyr Mazorchuk)

Symmetry and quasi-heredity are two important homological properties in the world of finite-dimensional algebras, which mutually exclude each other. However, moving from the world of finite-dimensional algebras to infinite-dimensional, but locally finite-dimensional, ones, there are many examples for algebras that satisfy both conditions. Here, locally finite-dimensional can roughly be understood as having finite-dimensional projectives and injectives, so the infinity of dimension comes from having a countable number of primitive orthogonal idempotents. The first examples were rhombal algebras studied in the PhD thesis of Peach [7], and subsequently cubist algebras defined by Chuang and Turner [1], both of which are related to weight 2 blocks of symmetric groups. On the other hand, Brundan and Stroppel [8, 9, 10] study quasi-hereditary covers of Khovanov diagram algebras, which are Morita equivalent to integral blocks of parabolic category $\mathcal{O}$ of associated to the parabolic subalgebra $\mathfrak{gl}_m \times \mathfrak{gl}_n$ in $\mathfrak{gl}_{m+n}$. Taking a certain limit, these algebras are also symmetric and quasi-hereditary. Special symmetric and quasi-hereditary algebras also play and important role in the representation theory of the algebraic group $GL_2(F)$ for an algebraically closed field F of positive characteristic [4, 5]. Assuming that all standard modules in the quasi-hereditary structure have the same Loewy length, all symmetric quasi-hereditary algebras of Loewy length 4 have been classified by Turner and the author as being in bijection with bipartite graphs [6].

Recalling a classical result of Dlab and Ringel [2], which states that every finite-dimensional algebra A can be realised as a centralizer subalgebra of a quasi-hereditary algebra B (that is, as eBe for some idempotent $e \in B$), we show that this can be roughly generalised to symmetric quasi-hereditary algebras [3]. Starting with a finite-dimensional algebra A, we construct a locally finite-dimensional quasi-hereditary algebra C with and idempotent e such that $A \cong eCe$. This is constructed explicitly as a $\mathbb{Z}$-indexed matrix algebra and has a very transparent quasi-hereditary structure. Assuming the algebra A is symmetric and (as a technical condition) the trace form induces a non-degenerate pairing between opposing subquotients in a filtration by two-sided ideals (i.e. highest with lowest, etc.), the algebra C is also symmetric, hence we have realised A as a centralizer subalgebra of a symmetric quasi-hereditary algebra. If A is not symmetric, it is easy to see that such a result cannot hold, but we define a symmetric quasi-hereditary algebra D with idempotents e and f such that $A \cong eDe/eDfDe$, by artificially attaching a new left socle to A, constructing C for this new algebra, and finally taking the trivial extension of this with its dual bimodule. This indicates that similarly to finite-dimensional quasi-hereditary algebras, symmetric quasi-hereditary algebras are in some sense 'universal', namely the module category of any finite-dimensional algebra can be nicely embedded into the module category of a symmetric quasi-hereditary one.

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Deligne's category $\underline{\text{Re}}\text{p}\,(S_t)$

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(joint work with Victor Ostrik)

In [3], Deligne defines a $\mathbb{C}$-linear tensor category $\underline{\text{Re}}\text{p}(S_t)$ for each $t \in \mathbb{C}$ which has the following properties: $\underline{\text{Re}}\text{p}(S_t)$ is semisimple if and only if t is not a nonnegative integer. Moreover, when t is a nonnegative integer, $\underline{\text{Re}}\text{p}(S_t)$ "interpolates" the category $\text{Rep}(S_t)$ of finite dimensional complex representations of the symmetric group S_t (in other words there exists a full tensor functor $\underline{\text{Re}}\text{p}(S_t) \to \text{Rep}(S_t)$ which is surjective on objects). One can use partition algebras (a generalization of the Temperley-Lieb algebras introduced by Martin in [4] and [5]) to construct $\underline{\text{Re}}\text{p}(S_t)$. More precisely, the category $\underline{\text{Re}}\text{p}(S_t)$ can be realized as the pseudo-abelian envelope of the partition category (a category whose objects are indexed by nonnegative integers with endomorphism rings equal to the partition algebras $\mathbb{C}P_n(t)$). Consequently, studying $\underline{\text{Re}}\text{p}(S_t)$ can be viewed as simultaneously studying finitely generated projective modules of the partition algebras $\mathbb{C}P_n(t)$ for $n = 0, 1, 2, \ldots$.

Blocks in $\underline{\text{Re}}\text{p}(S_t)$. Let $\mathcal{A}$ denote an arbitrary $\mathbb{C}$-linear category. Consider the weakest equivalence relation on the set of all indecomposable objects in $\mathcal{A}$ where two indecomposable objects are in the same equivalence class whenever there exists a nonzero morphism between them. We call the equivalence classes in this relation *blocks*. We will also use the term block to refer to a full subcategory of $\mathcal{A}$ generated by the indecomposable objects in a single block. We say a block is *trivial* if it contains only one indecomposable object and its endomorphism ring is $\mathbb{C}$. In particular, $\mathcal{A}$ is semisimple if and only if all the blocks in $\mathcal{A}$ are trivial. Finally, given a block $\mathcal{B}$ with countably many indecomposable objects and finite-dimensional Hom spaces between those objects, we associate a quiver with relations as follows: *Vertices are labelled by the indecomposable objects in $\mathcal{B}$. There are* $\dim_{\mathbb{C}} \text{Hom}_{\mathcal{A}}(L, L')$ *arrows from L to L' whose labels form a basis of* $\text{Hom}_{\mathcal{A}}(L, L')$. *The relations are given by compositions of morphisms.*

In [1] we explain how the set of indecomposable objects in $\underline{\text{Re}}\text{p}(S_t)$ up to isomorphism is in bijective correspondence with the set of Young diagrams of arbitrary size. Let $L(\lambda)$ denote an indecomposable object in $\underline{\text{Re}}\text{p}(S_t)$ labelled by Young diagram λ. The following results on the blocks of $\underline{\text{Re}}\text{p}(S_t)$ are proven in [1]. Their partition algebra counterparts can be found in [6]

Result 1. Suppose λ and μ are Young diagrams. $L(\lambda)$ and $L(\mu)$ are in the same block of $\underline{\mathrm{Rep}}(S_t)$ if and only if the sequence $t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \ldots$ is obtained from the sequence $t - |\mu|, \mu_1 - 1, \mu_2 - 2, \ldots$ by permuting finitely many of the terms. In particular, the blocks of $\underline{\mathrm{Rep}}(S_t)$ are all trivial unless t is a nonnegative integer, so we recover Deligne's result on the semisimplicity of $\underline{\mathrm{Rep}}(S_t)$.

Result 2. Suppose d is a nonnegative integer. There are finitely many nontrivial blocks in $\underline{\mathrm{Rep}}(S_d)$; these blocks are indexed by Young diagrams of size d. Moreover, all nontrivial blocks are equivalent as categories and the associated quiver is explicitly computed.

Deligne's conjecture. One advantage of studying $\underline{\mathrm{Rep}}(S_t)$ rather than studying partition algebras directly is the ability to ask interesting category theory questions which do not have obvious counterparts on the level of algebras. For instance, in [3], Deligne constructs an abelian tensor category $\underline{\mathrm{Rep}}^{ab}(S_t)$ which contains $\underline{\mathrm{Rep}}(S_t)$ as a tensor subcategory. Deligne conjectures that $\underline{\mathrm{Rep}}^{ab}(S_t)$ has the following universal property: Given an abelian tensor category $\mathcal{T}$ and a tensor functor $\mathcal{F} : \underline{\mathrm{Rep}}(S_t) \to \mathcal{T}$, either $\mathcal{F}$ is an extension of the inclusion functor $\underline{\mathrm{Rep}}(S_t) \to \underline{\mathrm{Rep}}^{ab}(S_t)$ by an exact tensor functor $\underline{\mathrm{Rep}}^{ab}(S_t) \to \mathcal{T}$, or $t \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F}$ factors through the "interpolation functor" $\underline{\mathrm{Rep}}(S_t) \to \mathrm{Rep}(S_t)$. In [2] we use our result on blocks from [1] to prove Deligne's conjecture.

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Total positivity for loop groups: intrinsic energy

Thomas Lam

(joint work with Pavlo Pylyavskyy)

The intrinsic energy function plays an important role in the path model for affine crystals [1]. From the point of view of representation theory, the energy function is used to calculat the affine weight of a crystal basis element. The energy function is also related to charge statistic of Lascoux-Schützenberger on semistandard tableaux (see [3]), which establishes a relation between one dimensional configuration sums arising in solvable lattice models and Kostka-Foulkes polynomials.

In my talk I discussed an explicit subtraction-free formula for the tensor product $B = B_1 \otimes \cdots \otimes B_m$ of $U'_q(\hat{\mathfrak{sl}}_n)$ Kirillov-Reshetikhin crystals, where each B_i is the crystal for a symmetric power of the standard representation. Identify B_i with the semistandard Young tableaux with row shape, filled with the numbers $1, 2, \ldots, n$. Let $b = b_1 \otimes \cdots \otimes b_m \in B$, and write $x_i^{(r+i-1)}$ for the number of r's in b_i. The upper index $(r+i-1)$ is to be considered as an element of $\mathbb{Z}/n\mathbb{Z}$. The main result of the talk is the following formula for the intrinsic energy function $\overline{D}_B$ of B.

Let $\delta_t = (t, t-1, \ldots, 1)$ denote the staircase shape of side-length t.

Theorem 1. *We have*

$$\overline{D}_B(b) = \min_T \left\{ \sum_{(i,j)\in(n-1)\delta_{m-1}} x_{T(i,j)}^{(i-j)} \right\}.$$

where the minimum is over all semistandard tableaux T of shape $(n-1)\delta_{m-1}$, and entries in $1, 2, \ldots, m$.

In the physical interpretation, each b_i represents a particle, and the intrinsic energy function $\overline{D}_B(b)$ is defined as the sum of $\binom{m}{2}$ local energies of interactions of particles. Theorem 1 thus has the following interpretation: each tableau T encodes a way for m particles to interact *simultaneously*, and intrinsic energy is equal to the minimum of these.

It is well known that the energy function $\overline{D}_B$ is invariant under the action of the combinatorial R-matrix on B, which generates an action of the symmetric group S_m on B. Our point of view is that $\overline{D}_B$ should be viewed as some kind of symmetric function. Indeed, the formula for $\overline{D}_B$ in Theorem 1 is a piecewise-linear analogue of a Schur function. Switching from the piecewise-linear world to the birational world, one is led to study the invariants of the birational R-matrix. This birational S_m-action was previously studied by Kirillov in the context of the Robinson-Schensted algorithm, by Noumi-Yamada in the context of discrete Painléve systems, by Etingof in the context Yang-Baxter equations, by Berenstein-Kazhdan in the context of geometric crystals, and by the authors in the context of total positivity of loop groups.

Instead of considering symmetric functions as the invariants of a symmetric group action on a polynomial ring $\mathbb{C}[a_1, a_2, \ldots, a_m]$, one can alternatively obtain them as the coefficients of a polynomial $p(t) = \prod_{i=1}^m (t - a_i)$ whose roots are $a_1, a_2, \ldots, a_m$. Galois Theory then connects the two points of view. Our generalization of symmetric functions, introduced in [2], is obtained by replacing the polynomial $p(t)$, by a polynomial map $f : \mathbb{C}^* \to GL_n(\mathbb{C})$. One then attemps to factorize the map f into "linear" factors, using the product structure of the image $GL_n(\mathbb{C})$. In [2], with the additional notion of total positivity, we were able to achieve such factorizations into factors we called whirls, denoted $M(a^{(1)}, a^{(2)}, \ldots, a^{(n)})$. These whrils do not commute in general, and the same Galois theory point of view leads to a non-trivial birational action of the symmetric group on a polynomial ring. We call this ring the ring of loop symmetric functions LSym.

Like usual symmetric functions, the ring LSym has distinguished elements which we call loop Schur functions. It turns out that the intrinsic energy $\overline{D}_B$ can be lifted to a *polynomial* invariant, and thus is an element of LSym. Indeed $\overline{D}_B$ lifts to exactly one such loop Schur function. The ring LSym appears to be of fundamental importance. For example, somewhat surprisingly it is a free polynomial ring. In addition, the crystal structure of B (which commutes with the R-matrix) can be completely described in terms of LSym. This and other connections we intend to study in future work.

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Hook length formulas for integer partitions

Guo-Niu Han

The hook lengths for integer partitions are widely studied in the Theory of Partitions, in Algebraic Combinatorics and Group Representation Theory. Let n be an integer, a *partition* λ of n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. Each partition can be represented by its Ferrers diagram. For each box v in the Ferrers diagram of a partition λ, define the *hook length* of v, denoted by $h_v(\lambda)$ or h_v, to be the number of boxes u such that $u = v$, or u lies in the same column as v and above v, or in the same row as v and to the right of v.

The classical hook length formula, due to Frame, Robinson and Thrall

$$f_\lambda = \frac{n!}{\prod_{v\in\lambda} h_v}, \tag{1}$$

where f_λ is the number of standard Young tableaux of shape λ, can be rewritten as the following equivalent forms:

$$\sum_\lambda x^{|\lambda|} \prod_{v\in\lambda} \frac{1}{h_v^2} = e^x \tag{2}$$

and

$$\sum_\lambda x^{|\lambda|} \prod_{v\in\lambda} \frac{1}{h_v} = e^{x+x^2/2}. \tag{3}$$

Formulas (2) and (3) are referred to as the basic *hook length formulas*. The purpose of this talk is to present the *hook length expansion* technique for discovering new hook length formulas.

Definition. Let $\rho : N^* \to K$ be a map of the set of positive integers to some field K. Also let $f(x) \in K[[x]]$ be a formal power series in x with coefficients in K such that $f(0) = 1$. If

$$\sum_{\lambda \in P} x^{|\lambda|} \prod_{h \in H(\lambda)} \rho(h) = f(x), \tag{4}$$

the series $f(x)$ is called the *generating function* for partitions by the weight function ρ. The left-hand side of (4) is called the *hook length expansion* of $f(x)$. Furthermore, when both ρ and $f(x)$ have simple forms, equation (4) is called a *hook length formula*.

It is easy to see that the generating function $f(x)$ is uniquely determined by the weight function ρ. Conversely, the weight function ρ can be uniquely determined by $f(x)$ in most cases. In the other cases (called *singular cases*), the weight function ρ does not exist, or is not unique. We next provide an algorithm for computing ρ when $f(x)$ is given.

Let $P_L(n)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n such that $\ell(\lambda) = 1$ or $\lambda_2 = 1$. The partitions in $P_L(n)$ are usually called *hooks*. The *hook length multi-set* $H(\lambda)$ of a hook λ of n is simply

$$H(\lambda) = \{1, 2, \cdots \ell(\lambda) - 1,\ 1, 2, \cdots, n - \ell(\lambda),\ n\}.$$

Let $P_Z(n)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n such that $\ell \geq 2$ and $\lambda_2 \geq 2$. It is easy to see that the hook length multi-set of each partition of $P_Z(n)$ does not contain the integer n. Since $P(n) = P_L(n) \cup P_Z(n)$ we have

$$\sum_{\lambda \vdash n} \prod_{h \in H(\lambda)} \rho(h) = \rho(n) \sum_{\lambda \in P_L(n)} \prod_{h=1}^{\ell(\lambda)-1} \rho(h) \prod_{h=1}^{n-\ell(\lambda)} \rho(h) + \sum_{\lambda \in P_Z(n)} \prod_{h \in H(\lambda)} \rho(h). \tag{5}$$

The value $\rho(n)$ of the weight function can be obtained by (5). Based on this observation we derive an explicit algorithm for computing the weight function ρ when $f(x)$ is given.

By using the hook length expansion algorithm we obtained several new hook length formulas. For example,

$$\sum_{\lambda \in P} x^{|\lambda|} \prod_{v \in \lambda} \rho(z; h_v) = e^{x + zx^2/2}, \tag{6}$$

where the weight function $\rho(z; n)$ is defined by

$$\rho(z; n) = \frac{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^k}{n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} z^k}$$

and

$$\sum_{\lambda \in P} x^{|\lambda|} \prod_{h \in H_t(\lambda)} \left(y - \frac{tyz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z}(1 - x^k)}, \tag{7}$$

where H_t is the multi-set of hook lengths that is multiple of t.

Note that identity (6) can be seen as an interpolation between (2) and (3), and that identity (7) unifies several formulas, including the Jacobi triple product identity, the Macdonald identities for $A_\ell^{(a)}$, the generating functions for partitions and for t-cores, the Nekrasov-Okounkov identity.

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Combinatorics of Kazhdan-Lusztig elements: Factorisation and fully packed loop models

Jan de Gier

(joint work with Alain Lascoux, Mark Sorrell)

The Hecke algebra $\mathcal{H}$ of the symmetric group $W = S_n$ generated by the simple reflections s_i, is the algebra over $\mathbb{C}$ defined in terms of generators $T_i \equiv T_{s_i}$, $i = 1, \ldots, n-1$, and relations

$$(T_i - t)(T_i + t^{-1}) = 0, \qquad T_iT_j = T_jT_i \quad \forall\, i,j:\, |i-j| > 1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

The "Baxterised element" $T_i(u) \in \mathcal{H}$ is defined by

$$T_i(u) = T_i + \frac{t^{-u}}{[u]}, \qquad [u] = \frac{t^u - t^{-u}}{t - t^{-1}}.$$

A polynomial representation is realised in terms of the divided difference operator ∂_i. The projector $T_i(1)$ induces the operator ∇_i,

$$\nabla_i = (tz_i - t^{-1}z_{i+1})\partial_i := \frac{tz_i - t^{-1}z_{i+1}}{z_i - z_{i+1}}(1 - s_i).$$

We will restrict ourselves to irreducible polynomial representations of maximal parabolic subalgebras.

1.1. **Parabolic Kazhdan-Lusztig and Young bases from the t-Vandermonde determinant.** Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression of a word $w \in W$, then define $T_w = T_{i_1} \cdots T_{i_l}$. The Kazhdan-Lusztig (KL) basis C_w, $w \in W$, of $\mathcal{H}$ is defined by

$$\overline{C_w} = C_w,$$
$$C_w - T_w \in \bigoplus_{v \in W} t^{-1}\mathbb{Z}[t^{-1}]\, T_v,$$

where the bar operator sends $T_i \mapsto T_i^{-1}$ and $t \mapsto t^{-1}$. Maximal parabolic basis elements can be labeled by partitions. The parabolic KL basis [3] can be obtained in the polynomial representation in the following way [6]. Let $\rho = (n-1, \ldots, 1, 0)$

be a code, the nonsymmetric Macdonald polynomial $M_\emptyset := M_{\rho\rho}$ degenerates at $q = t^6$ to the product of two t-Vandermonde determinants,

$$M_{\rho\rho}(x_1, \ldots, x_{2n}; q = t^6, t) = \Delta_t(x_1, \ldots, x_n)\Delta_t(x_{n+1}, \ldots, x_{2n}),$$
$$\Delta_t(x_1, \ldots, x_N) = \prod_{1\le i<j\le N} (tx_i - t^{-1}x_j),$$

which is a homogeneous polynomial. The action of the Hecke algebra on this polynomial generates an irreducible module of homogeneous polynomials. It was shown in [6, 2] that the KL basis in this maximal parabolic module can be generated by applying operators of the form $T_i(u)$ to $M_{\rho\rho}$, with appropriate arguments u. For example, for $n = 4$ we find $C_{[3,1]} = T_2(1)T_5(1)T_4(2)T_3(3) \cdot C_\emptyset$, and we represent this as a labeled partition, the index i of each operator $T_i(j)$ corresponding to the ith NW-SE diagonal of the Young diagram:

3	2	1
1		

The general rule for integer labels u_{ij} in each box is given recursively by $u_{ij} = \max\{u_{i+1,j}, u_{i,j+1}\} + 1$, and with initial condition that corner box is labeled 1.

							1
	u_{ij}						
				1			
		1					

$$u_{ij} = \max\{u_{i+1,j}, u_{i,j+1}\} + 1.$$

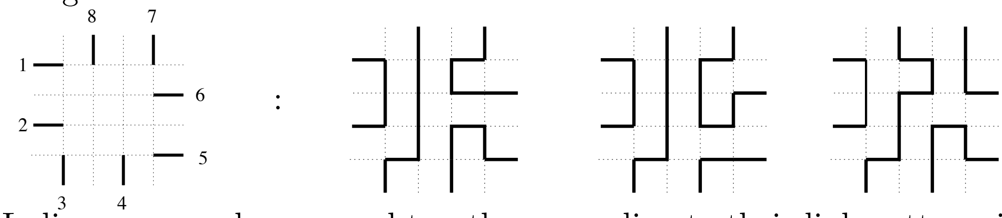
The degenerate nonsymmetric Macdonald polynomials (M_v) indexed by Yamanouchi words also form a basis for this module. These homogeneous successor polynomials of $M_{\rho\rho}$ retain their vanishing properties, and may also be indexed by labeled Young diagrams. For the element M_λ with $\lambda \subseteq \mu = [n-1, \ldots, 1, 0]$, the integer corresponding to box $b_{ij} \in \lambda$ is given $u_{ij} = n - i - j + 2$. The following was conjectured in [2].

Theorem 1. *The expansion of the maximal M basis element M_μ in terms of KL polynomials is given by*

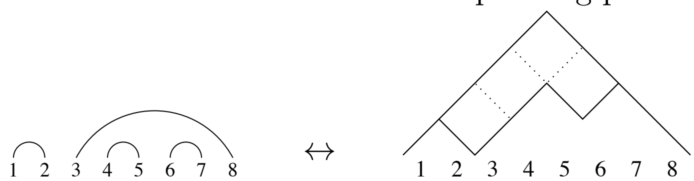
$$M_\mu = \sum_{\lambda\subseteq\mu} \tau^{-c_\lambda}\, C_\lambda, \qquad \tau = -[2].$$

Here c_λ is defined as the signed sum of boxes between the maximal staircase and the shape λ, where boxes along each SW-NE diagonal carry the same sign but the sign across diagonals alternates.

1.2. **Specialisations and fully packed loops.** The specialisations $x_i = 1$ of the degenerate Macdonald and KL polynomials, normalised by dividing by the t-Vandermonde determinants, correspond at $\tau = 1$ ($t = \mathrm{e}^{\mathrm{i}\pi/3}$) to the enumeration of fully packed loop (FPL) diagrams (or alternating sign matrices with which they are in simple bijection). These are diagrams of polygons on a square grid such that every site is visited by exactly one polygon. The figure below shows on the left the grid with prescribed boundary conditions, and also gives three example FPL configurations.



FPL diagrams can be grouped together according to their link pattern, i.e. the way they connect the boundary edges to each other. There furthermore is an easy bijection from link patterns to Dyck paths, and from Dyck paths to Young diagrams, see below. Hence, FPL diagrams can be labeled by partitions. For example, the three FPL diagrams above all have loops connecting one to two, three to eight, four to five and six to seven. The corresponding partition is [2,1,1,1]:



These are in fact the only three FPL diagrams corresponding to the partition [2,1,1,1]. The ratio $C_{[2,1,1,1]}/C_\emptyset\big|_{x_i=1} = \tau^2 + 2\tau^4$ which equals 3 at $\tau = -[2] = 1$, i.e. $t = \mathrm{e}^{2\mathrm{i}\pi/3}$. This is not a coincidence as we have the following theorem, which proof is scattered in the literature [6, 5, 4, 2, 1].

Theorem 2. *The evaluation* $C_\lambda(1,\dots,1)/C_\emptyset(1,\dots,1)\Big|_{\tau=1}$ *is equal to the number of FPL diagrams labeled by a partition of shape* λ.

Theorem 1 implies that the total number of FPL diagrams (which equals the known number of alternating sign matrices) is given by an evaluation of the nonsymmetric Macdonald polynomial $M_\mu\big|_{x_i=1,\tau=1} = 1, 7, 42, 429, \dots$ for $n = 1, 2, 3, 4, \dots$. This evaluation also correspond to the number of totally symmetric self complementary plane partitions with a weight τ [4]. It is further possible to define deformed polynomials by choosing the label u_{ij} of box b_{ij} of a partition λ as

$$u_{ij} = u_i + \lambda_i - j + 1.$$

Theorem 3. *The maximal deformed Macdonald polynomial* $M_\mu(u, x; t)$ *is equal to the sum*

$$M_\mu(u, x; t) = \sum_{\lambda \subseteq \mu} c_{\mu\lambda}\ C_\lambda(x; t),$$

where the coefficients $c_{\mu\lambda}$ *are monomials in* $y_i = -[u_i]/[u_i + 1]$, *of degree at most 1 in each variable, and each KL element appears in the sum.*

1.3. **Open problems.** Many questions remain unanswered. Some open problems are: Is it possible to find a combinatorial interpretation for evaluations of KL elements for other, non-maximal, parabolic modules? Is there a Cauchy summation formula for the KL elements corresponding to a maximally parabolic module?

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Graded cellularity and the quiver Hecke algebras of type A

Andrew Mathas
(joint work with Jun Hu)

In a groundbreaking series of papers Brundan and Kleshchev (and Wang) [2, 4, 3] have shown that the cyclotomic Hecke algebras of type $G(\ell, 1, n)$, and their rational degenerations, are graded algebras. Moreover, they have extended Ariki's categorification theorem [1] to show over a field of characteristic zero the graded decomposition numbers of these algebras can be computed using the canonical bases of the higher level Fock spaces.

The starting point for Brundan and Kleshchev's work was the introduction of certain graded algebras $\mathscr{R}_n^\Lambda$ which arose from Khovanov and Lauda's [6, §3.4] categorification of the negative part of quantum group of an arbitrary Kac-Moody Lie algebra and, independently, in work of Rouquier [9]. In type A Brundan and Kleshchev [2] proved that the (degenerate and non-degenerate) cyclotomic Hecke algebras are $\mathbb{Z}$-graded by constructing explicit isomorphisms to $\mathscr{R}_n^\Lambda$.

The **cyclotomic Khovanov-Lauda–Rouquier algebra** $\mathscr{R}_n^\Lambda$ is generated by certain elements $\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n \,\}$ which are subject to a long list of relations. Each of these relations is homogeneous, so it follows directly from the presentation that $\mathscr{R}_n^\Lambda$ is $\mathbb{Z}$-graded. Unfortunately, it is not at all clear from the relations how to construct a homogeneous basis of $\mathscr{R}_n^\Lambda$, even using the isomorphism from $\mathscr{R}_n^\Lambda$ to the cyclotomic Hecke algebras.

The main result of this paper gives an explicit homogeneous basis of $\mathscr{R}_n^\Lambda$. In fact, this basis is cellular so our Main Theorem also proves a conjecture of Brundan, Kleshchev and Wang [4, Remark 4.12].

To describe this basis let $\mathscr{P}_n^\Lambda$ be the set of multipartitions of n, which is a poset under the dominance order. For each $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux (these terms are defined in §3.3). For each $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ there is an idempotent $e_{\boldsymbol{\lambda}}$ and a homogeneous element $y_{\boldsymbol{\lambda}} \in K[y_1, \dots, y_n]$. Brundan, Kleshchev and Wang [4] have defined a combinatorial *degree* function $\deg : \coprod_{\boldsymbol{\lambda}} \operatorname{Std}(\boldsymbol{\lambda}) \longrightarrow \mathbb{Z}$ and for each $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ there is a well-defined element $\psi_{d(\mathfrak{t})} \in \langle \psi_1, \ldots, \psi_{n-1} \rangle$ and we set $\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{d(\mathfrak{s})^{-1}} e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{d(\mathfrak{t})}$. Our Main Theorem is the following.

Theorem 1 (Hu and Mathas [5])**.** *Suppose that $\mathcal{O}$ is a commutative integral domain such that e is invertible in $\mathcal{O}$, $e = 0$, or e is a non-zero prime number, and let $\mathscr{R}_n^\Lambda$ be the cyclotomic Khovanov-Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ over $\mathcal{O}$. Then $\mathscr{R}_n^\Lambda$ is a graded cellular algebra with respect to the dominance order and with homogeneous cellular basis $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$. Moreover,* $\deg(\psi_{\mathfrak{s}\mathfrak{t}}) = \deg \mathfrak{s} + \deg \mathfrak{t}$.

We prove our Main Theorem by considering the two really interesting cases where $\mathscr{R}_n^\Lambda$ is isomorphic to either a degenerate or a non-degenerate cyclotomic Hecke algebra over a field. In these two cases we show that $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is a homogeneous cellular basis of $\mathscr{R}_n^\Lambda$. We then use these results to deduce our main theorem

The main difficulty in proving this theorem is that the graded presentation of the cyclotomic Khovanov-Lauda–Rouquier algebras hides many of the relations between the homogeneous generators. We overcome this by first observing that the KLR idempotents $e(\mathbf{i})$, for $\mathbf{i} \in I^n$, are precisely the primitive idempotents in the subalgebra of the cyclotomic Hecke algebra which is generate by the Jucys-Murphy elements. Using results from [8] this allows us to lift $e(\mathbf{i})$ to an element $e(\mathbf{i})^{\mathcal{O}}$ which lives in an integral form of the Hecke algebra defined over a suitable discrete valuation ring $\mathcal{O}$. The elements $e(\mathbf{i})^{\mathcal{O}}$ can be written as natural linear combinations of the seminormal basis elements [7]. In turn this allows us to construct a family of non-zero elements $e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda}$ a multipartition, which form the skeleton of our cellular basis and hence prove our main theorem.

In fact, we give two graded cellular bases of the cyclotomic Khovanov-Lauda-Rouquier algebras $\mathscr{R}_n^\Lambda$. Intuitively, one of these bases is built from the *trivial* representation of the Hecke algebra and the other is built from its *sign* representation. We then show that these two bases are dual to each other, modulo more dominant terms. As a consequence, we deduce that the blocks of $\mathscr{R}_n^\Lambda$ are graded symmetric algebras, as conjectured by Brundan and Kleshchev[3, Remark 4.7].

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Quasisymmetric Schur functions

Stephanie van Willigenburg
(joint work with C. Bessenrodt, J. Haglund, K. Luoto, S. Mason)

Quasisymmetric functions were introduced as a source of generating functions for P-partitions [2] but since then they have impacted, and deepened the understanding of, other areas of mathematics in addition to combinatorics. For example, in category theory they are the terminal object in the category of graded Hopf algebras equipped with a zeta function [1] and in representation theory they arise as characters of a degenerate quantum group [5]. In order to define quasisymmetric functions, we first need to recall some combinatorial constructs.

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n, denoted $\alpha \vDash n$, is a list of positive integers whose sum is n. Given compositions α, β we say that α is a *coarsening* of β (or β is a *refinement* of α), denoted $\alpha \succeq \beta$, if we can obtain α by adding together adjacent parts of β, for example, $(3, 2, 4, 2) \succeq (3, 1, 1, 1, 2, 1, 2)$.

A *quasisymmetric* function is then a bounded degree formal power series in $\mathbb{Q}[[x_1, x_2, \ldots]]$ such that for all k and $i_1 < i_2 < \cdots < i_k$ the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ is equal to the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ for all compositions $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. The set of all quasisymmetric functions forms a graded Hopf algebra $\mathcal{Q} = \bigoplus_{n\geq 0} \mathcal{Q}_n$.

Two natural bases for quasisymmetric functions are the monomial basis $\{M_\alpha\}$ and the fundamental basis $\{F_\alpha\}$ indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. The *monomial* basis consists of $M_0 = 1$ and all formal power series

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The *fundamental* basis consists of $F_0 = 1$ and all formal power series

$$F_\alpha = \sum_{\alpha \succeq \beta} M_\beta.$$

Furthermore, $\mathcal{Q}_n = \mathrm{span}_{\mathbb{Q}}\{\mathrm{M}_\alpha | \alpha \vDash \mathrm{n}\} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{F}_\alpha | \alpha \vDash \mathrm{n}\}$.

Other bases for $\mathcal{Q}$ exist, in particular the basis of quasisymmetric Schur functions introduced in [3], which is defined in terms of composition tableaux.

Definition 1. [3]

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ let the *composition diagram* of α, also denoted by α, be the array of left-justified cells with α_i cells in row i from the top. Furthermore, given a composition diagram α, a *composition tableau* T of shape α is a filling of the cells with positive integers such that

(1) the entries in the first column strictly increase when read from top to bottom,

(2) the entries in each row weakly decrease when read from left to right, and
(3) for $i < j$

$$T(j, k+1) \leq T(i, k) \Rightarrow T(i, k+1) \text{ exists and } T(j, k+1) < T(i, k+1).$$

Example 2. $\begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 \\ \cline{1-1} 8 & 7 & 2 \\ \cline{1-3}\end{array}$ *is a composition tableau of shape* $(4, 1, 3)$.

Definition 3. [3]

Given a composition tableau, T, let

$$\mathbf{x}^T = \prod_{i \geq 1} x_i^{T(i)}$$

where $T(i)$ is the number of times i appears in T.

If α is a composition then the *quasisymmetric Schur function* $\mathcal{S}_\alpha$ is

$$\mathcal{S}_\alpha = \sum_T \mathbf{x}^T$$

where the sum is over all composition tableaux of shape α.

Example 4. *Restricting to three variables we calculate*

$$\mathcal{S}_{12} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3$$

from

$$\begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array} .$$

Refined properties of Schur functions. The set of quasisymmetric Schur functions not only forms a $\mathbb{Z}$-basis for $\mathcal{Q}$ [3], but also refines a number of properties of Schur functions including the following. In [3]

- the expression for Schur functions in terms of monomial symmetric functions refines to an expression for quasisymmetric Schur functions in terms of monomial quasisymmetric functions, giving rise to quasisymmetric Kostka coefficients;
- the expression for Schur functions in terms of fundamental quasisymmetric functions naturally refines to quasisymmetric Schur functions;
- the Pieri rule for multiplying a Schur function indexed by a row or a column with a generic Schur function refines to a rule for multiplying a quasisymmetric Schur function indexed by a row or a column with a generic quasisymmetric Schur function.

In [4]

- the Littlewood-Richardson rule for the product of two generic Schur polynomials expanded in terms of Schur polynomials refines to a rule for the product of a generic Schur polynomial with a generic quasisymmetric Schur polynomial expanded in terms of quasisymmetric Schur polynomials.

In joint work of the author with Bessenrodt and Luoto, yet to appear,

- expressions for skew Schur functions in terms of monomial symmetric functions and fundamental quasisymmetric functions refine to expressions for quasisymmetric skew Schur functions in terms of monomial and fundamental quasisymmetric functions;
- the Littlewood-Richardson rule for expressing skew Schur functions in terms of Schur functions refines to a rule for expressing quasisymmetic skew Schur functions in terms of quasisymmetric Schur functions.

Open problems. A clear open area of research is to determine which other properties of Schur functions refine to quasisymmetric Schur functions. Examples include

- discovering a Jacobi-Trudi or Giambelli formula;
- establishing a representation theoretic interpretation;
- finding a combinatorial rule for expressing the product of two quasisymmetric Schur functions as a linear combination of quasisymmetric Schur functions.

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A relative hook formula for character degrees of symmetric groups

Gunter Malle
(joint work with Gabriel Navarro)

We present a factorization of the well-known hook formula for character degrees of symmetric groups $\mathfrak{S}_n$. For $\lambda \vdash n$ a partition let χ_λ denote the corresponding irreducible character of $\mathfrak{S}_n$. Let p be a prime. Then the degree $\chi_\lambda(1)$ factors into the degree of χ_μ, where μ is the p-core of λ, and a second factor consisting of inverses of hook lengths of a symbol (abacus diagram) attached to the p-quotient of λ.

This apparently new factorization is obtained by specialization at $q = 1$ of a formula for character degrees of unipotent characters of general linear groups which was obtained in [1] and can be interpreted as an instance of Howlett-Lehrer theory for d-Harish-Chandra series. Its proof is purely combinatorial, but it would also follow directly from a conjecture of Broué and the author on the decomposition of Lusztig induction. This approach also yields, by specialization of q to a pth root of unity, a congruence mod p of $\chi_\lambda(1)/\chi_\mu(1)$ to the degree of the associated relative Weyl group $C_p \wr \mathfrak{S}_w$ where w is the weight of the p-block indexed by μ.

The relative hook formula was instrumental in showing the following result for blocks of alternating groups $\mathfrak{A}_n$ (see [2, Cor. 9.3]):

Theorem 1. *Let p a prime and B a p-block of $G = \mathfrak{A}_n$ with $n \geq 5$. Then either $p = 3$ and B is of weight at most 1, or B contains two height zero characters of different degrees.*

More generally we show ([2, Thm. 6.1]):

Theorem 2. *Let G be a quasi-simple finite group, B a p-block which is neither a spin block of the double cover of the symmetric group, nor a quasi-isolated block of an exceptional group of Lie type for p a bad prime. Then B is nilpotent if and only if all height zero degrees in B are equal.*

This investigation was motivated by the following:

Question 3. *Let G be a finite group, B a p-block of G. Is it true that:*

$$B \text{ is nilpotent} \iff \text{all height zero degrees in } B \text{ are equal.}$$

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Fusion algebra versus quantum cohomology: a combinatorial description

Catharina Stroppel
(joint work with Christian Korff)

1. Fusion ring versus quantum cohomology

Let $n \geq 3$ and k be fixed non-negative integral numbers and set $N = k+n$. The main goal of the talk is a combinatorial description of the relationship between the (small) quantum cohomology ring of the Grassmannian $\mathrm{Gr}(n, N)$ of k-planes in $\mathbb{C}^N$ and the fusion algebra of the affine Kac-Moody Lie algebra $\hat{\mathfrak{sl}}_n$ at level k. To motivate our discussion we recall the following theorem due to Witten [12], based on earlier work of Gepner [1], Vafa [11] and Intrilligator [4] which states that there is an isomorphism of rings

$$\mathcal{F}(\widehat{\mathfrak{gl}}(n))_k \cong qH^\bullet(\mathrm{Gr}_{n,N})_{q=1}$$

between the level k fusion ring of $\widehat{\mathfrak{gl}}(n)$ and the specialised quantum cohomology.

Let $\Lambda^{(k)} = \mathbb{Z}[e_1, \ldots, e_k]$ be the ring of symmetric polynomials in k variables and P_k^+ the natural basis of dominant integral weights of level k for $\hat{\mathfrak{sl}}_n$. As a consequence of our main theorem we obtain an explicit description of the fusion ring in terms of *commutative* symmetric functions:

Theorem 1. *The assignment $P_k^+ \ni \hat{\lambda} \mapsto s_{\lambda^t}$ defines an isomorphism of rings*

$$\mathcal{F}(\widehat{\mathfrak{sl}}(n))_k \cong \mathbb{Z}[e_1, \ldots, e_k]/\langle h_n - 1, h_{n+1}, \ldots, h_{n+k-1}, h_{n+k} + (-1)^k e_k \rangle.$$

This is naturally a quotient of $qH^\bullet(\mathrm{Gr}_{k,n+k})$ in its Siebert-Tian presentation [10] *by imposing the additional relations $q = e_k$ and $h_n = 1$.*

2. The combinatorial models

To understand the combinatorial models note first that the elements of P_k^+ are weights $\hat{\lambda} = \sum_{i=0}^{n-1} m_i \omega_i$ expressed in the basis of the fundamental weights with coefficients $m_i \in \mathbb{Z}_{\geq 0}$ whose sum is k. This set of weights can be identified with

- partitions whose Young diagram fits into a box of height $n-1$ and width k (such that m_i becomes the number of columns of height i).
- "bosonic" configurations of particles on a circle with n sites (such that there are m_i particles at the places i, see Figure 1).

On the other hand, the (integral) quantum cohomology $qH^\bullet(\mathrm{Gr}_{n,N})$ is a free $\mathbb{Z}[q]$-module over the ordinary (integral) cohomology $H^\bullet(\mathrm{Gr}_{n,N})$. Hence a basis is given by the Schubert classes or Borel orbits which can be labeled by

- partitions whose Young diagram fits into a box of height n and width k.
- words of length $n+k$ in the letters 0 and 1 with precisely k ones (which can be read off the diagram when walking along the boundary, where moving to the right translates into a 1 and moving up gives a 0).
- "fermionic" configurations of particles on a circle with $n+k$ sites, at most one particle at each place (indicated by the corresponding $\{0,1\}$-word).

In either of the two particle pictures, there is for a fixed place i the particle creation operator ψ_i^* and annihilation ψ_i. Let $H := \oplus_{i=0}^N H_i$ denote the $\mathbb{Z}[q]$-module underlying $\oplus_{i=0}^N qH^\bullet(\mathrm{Gr}_{i,N})$ with basis given by say the $\{0,1\}$-words as above. Then ψ_i^* is the linear endomorphism which sends a word with letter 0 at the i-th place to the word where this letter is changed to a 1, and annihilates all other words. Similarly for ψ_i with the role of 1 and zero swapped. These operators pass between different quantum cohomology rings, and satisfy the Clifford algebra relations

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij} \,. \tag{1}$$

For the bosonic picture, the operators are graphically displayed in Figure 1 and satisfy so-called phase algebra relations. Consider the basis element $\varnothing$ corresponding to the configuration with no particles and define a $\mathbb{C}[q]$-bilinear multiplication $\star$ on H by

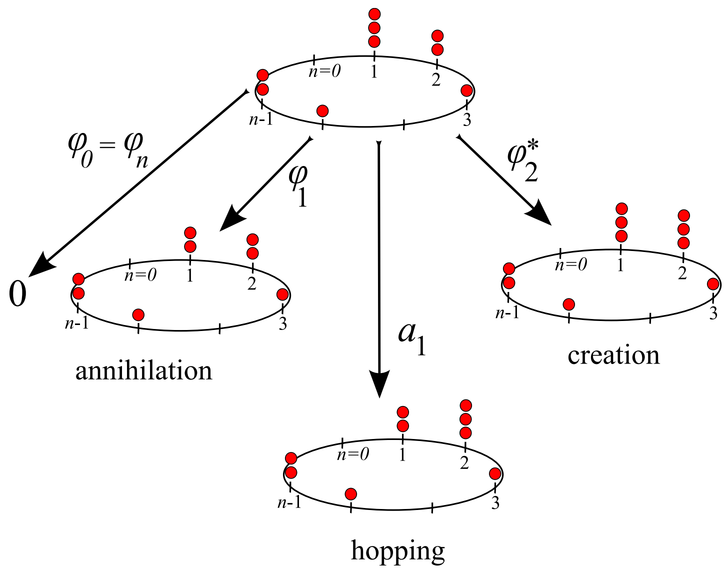

FIGURE 1. The dominant integral weight $(0,3,2,1,0,1,2)$ of level $k=9$ as particle configuration on a circle; the processes of creating and annihilating, and hopping applied to it.

$$(2)\quad \lambda\star\mu \;:=\; \sum_T \hat{\psi}^*_{\ell_n(\mu)+t_n}\psi^*_{\ell_{n-1}(\mu)+t_{n-1}}\psi^*_{\ell_{n-2}(\mu)+t_{n-2}}\psi^*_{\ell_{n-3}(\mu)+t_{n-3}}\cdots\varnothing,$$

where $\psi^*_{i+N}=(-1)^{n-1}q\psi^*_i$ for $1\le i\le N$ and $l_i(\mu)$ denotes the position of the i-th 1 from the left in the word μ. The sum runs over all semi-standard tableaux $T=T(\lambda)$ of shape λ with fillings from $[1,n]$, where i occurs precisely t_i times.

Theorem 2 (Fermion presentation of quantum cohomology)**.**
$(H,\star)$ is a commutative, associative, unital algebra. The multiplication agrees with the multiplication on $qH^*(\mathrm{Gr}_{n,N})$, so the structure constants $C^{\nu,d}_{\lambda\mu}$ in the product expansion

$$\lambda\star\mu=\sum_d\sum_\nu q^dC^{\nu,d}_{\lambda\mu}\nu$$

are the Gromov-Witten invariants with $C^{\nu,d}_{\lambda\mu}=0$ unless $|\lambda|+|\mu|-|\nu|=Nd$.

Remark 3 (Quantum Racah-Speiser and honeycombs)**.** When setting $q=0$ only the structure constants with $d=0$ survive and the formula specialises to an expression for the Littlewood-Richardson coefficients in terms of Kostka numbers. Recall that the Kostka number $K_{\lambda,\mu}$ gives the multiplicity of the weight μ in the $\mathfrak{sl}(n)$-representation $V(\lambda)$ of highest weight λ, while the Littlewood-Richardson coefficients coincide with the multiplicity of the highest weight representation $V(\nu)$ in the tensor product decomposition of $V(\lambda)\otimes V(\mu)$. Thus, this result can be interpreted as a combinatorial derivation of the Racah-Speiser algorithm. When specializing $q=0$, there is a direct connection to Knutson-Tao-Woodward puzzles and honeycombs [5], see [3]. Hence the above theorem might be viewed as a q-version of the latter.

Define now the following endomorphisms of H ("particle hopping"),

$$u_i = \psi_{i+1}^* \psi_i \ , \quad i = 1, \dots, N-1 \quad \text{and} \quad u_N = (-1)^{\mathrm{n}-1} q \psi_1^* \psi_N \ , \tag{3}$$

then the u_i's generate a (noncommutative) subalgebra in $\mathrm{End}(H)$ isomorphic to the *affine nil-Temperley-Lieb algebra*, characterized by the following relations

$$u_i^2 = u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} = 0, \qquad u_i u_j = u_j u_i \quad \text{if } |i-j| > 1 \bmod N \ , \tag{4}$$

where all indices are understood modulo N.

Following [7] and [2, Definition 9.4] one can define *non-commutative* elementary functions as endomorphisms of H, that is elementary functions $\{\boldsymbol{e}_r\}$ in the non-commuting variables u_i. Although the variables do not commute, the endomorphisms $\boldsymbol{e}_r$ do. This is proved in [2] by constructing an interesting simultaneous eigenbasis using Bethe Ansatz techniques. So it makes sense to define noncommutative Schur polynomials via the determinant formula

$$\boldsymbol{s}_\lambda = \det(\boldsymbol{e}_{\lambda_i^t - i + j})_{1 \le i,j \le N} \tag{5}$$

And they satisfy all the familiar relations from the ring of commutative symmetric functions. In particular, one has the special cases $\boldsymbol{s}_{(1^r)} = \boldsymbol{e}_r$.

Exactly the same construction works on the "bosonic" side. Instead of the affine nil-Temperley-Lieb algebra we obtain what we call the *affine local plactic algebra* in generators a_i, $0 \le i \le n-1$. This algebra is an affine version of a plactic algebra or generic Hall algebra ([6], [8]). In this case, a combinatorial proof that the Schur polynomials are well-defined is new and will appear in [9]. In [2] this result is again obtained by Bethe Ansatz techniques. Our two rings we are interested in can now be described in terms of non-commutative Schur polynomials as follows:

Theorem 4 (Combinatorial quantum cohomology ring). *Fix $n \in \mathbb{Z}_{\geq 0}$ and consider the n-particle subspace $H_n \subset H$ corresponding to the summand $qH(\mathrm{Gr}(n,N))$. The assignment*

$$(\lambda, \mu) \mapsto \lambda \star \mu := \boldsymbol{s}_\lambda \mu \tag{6}$$

for basis elements $\lambda, \mu \in \mathfrak{P}_{\leq n,k}$ defines an associative commutative unital ring structure equal to the quantum cohomology ring. The analogous statement holds for the fusion ring.

The main idea in the proof is the following: As mentioned above, using Bethe Ansatz techniques one can construct a simultaneous eigenbasis for the action of the non-commutative symmetric functions. Expressing the above product in terms of this basis recovers the so-called Bertram-Vafa intrilligator formulas on the quantum cohomology side and the Verlinde formula on the fusion algebra side. In this way one can prove that the combinatorially defined multiplication is indeed the multiplication we were looking for. The fermionic formulas in Theorem 1 are then deduced complete combinatorially. The presentation of the fusion ring from Theorem 2 appears naturally, with the ideal generated by *Bethe Ansatz equations.*

Using the same techniques we also obtain the Siebert-Tian [10] presentation of the quantum cohomology.

The details of the results outlined in this talk can be found in [2] and [3].

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Irreducible representations of Yangians via Howe duality

Maxim Nazarov

(joint work with Sergey Khoroshkin)

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$ arose from the theory of quantum integrable systems as one the first examples of affine quantum groups. Much later, the study of quantum integrable systems with boundary conditions produced the twisted Yangians $\mathrm{Y}(\mathfrak{so}_n)$ and $\mathrm{Y}(\mathfrak{sp}_n)$. The latter two are (one-sided) coideal subalgebras in the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$, corresponding to the orthogonal and symplectic subalgebras in the Lie algebra $\mathfrak{gl}_n$. In this talk I outlined a uniform construction of irreducible finite-dimensional representations of the Yangians $\mathrm{Y}(\mathfrak{gl}_n)$, $\mathrm{Y}(\mathfrak{so}_n)$ and $\mathrm{Y}(\mathfrak{sp}_n)$ based on the theory of reductive dual pairs due to Howe. I used the pairs of reductive Lie groups

$$(\mathrm{G},\mathrm{G}') = (\mathrm{GL}_m,\mathrm{GL}_n),\ (\mathrm{O}_{2m},\mathrm{O}_n),\ (\mathrm{Sp}_{2m},\mathrm{Sp}_n)$$

acting on the Clifford algebra of the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$. The Yangian $\mathrm{Y}(\mathfrak{g}')$ corresponds to the Lie algebra $\mathfrak{g}'$ of the second group in any of these dual pairs.

Let $\mathfrak{g}$ be the Lie algebra of the first group in of these dual pairs. Choose a triangular decomposition of $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$. In our construction, the irreducible representations of $\mathrm{Y}(\mathfrak{g}')$ correspond to the orbits in $\mathfrak{h}^* \times \mathfrak{h}^*$ by the (diagonal) shifted action of the Weyl group of $\mathfrak{g}$. The irreducible representations appear as quotients of tensor products of m representations of $\mathrm{Y}(\mathfrak{g}')$ in the exterior powers of $\mathbb{C}^n$. The quotient is taken by the kernel of a canonical $\mathrm{Y}(\mathfrak{g}')$-intertwining operator (called the R-matrix in the theory of quantum integrable systems) acting from the m-fold tensor product, and corresponding to the longest element of the Weyl group of $\mathfrak{g}$. This talk is based on my joint works with Khoroshkin [5, 6, 7, 8, 9] and extends the results obtained for $\mathrm{Y}(\mathfrak{gl}_n)$ in [1, 2, 3, 4].

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Finite dimensional irreducible modules for finite *W*-algebras associated to even multiplicity nilpotent orbits in classical Lie algebras

Simon Goodwin

(joint work with Jonathan Brown)

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and let $e \in \mathfrak{g}$ be nilpotent. The finite W-algebra $U(\mathfrak{g}, e)$ associated to the pair $(\mathfrak{g}, e)$ is a finitely generated algebra obtained from $U(\mathfrak{g})$ by a certain quantum Hamiltonian reduction. Finite W-algebras were introduced to the mathematical literature by Premet in [10] and have already found many striking applications. In particular, there is a close connection between finite dimensional irreducible $U(\mathfrak{g}, e)$-modules and primitive ideals of $U(\mathfrak{g})$, see [8, Thm. 1.2.2]. Consequently, there has been a great deal of research interest in the representation theory of finite W-algebras, see for example [3, 4, 6, 7, 11].

Despite the high level of recent interest, the representation theory of finite W-algebras is only well-understood in certain special cases. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ a thorough study of the representation theory of $U(\mathfrak{g}, e)$ was undertaken by Brundan and Kleshchev in [4]; in particular, they obtained a classification of finite dimensional irreducible modules. Recent work of Brown gives a classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules for *rectangular nilpotent orbits* when $\mathfrak{g}$ is of classical type, see [1].

We have considered finite W-algebras $U(\mathfrak{g}, e)$, where $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ or $\mathfrak{so}_{2n}(\mathbb{C})$ and $e \in \mathfrak{g}$ is an *even multiplicity* nilpotent element; this means all parts of the Jordan decomposition of e have even multiplicity. In this situation, we have classified all finite dimensional irreducible $U(\mathfrak{g}, e)$-modules.

In [3], Brundan, Kleshchev and the author developed a highest weight theory for finite W-algebras. This leads to a strategy for classifying finite dimensional irreducible $U(\mathfrak{g}, e)$-modules in the usual way. Using this highest weight theory, our classification is nicely encoded in terms of the *pyramid* associated to e – the classification is of the form "row equivalent to column strict" up to the action of the component group of the centralizer of e in the adjoint group of $\mathfrak{g}$, see [2] for a precise statement.

The key ingredients for our proof are:

- combinatorics of fillings of the pyramid associated to e;
- a conjecture in [3] relating the *category* $\mathcal{O}(e)$ for $U(\mathfrak{g}, e)$ to a category of generalized Whittaker modules that was proved by Losev in [9];
- the algorithm of Barbasch and Vogan from [5] for determining the associated variety of a primitive ideal of $U(\mathfrak{g})$;
- "Levi subalgebras" of $U(\mathfrak{g}, e)$;
- the type A case from [4];
- the case of rectangular nilpotent orbits from [1]; and
- combinatorics for the two sided cell of the Weyl group of G corresponding to e.

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Multiplicity Spaces in Symplectic Branching

Oded Yacobi

1. Motivation

The restriction of an irreducible representation of GL_n to GL_{n-1} uniquely decomposes into a direct sum of irreducible representations, i.e. the branching is multiplicity-free. Many combinatorial results about the representation theory of GL_n can be reduced to this fact. Thus, it is natural to ask whether such constructions work for other classical groups. Here we study the symplectic groups.

Let $G_n = Sp_{2n}$ and consider the embedding $G_{n-1} \subset G_n$. In this case the restriction of an irreducible representation of G_n to G_{n-1} is not multiplicity-free. Therefore many techniques that work for the general linear groups cannot be directly applied to this setting.

We use invariant theory to resolve the multiplicities that occur in symplectic branching. Our main result is that the multiplicity spaces that occur in symplectic branching carry a canonical irreducible action of a product of $SL_2's$. We prove an isomorphism of so-called branching algebras, which allows us to reduce questions about symplectic branching to ones about branching from GL_{n+1} to GL_{n-1}.

2. Preliminaries

Let Λ_n be the set of partitions of length n. Λ_n naturally indexes the following sets of irreducible representations:

$$\mathrm{Irr}_{\mathrm{poly}}(GL_n) \longleftrightarrow \Lambda_n \longleftrightarrow \mathrm{Irr}(G_n)$$

Here $\mathrm{Irr}_{\mathrm{poly}}(GL_n)$ is the set of irreducible polynomial representations of GL_n, and $\mathrm{Irr}(G_n)$ is the set of irreducible representations of G_n. For $\lambda \in \Lambda_n$ let V_λ (respectively W_λ) denote the corresponding irreducible representation of GL_n (respectively G_n).

For $\lambda \in \Lambda_n$ we write

$$Res^{GL_n}_{GL_{n-1}} V_\lambda \cong \bigoplus_{\mu \in \Lambda_{n-1}} V_\mu \otimes N^\lambda_\mu,$$

where N^λ_μ is the multiplicity space $Hom_{GL_{n-1}}(V_\mu, V_\lambda)$. Similarly, for $\lambda \in \Lambda_n$ we write

$$Res^{G_n}_{G_{n-1}} W_\lambda \cong \bigoplus_{\mu \in \Lambda_{n-1}} W_\mu \otimes M^\lambda_\mu,$$

where M_μ^λ is the multiplicity space $Hom_{G_{n-1}}(W_\mu, W_\lambda)$. That branching from G_n to G_{n-1} is not multiplicity free is equivalent to the fact that $dimM_\mu^\lambda$ is greater than one for some μ and λ. It's a classical result that:

$$dimM_\mu^\lambda \neq 0 \Leftrightarrow \mu \text{ double interlaces } \lambda.$$

The double interlacing condition here means that $\lambda_i \geq \mu_i \geq \lambda_{i+2}$ for all i.

3. Main Results

Let $\Lambda_{n-1,n} = \Lambda_{n-1} \times \Lambda_n$, and let $(\mu, \lambda) \in \Lambda_{n-1,n}$. Our starting point is the simple observation that M_μ^λ is naturally an SL_2-module. Indeed, there is a natural copy of SL_2 that centralizes $G_{n-1} \subset G_n$. This leads us to ask, what is the SL_2-module structure of M_μ^λ? We answer this by reducing the problem to an analogous one concerning the general linear groups.

Let $U_n \subset G_n$ be the unipotent radical of a Borel subgroup of G_n. Consider the ring of regular functions on G_n which are left-invariant with respect to U_n and right-invariant with respect to U_{n-1}:

$$\mathcal{M} = \mathcal{O}(U_n \backslash G_n / U_{n-1}).$$

By (algebraic) Peter-Weyl theory, $\mathcal{M}$ is $\Lambda_{n-1,n}$-graded:

$$\mathcal{M} = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}} M_\mu^\lambda.$$

In other words, the graded components are isomorphic to symplectic multiplicity spaces. $\mathcal{M}$ is an example of a branching algebra.

We want to compare $\mathcal{M}$ to a branching algebra corresponding to restriction from GL_{n+1} to GL_{n-1}. Now let $U_n \subset GL_n$ be the upper triangular unipotent matrices and $M_{m,n}$ the $m \times n$ matrices with complex entries. We use (GL_n, GL_{n+1})-duality to construct a branching algebra:

$$\mathcal{N} = \mathcal{O}(U_n \backslash M_{n,n+1} / U_{n-1}) = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}} N_\mu^{\lambda^+}.$$

Here if $\lambda \in \Lambda_n$ we set $\lambda^+ = (\lambda_1, ..., \lambda_n, 0) \in \Lambda_{n+1}$. Thus $\mathcal{N}$ is also $\Lambda_{n-1} \times \Lambda_n$-graded; its graded components are certain multiplicity spaces that occur in branching from GL_{n+1} to GL_{n-1}.

Notice that $\mathcal{M}$ and $\mathcal{N}$ are both graded by the same semigroup. Moreover, they both carry a natural action of SL_2. Let $f : G_n \to M_{n,n+1}$ be defined by taking g to its principal $n \times (n+1)$ cut-off. Consider the induced map

$$f^* : \mathcal{O}(M_{n,n+1}) \to \mathcal{O}(G_n)$$

on functions. It's not hard to show that $f^*(\mathcal{N}) \subset \mathcal{M}$.

Theorem 1 (Theorem 3.1, [2]). *$f^* : \mathcal{N} \to \mathcal{M}$ is an isomorphism of $\Lambda_{n-1} \times \Lambda_n$-graded SL_2-algebras.*

This theorem allows us to reduce questions about the branching of the symplectic groups to ones about branching from GL_{n+1} to GL_{n-1}. For example, if $(\mu, \lambda) \in \Lambda_{n-1,n}$, then by the above theorem M_μ^λ is isomorphic to $N_\mu^{\lambda^+}$ as SL_2-modules. Moreover, to determine the SL_2-module structure of $N_\mu^{\lambda^+}$ is easy since, by factoring through GL_n, we can simply write down the character. Let F_k be the $k+1$ dimensional irreducible representation of SL_2.

Proposition 2 (cf. [1]). *Suppose μ double interlaces λ. Then as SL_2-modules*

$$M_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)},$$

where SL_2 acts by the tensor product representation on the right hand side, and

$$r_i(\mu, \lambda) = \min(\mu_{i-1}, \lambda_i) - \max(\mu_i, \lambda_{i+1})$$

with $\lambda_{n+1} = \mu_0 = 0$.

This answers the question posed above, but it also suggests a deeper question. For $(\mu, \lambda) \in \Lambda_{n-1,n}$ consider the irreducible $L = \prod_{i=1}^n SL_2$-module $A_\mu^\lambda = \bigotimes_{i=1}^n F_{r_i(\mu,\lambda)}$. The above proposition states that $M_\mu^\lambda \cong Res_{SL_2}^L A_\mu^\lambda$, where $SL_2 \subset L$ is diagonally embedded. We therefore ask, is there a natural action of L on M_μ^λ such that $M_\mu^\lambda \cong A_\mu^\lambda$ as L-modules?

To answer this we investigate the double interlacing condition that characterizes symplectic branching. An **order type** σ is a word in the alphabet $\{\geq, \leq\}$ of length $n-1$. Suppose $(\mu, \lambda) \in \Lambda_{n-1,n}$ and $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ is an order type. Then we say (μ, λ) is **of order type** σ if for $i = 1, ..., n-1$,

$$\begin{cases} \sigma_i = \text{“} \geq \text{”} \implies \mu_i \geq \lambda_{i+1} \\ \sigma_i = \text{“} \leq \text{”} \implies \mu_i \leq \lambda_{i+1} \end{cases}$$

Let Σ be the set of order types, and for each $\sigma \in \Sigma$ let $\Lambda_{n-1,n}(\sigma)$ be the pairs (μ, λ) of order type σ. It's easy to check that $\Lambda_{n-1,n}(\sigma)$ is a semigroup, and therefore

$$\mathcal{M}(\sigma) = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}(\sigma)} M_\mu^\lambda$$

is a subalgebra of $\mathcal{M}$. Moreover, $\mathcal{M}(\sigma)$ is SL_2 invariant.

Our second result says that the natural SL_2 action on M_μ^λ can be extended canonically to an irreducible action of L, thereby resolving the multiplicities that occur in symplectic branching. We warn the reader here that the group L is not the n-fold product of $SL_2's$ which naturally embeds in G_n. Indeed, this latter n-fold product does not act on M_μ^λ.

Theorem 3 (Theorem 3.8, [2]). *There is a unique representation $(\Phi, \mathcal{M})$ of L satisfying the following two properties:*

(1) for all μ, λ, M_μ^λ is an irreducible L-invariant subspace of $\mathcal{M}$ isomorphic to A_μ^λ, and

(2) for all $\sigma \in \Sigma$, L acts as algebra automorphisms on $\mathcal{M}(\sigma)$.

Moreover, $Res^L_{SL_2}(\Phi)$ is the natural action of SL_2 on $\mathcal{M}$.

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Pieces of nilpotent cones for classical groups

Anthony Henderson

(joint work with Pramod Achar, Eric Sommers)

Any complex reductive group G acts with finitely many orbits in its *nilpotent cone*

$$\mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} = \mathrm{Lie}(G) \,|\, x \text{ is nilpotent}\}.$$

For example, it is well known that the GL_n-orbits in $\mathcal{N}(\mathfrak{gl}_n)$ are in bijection with $\mathcal{P}_n$, the set of partitions of n: for $\lambda \in \mathcal{P}_n$, the corresponding orbit $\mathcal{O}^A_\lambda$ consists of those $x \in \mathcal{N}(\mathfrak{gl}_n)$ whose Jordan form has blocks of sizes $\lambda_1, \lambda_2, \cdots$. Moreover, the closure ordering on orbits corresponds to the dominance order on partitions: for $\pi, \lambda \in \mathcal{P}_n$, $\mathcal{O}^A_\pi \subseteq \overline{\mathcal{O}^A_\lambda}$ if and only if λ dominates π.

For general reductive G, the *Springer correspondence* gives an injective map

$$G \setminus \mathcal{N}(\mathfrak{g}) \hookrightarrow \mathrm{Irr}(W),$$

where $\mathrm{Irr}(W)$ denotes the set of isomorphism classes of irreducible representations of the Weyl group W of G. If $\mathcal{O}$ is a nilpotent orbit on the left-hand side and $x \in \mathcal{O}$, the associated irreducible representation of W can be realized as $H^{\mathrm{top}}(\mathcal{B}_x)^{G_x}$, where $\mathcal{B}_x$ is the Springer fibre and we take invariants for the stabilizer G_x. The special property of $G = GL_n$ which makes this injective map bijective is that all these stabilizers are connected and hence act trivially on $H^*(\mathcal{B}_x)$. (In general, to construct all of $\mathrm{Irr}(W)$ one needs to consider not just G_x-invariants but other isotypic components for the action of G_x/G_x°.)

The groups SO_{2n+1} (of type B_n) and Sp_{2n} (of type C_n) are dual to each other and have the same Weyl group $W = \{\pm 1\} \wr S_n$. However, the relationship between their nilpotent orbits is not as simple as one might suppose. We identify $\mathrm{Irr}(W)$ in the usual way with $\mathcal{Q}_n$, the set of bipartitions of n. Shoji in [6] showed that the Springer parameters for the nilpotent orbits are as follows:

$$SO_{2n+1} \setminus \mathcal{N}(\mathfrak{so}_{2n+1}) \longleftrightarrow \mathcal{Q}^B_n := \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 2,\, \nu_i \geq \mu_{i+1}\},$$
$$Sp_{2n} \setminus \mathcal{N}(\mathfrak{sp}_{2n}) \longleftrightarrow \mathcal{Q}^C_n := \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 1,\, \nu_i \geq \mu_{i+1} - 1\}.$$

We write $\mathcal{O}^B_{\mu;\nu}$ for the orbit in $\mathcal{N}(\mathfrak{so}_{2n+1})$ corresponding to $(\mu;\nu) \in \mathcal{Q}^B_n$, and $\mathcal{O}^C_{\mu;\nu}$ for the orbit in $\mathcal{N}(\mathfrak{sp}_{2n})$ corresponding to $(\mu;\nu) \in \mathcal{Q}^C_n$.

To compare the two collections of nilpotent orbits, we consider

$$\mathcal{Q}^\circ_n = \mathcal{Q}^B_n \cap \mathcal{Q}^C_n = \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 1,\, \nu_i \geq \mu_{i+1}\},$$

which consists of the *special bipartitions* (those labelling Lusztig's special representations of the Weyl group). Spaltenstein observed that every non-special orbit in $\mathcal{N}(\mathfrak{so}_{2n+1})$ or $\mathcal{N}(\mathfrak{sp}_{2n})$ lies in the closure of a unique minimal special orbit; hence the orbits can be grouped into *special pieces* $\mathcal{S}^B_{\mu;\nu}$ and $\mathcal{S}^C_{\mu;\nu}$, indexed in each case by $\mathcal{Q}^\circ_n$. Lusztig proved in [5] that for every $(\mu;\nu) \in \mathcal{Q}^\circ_n$, the corresponding special pieces $\mathcal{S}^B_{\mu;\nu}$ and $\mathcal{S}^C_{\mu;\nu}$ have the same equivariant Betti numbers (defined using equivariant cohomology with rational coefficients, relative to the groups SO_{2n+1} and Sp_{2n} respectively).

Syu Kato has introduced in [3, 4] the *exotic nilpotent cone*

$$\mathfrak{N} := \{(v,x) \in \mathbb{C}^{2n} \times \mathfrak{gl}_{2n} \,|\, x \text{ is nilpotent}, \langle xu, u\rangle = 0, \forall u \in \mathbb{C}^{2n}\},$$

where $\langle\cdot,\cdot\rangle$ denotes a symplectic form on $\mathbb{C}^{2n}$. It carries an action of the symplectic group Sp_{2n}, but also has some affiliation with type B_n, as the results below indicate. Kato discovered an *exotic Springer correspondence*

$$Sp_{2n} \setminus \mathfrak{N} \longleftrightarrow \mathrm{Irr}(W) = \mathcal{Q}_n,$$

which is bijective: as with the ordinary Springer correspondence for GL_n, all stabilizers are connected. Let $\mathbb{O}_{\mu;\nu}$ be the orbit in $\mathfrak{N}$ corresponding to $(\mu;\nu) \in \mathcal{Q}_n$.

The closure ordering on orbits in $\mathfrak{N}$ was determined by Achar and myself:

Theorem 1. [1, Theorem 6.3] *For* $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$,

$$\mathbb{O}_{\rho;\sigma} \subseteq \overline{\mathbb{O}_{\mu;\nu}} \iff \begin{array}{rcl} \rho_1 & \le & \mu_1, \\ \rho_1+\sigma_1 & \le & \mu_1+\nu_1, \\ \rho_1+\sigma_1+\rho_2 & \le & \mu_1+\nu_1+\mu_2, \\ \rho_1+\sigma_1+\rho_2+\sigma_2 & \le & \mu_1+\nu_1+\mu_2+\nu_2, \\ \vdots \quad \vdots \quad \vdots & & \vdots \quad \vdots \end{array}$$

Achar, Sommers, and I show in [2] that the induced partial order on the subsets $\mathcal{Q}^B_n$ and $\mathcal{Q}^C_n$ corresponds to the closure ordering on orbits in $\mathcal{N}(\mathfrak{so}_{2n+1})$ and $\mathcal{N}(\mathfrak{sp}_{2n})$.

We can define three partitions of $\mathfrak{N}$ by grouping orbits as dictated by the closure ordering: a partition into *type-B pieces* $\mathbb{T}^B_{\mu;\nu}$ indexed by $\mathcal{Q}^B_n$, a partition into *type-C pieces* $\mathbb{T}^C_{\mu;\nu}$ indexed by $\mathcal{Q}^C_n$, and (coarser than either of these) a partition into *special pieces* $\mathbb{S}_{\mu;\nu}$ indexed by $\mathcal{Q}^\circ_n$. One of our main results gives a numerical relationship between these pieces and the orbits in $\mathcal{N}(\mathfrak{so}_{2n+1})$ and $\mathcal{N}(\mathfrak{sp}_{2n})$:

Theorem 2. [2, Theorem 2.23]

(1) For $(\mu;\nu) \in \mathcal{Q}^B_n$, $\mathbb{T}^B_{\mu;\nu}$ *has the same equivariant Betti numbers as* $\mathcal{O}^B_{\mu;\nu}$.

(2) For $(\mu;\nu) \in \mathcal{Q}^C_n$, $\mathbb{T}^C_{\mu;\nu}$ *has the same equivariant Betti numbers as* $\mathcal{O}^C_{\mu;\nu}$.

(3) For $(\mu;\nu) \in \mathcal{Q}^\circ_n$, $\mathbb{S}_{\mu;\nu}$ *has the same equivariant Betti numbers as* $\mathcal{S}^B_{\mu;\nu}$ *on the one hand, and* $\mathcal{S}^C_{\mu;\nu}$ *on the other.*

Part (3) follows from (1) and (2), and offers a new explanation of Lusztig's result.

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Projective and involutory reflection groups

Fabrizio Caselli

1. Introduction

The relationship between the combinatorics and the (invariant) representation theory of symmetric groups is fascinating from both combinatorial and algebraic points of view, and the problem of generalizing these sort of results to all (complex) reflection groups has been faced in many ways. Besides several results that hold in the full generality of reflection groups, there are some relevant generalizations which are quite clear for wreath products only, and remain open in the more general context of reflection groups. For example no basis is known for the free module of diagonal invariants over the tensor invariants (this is known in type A and B only [20, 26, 11, 9]), there is no Robinson-Schensted type correspondence (the case of wreath products was carried out in [28]), no major index statistics is known that produces nice Hilbert series (wreath products and classical Weyl groups were studied in [20, 29, 4, 1, 13]), and no general Gelfand model is known (the case of symmetric groups is studied in [3, 5, 21, 22, 23, 24] and other special classes of complex reflection groups in [2, 6, 7, 8]). Some attempts to extend these results to other reflection groups have been made, in particular for Weyl groups of type D, (see, e.g., [13, 14, 9]) though they are probably not completely satisfactory as in the case of wreath products. Here we present a unified solution to all these problems for generic reflection groups $G(r,p,n)$ (with the exception of the construction of a Gelfand model where we have to assume that $\mathrm{GCD}(p,n)=1,2$.)

With these problems in mind we introduce a new class of groups, the projective reflection groups, which are a generalization of reflection groups. In this report I will focus on the infinite family $G(r,p,q,n)$ of projective reflection groups, which includes all the groups $G(r,p,n)$ (in fact $G(r,p,1,n)=G(r,p,n)$). Fundamental in the theory of these groups is the following notion of duality: if $G=G(r,p,q,n)$ then we denote by $G^*=G(r,q,p,n)$. We note in particular that reflection groups G satisfying $G=G^*$ are exactly the wreath products $G(r,n)=G(r,1,1,n)$ and that in general if G is a reflection group then G^* is not. We show that, in many cases, the combinatorics of a projective reflection group G of the form $G(r,p,q,n)$ is strictly related to the (invariant) representation theory of G^*, generalizing several known results for wreath products in a very natural way.

2. Two results

We start introducing the groups $G(r,p,q,n)$. Let $r,p,q,n \in \mathbb{N}$ be such that $p|r$, $q|r$ and $pq|rn$. Then we let

$$G(r,p,q,n) := G(r,p,n)/C_q,$$

where C_q is the cyclic group of scalar matrices of order divided by q.

Here the groups $G(r,p,n)$ are those appearing in the classification of Shephard-Todd of complex reflection groups. If $g \in G(r,p,q,n)$ one can define, in a simple combinatorial way (see [16]) a partition $\lambda(g)$ having length at most n. If $q = p = 1$ the partition $\lambda(g)$ gives an alternative definition for the flag-major index of Adin and Roichman (see [4]) for wreath products $G(r,n)$: in this case we have $\mathrm{fmaj}(g) = |\lambda(g)|$.
The group $G = G(r,p,q,n)$ acts in a natural way on $S_q[X]$ the subalgebra of $\mathbb{C}[x_1,\ldots,x_n]$ generated by all homogeneous polynomials of degree q. We consider the actions of G^k and of its diagonal subgroup ΔG on the k-fold tensor product $S_q[X]^{\otimes k}$. Then, using the partitions $\lambda(g)$ we can define a polynomial $a_{g_1,\ldots,g_k} \in (S_q[X]^{\otimes k})^{\Delta G}$, where $(g_1,\ldots,g_k)$ range among all k-tuples in G^* whose product is the identity.

Theorem 1. *The set of polynomials*

$$\{a_{g_1,\ldots,g_k} : g_1,\ldots,g_k \in G^* \text{ and } g_1\cdots g_k = 1\},$$

is a basis for $(S_q[X]^{\otimes k})^{\Delta G}$ as a free module over $(S_q[X]^{\otimes k})^{G^k}$.

As a further important application of the duality we present another type of result. If $\mathrm{GCD}(p,n) = 1, 2$, which implies that the group $G = G(r,p,q,n)$ is involutory (see [17]), we can construct an explicit (monomial) representation ρ of G on the vector space $V(G^*)$ spanned by the absolute involutions of G^*. Here absolute involution means elements g satisfying $g\bar{g} = 1$.

Theorem 2. *The representation $(V(G^*),\rho)$ is a Gelfand model for G, i.e. it is isomorphic to the direct sum of all irreducible representations of G with multiplicity 1.*

This result has been much refined in [18], by showing a finer decomposition which is extremely well-behaved with respect to the projective Robinson-Schensted correspondence (see [16]) and the so-called split representations of G.

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Representation Theory of Quiver Hecke Algebras via Lyndon Bases

David Hill

(joint work with George Melvin, Damien Mondragon)

Recently, Khovanov and Lauda [KL1, KL2] and Rouquier [Rq] have independently introduced a remarkable family of graded algebras, $H(\Gamma)$, defined in terms of quivers associated to the Dynkin diagram, Γ, of a symmetrizable Kac-Moody algebra, $\mathfrak{g}$, known as 'quiver Hecke algebras'. These algebras categorify 'one-half' of the quantum group associated to the Dynkin diagram Γ. That is, there is an isomorphism of *twisted* bialgebras

$$K(\Gamma) \cong \mathcal{U}_{\mathcal{A}}^*(\mathfrak{n}), \tag{1}$$

where $K(\Gamma)$ is the Grothendieck group of the full subcategory, $\mathrm{Rep}(\Gamma)$, of finite dimensional graded $H(\Gamma)$-modules, $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and $\mathcal{U}_{\mathcal{A}}^*(\mathfrak{n})$ is an integral form of the quantized enveloping algebra of $\mathfrak{n}$.

Further evidence of the importance of these algebras was obtained in [BK2]. In this work, Brundan and Kleshchev showed that when Γ is of type A_∞ or $A_{\ell-1}^{(1)}$, there is an isomorphism between blocks of cyclotomic Hecke algebras of symmetric groups, and blocks of a corresponding cyclotomic quotient of $H(\Gamma)$. Moreover, this isomorphism applies equally well to quotients of the affine Hecke algebra and its rational degeneration, depending only on Γ and the underlying ground field. In light of the work [BK1], it is expected that a similar relationship should hold between cyclotomic quotients of $H(\Gamma)$ and cyclotomic Hecke-Clifford algebras when Γ is of type B_∞ and $A_{2\ell}^{(2)}$ (see also [HKS]).

As in the classical case of the affine Hecke algebra, the cyclotomic quotients $H^\Lambda(\Gamma)$ of $H(\Gamma)$ are in natural correspondence with dominant integral weights, Λ, of $\mathfrak{g}$. Let $K^\Lambda(\Gamma)$ denote the Grothendieck group of the category $\mathrm{Rep}H^\Lambda(\Gamma)$ of graded finite dimensional $H^\Lambda(\Gamma)$-modules. Then, we have the following conjecture of Khovanov and Lauda:

Khovanov-Lauda Conjecture: [KL1, §3.4] There is an isomorphism of $U_q(\mathfrak{g})$-modules

$$\mathbb{Q}(q) \otimes K^\Lambda(\Gamma) \cong V(\Lambda),$$

where $V(\Lambda)$ is the irreducible $U_q(\mathfrak{g})$-module of highest weight Λ. Under this isomorphism, $K^\Lambda(\Gamma)$ corresponds to the minimal admissible lattice inside $V(\Lambda)$ and the isomorphism classes of simple modules correspond to the dual canonical basis.

There has been some progress toward the Khovanov-Lauda Conjecture. For Γ of types A_∞ and $A_{\ell-1}^{(1)}$, Brundan and Kleshchev [BK3] used the isomorphism they established in [BK2] and the graded Specht modules constructed with Wang [BKW] to lift Ariki's theorem [A] to the graded setting, thereby proving the result in this case. For general Γ, Lauda and Vazirani [LV] have shown that $K^\Lambda(\Gamma)$ has the structure of a crystal, where the crystal operators are defined in terms of induction and restriction functors. Moreover, this crystal coincides with the crystal

$B(\Lambda)$ of $V(\Lambda)$, providing significant combinatorial evidence for the Khovanov-Lauda Conjecture.

At the same time, Kleshchev and Ram [KR2] completed their investigation of the representation theory of $H(\Gamma)$, for Γ of finite type, with the goal of constructing irreducible representations. The parametrization of these representations is given by the combinatorics of Lyndon words as developed in [LR]. The observation that these combinatorics should have a representation theoretic interpretation is due to Leclerc [Le, Sections 6-7] (see [HKS, Section 8] for another application of these ideas). In particular, Kleshchev and Ram reduced the study of this representation theory to generalizations of the segment representations of Bernstein and Zelevinsky [BZ, Z], which they called *cuspidal representations*.

Cuspidal representations are in one-to-one correspondence with the set of positive roots, Δ^+, in the root system associated to $\mathfrak{g}$, and their definition depends on a choice of total ordering of the index set I labeling the set of simple roots. The corresponding lexicographic ordering on the set $W = \bigsqcup_d I^d$ of *words* in I induces a *convex* ordering on Δ^+. Corresponding to this convex ordering is a reduced expression for the longest word in the Weyl group, and a dual PBW and dual canonical basis for $\mathcal{U}_{\mathcal{A}}^*(\mathfrak{n})$ via an action of the braid group. These bases are naturally labeled by a set of *dominant* words $W_+ \subset W$.

For each $\beta \in \Delta^+$, the associated cuspidal representation corresponds under the isomorphism (1) to a dual canonical basis element labeled by a minimal element of weight β in W_+, [KR2, Lemma 6.4]. The dual PBW basis corresponds to a basis for $K(\Gamma)$ given by *standard* modules obtained from cuspidal representations by parabolic induction. The standard modules have unique irreducible quotients and these quotients are precisely the simple modules up to isomorphism and grading shift, [KR2, Theorem 7.2]. It is expected that these modules correspond under the isomorphism (1) to the dual canonical basis. This conjecture is known in the simply laced case due to the work of Varagnolo-Vasserot [VV]. A more general result has been announced by Rouquier which would imply the conjecture in all types, but the details are unavailable.

Kleshchev and Ram produced cuspidal representations for $H(\Gamma)$ in most cases relative to a fixed ordering on I, the exceptions being the nontrivial cases in type F_4 and 12 cases in type E_8. Additionally, they constructed cuspidal representations for all orderings in type A. Subsequently, Hill, Melvin and Mondragon [HMM] completed their own investigation of quiver Hecke algebras in finite type following an observation of Hill, Kujawa and Sussan [HKS]. In this paper they constructed cuspidal representations of $H(\Gamma)$ in all finite types, including F_4 and E_8, with respect to a different ordering on I. We believe that the cuspidal representations in [HMM] are as small as possible. For example, in classical type, the dimensions of these modules are almost always bounded by 2 (the exception being the long roots in type C). In contrast, the modules constructed in [KR2] typically have dimensions that grow with the height of the associated positive root. Additionally, in type E_8, all cuspidal representations in [HMM] are homogeneous in the sense of [KR1] and therefore can be constructed using the machinery developed there.

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Factorization of the canonical bases for higher level Fock spaces

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(joint work with Susumu Ariki and Cédric Lecouvey)

Let $\mathcal{H}_{\mathbb{C}}$ be an Ariki-Koike algebra (*i.e* an Hecke algebra of the complex reflection group $G(l,1,n)$) over $\mathbb{C}$ where the parameters are power of the same e-root of unity. In general, this algebra is split but non semisimple. The representations of this algebra are encoded in a matrix D_e which is called the decomposition matrix for $\mathcal{H}_{\mathbb{C}}$. By a result of M.Geck and R.Rouquier [2], we know that one can factorize this matrix by another decomposition matrix D_∞ which does not depend on e. As a consequence, we have $D_e = D_\infty.A$ where A is a matrix with non negative integers coefficients. When $l = 2$ that is when $\mathcal{H}_{\mathbb{C}}$ is an Iwahori-Hecke algebra of type B_n, one has closed formulae for the matrices D_∞. In this case, they correspond to the matrices of constructible representations as defined by Lusztig.

By Ariki's theorem (generalization of the LLT's conjecture), the decomposition matrices D_e and D_∞ have natural quantification: they are the specializations of the matrices of the canonical bases $D_e(v)$ and $D_\infty(v)$ for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules at $v = 1$. In this context, it is then natural to ask if there exists an analogue of the above factorization for these "v-decomposition matrices".

In a common work with S. Ariki and C. Lecouvey [1], we prove that the "adjustement matrix" A has also a natural quantization, namely that there exists a matrix $A(v)$ with entries in $\mathbb{N}[v]$ such that $D_e(v) = D_\infty(v).A(v)$. In fact, such a result is not only true for the canonical bases of irreducible highest weight modules but also true for the canonical bases of the Fock spaces as defined by Uglov [4]. To do this, we study the matrices of the involution on the Fock space and combine this with results by Grojnowski and Haiman [3] on the positivity of the structure constants in the affine Hecke algebra of type A.

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From Macdonald polynomials to a charge statistic in classical types

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There has been considerable interest recently in the combinatorics of Macdonald polynomials $P_\mu(x; q, t)$ [21]. This stems in part from the Haglund-Haiman-Loehr (HHL) combinatorial formula for the ones corresponding to type A [6], which is in terms of fillings of Young diagrams. This formula found important applications to the *two-parameter Kostka-Foulkes polynomials* [1, 5]. More recently, Ram and Yip [23] gave a formula (described below) for the Macdonald polynomials of arbitrary type based on calculations in the double affine Hecke algebra. This formula is in terms of so-called *alcove walks*, which originate in the work of Gaussent-Littelmann [4] and of myself and Postnikov [16, 17] on discrete counterparts to the celebrated *Littelmann path model* [18, 19].

Given a dominant weight μ, we associate with it a sequence of roots, called a *μ-chain*, namely $\Gamma = (\beta_1, \ldots, \beta_m)$, where $m = 2\langle \mu, \rho^\vee \rangle$. The Ram-Yip formula is in terms of pairs $(w, J) \in W \times 2^{[m]} =: \mathcal{F}(\Gamma)$, which we call *folding pairs*, where

W is the Weyl group. Let $r_i := s_{\beta_i}$ be the corresponding reflection. Given (w, J) with $J = \{j_1 < \ldots < j_s\}$, we identify it with the chain of Weyl group elements

$$w = w_0, \ \ldots, \ w_i := w r_{j_1} \ldots r_{j_i}, \ \ldots, \ w_s .$$

Theorem 1. [23] *We have*

$$P_\mu(X; q, t) = \sum_{(w,J)\in\mathcal{F}(\Gamma)} RY(w, J; q, t)\, x^{\mathrm{weight}(w,J)}\,, \tag{1}$$

where $RY(w, J; q, t)$ is a rational function in q, t, and weight(w, J) *is a weight.*

My goal is to derive from the above Ram-Yip formula simpler and more explicit formulas in classical types, in terms of fillings of the Young diagram μ. The main ingredient is the so-called *filling map* f from $\mathcal{F}(\Gamma)$ to fillings of μ, cf. [14]. One of my results is a derivation of the HHL formula from the type A instance of the Ram-Yip formula via a "compression" procedure, as explained below.

Theorem 2. [14] *Let F be any filling of μ with $1, \ldots, n$. We have*

$$\sum_{(w,J)\in f^{-1}(F)} RY(w, J; q, t)\, x^{\mathrm{wt}(w,J)} = HHL(F; q, t)\, x^{\mathrm{content}(F)}\,, \tag{2}$$

where the right-hand side is a term in the HHL formula.

In (2), $HHL(F; q, t)$ is a rational function defined in terms of statistics "inv" and "maj" on the filling F. The compression phenomenon described in Theorem 2 explains the way in which the statistics "inv" and "maj", originally discovered via computer experiments, follow naturally from more general concepts. I have done related work, which also refers to types B and C [12, 13, 15].

Let us now consider the case when the parameter $t = 0$. The Ram-Yip formula takes the simpler form

$$P_\mu(X; q, 0) = \sum_{(w,J)\in\mathcal{A}(\Gamma)} q^{\mathrm{level}(w,J)}\, x^{\mathrm{weight}(w,J)}\,, \tag{3}$$

where $\mathcal{A}(\Gamma)$ is the set of $(w, J) \in \mathcal{F}(\Gamma)$ corresponding to nonzero terms in (1).

Proposition 3. *Let $(w, J) = (w, \{j_1 < j_2 < \ldots < j_s\})$ in a classical type. We have $(w, J) \in \mathcal{A}(\Gamma)$ if and only if we have a path*

$$w = w_0 \stackrel{\beta_{j_1}}{\longleftarrow} w_1 \stackrel{\beta_{j_2}}{\longleftarrow} \ldots \stackrel{\beta_{j_s}}{\longleftarrow} w_s = Id$$

in the corresponding quantum Bruhat graph.

The quantum Bruhat graph, which first arose in connection with the quantum cohomology of G/B, is the directed graph on W with labeled edges $w \stackrel{\alpha}{\longrightarrow} ws_\alpha$ for each α such that $\ell(ws_\alpha) = \ell(w) + 1$ or $\ell(ws_\alpha) = \ell(w) - 2\langle \rho^\vee, \alpha\rangle + 1$.

The following theorem gives a more explicit description of $\mathcal{A}(\Gamma)$ in types A and C in terms of the corresponding *Kirillov-Reshetikhin (KR) crystals* $B^{k,1}$ indexed by columns of height k. It is conjectured that the same result holds in types B and D. There are complications in these cases because the corresponding KR crystals, viewed as classical crystals, split as direct sums of fundamental crystals $B^{k,1} = B(\omega_k) \oplus B(\omega_{k-2}) \oplus \ldots$, whereas in type C we have $B^{k,1} = B(\omega_k)$.

Theorem 4. *Let $B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1}$, where μ' is the conjugate partition to μ. In types A and C there are bijections (f being the filling map above)*

$$\mathcal{A}(\Gamma) \xrightarrow{f} f(\mathcal{A}(\Gamma)) \xrightarrow{r} B_\mu \,. \tag{4}$$

The crystal B_μ is realized in terms of *Kashiwara-Nakashima columns* [9]. The map r in (4) is simply sorting the columns. The construction of the map r^{-1} and implicitly the combinatorics of the quantum Bruhat order lead to the definition in types A and C of a *charge* statistic "ch" on fillings F in B_μ such that $\mathrm{ch}(F) = \mathrm{level}(w, J)$ for $F = r(f(w, J))$, cf. (3). From (3) we deduce

$$P_\mu(X; q, 0) = \sum_{F \in B_\mu} q^{\mathrm{ch}(F)}\, x^{\mathrm{content}(F)}\,.$$

The above charge coincides with the one due to Lascoux-Schützenberger in type A [10]. Generalizing the latter to all classical types has been a long-standing problem, for which we described an approach using Macdonald polynomials.

We now mention the relationship of charge with the *one-dimensional (1-d) sums* $X_{\lambda\mu}(q)$ in the theory of solvable lattice models. These can be defined as graded tensor product multiplicities for B_μ, where the grading on B_μ is given by the *energy function* D_{B_μ} [7]. Nakayashiki and Yamada proved that the Lascoux-Schützenberger charge coincides with the energy function in type A [22]. A joint project with A. Schilling consists of showing that a similar result holds in type C (based on the newly defined charge), namely that $D_{B_\mu}(F) = \mathrm{ch}(F)$ for all F in B_μ. This conjecture, which is believed to generalize to all classical types, is supported in type D by results in [2, 3, 8] and by recent work of Schilling. These results are based on the interpretation of $P_\mu(x; q, 0)$ as an affine Demazure character.

The 1-d sums are related to *Lusztig's q-analogue of weight multiplicities* (or Kostka-Foulkes polynomials) [20] in all classical types, by [11]. Therefore, I expect that the newly defined charge can be used to express Lusztig's q-analogue as well.

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Kazhdan–Lusztig polynomials, tight quotients and Dyck superpartitions

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(joint work with Francesco Brenti, Federico Incitti)

In 1979, Kazhdan and Lusztig [5] introduced a combinatorial way to construct representations of the Hecke algebra of an arbitrary Coxeter group W. The main ingredient is a family $\{P_{u,v}(q)\}_{u,v\in W}$ of polynomials with integer coefficients, indexed by pairs of elements in W, which are now known as the Kazhdan–Lusztig polynomials of W and play an important role in various areas of mathematics (see, e.g., [1], [4] and the references cited there). In 1987, Deodhar [3] developed an analogous theory for the parabolic setup. Given any parabolic subgroup W_J of W, Deodhar introduced two Hecke algebra modules (one for each of the two roots

q and -1 of the polynomial $x^2 - (q-1)x - q$) and two families of polynomials $\{P^{J,q}_{u,v}(q)\}_{u,v \in W^J}$ and $\{P^{J,-1}_{u,v}(q)\}_{u,v \in W^J}$ indexed by pairs of elements in the set W^J of minimal coset representatives. These polynomials are the parabolic analogues of the Kazhdan–Lusztig polynomials: while they are related to their ordinary counterparts in several ways, they also play a direct role in several areas such as the geometry of partial flag manifolds, the theory of Macdonald polynomials, tilting modules, generalized Verma modules, canonical bases, the representation theory of the Lie algebra $\mathfrak{gl}_n$, and quantized Schur algebras. The ordinary Kazhdan–Lusztig polynomials are obtained as a special case of the parabolic Kazhdan–Lusztig polynomials for $J = \emptyset$.

The purpose of this work is to study the parabolic Kazhdan–Lusztig polynomials for the tight quotients of the symmetric group S_n. The tight quotients have been introduced by Stembridge in [7] who classified them for finite Coxeter groups [7, Theorem 3.8]. For the symmetric group, the non-trivial tight quotients are obtained by taking either $J = [n-1] \setminus \{i\}$, $i \in [n-1]$ (maximal quotients), or $J = [n-1] \setminus \{i-1, i\}$, $i \in [2, n-1]$. The parabolic Kazhdan–Lusztig polynomials for the maximal quotients have been studied in [2]. The parabolic Kazhdan–Lusztig polynomials of $S_n^{[n-1]\setminus\{i-1,i\}}$ of type -1 can be computed using the argument in [6], since they are equal to ordinary Kazhdan–Lusztig polynomials indexed by vexillary permutations. We complete the study for all tight quotients of the symmetric group giving an explicit closed combinatorial formula for the parabolic Kazhdan–Lusztig polynomials of $S_n^{[n-1]\setminus\{i-1,i\}}$ of type q. The formula, which implies that these polynomials are always either zero or a monic monomial, can be used to give another proof of the formula found in [2] for the maximal quotients and to compute the function μ for the tight quotients (μ gives the labels of the graphs from which Kazhdan–Lusztig representations can be constructed), and involves a new class of superpartitions, which we call Dyck.

Although superpartitions can be traced back to MacMahon diagrams, it is especially in recent years that they attracted much attention, since they have been shown to arise in several contexts. Superpartitions (or strictly related concepts) have been extensively studied, sometimes under different names such as dotted partitions, joint partitions, colored partitions, jagged partitions, and overpartitions. This work provides a Lie theoretic application of the concept of superpartition.

The proof of the formula is obtained by describing the combinatorics of the tight quotients $S_n^{[n-1]\setminus\{i-1,i\}}$ in terms of 1-superpartitions (superpartitions with fermionic degree equal to 1). In fact, there is a bijection between $S_n^{[n-1]\setminus\{i-1,i\}}$ and the set of 1-superpartitions contained in $((n-i+1)^i)$. Superpartitions not only encode campactly the elements of the quotients but also their algebraic properties and their poset structure. For instance, we can describe the length function, the descent sets, the effect of multiplications by generators, the covering relations and the Bruhat order in terms of superpartitions. In particular, for every pair of minimal coset representatives u, v with $u \leq v$ in the Bruhat order, the superpartition associated with u is contained in the superpartition associate with v. We show that the parabolic Kazhdan–Lusztig polynomial indexed by u and v is encoded in the two superpartitions associated with u and v. More precisely, the polynomial $P^{J,q}_{u,v}(q)$ is non-zero if and only if the two superpartitions form a Dyck skew superpartition and, in this case, it is a power of q whose exponent is an explicit statistic of the Dyck skew superpartition. The formula is obtained by showing that Dyck superpartitions have properties that mirror the complicated recursion satisfied by the parabolic Kazhdan–Lusztig polynomials (see [3]).

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One-skeleton galleries, Hall-Littlewood polynomials and the path model

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(joint work with Peter Littelmann)

The aim of our work is twofold: we want to give a direct geometric interpretation of the path model for representations and the associated Weyl group combinatorics, and we want to get a geometric compression for Schwer's formula for Hall-Littlewood polynomials.

Concerning the connection with the path model, a first step in this direction was done in our first paper using galleries of alcoves. The advantage of the new approach is that galleries in the one-skeleton of the apartment can directly be identified with piecewise linear paths running along the one-skeleton, and they can be concatenated. The goal now is to show that the original approach by Lakshmibai, Musili and Seshadri towards what later became the path model has an intrinsic geometric interpretation in the geometry of the affine Grassmannian, respectively in the geometry of the associated affine building.

To give a more precise description of both aims, let G be a semisimple algebraic group defined over $\mathbb{C}$, fix a Borel subgroup B and a maximal torus T. Let U^- be the unipotent radical of the opposite Borel subgroup. Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of complex formal power series and let $\mathcal{K} = \mathbb{C}((t))$ be the quotient field.

For a dominant coweight λ and an arbitrary coweight μ consider the following intersection in the affine Grassmannian $G(\mathcal{K})/G(\mathcal{O})$:

$$Z_{\lambda,\mu} = G(\mathcal{O}).\lambda \cap U^-(\mathcal{K}).\mu.$$

Let $\mathbb{F}_q$ be the finite field with q elements and replace the field of complex numbers by the algebraic closure K of $\mathbb{F}_q$. Assume that all groups are defined and split over $\mathbb{F}_q$. Replace $\mathcal{K}$ by $\mathcal{K}_q = \mathbb{F}_q((t))$ and $\mathcal{O}$ by $\mathcal{O}_q = \mathbb{F}_q[[t]]$; the Laurent polynomials $L_{\lambda,\mu}$ defined by $L_{\lambda,\mu}(q) = |Z^q_{\lambda,\mu}|$ show up as coefficients in the Hall-Littlewood polynomial: $P_\lambda = \sum_{\mu \in X^\vee_+} q^{-\langle \rho, \lambda+\mu\rangle} L_{\lambda,\mu} m_\mu$.

Based on the description of $Z_{\lambda,\mu}$ we obtained, Schwer gives a decomposition $Z^q_{\lambda,\mu} = \bigcup S_\delta$, where the δ are certain galleries of alcoves in the standard apartment of the associated affine building. The structure of the S_δ is quite simple and hence $|S_\delta|$ is easy to compute, but the decomposition has the disadvantage that the sum $|Z^q_{\lambda,\mu}| = \sum |S_\delta|$ has many terms.

There are other formulas, for example, in type $\mathtt{A}$, one can specialize the Haglund-Haiman-Loehr formula for Macdonald polynomials. By analyzing the combinatorics involved in the formulas, Lenart has shown that certain terms in Schwer's formula can be naturally grouped together such that the resulting formula coincides with the specialisation of the Haglung-Haiman-Loehr formula, he calls this the compression phenomenon.

Our approach to "compression" is geometric and independent of the type of the group. We replace the desingularisation of the Schubert variety X_λ we used before by a Bott-Samelson type variety Σ which is a fibered space having as factors varieties of the form H/Q, where H is a semisimple algebraic group and Q is a maximal parabolic subgroup. In terms of the affine building, a point in this variety is a sequence of parahoric subgroups of $G(\mathcal{K})$ reciprocative contained in each other, or, more precisely, in terms of faces, it is a sequence of closed one-dimensional faces, where successive faces have (at least) a common zero-dimensional face (i.e. a vertex).

The Białynicki-Birula decomposition of Σ can be used to define a decomposition of $Z_{\lambda,\mu}$, the indexing set of the strata are positively folded one-skeleton galleries. For G of type $\mathtt{A}_n$ for example, the galleries can be translated into the language of Young tableaux and the strata of $Z_{\lambda,\mu}$ can be indexed by the semi-standard Young tableaux of shape λ and weight μ. Keeping in mind that these diagrams are in bijection with the components of maximal dimension of $Z_{\lambda,\mu}$, this can be viewed as the optimal geometric decomposition. The reason for this compression is that in Schwer's picture there are the LS-galleries (see below), which determine the power of the leading term in $|Z^q_{\lambda,\mu}|$, and there are many positively folded galleries which are not LS-galleries. In our new approach the variety Σ has less Białynicki-Birula cells, hence there are less combinatorial galleries of fixed type and there are also less positively folded galleries. In the case of type $\mathtt{A}_n$ considered above, it turns out that in fact all positively folded galleries are LS-galleries.

We introduce the notion of a minimal one-skeleton gallery and of a positively folded combinatorial gallery in the one-skeleton. Geometrically the two notions have the following meaning: the points in Σ corresponding to the points in the open orbit $G(\mathcal{O}).\lambda \subset X_\lambda$ are exactly the minimal galleries. Since Σ is smooth, by choosing a generic one parameter subgroup of T in the anti-dominant Weyl chamber, we get a Białynicki-Birula decomposition, the centers δ of the cells C_δ correspond exactly to the combinatorial one-skeleton galleries δ, i.e. the galleries lying in the standard apartment of the building.

To decide whether a cell C_δ contains minimal galleries, we need to unfold the folded gallery. We prove that the cell C_δ contains minimal galleries if and only if δ is positively folded. We get the following formula for the coefficients of the Hall-Littlewood polynomials, the summands below counting the number of points in the intersection of $Z^q_{\lambda,\mu} \cap C_\delta$ for δ being positively folded:

Theorem.
$L_{\lambda,\mu}(q) = \sum_{\delta\in\Gamma^+(\gamma_\lambda,\mu)} q^{\ell(w_{D_0})} \left(\prod_{j=1}^r \sum_{\mathbf{c}\in\Gamma^+_{{}^{s}V_j^j}(\mathbf{i}_j,op)} q^{t(\mathbf{c})}(q-1)^{r(\mathbf{c})} \right).$

The positively folded one-skeleton galleries having $q^{\langle\lambda+\mu,\rho\rangle}$ as a leading term in the counting formula for $|Z^q_{\lambda,\mu}\cap C_\delta|$, are called *LS-galleries*; this is an abbreviation for Lakshmibai-Seshadri galleries.

An important notion introduced in the theory of standard monomials is the defining chain, which was a breakthrough on the way for the definition of standard monomials and generalized Young tableaux. In the context of the crystal structure of the path theory this notion again turned up to be an important combinatorial tool to check whether a concatenation of paths is in the Cartan component or not. Still, the definition had the air of an ad hoc combinatorial tool. But in the context of Białynicki-Birula cells, the folding of a minimal gallery by the action of the torus occurs naturally: during the limit process (going to the center of the cell) the direction (= the sector) attached to a minimal gallery is transformed into the weakly decreasing sequence of Weyl group elements, the defining chain for the positively folded one-skeleton gallery in the center of the cell.

The connection between the path model theory and the one-skeleton galleries is summarized in the following corollary. For a fundamental coweight ω let π_{ω_i} : $[0,1] \to X^\vee_\mathbb{R}$, $t \mapsto t\omega$ be the path which is just the straight line joining $\mathfrak{o}$ with ω and let γ_ω be the one-skeleton gallery obtained as the sequence of edges and vertices lying on the path.

Corollary. *Write a dominant coweight $\lambda = \omega_{i_1} + \ldots + \omega_{i_r}$ as a sum of fundamental coweights, write $\underline{\lambda}$ for this ordered decomposition. Let $\mathcal{P}_{\underline{\lambda}}$ be the associated path model of LS-paths of shape $\underline{\lambda}$ having as starting path the concatenation $\pi_{\omega_{i_1}} * \ldots * \pi_{\omega_{i_r}}$. For a path π in the path model denote by γ_π the associated gallery in the one-skeleton of $\mathbb{A}$ obtained as the sequence of edges and vertices lying on the path. The one-skeleton galleries γ_π obtained in this way are precisely the LS-galleries of the same type as $\gamma_{\omega_{i_1}} * \ldots * \gamma_{\omega_{i_r}}$.*

In fact, the notion of a *defining chain for LS-paths* coincides in this case with the notion of a defining chain for the associated gallery.

Since the number of the LS-galleries is the coefficient of the leading term of $L_{\lambda,\mu}$, and since $P_\lambda \to s_\lambda$ for $q \to \infty$, we get as an immediate consequence of the previous Theorem the following character formula. In combination with the above Corollary, this provides a geometric proof of the path character formula, first conjectured by Lakshmibai and proved by Littelmann:

Corollary. Char $V(\lambda) = \sum_\delta e^{target(\delta)}$, *where the sum runs over all LS-galleries of the same type as* γ_λ*.*

The crystal graph structure on simple modules of KLR algebras

Monica Vazirani

(joint work with Aaron D. Lauda)

Khovanov-Lauda-Rouquier (KLR) algebras $R = \bigoplus_{\nu \in Q_+} R(\nu)$ were invented to categorify quantum groups of arbitary type. See [KL09, KL08a, Rou08]. Khovanov-Lauda in fact showed this algebra categorifies the integral form ${}_{\mathcal{A}}\mathbf{U}_q^- := {}_{\mathcal{A}}\mathbf{U}_q^-(\mathfrak{g})$ of the negative half of the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ associated to a symmetrizable Kac-Moody algebra $\mathfrak{g}$. In joint work with Aaron Lauda [LV], we show that the simple R-modules carry the structure of the crystal graph $B(\infty)$, which is the crystal of $\mathbf{U}_q^-$.

Set $\mathcal{B}$ to be the I-colored directed graph with nodes indexed by isomorphism classes of simple R-modules, up to grading shift, and edges corresponding to socle of i-restriction. Let $\mathcal{B}^\Lambda$ be the subgraph whose nodes are simple modules for a given cyclotomic quotient R^Λ depending on $\Lambda \in P^+$.

Theorem 1. *(1)* $\mathcal{B}$ *is a crystal graph and is isomorphic to* $B(\infty)$*.*
(2) $\mathcal{B}^\Lambda$ *is a crystal graph and is isomorphic to* $B(\Lambda)$*.*

Another consequence of this theorem is that it computes the rank of the Grothendieck group $K(R^\Lambda(\nu)\text{−mod})$.

1. Generators and Relations

For $\nu = \sum_{i\in I} \nu_i \cdot \alpha_i \in Q_+ \simeq \mathbb{N}[I]$ let Seq(ν) be the set of all sequences of vertices $\boldsymbol{i} = i_1 \dots i_m$ where $i_r \in I$ for each r and vertex i appears ν_i times in the sequence. The length m of the sequence is equal to $|\nu| = \sum_{i\in I} \nu_i$. It is sometimes convenient to identify $\nu = \sum_{i\in I} \nu_i \cdot i \in \mathbb{N}[I]$. We also sometimes write $\langle i, j\rangle = \langle h_i, \alpha_j\rangle$ to save space.

For $\nu \in Q_+$ with $|\nu| = m$, let $R(\nu)$ denote the associative, $\mathbb{K}$-algebra on generators

$$
\begin{aligned}
&1_{\boldsymbol{i}} &&\text{for } \boldsymbol{i} \in \mathrm{Seq}(\nu)\\
&x_r &&\text{for } 1 \le r \le m\\
&\psi_r &&\text{for } 1 \le r \le m-1
\end{aligned}
$$

subject to the following relations for $\boldsymbol{i}$, $\boldsymbol{j} \in \mathrm{Seq}(\nu)$:

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}},$$
$$x_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_r,$$
$$\psi_r 1_{\boldsymbol{i}} = 1_{s_r(\boldsymbol{i})} \psi_r,$$
$$x_r x_t = x_t x_r,$$
$$\psi_r \psi_t = \psi_t \psi_r \qquad \text{if } |r-t| > 1,$$
$$\psi_r \psi_r 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ 1_{\boldsymbol{i}} & \text{if } (\alpha_{i_r}, \alpha_{i_{r+1}}) = 0 \\ \left(x_r^{-\langle i_r, i_{r+1} \rangle} + x_{r+1}^{-\langle i_{r+1}, i_r \rangle} \right) 1_{\boldsymbol{i}} & \text{if } (\alpha_{i_r}, \alpha_{i_{r+1}}) \neq 0 \text{ and } i_r \neq i_{r+1}, \end{cases}$$
$$(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) 1_{\boldsymbol{i}} =$$
$$= \begin{cases} \sum_{t=0}^{-\langle i_r, i_{r+1} \rangle - 1} x_r^t x_{r+2}^{-\langle i_r, i_{r+1} \rangle - 1 - t} 1_{\boldsymbol{i}} & \text{if } i_r = i_{r+2} \text{ and } (\alpha_{i_r}, \alpha_{i_{r+1}}) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$
$$\left(\psi_r x_t - x_{s_r(t)} \psi_r \right) 1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } t = r \text{ and } i_r = i_{r+1} \\ -1_{\boldsymbol{i}} & \text{if } t = r+1 \text{ and } i_r = i_{r+1} \\ 0 & \text{otherwise.} \end{cases}$$

We note that in the literature the idempotent $1_{\boldsymbol{i}}$ is often denoted $e(\boldsymbol{i})$.

Define the character $\mathrm{ch}(M)$ of a graded finitely-generated $R(\nu)$-module M as

$$\mathrm{ch}(M) = \sum_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} \mathrm{gdim}(1_{\boldsymbol{i}} M) \cdot \boldsymbol{i}.$$

When M is finite dimensional, its character $\mathrm{ch}(M)$ is an element of the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\mathrm{Seq}(\nu)$. We remark that all simple R-modules are finite dimensional.

In this talk, we take the point of view that in retrospect, given that we want Theorem 1 to hold, one could reconstruct the above relations. Following work of [Gro99, Ari96] we expect the representation theory of $R(k\alpha_i)$ to correspond to that of the block of the affine Hecke algebra H_k on which all polynomial generators act as q^i, which has a unique simple module of dimension $k!$. It is no surprise we have the above generators and local relations in the case $i_r = i_{r+1} = i$.

$R(k\alpha_i)$ has a unique (up to grading shift) simple module denoted $L(i^k)$ with graded character $[k]_i! i^k$.

Let M be a simple $R(\nu)$-module and $i \in I$. We set $\tilde{f}_i M := \mathrm{cosoc}\ \mathrm{Ind}_{\nu,i}^{\nu+i} M \boxtimes L(i)$ and $\tilde{e}_i M := \mathrm{soc}\ e_i M$ where $e_i M := \mathrm{Res}_{\nu-i}^{\nu-i,i} \circ \mathrm{Res}_{\nu-i,i}^{\nu} M$. Likewise we can define $\tilde{e_i}^*$ where $e_i^* := \mathrm{Res}_{\nu-i}^{i,\nu-i} \circ \mathrm{Res}_{i,\nu-i}^{\nu} M$. In [KL09] it is shown $\tilde{e_i} M$ and $\tilde{e_i}^* M$ are simple or zero. Set

$$\epsilon_i(M) := \max\{n \geq 0 \mid \tilde{e_i}^n M \neq \bar{0}\} \quad \text{and} \quad \epsilon_i^*(M) := \max\{n \geq 0 \mid \tilde{e_i}^{*n} M \neq \bar{0}\}.$$

We note that we can read ϵ_i (resp. ϵ_i^*) off the character of M as the maximal k such that $i_m = i_{m-1} = \cdots = i_{m-k+1}$ (resp. $i_1 = i_2 = \cdots = i_k$) as we range over all $\boldsymbol{i} \in \mathrm{Seq}$ with $1_{\boldsymbol{i}} M \neq \bar{0}$.

The next task is to understand relations for which $i_r = i \neq j = i_{r+1}$. To do this, we examine simple modules of $R(c\alpha_i + \alpha_j)$.

Theorem 2. *(1) Let $c \leq a$ and let $\nu = c\alpha_i + \alpha_j$. Up to isomorphism and grading shift, there exists a unique simple $R(\nu)$-module denoted $\mathcal{L}(i^{c-n} j i^n)$ with $\epsilon_i(\mathcal{L}(i^{c-n} j i^n)) = n$ for each n with $0 \leq n \leq c$. Furthermore, $\epsilon_i^*(\mathcal{L}(i^{c-n} j i^n)) = c - n$ and*

$$\mathrm{ch}(\mathcal{L}(i^{c-n} j i^n)) = [c-n]_i! [n]_i! i^{c-n} j i^n.$$

(2) Let $c > a$. Let N be a simple $R(ci + j)$-module with $\epsilon_i(N) = n$. Then $c - a \leq n \leq c$ and up to grading shift

$$N \cong \mathrm{Ind}\, \mathcal{L}(n - (c-a)) \boxtimes L(i^{c-a}).$$

2. Crystal graphs

We recall some facts and definitions from the tensor category of crystals following Kashiwara [Kas95, KS97].

A *crystal* is a set B together with maps $\mathrm{wt}\colon B \longrightarrow P$, $\epsilon_i, \phi_i\colon B \longrightarrow \mathbb{Z} \sqcup \{\infty\}$ for $i \in I$, $\tilde{e_i}, \tilde{f_i}\colon B \longrightarrow B \sqcup \{0\}$ for $i \in I$, satisfying certain properties.

Definition 3 (B_i $(i \in I)$). *$B_i = \{b_i[n]; n \in \mathbb{Z}\}$ with $\mathrm{wt}(b_i[n]) = n\alpha_i$,*

$$(1) \qquad \epsilon_j(b_i[n]) = \begin{cases} -n & \text{if } i = j \\ -\infty & \text{if } j \neq i, \end{cases} \qquad \phi_j(b_i[n]) = \begin{cases} n & \text{if } i = j \\ -\infty & \text{if } j \neq i, \end{cases}$$

$$(2) \qquad \tilde{e_j} b_i[n] = \begin{cases} b_i[n+1] & \text{if } i = j \\ 0 & \text{if } j \neq i, \end{cases} \qquad \tilde{f_j} b_i[n] = \begin{cases} b_i[n-1] & \text{if } i = j \\ 0 & \text{if } j \neq i. \end{cases}$$

Proposition 4 ([KS97] Proposition 3.2.3)**.** *Let B be a crystal and b_0 an element of B with weight zero. Assume the following conditions.*

(B1) $\mathrm{wt}(B) \subset Q_-$.
(B2) *b_0 is the unique element of B with weight zero.*
(B3) *$\epsilon_i(b_0) = 0$ for every $i \in I$.*
(B4) *$\epsilon_i(b) \in \mathbb{Z}$ for any $b \in B$ and $i \in I$.*
(B5) *For every $i \in I$, there exists a strict embedding $\Psi_i\colon B \to B \otimes B_i$.*
(B6) $\Psi_i(B) \subset B \times \{\tilde{f_i}^n b_i[0]; n \geq 0\}$.
(B7) *For any $b \in B$ such that $b \neq b_0$, there exists i such that $\Psi_i(b) = b' \otimes \tilde{f_i}^n b_i[0] = b' \otimes b_i[-n]$ with $n > 0$.*

Then B is isomorphic to $B(\infty)$.

The module-theoretic strict embedding is given by

$$\begin{aligned} \Psi_i\colon \mathcal{B} \quad &\to \quad \mathcal{B} \otimes B_i \\ M \quad &\mapsto \quad (\tilde{e_i}^*)^c(M) \otimes b_i[-c], \end{aligned}$$

where $c = \epsilon_i^*(M)$.

Let $a = -\langle h_i, \alpha_j \rangle$. Let $n \le c \le a$. Using the strict embedding, one can show that the graph $B(\infty)$ has nodes of the form

$$\mathbf{b_0} \xrightarrow{i} \circ \xrightarrow{i} \cdots \xrightarrow{i} \mathbf{p} \xrightarrow{j} \mathbf{q} \xrightarrow{i} \circ \xrightarrow{i} \cdots \xrightarrow{i} \mathbf{r}$$

where $\epsilon_i^*(\mathbf{p}) = \epsilon_i^*(\mathbf{q}) = \epsilon_i^*(\mathbf{r}) = c - n$ but $\epsilon_i(\mathbf{p}) = c - n, \epsilon_i(\mathbf{q}) = 0, \epsilon_i(\mathbf{r}) = n$.

This tells us for $c \le a$ that $R(c\alpha_i + \alpha_j)$ has simple modules $\mathcal{L}(i^{c-n}ji^n)$ with ϵ_i, ϵ_i^* data the same as that of node $\mathbf{r}$. Taking into account we understand the unique (up to grading shift) simple module $L(i^k)$ of $R(k\alpha_i)$, we see $\mathrm{ch}(\mathcal{L}(i^{c-n}ji^n)) = [c-n]![n]!i^{c-n}ji^n$.

However, when $c > a$, the ϵ_i^* data changes differently. In particular, when $c = a$, whereas $\epsilon_i(\mathbf{r}) = n$, $\epsilon_i^*(\mathbf{r}) = a - n$, we have $\epsilon_i(\tilde{f}_i\mathbf{r}) = n + 1$, $\epsilon_i^*(\tilde{f}_i\mathbf{r}) = a - n + 1 = \epsilon_i^*(\mathbf{r}) + 1$. Module-theoretically, this means that up to grading shift $\mathrm{Ind}\, \mathcal{L}(i^{a-n}ji^n) \boxtimes L(i) \simeq \mathrm{Ind}\, L(i) \boxtimes \mathcal{L}(i^{a-n}ji^n)$ is simple, with character $[a-n]![n]!(i^{a-n}ji^n \sqcup\!\sqcup\, i)$, where $\sqcup\!\sqcup$ is the quantum shuffle.

We obviously have a surjection

$$\mathrm{Ind}\, L(i^{c-n}) \boxtimes L(j) \boxtimes L(i^n) \to \mathcal{L}(i^{c-n}ji^n).$$

Comparing characters, when $c \le a$ the induced module must have maximal submodule corresponding to all $\psi_{\hat{w}}$ with $\ell(w) \ne 0$, $w \in S_{c+1}/S_{c-n} \times S_1 \times S_n$ and $\hat{w}$ a fixed reduced expression for w. However, when $c > a$, this space fails to be a proper submodule. This, combined with the fact $x_r^{n-1}L(i^n) \ne \bar{0}$ but $x_r^n L(i^n) = \bar{0}$, in part explains the braidlike relation in the case $i_r = i, i_{r+1} = j, i_{r+2} = i$. In other words, in our situation it yields $\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1}$ acts as 0 when $c \le a$, but equals a length 0 term that acts nonzero when $c > a$. It similarly yields $\psi_r\psi_r$ acts as 0 when $c \le a$, but equals a length 0 term that acts nonzero when $c > a$. This corresponds to the exponents on x_r, x_{r+1}, x_{r+2} in the right-hand sides of the corresponding cubic and quadratic relations.

In retrospect, arguments like these help to explain the relations given by KLR. However, in actuality, [LV] uses the relations extensively to prove Theorem 2 holds, and then uses it, following the ideas of [Gro99], to prove Theorem 1.

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Generalized trace and modified dimension functions on ribbon categories

Jonathan Kujawa
(joint work with Nathan Geer, Bertrand Patureau-Mirand)

1.1. In many categories which arise in representation theory the dimension of an object and trace of a morphism are essential tools. In particular, it is often of interest to know when these are zero. For example, the vanishing of dimensions arises in the Kac-Weisfeiler conjecture for Lie algebras (proved by Premet in [9]), the DeConcini, Kac and Procesi conjecture for quantum groups at a root of unity, 2 and p divisibility for representations of Lie superalgebras [3, 11], and well known p divisibility results for modular representations of finite groups. The vanishing of the trace function allows one to define the *radical* of the category. The resulting quotient category plays an important role in representation theory. Andersen constructed a three dimensional quantum field theory from the category of tilting modules for a quantum group at a root of unity via this technique [1]. In recent work Deligne [4] and Knop [8] used this approach to show how to construct categories which interpolate among the representation categories of the symmetric groups and $GL(n, \mathbb{F}_q)$, respectively.

These functions also are intrinsic to the definition of knot invariants [10]. However, in the case when these functions vanish one obtains trivial invariants. Tackling this problem, Geer and Patureau-Mirand introduced a modified dimension for representations of quantum supergroups [5], and with Turaev a generalization of this construction to include, for example, the quantum group for $\mathfrak{sl}(2)$ at a root of unity [6].

In joint work with Geer and Patureau-Mirand, we generalize their construction to the setting of ribbon categories. Our approach generalizes well known results from representation theory as well as providing new insights and conjectures.

1.2. The setting of our results is within a ribbon category $\mathcal{C}$. That is, roughly speaking, within a category $\mathcal{C}$ with a tensor product bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbb{1}$, and a braiding; that is, for all V and W in $\mathcal{C}$ we have canonical isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$. Furthermore, $\mathcal{C}$ admits a duality functor $V \mapsto V^*$, and morphisms

$$b_V : \mathbb{1} \to V \otimes V^* \qquad d_V : V^* \otimes V \to \mathbb{1},$$
$$b'_V : \mathbb{1} \to V^* \otimes V \qquad d'_V : V \otimes V^* \to \mathbb{1}.$$

Many categories which naturally arise in representation theory are ribbon categories. For example, finite dimensional representations of groups, Lie algebras, and superalgebras over a field of arbitrary characteristic, and finite dimensional representations of quantum groups. The also arise in topology, algebraic geometry, physics, and quantum computing.

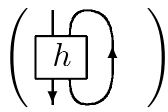
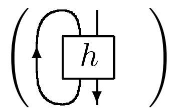
1.3. Assume $K := \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a field, and that J in $\mathcal{C}$ is a simple object which admits a linear map

$$\mathsf{t}_J : \mathrm{End}_{\mathcal{C}}(J) \to K$$

which satisfies

$$\mathsf{t}_J\left((d_J \otimes \mathrm{Id}_J) \circ (\mathrm{Id}_{J^*} \otimes h) \circ (b'_J \otimes \mathrm{Id}_J)\right) = \mathsf{t}_J\left((\mathrm{Id}_J \otimes d'_J) \circ (h \otimes \mathrm{Id}_{J^*}) \circ (\mathrm{Id}_J \otimes b_J)\right),$$

for all $h \in \mathrm{End}_{\mathcal{C}}(J \otimes J)$. That is, in the graphical calculus of ribbon categories we have

t_J  $= \mathsf{t}_J$ 

for all $h \in \mathrm{End}_{\mathcal{C}}(J \otimes J)$. Such a linear map is called an *ambidextrous trace on J* and J is called *ambidextrous.*

Given J in $\mathcal{C}$, let $\mathcal{I}_J$ be the full subcategory of all objects of $\mathcal{C}$, W, such that there is an object X in $\mathcal{C}$ and morphisms $\alpha : W \to J \otimes X$ and $\beta : J \otimes X \to W$ such that $\beta \circ \alpha = \mathrm{Id}_W$ (that is, roughly speaking, W is a direct summand of $J \otimes X$ for some object X). Then, for example, we have the following results.

Theorem 1. *If J in $\mathcal{C}$ admits an ambidextrous trace, then there is a unique family of ambidextrous trace functions $\{\mathsf{t}_V\}_{V \in \mathcal{I}_J}$ on $\mathcal{I}_J$ determined by that ambidextrous trace.*

We can then define a generalized dimension function on $\mathcal{I}_J$ via

$$\mathsf{d}_J(V) = \mathsf{t}_V(\mathrm{Id}_V),$$

for every V in $\mathcal{I}_J$. When $J = \mathbb{1}$ we recover the ordinary trace and dimension functions.

Theorem 2. *Let $\mathcal{C}$ be an abelian category, J be ambidextrous and let V be an object in $\mathcal{I}_J$ with $\mathrm{End}_{\mathcal{C}}(V) = K$ and $d_V : V^* \otimes V \to \mathbb{1}$ is an epimorphism. We then have:*

(1) Let $U \in \mathcal{I}_V \subseteq \mathcal{I}_J$. If $\mathsf{d}_J(V) = 0$, then $\mathsf{d}_J(U) = 0$.
(2) The canonical epimorphism $d_V \otimes \mathrm{Id}_J : V^* \otimes V \otimes J \to J \to 0$ splits if and only if $\mathsf{d}_J(V) \neq 0$.
(3) If J is not projective in $\mathcal{C}$ and P is projective in $\mathcal{C}$, then P is an object of $\mathcal{I}_J$ and $\mathsf{d}_J(P) = 0$.

The above results may become more recognizable when we specialize to the particular case of finite dimensional representations of a finite group over an algebraically closed field of characteristic p and take J to be the trivial module. In this setting the first statement of the theorem becomes: if p divides the dimension of

V, then p divides the dimension of any direct summand of $V \otimes X$ for any module X. The second statement becomes: the trivial module is a direct summand of $V^* \otimes V$ if and only if p does not divide the dimension of V. In this particular context these results were proven by Benson and Carlson [2]. The third statement becomes the well known result that for a finite group whose order is divisible by p, the projective modules over a field of characteristic p all have dimension divisible by p. If instead we specialize Theorem 2(1) to when $\mathcal{C}$ is the finite dimensional representations of a quantum group at a root of unity and again J is the trivial module, we recover a result of Andersen [1]. Thus these results fit within a common categorical framework.

1.4. To give another example, in the setting of basic classical Lie superalgebras we have combinatorially defined integers called the defect of $\mathfrak{g}$ and the atypicality of a simple supermodule of $\mathfrak{g}$. Let us write $\operatorname{def}(\mathfrak{g})$ for the defect of $\mathfrak{g}$ and $\operatorname{atyp}(L)$ for the atypicality of a simple supermodule L. In general, the atypicality of a simple supermodule is among $0, 1, 2, \ldots, \operatorname{def}(\mathfrak{g})$ and $\operatorname{def}(\mathfrak{g}) = \operatorname{atyp}(\mathbb{C})$, where $\mathbb{C}$ is the trivial supermodule. Also, recall that if $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a supermodule, then the superdimension is given by: $\operatorname{sdim}(L) = \dim_{\mathbb{C}}(L_{\bar{0}}) - \dim_{\mathbb{C}}(L_{\bar{1}})$. Kac and Wakimoto stated the following intriguing conjecture [7].

Conjecture 3. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra and let L be a simple $\mathfrak{g}$-supermodule. Then*

$$\operatorname{atyp}(L) = \operatorname{def}(\mathfrak{g}) \text{ if and only if } \operatorname{sdim}(L) \neq 0.$$

Partial results are known and recently Serganova has announced a proof for the classical contragradiant Lie superalgebras. Our framework suggests that this conjecture is but the "top level" (when $J = \mathbb{C}$ and $\operatorname{atyp}(J) = \operatorname{def}(\mathfrak{g})$) of the following conjecture.

Conjecture 4. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra, let J be a simple $\mathfrak{g}$-supermodule which admits an ambidextrous trace and let $L \in \mathcal{I}_J$ be a simple $\mathfrak{g}$-supermodule. Then*

$$\operatorname{atyp}(L) = \operatorname{atyp}(J) \text{ if and only if } \mathsf{d}_J(L) \neq 0.$$

Note that Serganova believes that one can use her work with Duflo on associated varieties to show that all simple modules for the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ admit ambidextrous traces. In the case of $\mathfrak{gl}(m|n)$ we can provide the following evidence for the generalized Kac-Wakimoto conjecture. Recall that a simple $\mathfrak{g}$-supermodule is, by definition, polynomial if it appears as a composition factor of some tensor power of the natural module.

Theorem 5. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, let J be a simple $\mathfrak{g}$-supermodule which admits an ambidextrous trace, and let $L \in \mathcal{I}_J$ be a simple $\mathfrak{g}$-supermodule. Then the following are true.*

1. *One always has* $\operatorname{atyp}(L) \leq \operatorname{atyp}(J)$.
2. *If* $\mathsf{d}_J(L) \neq 0$, *then* $\operatorname{atyp}(L) = \operatorname{atyp}(J)$.

(3) If $\text{atyp}(J) = 0$, *then* $\text{atyp}(L) = \text{atyp}(J)$ *and* $\mathsf{d}_J(L) \neq 0$.

(4) If J *and* L *are polynomial, then* J *necessarily admits an ambidextrous trace (ie. it does not have to be assumed), and* $\mathsf{d}_J(L) \neq 0$ *if and only if* $\text{atyp}(L) = \text{atyp}(J)$.

That is, for $\mathfrak{gl}(m|n)$ we can prove one direction of the generalized Kac-Wakimoto conjecture in general, and both directions for both atypicality zero and polynomial representations.

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The Murnaghan–Nakayama rule for k-Schur functions

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(joint work with Jason Bandlow, Mike Zabrocki)

The Murnaghan–Nakayama rule [9, 10] is a combinatorial formula for the characters $\chi_\lambda(\mu)$ of the symmetric group in terms of ribbon tableaux. Under the Frobenius characteristic map, there exists an analogous statement on the level of symmetric functions, which follows directly from the formula

$$p_r s_\lambda = \sum_\mu (-1)^{\text{ht}(\mu/\lambda)} s_\mu. \tag{1}$$

Here p_r is the r-th power sum symmetric function, s_λ is the Schur function labeled by partition λ, and the sum is over all partitions $\lambda \subseteq \mu$ for which μ/λ is a border strip of size r. Recall that a border strip is a connected skew shape without any

2×2 squares. The height $\mathrm{ht}(\mu/\lambda)$ of a border strip μ/λ is one less than the number of rows.

In [2], Fomin and Greene develop the theory of Schur functions in noncommuting variables. In particular, they derive a noncommutative version of the Murnaghan–Nakayama rule [2, Theorem 1.3] for the nilCoxeter algebra (or more generally the local plactic algebra)

$$\mathbf{p}_r \mathbf{s}_\lambda = \sum_w (-1)^{\mathrm{asc}(w)} w \mathbf{s}_\lambda \tag{2}$$

where w is a hook word of length r. Here $\mathbf{p}_r$ and $\mathbf{s}_\lambda$ are the noncommutative analogues of the power sum symmetric function and the Schur function. The word w is a hook word if $w = b_l b_{l-1} \dots b_1 a_1 a_2 \dots a_m$ where

$$b_l > b_{l-1} > \cdots > b_1 > a_1 \leq a_2 \leq \cdots \leq a_m \tag{3}$$

and $\mathrm{asc}(w) = m - 1$ is the number of ascents in w. Actually, by [2, Theorem 5.1] it can further be assumed that the support of w is an interval.

We derive a (noncommutative) Murnaghan–Nakayama rule for k-Schur functions of Lapointe and Morse [7]. k-Schur functions form a basis for the ring $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$ spanned by the first k complete homogeneous symmetric functions h_r, which is a subspace of ring of symmetric functions Λ. Lapointe and Morse [7] gave a formula for a homogeneous symmetric function h_r times a k-Schur function (at $t = 1$) as

$$h_r s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)}, \tag{4}$$

where the sum is over all k-bounded partitions μ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip. Here $\lambda^{(k)}$ denotes the k-conjugate of λ. Equation (4) is a simple analogue of the Pieri rule for usual Schur functions, called the k-Pieri rule. This formula can in fact be taken as the definition of k-Schur functions from which many of their properties can be derived. Conjecturally, the k-Pieri definition of the k-Schur functions is equivalent to the original definition by Lapointe, Lascoux, and Morse [4] in terms of atoms.

Lam [3] defined a noncommutative version of the k-Schur functions in the affine nilCoxeter algebra as the dual of the affine Stanley symmetric functions

$$F_w(X) = \sum_{a=(a_1,\dots,a_t)} \langle \mathbf{h}_{a_t}(u) \mathbf{h}_{a_{t-1}}(u) \cdots \mathbf{h}_{a_1}(u) \cdot 1, w \rangle \; x_1^{a_1} \cdots x_t^{a_t},$$

where the sum is over all compositions of $\mathrm{len}(w)$ satisfying $a_i \in [0, k]$. Here

$$\mathbf{h}_r(u) = \sum_A u_A^{\mathrm{dec}}$$

are the analogues of homogeneous symmetric functions in noncommutative variables where the sum is over all r-subsets A of $[0, k]$ and u_A^{dec} is the product of the generators of the affine nilCoxeter algebra in cyclically decreasing order with indices appearing in A. We denote the noncommutative analogue of $\Lambda_{(k)}$ by $\mathbf{\Lambda}_{(k)}$.

Denote by $\mathbf{s}_\lambda^{(k)}$ the noncommutative k-Schur function labeled by the k-bounded partition λ and $\mathbf{p}_r$ the noncommutative power sum symmetric function in the affine nilCoxeter algebra. There is a natural bijection between k-bounded partitions λ and $(k+1)$-cores, denoted $\mathrm{core}_{k+1}(\lambda)$. We define a *vertical domino* in a skew-partition to be a pair of cells in the diagram, with one sitting directly above the other. For the skew of two k-bounded partitions $\lambda \subseteq \mu$ we define the height as

$$\mathrm{ht}(\mu/\lambda) = \text{number of vertical dominos in } \mu/\lambda \tag{5}$$

Definition 1. The skew of two k-bounded partitions, μ/λ, is called a *k-ribbon of size r* if μ and λ satisfy the following properties:

(0) (containment condition) $\lambda \subseteq \mu$ and $\lambda^{(k)} \subseteq \mu^{(k)}$;
(1) (size condition) $|\mu/\lambda| = r$;
(2) (ribbon condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is a ribbon (in the usual sense of containing no 2×2 squares);
(3) (connectedness condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is k-connected;
(4) (height statistics condition) $\mathrm{ht}(\mu/\lambda) + \mathrm{ht}(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Our main result is the following theorem.

Theorem 2. *For $1 \le r \le k$ and λ a k-bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} \mathbf{s}_\mu^{(k)},$$

where the sum is over all k-bounded partitions μ such that μ/λ is a k-ribbon of size r.

Let λ, μ be k-bounded partitions of the same size, and ℓ be the length of μ. A k-ribbon tableau of shape λ and type μ is a filling, T, of the cells of λ with the labels $\{1, 2, \ldots, \ell\}$ which satisfies the following conditions for all i:

(i) the shape of the restriction of T to the cells labeled $1, \ldots, i$ is a partition, and
(ii) the skew shape which is the restiriction of T to the cells labeled i is a k-ribbon of size μ_i, denoted by r_i.

The *weight* of a k-ribbon tableau T is

$$\mathrm{wt}(T) = \prod_{i=1}^{\ell} (-1)^{\mathrm{ht}(r_i)}.$$

We also define

$$\chi_{\lambda,\mu}^{(k)} = \sum_T \mathrm{wt}(T)$$

where the sum is over all k-ribbon tableaux T of shape λ and weight μ.

Iterating Theorem 2 gives the following corollary. We remark that this formula may also be considered as a definition of the k-Schur functions.

Corollary 3. *For μ a k-bounded partition, we have*

$$\mathbf{p}_\mu = \sum_\lambda \chi^{(k)}_{\lambda,\mu} \mathbf{s}^{(k)}_\lambda.$$

There is a ring isomorphism

$$\iota : \mathbf{\Lambda}_{(k)} \to \Lambda_{(k)}$$

sending the noncommutative symmetric functions to their symmetric function counterpart. This leads us to the following corollary.

Corollary 4. *Theorem 2 and Corollary 3 also hold when replacing $\mathbf{p}_r$ by the power sum symmetric function p_r and $\mathbf{s}^{(k)}_\lambda$ by the k-Schur function $s^{(k)}_\lambda$.*

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Irreducible Specht modules

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(joint work with Sinéad Lyle)

For any finite group G and any prime p, it is a natural question to ask which ordinary irreducible representations of G remain irreducible in characteristic p. For the symmetric groups, this amounts to classifying the irreducible *Specht modules*. This problem is actually solved, but a slight generalisation to Iwahori–Hecke algebras of type A is still unsolved. We report on some recent progress with the outstanding case, as well as some work in progress on the type B case.

To be precise, let $\mathbb{F}$ be a field of characteristic p, and let q be a non-zero element of $\mathbb{F}$. Define the *quantum characteristic* e to be 0 if q is not a root of unity in $\mathbb{F}$, to be p if $q = 1$, or to be the multiplicative order of q in $\mathbb{F}$ otherwise. Let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of type A_{n-1} (with quadratic relations $(T_i - q)(T_i + 1) = 0$). For any partition λ of n, there is an $\mathcal{H}_n$-module S^λ called the *Specht module*. In the semisimple case (in particular, if $e = 0$), then the Specht modules are precisely the irreducible modules of $\mathcal{H}_n$ up to isomorphism. The problem under consideration is to classify the triples $(\lambda, \mathbb{F}, q)$ for which S^λ is irreducible.

In fact, the reducibility of S^λ depends only on λ, e, p, not on $\mathbb{F}$ or q. So we say that λ is *(e,p)-irreducible* if S^λ is irreducible for a choice of $\mathbb{F}, q$ which yield these values of e, p. The classification of (e,p)-irreducible partitions is complete in all cases where $e > 2$, and also when λ or its conjugate λ' is 2-regular (i.e. does not have equal non-zero parts); this result is known as the *Generalized Carter Criterion*, and is the result of a series of papers [6, 9, 1, 2, 5, 10]. So we can restrict attention to the case where $e = 2$ (i.e. $q = -1$) and λ is *doubly-singular*.

In 2004, the speaker and Andrew Mathas carried out some computations in the case where $p = 0$; in this case, one can compute the decomposition matrix for any $\mathcal{H}_n$, and therefore determine whether any given Specht module is irreducible. From these computations arose a conjecture, but this has not yielded to any attempts to prove it. A precise statement of this conjecture can be found in [3, Conjecture 2.2], but roughly speaking in says that $(2,0)$-irreducible partitions can be obteind by modifying rectangular partitions in certain ways.

More recent work has concentrated on the case of positive characteristic. This case is usually more difficult than characteristic zero, but in fact the problem under consideration is solved in characteristic 2, when $\mathcal{H}_n$ is simply the group algebra of the symmetric group. This result [7] says that there is only one doubly-singular $(2,2)$-irreducible partition. A recent result of the speaker [4] extends this result to the statement that in any given positive characteristic p there are only finitely many doubly-singular $(2,p)$-irreducible partitions. The proof of this requires three ingredients:

(1) a *decomposition map* which compares $\mathcal{H}_n$ with an Iwahori–Hecke algebra at a primitive $2p$th root of unity in a field of characteristic 0 (and implies that a $(2,p)$-irreducible partition must be $(2p,0)$-irreducible);
(2) a recent theorem of the speaker and Lyle [3], which says that a partition which is *broken* (i.e. satisfies a certain very simple criterion on the Young diagram of a partition) is necessarily $(2,p)$-reducible;
(3) a combinatorial lemma which says that for any p there are only finitely many doubly-singular unbroken $(2p,0)$-irreducible partitions.

Now let $\mathcal{B}_n$ denote the Iwahori–Hecke algebra of type B_n, with $\mathcal{H}_n$ as a subalgebra, and with the extra generator T_0 satisfying the relation $(T_0 - Q)(T_0 + 1) =$

0 for some $Q \in \mathbb{F}$. Using a result of Dipper and James, it suffices for most representation-theoretic purposes to assume that $Q = -q^r$ for some $r \in \mathbb{Z}$.

$\mathcal{B}_n$ has Specht modules indexed by *bipartitions* of n. Again, in the semisimple case these give the irreducible representations, and in general one would like to know which Specht modules are irreducible. The case $e = 0$ is now non-trivial, since this condition no longer implies that the algebra is semisimple. However, it is easy to classify the irreducible Specht modules in this case, since the decomposition numbers are known explicitly; these were computed by Leclerc and Miyachi [8] following work of Lusztig. Using this, one finds a simple criterion for a Specht modules to be irreducible in the case $e = 0$, using the *beta-sets* of the components partitions.

We also look at irreducible Specht modules for $\mathcal{B}_n$ in the case $e = 2$; in this case, we can reduce to the type A problem. The result here essentially says that a Specht module can only be irreducible if the labelling bipartition can be reduced in a certain well-defined way (corresponding to i-restriction functors from $\mathcal{B}_n$ to $\mathcal{B}_{n-k}$) until one of the components partitions is empty (at which point the type A result applies). The proof of the reducibility uses decomposition maps from Iwahori–Hecke algebras of type B with $e = 0$ (for a given case there is a variety of ways to make this 'lift', since the equation $Q = -(-1)^r$ only determines the parity of r).

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