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Geometrie

Organised by
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Abstract. The program of this meeting covered a wide range of recent developements in geometry such as geometric flows and connections to transport problems, metric and Alexandrov geometry, positively curved manifolds and the positive energy theorem.

Mathematics Subject Classification (2000): 53-xx.

Introduction by the Organisers

The official program consisted of 18 lectures and therefore left plenty of space for fruitful informal collaboration of the participants.

Analytic aspects cover the use of geometric flows, optimal transport, spectral properties and the investigation of submanifolds. A central problem for all methods based on deformations is to make sure that singularities which arise can be controlled. Exciting progress could be reported on

- new results about Ricci flows. Several talks investigated new properties about the Kähler-Ricci flow, Ricci flows with non-smooth initial data and uniqueness questions.
- locally collapsed 3-manifolds. They arise in the final step of Perelman's proof of the geometrization conjecture.
- Special geometries: Einstein manifolds and Calabi-Yau equation on four manifolds and manifolds admitting and manifolds admitting and affine flat structure.
- minimal surfaces with one end.
- manifolds with positive scalar curvature.

- the investigation of rigidity and flexibility questions of embeddings.

The second main part covers metric aspects. Important new developements were presented on

- the investigation of the relationship between Ptolemy spaces and Möbius structures.
- the injective hull of hyperbolic groups
- gradient flows on Alexandrov geometry
- the use of optimal transport methods. Those methods are characterized by an interesting interplay between geometry, analysis and probability theory. Manifolds are replaced by the more general framework of metric measure spaces. Recent questions concern for example the definition of Ricci curvature and stability properties of those spaces.

Workshop: Geometrie

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Abstracts

Möbius geometry of the boundary at infinity of complex hyperbolic spaces

Sergei Buyalo
(joint work with Viktor Schroeder)

1. Möbius structures and Ptolemy spaces

Two metrics d, d' on a set X are *Möbius equivalent* if for any quadruple $Q = (x, y, z, u) \subset X$ of pairwise distinct points the respective *cross-ratio triples* coincide, $\operatorname{crt}_d(Q) = \operatorname{crt}_{d'}(Q)$, where

$$\operatorname{crt}_d(Q) = (d(x, y) \cdot d(z, u) : d(x, z) \cdot d(y, u) : d(x, u) \cdot d(y, z)) \in \mathbb{R}P^2.$$

We consider *extended* metrics on X for which existence of an *infinitely remote* point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if exists, and that $d(\omega, \omega) = 0$. We use notation $X_\omega := X \setminus \omega$ and the standard conventions for the calculation with $\omega = \infty$. If ∞ occurs in Q, say $u = \infty$, then $\operatorname{crt}(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$.

A *Möbius structure* on a set X is a maximal collection $\mathcal{M} = \mathcal{M}(X)$ of metrics on X which are pairwise Möbius equivalent. A topology on X is well defined by a Möbius structure. When a Möbius structure $\mathcal{M}$ on X is fixed, we say that $(X, \mathcal{M})$ or simply X is a *Möbius space.*

A map $f : X \to X'$ between two Möbius spaces is called *Möbius*, if f is injective and for all quadruples $Q \subset X$ of pairwise distinct points

$$\operatorname{crt}(f(Q)) = \operatorname{crt}(Q),$$

where the cross-ratio triples are taken with respect to some (and hence any) metric of the Möbius structures of X, X'. Möbius maps are continuous. If a Möbius map $f : X \to X'$ is bijective, then f^{-1} is Möbius, f is homeomorphism, and the Möbius spaces X, X' are said to be *Möbius equivalent.*

In general different metrics in a Möbius structure $\mathcal{M}$ can look very different. However if two metrics have the same infinitely remote point, then they are homothetic.

A classical example of a Möbius space is the extended $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 1$, where the Möbius structure is generated by some extended Euclidean metric on $\widehat{\mathbb{R}}^n$. Euclidean metrics which are not homothetic to each other generate different Möbius structures which however are Möbius equivalent.

A Möbius space X is called a *Ptolemy space*, if it satisfies the Ptolemy property, that is, for all quadruples $Q \subset X$ of pairwise distinct points the entries of the respective cross-ratio triple $\operatorname{crt}(Q) \in \mathbb{R}P^2$ satisfies the triangle inequality. The importance of the Ptolemy property comes from the following fact: A Möbius structure $\mathcal{M}$ on a set X is Ptolemy if and only if for all $z \in X$ there exists a metric $d_z \in \mathcal{M}$ with infinitely remote point z.

The classical example of Ptolemy space is $\widehat{\mathbb{R}}^n$ with a standard Möbius structure.

We list some known results on Ptolemy spaces. A real normed vector space, which is ptolemaic, is an inner product space (Schoenberg, 1952); a Riemannian locally ptolemaic space is nonpositively curved (Kay, 1963); all Bourdon and Hamenstädt metrics on $\partial_\infty X$, where X is CAT(-1), generate a Ptolemy space (Foertsch-Schroeder, 2006); a geodesic metric space is CAT(0) if and only if it is ptolemaic and Busemann convex, a ptolemaic proper geodesic metric space is uniquely geodesic (Foertsch-Lytchak-Schroeder, 2007); any Hadamard space ptolemaic, a complete Riemannian manifold is ptolemaic if and only if it is a Hadamard manifold, any Finsler ptolemaic manifold is Riemannian (Buckley-Falk-Wraith, 2009);

A (Ptolemy) circle in a Ptolemy space X is a subset $\sigma \subset X$ homeomorphic to S^1 such that for every quaruple $(x, y, z, u) \in \sigma$ of distinct points the equality

$$|xz||yu| = |xy||zu| + |xu||yz| \tag{1}$$

holds, where it is supposed that the pair (x, z) separates the pair (y, u), i.e. y and u are in different components of $\sigma \setminus \{x, z\}$. Recall the classical Ptolemy theorem that four points x, y, z, u of the Euclidean plane lie on a circle (in this order) if and only if their distances satisfy the Ptolemy equality (1). Let σ be a circle passing through the infinitely remote point ω and let $\sigma_\omega = \sigma \setminus \omega$. Then for x, y, $z \in \sigma_\omega$ (in this order) we have $|xy| + |yz| = |xz|$, i.e. it implies that σ_ω is a geodesic, actually a complete geodesic isometric to $\mathbb{R}$.

A Möbius characterization of the boundary at infinity of real hyperbolic spaces $\partial_\infty \mathrm{H}^{n+1}$ is obtained by T. Foertsch and V. Schroeder, 2009.

Theorem 1. *Let X be a compact Ptolemy space such that through any three points there is a circle. Then X is Möbius equivalent to $\widehat{\mathbb{R}}^n = \partial_\infty \mathrm{H}^{n+1}$.*

2. Ptolemy spaces with many circles and many automorphisms

We are interested in Möbius characterization of the boundary at infinity of rank one symmetric space different from real hyperbolic spaces, for which the answer is given by Theorem 1. Such a boundary is a compact Ptolemy space with many circles and automorphisms, the property, which we formalize in the following four basic axioms. It is convenient to use term a $\mathbb{R}$-*circle* for a Ptolemy circle.

1. Existence axiom: through every two points in X there is a $\mathbb{R}$-circle.
2. Uniqueness axiom: given a quadruple of points $Q \subset X$ such that the Ptolemy equality holds for Q, and three points of Q lie on a $\mathbb{R}$-circle $\sigma \subset X$, then the fourth point of Q lies also on σ.
3. Self-duality axiom: given a $\mathbb{R}$-circle $\sigma \subset X$, let $\psi : (X \setminus \sigma) \times \sigma \to \sigma$ be a map defined by $\psi(x, \omega) \in \sigma$ is the closest to x point in the space X_ω (by Axiom 2, ψ is well defined). Then $\psi(x, \psi(x, \omega)) = \omega$ for all $x \in X \setminus \sigma$, $\omega \in \sigma$.
4. Extension axiom: any Möbius map between any $\mathbb{R}$-circles in X extends to a Möbius automorphism of X.

Conjecture 1. *Let X be a compact Ptolemy space which satisfies Axioms (1)–(4). Then X is Möbius equivalent to the boundary at infinity of rank one symmetric space of noncompact type.*

As an important step towards Conjecture 1, we have the following conjecture. For $\omega \in X$, we consider $X_\omega = X \setminus \omega$ as a metric space with a metric d from the Möbius structure of X with infinitely remote point ω.

Conjecture 2. *Let X be a compact Ptolemy space which satisfies Axiom 1–4. Then for every $\omega \in X$ there is a submetry $\pi_\omega : X_\omega \to B_\omega$ with the base B_ω isometric to an Euclidean space $\mathbb{R}^k$, $k \leq \partial_\infty mX$, such that any Möbius automorphism $\phi : X \to X$ with $\phi(\omega) = \omega'$ induces a homothety $\overline{\phi} : B_\omega \to B_{\omega'}$ with $\pi_{\omega'} \circ \phi = \overline{\phi} \circ \pi_\omega$. Completed fibers $\widehat{F} = F \cup \omega$ of π_ω, called* $\mathbb{K}$-circles*, are homeomorphic to the sphere S^p, $k + p = \partial_\infty mX$, and the following properties hold*
($1_{\mathbb{K}}$) through any two distinct points in X there is a unique $\mathbb{K}$-circle;
($2_{\mathbb{K}}$) any $\mathbb{K}$-circle and any $\mathbb{R}$-circle in X have at most two points in common;
($3_{\mathbb{K}}$) given a $\mathbb{K}$-circle $\widehat{F} = F \cup \omega$ through $\omega \in X$, and $x \in X \setminus \widehat{F}$, there is a unique $\mathbb{R}$-circle $\sigma \subset X$ through x, ω that intersects F;
($4_{\mathbb{K}}$) given distinct $\mathbb{K}$-circles $\widehat{F} = F \cup \omega$, $\widehat{F}' = F' \cup \omega$ through $\omega \in X$ and two $\mathbb{R}$-circles through ω that intersect F, F', for any other $\mathbb{K}$-circle $\widehat{F}'' = F'' \cup \omega$ if F'' intersects one of the $\mathbb{R}$-circles, then it necessarily intersects the other.

This conjecture is much plausible, at the moment we are able to prove all properties ($1_{\mathbb{K}}$)–($4_{\mathbb{K}}$) except the existence in ($3_{\mathbb{K}}$). Our main result is the following.

Theorem 2. *Let X be a compact Ptolemy space which satisfies Axioms (1)–(4). Assume in addition that $p = 1$ in the conclusion of Conjecture 2, that is, X also has properties ($1_{\mathbb{K}}$)–($4_{\mathbb{K}}$) with $p = 1$ and $\mathbb{K} = \mathbb{C}$. Then X is Möbius equivalent to the boundary at infinity of a complex hyperbolic space.*

Injective hulls of word hyperbolic groups

Urs Lang

A metric space Y is called *injective* if for every metric space X and every 1-Lipschitz map $f : A \to Y$ with $A \subset X$ there exists a 1-Lipschitz extension $\bar{f} : X \to Y$ of f. The simplest examples of injective metric spaces are $\mathbb{R}$, all complete metric trees, and $l^\infty(I)$ for an arbitrary index set I. In the 1960es, J. R. Isbell [3] showed by means of an explicit construction that every metric space X possesses a uniquely determined *injective hull*: There exist an injective metric space $\mathbf{E}X$ and an isometric inclusion $X \subset \mathbf{E}X$ such that

- every isometric embedding $f : X \to Y$ into some injective metric space Y extends to an isometric embedding $\bar{f} : \mathbf{E}X \to Y$, and
- the only isometric embedding of $\mathbf{E}X$ into itself that leaves X pointwise fixed is the identity on $\mathbf{E}X$.

Isbell's construction was introduced independently under the name *tight span* by A. Dress [2] in 1984. Nowadays it is used mainly in discrete optimization, metric fixed point theory, and as a tool to recognize tree-like structures in the natural sciences, but besides this it is still little known. The injective hull of a compact metric space is compact, and the hull of a finite metric space consisting of n

points is a polyhedral complex of dimension at most $n/2$ with l^∞-metrics on the cells. A large part of the literature about injective hulls refers to this last case. Another part regards normed real vectorspaces X, for which $\mathbf{E}X$ agrees with the linearly injective hull (the linear injectivity of $\mathbb{R}$ corresponds to the Hahn–Banach Theorem).

The aim of our work, carried out partly in collaboration with A. Moezzi, is to apply the construction of the injective hull in the context of geometric group theory to finitely generated (infinite) groups, in order to possibly obtain new geometric models of such groups. Our interest in this comes from the fact that injective metric spaces share a number of properties with simply connected spaces of nonpositive curvature. Every injective metric space is complete, geodesic, contractible, and admits a convex geodesic bicombing. As a consequence of [5], injective spaces satisfy isoperimetric inequalities of Euclidean type for integral cycles in any dimension. Furthermore, like in a tree, every triple of points spans a geodesic tripod.

Let now Γ be a finitely generated group, endowed with the word metric d_S with respect to some finite generating system S. The isometric action of Γ by left multiplication on $\Gamma_S = (\Gamma, d_S)$ extends canonically to an isometric action on the injective hull $\mathbf{E}\Gamma_S$. This action is obviously proper, in the sense that $\{\gamma \in \Gamma : \gamma B \cap B \neq \emptyset\}$ is finite for every bounded set $B \subset \mathbf{E}\Gamma_S$. In view of the polyhedral structure of $\mathbf{E}X$ for a finite metric space X it is natural to ask under what conditions on Γ and S the injective hull $\mathbf{E}\Gamma_S$ is still a finite dimensional, locally finite polyhedral complex (with l^∞-metrics on the cells) on which Γ acts cocompactly. A simple necessary condition is that Γ be combable, but this condition is clearly not sufficient. For instance, $\mathbb{Z}^n$ with the standard generating set is combable, and $\mathbf{E}\mathbb{Z}^n$ turns out to be isometric to $\mathbb{R}^{2^{n-1}}$ with the l^∞-metric; thus the action is not cocompact unless $n \in \{1, 2\}$. See [4] for a detailed discussion of these and a number of further results on injective spaces and hulls.

Our main focus is on word hyperbolic groups, for which the above question is of particular interest. The following observation served as a starting point.

Theorem 1 (L. 2004). *If X is a Gromov hyperbolic geodesic metric space, the injective hull $\mathbf{E}X$ is within bounded distance of X, so $\mathbf{E}X$ is also Gromov hyperbolic.*

The distance bound depends only on the hyperbolicity constant of X. The next theorem was first established in [4] under an extra condition on Γ, "Axiom Y", which could be verified for various classes of groups, but which also turned out to be violated by some hyperbolic groups. By now we have the general result.

Theorem 2 (L. 2010). *Let (Γ, S) be a word hyperbolic group. Then $\mathbf{E}\Gamma_S$ is a proper, locally finite polyhedral complex with finitely many isometry types of cells, and Γ acts properly and cocompactly on $\mathbf{E}\Gamma_S$ by cellular isometries.*

The proof relies on Theorem 1 and J. Cannon's result on the finiteness of cone types [1] for hyperbolic groups. Further properties of this injective polyhedral complex $\mathbf{E}\Gamma_S$ will be discussed in a forthcoming article.

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Uniqueness and nonuniqueness in Ricci flow

Peter Topping

The Ricci flow takes a Riemannian metric g on a manifold M and deforms it under the nonlinear PDE

$$\frac{\partial g}{\partial t} = -2Ric(g).$$

A good well-posedness theory has been developed by Hamilton [4], Shi [5] and Chen-Zhu [1] which gives the short-time existence and uniqueness of a complete bounded curvature flow $g(t)$ for each complete bounded curvature initial metric g_0.

In this talk we discussed some extensions of this theory. In particular, we discussed recent advances in the following two subjects:

Instantaneously complete Ricci flows.

Consider an arbitrary smooth metric g_0 (possibly incomplete and possibly of unbounded curvature) on a surface M^2.

Theorem 1. *(T.- [7] and Giesen and T.- [3]) There exists a smooth Ricci flow $g(t)$ for $t \in [0, T)$ such that $g(0) = g_0$ and $g(t)$ is complete for all $t \in (0, T)$.*

We also described how to specify the maximal existence time $T \in (0, \infty]$ exactly, and determined the infinite-time behaviour in certain cases.

It was conjectured in [7] that this Ricci flow should also be *unique*. In the talk we described an almost-complete resolution of this conjecture - joint work with G. Giesen.

Contracting cusps.

We then presented a generalised way of posing Ricci flow and demonstrated how it leads to examples of nonunique Ricci flows, and flows which could change their underlying domain manifold. In particular, we showed how a surface with a cusp-like end can evolve by adding in a point at infinity on the end of the cusp and then contracting the cusp. On the other hand we demonstrated how in the case that the injectivity radius is controlled from below by a positive number, this behaviour cannot occur and we get uniqueness. The results appear in [6].

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On 4-Dimensional Einstein Manifolds

Claude LeBrun

A Riemannian manifold (M, g) is said to be *Einstein* if it has constant Ricci curvature. This condition is equivalent to requiring that

$$r = \lambda g$$

where r is the Ricci tensor of g, and $\lambda \in \mathbb{R}$ is some arbitrary constant. When this holds, λ is then called the *Einstein constant* [1] of the Einstein metric g.

In dimensions 2 and 3, a metric is Einstein iff it has constant constant sectional curvature. This fact plays a fundamental role in the study of 2- and 3-manifolds via geometrization, and in particular in the Ricci-flow proof of the classical Poincaré conjecture [7, 16]. By contrast, when $n \geq 5$, S^n admits [3, 4] unit-volume Einstein metrics for many different values of λ, making Einstein metrics ineffective as a tool for recognizing whether or not a given n-manifold is the n-sphere. While the corresponding question for S^4 remains open, there are several interesting classes of 4-manifolds for which the moduli space of Einstein metrics is now known to be connected [1, 2, 8, 9], and it therefore seems quite plausible that Einstein metrics might eventually play an important role in the geometrization of 4-manifolds.

However, not every 4-manifold admits Einstein metrics [1, 8, 10, 11, 18]. In fact, the corresponding existence problem turns out to be deeply related to fundamental issues in 4-dimensional differential topology. At present, our most powerful techniques for proving the existence of Einstein metrics in four dimensions come from Kähler geometry. While this tool-kit is only applicable to 4-manifolds which admit both complex and symplectic structures, this precisely puts one in the arena where Seiberg-Witten theory gives rise to striking relations between curvature and differential topology.

The following result, proved in joint work [5] with Chen and Weber, illustrates the interplay of all these phenomena:

Theorem 1. *Suppose that M is a smooth compact oriented 4-manifold which admits a* complex structure J*. Then M also admits an (*a priori *unrelated) Einstein metric g with $\lambda > 0 \iff$ it appears on the following list of diffeomorphism types:*

$$M \approx \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad \textit{or} & \\ S^2 \times S^2. & \end{cases}$$

Here # indicates the connected sum of smooth oriented manifolds, and $\overline{\mathbb{CP}}_2$ denotes the reverse-oriented version of the complex projective plane. The above diffeotypes are exactly those realized by the Del Pezzo surfaces [6, 13].

The existence part of the story largely relies on the theory of Kähler-Einstein metrics [17, 19, 20]. However, no Kähler-Einstein metric can exist [14] on either $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$. Nonetheless, these manifolds *do* admit Einstein metrics which are conformal rescalings of extremal Kähler metrics. For $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$, our existence proof involves the Bach tensor, the Futaki invariant, gluing theorems, Gromov-Hausdorff convergence, and classification results for ALE scalar-flat Kähler surfaces. By contrast, the relevant metric on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ can actually be written down [15] in closed form!

Showing that the above list actually exhausts all the possibilities involves surface classification, Seiberg-Witten theory and the Hitchin-Thorpe inequality. A similar argument also proves an analogous statement for symplectic manifolds:

Theorem 2. *Suppose that M is a smooth compact oriented 4-manifold which admits a* symplectic form ω*. Then M also admits an (*a priori *unrelated) Einstein metric g with $\lambda > 0 \iff$ it appears on the same list of diffeomorphism types as appeared above in Theorem 1.*

It is not hard to extend these results to also cover [12] the case of $\lambda = 0$. However, our understanding of the analogous question for $\lambda < 0$ is distinctly limited at present. For some partial results, see [10, 11].

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Geometric flows with rough initial data

Tobias Lamm

(joint work with Herbert Koch)

In this talk we discuss the recent results of [3] on the existence and uniqueness (in certain Banach spaces) of global solutions of geometric flows with non-smooth initial data. More precisely, we consider the graphical mean curvature flow, the Ricci-DeTurck flow on $\mathbb{R}^n$ and the harmonic map flow for maps from $\mathbb{R}^n$ into a compact target manifold.

The initial data we are interested in are Lipschitz functions for the mean curvature flow, and L^∞ metrics (respectively maps) for the Ricci-DeTurck and harmonic map flow. We say that a function f is Lipschitz if it belongs to the homogeneous Lipschitz space $C^{0,1}(\mathbb{R}^n)$ with norm $||f||_{C^{0,1}(\mathbb{R}^n)} = ||\nabla f||_{L^\infty(\mathbb{R}^n)}$. We construct the solutions of the flows via a fixed point argument and therefore we require the initial data to be small in the corresponding spaces. Our result for the graphical mean curvature partially extends the result of Ecker and Huisken [1, 2] to arbitrary codimensions. Moreover, our result for the Ricci-DeTurck flow extends a recent result of Schnürer, Schulze and Simon [5].

Crucial in the construction are norms based on space-time cylinders, similar to the Carleson weight characterization of BMO (see [6, 4]). This point of view has been introduced by Koch and Tataru [4] in the context of the Navier-Stokes equations.

We show that a similar approach also works for quasilinear equations and we obtain new and possibly optimal results in terms of the regularity of the initial data and the regularity of the solution.
Moreover our method to construct the solutions allows a uniform and efficient treatment of the three geometric evolution equations.

In [4] a fixed point argument was used in order to show the existence of a unique global solution of the Navier-Stokes equations for any initial data which is divergence free and small in BMO^{-1} (the space of distributions which are the divergence of vector fields with BMO components). By localizing their construction the authors were also able to show the existence of a unique local solution of the Navier-Stokes equations for any initial data which is divergence free and in VMO^{-1}.

The novelty of the approach sketched in the talk consists in the use of scale invariant L^∞ and Lipschitz spaces for the initial data, even for quasilinear equations.

At the end of the talk we also mention recent more general results on the local well-posedness of parabolic systems on certain Riemannian manifolds, which are modelled on the above mentioned geometric flows.

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Milnor-Wood inequalities and affine manifolds

Michelle Bucher-Karlsson
(joint work with Tsachik Gelander)

A classical inequality of Milnor says that the Euler number of flat oriented vector bundles over surfaces (different from the sphere) are at most half the Euler characteristic of the surface. It follows that surfaces with nonzero Euler characteristic cannot admit a flat, nor an affine structure. More generally, it is an old conjecture that closed aspherical manifolds with nonzero Euler characteristic admit no flat structure, and the special case that such manifolds are not affine is known as the Chern conjecture.

We prove sharp Milnor-Wood inequalities for Riemannian manifolds which are locally a product of hyperbolic planes. As a consequence, we confirm the conjecture that such manifolds (which have nonzero Euler characteristic) cannot admit a flat nor an affine structure.

Deformations of the hemisphere that increase scalar curvature

Fernando Coda Marques
(joint work with Simon Brendle, Andre Neves)

It is well-known that the Euclidean space is rigid under compactly supported and scalar curvature increasing deformations. More precisely, any metric g on $\mathbb{R}^n$ that has nonnegative scalar curvature, and that coincides with the Euclidean metric outside some compact set, is necessarily flat. This is a consequence of Witten's proof of the Positive Mass Theorem in the spin case ([7]), and it also follows from earlier work of Schoen and Yau if $n \leq 7$ ([6]). (We note that there is recent work of Lohkamp on the full version of the Positive Mass Theorem [3].)

Inspired by that, Min-Oo proved the analogous rigidity statement for the hyperbolic space in 1989 ([4]), and later made the following conjecture for the spherical setting: if g is a metric of scalar curvature greater than or equal to $n(n-1)$ on the standard hemisphere S^n_+, and g coincides with the standard metric in a neighborhood of the equator, then g has constant sectional curvature 1 ([5]). Since then, partial results have been obtained by several people.

In this talk we will describe our construction of counterexamples to Min-Oo's conjecture in any dimension greater than or equal to three ([2]). We shall also comment on the infinitesimal scalar curvature rigidity of some smaller geodesic balls in S^n ([1]).

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Optimal Transport and Geometry

Karl-Theodor Sturm

The theory of optimal transports, originating in the classical problem of mass transportation due to Monge (1781) and its weak reformulation by Kantorovich (1942), has undertaken an impressive development in the last decade, initiated by fundamental works of Brenier, McCann, Otto, Villani and others. One of the remarkable observations is that applying geometric concepts to the space of probability measures – the so-called Wasserstein space – allows to derive surprising new results and, vice versa, properties of the Wasserstein space can be used to analyze the geometry of the underlying space.

In the talk, we will first give a brief introduction to the theory of optimal transports. In particular, we will present the Riemannian structure on the space of probability measures induced by the optimal transports on Euclidean space or on a given Riemannian manifold. This picture, for instance, allows to identify the heat equation as the gradient flow for the relative entropy. Indeed, this identification holds true in great generality including Finsler spaces, sub-Riemannian geometries like the Heisenberg group and infinite dimensional spaces like the Wiener space.

Then we present the concept of generalized lower Ricci curvature bounds for metric measure spaces (M, d, m), introduced by Lott & Villani and the author. These curvature bounds are defined in terms of optimal transports, more precisely, in terms of convexity properties of the relative entropy regarded as function on the Wasserstein on the given space M. For Riemannian manifolds, this turns out to characterize lower Ricci bounds. Other important examples covered by this concept are Alexandrov spaces and Finsler manifolds.

One of the main results is that these generalized lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence. In combination with an upper bound on the dimension, they imply sharp versions of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem.

Finally, we explain how convexity properties of the relative entropy on the Wasserstein space are related with functional inequalities (e.g. logarithmic Sobolev inequalities), with concentration of measure and with contraction properties of the evolution semigroups. We also sketch some recent applications and links to Ricci flow which allow to derive or reformulate various monotonicity formulas of Perelman in terms of optimal transports.

Locally collapsed 3-manifolds

Bruce Kleiner
(joint work with John Lott)

The lecture concerned a local collapsing result that arises in the last part of Perelman's proof of Thurston's Geometrization Conjecture.

We begin with several definitions.

Definition 1. An orientable 3-manifold M is a **graph manifold** if there is a finite disjoint collection of embedded tori $\{T_1, \ldots, T_k\}$, such that each connected component of the complement $M \setminus \cup_i T_i$ is a circle bundle over a surface.

Definition 2. If M is a Riemannian manifold and $p \in M$, the **curvature scale at** p, denoted $\mathcal{R}_p$, is the supremum of the set of radii $r \in (0, \infty)$ such that the sectional curvature of the ball $B(p, r)$ is at least $-r^{-2}$. This is finite if and only if the connected component of p has a 2-plane with strictly negative sectional curvature.

Definition 3. Pick $w \in (0, \infty)$. A complete Riemannian manifold M is **locally** w**-collapsed with a lower curvature bound** if for every $p \in M$ with $\mathcal{R}_p < \infty$, we have $\mathrm{vol}(B(p, \mathcal{R}_p)) \leq w\mathcal{R}_p^3$.

Our main result is:

Theorem 1. *There is a $w_0 \in (0, 1)$ such that any 3-manifold which is locally w_0-collapsed with a lower curvature bound is a graph manifold.*

Variants of this theorem, with different proofs, were given earlier by Shioya-Yamaguchi, Morgan-Tian, and Cao-Ge. Perelman stated a related result without proof in his second Ricci flow preprint.

Lower Ricci Curvature, Convexity and Applications

Aaron Naber
(joint work with Toby Colding)

We prove new estimates for tangent cones along minimizing geodesics in GH limits of manifolds with lower Ricci curvature bounds. We use these estimates to show convexity results for the regular set of such limits. Applications include the proofs of several conjectures dating back to the work of Cheeger/Colding and the ruling out of certain limit spaces, including the so called generalized trumpet spaces. We construct new examples which exhibit various new behaviors and show sharpness of the new theorems.

The Calabi-Yau equation on symplectic four-manifolds

Valentino Tosatti
(joint work with Ben Weinkove, S.-T. Yau)

Perhaps the most striking result in Kähler geometry is the Calabi conjecture, proved by Yau in the seventies [11]. This says that on a compact Kähler manifolds it is possible to prescribe the Ricci curvature of a Kähler metric, once the natural cohomological constraint is satisfied, and the resulting metric is unique in its cohomology class. Moreover prescribing the Ricci curvature is equivalent to prescribing the volume form of the metric. This theorem is proved by reducing the problem to a highly nonlinear partial differential equation, of complex Monge-Ampère type, and then by showing that the required *a priori* estimates on the solution indeed hold. This means that any given Kähler metric can be estimated (together with all its derivatives) depending only on its volume form, on its cohomology class and on the geometry of the underlying complex manifold.

Donaldson has recently proposed an extension of this result in the context of symplectic geometry. Given a compact symplectic 4-manifold (M,ω) (so by definition the 2-form ω is closed and the wedge product ω^2 is a nowhere vanishing volume form) we can always fix an almost-complex structure J which is compatible with the symplectic form. This means that the following two conditions hold

$$\omega(X, JX) > 0 \quad \text{for all } X \neq 0, \tag{1}$$

$$\omega(JX, JY) = \omega(X, Y) \quad \text{for all } X, Y. \tag{2}$$

Whenever only (1) holds, we say that ω tames J. In this case we can define a Riemannian metric by $g(X,Y) = \frac{1}{2}(\omega(X,JY)+\omega(Y,JX))$. If the almost-complex structure is integrable then we exactly have a Kähler surface, but in general a symplectic manifold is not Kähler. Donaldson then asked whether any other symplectic form $\tilde{\omega}$ on M, compatible with J and cohomologous to ω, could be estimated in terms only of its volume form $\tilde{\omega}^2$ and the geometry of (M,ω,J). More precisely, in [1] he posed the following

Conjecture 3 (Donaldson [1]). *Let (M,ω) be a compact symplectic four-manifold equipped with an almost complex structure J tamed by ω. Let σ be a smooth volume form on M. If $\tilde{\omega}$ is a symplectic form on M which is cohomologous to ω, compatible with J and solves the Calabi-Yau equation*

$$\tilde{\omega}^2 = \sigma, \tag{3}$$

then there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on ω, J and σ.

He then went on to show that this conjecture, if solved, would have striking applications in symplectic topology. In fact, it would provide a new and powerful tool to construct symplectic forms on 4-manifolds as solutions of the nonlinear partial differential equation (3). For example, he showed that Conjecture 3 would imply the following basic

Conjecture 4 (Donaldson [1]). *Let (M^4, ω) be a compact syplectic 4-manifold with $b^+(M) = 1$, and let J be an almost-complex structure on M tamed by ω. Then there exists another symplectic form $\tilde{\omega}$ which is compatible with J.*

Conjecture 4 is known to hold when M is the complex projective plane and ω is the standard Fubini-Study metric, by a result of Gromov [2]. It is also known in the case when J is integrable [3] without any assumption on $b^+(M)$, thanks to the Kodaira-Enriques classification. Very recently, Taubes [5] has extended Gromov's method and proved Conjecture 4 for generic J. Other applications of Conjecture 3 to topology can be found in [1].

Let us now discuss the present state of Conjecture 3. In the case when (M, ω, J) is Kähler, Conjecture 3 reduces precisely to the Calabi-Yau theorem proved by Yau [11]. In the paper [1] where he formulated the conjecture, Donaldson reduced it to proving a uniform BMO-type bound for $\tilde{\omega}$ (which would be implied for example by a Hölder C^α bound on $\tilde{\omega}$). Subsequently Weinkove [10], in the case when ω is compatible with J, proved Conjecture 3 if J is close to being integrable and moreover reduced all the estimates to an L^∞ bound on a scalar function (see (4) below). Further progress was done in [9], where we proved that:

Theorem 1 (T., Weinkove, Yau [9]). *Conjecture 3 holds if (M, ω, J) is nonnegatively curved in a suitable sense.*

In the case when (M, ω, J) is the complex projective plane with the Fubini-Study metric, the relevant curvature is strictly positive, and hence Conjecture 3 holds whenever the data (ω, J) is sufficiently close to the Fubini-Study metric. To describe our next result, let us introduce some notation. It is possible to define a scalar function φ by the equation $\tilde{\Delta}\varphi = 4 - \mathrm{tr}_{\tilde{g}} g$, together with the normalization $\sup_M \varphi = 0$, where $\tilde{\Delta}$ is the Laplacian of the metric $\tilde{g}$ determined by $\tilde{\omega}$ and J, and g is the metric associated to ω and J. We note that in the Kähler case, such a φ would satisfy $\tilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$, but this fails in general. We have:

Theorem 2 (T., Weinkove, Yau [9], T., Weinkove [8]). *In the setting of Conjecture 3, there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on the fixed data ω, J, σ and on $|\tilde{\omega}|_g$. Moreover there are uniform constants A, C such that*

$$|\tilde{\omega}|_g \leq Ce^{A(\varphi - \inf_M \varphi)}, \tag{4}$$

$$|\varphi|_{L^\infty} \leq C + C\int_M |\varphi| \, dV_g. \tag{5}$$

This reduces Conjecture 3 to establishing a uniform bound on the L^1 norm of φ. In the Kähler case such a bound is an easy classical result.

Another case when Conjecture 3 is solved is the Kodaira-Thurston manifold [7], where we restrict to certain almost-complex structures with T^2-symmetry. This provided the first example of a symplectic manifold without Kähler metrics where Conjecture 3 holds.

There are also links between the Calabi-Yau equation and the theory of harmonic maps [4], which were described in [6]. For simplicity assume that ω is compatible with J, and consider the identity map $\mathrm{Id} : (M, g) \to (M, \tilde{g})$. By assumption it is J-holomorphic and therefore harmonic, with energy density $\mathrm{tr}_g \tilde{g}$. Moreover the energy of Id is uniformly bounded $\int_M \mathrm{tr}_g \tilde{g}\, dV_g \leq C$. We also have a monotonicity formula that says that the quantity $r^{-2} \int_{B_g(p,r)} \mathrm{tr}_g \tilde{g}\, dV_g$ is essentially increasing for all r small and for all points $p \in M$. Analogously to the sitatuation for harmonic maps, we conjecture the following ε-regularity theorem:

Conjecture 5. *In the setting of Conjecture 3, there are uniform $\varepsilon, C > 0$ such that if*

$$\frac{1}{r^2} \int_{B_g(p,r)} tr_g \tilde{g}\, dV_g \leq \varepsilon, \tag{6}$$

for some point $p \in M$ and small radius r, then

$$\sup_{B_g(p,r/2)} tr_g \tilde{g} \leq \frac{C}{r^4} \int_{B_g(p,r)} tr_g \tilde{g}\, dV_g. \tag{7}$$

If this is proved, then one should be able to show that if the estimates of Conjecture 3 don't hold then the blow-up set will be a union of J-holomorphic curves, and developing this connection further might result in a solution of Conjecture 3.

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Contracting exceptional divisors by the Kähler-Ricci flow

Ben Weinkove
(joint work with Jian Song)

We consider a compact Kähler manifold X of complex dimension n, with Kähler metric g_0. Associated to g_0 is the Kähler form $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz^i \wedge \overline{dz^j}$, which we also sometimes refer to as a metric.

We consider the Kähler-Ricci flow starting at g_0, namely:

$$\text{(KRF)} \qquad \frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}, \qquad g_{ij}|_{t=0} = (g_0)_{i\bar{j}}.$$

It is known by a result of Tian-Zhang (extending work of H.-D. Cao) that a smooth solution of (KRF) exists for $[0, T)$ with T given by

$$T = \sup\{t > 0 \mid [\omega_0] - tc_1(X) > 0\}.$$

If $T < \infty$ then (KRF) develops a singularity at time T. The basic question we ask is: what is the behavior of the Kähler-Ricci flow as $t \to T$? Does the Kähler-Ricci flow 'pass through' the singularity and continue on a new manifold?

In [1] we consider the case where the Kähler-Ricci flow carries out contractions of exceptional divisors. Before we state our first theorem, we describe our results in a basic example.

Let X be $\mathbb{P}^n$ blown up at one point, y_0. Let $\pi : X \to \mathbb{P}^n$ be the blow-down map, and write $E = \pi^{-1}(y_0) \cong \mathbb{P}^{n-1}$ for the exceptional divisor. Then $\pi|_{X \setminus E} : X \setminus E \to \mathbb{P}^n \setminus \{y_0\}$ is a biholomorphism. A general Kähler class on X is of the form

$$[\omega_0] = b_0 \pi^*[H] - a_0[E], \quad 0 < a_0 < b_0,$$

where $[H]$ is the hyperplane bundle on $\mathbb{P}^n$. We now assume

$$a_0(n+1) < b_0(n-1).$$

In this case, $T = a_0/(n-1)$ and a simple cohomological computation shows that the volume of E with respect to $g(t)$ tends to zero as $t \to T$. In addition, the limiting class along the Kähler-Ricci flow is given by the pull-back of the Fubini-Study metric ω_{FS} on $\mathbb{P}^n$:

$$[\omega_0] - Tc_1(X) = \kappa[\pi^* \omega_{\mathrm{FS}}],$$

for some $\kappa > 0$.

Then we show [1]:

(i) $g(t) \to g_T$ as $t \to T^-$, a smooth Kähler metric on $X \setminus E$, where the convergence is in C^∞ on compact subsets of $X \setminus E$.

(ii) g_T defines a distance function d_{g_T} on $\mathbb{P}^n \setminus \{y_0\}$ via π. The metric completion of $(\mathbb{P}^n \setminus \{y_0\}, d_{g_T})$ is a compact metric space $(\mathbb{P}^n, d_T)$.

(iii) $(X, g(t))$ converges in the Gromov-Hausdorff sense to $(\mathbb{P}^n, d_T)$ as $t \to T^-$.

(iv) There exists a smooth solution $g(t)$ of (KRF) on $\mathbb{P}^n$ for $t \in (T, T')$, with $T' > T$, such that $g(t) \to g_T$ as $t \to T^+$, where convergence is in C^∞ on compact subsets of $\mathbb{P}^n \setminus \{y_0\}$.

(v) $(\mathbb{P}^n, g(t))$ converges in the Gromov-Hausdorff sense to $(\mathbb{P}^n, d_T)$ as $t \to T^+$.

Thus we see that the Kähler-Ricci flow 'contracts' the exceptional divisor E and continues on the new manifold $\mathbb{P}^n$. The convergence is smooth away from the exceptional divisor, and we obtain Gromov-Hausdorff convergence as $t \to T^-$ and $t \to T^+$. We say that the Kähler-Ricci flow carries out a *canonical surgical contraction.* This was conjectured by Feldman-Ilmanen-Knopf, who constructed self-similar solutions of (KRF) exhibiting this behavior. Also we remark that part (i) above was already known by results of Tian-Zhang, and part (iv) makes use of a result of Song-Tian.

Our main theorem is:

Theorem 1. *Let (X, g_0) be a compact Kähler manifold. Suppose there exists a holomorphic map $\pi : X \to Y$ blowing down disjoint exceptional divisors $E_1, \dots, E_k$ on X, where Y is a compact Kähler manifold. In addition assume that*

$$[\omega_0] - Tc_1(X) = [\pi^* \omega_Y]$$

for some Kähler metric ω_Y on Y. Then (KRF) performs a canonical surgical contraction of $E_1, \dots, E_k$ and continues on Y, in the sense above.

Thus we see that the Kähler-Ricci flow will perform a canonical surgical contraction whenever the cohomological information tells us that one should occur. We can apply this in the case of projective algebraic surfaces (complex dimension 2):

Theorem 2. *Assume now X is a projective algebraic surface, with $[\omega_0]$ a rational class. Then the Kähler-Ricci flow performs a finite sequence of canonical surgical contractions on manifolds $X = X_0, X_1, \dots, X_k$ for $t \in [0, T_0), (T_0, T_1), \dots, (T_{k-1}, T_k)$ each of the type of Theorem 1. And either*

(i) *$T_k < \infty$, in which case $Vol_{g(t)} X_k \to 0$ as $t \to T_k^-$. In this case X_k is Fano or a ruled surface.*

(ii) *Or $T_k = \infty$ and X_k has no exceptional curves of the first kind (i.e. X_k is minimal).*

Moreover in the case (ii) one can see that X must have nonnegative Kodaira dimension $\mathrm{Kod}(X)$. There are three possible outcomes. If $\mathrm{Kod}(X) = 0$ then the flow on X_k converges to a Ricci-flat metric (applying the results of Cao, Yau). If $\mathrm{Kod}(X) = 1$ then Song-Tian showed $\frac{1}{t} g(t)$ converges in the sense of currents to the pull-back of a 'generalized Kähler-Einstein metric' on the canonical model. In the case of $\mathrm{Kod}(X) = 2$, it was shown by Tsuji, Tian-Zhang, that the normalized metrics $\frac{1}{t} g(t)$ converge to the pull-back of a Kähler-Einstein metric (possibly of orbifold type) from the canonical model.

Thus we see that, roughly speaking, the Kähler-Ricci flow sees the algebraic type of the manifold and behaves accordingly.

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Ancient solutions to the Ricci flow

Natasa Sesum

We will consider the Ricci flow equation

$$\frac{\partial}{\partial t}g - ij = -2R_{ij}.$$

We are interested in studying the ancient solutions to the RF equation since they appear as singularity models in studying the finite time singularities. In the talk we gave the classification of ancient solutions to the Ricci flow in 2 dimensions in which case the flow can be parametrized over the limiting sphere, that is, $g = ug_{s^2}$, where $v = \frac{1}{u}$ satisfies

$$v_t = v\Delta_{S^2}v - |\nabla v|^2 + 2v^2 = Rv.$$

We show that the only such compact solutions are either the contracting spheres or one of the King Rosenau solutions. This is a joint work with Daskalopoulos and Hamilton.

In the second part of the talk we focused on the long time behaviour of the RF on complete surfaces and we showed that in the case of a cusp metric on a manifold with negative Euler characteristics the RF exists for all times and converges to a unique hyperbolic (uniformizing) metric in the considered conformal class. We also consider the RF on surfaces with finitely many conic ends and negative Euler characteristics in which case after rescaling the flow it converges to a cusp metric of constant negative curvature (which implies the asymptotic geometry drastically changes in the limit).

Rigidity and flexibility of isometric ebeddings

Camillo De Lellis

(joint work with Sergio Conti, László Székelyhidi)

Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An isometric immersion (resp. embedding) of (M^n, g) into $\mathbb{R}^m$ is a (injective) map $u \in C^1(M^n; \mathbb{R}^m)$ such that the induced metric agrees with g. In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{1}$$

consisting of $n(n+1)/2$ equations in m unknowns. A short immersion (resp. embedding) is instead a map $u : M^n \to \mathbb{R}^m$ such that the metric induced on M by u is shorter than g, that is $(\partial_i u \cdot \partial_j u) \leq (g_{ij})$ in the sense of quadratic forms.

Assume for the moment that $g \in C^\infty$. The two classical theorems concerning the solvability of (1) are:

(A) if $m \geq (n+2)(n+3)/2$, then any short embedding can be uniformly approximated by isometric embeddings of class C^∞ (Nash [22], Gromov [17]);
(B) if $m \geq n+1$, then any short embedding can be uniformly approximated by isometric embeddings of class C^1 (Nash [21], Kuiper [20]).

(A) and (B) are not merely existence theorems, they show that there exists a very large set of solutions. This abundance is a central aspect of Gromov's h-principle, for which the isometric embedding problem is a primary example (see [17, 13]). Naively, this type of flexibility could be expected for high codimension as in (A), since then there are many more unknowns than equations in (1). The h-principle for C^1 isometric embeddings is on the other hand rather striking, especially when compared to the classical rigidity result concerning the Weyl problem: if (S^2, g) is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion ([8, 18], see also [29] for a thorough discussion). Thus it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

The proof of (B) involves an iteration technique called convex integration. This technique was developed by Gromov [16, 17] into a very powerful tool to prove the h-principle in a wide variety of geometric problems (see also [13, 31]). In general the regularity of solutions obtained using convex integration agrees with the highest derivatives appearing in the equations (see [30]). Thus, an interesting question raised in [17] p219 is how one could extend the methods to produce more regular solutions. Essentially the same question, in the case of isometric embeddings, is also mentioned in [32] (see Problem 27). For high codimension this is resolved in [19]. We focus here on the case $m = n+1$. In [6] Borisov announced that if g is analytic, then the h-principle holds for local isometric embeddings $u \in C^{1,\alpha}$ for $\alpha < \frac{1}{1+n+n^2}$. A proof for the case $n = 2$ appeared in [7]. In [10] we provide a proof of the h-principle in this range for g which is not necessarily analytic. The novelty of our approach, compared to Borisov's, is that only a finite number of derivatives need to be controlled. This is achieved by introducing a smoothing operator in the iteration step, analogous to the device of Nash used to overcome the loss of derivative problem in [22]. A similar method was used by Källen in [19].

Concerning rigidity in the Weyl problem, it is known from the work of Pogorelov and Sabitov that closed C^1 surfaces with positive Gauss curvature and bounded extrinsic curvature are convex (see [25]; we refer to this book for the notion of bounded extrinsic curvature) and that closed convex surfaces are rigid in the sense that isometric immersions are unique up to rigid motion [24]. Thus, extending the rigidity in the Weyl problem to $C^{1,\alpha}$ isometric immersions can be reduced to showing that the image of the surface has bounded extrinsic curvature. Using geometric arguments, in a series of papers [1, 2, 3, 4, 5] Borisov proved that for $\alpha > 2/3$ the image of surfaces with positive Gauss curvature has indeed bounded extrinsic curvature. Consequently, rigidity holds in this range and in particular 2/3 is an upper bound on the range of Hölder exponents that can be reached using convex integration. In [10] we provide a short and self-consistent analytic proof of this result.

We next state in full detail our main existence results for $C^{1,\alpha}$ isometric immersions. One is of local nature, whereas the second is global. Note that for the local result the exponent matches the one announced in [6]. In what follows, we denote by sym_n^+ the cone of positive definite symmetric $n \times n$ matrices. Moreover, given an immersion $u : M^n \to \mathbb{R}^m$, we denote by u^*e the pullback of the standard Euclidean metric through u, so that in local coordinates

$$(u^*e)_{ij} = \partial_i u \cdot \partial_j u.$$

Finally, let

$$n_* = \frac{n(n+1)}{2}.$$

Theorem 1 (Local existence). *Let $n \in \mathbb{N}$ and $g_0 \in \mathrm{sym}_n^+$. There exists $r > 0$ such that the following holds for any smooth bounded open set $\Omega \subset \mathbb{R}^n$ and any Riemannian metric $g \in C^\beta(\overline{\Omega})$ with $\beta > 0$ and $\|g - g_0\|_{C^0} \leq r$. There exists a constant $\delta_0 > 0$ such that, if $u \in C^2(\overline{\Omega}; \mathbb{R}^{n+1})$ and α satisfy*

$$\|u^*e - g\|_0 \leq \delta_0^2 \qquad \text{and} \qquad 0 < \alpha < \min\left\{\frac{1}{1+2n_*}, \frac{\beta}{2}\right\},$$

then there exists a map $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$ with

$$v^*e \;=\; g \qquad \text{and} \qquad \|v - u\|_{C^1} \;\leq\; C\,\|u^*e - g\|_{C^0}^{1/2}.$$

Corollary 1 (Local h-principle). *Let $n, g_0, \Omega, g, \alpha$ be as in Theorem 1. Given any short map $u \in C^1(\overline{\Omega}; \mathbb{R}^{n+1})$ and any $\varepsilon > 0$ there exists an isometric immersion $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$ with $\|u - v\|_{C^0} \leq \varepsilon$.*

Remark 1. In the last corollary, if u is an embedding, then there exists a corresponding v which in addition is an embedding.

Coming to the rigidity statements, a crucial point of our proof is the following estimate for the metric pulled back by standard regularizations of a given map.

Proposition 1 (Quadratic estimate). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $v \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$ with $v^*e \in C^2$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ a standard symmetric convolution kernel. Then, for every compact set $K \subset \Omega$,*

$$\|(v * \varphi_\ell)^*e - v^*e\|_{C^1(K)} \;=\; O(\ell^{2\alpha-1}). \tag{2}$$

In particular, fix a map u and a kernel φ satisfying the assumptions of the Proposition with $\alpha > 1/2$. Then the Christoffel symbols of $(v * \varphi_\ell)^*e$ converge to those of v^*e. This corresponds to the results of Borisov in [1, 2], and hints at the absence of h–principle for $C^{1,\frac{1}{2}+\varepsilon}$ immersions. Relying mainly on this estimate we can give a fairly short proof of Borisov's theorem:

Theorem 2. *Let (M^2, g) be a surface with C^2 metric and positive Gauss curvature, and let $u \in C^{1,\alpha}(M^2; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then $u(M)$ is a surface of bounded extrinsic curvature.*

This leads to the following corollaries, which follow from the work of Pogorelov and Sabitov.

Corollary 2. *Let (S^2, g) be a closed surface with $g \in C^2$ and positive Gauss curvature, and let $u \in C^{1,\alpha}(S^2; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then, $u(S^2)$ is the boundary of a bounded convex set and any two such images are congruent. In particular if the Gauss curvature is constant, then $u(S^2)$ is the boundary of a ball $B_r(x)$.*

Corollary 3. *Let $\Omega \subset \mathbb{R}^2$ be open and $g \in C^{2,\beta}$ a metric on Ω with positive Gauss curvature. Let $u \in C^{1,\alpha}(\Omega; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then $u(\Omega)$ is $C^{2,\beta}$ and locally uniformly convex (that is, for every $x \in \Omega$ there exists a neighborhood V such that $u(\Omega) \cap V$ is the graph of a $C^{2,\beta}$ function with positive definite second derivative).*

There is an interesting analogy between isometric immersions in low codimension (in particular the Weyl problem) and the incompressible Euler equations. In [11] a method, which is very closely related to convex integration, was introduced to construct highly irregular energy-dissipating solutions of the Euler equations. Being in conservation form, the "expected" regularity space for convex integration for the Euler equations should be C^0. This is still beyond reach, and in [11] a weak version of convex integration was applied instead, to produce solutions in L^∞ (see also [12] for a slightly better space) and, moreover, to show that a weak version of the h-principle holds.

Nevertheless, just like for isometric immersions, for the Euler equations there is particular interest to go beyond C^0: in [23] L. Onsager, motivated by the phenomenon of anomalous dissipation in turbulent flows, conjectured that there exist weak solutions of the Euler equations of class C^α with $\alpha < 1/3$ which dissipate energy, whereas for $\alpha > 1/3$ the energy is conserved. The latter was proved in [14, 9], but on the construction of energy-dissipating weak solutions nothing is known beyond L^∞ (for previous work see [26, 27, 28]). The critical exponent 1/3 is very natural - it agrees with the scaling of the energy cascade predicted by Kolmogorov's theory of turbulence (see for instance [15]). For the analogous problem for isometric immersions there does not seem to be a universally accepted critical exponent (cp. with Problem 27 of [32]), even though 1/2 seems likely. In fact, the regularization and the commutator estimates used in our proof of Proposition 1 and Theorem 2 have been inspired by (and are closely related to) the arguments of [9].

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Minimal Surfaces with One End

Christine Breiner
(joint work with Jacob Bernstein)

The conformal type and asymptotic geometry of complete, embedded, minimal surfaces with finite topology (in $\mathbb{R}^3$) and two or more ends is well understood. In this talk we address recent results concerning such surfaces when they have one end. Work by Colding-Minicozzi on the structure of compact, embedded minimal surfaces with connected boundary paved the way for Meeks and Rosenberg's proof of the uniqueness of the helicoid as well as an understanding of the conformal type of these surfaces in the complete case. In particular, we outline the proof of the following theorem, which is proven in [2].

Theorem 1. *Let $\Sigma \in \mathbb{R}^3$ be a complete, embedded minimal surface with finite topology and one end. Then Σ is conformally a once punctured, compact Riemann surface. Moreover, if the surface is not flat, then it is C^0 asymptotic to some helicoid.*

The proof of this result draws heavily on the fundamental work of Colding and Minicozzi on the geometric structure of embedded minimal surfaces in $\mathbb{R}^3$ [3, 4, 5, 6, 7]. Assuming only mild conditions on the boundaries, they give a description of the geometric structure of essentially all embedded minimal surfaces with finite genus. From this structure, they deduce a number of important consequences. We highlight, in particular, two results of Colding-Minicozzi. The first concerns embedded minimal disks with large curvature at their center, which are thus known to be not graphical on a sufficiently large scale.

Theorem 2. *Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. Then there exist $C, \Omega > 1$ such that:*
if $|A|^2(0) > CR^{-2}$ then the component of $B_{R/\Omega} \cap \Sigma$ containing 0 is the union of two multi-valued graphs that spiral together.

"Multi-valued graph" should be thought of as looking roughly like one half of the helicoid. Another important result is the "one sided curvature estimate". One can think of it as a local version of the strong half-space theorem. Precisely, it is as follows:

Theorem 3. *There exists $\epsilon > 0$ such that if $\Sigma \subset B_{2R} \cap \{x_3 > 0\} \subset \mathbb{R}^3$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R}$, then every component Σ' of $\Sigma \cap B_R$ that intersects $B_{\epsilon R}$ has*

$$\sup_{\Sigma'} |A|^2 \leq R^{-2}.$$

Colding and Minicozzi's work is also an essential ingredient in understanding minimal surfaces with infinite total curvature, i.e., complete surfaces with one end. Prior to their work, the study of these surfaces required very strong assumptions on the conformal structure and behavior of the Gauss map at the end. For example, Hauswirth, Perez and Romon [9] consider $E \subset \mathbb{R}^3$, a complete embedded minimal annulus with one compact boundary component and one end with infinite total curvature. They assume, in addition, that E is conformal to a punctured disk, the Weierstrass data (g, dh) has the property that dg/g and dh extend across the puncture, and the flux over the boundary of E has zero vertical component. They then deduce more precise information about the asymptotic geometry of E. Indeed, their result immediately gives the asymptotic behavior of the Σ of interest in Theorem 1, once the conditions on the Weierstrass data are shown.

By using Colding and Minicozzi's work, in particular the compactness result of [6], Meeks and Rosenberg were able to remove such strong assumptions for disks. Indeed, in [10], they resolve the question of the uniqueness of the helicoid.

Theorem 4. *Let Σ be a complete, embedded minimal disk in $\mathbb{R}^3$. Then Σ is the plane or the helicoid.*

Let us now recall the argument of [1], where we provide an alternative proof to the uniqueness of the helicoid. There it is shown that any complete, non-flat, properly embedded minimal disk can be decomposed into two regions: one a region of strict spiraling, i.e. the union of two strictly spiraling multi-valued graphs over the $x_3 = 0$ plane (after a rotation of $\mathbb{R}^3$), and the other a neighborhood of the region where the graphs are joined and where the normal has small vertical component. By strictly spiraling, we mean that each sheet of the graph meets any (appropriately centered) cylinder with axis parallel to the x_3-axis in a curve along which x_3 strictly increases (or decreases). This follows from existence results for multi-valued minimal graphs in embedded disks found in [4] and an approximation result for such minimal graphs from [8]. The strict spiraling is then used to see that $\nabla_\Sigma x_3 \neq 0$ everywhere on the surface; thus, the Gauss map is not vertical and the holomorphic map $z = x_3 + i x_3^*$ is a holomorphic coordinate. By looking at the log of the stereographic projection of the Gauss map, the strict spiraling is used to show that z is actually a proper map and thus, conformally, the surface is the plane. Finally, this gives strong rigidity for the Weierstrass data, implying the surface is a helicoid.

For Σ as in Theorem 1, as there is finite genus and only one end, the topology of Σ lies in a ball in $\mathbb{R}^3$, and so, by the maximum principle, all components of the intersection of Σ with a ball disjoint from the genus are disks. Hence, outside of a large ball, one may use the local results of [3, 4, 5, 6] about embedded minimal disks. For Σ non-simply connected , the presence of non-zero genus complicates matters. Nevertheless, the global structure will follow from the far reaching description of embedded minimal surfaces given by Colding and Minicozzi in [7]. In particular, as Σ has one end, globally it looks like a helicoid. We first prove a sharper description of the global structure; indeed, one may generalize the decomposition of [1] as:

Theorem 5. *There exist $\epsilon_0 > 0$ and a decomposition of Σ into disjoint subsets $\mathcal{R}_A$, $\mathcal{R}_S$, and $\mathcal{R}_G$ such that:*

(1) *$\mathcal{R}_G$ is compact, connected, has connected boundary and $\Sigma \backslash \mathcal{R}_G$ has genus 0;*
(2) *after a rotation of $\mathbb{R}^3$, $\mathcal{R}_S$ can be written as the union of two (oppositely oriented) strictly spiraling multi-valued graphs Σ^1 and Σ^2;*
(3) *in $\mathcal{R}_A$, $|\nabla_\Sigma x_3| \geq \epsilon_0$.*

Remark 1. We say Σ^i $(i = 1, 2)$ is a multi-valued graph if Σ^i is the graph, Γ_{u^i}, of a single function u^i with $u^i_\theta \neq 0$.

To prove this decomposition, we first find the region of strict spiraling, $\mathcal{R}_S$. The strict spiraling controls the asymptotic behavior of level sets of x_3 which, as x_3 is harmonic on Σ, gives information about x_3 in all of Σ.

By Stokes' Theorem, x_3^* (the harmonic conjugate of x_3) exists on Γ and thus there is a well defined holomorphic map $z : \Gamma \to \mathbb{C}$ given by $z = x_3 + i x_3^*$. Using Theorem 5 and a Rado type theorem, we deduce that z is a holomorphic coordinate on Γ. We claim that z is actually a proper map and so Γ is conformally a punctured disk. This can be shown by studying the Gauss map. On Γ, the stereographic projection of the Gauss map, g, is a holomorphic map that avoids the origin. Moreover, the minimality of Σ and the strict spiraling in $\mathcal{R}_S$ imply that the winding number of g around the inner boundary of Γ is zero. Hence, by monodromy there exists a holomorphic map $f : \Gamma \to \mathbb{C}$ with $g = e^f$. Then the strict spiraling in $\mathcal{R}_S$ imposes strong control on f which is sufficient to show that z is proper. Further, once we establish Γ is conformally a punctured disk, the properties of the level sets of f imply that it extends meromorphically over the puncture with a simple pole. This gives Theorem 1.

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The first eigenvalue of the Dirichlet-to-Neumann map and minimal surfaces

Ailana Fraser
(joint work with Richard Schoen)

We consider the relationship of the geometry of compact Riemannian manifolds with boundary to the first nonzero eigenvalue σ_1 of the Dirichlet-to-Neumann map (Steklov eigenvalue). For surfaces with boundary we obtain the upper bound:

Theorem 1 ([FS]). *Let Σ be a compact Riemannian surface of genus γ with k boundary components. Then*

$$\sigma_1 L(\partial\Sigma) \leq 2(2\gamma + k)\pi.$$

For $\gamma = 0$ and $k = 1$ this result was obtained by Weinstock in 1954, and is sharp. This type of eigenvalue estimate is analogous to a result of Szegö [Sz] for the first nonzero Neumann eigenvalue. In general the bound given above is not sharp, and we attempt to determine the sharp bound for the case of surfaces homeomorphic to an annulus. For rotationally symmetric metrics we show that the best constant is achieved by the induced metric on the portion of the catenoid centered at the origin which meets a sphere orthogonally. Moreover, for this surface, σ_1 has multiplicity 3 and the eigenspace is spanned by the embedding functions. We let T_0 denote the unique value of T such that the critical catenoid is biholomorphic to $[-T_0, T_0] \times S^1$ and we refer to an annulus biholomorphic to $[-T, T] \times S^1$ with $T \leq T_0$ (resp. $T \geq T_0$) as *subcritical* (resp. *supercritical*). For supercritical annuli we obtain the improved sharp bound:

Theorem 2 ([FS]). *For any supercritical Riemannian annulus Σ we have*

$$\sigma_1 L \leq \sigma_{\text{crit}} L_{\text{crit}} \approx 4\pi - 2.09,$$

where σ_{crit} denotes the first eigenvalue for the critical catenoid, and L_{crit} denotes the length of its boundary. Moreover, equality is achieved if and only if Σ is conformally equivalent to the critical catenoid by a conformal transformation which is an isometry on the boundary.

Motivated by the case of annuli we then explored the connection to minimal submanifolds Σ^k lying in the unit ball B^n with boundary contained in the boundary of the ball and with conormal vector equal to the position vector at boundary points. Such minimal submanifolds are critical for the free boundary problem of extremizing the volume among deformations which preserve the ball. Examples include equatorial disks, the critical catenoid discussed above, as well as the cone over any minimal submanifold of the sphere. We show that a proper submanifold of the ball is a free boundary solution if and only if it is immersed by Steklov eigenfunctions. It is then natural to ask whether free boundary solutions generally solve extremal problems for Steklov eigenvalues in natural classes of manifolds. We prove general upper bounds for conformal metrics on manifolds of any dimension which can be properly conformally immersed into the unit ball in terms of certain conformal volume quantities, analogous to the Li-Yau conformal volume [LY]. Moreover, we show that these bounds are only achieved when the manifold is minimally immersed by first Steklov eigenfunctions.

Theorem 3 ([FS]). *Let Σ be a compact k-dimensional Riemannian manifold with nonempty boundary. Then*

$$\sigma_1 \, V(\partial\Sigma) \, V(\Sigma)^{\frac{2-k}{k}} \leq k \, V_{rc}(\Sigma, n)^{\frac{2}{k}}$$

for all n for which the relative conformal volume $V_{rc}(\Sigma, n)$ is defined (i.e. such that there exists a proper conformal immersion $\varphi : \Sigma \to B^n$). Equality implies that there exists a conformal harmonic map $\varphi : \Sigma \to B^n$ which (after rescaling the metric) is an isometry on $\partial\Sigma$, with $\varphi(\partial\Sigma) \subset \partial B^n$ and such that $\varphi(\Sigma)$ meets ∂B^n orthogonally along $\varphi(\partial\Sigma)$. For $k > 2$ this map is an isometric minimal immersion of Σ to its image. Moreover, the immersion is given by a subspace of the first eigenspace.

For $k = 2$ this reduces to the bound $\sigma_1 \cdot L(\partial\Sigma) \leq 2V_{rc}(\Sigma, n)$. In an application related to these ideas we show that when $k = 2$, a free boundary solution has boundary length which is a maximum over the boundary lengths of its conformal images in the ball. We use this to show that any free boundary solution has area at least π:

Theorem 4 ([FS]). *Let Σ be a minimal surface in B^n, with (nonempty) boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$. Then*

$$A(\Sigma) \geq \pi.$$

We observe that this implies the sharp isoperimetric inequality for free boundary solutions in the two dimensional case.

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A guided tour to Alexandrov geometry

Anton Petrunin

I discuss the following constructions in Alexandrov geometry:

- Gradient flow;
- Struts and open map theorem;
- Plaut's theorem;
- Distance charts;
- Perelman's lemma and Kliner's spine.

As an illustration, I show how one can use all above to prove the following result:

Theorem. *Let $\mathcal{A}$ be m-dimensional Alexandrov space with curvature bounded below and S be the set of singular points*[1] *in $\mathcal{A}$. Then the set S can be presented as a countable union of $(m-1)$-rectifiable sets.*

Scalar Curvature, Energy and Large Manifolds

Joachim Lohkamp

The study of scalar curvature on arbitrary dimensional (non-spin) manifolds is currently based on one particular approach initiated in the late 70ties. One considers area minimizing hypersurfaces within a manifold of, let us say, positive scalar curvature and notices that this area minimizer also admits a metric of positive scalar curvature. If this hypersurface (all those that follow) are smooth one can inductively iterate the argument. In fact this way one obtains easily controllable obstructions for positive scalar curvature.
However an initially fairly underestimated problem arises in dimensions ≥ 8 and stopped the development of this method at an early stage in the 80ties. Namely, an area minimizer H in a manifold of dimension ≥ 8 may have a *not* necessarily resolvable singular set Σ:
H is a smooth hypersurface outside a set Σ of codimension 8 and this high codimension was a common source for the misjudgement the occurrence of Σ might be a minor issue. However, the problem is not primarily the singular set but the fact that the singularities usually induce a rampancy of topology and of diverging metric distortions of H when one approaches the - mostly unknown - point-set Σ. Since any analytic treatments of singular minimizers will also register a full neighborhood of Σ these singular structures may have a considerable impact.
In fact, despite many attempts, virtually no progress was made to incorporate singular minimal hypersurfaces in the study of scalar curvature geometry - during a period of more than three decades. For this only reason, but also in view of more recent counterexamples for several related problems, this eventually became the critical challenge in this area.
We presented the rough scheme of our recent solution of this problem which allows

[1] A point $p \in \mathcal{A}$ is singular if its tangent cone is not isometric to the Euclidean space.

us to overcome the dimensional barrier and comprises many new concepts and constructions. In particular we mentioned some new smoothing concepts within H, their relation to boundary Harnack inequalities and briefly noted how arguments adapted from the theory of metrics with negative Ricci curvature imply the fact that enlargeable manifolds do not admit $scal > 0$-metrics.

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