Mathematisches Forschungsinstitut Oberwolfach

Report No. 29/2010

DOI: 10.4171/OWR/2010/29

Homogeneous Dynamics and Number Theory

Organised by
Manfred Einsiedler, Zürich
Dmitry Kleinbock, Waltham
Elon Lindenstrauss, Jerusalem/Princeton
Hee Oh, Providence/Seoul

July 4th – July 10th, 2010

Abstract. The theory of flows on homogeneous spaces of Lie groups has emerged as a distinct, rapidly advancing subject over the last few decades incorporating ergodic theory, geometry and number theory. The workshop showcased the latest advances in the subject as well as a wide range of applications.

Mathematics Subject Classification (2000): 37D40, 37A25, 22E40, 11N35.

Introduction by the Organisers

The workshop "Flows on homogeneous spaces and arithmetic", organised by Manfred Einsiedler (ETH Zürich), Dmitry Kleinbock (Brandeis), Elon Lindenstrauss (Hebrew University) and Hee Oh (Brown) was held July 4–10, 2010 and was attended by 52 participants from around the world. The participants ranged from senior leaders in the field to young post-doctoral fellows and PhD students; their range of expertise covered areas from ergodic theory and dynamical systems to automorphic forms, Diophantine approximation and additive combinatorics.

Flows on homogeneous spaces are a class of concrete dynamical systems intimately connected to number theory. Many problems can be approached both via dynamics and via number theoretic and spectral techniques. Recently the interconnection between the dynamics and the arithmetic has flourished, as in many case these two approaches are complementary. The idea of the workshop was to bring together experts in these fields to discuss and collaborate on problems related to homogeneous dynamics, equidistribution, counting integer and rational points, diophantine approximations, and automorphic forms.

The theory of flows on homogeneous spaces received a major impetuous in the late 1980's when Margulis used these dynamical techniques to settle a longstanding conjecture by Oppenheim. Ratner's well known theorems on rigidity of unipotent flows have found numerous arithmetic applications, often unexpected, e.g. in the study of the number and distribution of integer points in symmetric varieties, values and representations of integer and irrational quadratic forms and even to nonvanishing of certain L-functions. A technique developed by Margulis and others to study nondivergence properties of unipotent flows found further applications in metric Diophantine approximation.

Unlike unipotent flows which are well studied, multidimensional diagonal actions are still quite mysterious, though in recent years a substantial progress has been made in their understanding. The talks by Shapira, Tomanov, Wang and Katok dealt with various new phenomena arising from studying such actions, including new examples of irregular orbit closures, while Zamojski talked about applying diagonal actions to a certain counting problem.

One of the central recent events in the area has been a breakthrough work of Benoist and Quint on classification of invariant and stationary measures for actions of Zariski dense subgroups of Lie groups, not necessarily generated by unipotent elements. Quint gave a series of two talks on this subject, and in addition Eskin spoke about a new development in classification of invariant measures for some actions on the moduli space of Riemann surfaces utilizing some ideas due to Benoist and Quint.

Another area which has attracted a lot of attention during the workshop was equidistribution of horocycles and their geodesic translates on homogeneous spaces of infinite volume. Talks of Oh, Shah, Marklof, Roblin, Schapira, Paulin discussed such results, and in many cases – applications to counting problems. The theme of counting and equidistribution can be also approached by methods coming from the theory of automorphic forms, or by studying the spectral gap of certain groups. This has been highlighted in talks by Fuchs, Kontorovich, Kowalski, Ghosh, Gorodnik, while Breuillard and Varju discussed families of expanders arising from certain groups. Green's talk on his joint work with Tao centered on a connection between problems in additive combinatorics and equidistribution of nilflows. Another approach to counting problems, based on integral inequalities on the space of lattices, was surveyed by Margulis.

The workshop went very well; in order to leave enough time for fruitful discussions, the number of talks (50 minutes long) was limited to five per day, and to 23 altogether. On Thursday evening we had a session for short communications (5 minute long talks and five minute intervals for discussion), which allowed many young participants to introduce themselves and the circle of problems they have been working on. The traditional Wednesday afternoon hike has successfully contributed to cheerful and productive atmosphere of the workshop!

Workshop: Homogeneous Dynamics and Number Theory

Table of Contents

Emmanuel Breuillard (joint with Ben Green, Bob Guralnick and Terence Tao)
Product theorems and expander graphs 1707

Alex Eskin (joint with Maryam Mirzakhani)
On Invariant measures for the $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space* 1709

Elena Fuchs
Positive Density in Apollonian Circle Packings 1712

Anish Ghosh (joint with Alexander Gorodnik and Amos Nevo)
Density, Duality, Spectra 1716

Alexander Gorodnik (joint with Amos Nevo)
Property tau, counting lattice points, and applications 1719

Ben Green
Distribution of flows on nilmanifolds and applications to additive combinatorics 1721

Alex V. Kontorovich
Progress on Affine Sieves 1724

Emmanuel Kowalski
Equidistribution and the large sieve 1726

Gregory Margulis
Homogeneous dynamics and quadratic forms 1728

Jens Marklof
Horospheres, Farey fractions and Frobenius numbers 1730

Hee Oh
Distribution of circles in Apollonian circle packings and beyond 1734

Jouni Parkkonen and Frédéric Paulin
Equidistribution, Counting and Arithmetic Applications 1737

Jean-Francçis Quint (joint with Yves Benoist)
Stationary measures and invariant subsets of homogeneous spaces 1739

Barbara Schapira
Equidistribution of half-horocycles on infinite volume hyperbolic surface . 1741

Nimish Shah (joint with Hee Oh)
Integral points on one-sheeted hyperboloids and limits of translates of divergent geodesics 1744

Uri Shapira (joint with Manfred Einsiedler and Lior Fishman and with Menny Aka)
Dynamics and continued fractions . . . 1746

George Tomanov
Margulis' conjecture for the locally divergent orbits on real Hilbert modular space forms . . . 1746

Péter P. Varjú (joint with Jean Bourgain and with Alireza Salehi Golsefidy)
Expansion in Arithmetic Groups . . . 1749

Zhiren Wang (joint with Elon Lindenstrauss)
Topological self-joinings of full Cartan actions by toral automorphisms . . . 1750

Thomas Zamojski
Counting Rational Matrices of a Fixed Irreducible Characteristic Polynomial . . . 1752

Abstracts

Product theorems and expander graphs

Emmanuel Breuillard
(joint work with Ben Green, Bob Guralnick and Terence Tao)

Let G be a group and A a finite subset of G. Let $AA = \{a_1a_2|a_1, a_2 \in A\}$ be the 'product set' of A. Similarly we define $A^k = A \cdot ... \cdot A$ the k-fold product set of A. Let $|A|$ be the cardinality of A. Obviously $|A| \leq |AA| \leq |A|^2$. We are interested in the following question, sometimes called *the non-commutative Freiman problem* (see [12] and Tao's blog):

Question: What can be said about A if $|AA|$ is small compared to $|A|^2$, that is if $|AA| \leq K|A|$, where K is a fixed parameter ?

Here are some examples (=exercises) that show what can happen in some cases:

(1) $|AA| = |A|$ if and only if $A = aH$ is a normalizing coset of a finite subgroup H of G.
(2) $|AA| < \frac{3}{2}|A|$ if and only if $A \subset aH$ is contained in a normalizing coset of a finite subgroup H of G of size $|H| < \frac{3}{2}|A|$.
(3) If $A = \{0, ..., n\} \subset \mathbb{Z}$, $A+A = \{0, ..., 2n\}$ and thus $|A+A| = 2|A|-1 < 2|A|$.
(4) If A is the ball of radius n in the Cayley graph of a nilpotent group G generated by some finite set S, then $|AA| \leq K|A|$, where K depends only on G, S but not on n.

Note that the $\frac{3}{2}$ in item (ii) above is sharp (take $A = \{0, 1\} \subset \mathbb{Z}$). It is a rather remarkable feature that from such a weak assumption on the size of AA one can prove the existence of an underlying algebraic structure (i.e. the group H).

When G is non-commutative, it is somewhat easier to study sets A for which the triple product is small, i.e. $|AAA| \leq K|A|$. In fact, arguments inspired by Ruzsa (see [12]) show that $|AAA| \leq K|A|$ imply that $|A^n| \leq K^{2n-2}|A|$. However a set A may have $|AA| < 3|A|$ while $|AAA| \geq (|A| - 1)^2$ (take $A = H \cup \{x\}$ where H is a finite subgroup and x such that $H \cap xHx^{-1} = \{1\}$).

These examples give a flavour of what we expect to happen in a general ambient group G. When G is a simple algebraic group, together with Ben Green and Terence Tao, we prove the following 'product theorem':

Theorem 1 ([2]). *Let $\mathbb{G}$ be an almost simple algebraic group over an algebraically closed field k and $d = \dim \mathbb{G}$. Let $A \subset \mathbb{G}(k)$ be a finite set. Then*

(1) *either $A \subset \mathbf{H}(k)$, where $\mathbf{H}$ is a proper algebraic subgroup of $\mathbb{G}$ with at most $c(d)$ connected components.*
(2) *or $|AAA| \geq \min\{|\langle A \rangle|, |A|^{1+\epsilon}\}$, where $\epsilon = \epsilon(d) > 0$ and $\langle A \rangle$ the subgroup generated by A.*

Remarks: (a) In the case when $k = \overline{\mathbb{F}_p}$, and A is assumed to generate $\mathbb{G}(\mathbb{F}_q)$ for q some power of a prime p, then this theorem was proved independently by Pyber and Szabo in [11]. This case is in fact the hardest case.

(b) The above theorem is only an improvement of a theorem of Hrushovski [8], which under the same hypotheses asserted the somewhat weaker conclusion that $|AAA| \geq \min\{|\langle A\rangle|, |A|f(|A|)\}$ for some function $f : [0, +\infty) \to [0, +\infty)$ tending to $+\infty$ in $+\infty$. Our theorem can be viewed as a quantitative version of Hrushovski's, where we show that $f(x)$ can be taken to grow at least like x^ϵ for some $\epsilon > 0$. This improvement is crucial for the applications to expanders below.
(c) The result extends prior work of Helfgott [6] and [7] and Helfgott and Gill [4], which where concerned respectively with the special cases of $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{SL}_3(\mathbb{F}_p)$ and of $\mathrm{SL}_n(\mathbb{F}_p)$ for sets A of small size. Although some features of our proof (such as considering maximal tori) are common with some of Helfgott's prior work, our methods are largely different and are closer in spirit to the work of Larsen-Pink [10] on finite subgroups of algebraic groups as used by Hrushovski in [8].

In a breakthrough paper [1] Bourgain and Gamburd used the above product theorem in the case of $\mathrm{SL}_2(\mathbb{F}_p)$ (i.e. Helfgott's theorem) to give many new examples of Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$ that are ϵ-expanders, i.e. have a uniform lower bound $\epsilon > 0$ on the first eigenvalue of their discrete laplacian. They also proved that random Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$ form a family of expanders. Together with R. Guralnick, we managed to generalize this last result to the higher rank case and all finite fields:

Theorem 2 ([3])**.** *Suppose that G is a finite simple group of Lie type of fixed rank and that $a, b \in G$ are selected uniformly at random. Then with probability $1 - o_{|G|\to\infty}(1)$, $\{a, b\}$ generates G and its Cayley graph is an ϵ-expander for some $\epsilon > 0$ depending only on the rank of G.*

Fairly recently Kassabov-Lubotzky-Nikolov [9] showed that the family of all finite simple groups can be turned into a family of expanders. This means in each finite simple group, one can find a generating set with at most k elements (they showed $k \leq 1000$) whose associated Cayley graph is an ϵ-expander with $\epsilon > 10^{-10}$. Theorem 2 above proves that one can in fact find pairs of generators that produce an expanding family for all finite simple groups of Lie type of given rank, and that a random pair will do. It does not say anything however when the rank grows to infinity.

The proof follows the original Bourgain-Gamburd strategy (see [5] for a nice exposition), making key use of the product theorem (i.e. Theorem 1 above). A new difficulty arises in the higher case, where the subgroups structure is richer. This difficulty is overcome by proving the following result about free subgroups of simple algebraic groups and its finite version. A free subgroup of an algebraic group is said to be *strongly dense* if all of its subgroups are either trivial, cyclic, or themselves Zariski-dense.

Theorem 3 ([3])**.** *Suppose that $\mathbb{G}(k)$ is a semisimple algebraic group over an algebraically closed field k, and suppose that k has transcendence degree at least $2\dim(\mathbb{G})$ over the field k_0 of definition of $\mathbb{G}$. Then there exists a non-abelian free subgroup Γ of $\mathbb{G}(k)$ on two generators which is strongly dense.*

Corollary 1. *Fix a positive integer r. Let $G(q)$ be a finite simple group of Lie type of rank r over the finite field $\mathbb{F}_q$ (including twisted groups and Ree and Suzuki groups). Let F_2 be the free group on generators x, y. Let w_1 and w_2 be non-commuting words in F_2. Then the probability that $w_1(a,b)$ and $w_2(a,b)$ generate $G(q)$ tends to 1 as $q \to \infty$.*

References

[1] J. Bourgain and A. Gamburd, *Uniform expansion bounds for Cayley graphs of* $\mathrm{SL}_2(\mathbb{F}_p)$, Ann. of Math. (2) **167** (2008), no. 2, 625–642.

[2] E. Breuillard, B. J. Green and T. C. Tao, *Approximate subgroups of linear groups,* preprint, available at `arxiv:math/1005.1881`.

[3] E. Breuillard, B. J. Green, R. Guralnick, and T. C. Tao, *Expansion in simple groups of Lie type*, in preparation.

[4] N. Gill and H. A. Helfgott, *Growth of small generating sets in* $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$, preprint, arXiv:1002.1605.

[5] B. J. Green, *Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak,* Current Events Bulletin of the AMS, 2010.

[6] H. A. Helfgott, *Growth and generation in* $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, Ann. of Math. (2) **167** (2008), no. 2, 601–623.

[7] H. A. Helfgott, *Growth in* $\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z})$, J. Eur. Math. Soc, to appear, arXiv:0807.2027.

[8] E. Hrushovski, *The elementary theory of the Frobenius automorphisms,* available at `arXiv:math/0406514`.

[9] M. Kassabov, A. Lubotzky and N. Nikolov, *Finite simple groups as expanders,* Proc. Natl. Acad. Sci. USA **103** (2006), no. 16, 6116–6119.

[10] M. Larsen and R. Pink, *Finite subgroups of algebraic groups*, preprint (1995).

[11] L. Pyber and E. Szabo, *Growth in finite simple groups of Lie type of bounded rank,* preprint, available at `arxiv:math/1005.1858`.

[12] T. C Tao, *Product set estimates in noncommutative groups*, Combinatorica **28** (2008), 547–594.

On Invariant measures for the $\mathbf{SL(2,\mathbb{R})}$ action on moduli space

Alex Eskin

(joint work with Maryam Mirzakhani)

Let $\mathcal{H}(m_1 \dots, m_n)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $m_1 \dots m_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(m_1 \dots, m_n)$ as *flat surfaces* or *translation surfaces.*

The space $\mathcal{H}(m_1 \dots, m_n)$ admits an action of the group $SL(2,\mathbb{R})$ which generalizes the action of $SL(2,\mathbb{R})$ on the space $GL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of flat tori. In this note, we announce some ergodic-theoretic rigidity properties of this action.

Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \dots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(m_1 \dots m_n) \to \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} w \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(m_1 \dots m_n)$. We will call these coordinates *period coordinates.*

We can consider the measure λ on $\mathcal{H}(m_1 \dots m_n)$ which is given by the pullback of the Lebesque measure on $H^1(M, \Sigma, \mathbb{C}) \approx \mathbb{C}^k$. The measure λ is independent of the choice of basis $\{\gamma_1, \dots, \gamma_k\}$, and is easily seen to be $SL(2, \mathbb{R})$-invariant. We call λ the *Lebesque* measure on $\mathcal{H}(m_1, \dots, m_n)$.

The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(m_1 \dots, m_n)$ is defined as a subset of translation surfaces in $\mathcal{H}(m_1, \dots, m_n)$ of area one. The SL-invariant Lebesque measure λ_1 on $\mathcal{H}_1(m_1, \dots, m_n)$ is defined by disintegration of the Lebesque measure λ on $\mathcal{H}_1(m_1 \dots, m_n)$, namely

$$d\lambda = d\lambda_1 \, da$$

A fundamental result of Masur [Mas1] and Veech [Ve1] is that $\lambda_1(\mathcal{H}_1(m_1, \dots, m_n)) < \infty$. In this paper, we normalize λ_1 so that $\lambda_1(\mathcal{H}_1(m_1, \dots, m_n)) = 1$ (and so λ_1 is a probability measure).

For a subset $\mathcal{M}_1 \subset \mathcal{H}_1(m_1 \dots, m_n)$ we write

$$\mathbb{R}\mathcal{M}_1 = \{(M, t\omega) \mid (M, \omega) \in \mathcal{M}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(m_1 \dots, m_n).$$

Definition 1. *An ergodic $SL(2, \mathbb{R})$-invariant probability measure ν_1 on $\mathcal{H}_1(m_1, \dots, m_n)$ is called* affine *if the following hold:*

(i) *The support $\mathcal{M}_1$ of ν_1 is an suborbitfold of $\mathcal{H}_1(m_1, \dots, m_n)$. Locally $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ is given by a complex linear subspace in the period coordinates.*

(ii) *Let ν be the measure supported on $\mathcal{M}$ so that $d\nu = d\nu_1 da$. Then ν is an affine linear measure in the period coordinates on $\mathcal{M}$, i.e. it is (up to normalization) the restriction of the Lebesque measure λ to the subspace $\mathcal{M}$.*

Definition 2. *We say that any suborbitfold $\mathcal{M}_1$ for which there exists a measure ν_1 such that the pair $(\mathcal{M}_1, \nu_1)$ satisfies (i) and (ii) an* affine invariant submanifold.

We also consider the entire stratum $\mathcal{H}(m_1, \dots, m_n)$ to be an (improper) affine invariant submanifold.

The classification of the affine invariant submanifolds is complete in genus 2 by the work of McMullen [Mc1] [Mc2] [Mc3] [Mc4] [Mc5] and Calta [Ca]. In genus 3 or greater it is an important open problem.

The main theorem is the following:

Theorem 1. *Let ν be any ergodic $SL(2, \mathbb{R})$-invariant probability measure on $\mathcal{H}_1(m_1, \dots, m_n)$. Then ν is affine.*

For the case of strata in genus 2, Theorem 1 is proved using a different method by Curt McMullen [Mc6].

1. Some notes on the proof

The main theorem inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work. In particular, the analogue of Theorem 1 in homogeneous dynamics is due to Ratner [Ra4], [Ra5], [Ra6], [Ra7]. These results are based in part on the "polynomial divergence" of the unipotent flow on homogeneous spaces.

However, in our setting, the dynamics of the unipotent flow (i.e. the action of N) on $\mathcal{H}_1(m_1,\dots,m_n)$ is poorly understood, and plays no role in our proofs. The main strategy is to replace the "polynomial divergence" of unipotents by the "exponential drift" idea in the recent breakthrough paper by Benoist and Quint [BQ]. Also, in our setting, since we have assumed invariance under A, we can (and do) use entropy arguments throughout the proof, and not just at the end.

References

[BQ] Y. Benoist and J-F Quint. Mesures Stationnaires et Fermés Invariants des espaces homogènes. *Preprint.*

[Ca] K. Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.* **17** (2004), no. 4, 871–908

[Dan1] S.G. Dani, On invariant measures, minimal sets and a lemma of Margulis, *Invent. Math.* **51** (1979), 239–260.

[Dan2] S.G. Dani, Invariant measures and minimal sets of horoshperical flows, *Invent. Math.* **64** (1981), 357–385.

[Dan3] S.G. Dani, On orbits of unipotent flows on homogeneous spaces, *Ergod. Theor. Dynam. Syst.* **4** (1984), 25–34.

[Dan4] S.G. Dani, On orbits of unipotent flows on homogenous spaces II, *Ergod. Theor. Dynam. Syst.* **6** (1986), 167–182.

[DM1] S.G. Dani and G.A. Margulis, Values of quadratic forms at primitive integral points, *Invent. Math.* **98** (1989), 405–424.

[DM2] S.G. Dani and G.A. Margulis, Orbit closures of generic unipotent flows on homogeneous spaces of $SL(3,\mathbb{R})$, *Math. Ann.* **286** (1990), 101–128.

[DM3] S.G. Dani and G.A. Margulis. Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces, *Indian. Acad. Sci. J.* **101** (1991), 1–17.

[DM4] S.G. Dani and G.A. Margulis, Limit distributions of orbits of unipotent flows and values of quadratic forms, in: *I. M. Gelfand Seminar*, Amer. Math. Soc., Providence, RI, 1993, pp. 91–137.

[EMM] A. Eskin, J. Marklof, D. Morris. Unipotent flows on the space of branched covers of Veech surfaces. *Ergodic Theory Dynam. Systems* **26** (2006), no. 1, 129–162.

[EKL] M. Einsiedler, A. Katok, E. Lindenstrauss. Invariant measures and the set of exceptions to Littlewood's conjecture. *Ann. of Math.* (2) **164** (2006), no. 2, 513–560.

[Mar1] G.A. Margulis, *On the action of unipotent groups in the space of lattices*, In *Lie Groups and their representations*, Proc. of Summer School in Group Representations, Bolyai Janos Math. Soc., Akademai Kiado, Budapest, 1971, p. 365–370, Halsted, New York, 1975.

[Mar2] G.A. Margulis, Formes quadratiques ind'efinies et flots unipotents sur les spaces homog'enes, *C. R. Acad. Sci. Paris Ser. I* **304** (1987), 247–253.

[Mar3] G.A. Margulis, em Discrete Subgroups and Ergodic Theory, in: *Number theory, trace formulas and discrete subgroups*, a symposium in honor of A Selberg, pp. 377–398. Academic Press, Boston, MA, 1989.

[Mar4] G.A.Margulis, *Indefinite quadratic forms and unipotent flows on homogeneous spaces*, In *Dynamical systems and ergodic theory*, Vol. 23, pp. 399–409, Banach Center Publ., PWN – Polish Scientific Publ., Warsaw, 1989.

[MaT] G. A. Margulis and G. M. Tomanov. Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. *Invent. Math.* **116** (1994), 347–392.

[Mas1] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982), no. 1, 169–200.

[Mc1] C. McMullen. Billiards and Teichmller curves on Hilbert modular surfaces. *J. Am. Math. Soc.* **16**, 857–885 (2003)

[Mc2] C. McMullen. Teichmller geodesics of infinite complexity. *Acta Math.* **191**, 191–223 (2003)

[Mc3] C. McMullen. Teichmller curves in genus two: Discriminant and spin. *Math. Ann.* **333**, 87–130 (2005)

[Mc4] C. McMullen. Teichmller curves in genus two: The decagon and beyond. *J. Reine Angew. Math.* **582**, 173–200.

[Mc5] C. McMullen. Teichmller curves in genus two: torsion divisors and ratios of sines. *Invent. Math.* **165** (2006), no. 3, 651–672.

[Mc6] C. McMullen. Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two. *Ann. of Math.* (2) **165** (2007), no. 2, 397–456.

[Mor] D. W. Morris. *Ratner's Theorems on Unipotent Flows.* Univ. of Chicago Press: Chicago, 2005. http://arxiv.org/math.DS/0310402

[Ra1] M. Ratner. Rigidity of horocycle flows. *Ann. of Math.* **115** (1982), 597–614.

[Ra2] M. Ratner. Factors of horocycle flows. *Ergodic Theory Dynam. Systems* **2** (1982), 465–489.

[Ra3] M. Ratner. Horocycle flows, joinings and rigidity of products. *Ann. of Math.* **118** (1983), 277–313.

[Ra4] M. Ratner. Strict measure rigidity for unipotent subgroups of solvable groups. *Invent. Math.* **101** (1990), 449–482.

[Ra5] M. Ratner. On measure rigidity of unipotent subgroups of semisimple groups. *Acta Math.* **165** (1990), 229–309.

[Ra6] M. Ratner. On Raghunathan's measure conjecture. *Ann. of Math.* **134** (1991), 545–607.

[Ra7] M. Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.* **63** (1991), no. 1, 235–280.

[Ve1] W. Veech, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.*, **15** (1982), 201–242.

Positive Density in Apollonian Circle Packings

Elena Fuchs

In the first picture in Figure 1 there are three mutually tangent circles packed in a large circle on the outside, with four curvilinear triangles inbetween. By an old theorem (circa 200 BC) of Apollonius of Perga, there are precisely two circles tangent to all of the circles in a triple of mutually tangent circles. One can therefore inscribe a unique circle into each curvilinear triangle as in the second picture in Figure 1. Since this new picture has many new curvilinear triangles, we can continue packing circles in this way – this process continues indefinitely, and we thus get an infinite packing of circles known as the Apollonian circle packing (ACP).

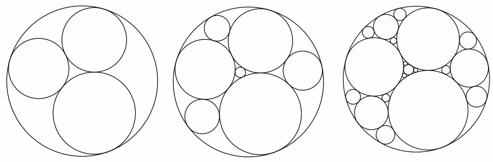


FIGURE 1. Apollonian Circle Packings

A remarkable feature of these packings is that, given a packing in which the initial four mutually tangent circles have integer curvature (reciprocal of the radius), all of the circles in the packing will have integer curvature as well – we refer to such packings as integer ACP's. There are various problems associated to the diophantine properties of integer ACP's which are addressed in [GLMWY] by the five authors Graham, Lagarias, Mallows, Wilks, and Yan. They make considerable progress in treating the problem, and ask several fundamental questions many of which are now solved and discussed further in [S1], [F2], [FS], [KO], and [S2].

In most of these papers, ACP's are studied using a convenient representation of the curvatures appearing in an ACP as maximum-norms of vectors in an orbit of a specific subgroup A of the orthogonal group $\mathrm{O}(3,1)$. This group-orbit description of ACP's was derived by Hirst in [H] from the following theorem:

Theorem 1. *(Descartes, 1643): Let a, b, c, and d denote the curvatures of four mutually tangent circles, where a circle has negative curvature iff it is internally tangent to the other three. Then*

$$Q(a,b,c,d) := 2(a^2+b^2+c^2+d^2) - (a+b+c+d)^2 = 0. \tag{1}$$

For a proof of this, see [Cx]. The Apollonian group A is then a subgroup of $\mathrm{O}_Q(\mathbb{Z})$, generated by the involutions

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix},$$

The curvatures appearing in any given ACP are then simply the coordinates of the vectors in an orbit $A\mathbf{v}$, and so most number-theoretic problems regarding ACP's are reduced to counting points in such orbits. Two crucial properties of A are that

1) A is an infinite index subgroup of $\mathrm{O}_Q(\mathbb{Z})$, and so there are infinitely many integer ACP's. As a subgroup of isometries of hyperbolic 3-space, A acts with an infinite volume fundamental domain on $\mathbb{H}^3$.
2) A is Zariski dense in $\mathrm{O}_Q(\mathbb{C})$.

The first of these properties makes counting points in the orbit of A very difficult using classical methods such as the theory of automorphic forms. The second property, however, indicates a certain richness in the group's orbits – in particular, it allows for the use of the affine linear sieve as developed in [BGS] to tackle related counting problems. In order to carry out this sieve, it is necessary to understand the structure of the orbits of A modulo square free integers $d > 1$ – namely, we require an analog of the Chinese Remainder Theorem for this orbit in affine space. The analysis of orbits modulo integers d (not necessarily square free) is carried out in [F2], and relies heavily on the second property above. It is summarized in the following theorem.

Theorem 2. *(F, '09): Let $\mathcal{O}$ be an orbit of A acting on a root quadruple*[1] *of a packing, and let $\mathcal{O}_d$ be the reduction of this orbit modulo an integer $d > 1$. Let $C = \{\mathbf{v} \neq \mathbf{0}\,|Q(\mathbf{v}) = 0\}$ denote the cone of solutions to (1) without the origin, and let C_d be C over $\mathbb{Z}/d\mathbb{Z}$:*

$$C_d = \{\mathbf{v} \in \mathbb{Z}/d\mathbb{Z}\,|\mathbf{v} \not\equiv \mathbf{0}\,(d),\, Q(\mathbf{v}) \equiv 0\,(d)\}$$

Write $d = d_1 d_2$ with $(d_2, 6) = 1$ and $d_1 = 2^n 3^m$ where $n, m \geq 0$. Write $d_1 = v_1 v_2$ where $v_1 = gcd(24, d_1)$. Then

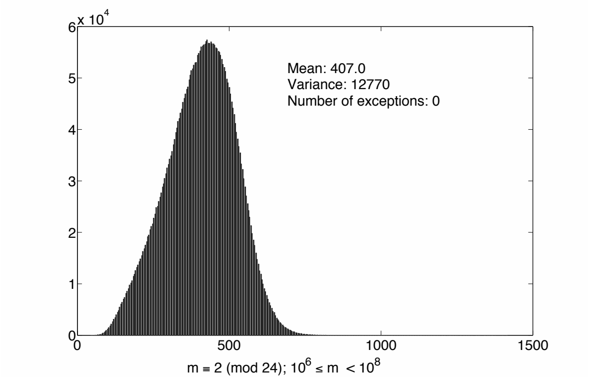
(i) *The natural projection $\mathcal{O}_d \longrightarrow \mathcal{O}_{d_1} \times \mathcal{O}_{d_2}$ is surjective.*
(ii) *Let $\pi : C_{d_1} \to C_{v_1}$ be the natural projection. Then $\mathcal{O}_{d_1} = \pi^{-1}(\mathcal{O}_{v_1})$.*
(iii) *The natural projection $\mathcal{O}_{d_2} \longrightarrow \prod_{p^r || d_2} \mathcal{O}_{p^r}$ is surjective and $\mathcal{O}_{p^r} = C_{p^r}$.*

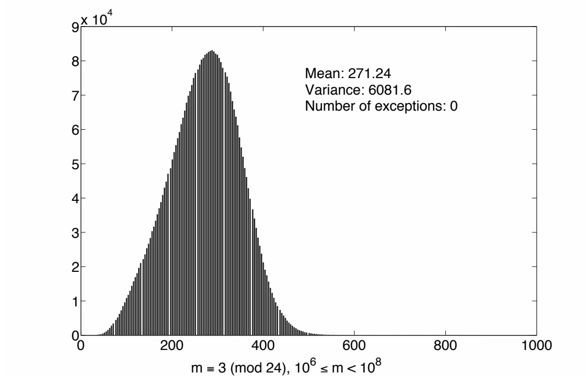
This is precisely the analog of the Chinese Remainder Theorem needed to sieve over the orbit of A. It also implies that the only local obstruction for any orbit of A is modulo 24 – knowing a given orbit modulo 24 is enough to derive the orbit modulo any integer from the structure of the cone C above. A conjecture which was first formulated by Graham et.al. in [GLMWY] and made more precise in [FS] is the following local to global principle for ACP's:

Conjecture 1. *(Graham-Lagarias-Mallows-Wilks-Yan '04, F-Sanden '10): Let P be an integral ACP and let P_{24} be the set of residue classes mod 24 of curvatures in P. Then there exists $X_P \in \mathbb{Z}$ such that any integer $x > X_P$ whose residue mod 24 lies in P_{24} is in fact a curvature of a circle in P.*

This conjecture seems very difficult to prove at this time – it is comparable in difficulty to Hilbert's 11th problem, yet experimental data suggests it is believable. The following histograms illustrate the distribution of the frequencies with which each integer in the given range satisfying the specified congruence condition occurs as a curvature in the packing with root quadruple $(-1, 2, 2, 3)$. Note that there are no exceptions to the local to global principle in this range whenever 0 is not a frequency represented in the histogram (i.e. each integer occurs at least once).

[1] A root quadruple of a packing P is essentially the 4-tuple of the curvatures of the largest four circles in P. It is well defined and its properties are discussed in [GLMWY].





A more feasible task is to show that the integers appearing as curvatures in any integer ACP make up a positive fraction of all integers. This is done in [BF] and is summarized in the following theorem.

Theorem 3. *(Bourgain, F '10): Let $\kappa(P, X)$ denote the number of distinct curvatures less than X of circles in an integer Apollonian packing P. Then for X large we have*

$$\kappa(P, X) \gg X$$

where the implied constant depends on the packing P.

This is shown by counting curvatures in different "subpackings" of an ACP. Namely, we fix a circle C_0 of curvature a_0 and investigate which integers occur as curvatures of circles tangent to C. This gives the preliminary lower bound $\kappa(P, X) \gg X/\sqrt{\log X}$ which was first proven by Sarnak in [S1]. The essential observation which leads to this lower bound is that the set of integers appearing as curvatures of circles tangent to C_0 contain the integers represented by an inhomogeneous binary quadratic form

$$f_{a_0}(x, y) - a_0$$

of discriminant $-4a_0^2$. To prove Theorem 3, we repeat this method for a subset of the circles which we find are tangent to C_0 in this way. For every circle C of curvature a tangent to C_0, we can produce a shifted binary quadratic form

$$f_a(x, y) - a$$

where f_a has discriminant $-4a^2$ and consider the integers represented by f_a. We consider a in a suitably reduced subset of $[(\log X)^2, (\log X)^3]$ and count the integers represented by $f_a - a$ for a in this subset. It is important to note that the integers represented by f_a and $f_{a'}$ for $a \neq a'$ are a subset of integers which can be written as a sum of two squares since both forms have discriminant of the form $-\delta^2$. In fact, f_a and $f_{a'}$ represent practically the same integers (see [F3] for a more detailed discussion). It is rather the *shift* of each form f_a by a that makes the integers found in this way vary significantly. Our final step is to give an upper bound on the number of integers in the intersection

$$\{m \text{ represented by } f_a - a\} \cap \{m' \text{ represented by } f_{a'} - a'\}$$

In obtaining this upper bound, we count integers with multiplicity using a version of the circle method from [N], which is a sacrifice we can afford to make for our purposes. This method is easily generalizable to many discrete linear algebraic groups acting on $\mathbb{H}^3$ with an integral orbit. If the group's orbit contains certain nice suborbits as in the Apollonian case, one may restrict to counting in these suborbits as explained above and thus yield a comparable lower bound on the number of integers less than X in the full orbit of the group (counted without multiplicity).

References

[Be] P. Bernays, *Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht quadratischen Diskriminante*, Ph.D. dissertation, Georg-August-Universität, Göttingen, Germany (1912)

[BG] V. Blomer, A. Granville, *Estimates for representation numbers of quadratic forms*, Duke Mathematical Journal, Vol. 135, No 2, pp. 261-302 (2006)

[BF] J. Bourgain, E. Fuchs, *The positive density conjecture for integer Apollonian circle packings*, http://www.math.princeton.edu/~efuchs

[BGS] J. Bourgain, A. Gamburd, P. Sarnak, *Affine linear sieve, expanders, and sum-product*, Invent. Math. **179** No. 3, pp. 559-644 (2010)

[Cx] H.S.M. Coxeter, *An absolute property of four mutually tangent circles*, *Non-Euclidean Geometries, Jaános Bolyai Memorial Volume* (eds. A. Prékopa and E. Molnàr), Kluwer Academic Pub. (2005)

[F1] E. Fuchs, *Arithmetic properties of Apollonian circle packings*, Ph. D. Thesis, Princeton (2010)

[F2] E. Fuchs, *Strong approximation in the Apollonian group*, in preparation

[F3] E. Fuchs, *A note on the density of curvatures in integer Apollonian circle packings*, preprint, http://www.math.princeton.edu/~efuchs (2009)

[FS] E. Fuchs, K. Sanden, *Some experiments with integral Apollonian circle packings*, preprint, http://www.math.princeton.edu/~efuchs (2010)

[GLMWY] R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H. Yan, *Apollonian circle packings: number theory*, J. of Number Theory, vol 100 (1) pp. 1-45 (2003)

[H] K.E. Hirst, *The Apollonian packing of circles*, Proc. Nat. Acad. Sci. USA, 29, pp. 378-384 (1943)

[KO] A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*, preprint, http://arxiv.org/pdf/0811.2236 (2008)

[N] N. Niedermowwe, *A version of the circle method for the representation of integers by quadratic forms*, preprint arXiv:0905.1229v1 (2009)

[S1] P. Sarnak, *Letter to Lagarias*, http://www.math.princeton.edu/sarnak (2008)

[S2] P. Sarnak, *MAA Lecture on Apollonian Circle Packings*, http://www.math.princeton.edu/sarnak (2009)

Density, Duality, Spectra

Anish Ghosh

(joint work with Alexander Gorodnik and Amos Nevo)

We describe joint work with Alexander Gorodnik and Amos Nevo ([1], [2]). We consider the very general problem of Diophantine approximation on homogeneous varieties of semisimple groups and obtain bounds which are optimal in a variety of cases. The motivation for our work comes from results and conjectures of M. Waldschmidt ([5], [6]) about the effective density of rational points on abelian varieties.

Our main tools are the ergodic theory of group actions, bounds on the automorphic spectrum of the ambient group, and a method we have christened "duality" which provides a tight connection between Diophantine approximation and the behaviour of certain orbits on adelic quotients of semisimple groups. We refer the reader to [1] for precise statements, which are necessarily technical. Further, in [2] we study metric Diophantine approximation on homogeneous varieties, proving in particular a version of the Khintchine-Groshev[1] theorem. Another noteworthy and in our opinion surprising aspect of our work is that it provides a precise relationship between automorphic bounds and Diophantine approximation. In particular, we are able to obtain lower bounds for the automorphic spectrum of semisimple groups which are sharp in a number of cases (e.g. SL_n), using Diophantine approximation!

The short nature of this report necessitates brevity. With this in mind, we have chosen to focus on a special case of one of our results and a couple of examples with the intention of conveying the theme of our work. As mentioned above, we invite the reader to peruse our preprint for more general statements and proofs.

1. Example 1

Let K be a totally real number field, V its set of places O its ring of integers, and S a proper subset of the set of infinite places of K. The Hilbert modular group $\mathrm{SL}_2(O)$ is a dense subgroup of $\prod_{v\in S}\mathrm{SL}(2,K_v)$ and we would like to quantify this density. To do so, we introduce the following height on the rational points of $\mathrm{SL}(2,K)$ (in fact the same definition works for any affine variety, in particular Example 2 below):

$$\mathrm{H}(g) := \prod_{v\in V}\max(1,|g_i|_v)$$

and the following distance on $\mathrm{SL}_2(K_v)$:

$$\|x-y\|_v := \max_i |x_i - y_i|_v.$$

Assuming the Ramanujan-Petersson conjectures [4], we prove that for almost every $x_v \in \mathrm{SL}_2(K_v)$ with $v\in S, \delta>0$, and $\epsilon_v \in (0,\epsilon_0(x,\delta))$, there exists $z\in \mathrm{SL}_2(O)$ such that

$$\|x_v - z\|_v \le \epsilon_v \text{ for } v\in S \text{ and } \mathrm{H}(z) \le \prod_{v\in S}\epsilon_v^{-\frac{3}{2}-\delta}.$$

Moreover, the exponent $\frac{3}{2}$ is the best possible in the sense that we can (using elementary covering arguments) deduce a lower bound with the same exponent. Using the best currently known estimates towards the Ramanujan-Petersson conjecture (see [3]), we get unconditional solutions to the above inequalities with

$$\mathrm{H}(z) \le \epsilon^{-\frac{27}{14}-\delta}.$$

[1] In fact, we prove weak versions of the Duffin-Schaeffer conjecture, i.e. no monotonicity condition on the approximating function is assumed.

For *every* $x_v \in \mathrm{SL}_2(K_v)$ with $v \in S$ such that $\|x_v\|_v \le r, \delta > 0$, and $\epsilon_v \in (0, \epsilon_v(r,\delta))$, there exists $z \in \mathrm{SL}_2(O)$ such that

$$\|x - z\|_v \le \epsilon \text{ for } v \in S \text{ and } \mathrm{H}(z) \le \prod_{v \in S} \epsilon_s^{-\frac{27}{7}-\delta}.$$

We now state a special case of our results for group varieties. Let K be a number field, V the set of places of K and $\mathbb{A}$ the ring of adeles of K. Let G be a connected almost simple algebraic K-group. We fix a maximal compact subgroup U_v of $G(K_v)$ which is good for almost all places v. The group $G(K)$ is a lattice in $G(\mathbb{A})$. We denote by $L^2_{00}(G(\mathbb{A})/G(K))$ the subspace of $L^2(G(\mathbb{A})/G(K))$ orthogonal to all automorphic characters. The translation action of the group G_v on $G(\mathbb{A})/G(K)$ defines the unitary representation π_v of G_v on $L^2_{00}(G(\mathbb{A})/G(K))$. We define the spherical integrability exponent of π_v w.r.t. U_v as follows

$$q_v(G) := \inf\{q > 0 \ : \ \forall\, U_v-\text{invariant } w \in L^2_{00}(G(\mathbb{A})/G(K)), \langle \pi_v(g)w, w\rangle \in L^q(G(K_v))\}.$$

This definition extends naturally to a subset S of V:

$$q_S(G) := \sup_{v \in S} q_v(G).$$

Given a subset S of V, we define also the exponent $\mathfrak{a}_S(G)$ by

$$\mathfrak{a}_S(G) := \sup \inf_{O \supset Y} \limsup_{h\to\infty} \log \frac{A_S(O,h)}{\log h}.$$

Here the supremum is taken over bounded $Y \subset \prod_{v\in S} G(K_v)$ and

$$A_S(O,h) := |\{z \in G(K) \ : \ \mathrm{H}(z) \le h, z \in O\}|.$$

Given $S \subset V$, a finite subset S' of S, $x \in \prod_{v \in S} G(K_v)$ and $(\epsilon_v)_{v\in S'}$ we define

$$\omega_S(x, (\epsilon_v)_{v\in S'}) := \min\{\mathrm{H}(z) : z \in G(O_{(V\setminus S)\cup S'}), |x_v - z|_v \le \epsilon_v, v \in S'\}.$$

The function ω_S quantifies the approximation property on the group variety with respect to S. Thus, the problem of Diophantine approximation on such varieties can be recast as the problem of obtaining bounds for ω_S.

Theorem 1. [1] *Let G be a connected simply connected almost simple algebraic K-group and S a finite subset of V such that G is isotropic over $V\setminus S$. Then there exists a subset Y of full measure in $\prod_{v\in S} G(K_v)$ such that for every $\delta > 0$, finite $S' \subset S$, $x \in Y$, and $(\epsilon_v)_{v\in S'} \in (0, \epsilon_0(x, v, \delta))$ we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \le \prod_{v\in S'} \left(\epsilon_v^{-\frac{\dim G}{\mathfrak{a}_S(G)}-\delta}\right)^{q_{V\setminus S}/2}.$$

2. Example 2

We end with an example in the setting of homogeneous varieties of groups. Let S^d denote the d dimensional unit sphere in $\mathbb{R}^{d+1}$. Then for almost every $x \in S^2(\mathbb{R}), \delta > 0$, and $\epsilon \in (0, \epsilon_0(x,\delta))$, there exists $z \in S^2(\mathbb{Z}[1/p])$ such that

$$\|x - z\|_\infty \le \epsilon \text{ and } \mathrm{H}(z) \le \epsilon^{-2-\delta}.$$

Moreover this exponent is again the best possible. To cast these estimates in terms of group actions, we consider the group $G \backsimeq D^\times/Z^\times$, where D denotes Hamilton's quaternion algebra and Z the centre of D. This group naturally acts on the variety of pure quaternions of norm one, which can be identified with S^2.

References

[1] A. Ghosh, A. Gorodnik and A. Nevo, *Diophantine approximation and automorphic spectrum*, preprint, http://front.math.ucdavis.edu/1007.0593.
[2] A. Ghosh, A. Gorodnik and A. Nevo, *Density, Duality, Spectra*, in preparation.
[3] H. Kim, *Functoriality for the exterior square of* GL(4) *and the symmetric fourth of* GL(2), with appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P.Sarnak. J. Amer. Math. Soc. 16 (2003), no. 1, 139–183.
[4] P. Sarnak, *Notes on the generalized Ramanujan conjectures*, In Harmonic analysis, the trace formula, and Shimura varieties, volume 4 of Clay Math. Proc., pages 659685. Amer. Math. Soc., Providence, RI, 2005.
[5] M. Waldschmidt, *Density measure of rational points on Abelian varieties*, Nagoya. Math. J. 155 (1999), 27-53.
[6] M. Waldschmidt, *Open Diophantine problems.* Mosc. Math. J. 4 (2004), no. 1, 245–305.

Property tau, counting lattice points, and applications

Alexander Gorodnik
(joint work with Amos Nevo)

Let F denote a number field equipped with the set V_F of the absolute values of F extended the standard normalised absolute values of the rational numbers and F_v, $v \in V_F$, denote the corresponding local fields.

We introduce local and global heights. For Archimedean $v \in V_F$, and for $x = (x_1, \dots, x_d) \in F_v^d$, we set

$$\mathrm{H}_v(x) = \left(|x_1|_v^2 + \cdots + |x_d|_v^2\right)^{1/2},$$

and for non-Archimedean v,

$$\mathrm{H}_v(x) = \max\{|x_1|_v, \dots, |x_d|_v\}.$$

For $x = (x_1, \dots, x_d) \in F^d$, we set

$$\mathrm{H}(x) = \prod_{v \in V_F} \mathrm{H}_v(x).$$

Let S be a finite subset of V_F containing all Archimedean absolute values, and

$$O_S = \{x \in F : |x|_v \le 1 \text{ for } v \notin S\}$$

is the ring of S-integers in F. We consider a system X of polynomial equations with coefficients in O_S. Given an ideal $\mathfrak{a}$ of O_S, we denote by $\mathsf{X}^{(\mathfrak{a})}$ the system of polynomial equations over the factor-ring $O_S/\mathfrak{a}$ obtained by reducing X modulo $\mathfrak{a}$. There is a natural reduction map

$$\pi_{\mathfrak{a}} : \mathsf{X}(O_S) \to \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a}).$$

The question whether a solution in $\mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ can be lifted to an integral solution in $\mathsf{X}(O_S)$ is of fundamental importance in number theory. It is closely related to the strong approximation property for algebraic varieties (see [PR]). For instance, if G is a connected F-simple simply connected algebraic group which is isotropic over S, then it satisfies the strong approximation property (see [PR]) and, in particular, the map $\pi_{\mathfrak{a}}$ is surjective in this case. For more general homogeneous varieties, the map $\pi_{\mathfrak{a}}$ does not have to be surjective, but the image $\pi_{\mathfrak{a}}(\mathsf{X}(O_S))$ can be described using the Brauer–Manin obstructions (see [Hr, CTX, BD]).

Theorem 1. *Let X be an affine variety defined over F and equipped with a transitive F-action of a connected F-simple algebraic group. Then there exists a finite subset of V_F, containing all Archimedean absolute values, such that for every finite set $S \supset S_0$ there exist $c_S, \sigma_S > 0$ such that for every ideal $\mathfrak{a}$ of O_S and every $\bar{x} \in \pi_{\mathfrak{a}}(\mathsf{X}(O_S))$, there exists $x \in \mathsf{X}(O_S)$ such that*

$$\pi_{\mathfrak{a}}(x) = \bar{x} \quad \textit{and} \quad \mathrm{H}(x) \leq c_S\, |O_S/\mathfrak{a}|^{\sigma_S}.$$

Given an affine variety X defined over a number field F, we set

$$N_T(\mathsf{X}(O_S)) = |\{x \in \mathsf{X}(O_S) :\, \mathrm{H}(x) \leq T\}|$$

where O_S is a ring of S-integers in F. We will be interested in producing an upper estimate on $N_T(\mathsf{Y}(O_S))$ for proper affine subvarieties Y of X. Although one might naively expect that for irreducible X,

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{X},\deg(\mathsf{Y})} N_T(\mathsf{X}(O_S))^{1-\sigma_{\mathsf{Y}}}$$

with $\sigma_{\mathsf{Y}} > 0$, where we write $\deg(\mathsf{Y})$ for the degree of the projective closure of Y, this estimate is false in general as can be demonstrated by the variety $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$, where most of rational point lie on lines. Nonetheless, in the case of group varieties we have the following:

Theorem 2. *Let G be a connected F-simple simply connected algebraic group defined over a number field F. Let $S \subset V_F$ be a finite subset containing all Archimedean absolute values such that G is isotropic over S. Then there exists $\sigma = \sigma(\mathsf{G}, S, \dim(\mathsf{X})) \in (0,1)$ such that for every absolutely irreducible proper affine subvariety Y of G defined over F, we have*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\deg(\mathsf{Y})} N_T(\mathsf{G}(O_S))^{1-\sigma}$$

as $T \to \infty$.

Let G be a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$ and $\mathsf{G} \to \mathrm{GL}_n$ a representation of G which is also defined over $\mathbb{Q}$. Fix $v \in \mathbb{Z}^n$. We assume that $\mathsf{X} = \mathsf{G}v$ is Zariski close, and $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is connected and has no nontrivial characters. Then the coordinate ring $\mathbb{C}[\mathsf{X}]$ is a unique factorisation domain. Let f be a regular function on X defined over $\mathbb{Q}$ such that it has a decomposition into irreducible factors $f = f_1 \cdots f_t$ where all f_i's are distinct and defined over $\mathbb{Q}$. Let $\mathcal{O} = \Gamma v$ be the orbit of $\Gamma = \mathsf{G}(\mathbb{Z})$. We assume that f takes integral values on $\mathcal{O}$ and is weakly primitive (that is, $\gcd(f(x) : x \in \mathcal{O}) = 1$). The saturation number $r_0(\mathcal{O}, f)$ of the pair $(\mathcal{O}, f)$ is the least r such that the set of

$x \in \mathcal{O}$ for which $f(x)$ has at most r prime factors is Zariski dense in X, which is the Zariski closure of $\mathcal{O}$ by the Borel density theorem. It is natural to ask whether the saturation number $r_0(\mathcal{O}, f)$ is finite and establish quantitative estimates on the set $\{x \in \mathcal{O} : f(x) \text{ has at most } r \text{ prime factors}\}$.

We fix a norm on $\mathbb{R}^n$ and set $\mathcal{O}(T) = \{w \in \mathcal{O} : \|w\| \le T\}$. It was shown in [NS] that for $\mathsf{X} \simeq \mathsf{G}$, the saturation number is finite and there exists explicit $r \ge 1$ such that

$$(1) \qquad |\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg_{f,\mathcal{O}} \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$. As remarked in [BGS, NS], the assumption that $\mathsf{X} \simeq \mathsf{G}$ is not crucial if only finiteness of the saturation number is concerned, and $r_0(\mathcal{O}, f)$ is finite for general orbits. However, the sharp lower estimate (1) seems to be more delicate, and so far it has only been established for 2-dimensional quadratic surfaces [LS] and for group varieties [NS]. Our goal here is to prove (1) for general symmetric varieties.

Theorem 3. *Let $\mathcal{O}$ and f be as above and assume in addition that $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is symmetric (that is, L is the set of fixed points of an involution of G). Then there exists $r \ge 1$ such that*

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg_{f,\mathcal{O}} \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$.

References

[BD] M. Borovoi and C. Demarche, Manin obstruction to strong approximation for homogeneous spaces. ArXiv:0912.0408v1.

[BGS] J. Bourgain, A. Gamburd, and P. Sarnak, Affine linear sieve, expanders, and sum-product. Invent. Math. 179 (2010), no. 3, 559–644.

[CTX] J.-L. Colliot-Thélène and F. Xu, Brauer–Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms. Compositio Math. 145 (2009), no. 2, 309–363.

[Hr] D. Harari, Le défaut d'approximation forte pour les groupes algébriques commutatifs. Algebra Number Theory 2 (2008), no. 5, 595–611.

[LS] J. Liu and P. Sarnak, Almost primes on quadrics in 3 variables. Preprint.

[NS] A. Nevo and P. Sarnak, Prime and Almost Prime Integral Points on Principal Homogeneous Spaces. ArXiv:0902.0692v1.

[PR] V. Platonov and A. Rapinchuk, Algebraic groups and number theory. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, 1994.

Distribution of flows on nilmanifolds and applications to additive combinatorics

Ben Green

The talk was divided into two parts, aimed at giving brief answers to the following two questions:

Q1. Why are flows on nilmanifolds relevant to the subject of additive combinatorics, which is concerned with questions about arithmetic progressions and suchlike?

Q2. What specific dynamical facts about flows on nilmanifolds are required in order to study Q1 effectively?

Let us begin with Q1. For the sake of illustration, let us suppose that we have some set $A \subseteq [N]$ with $|A| = \alpha N$ and that we wish to enumerate the number of four-term arithmetic progressions $x, x+d, x+2d, x+3d$ in A. In this regard it is natural to introduce the quartilinear operator

$$T(f_1, f_2, f_3, f_4) := \mathbb{E}_{x,d \in [N]} f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d),$$

the thought perhaps being that we might split $1_A = \alpha + f$ and write $T(1_A, 1_A, 1_A, 1_A)$ as a "main term" α^4 plus fifteen other terms, each involving at least one copy of the function f, which has average value 0 and is presumably, in a typical situation, highly oscillatory and likely to give rise to substantial cancellation.

For this reason it is important to have a way of bounding $T(f_1, f_2, f_3, f_4)$ above. A relatively simple way of doing this is to invoke the *Generalised von Neumann inequality*, which states that

$$|T(f_1, f_2, f_3, f_4)| \leq \|f_i\|_{U^3}$$

for each $i = 1, 2, 3, 4$, provided that $|f_i(x)| \leq 1$ for $i = 1, 2, 3, 4$. Here, $\|f\|_{U^3}$ is an important object in additive combinatorics called the Gowers U^3-norm, and it is defined by

$$\|f\|_{U^3}^8 := \mathbb{E}_{x,h_1,h_2,h_3} f(x)\overline{f(x+h_1)f(x+h_2)f(x+h_3)}\times$$

$$\times f(x+h_1+h_2)f(x+h_1+h_3)f(x+h_2+h_3)\overline{f(x+h_1+h_2+h_3)},$$

a kind of average of f over three-dimensional parallelepipeds. It is true, but not completely obvious, that this does indeed define a norm. The generalised von Neumann theorem is merely a matter of three (judicious) applications of the Cauchy-Schwarz inequality, and as such is not hard to prove. It has the effect of raising a new, key, question:

Question. Suppose that $|f(x)| \leq 1$ for all x. Is there a useful criterion which allows us to assert that $\|f\|_{U^3}$ is "small", say less than δ?

In studying this question it is instructive to first consider the extreme version of it:

Question. When is $\|f\|_{U^3} = 1$?

It is a pleasant exercise to show that this is so if and only if $f(x) = e^{2\pi i\phi(x)}$, where $\phi(x) = \alpha x^2 + \beta x + \gamma$ is a quadratic phase function. This observation immediately reveals that there is quadratic structure underlying four term arithmetic progressions, a fact not obvious at first sight.

It turns out that slightly more exotic types of structure are present too. Indeed it can be checked that the "fake quadratic" $f(x) = e^{2\pi i\alpha x[\beta x]}$ has Gowers U^3 norm at least 0.1; here, $[t]$ denotes the integer part of t. Perhaps surprisingly, this is the only type of structure that can be responsible for having large Gowers norm.

Theorem 1 (Gowers, Green-Tao)**.** [1] *Suppose that $f : [N] \to \mathbb{C}$ is a function with $|f(x)| \leq 1$ for all x, and that $\|f\|_{U^3} \geq \delta$. Then there is a fake quadratic*

$$\phi(x) = \alpha_1 x[\beta_1 x] + \cdots + \alpha_k[\beta_k x],$$

$k = O_\delta(1)$, such that

$$|\mathbb{E}_{x\in[N]} f(x) e^{2\pi i\phi(x)}| \gg_\delta 1.$$

A very important realisation, coming from dynamics, is that fake quadratics can be interpreted as functions on nilmanifolds. To give a relevant example, let $G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$ be the Heisenberg group, and let $\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ be the natural lattice. The homogeneous space G/Γ is a nilmanifold. The key observation is that one can write

$$\alpha n[\beta n] = F(g(n)\Gamma),$$

where $g : \mathbb{Z} \to G$ is the "polynomial map" defined by

$$g(n) := \begin{pmatrix} 1 & \alpha n & 0 \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix}$$

and $F : G/\Gamma \to \mathbb{C}$ is the piecewise Lipschitz function defined by

$$F(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\Gamma) = e(-z)$$

whenever $0 < x, y, z < 1$.

In words, "nilsequences" such as $F(g(n)\Gamma)$ are roughly the same thing as fake quadratics. Indeed there is an alternative formulation of Theorem 1, as follows.

Theorem 2 (Green-Tao)**.** [1] *Suppose that $f : [N] \to \mathbb{C}$ is a function with $|f(x)| \leq 1$ for all x, and that $\|f\|_{U^3} \geq \delta$. Then there is a nilsequence $F(g(n)\Gamma)$ of complexity $O_\delta(1)$ such that*

$$|\mathbb{E}_{x\in[N]} f(x) F(g(x)\Gamma)| \gg_\delta 1.$$

We will not discuss the notion of "complexity" here, but roughly speaking it is a measure of how complicated the Lie group G (which might be more general than the Heisenberg group) is, and how smooth F is.

This theorem provides a bridge between additive combinatorics and dynamical systems on nilmanifolds. It has recently been generalised to higher Gowers norms (defined in the obvious way) by the author, Tao and Ziegler; here, higher-step nilpotent groups are required.

The theorem motivates the study of sequences $(g(n)\Gamma)_{n\in[N]} \subseteq G/\Gamma$, and in particular their distribution properties. In joint work with Tao we prove the following two results, stated here in a very rough form:

Theorem 3 (Green-Tao)**.** [2] *We have the following statements.*

(1) $(g(n)\Gamma)_{n\in[N]}$ *is close to equidistributed if and only if its abelianization, the projection to* $G/[G,G]\Gamma$*, is equidistributed in that torus.*
(2) *Possibly after passing to a large subprogression* $P \subseteq [N]$, $(g(n)\Gamma)_{n\in P}$ *is close to equidistributed on some closed subnilmanifold* $H/H\Gamma$ *of* G/Γ*.*

This theorem (when properly formulated of course) represents a quantitative version of results of Leon Green and Sasha Leibman in dynamical systems.

Using this result and some techniques from analytic number theory, we establish the following.

Theorem 4 (Green-Tao)**.** [3] *The Möbius function is asymptotically orthogonal to all nilsequences, in the sense that*

$$\mathbb{E}_{n\leq N}\mu(n)F(g(n)\Gamma) \to 0$$

as $N \to \infty$*.*

Using this, we establish some conjectures of Hardy and Littlewood concerning the frequency of various linear patterns in the prime numbers.

None of this work would have been possible without insights coming from dynamics and, in particular, the work of Furstenberg, Weiss, Host, Kra, Leibman and Ziegler.

References

[1] B. J. Green and T. C. Tao, *An inverse theorem for the Gowers* U^3*-norm,* Proc. Edinburgh Math. Soc. **51** (2008), no. 1, 73-153.
[2] B. J. Green and T. C. Tao, *The quantitative distribution of polynomial orbits on nilmanifolds, with applications,* to appear in Annals of Math.
[3] B. J. Green and T. C. Tao, *The Möbius function is asymptotically orthogonal to nilsequences,* to appear in Annals of Math.

Progress on Affine Sieves

Alex V. Kontorovich

Recently Bougain, Gamburd and Sarnak [BGS06, BGS08] introduced the Affine Linear Sieve, which concerns the application of various sieve methods to the setting of (possibly thin) orbits of groups of morphisms of affine n-space. Applications include the following

Theorem 1 ([Kon09, KO09])**.** *Let* Q *be the indefinite ternary quadratic form* $Q(\mathbf{x}) = x^2+y^2-z^2$*, let* $G = SO_{\mathbb{R}}(Q)$ *be the real special orthogonal group preserving* Q*, and let* C *be the cone defined by* $Q = 0$*. Take some primitive Pythagorean triple* $v_0 \in C(\mathbb{Z})$*, say* $v_0 = (3,4,5)$*, and let* $\Gamma < G(\mathbb{Z})$ *be a discrete, finitely-generated,* *non-elementary subgroup whose critical exponent δ is sufficiently close to* 1. *Let* $\mathcal{O} := v_0 \cdot \Gamma$ *be the orbit of* v_0 *under* Γ. *Then*

(1) *There is an infinitude of* $(x, y, z) \in \mathcal{O}$ *such that the hypotenuse* z *is the product of at most* 14 *primes.*
(2) *There is an infinitude of* $(x, y, z) \in \mathcal{O}$ *such that the area* $\frac{1}{2}xy$ *is the product of at most* 25 *primes.*
(3) *There is an infinitude of* $(x, y, z) \in \mathcal{O}$ *such that the value* xyz *is the product of at most* 29 *primes.*

A standard conjecture on the cancellation of the Möbius function predicts that the number 14 above can be replaced by 1, that is, the set of hypotenuses should contain an infinitude of primes. That said, Selberg identified the so-called "parity barrier," that sieves alone cannot distinguish between sets having an even or odd number of prime factors. Hence the best one can hope for using only sieve techniques is to replace 14 by 2. In his resolution of the ternary Goldbach problem, Vinogradov introduced methods using bilinear forms estimates, thereby overcoming the parity barrier. In joint work with Jean Bourgain [BK10], we have introduced bilinear forms into the Affine Sieve to produce primes in thin orbits.

Theorem 2 ([BK10])**.** *Let* $\mathcal{O}$ *be as above. Then there exists an infinitude of* $(x, y, z) \in \mathcal{O}$ *such that* $\sqrt{y + z}$ *is prime.*

Note that the quantity $\sqrt{y + z}$ above is integral, as follows from the ancient parametrization that, for a primitive $(x, y, z) \in \mathcal{O}$, there exists a primitive $(u, v) \in \mathbb{Z}^2$ of opposite parity such that

$$x = u^2 - v^2, \qquad y = 2uv, \qquad z = u^2 + v^2.$$

More is true.

Theorem 3 ([BK10])**.** *Let* $\mathcal{O}$ *be as above, and let* S *denote the set of all* $n = \sqrt{y + z}$ *for* $(x, y, z) \in \mathcal{O}$. *Then* S *almost satisfies a local-global principle, with a power savings in the exceptional set, by which we mean the following. For* $N > 1$, *let* $E(N)$ *denote the set of exceptions up to* N, *that is,* $E(N)$ *contains the set of all* $n \in [-N, N]$ *such that* n *is admissible (meaning* $n \in S \pmod{q}$ *for all* $q \geq 1$*) but nevertheless,* n *is not in* S. *Then for some* $\eta > 0$,

$$|E(N)| \ll N^{1-\eta},$$

as $N \to \infty$.

A key input in the above is a counting statement for sectors in infinite volume with power savings error terms and uniformity over cosets of congruence subgroups. This is carried out in joint work with Jean Bourgain and Peter Sarnak in [BKS10].

References

[BGS06] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Sieving and expanders. *C. R. Math. Acad. Sci. Paris*, 343(3):155–159, 2006.

[BGS08] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Affine linear sieve, expanders, and sum-product, 2008.

[BK10] J. Bourgain and A. Kontorovich. On representations of integers in thin subgroups of $\mathrm{SL}(2, \mathbf{Z})$, 2010. To appear, GAFA, *http://arxiv.org/abs/1001.4534*.

[BKS10] J. Bourgain, A. Kontorovich, and P. Sarnak. Sector estimates for hyperbolic isometries, 2010. To appear, GAFA. *http://arxiv.org/abs/1001.4541*.

[KO09] A. Kontorovich and H. Oh. Almost prime Pythagorean triples in thin orbits, 2009. Preprint, *http://arxiv.org/abs/1001.0370*.

[Kon09] A. V. Kontorovich. The hyperbolic lattice point count in infinite volume with applications to sieves. *Duke J. Math.*, 149(1):1–36, 2009. *http://arxiv.org/abs/0712.1391*.

Equidistribution and the large sieve

Emmanuel Kowalski

Although sieve methods have traditionally been used to study prime numbers and other sets of integers defined by multiplicative conditions, they can be used in principle in much greater generality. In the book [8], for instance, general forms of "sifted" sets are considered:

$$S = \{x \in X \mid \text{for all } \ell \in \mathcal{L},\ \rho_\ell(F(x)) \notin \Omega_\ell\}$$

where

- X is a set of "global" objects of interest; the emphasis is in obtaining quantitative estimates, and therefore it is assumed that X carries a finite measure μ, so that $\mu(S)$ becomes the quantity of most interest;
- $\mathcal{L}$ is a finite set of parameters used to describe "local" conditions satisfied by the global objects when sifted;
- F is a map from X to a set Y, typically infinite, which represents the data for which we have local information;
- for each ℓ, ρ_ℓ is a surjective map from Y to a finite set Y_ℓ, representing the local information;
- $\Omega_\ell \subset Y_\ell$ is any set that defines what local information are excluded from the sifted objects.

This generalizes the simplest examples, such as

$$S_2 = \{n \leq N \mid \text{for all primes } p \leq z,\ n \,(\mathrm{mod}\, p) \notin \{0, -2\} \subset \mathbb{Z}/p\mathbb{Z}\},$$

which, when z is $\sqrt{N}$, is precisely the set of twin primes $> \sqrt{N}$. Here X is the set of integers up to N, μ the counting measure, $Y = \mathbb{Z}$ and the local information is given by reduction modulo primes.

A more interesting example, considered in joint works with F. Jouve and D. Zywina [7], is the following:

- (X, μ) is a probability space;
- $Y = \mathbb{G}(\mathbb{Z})$ is the arithmetic group of integral points of a split semisimple, simply-connected, Chevalley group $\mathbb{G}/\mathbb{Z}$, for instance $SL(n, \mathbb{Z})$;
- $F = \gamma_k = \xi_1 \cdots \xi_k$ is the random variable on X which is the k-th step of a random walk on Y with steps (ξ_i) which are independent and, for instance, uniformly distributed on a fixed finite symmetric generating set T of Y (note that here F is not injective);

- the local information are given by reduction modulo primes $\rho_l : Y \to Y_l = \mathbb{G}(\mathbb{F}_l)$.

In that case, fixing a faithful representation

$$\pi : \mathbb{G} \to GL(N),$$

one can construct suitable choices of Ω_l, related to the conjugacy classes of the Weyl group W of $\mathbb{G}$, lead to an inclusion of the type

$$\{x \in X \mid \det(T - \pi(\gamma_k)) \text{ has Galois group not } W\} \subset \bigcup_c S_c$$

where c runs over those conjugacy classes and S_c is a sifted set depending on c. Thus a good upper bound for the measure of these S_c will lead to an upper bound on the probability that the splitting field of $\pi(\gamma_k)$ has Galois group different from W (it follows from simple facts about the structure theory of $\mathbb{G}$ that W is the "biggest" possible Galois group here).

These examples are typical of large-sieve questions because, for each c, the sets $\Omega_{c,l}$ defining the sieve have the property that

$$\limsup_{l \to +\infty} \frac{|\Omega_l|}{|Y_l|} > 0,$$

whereas for twin primes $\Omega_p/p \to 0$.

The heuristic underlying estimates for $\mu(S)$ for general sifted sets is that, for a single condition, the local information $\rho_l(F(x))$ is "close" to being equidistributed on Y_l, with respect to some natural probability measure ν_l, so that we expect that

$$\mu(\{x \in X \mid \rho_l(F(x)) \notin \Omega_l\}) \approx \mu(X)(1 - \nu_l(\Omega_l)),$$

and furthermore that distinct conditions imposed for $l \neq l'$ are "independent", so that one should compare $\mu(S)$ with

$$\mu(X) \prod_l (1 - \nu_l(\Omega_l)).$$

In the example above, and in many others related, e.g., to the "affine linear sieve" of Bourgain-Gamburd-Sarnak [2], these facts are obtained from a form of "strong approximation" theorems for subgroups of arithmetic groups over number fields, the independence being crucially linked to the simple-connectedness of the group.

The most crucial ingredient in implementing the general large sieve inequality [8, Ch. 7] in these contexts of groups with exponential growth or hyperbolic nature is the condition that the family of Cayley graphs of Y_l, or indeed of

$$\mathbb{G}(\mathbb{Z}/q\mathbb{Z}) = \prod_{l \mid q} Y_l, \qquad q \text{ squarefree},$$

(with respect to the generating set T) forms an *expander family.* This provides many interesting and sometimes surprising applications of expanders.

Proving the expansion property is often difficult; there are however a number of tools available, which promise many applications in the future:

- If $\mathbb{G}$ has real rank at least 2, this follows from the stronger condition that Y has Property (T); indeed, the first explicit examples of expanders were constructed by Margulis using this link;
- If $Y = \mathbb{G}(\mathbb{Z})$, $Y_l = \mathbb{G}(\mathbb{F}_l)$ in general, this follows from Clozel's proof of Conjecture (τ), generalizing Selberg's proof of the spectral gap for congruence subgroups of $SL(2, \mathbb{Z})$;
- If Y is taken to be a Zariski-dense finitely-generated subgroup of $\mathbb{G}(\mathbb{Z})$, not necessarily of finite index, this is related to the remarkable new works of Helfgott [4], Bourgain-Gamburd-Sarnak [2], Pyber-Szabò [9], Breuillard-Green-Tao [3] and Bourgain-Varjù [1] (among others).

The use of random walks in these problems is technically very convenient, but other natural choices for the set Y are possible:

(1) One may wish to take Y to be the set of elements of norm $\leq x$ for some large x, with respect to some archimedean norm or height arising from an embedding in $GL(n, \mathbb{R})$, for instance; what typically arise then are lattice-point counting problems of hyperbolic nature where it is crucial to obtain a large range of uniformity with respect to the subgroup involved;

(2) One may wish to take Y to be a ball of radius x with respect to a combinatorial distance defined using a generating set T as before; in that case, although the expansion property remains crucial, asymptotic equidistribution of the image modulo l is hard to prove if Y is not free; however, by restricting to a free subgroup, it is possible to obtain good results, as in [2].

References

[1] J. Bourgain and P. Varjù: *Expansion in $SL_d(\mathbb{Z}/q\mathbb{Z})$, q arbitrary*, preprint (2010).
[2] J. Bourgain, A. Gamburd and P. Sarnak: *Affine linear sieve, expanders and sum-product*, Invent. math. 179 (2010), 559–644.
[3] E. Breuillard, B.J. Green and T. Tao: *Linear approximate groups*, preprint (2010).
[4] H. Helfgott: *Growth and generation in $SL_2(\mathbb{Z}/pp\mathbb{Z})$*, Ann. of Math. 167 (2008), 601–623.
[5] E. Hrushovski and A. Pillay: *Definable subgroups of algebraic groups over finite fields*, J. reine angew. Math 462 (1995), 69–91.
[6] H. Iwaniec and E. Kowalski: *Analytic Number Theory*, A.M.S Colloq. Publ. 53, A.M.S (2004).
[7] F. Jouve, E. Kowalski and D. Zywina: *Splitting fields of characteristic polynomials of random elements in arithmetic groups*, in progress.
[8] E. Kowalski: *The large sieve and its applications*, Cambridge Tract. in Math. 175, Cambridge Univ. Press, 2008.
[9] L. Pyber, L. and E. Szabó: *Growth in finite simple groups of Lie type of bounded rank*, preprint (2010).

Homogeneous dynamics and quadratic forms

Gregory Margulis

Let Q be a real nondegenerate indefinite quadratic form in n variables. We say that Q is *rational* if it is a multiple of a form with rational coefficients and *irrational* otherwise. Let us set

$$m(Q) = \inf\{\mid Q(x) \mid : x \in \mathbb{Z}^n, x \neq 0\}.$$

According to the classical Meyer theorem, if Q is rational and $n \geq 5$ then $m(Q) = 0$. In 1929, A. Oppenheim conjectured that if $n \geq 5$ then $m(Q) = 0$ also for irrational Q. Later it was realized that $m(Q)$ should be equal to 0 under a weaker condition $n \geq 3$ (for irrational forms (Q). I proved the Oppenheim conjecture in 1986 by studying orbits of the orthogonal group $SO(2,1)$ on the space $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^3$ (see [5]; preliminary oral announcements had been made already in 1984).
The proof was inspired by a remarkable observation due to M. S. Raghunathan that the Oppenheim conjecture is related to the theory of unipotent flows. Actually, M.S.Raghunathan noticed that the Oppenheim conjecture would follow from a conjecture about closures of orbits of unipotent subgroups. The Raghunathan conjecture states that if H is a connected Lie group, Γ a lattice in G, and $U \subset G$ a connected unipotent subgroup (or, more generally, a connected subgroup generated by unipotent elements), then for any $x \in G/\Gamma$ there exists a closed connected subgroup $L = L(x)$ containing U such that the closure of the orbit Ux coincides with Lx. The main dynamical result in [5] can be considered as the proof of the Raghunathan conjecture in a special case. The technique introduced in [5] and developed later in joint papers of S.G.Dani and myself and in papers by N.Shah gives the proof of Raghunathan's conjecture in many other special cases. It also suggests an approach for proving the Raghunathan conjecture in general. This approach is based on the technique which involves finding orbits of larger subgroups inside closed sets invariant under unipotent subgroups by studying the minimal invariant sets, and the limits of orbits of sequences of points tending to a minimal invariant set.

The Raghunathan conjecture was eventually proved in 1990 in complete generality by M.Ratner (see [9]). Her proof is based on measure rigidity for unipotent flows which she established in 1989 and published in a series of three papers, the last one of which is [8]. The measure rigidity for unipotent flows proves a conjecture by S.G.Dani and gives the classification of finite U-ergodic U-invariant measure on G/Γ for unipotent subgroups U of G (more precisely, any such measure is Haar measure on a closed orbit of a subgroup containing U). Using this classification and some other technique, a number of results on the asymptotic behavior of $N_{a,b}(T)$ were obtained in [1], [2] and [3] where

$$N_{a,b}(T) = \#\{x \in \mathbb{Z}^n : a < Q(x) < b, x \in T\Omega\}$$

and Ω is a star-like bounded domain in $\mathbb{R}^n$, $n \geq 3$, and Q is irrational. Similar results for inhomogeneous forms were obtained in [6] and [7]. It should be mentioned that the approach in [7] is very different from the approach in [6], though it also used the measure rigidity for unipotent flows.

The methods developed in [1], [2], [3], [5], [6], and [7] are not "effective" because they use such notions as minimal set and ergodic measure. Another and completely different approach is used in [4]. This approach is "effective" but it works only for $n \geq 5$. It is based on estimates for certain theta series, and it combines methods introduced by F.Götze for related problems and methods for obtaining some integrability estimates and introduced in [2]. One of the results in [4] is a polynomial type estimate for the size of the smallest nontrivial integral solution of the inequality $| Q(x) | < \varepsilon$ (for the case $n \geq 5$).

References

[1] S. G. Dani and G. Margulis, *Limit distribution of orbits of unipotent flows and values of quadratic forms*, Adv. in Soviet Math 16 (1993), 91-137.

[2] A. Eskin, G. Margulis and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. 147 (1998), 93-141.

[3] ———, *Quadratic forms of signature* $(2,2)$ *and eigenvalue spacings on rectangular* 2*-tori*, Ann. of Math. 161 (2005), 679-725.

[4] F. Götze and G. Margulis, *Distribution of values of quadratic forms at integral points*, preprint arXiv: 1004.5123.

[5] G. Margulis, *Formes quadratiques indéfinies et flots unipotents sur les spaces homogénes*, C.R. Acad. Sci. Paris 304 (1987), 247-253.

[6] G. Margulis and A. Mohammadi, *Quantitative version of the Oppenheim conjecture for inhomogeneous quadratic forms*, preprint arXiv: 1001.2756.

[7] J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math. 158 (2003), 419-471.

[8] M.Ratner, *On Raghunathan's measure conjectue*, Ann. of Math. 134 (1991), 545-607.

[9] ———, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. 77 (1991) 275-382.

Horospheres, Farey fractions and Frobenius numbers

Jens Marklof

Frobenius numbers. Let $\widehat{\mathbb{Z}}^d = \{\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d : \gcd(a_1, \ldots, a_d) = 1\}$ be the set of primitive lattice points, and $\widehat{\mathbb{Z}}^d_{\geq 2}$ the subset with coefficients $a_j \geq 2$. Given $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2}$, the Frobenius number $F(\boldsymbol{a})$ is defined as the largest integer that does not have a representation of the form $\boldsymbol{m} \cdot \boldsymbol{a}$ with $\boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}$. In the case of two variables ($d = 2$) Sylvester showed that $F(\boldsymbol{a}) = a_1 a_2 - a_1 - a_2$. No such explicit formulas are known in higher dimensions [10]. In his studies of "arithmetic turbulence", Arnold [2] conjectured that $F(\boldsymbol{a})$ should fluctuate wildly as a function of $\boldsymbol{a}$. The following theorem establishes the existence of a limit distribution for these fluctuations. As we shall see, the key in the proof of this statement uses a novel interpretation of the Frobenius number in terms of the dynamics of a certain flow Φ^t on the space of lattices $\Gamma\backslash G$, with $G := \mathrm{SL}(d, \mathbb{R})$, $\Gamma := \mathrm{SL}(d, \mathbb{Z})$.

Theorem 1 ([7])**.** *Let $d \geq 3$. There exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ with boundary of Lebesgue measure zero, and any $R \geq 0$,*

$$\lim_{T\to\infty} \frac{1}{T^d} \#\left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} = \frac{\mathrm{vol}(\mathcal{D})}{\zeta(d)}\, \Psi_d(R). \tag{1}$$

Variants of Theorem 1 were previously known only in dimension $d = 3$ in the work of Bourgain and Sinai [4], and Shur, Sinai and Ustinov [13]. For $d = 3$

Ustinov [14] derived an explicit formula for the limit density,
(2)

$$-\Psi_3'(t) = \begin{cases} 0 & (0 \le t \le \sqrt{3}) \\ \frac{12}{\pi}\left(\frac{t}{\sqrt{3}} - \sqrt{4-t^2}\right) & (\sqrt{3} \le t \le 2) \\ \frac{12}{\pi^2}\left(t\sqrt{3}\arccos\left(\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}\right) + \frac{3}{2}\sqrt{t^2-4}\log\left(\frac{t^2-4}{t^2-3}\right)\right) & (2 \le t). \end{cases}$$

For arbitrary $d \ge 3$, the limit distribution $\Psi_d(R)$ is given by the distribution of the covering radius of the simplex $\Delta = \{\boldsymbol{x} \in \mathbb{R}^{d-1}_{\ge 0} : \boldsymbol{x} \cdot \boldsymbol{e} \le 1\}$, $\boldsymbol{e} := (1, 1, \ldots, 1)$, with respect to a random lattice in $\mathbb{R}^{d-1}$ [7]. Here, the *covering radius* (sometimes also called *inhomogeneous minimum*) of a set $K \subset \mathbb{R}^{d-1}$ with respect to a lattice $\mathcal{L} \subset \mathbb{R}^{d-1}$ is defined as the infimum of all $\rho > 0$ with the property that $\mathcal{L} + \rho K = \mathbb{R}^{d-1}$. To state this result precisely, let $\mathbb{Z}^{d-1}A$ be a lattice in $\mathbb{R}^{d-1}$ with $A \in G_0 := \mathrm{SL}(d-1, \mathbb{R})$. The *space of lattices* (of unit covolume) is $\Gamma_0 \backslash G_0$ with $\Gamma_0 := \mathrm{SL}(d-1, \mathbb{Z})$. We denote by μ_0 the unique G_0-right invariant probability measure on $\Gamma_0 \backslash G_0$.

Theorem 2 ([7])**.** *Let $\rho(A)$ be the covering radius of the simplex Δ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then $\Psi_d(R) = \mu_0(\{A \in \Gamma_0 \backslash G_0 : \rho(A) > R\})$.*

The connection between Frobenius numbers and lattice free simplices is well understood [6], [12]. In particular, Theorem 2 connects nicely to the sharp lower bound of [1] (see also [11]): $F(\boldsymbol{a}) + \boldsymbol{e} \cdot \boldsymbol{a} \ge \rho_*(a_1 \cdots a_d)^{1/(d-1)}$, with $\rho_* := \inf_{A \in \Gamma_0 \backslash G_0} \rho(A)$. It is proved in [1] that $\rho_* > ((d-1)!)^{1/(d-1)} > 0$, and so in particular $\Psi_d(R) = 1$ for $0 \le R < \rho_*$.

Horospheres. Let $G := \mathrm{SL}(d, \mathbb{R})$ and $\Gamma := \mathrm{SL}(d, \mathbb{Z})$, and define
(3)

$$n_+(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{x} & 1 \end{pmatrix}, \qquad n_-(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{x} \\ \boldsymbol{0} & 1 \end{pmatrix}, \qquad \Phi^t = \begin{pmatrix} \mathrm{e}^{-t} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & \mathrm{e}^{(d-1)t} \end{pmatrix}.$$

The right action $\Gamma \backslash G \to \Gamma \backslash G$, $\Gamma M \mapsto \Gamma M \Phi^t$, defines a flow on the space of lattices $\Gamma \backslash G$. The horospherical subgroups generated by $n_+(\boldsymbol{x})$ and $n_-(\boldsymbol{x})$ parametrize the stable and unstable directions of the flow Φ^t as $t \to \infty$. Let us now identify a function W_δ on $\Gamma \backslash G$ that, when evaluated along a specific orbit of the flow Φ^t, produces the Frobenius number. Brauer and Shockley [5] proved that $F(\boldsymbol{a}) = \max_{r \bmod a_d} N_r(\boldsymbol{a}) - a_d$, where N_r is the smallest positive integer that has a representation in $r \bmod a_d$. A short calculation shows that

(4) $$N_r(\boldsymbol{a}) = \begin{cases} a_d & (r \equiv 0 \bmod a_d) \\ \min\{\boldsymbol{m}' \cdot \boldsymbol{a}' : \boldsymbol{m}' \in \mathbb{Z}^{d-1}_{\ge 0},\ \boldsymbol{m}' \cdot \boldsymbol{a}' \equiv r \bmod a_d\} & (r \not\equiv 0 \bmod a_d) \end{cases}$$

with $\boldsymbol{a}' = (a_1, \ldots, a_{d-1})$. This formula is the starting point in [7] of the construction of the function $W_\delta : \mathbb{R}^{d-1}_{\ge 0} \times G \to \mathbb{R}$, $(\boldsymbol{\alpha}, M) \mapsto W_\delta(\boldsymbol{\alpha}, M)$, given by

(5) $$W_\delta(\boldsymbol{\alpha}, M) = \sup_{\boldsymbol{\xi} \in \mathbb{T}^d} \min\nolimits_+ \left\{(\boldsymbol{m} + \boldsymbol{\xi})M \cdot (\boldsymbol{\alpha}, 0) : \boldsymbol{m} \in \mathbb{Z}^d,\ (\boldsymbol{m} + \boldsymbol{\xi})M \in \mathcal{R}_\delta\right\}$$

where $\mathcal{R}_\delta = \mathbb{R}_{\geq 0}^{d-1} \times (-\delta, \delta)$. Note that for every $\gamma \in \Gamma$, we have $W_\delta(\boldsymbol{\alpha}, \gamma M) = W_\delta(\boldsymbol{\alpha}, M)$, and thus W_δ can be viewed as a function on $\mathbb{R}_{\geq 0}^{d-1} \times \Gamma\backslash G$. The relation with the Frobenius number is as follows:

Theorem 3. *Let* $\boldsymbol{a} = (a_1, \ldots, a_d) \in \widehat{\mathbb{Z}}_{\geq 2}^d$ *with* $a_1, \ldots, a_{d-1} \leq a_d \leq \mathrm{e}^{(d-1)t}$, *and* $0 < \delta \leq \frac{1}{2}$. *Then* $F(\boldsymbol{a}) = \mathrm{e}^t W_\delta\big(\boldsymbol{a}', n_-(\widehat{\boldsymbol{a}})\Phi^t\big) - \boldsymbol{e} \cdot \boldsymbol{a}$, *where* $\widehat{\boldsymbol{a}} := \frac{\boldsymbol{a}'}{a_d} = \big(\frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}\big)$.

By exploiting standard probabilistic arguments [7], Theorem 1 now follows from Theorem 3 and the below equidistribution theorem for Farey fractions on a certain embedded submanifold of the space of lattices $\Gamma\backslash G$.

Farey fractions. Denote by $\mu = \mu_G$ the Haar measure on $G = \mathrm{SL}(d, \mathbb{R})$, normalized so that it represents the unique right G-invariant probability measure on the homogeneous space $\Gamma\backslash G$, where $\Gamma = \mathrm{SL}(d, \mathbb{Z})$. We will use the notation μ_0 for the right G_0-invariant probability measure on $\Gamma_0\backslash G_0$, with $G_0 = \mathrm{SL}(d-1, \mathbb{R})$ and $\Gamma_0 = \mathrm{SL}(d-1, \mathbb{Z})$ Consider the subgroups $H = \left\{ \begin{pmatrix} A & {}^\mathrm{t}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : A \in G_0,\ \boldsymbol{b} \in \mathbb{R}^{d-1} \right\}$ and $\Gamma_H = \Gamma \cap H$. We normalize the Haar measure μ_H of H so that it becomes a probability measure on $\Gamma_H\backslash H$; explicitly: $d\mu_H(M) = d\mu_0(A)\, d\boldsymbol{b}$.

Let us denote the Farey sequence of level Q by

$$\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ 0 < q \leq Q \right\}. \tag{6}$$

Note that $|\mathcal{F}_Q| \sim \frac{Q^d}{d\,\zeta(d)}$ as $Q \to \infty$.

Theorem 4 ([7])**.** *Let* $f : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ *be bounded continuous. Then, for* $Q = \mathrm{e}^{(d-1)t}$,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{\boldsymbol{r} \in \mathcal{F}_Q} & f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) \\ &= d(d-1) \int_0^\infty \int_{\mathbb{T}^{d-1} \times \Gamma_H\backslash H} \widetilde{f}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \end{aligned} \tag{7}$$

with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^\mathrm{t}M^{-1})$.

This statement can be established as a consequence of the mixing property of the flow Φ^t on $\Gamma\backslash G$, see [7] for details. It is interesting to note that, if one replaces $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ with a lattice Γ not commensurable with $\mathrm{SL}(d, \mathbb{Z})$, the Farey sequence becomes uniformly distributed in all of $\Gamma\backslash G$ with respect to Haar measure [8].

Open problems. In the case $d = 2$ the proof of Theorem 4 is very simple. In fact one can prove a stronger statement on the equidistribution of rationals with denominator $= q$. For every bounded continuous $f : \mathbb{T} \times \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ ($\mathbb{H}$ is the upper half plane, and $\mathrm{SL}(2,\mathbb{R})$ acts by fractional linear transformations)

$$\lim_{q\to\infty}\frac{1}{\varphi(q)}\sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q},\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}\right)=\int_0^1\int_0^1 f(\xi,x+\mathrm{i}\sigma^{-1})\,d\xi\,dx, \tag{8}$$

where $\varphi(q)$ is Euler's totient function. To prove this notice that $\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}$ is mapped by a suitable element from $\mathrm{SL}(2,\mathbb{Z})$ to the point $-\frac{\overline{p}}{q}+\mathrm{i}\frac{1}{\sigma}$, where $\overline{p}$ denotes the inverse of $p \bmod q$. Eq. (8) then follows from Fourier expanding f and applying standard bounds on Kloosterman sums. In analogy with the Corollary of Theorem 2 in [8], I conjecture that for every $\alpha\notin\mathbb{Q}$ and f as above,

$$\lim_{q\to\infty}\frac{1}{\varphi(q)}\sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q},\alpha\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}\right)=\frac{3}{\pi}\int_0^1\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\xi,x+\mathrm{i}y)\,d\xi\,\frac{dx\,dy}{y^2}. \tag{9}$$

(Here $\pi/3$ is the area of the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.) It is not hard to see that for bounded continuous f, eq. (9) implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N},\alpha\frac{n}{N}+\mathrm{i}\frac{\sigma}{N^2}\right)=\frac{3}{\pi}\int_0^1\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\xi,x+\mathrm{i}y)\,d\xi\,\frac{dx\,dy}{y^2}. \tag{10}$$

If (10) could be shown also for unbounded continuous functions with $|f(\xi,x+\mathrm{i}y)|\leq Cy^{1/2}$ for all $y\geq 1$ (presumably under some additional diophantine condition on α), then (10) would imply that the pair correlation function of the fractional parts of $n^2\alpha/N$ converges to that of independent random variables (see [9] for details of the analogous argument for the fractional parts of $n^2\alpha$). This in turn would prove a special instance of the Berry-Tabor conjecture in quantum chaos for the eigenvalues of the "boxed oscillator" [3, 15]!

References

[1] I.M. Aliev and P.M. Gruber, An optimal lower bound for the Frobenius problem. J. Number Theory 123 (2007) 71–79.

[2] V.I. Arnold, Arithmetical turbulence of selfsimilar fluctuations statistics of large Frobenius numbers of additive semigroups of integers. Mosc. Math. J. 7 (2007), 173–193.

[3] M.V. Berry and M. Tabor, Level clustering in the regular spectrum, Proc. Roy. Soc. A 356 (1977) 375–394.

[4] J. Bourgain and Ya.G. Sinai, Limit behavior of large Frobenius numbers, Russian Math. Surveys 62 (2007), 713–725.

[5] A. Brauer and J.E. Shockley, On a problem of Frobenius. J. Reine Angew. Math. 211 (1962), 215–220.

[6] R. Kannan, Lattice translates of a polytope and the Frobenius problem. Combinatorica 12 (1992) 161–177.

[7] J. Marklof, The asymptotic distribution of Frobenius numbers, Invent. Math. 181 (2010) 179–207

[8] J. Marklof, Horospheres and Farey fractions, Contemporary Mathematics, to appear

[9] J. Marklof and A. Strömbergsson, Equidistribution of Kronecker sequences along closed horocycles, GAFA 13 (2003) 1239–1280

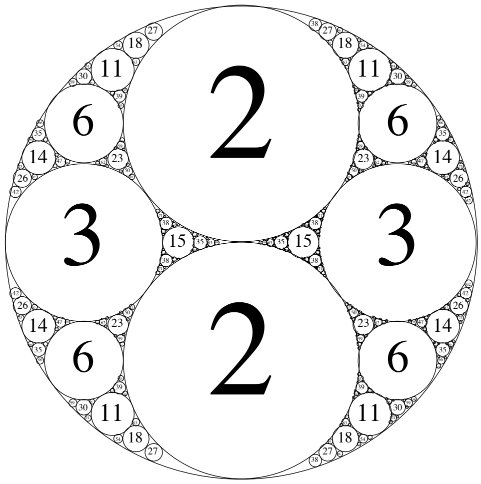


Figure 1. An Apollonian circle packing labeled by curvatures.

[10] J.L. Ramirez Alfonsin, The Diophantine Frobenius problem. Oxford University Press, Oxford, 2005.
[11] Ø. Rødseth, An upper bound for the h-range of the postage stamp problem. Acta Arith. 54 (1990) 301–306.
[12] H.E. Scarf and D.F. Shallcross, The Frobenius problem and maximal lattice free bodies. Math. Oper. Res. 18 (1993) 511–515.
[13] V. Shur, Ya.G. Sinai and A. Ustinov, Limiting distribution of Frobenius numbers for $n = 3$, J. Number Theory 129 (2009), 2778–2789.
[14] A. Ustinov, The solution of Arnold's problem on the weak asymptotics of Frobenius numbers with three arguments, Sbornik Mathematics 200 (2009) 597–627.
[15] S. Zelditch, Level spacings for integrable quantum maps in genus zero, Commun. Math. Phys. 196 (1998) 289–329.

Distribution of circles in Apollonian circle packings and beyond

Hee Oh

Given a set of four mutually tangent circles in the plane $\mathbb{C}$ with distinct points of tangency, one can construct four new circles, each of which is tangent to three of the given ones. Continuing to repeatedly fill the interstices between mutually tangent circles with further tangent circles, we obtain an infinite circle packing, called an *Apollonian circle packing*, after the great geometer Apollonius of Perga (262-190 BC).

Let $\mathcal{P}$ be an Apollonian circle packing. For $\mathcal{P}$ bounded and $T > 0$, denote by $N_T(\mathcal{P})$ the number of circles in $\mathcal{P}$ whose curvature (=the reciprocal of its radius) is at most T. Note that $N_T(\mathcal{P}) = \infty$ for a general unbounded packing. However in the special case of unbounded packing $\mathcal{P}$ which lies between two parallel lines, the altered definition of $N_T(\mathcal{P})$ to count circles in a fixed period is a well-defined finite number for any $T > 0$.

Theorem 1 (Kontorovich-O.). [1] *Let $\mathcal{P}$ be either bounded or between two parallel lines. There exists $c_{\mathcal{P}} > 0$ such that as $T \to \infty$,*

$$N_T(\mathcal{P}) \sim c_{\mathcal{P}} \cdot T^{\alpha}$$

where α is the Hausdorff dimension of the residual set of $\mathcal{P}$.

The residual set of $\mathcal{P}$ is defined to be the complement in $\mathbb{C}$ of the all open disks enclosed by circles in $\mathcal{P}$. McMullen computed that α is approximately 1.30568(8).

Two natural questions arising from the above result are the following:

Question 1. (1) *For an arbitrary Apollonian circle packing, can we describe the asymptotic distribution of circles in $\mathcal{P}$. That is, is there a Borel measure $\omega_{\mathcal{P}}$ on $\mathbb{C}$ such that for all nice bounded Borel subset $E \subset \mathbb{C}$,*

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \mathrm{Curv}(C) < T\} \sim \omega_{\mathcal{P}}(E) \cdot T^{\alpha}$$

as $T \to \infty$.

(2) *How about other circle packings (=countable union of circles in $\mathbb{C}$) beyond Apollonian circle packings?*

Example 2. Here is one simple way of constructing circle packings: consider the Möbius transformation action of $\mathrm{PSL}_2(\mathbb{C})$ on the extended complex plane $\hat{\mathbb{C}}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ and $z \in \hat{\mathbb{C}}$. A Möbius transformation maps a circle to a circle; here Euclidean lines are thought of circles passing through the point at infinity. Therefore, for a discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ and a circle C in the plane, the Γ-orbit $\mathcal{P} := \Gamma(C)$ of C is a circle packing in our sense.

Let $\mathcal{P}$ be an infinite circle packing invariant under a non-elementary discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ with finite many Γ-orbits, i.e.,

$$\mathcal{P} = \cup_{i=1}^{m} \Gamma(C_i).$$

Assume that $\mathcal{P}$ is locally finite in the sense that

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \mathrm{Curv}(C) < T\}$$

is finite for any bounded subset $E \subset \mathbb{C}$ and $T > 1$.

Recalling that the limit set $\Lambda(\Gamma)$ is defined to be the set of all accumulation points of $\Gamma(z)$, $z \in \hat{\mathbb{C}}$, we expect that smaller and smaller circles in $\mathcal{P}$ are more and more concentrated toward the limit set of Γ. Hence the measure $\omega_{\mathcal{P}}$, if exists, must be supported in the limit set of Γ.

Our approach to investigating the asymptotic of $N_T(\mathcal{P}, E)$ is via the study of the 3-dimensional hyperbolic geometry and we construct $\omega_{\mathcal{P}}$ with the help of the Patterson-Sullivan theory for Kleinian groups.

The group $G := \mathrm{PSL}_2(\mathbb{C})$ identifies with the group of all orientation preserving isometries of $\mathbb{H}^3$. Considering the upper-half space model $\mathbb{H}^3 = \{(z, r) : z \in \mathbb{C}, r > 0\}$ the geometric boundary $\partial_\infty(\mathbb{H}^3)$ is naturally identified with $\hat{\mathbb{C}}$. For

$j = (0,1) \in \mathbb{H}^3$, let ν_j denote the Patterson-Sullivan measure on $\Lambda(\Gamma)$ viewed from j.

Definition 3. *Define a Borel measure* ω_Γ *on* $\mathbb{C}$ *by*

$$d\omega_\Gamma = (|z|^2+1)^{\delta_\Gamma} d\nu_j$$

where $0 < \delta_\Gamma \le 2$ *is the critical exponent of* Γ.

Definition 4 (The Γ-skinning size of $\mathcal{P}$)**.** *Define* $0 \le \mathrm{sk}_\Gamma(\mathcal{P}) \le \infty$ *by the following:*

$$\mathrm{sk}_\Gamma(\mathcal{P}) := \sum_{i=1}^{m} \int_{s \in \mathrm{Stab}_\Gamma(C_i^\dagger)\backslash C_i^\dagger} e^{\delta_\Gamma \beta_{s^+}(j,\pi(s))} d\nu_j(s^+)$$

where $\pi : \mathrm{T}^1(\mathbb{H}^3) \to \mathbb{H}^3$ is the canonical projection, $C_i^\dagger \subset \mathrm{T}^1(\mathbb{H}^3)$ is the set of unit normal vectors to the convex hull $\hat{C}_i$ of C_i, and $u^+ \in \hat{\mathbb{C}}$ (resp. $u^- \in \hat{\mathbb{C}}$) denotes the forward (resp. backward) end point of the geodesic determined by $u \in \mathrm{T}^1(\mathbb{H}^3)$.

Theorem 2 (O.- Shah)**.** [2] *Let* Γ *be geometrically finite.*
Suppose one of the following conditions hold:

1. Γ *is a lattice;*
2. Γ *is convex co-compact;*
3. *all circles in* $\mathcal{P}$ *are mutually disjoint;*
4. $\cup_{i \in I} C_i^\circ \subset \Omega(\Gamma)$, *where* C_i° *denotes the open disk enclosed by* C_i *and* $\Omega(\Gamma) = \hat{\mathbb{C}} - \Lambda(\Gamma)$ *is the domain of discontinuity for* Γ.

For any bounded Borel subset E *of* $\mathbb{C}$ *with* $\omega_\Gamma(\partial(E)) = 0$, *we have, as* $T \to \infty$,

$$N_T(\mathcal{P}, E) \sim \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \omega_\Gamma(E) \cdot T^{\delta_\Gamma}$$

where $0 < \mathrm{sk}_\Gamma(\mathcal{P}) < \infty$ *and* $0 < |m_\Gamma^{\mathrm{BMS}}| < \infty$ *is the total mass of the Bowen-Margulis-Sullivan measure associated to* ν_j.

For Γ geometrically finite, Sullivan showed that $|m_\Gamma^{\mathrm{BMS}}| < \infty$. The above conditions (1)-(4) are made so as to ensure that $0 < \mathrm{sk}_\Gamma(\mathcal{P}) < \infty$. Indeed, the above theorem holds in a much more general setting where we only need to assume that Γ admits a finite Bowen-Margulis-Sullivan measure and that the Γ-skinning size of $\mathcal{P}$ is finite.

Remark 5.
1. Note that δ_Γ is positive, as Γ is non-elementary, and equal to the Hausdorff dimension of the limit set $\Lambda(\Gamma)$, as Γ is geometrically finite.
2. We have $\omega_\Gamma(\text{finite subset}) = 0$
3. If Γ is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$ considered as a real algebraic group, then $\omega_\Gamma(\text{any real algebraic curve}) = 0$.
4. On the contracting horosphere $H_\infty^-(j) \subset \mathrm{T}^1(\mathbb{H}^3)$ consisting of upward unit normal vectors on $\mathbb{C} + j = \{(z,1) : z \in \mathbb{C}\}$, the normal vector based at $z + j$ is mapped to z via the map $u \mapsto u^-$. Under this correspondence, ω_Γ coincides with the conditional of the Bowen-Margulis-Sullivan measure on the contracting horosphere $H_\infty^-(j)$.

References

[1] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. *Preprint, 2009.*
[2] Hee Oh and Nimish Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Preprint, 2010.*

Equidistribution, Counting and Arithmetic Applications

Jouni Parkkonen and Frédéric Paulin

Let M be a finite volume hyperbolic manifold of dimension n at least 2. Let $T^1M \to M$ be the unit tangent bundle of M, where T^1M is endowed with its usual Riemannian metric, whose induced measure is the Liouville measure vol_{T^1M}. Let $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow of M. Let C_0 be a finite volume immersed totally geodesic submanifold of M of dimension k with $0 < k < n$, and let $\nu^1 C_0$ be its unit normal bundle, so that $g^t\nu^1C_0$ is, for every $t \geq 0$, an immersed submanifold of T^1M.

Theorem 1. *The induced Riemannian measure of $g^t\nu^1C_0$ equidistributes to the Liouville measure as $t \to +\infty$:*

$$\mathrm{vol}_{g^t\nu^1C_0} / \| \mathrm{vol}_{g^t\nu^1C_0} \| \ \overset{*}{\rightharpoonup}\ \mathrm{vol}_{T^1M} / \| \mathrm{vol}_{T^1M} \| \,.$$

This theorem can be deduced from [EM, Theo. 1.2]. Our (short and direct) proof also uses, as in Margulis' equidistribution result for horospheres, the mixing property of the geodesic flow of M.

Let $\mathcal{H}_\infty$ be a small enough Margulis neighbourhood of an end of M, that is a connected component of the set of points of M at which the injectivity radius of M is at most ϵ_0, for some $\epsilon_0 > 0$ small enough. We use the above equidistribution theorem, and the fact that the submanifold $g^t\nu^1C_0$ is locally close to an unstable leaf in T^1M of the geodesic flow of M, to prove the following counting result.

Theorem 2. *The number of common perpendicular locally geodesic arcs between $\partial\mathcal{H}_\infty$ and C_0 with length at most t is equivalent, as t tends to $+\infty$, to*

$$\frac{\mathrm{Vol}(\mathbb{S}_{n-k-1})\,\mathrm{Vol}(\mathcal{H}_\infty)\,\mathrm{Vol}(C_0)}{\mathrm{Vol}(\mathbb{S}_{n-1})\,\mathrm{Vol}(M)}\, e^{(n-1)t} \,.$$

We refer to [PP1] for the proofs of the above theorems, as well as for references to other works and many geometric complements, and we now give a sample of their arithmetic applications, extracted from [PP1] except for the last corollary.

Counting quadratic irrationals. Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Endow the set of quadratic irrationals over K with the action by homographies of $\mathrm{PSL}_2(\mathcal{O}_K)$, and note that it is not transitive. We denote by α^σ the Galois conjugate over K of a quadratic irrational α over K. There are many works (see for instance [Bug]) on the approximation of real or complex numbers by algebraic numbers, and approximating them by elements in orbits of algebraic numbers under natural group actions for appropriate complexities seems to be interesting.

Starting with $K = \mathbb{Q}$, our first result is a counting result in orbits of real quadratic irrationals over $\mathbb{Q}$ for a natural complexity (see [PP1] for a more algebraic expression in terms of discriminants).

Corollary 1. *Let $\alpha_0 \in \mathbb{R}$ be a quadratic irrational over $\mathbb{Q}$, and let G be a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Then as s tends to $+\infty$,*

$$\mathrm{Card}\{\alpha \in G \cdot \{\alpha_0, \alpha_0^\sigma\} \bmod \mathbb{Z} \;:\; \frac{1}{|\alpha - \alpha^\sigma|} \leq s\} \sim \frac{24\, q_G\, \operatorname{argcosh} \frac{|\operatorname{tr}\gamma_0|}{2}}{\pi^2\, [\mathrm{PSL}_2(\mathbb{Z}) : G]\, n_0}\; s\,,$$

where q_G is the smallest positive integer q such that $z \mapsto z + q$ belongs to G, $\gamma_0 \in G - \{1\}$ fixes α_0 and n_0 is the index of ${\gamma_0}^{\mathbb{Z}}$ in the stabilizer of $\{\alpha_0, \alpha_0^\sigma\}$ in G (and note that q_G, γ_0, n_0 do exist).

For instance, if α_0 is the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ (which is reciprocal in Sarnak's terminology) and $G = \mathrm{PSL}_2(\mathbb{Z})$, we get $\mathrm{Card}\{\alpha \in G \cdot \phi \mod \mathbb{Z} : \frac{1}{|\alpha-\alpha^\sigma|} \leq s\} \sim \frac{24\,\log\phi}{\pi^2}\; s$. With $\mathbb{H}^2_{\mathbb{R}}$ the upper halfplane model of the real hyperbolic plane, the proof applies Theorem 2 to M the orbifold $G\backslash\mathbb{H}^2_{\mathbb{R}}$, to C_0 the image in M of the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ with endpoints α_0 and α_0^σ, and to $\mathcal{H}_\infty$ the image in M of the set of points in $\mathbb{H}^2_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that if a and b are close enough distinct real numbers, then the hyperbolic length of the perpendicular arc between the horizontal line at Euclidean height 1 and the geodesic line with endpoints a and b is exactly $-\log|b-a|$.

Assume K is imaginary quadratic, with discriminant D_K. We proved a general statement analogous to the previous corollary, but we only give here a particular case for ϕ.

Corollary 2. *Let $\mathfrak{a}$ be a non zero ideal in $\mathcal{O}_K$ and $\Gamma_0(\mathfrak{a}) = \left\{\pm\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) \;:\; c \in \mathfrak{a}\right\}$. Assume for simplicity that $D_K \neq -4$ and $\phi^\sigma \notin \Gamma_0(\mathfrak{a}) \cdot \phi$. Then as s tends to $+\infty$, the cardinality of $\{\alpha \in \Gamma_0(\mathfrak{a})\cdot\{\phi, \phi^\sigma\} \bmod \mathcal{O}_K \;:\; \frac{1}{|\alpha-\alpha^\sigma|} \leq s\}$ is equivalent to*

$$\frac{8\pi^2\, k_{\mathfrak{a}}\, \log\phi}{|D_K|\, \zeta_K(2)\, N(\mathfrak{a}) \prod_{\mathfrak{p} \text{ prime},\, \mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})}\right)}\; s^2\,,$$

with $k_{\mathfrak{a}}$ the smallest $k \in \mathbb{N} - \{0\}$ such that the $2k$-th term of the standard Fibonacci sequence belongs to $\mathfrak{a}$ (and note that $k_{\mathfrak{a}}$ does always exist, contrarily to the odd case).

Counting representations of integers by binary forms. Recall that a binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ is primitive integral if $a, b, c \in \mathbb{Z}$ are relatively prime, and indefinite non product if its discriminant $D = b^2 - 4ac$ is positive and not a square. Using the well known correspondence between pairs of Galois conjugated quadratic irrationals over $\mathbb{Q}$ and the set of such Q's up to sign, we prove the following counting result for the number of values of a fixed such Q on couples of relatively prime integers satisfying some congruence relations. Let (t, u)

be the minimal solution to the Pell-Fermat equation $t^2 - Du^2 = 4$ and $\epsilon = \frac{t+u\sqrt{D}}{2}$ the corresponding fundamental unit.

Corollary 3. *Let Q be as above, and let n be an integer at least 3. Then the number of couples $(x, y) \in \mathbb{Z}^2$, relatively prime, with $x \equiv 1 \mod n$ and $y \equiv 0 \mod n$, such that $|Q(x, y)| \leq s$, modulo the linear action of $\mathrm{SL}_2(\mathbb{Z})$, is equivalent, as s tends to $+\infty$, to*

$$\frac{24\ \log\epsilon}{\pi^2 n^2\sqrt{D}} \prod_{p \text{ prime},\ p|n} \left(1 - \frac{1}{p^2}\right)^{-1} s\ .$$

The final result, for a quadratic imaginary number field K, is proved in [PP2], along with extensions to representations satisfying congruence properties.

Corollary 4. *Let $f : (u, v) \mapsto a\,|u|^2 + 2\,\mathrm{Re}(b\,u\,\overline{v}) + c\,|v|^2$ be a binary Hermitian form, indefinite (that is $\Delta = |b|^2 - ac > 0$) and integral over K (that is $a, c \in \mathbb{Z}, b \in \mathcal{O}_K$). Let $\mathrm{SU}_f(\mathcal{O}_K) = \{g \in \mathrm{SL}_2(\mathcal{O}_K)\ :\ f \circ g = g\}$ be the group of automorphs of f. Then the number of orbits under $\mathrm{SU}_f(\mathcal{O}_K)$ of couples (u, v) of relatively prime elements of $\mathcal{O}_K$ such that $|f(u, v)| \leq s$ is equivalent, as s tends to $+\infty$, to*

$$\frac{\pi\ \mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K))}{2\ |D_K|\ \zeta_K(2)\ \Delta}\ s^2\ .$$

With $\mathbb{H}^3_{\mathbb{R}}$ the upper halfspace model of the real hyperbolic 3-space, the proof applies Theorem 2 to M the orbifold $\mathrm{PSL}_2(\mathcal{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$, to C_0 the image in M of the unique hyperbolic plane $P(f)$ in $\mathbb{H}^3_{\mathbb{R}}$ preserved by $\mathrm{PSU}_f(\mathcal{O}_K)$, and to $\mathcal{H}_\infty$ the image in M of the set of points in $\mathbb{H}^3_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that, for every $\gamma \in \mathrm{PSL}_2(\mathcal{O}_K)$, the hyperbolic plane $P(f \circ \gamma)$ is an Euclidean hemisphere whose diameter is $\frac{\sqrt{\Delta}}{f\circ\gamma(1,0)}$, hence whose perpendicular arc to the horizontal plane at Euclidean height 1 has (signed) hyperbolic length $\log\frac{f\circ\gamma(1,0)}{\sqrt{\Delta}}$, and that $SL_2(\mathcal{O}_K)$ acts transitively on the couples of relatively prime elements of $\mathcal{O}_K$.

References

[Bug] Y. Bugeaud, *Approximation by algebraic numbers*, Camb. Tracts Math. **160**, Cambridge Univ. Press, 2004.

[EM] A. Eskin, C. McMullen, *Mixing, counting, and equidistribution in Lie groups*, Duke Math. J. **71** (1993) 181–209.

[PP1] J. Parkkonen and F. Paulin. *Équidistribution, comptage et approximation par irrationnels quadratiques*, Preprint [arXiv:1004.0454].

[PP2] J. Parkkonen, F. Paulin, *On the representations of integers by indefinite binary Hermitian forms*, preprint 2010 [arXiv:1004.3211].

Stationary measures and invariant subsets of homogeneous spaces

Jean-Francqis Quint
(joint work with Yves Benoist)

Let G be a Lie group, Λ be a lattice in G and Γ be a closed subgroup of G. We are interested in the description of the Γ-orbit closures in the homogeneous space $X = G/\Lambda$. More precisely, we want to find assumptions on Γ which ensure that these orbit closures enjoy rigidity properties.

Let Y be a closed subset of X. Then Y is said to be homogeneous if there exists a closed subgroup S of G and some x in X with $Y = Sx$ and such that Y carries a S-invariant probability measure. In other terms, the stabilizer of x in S is a lattice in S. In the same way, a Borel probability measure ν on X is said to be homogeneous if there exists a closed subgroup S of G and an element x of X such that ν is S-invariant and $\nu(Sx) = 1$.

Finally, a one-parameter subgroup $(u_t)_{t\in\mathbb{R}}$ of G is said to be Ad-unipotent if, for any t in $\mathbb{R}$, $\mathrm{Ad}u_t$ is a unipotent automorphism of the Lie algebra of G. One has the following strong theorem:

Theorem 1 (Ratner, 1991). *Assume Γ is spanned by one-parameter* Ad-*unipotent subgroups of G. Then, any Γ-orbit closure in X is homogeneous. In the same way, every ergodic Γ-invariant Borel probability measure on X is homogeneous.*

This theorem is still conjectured to hold in case the Zariski closure of Γ (in some meaning that should be made precise) satisfies the same assumption. We shall deal with the case where this Zariski closure is semisimple. If H is a semisimple subgroup of G which contains Γ, we shall say that Γ is Zariski dense in H if its image under the adjoint representation is Zariski dense in the adjoint group of H. We then have the following

Theorem 2 (Benoist-Quint, 2010). *Assume there exists a semisimple subgroup H of G such that Γ is Zariski dense in H. Then, any Γ-orbit closure in X is homogeneous.*

In Ratner's theorem the classification of orbit closures follows from the one of invariant measures: indeed, the main difficulty consists in dealing with the case where Γ is Ad-unipotent. Then, in particular, Γ is nilpotent, thus amenable, and one can prove that every closed invariant subset carries a Γ-invariant probability measure. In our case, Γ is not amenable and the classification of invariant measures does not give any results on the one of orbit closures. We shall therefore use a weaker notion of invariance.

Let μ be a Borel probability measure on G. Then define $\mu * \nu$ to be the Borel probability measure $\int_G g_*\nu \mathrm{d}\mu(\mathrm{g})$. The probability measure ν is said to be μ-stationary if $\mu * \nu = \nu$. It is said to be ergodic if it is extremal among μ-stationary measures.

Theorem 3 (Benoist-Quint, 2010). *Let μ be a compactly supported Borel probability measure on G. Assume there exists a semisimple subgroup H of G such that* *the subgroup spanned by the support of μ is Zariski dense in H. Then, any ergodic μ-stationary Borel probability measure on X is homogeneous (and invariant).*

The proof of this theorem relies, on one hand, on arguments from abstract ergodic theory and martingale convergence properties (using in particular an idea which is originally due to A. Bufetov) and, on the other hand, on limit laws for the measures μ^{*n}, $n \in \mathbb{N}$, established by H. Furstenberg, H. Kesten, Y. Guivarc'h, E. Le Page, A. Raugi, E. Breuillard,...

In Ratner's theorem, to get the topological classification from the metric one, one needs to use Result by S.G. Dani and G. Margulis on the non-divergence of orbits of one-parameter Ad-unipotent flows. In our case, the analoguous non-divergence results are also used in the metric classification (under a quantitative version). More precisely, we get the following theorem, which extends previous work by A. Eskin and G. Margulis (and which was conjectured to hold by these authors):

Theorem 4 (Benoist-Quint, 2010)**.** *Let μ be a compactly Borel probability measure on G which admits exponential moments (that is, there exists $\delta > 0$ with $\int_G \{\|\mathrm{Ad} g^{\delta}\|\} \mathrm{d}\mu(\mathrm{g}) < \infty$). Assume there exists a semisimple subgroup H of G such that the subgroup spanned by the support of μ is Zariski dense in H. Then, for any x in X and $\varepsilon > 0$, there exists a compact subset K of X such that, for any n in $\mathbb{N}$, one has $\mu^{*n} * \delta_x(K) \geq 1 - \varepsilon$.*

In other terms, the Markov chain with state space X and transition probabilities $\mu * \delta_x$, $x \in X$, is recurrent.

To prove this result, we adapt the strategy by Eskin and Margulis and use some ideas from the representation theory of semisimple groups.

References

[1] Y. Benoist and J. Quint, *Mesures stationnaires et fermes invariants des espaces homogenes*, preprint.

Equidistribution of half-horocycles on infinite volume hyperbolic surface

Barbara Schapira

The study of ergodic properties of geodesic and horocycle flows acting on the unit tangent bundle of negatively curved manifolds is now a very classical subject : existence of invariant (finite or σ-finite) measures, entropy, unique ergodicity, equidistribution properies, generic vectors, ... This subject was intensively studied during the last century, in the case of geodesic and horocyclic flows on compact or finite volume surfaces of constant negative curvature.

I am particularly interested in situations where the classical powerful methods issued from harmonic analysis or lattices of Lie groups do not work : non compact manifolds, infinite volume, infinitely generated groups, infinite measure of maximal entropy, variable curvature (and even nonpositive curvature), Let us describe first the classical and well known situation.

Hyperbolic surfaces of finite volume. If S is a hyperbolic surface, its unit tangent bundle identifies with $PSL(2,\mathbb{R})/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$. The geodesic flow $(g^t)_{t\in\mathbb{R}}$ acts on the left as the one parameter group $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t\in\mathbb{R}\}$, whereas the (unstable) horocycle flow $(h^s)_{s\in\mathbb{R}}$ corresponds to the unipotent group $\{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ s\in\mathbb{R}\}$.

The geodesic flow is the typical geometrical example of a hyperbolic flow, with positive entropy, exponential mixing, dense orbits, dense periodic orbits, infinitely many invariant measures, ...

When S is compact, the horocyclic flow is minimal (Hedlund), ergodic with respect to the Liouville measure, and even uniquely ergodic (Furstenberg). In particular, all orbits $(h^s u)_{0\le s\le S}$ are equidistributed towards the Liouville measure when $S\to+\infty$. On finite volume surfaces appear periodic horocyclic orbits due to the presence of cusps (thin ends). But except the Dirac measures supported on periodic orbits, the Liouville measure is the unique ergodic probability measure on T^1S. And all nonperiodic orbits are equidistributed towards the Liouville measure. However, due to the presence of periodic orbits, this result becomes highly non trivial, and needs to understand the time spent by horocycles inside the cusps of the surface (Dani, and Dani-Smillie).

Geometrically finite negatively curved surfaces. In the particular case of hyperbolic surfaces, the notion of geometrical finiteness coincides with the fact that the fundamental group $\Gamma=\pi_1(S)$ is finitely generated.

If the surface has infinite volume, it has *funnels*, that is big ends of infinite volume, topologically homeomorphic to cylinders, separated from the compact part of the surface by a closed geodesic. Of course, the surface can still have *cusps*, that is thin ends of finite volume (also topologically homeomorphioc to cylinder). A geodesic orbit can enter a cusp and come back, and can do it even infinitely many times, during unbounded times, ... By contrast, if it enters a funnel, it never comes back.

The study must therefore restricted to the *nonwandering set Ω of the geodesic flow*. On this set, $(g^t)_{t\in\mathbb{R}}$ has the same qualitative properties as on a finite volume surface.

However, Ω is not invariant by the horocyclic flow, so that we need to consider also the *nonwandering set $\mathcal{E}$ of the horocyclic flow*, which consists of all horocyclic orbits intersecting Ω. Of course, $\Omega\subset\mathcal{E}\subset T^1S$, and all inclusions are strict.

As consequences, the measure of maximal entropy of the geodesic flow, supported on Ω, does not coincide anymore with the Liouville measure; the Liouville measure, supported on T^1S, is no more ergodic under any of the two flows; there is no measure invariant and ergodic under both flows.

However, most results true on finite volume surfaces can be extended in this context. For example, all horocyclic orbits of $\mathcal{E}$ are periodic or dense in $\mathcal{E}$. The horocyclic flow has a unique invariant ergodic measure on $\mathcal{E}$ except the Dirac measures supported on periodic orbits. This result was proved by Burger [**Bu**] in the case of geometrically finite surfaces without cusps, and with a critical exponent strictly larger than 1/2, by using harmonic analysis, and later by Roblin [**Ro**] in a more general context ($CAT(-1)$-spaces of any dimension). It is important to note that this measure is infinite.

In [**Scha1**], I proved that in a certain sense, horocycles of $\mathcal{E}$ do not spend to much time inside the cusps of the surface. (The article was written in the context of manifolds of any dimension and variable negative curvature).

As a consequence, in [**Scha2**], I obtained the equidistribution of nonperiodic horocyclic orbits of $\mathcal{E}$ towards the unique nonperiodic invariant ergodic measure on $\mathcal{E}$, which is infinite.

This result was obtained through other intermediate equidistribution results.

Half-horocycles. Usually, in classical ergodic theory, if a dynamical system is invertible, properties of the system and of its inverse are the same. For equidistribution property, for example, the usual statement, on a finite volume surface, says that for all nonperiodic vectors $v \in T^1S$, and all continuous functions f defined on T^1S, the Birkhoff average of f along the orbit $(h^s v)_{0\leq s\leq S}$ converges to the integral of f w.r.t. the Liouville measure when $S \to +\infty$.

It turns out that in the statement of my equidistribution result mentioned above, as well as in an analogous result previously obtained by M. Burger, I consider symmetric orbits $(h^s v)_{-S\leq s\leq S}$ and their behaviour when $S \to +\infty$.

In [**Scha3**], I clarify the cases where one needs to consider symmetric orbits. Once again, it is mainly due to the presence of funnels. In such situations, it can happen that a half-orbit will enter a funnel and never come back, whereas the other half orbit is recurrent and even dense in $\mathcal{E}$. But I prove that this is the only obstruction which makes a half-horocycle dense and the other not, on geometrically finite manifolds.

And I also prove that non-density is the only obstruction to equidistribution. In other words, as soon as a half-horocycle $(h^s v)_{s\geq 0}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow, it is equidistributed towards the unique (h^s)-invariant ergodic measure of full support in $\mathcal{E}$.

Geometrically infinite surfaces. We can ask whether it is reasonable or not to hope for density and equidistribution of (half-)horocycles on geometrically infinite surfaces. From the topological point of view, the picture is almost complete (see [**H**], [**Da**])

In a joint work [**Sa-Scha**] with Omri Sarig, we studied the question of equidistribution of orbits in the simplest case of geometrically infinite surfaces, the case of abelian covers of compact hyperbolic surfaces. In this situation, the invariant ergodic measures under the horocyclic flow were described and classified (Babillot-Ledrappier [**BL**], Sarig [**Sa**]), using the notion of *asymptotic cycle* of a vector. Roughly speaking, if you see a $\mathbb{Z}^d$-cover of a compact surface from very far, you see no details on the local geometry, but only $\mathbb{Z}^d$, and the asymptotic cycle of a vector v describes the asymptotic average displacement of $(g^t v)_{t\leq 0}$ in $\mathbb{Z}^d$. Babillot-Ledrappier and Sarig showed that (h^s)-invariant ergodic measures are classified by asymptotic cycles, and we also classify generic vectors using this notion of asymptotic cycle.

References

[**BL**] M. Babillot and F. Ledrappier: *Geodesic paths and horocycle flows on Abelian covers*, Lie groups and ergodic theory (Mumbai, 1996), 1–32, Tata Inst. Fund. Res. Stud. Math. **14**, Tata Inst. Fund. Res., Bombay, 1998.

[**Bu**] M. Burger: *Horocycle flows on geometrically finite surfaces*, Duke Math. J. **61** (1990), 779–803.

[**Da**] F. Dal'bo *Topologie du feuilletage fortement stable*, Ann. Inst. Fourier (Grenoble) **50** (3) (2000), 981–993.

[**H**] G.A. Hedlund *Fuchsian groups and transitive horocycles*, Duke Math. J. **2** (1936), 530-542.

[**Ro**] T. Roblin *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. France (2003), **95**.

[**Sa**] O. Sarig: *Invariant measures for the horocycle flow on Abelian covers.* Inv. Math. **157**, 519–551 (2004).

[**Sa-Scha**] O. Sarig, B. Schapira *The generic points for the horocycle flow on a class of hyperbolic surfaces with infinite genus* , (2008).

[**Scha1**] B. Schapira: *Lemme de l'ombre et non divergence des horosphères d'une variété géométriquement finie*, Ann. Inst. Fourier **54** (4) (2004), 939–987.

[**Scha2**] B. Schapira, *Equidistribution of the horocycles of a geometrically finite surface*, Inter. Math. Res. Not. **40** (2005), 2447–2471.

[**Scha3**] B. Schapira, *Density and Equidistribution of half-horocycles of a geometrically finite surface*, preprint (2009).

Integral points on one-sheeted hyperboloids and limits of translates of divergent geodesics

Nimish Shah

(joint work with Hee Oh)

Let $Q(x_1, x_2, x_3)$ be a real quadratic form of signature $(2, 1)$. For $d > 0$, the variety $V := \{X \in \mathbb{R}^3 : Q(X) = d\}$ is a one sheeted hyperboloid. Denote by G the identity component of the special orthogonal group $\mathrm{SO}_Q(\mathbb{R})$. Let $\Gamma < G$ be a lattice, that is, a discrete subgroup of finite co-volume, and $v_0 \in V$ be such that the orbit $v_0\Gamma$ is discrete.

For a norm $\|\cdot\|$ on $\mathbb{R}^3$, we consider the following counting function: for $T > 1$,

$$N_T(v_0\Gamma, \|\cdot\|) := \#\{w \in v_0\Gamma : \|w\| < T\}.$$

Denote by $H \simeq \mathrm{SO}(1, 1)^\circ$ the one-dimensional stabilizer subgroup of v_0 in G and set $B_T := \{w \in v_0G : \|w\| < T\}$ for $T > 1$. Extending the 1993 result of Duke, Rudnick and Sarnak [1], where one assumes that $H \cap \Gamma$ is a lattice in H, we show the following:

Theorem 1. *Suppose that $H \cap \Gamma$ is finite. Then as $T \to \infty$,*

$$N_T(v_0\Gamma, \|\cdot\|) \sim \frac{\int_{-\log T}^{\log T} 1 ds}{\operatorname{vol}_G(\Gamma\backslash G)} \operatorname{vol}_{H\backslash G}(B_T)$$

where $d\operatorname{vol}_G = ds \times d\operatorname{vol}_{H\backslash G}$ locally. Furthermore, for some $c > 0$

$$N_T(v_0\,\mathrm{G}, \|\cdot\|) = c \cdot T \cdot \log T(1 + O((\log T)^{-2/7}(\log\log T)^{2/7}).$$

Integral binary quadratic forms of a fixed discriminant For a binary quadratic form $q(x,y) = ax^2 + bxy + cy^2$, its discriminant $\operatorname{disc}(q)$ is defined to be $b^2 - 4ac$. For $d \in \mathbb{Z}$, denote by $\mathcal{B}_d(\mathbb{Z})$ the space of integral binary quadratic forms $q(x,y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{Z}$ with discriminant d. It is easy to see that $\mathcal{B}_d(\mathbb{Z}) \neq \emptyset$ if and only if d congruent to 0 or 1 modulo 4. For $d \neq 0$, a classical result of Gauss says that $\mathcal{B}_d(\mathbb{Z})$ consists of finitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits. If d is not a square, then the stabilizer of every $q \in \mathcal{B}_d(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z})$ is infinite. On the other hand, when d is a square, $\mathcal{B}_d(\mathbb{Z})$ contains a quadratic form q (e.g., $q(x,y) = x^2 + \sqrt{d}\,xy$) whose stabilizer in $\mathrm{SL}_2(\mathbb{Z})$ is finite.

Therefore by Theorem 1

Theorem 2. *For any non-zero square $d \in \mathbb{Z}$ and for any norm $\|\cdot\|$ on $\mathbb{R}^3$, there exists $c > 0$ such that*

$$\#\{q \in \mathcal{B}_d(\mathbb{Z}) : \operatorname{disc}(q) = d, \|q\| < T\} = c \cdot T\log T(1 + O((\log T)^{-2/7}(\log\log T)^{2/7}),$$

where $\|ax^2 + bxy + cy^2\| = \|(a, b, c)\|$.

The proof of Theorem 1 is based on the methods developed by Duke, Rudnick and Sarnak [1] and Eskin and McMullen [2] and the following result.

Orthogonal translates of a divergent geodesic Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a (non-uniform) lattice in G. Let μ be the G-invariant probability measure on $\Gamma\backslash G$. For $s \in \mathbb{R}$, define

$$(1) \qquad h(s) = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix} \quad \text{and} \quad a(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}.$$

Let $H = \{h(s) : s \in \mathbb{R}\}$. Understanding the limit of the translates $\Gamma\backslash\Gamma Ha(T)$ when $\Gamma\backslash\Gamma H$ is divergent in both directions is the new main ingredient of our proof of Theorem 1.

Theorem 3. *Let $x_0 \in \Gamma\backslash G$ and suppose that $x_0 h(s)$ diverges as $s \to +\infty$, that is, $x_0 h(s)$ leaves every compact subset for all sufficiently large $s \gg 1$. For a given compact subset $\mathcal{K} \subset \Gamma\backslash G$, there exists $c = c(\mathcal{K}) > 0$ such that for any $f \in C^\infty(\Gamma\backslash G)$ with support in $\mathcal{K}$, we have, as $|T| \to \infty$,*

$$\frac{1}{|T|}\int_0^\infty f(x_0 h(s)a(T))ds = \int_{\Gamma\backslash G} f\,d\mu + O(|T|^{-1}\log|T|)$$

where the implied constant depends only on $\mathcal{K}$ and a Sobolev norm of f.

Corollary 1. *Suppose that x_0H is closed and non-compact. For any $f \in C_c(\Gamma\backslash G)$,*

$$\lim_{T\to\pm\infty} \frac{1}{2|T|}\int_{-\infty}^{\infty} f(x_0h(s)a(T))\,ds = \int_{\Gamma\backslash G} f\,d\mu.$$

References

[1] W. Duke, Z. Rudnick and P. Sarnak, *Density of integer points on affine homogeneous varieties*, Duke Math. J. 71 (1993), no. 1, 143 -179.

[2] A. Eskin and C. Mcmullen, *Mixing, counting, and equidistribution in Lie groups*, Duke Math. J. 71 (1993), no. 1, 181 -209.

Dynamics and continued fractions

Uri Shapira

(joint work with Manfred Einsiedler and Lior Fishman and with Menny Aka)

We report several results regarding the continued fraction expansion (c.f.e) of the numbers composing a sequence of the form $\{n\alpha\}$, as n ranges over the natural numbers, or over the powers of some fixed prime. More precisely, we prove that for any real number α, if we let

$$c(\alpha) = \limsup a_n(\alpha),$$

where $a_n(\alpha)$ denotes the n'th coefficient in the c.f.e of α, then the sequence $c(n\alpha), n \in \mathbb{N}$, is unbounded. This is joint work with Manfred Einsiedler and Lior Fishman [EFS]. We also describe in some detail the proof of the following result obtained recently in a joint work with Menny Aka. Let α be a quadratic irrational and let p be a prime. Then the period of the c.f.e of $p^n\alpha$ exhibits statistics which converge to the one given by the Gauss measure. More precisely, if we denote for any finite word of natural numbers $b = (b_1, \ldots, b_\ell)$,

$$D(\alpha, b) = \liminf \frac{\text{number of times } b \text{ appears in the word } a_1(\alpha)\ldots a_N(\alpha)}{N}.$$

Then $\lim D(p^n\alpha, b)$ exists and is equal to the Gauss measure of the set $\{x \in [0,1] : a_1(x)\ldots a_\ell(x) = b\}$.

The proofs of the above results are dynamical. For the first result we rely on a rigidity result of Elon Lindenstrauss regarding the unique ergodicity of the action of the adelic points of the diagonal group on $\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})$. The second result uses a mixing argument on an S-arithmetic version of the above space. All the arguments are based on the tight connection between c.f.e and the geodesic flow on the unit tangent bundle of the modular surface, which is a factor of the above mentioned adelic and S-arithmetic spaces.

References

[EFS] M. Einsiedler, L. Fishman, U. Shapira, *Diophantine approximations on fractals*, preprint.

Margulis' conjecture for the locally divergent orbits on real Hilbert modular space forms

George Tomanov

Let $G = \underbrace{\mathrm{SL}(2,\mathbb{R}) \times \cdots \times \mathrm{SL}(2,\mathbb{R})}_{r}$, $r \geq 2$, and Γ be an irreducible non-uniform lattice in G. According to Selberg arithmeticity theorem Γ coincides (up to conjugation and commensurability) with the diagonal imbedding of $\mathrm{SL}(2,\mathcal{O})$ in G, where $\mathcal{O}$ is the ring of integers of K. Denote by D_i the group of diagonal matrices in the i-th copy of $\mathrm{SL}(2,\mathbb{R})$ and for every non-empty $I \subset \{1,\cdots,r\}$ denote by D_I the (direct) product of all $D_i, i \in I$. We write D instead of $D_{\{1,\cdots,r\}}$. The quotient space G/Γ is called *real Hilbert modular space form.* We denote by $\pi : G \to G/\Gamma$ the natural projection. The torus D_I is acting on G/Γ by left translations.

We are interested in the structure of the closures of the orbits $D_I\pi(g)$ when $\#I \geq 2$. The assumption $\#I \geq 2$ is very essential. Actually, the cases $\#I = 1$ and $\#I \geq 2$ represent different phenomena in many aspects. For instance, it is an unpublished result of Furstenberg and Benjamin Weiss that for any $\alpha \in [1,3]$ there is a point $x \in \mathrm{SL}(2,\mathbb{R})/\Gamma$ such that the closure $\overline{Dx}$ has Hausdorff dimension α. This is in sharp contrast to the theorem of Marina Ratner (proved for arbitrary Lie groups) which says that the closure of an orbit of subgroup generated by unipotent elements is *homogeneous*, i.e., it coincides with an orbit of a closed subgroup. A conjecture in this regard has been formulated by Gregory Margulis saying that if the action of an $\mathbb{R}$-split torus on a homogeneous space does not factor (in a natural way) to the action of a 1-dimensional split torus then its closure is homogeneous. (We refer to [M, Conjecture 1] for the precise formulation of the conjecture.) For action of a split torus T on $\mathrm{SL}(n,\mathbb{R})/\Gamma, n \geq 3$, T-orbits with non-homogeneous closures have been constructed by François Maucourant [Ma] if $n \geq 6$ and $\dim T = n-2$ and by Uri Shapira [Sh] if $n = 3$ and T is the full diagonal group.

In the present paper we explicitly describe the closures of the locally divergent D_I-orbits on G/Γ. An orbit $D_I\pi(g)$ is called *locally divergent* if $D_i\pi(g)$ is divergent (equivalently, closed) for all $i \in I$.

We prove the following:

Theorem 1. *With the above notation, let $\#I = 2$ and $D_I\pi(g)$ be a locally divergent orbit. The following propositions hold:*

(a) *if $g \in \mathcal{N}_G(D_I)G_K$ then $\overline{D_I\pi(g)} = T\pi(g)$, where T is a torus containing D_I;*

(b) *if $g \notin \mathcal{N}_G(D_I)G_K$ then $\overline{D_I\pi(g)} = D_I\pi(g) \cup \bigcup_{i=1}^{s} T_i\pi(h_i)$, where $2 \leq s \leq 4$, T_i are tori containing D_I and $T_i\pi(h_i)$ are pairwise different closed orbits. In particular, the closure of $D_I\pi(g)$ is not homogeneous which contradicts Margulis' conjecture.*

In the course of the proof of the theorem the pairwise different closed orbits $T_i\pi(h_i)$, $2 \leq s \leq 4$, are explicitly described in terms of g and there are examples when $s = 4$.

For action of maximal tori the above theorem implies:

Corollary 1. *Let $r = 2$. Then a locally divergent orbit $D\pi(g)$ is either closed or $\overline{D\pi(g)} = D\pi(g) \cup \bigcup_{i=1}^{s} D\pi(h_i)$, where $2 \leq s \leq 4$ and $D\pi(h_i)$ are pairwise different closed orbits. In particular, when $r = 2$ there are not dense locally divergent orbits.*

The situation differs drastically when $\#I > 2$.

Theorem 2. *Let $\#I > 2$ and $D_I\pi(g)$ be a locally divergent orbit. Then the following dichotomy holds:*

(a) *$g \in \mathcal{N}_G(D_I)G_K$ and $\overline{D_I\pi(g)} = T\pi(g)$, where T is a torus containing D_I;*
(b) *$g \notin \mathcal{N}_G(D_I)G_K$ and $D_I\pi(g)$ is a dense orbit.*

In the classical case of maximal tori Theorem 2 immediately implies:

Corollary 2. *Let $r > 2$. Then every locally divergent orbit D-orbit is either close or dense.*

Theorems 1 and 2 apply to the study of the values of rational binary quadratic forms at integral points. Namely, for every archimedean place v_i of K we denote by $f_i(X,Y) \in K[X,Y]$ a K-split non-degenerate quadratic form, i.e., $f_i(X,Y) = l_{i,1}(X,Y) \cdot l_{i,2}(X,Y)$ where $l_{i,1}$ and $l_{i,2} \in K[X,Y]$ are linearly independent over K linear forms. Let $A = \prod_{i=1}^{r} K_i$ and $A^* = \prod_{i=1}^{r} K_i^*$. We may (and will) regard $f := (f_i)_{i\in\overline{1,r}}$ as a polynomial in $A[X,Y]$ so that if $(\alpha,\beta) \in \mathcal{O}^2$, $f(\alpha,\beta)$ is an element in A with i-th coordinate equal $f_i(\alpha,\beta)$). It is trivial to see that if all $f_i, 1 \leq i \leq r$, are proportional over K then $f(\mathcal{O}^2)$ is discrete in A. In this case f is called *K-rational.* Theorem 1.8 from [T] implies the inverse: if $f(\mathcal{O}^2)$ is discrete in A then f is K-rational, that is, there exists a $\phi \in K[X,Y]$ such that $f_i = \alpha_i\phi, 1 \leq i \leq r$, where $\alpha_i \in K$. Theorems 1 and 2 allow to reinforce this result as follows.

Theorem 3. *With the above notation and assumptions, suppose that $f(\mathcal{O}^2)$ is not discrete in A, equivalently, that f is not K-rational. Then*

(a) *if $r = 2$, $\overline{f(\mathcal{O}^2)} \cap A^* = (f(\mathcal{O}^2) \cup \bigcup_{j=1}^{4} \phi^{(j)}(\mathcal{O}^2)) \cap A^*$ where $\phi^{(j)}$ are K-rational quadratic forms in $K[X,Y]$. In particular, $\overline{f(\mathcal{O}^2)} \cap A^*$ is a countable set;*
(b) *if $r > 2$, $f(\mathcal{O}^2)$ is dense in A.*

In the context of tori actions on G/Γ where G is an arbitrary real semisimple algebraic group the following conjecture seems plausible[1]:

[1]I am grateful to Elon Lindenstrauss for the useful discussion with him during the conference on the conjecture.

Conjecture 2. *Let G be a real $\mathbb{R}$-split semisimple algebraic group of rank > 1, Γ an irreducible lattice in G, D a maximal R-split torus of G and $x \in G/\Gamma$. Then either*

(1) $\overline{Dx} = G/\Gamma$, *or*

(2) $\overline{Dx} \setminus Dx \subset \bigcup_{i=1}^{N} H_i x_i$ *where H_i are closed proper subgroups of G containing D and $H_i x_i$ are closed orbits.*

References

[M] G. A. Margulis, *Problems and Conjectures in Rigidity Theory*, in **Mathematics: Frontiers and Perspectives** 161–174, Amer. Math. Soc. (2000).

[Ma] F. Maucourant, *A non-homogeneous orbit closure of a diagonal subgroup*, Ann. of Math.**171** (2010), 557-570.

[Sh] U.Shapira, *A solution to a problem of Cassels and Diophantine properties of cubic numbers*, preprint 2008 (to appear in Ann. of Math.).

[T] G. Tomanov, *Values of Decomposable Forms at S-integral Points and Orbits of Tori on Homogenesus Spaces*, Duke Math. Journal **138**, (2007) 533-562.

Expansion in Arithmetic Groups

Péter P. Varjú

(joint work with Jean Bourgain and with Alireza Salehi Golsefidy)

Let $\mathcal{G}$ be a graph, and for a set of vertices $X \subset V(\mathcal{G})$, denote by ∂X the set of edges that connect a vertex in X to one in $V(\mathcal{G}) \setminus X$. Define

$$c(\mathcal{G}) = \min_{X \subset V(\mathcal{G}),\ |X| \le |V(\mathcal{G})|/2} \frac{|\partial X|}{|X|},$$

where $|X|$ denotes the cardinality of the set X. A family of graphs is called a family of expanders, if $c(\mathcal{G})$ is bounded away from zero for graphs $\mathcal{G}$ that belong to the family. Expanders have a wide range of applications in computer science (see e.g. Hoory, Linial and Widgerson [8] for a recent survey on expanders) and recently they found remarkable applications in pure mathematics as well (see Bourgain, Gamburd and Sarnak [5] and Long, Lubotzky and Reid [9]). For further motivation, we refer to these papers.

Let G be a group and let $S \subset G$ be a symmetric (i.e. closed for taking inverses) set of generators. The Cayley graph $\mathcal{G}(G, S)$ of G with respect to the generating set S is defined to be the graph whose vertex set is G, and in which two vertices $x, y \in G$ are connected exactly if $y \in Sx$. Let q be a positive integer, and denote by $\pi_q : \mathbf{Z} \to \mathbf{Z}/q\mathbf{Z}$ the residue map. π_q induces maps in a natural way in various contexts, we always denote these maps by π_q. Consider a fixed symmetric $S \subset SL_d(\mathbf{Z})$ and assume that it generates a group G which is Zariski-dense in SL_d. We study the problem whether for a fixed set S and for q running through the integers, the family of Cayley graphs $\mathcal{G}(SL_d(\mathbf{Z}/q\mathbf{Z}), \pi_q(S))$ is an expander family or not. More precisely we, we report on the following two results:

Theorem 1 (Bourgain, V [6, Theorem 1]). *Let $S \subset SL_d(\mathbf{Z})$ be finite and symmetric. Assume that S generates a subgroup $G < SL_d(\mathbf{Z})$ which is Zariski dense in SL_d.*

Then $\mathcal{G}(\pi_q(G), \pi_q(S))$ form a family of expanders, when S is fixed and q runs through the integers. Moreover, there is an integer q_0 such that $\pi_q(G) = SL_d(\mathbf{Z}/q\mathbf{Z})$ if q is coprime to q_0.

Theorem 2 (Salehi Golsefidy, V [11, Theorem 1]). *Let $\Gamma \subseteq GL_d(\mathbf{Z}[1/q_0])$ be the group generated by a symmetric set $S \subset SL_d(\mathbf{Q})$. Then $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$ form a family of expanders when q ranges over square-free integers coprime to q_0 if and only if the connected component of the Zariski-closure of Γ is perfect.*

Both proofs follows the same lines as any of the papers [2]–[5], [12] which contain similar results in less generality. In the course of the proof one needs to show that for certain sets $A \subset SL_d(\mathbf{Z}/q\mathbf{Z})$ related to the generating set S we have $|A.A.A| > |A|^{1+\varepsilon}$ for some $\varepsilon > 0$. Here $A.A.A$ denotes the set of products of any three elements of A. This ingredient is the main part of the proofs and the only essential difference between results of this type. For Theorem 1 we use the square-free case (already proven combining [7], [10] with [12]) together with the result in the paper [1]. For Theorem 2 we first prove that we only need to consider sets A which are sufficiently well-distributed among the cosets of large index subgroups, and then we show the required inequality for sets A satisfying this condition.

References

[1] J. Bourgain, A. Furman, E. Lindenstrauss, S. Moses, *Stationary measures and equidistribution for orbits of non-abelian semigroups on the torus,* preprint available online: http://www.math.princeton.edu/~elonl/Publications/BFLM1.PDF

[2] J. Bourgain and A. Gamburd, *Uniform expansion bounds for Cayley graphs of $SL_2(\mathbf{F}_p)$*, Ann. of Math. **167** (2008), 625–642.

[3] J. Bourgain and A. Gamburd, *Expansion and random walks in $SL_d(\mathbf{Z}/p^n\mathbf{Z})$:I*, J. Eur. Math. Soc. **10** (2008), 987–1011.

[4] J. Bourgain and A. Gamburd, *Expansion and random walks in $SL_d(\mathbf{Z}/p^n\mathbf{Z})$:II. With an appendix by J. Bourgain*, J. Eur. Math. Soc. **11** No. 5. (2009), 1057–1103.

[5] J. Bourgain, A. Gamburd and P. Sarnak, *Affine linear sieve, expanders, and sum-product*, preprint, available at http://www.math.princeton.edu/sarnak/sespM8.pdf

[6] J. Bourgain and P.P. Varjú, *Expansion in $SL_d(\mathbf{Z}/q\mathbf{Z})$, q arbitrary*, preprint avaliable online: http://arxiv.org/abs/1006.3365

[7] E. Breuillard, B. Green and T. Tao, *Approximate subgroups of linear groups* preprint available online: http://arxiv.org/abs/1005.1881

[8] S. Hoory, N. Linial and A. Widgerson, *Expander graphs and their applications*, Bull. Amer. Math. Soc., **43** No. 4 (2006), 439–561.

[9] D. D. Long, A. Lubotzky and A. W. Reid, *Heegaard genus and property τ for hyperbolic 3-manifolds*, J. Topol., **1** (2008), 152–158.

[10] L. Pyber, E. Szabó, *Growth in finite simple groups of Lie type of bounded rank*, preprint available online: http://arxiv.org/abs/1005.1858

[11] A. Salehi Golsefidy, P. P. Varjú, *Expansion in perfect groups*, preprint

[12] P. P. Varjú, *Expansion in $SL_d(O_K/I)$, I square-free,* preprint available online: http://arxiv.org/abs/1001.3664

Topological self-joinings of full Cartan actions by toral automorphisms

Zhiren Wang
(joint work with Elon Lindenstrauss)

As a higher-dimensional extension to Furstenberg's famous theorem on $\times 2$, $\times 3$-invariant sets, the following was proved by Berend:

Theorem 1. *(Berend, 1983) For an abelian group G of $SL_d(\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^d)$, if the following assumptions hold:*
(1). G contains a totally irreducible toral automorphism;
(2). For any common eigenvector $v \in \mathbb{C}^d$ of G, there exists $g \in G$ such that $|g.v| > |v|$;
(3). $\mathrm{rank}(G) \geq 2$.
then $\forall x \in \mathbb{T}^d$, the orbit $G.x$ is dense in $\mathbb{T}^d$ unless x is a rational point.

Here a toral automorphism $g \in SL_d(\mathbb{Z})$ is said to be irreducible if there is no non-trivial g-invariant subtorus in $\mathbb{T}^d$, it is totally irreducible if g^n is irreducible for all $n \neq 0$.

In particular, Berend's theorem covers the special case of full Cartan actions, which is defined by the following condition:

Condition 1. *G is an abelian subgroup of $SL_d(\mathbb{Z})$ with* $\mathrm{rank}(G) \geq 2$*, such that:*
(1). G contains a totally irreducible toral automorphism;
(2). G is maximal in rank: there is no intermediate abelian subgroup G_1 in $SL_d(\mathbb{Z})$ containing G such that $\mathrm{rank}(G) < \mathrm{rank}(G_1)$

Condition 1 is of particular number-theoretical interest because the action of such a group G on $\mathbb{T}^d$ is, up to passing to a finite index subgroup, conjugate to the multiplicative action of U_K, the group of units of a non-CM number field K of degree d, on some arithmetic compact quotient of $K \otimes_{\mathbb{Q}} \mathbb{R}$. Notice this implicitly requires $\mathrm{rank}(G) \leq d-1$.

We try to understand what happens if the action is no longer irreducible. More precisely, consider the diagonal action of a group G satisfying Condition 1 on $\mathbb{T}^d \times \mathbb{T}^d$. We ask how orbit closures look like in this case. It is not hard to see the following three types of subsets of $\mathbb{T}^d \times \mathbb{T}^d$ can be orbit closures:

(I). A finite G-invariant set consisting of rational points, on which G acts transitively.

(II). A finite union of d-dimensional parallel subtori T_k, each of which is of the form $\{(x_1, x_2) \in \mathbb{T}^d \times \mathbb{T}^d | A_1 x_1 + A_2 x_2 = z_k\}$. Where A_1, A_2 are toral endomorphisms from $M_d(\mathbb{Z})$ that commute with G and don't depend on k; and the G-action permutes the T_k's in a transitive way.

(III). $\mathbb{T}^d \times \mathbb{T}^d$ itself.

It is natural to guess that these exhaust all the possibilities. Such a guess is true if $\mathrm{rank}(G) \geq 3$, but turns out to be false when $\mathrm{rank}(G) = 2$.

Theorem 2. *(L.-W., 2010) Suppose G satisfies Condition 1 and let it act diagonally on $\mathbb{T}^d \times \mathbb{T}^d$.*

(1). If $\mathrm{rank}(G) = 2$ then there exists a point $x \in \mathbb{T}^d \times \mathbb{T}^d$ and three d-dimensional subtori T_1, T_2, T_3 which are not parallel to each other, such that $\overline{G.x} = (G.x) \bigsqcup (\sqcup_{i=1}^3 T_i)$.

(2). If $\mathrm{rank}(G) \geq 3$, then $\forall x \in \mathbb{T}^d \times \mathbb{T}^d$, $\overline{G.x}$ is homogeneous, i.e. it belongs to one of the three classes (I), (II) and (III) described above.

The rank two case is proved in a constructive way, while the proof of case (2) relies on the following variation of Berend's original theorem:

Theorem 3. *Suppose G satisfies Condition 1 and has rank at least 3. Let $v \in \mathbb{C}^d$ be a common eigenvector of G. Then $\forall x \in \mathbb{T}^d$, $\forall \epsilon > 0$, the set $\left\{g.x \middle| \frac{|g.v|}{|v|} \in (1-\epsilon, 1+\epsilon)\right\}$ is dense in $\mathbb{T}^d$ unless:*

(i). $x = x_0 + rv$ for some rational point $x_0 \in \mathbb{T}^d$ and $r \in \mathbb{R}$ if v is real;

(ii). $x = x_0 + r\mathrm{Re}v + s\mathrm{Im}v$ for some rational point $x_0 \in \mathbb{T}^d$ and $r, s \in \mathbb{R}$ if v is imaginary.

This can be interpreted as the action of the subset $\left\{g \middle| \frac{|g.v|}{|v|} \in (1-\epsilon, 1+\epsilon)\right\} \subset G$ is approximately a group action, whose rank is roughly $\mathrm{rank}(G) - 1$. Therefore heuristically, the subaction in question has rank at least 2 and satisfies assumption (1) in Theorem 1 if $\mathrm{rank}(G) \geq 3$. However remark that the hyperbolicity assumption (3) in Theorem 1 fails in this setting, which gives rise to the exceptional cases in Theorem 3.

References

[1] D. Berend, *Multi-invariant sets on tori*, Trans. Amer. Math. Soc. 280 (1983), no. 2, 509-532.

Counting Rational Matrices of a Fixed Irreducible Characteristic Polynomial

Thomas Zamojski

As part of the study of rational points on algebraic varieties, we establish an asymptotic formula for the number of rational matrices of bounded height and of a fixed characteristic polynomial. This solves a new case of Manin's Conjecture (see [1] or see the survey article [9] for the conjecture).

Counting Estimate

Let $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0$ be a monic polynomial with rational coefficients and irreducible over the rationals $\mathbb{Q}$. For simplicity, we also assume the polynomial splits over the reals, although this is not necessary. Let V be the affine variety of $n \times n$ matrices of characteristic polynomial $P(\lambda)$, which is a variety defined over $\mathbb{Q}$. To a rational point $v \in V_{\mathbb{Q}}$ whose coordinates are over a common denominator and in lowest form, we define the height of v as the maximum in absolute value of the numerators and the denominator. We will denote this number by $Ht(v)$. It has the property that the number of rational points of height bounded by R is finite, and we denote this number by N_R.

Theorem 1 (counting estimate)**.** *There exists a constant C such that*

$$\lim_{R\to\infty} \frac{N_R}{R^{n(n-1)/2+1}} = C.$$

Homogeneity

The variety V is homogeneous: the affine algebraic group $G = PGL_n$ acts transitively on V by conjugation of matrices. By the existence rational canonical form for matrices, the rational points $V_{\mathbb{Q}}$ is a single orbit of $G_{\mathbb{Q}} = PGL_n(\mathbb{Q})$. The group $G_{\mathbb{Q}}$ is a lattice in the adelic group $G_{\mathbb{A}}$ via its diagonal embedding. Therefore the counting problem is one of counting the points on the orbit of a lattice.

Choosing a base point $v_0 \in V_{\mathbb{Q}}$, we let H be the stabiliser of v_0 in G. The group H is a $\mathbb{Q}$-anisotropic maximal torus, which implies that $H_{\mathbb{Q}}$ is cocompact in $H_{\mathbb{A}}$. Let μ_G be a Haar measure on $G_{\mathbb{Q}}\backslash G_{\mathbb{A}}$, μ_H be a Haar measure on $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}$ and vol be a $G_{\mathbb{A}}$-invariant measure on $V_{\mathbb{A}}$, satisfying the compatibility condition $d\mu_G = d\mu_H\, dvol$. Also, the height function extends to $V_{\mathbb{A}}$, and we denote B_R the height-ball of radius R, that is the set of elements in $V_{\mathbb{A}}$ of height bounded by R. Two functions $f(R)$ and $g(R)$ are said to be asymptotically the same, written $f \sim g$, if their ratio goes to 1 as $R \to \infty$. We prove that

Theorem 2.

$$N_R \sim \frac{\mu_H(H_{\mathbb{Q}}\backslash H_{\mathbb{A}})}{\mu_G(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})} vol(B_R).$$

Equidistribution

Any element $g \in G_{\mathbb{A}}$ acts on $X_{\mathbb{A}} = G_{\mathbb{Q}}\backslash G_{\mathbb{A}}$ by translation, and thus induces an action on measures on $X_{\mathbb{A}}$. The measure μ_H can be viewed as a measure on $X_{\mathbb{A}}$ since the orbit $G_{\mathbb{Q}}.H_{\mathbb{A}}$ in $X_{\mathbb{A}}$ is $H_{\mathbb{A}}$-equivariantly homeomorphic to $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}$. In that sense, we can state the following distribution theorem.

Theorem 3 (adelic mean ergodicity)**.** *With notation as above, as $R \to \infty$,*

$$\frac{1}{vol(B_R)} \int_{B_R} \mu_H \cdot g \; dvol(g) \to \mu_G \;,$$

where convergence is meant in the weak--topology.*

On the Proofs

Theorem 3 implies Theorem 2 via a classical application of unfolding [3, 6]. Theorem 2 implies Theorem 1 amounts to the computation of an asymptotic formula for $vol(B_R)$. Recent work of Chambert-Loir and Tschinkel establishes such a formula [2]. This aspect will be ignored here. Instead, we refer the reader to their paper and [10].

For the proof of Theorem 3, inspired by the work of Eskin, Mozes and Shah, we would like to apply Ratner's measure rigidity theorem. However, there is an important distinction in the adelic setting that makes this impossible. In general, as $H_{\mathbb{A}}g$ leaves compact sets, the tori $g^{-1}H_{\mathbb{A}}g$ do not stretch enough to give in the limit invariance under unipotent elements. This is because generically, $H_{\mathbb{A}}g$ tends to infinity in the places, but remains bounded for any given place. Therefore, a limit will be invariant under a real torus, but no unipotent elements a priori.

Consider now a limit μ of the averages of Theorem 3 as $R \to \infty$. It follows from [5] that μ is a probability measure given by a (continuous) linear combination of measures ν invariant under a maximal real split torus $T_{\mathbb{R}}$, together with a suitable positive entropy condition. From the measure rigidity theorem of Einsiedler, Katok and Lindenstrauss [4], each such measure ν is of the form $\sum \nu_L$, where the sum is over a countable class of subgroups L of $G_{\mathbb{R}}$ containing $T_{\mathbb{R}}$,and ν_L is supported on a subvariety consisting of translated $L \times G_{\hat{\mathbb{Z}}}$-orbits. If $L = G_{\mathbb{R}}$, we retrieve the Haar measure μ_G. Therefore, to prove Theorem 3 we have to prove that on average, the support of $\mu_H g$ spends little time in a neighbourhood of the support of the intermediary measures μ_L, $L \neq G_{\mathbb{R}}$. The lack of linearisation in the adelic setting is circumvented by using directions coming from the translations by g transverse to the $H_{\mathbb{A}}$-orbits, together with non-divergence.

Related Literature

As mentioned before, our solution is inspired by [7], but with two major differences coming from the adeles: no invariance under unipotent elements a priori, and no linearisation technique. However, the method of proof remains dynamical.

To our knowledge, the first application of measure rigidity to the problem of counting rational points on varieties is in the work of Gorodnik and Oh [8]. Varieties considered are equivariant compactifications of the variety $H\backslash G$, where G is a semi-simple algebraic group, and the stabiliser H is a semisimple maximal subgroup of G. In this case, the two difficulties of the previous paragraph do not appear. Nonetheless, their solution was influential to our work.

Finally, in the special case that the size n of the square matrices is a prime number, Theorem 3 follows from the work of Einsiedler, Lindenstrauss, Michel and Venkatesh on Duke's Theorem for cubic fields [5]. However, their method does not address the issue of the possible presence of intermediary measures, which is not needed when n is prime.

References

[1] V. V. Batyrev and Yu. I. Manin *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Math. Ann. **286** (1990), no.1-3, 27-43.

[2] A. Chambert-Loir and Y. Tschinkel *Igusa integrals and volume asymptotics in analytic and adelic geometry*, (2009), http://arxiv.org/abs/0909.1568v1.

[3] W. Duke, Z. Rudnick, and P. Sarnak *Density of integer points on affine homogeneous varieties*, Duke Math. J. **71** (1993), 143-179.

[4] M. Einsiedler, A. Katok, and E. Lindenstrauss *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. (2) **164** (2006), no. 2, 513-560.

[5] M. Einsiedler, E. Lindenstrauss, PH. Michel, and A. Venkatesh *Distribution of periodic torus orbits and Duke's theorem for cubic fields*, to appear in Annals of Mathematics.

[6] A. Eskin and C. McMullen *Mixing, counting and equidistribution in Lie groups*, Duke Math. J. **71** (1993), 181-209.

[7] A. Eskin, S. Mozes, and N. Shah *Unipotent flows and counting lattice points on homogeneous varieties*, Ann. of Math. **143** (1996), 253-299.

[8] A. Gorodnik and H. Oh *Rational points on homogeneous varieties and equidistribution of adelic periods*, preprint 2008, arXiv:0803.1996.

[9] Yuri Tschinkel *Geometry over nonclosed fields*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, (2006), 637-651.

[10] Thomas Zamojski *Counting Rational Matrices of a Fixed Irreducible Characteristic Polynomial*, In preparation, 2010.

Reporter: Anish Ghosh

Participants

Prof. Dr. Herbert Abels
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Dr. Jayadev S. Athreya
Department of Mathematics
University of Illinois at
Urbana-Champaign
1409 West Green Street
Urbana IL 61801
USA

Prof. Dr. Yves Benoist
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Emmanuel Breuillard
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Francesco Cellarosi
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Manfred Einsiedler
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Alex Eskin
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Elena Fuchs
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Prof. Dr. Alex Furman
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago , IL 60607-7045
USA

Dr. Anish Ghosh
School of Mathematics
University of East Anglia
GB-Norwich NR4 7TJ

Prof. Dr. Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Prof. Dr. Alexander Gorodnik
Department of Mathematics
University of Bristol
University Walk
GB-Bristol BS8 1TW

Prof. Dr. Ben Green
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WA

Prof. Dr. Antonin Guilloux
Institut de Mathematiques
Universite Paris VI
UFR 929
4, Place Jussieu
F-75005 Paris Cedex

Prof. Dr. Yves Guivarch
U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Dr. Alan K. Haynes
Department of Mathematics
University of York
GB-Heslington, York YO10 5DD

Prof. Dr. Dimitar Jetchev
Section de Mathematiques
Station 8
Ecole Polytechnique Federale de Lausanne
CH-1015 Lausanne

Shirali Kadyrov
Howard House 4
Queens Ave.
GB-Bristol BS8 1TW

Dr. Fanny Kassel
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Anatole B. Katok
Department of Mathematics
Pennsylvania State University
303 McAllister Building
University Park , PA 16802
USA

Prof. Dr. Svetlana Katok
Department of Mathematics
Pennsylvania State University
University Park , PA 16802
USA

Prof. Dr. Dmitry Kleinbock
Department of Mathematics
Brandeis University
Waltham , MA 02454-9110
USA

Prof. Dr. Bruno Klingler
Universite Paris VII
Institut de Mathematiques de Jussieu
175, rue de Chevaleret
F-75013 Paris

Prof. Dr. Alex Kontorovich
School of Mathematics
Institute for Advanced Study
Einstein Drive
Princeton , NJ 08540
USA

Prof. Dr. Emmanuel Kowalski
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Francois Ledrappier
LPMA / UMR 7599
Universite Pierre & Marie Curie
Paris VI
Boite Courrier 188
F-75252 Paris Cedex 05

Han Li
Department of Mathematics
Yale University
Box 208 283
New Haven , CT 06520
USA

Prof. Dr. Elon Lindenstrauss
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
91904 Jerusalem
ISRAEL

Beverly Lytle
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Grigorii A. Margulis
Department of Mathematics
Yale University
Box 208 283
New Haven , CT 06520
USA

Prof. Dr. Jens Marklof
School of Mathematics
University of Bristol
University Walk
GB-Bristol BS8 1TW

Prof. Dr. Francois Maucourant
Departement de Mathematiques
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Dave Witte Morris
Department of Mathematics and
Computer Science
University of Lethbridge
4401 University Drive
Lethbridge AB T1K 3M4
CANADA

Prof. Dr. Shahar Mozes
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Hee Oh
Department of Mathematics
Brown University
Box 1917
Providence , RI 02912
USA

Prof. Dr. Frederic Paulin
DMA UMR 8553 CNRS
Ecole Normale Superieure
45 rue d'Ulm
F-75230 Paris Cedex 05

Prof. Dr. Jean-Francois Quint
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. Thomas Roblin
Laboratoire de Probabilités et
Modèles Aléatoires
Université Paris VI, Boite 188
4, Place Jussieu
F-75252 Paris Cedex 05

Prof. Dr. Barbara Schapira
UFR de Sciences
Universite de Picardie Jules Verne
33 rue Saint Leu
F-80039 Amiens Cedex

Prof. Dr. Nimish A. Shah
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Uri Shapira
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Ralf J. Spatzier
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor , MI 48109-1043
USA

Prof. Dr. Andreas Strömbergsson
Department of Mathematics
University of Uppsala
Thunbergsvägen 3
S-752 38 Uppsala

Prof. Dr. Georges Tomanov
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Adrian Ubis Martinez
Departamento de Matematicas
Universidad Autonoma de Madrid
Ciudad Universitaria de Cantoblanco
E-28049 Madrid

Peter Varju
Mathematics Department
Princeton University
Fine Hall
Washington Road
Princeton NJ 08544-1000
USA

Zhiren Wang
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Tomasz Zamojski
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Prof. Dr. Tamar Ziegler
Department of Mathematics
Technion - Israel Institute of
Technology
Haifa 32000
ISRAEL