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Representation Theory and Harmonic Analysis

Organised by
Toshiyuki Kobayashi, Tokyo
Bernhard Krötz, Hannover

November 14th – November 20th, 2010

Abstract. The workshop gave an overview of current research in the representation theory and harmonic analysis of reductive Lie groups and its relation to algebraic number theory. Some particular topics covered in the 17 talks related to unitarity questions and globalizations for Harish–Chandra modules, Fourier transformation on symmetric spaces and p–adic groups, affine Hecke algebras or the spectral theory of automorphic forms and trace formulas.

Mathematics Subject Classification (2000): 22xx, 43xx.

Introduction by the Organisers

The international conference *Representation Theory and Harmonic Analysis*, organized by Toshiyuki Kobayashi (Tokyo) and Bernhard Krötz (Hannover) was held November 14th – November 20th, 2010. The general theme was representation theory of real and p-adic reductive Lie groups, also in connection to automorphic forms. Further talks included relations to other fields such as lattice counting, infinite symmetric groups or quantum ergodicity.

The format of the workshop consisted of 2–hour lectures by leading specialists supplemented by shorter presentations, many of which were given by younger participants. In between the talks the schedule reserved plenty of time for informal discussions. Thursday was reserved for a joint session with the parallel workshop on infinite–dimensional representation theory.

Topics covered in the presentations included unitarity questions and globalizations for Harish–Chandra modules, Fourier and Radon transforms on symmetric spaces or p–adic groups, affine Hecke algebras or the spectral theory of automorphic forms and trace formulas.

More specifically, as major topics of the workshop Schmid proposed Hodge–theory as an ingredient to understand the unitary dual of a real reductive Lie group in terms of its Harish–Chandra modules, and Bernstein gave an alternative approach to Casselman–Wallach's theorem about the unique smooth globalization of Harish–Chandra modules. The latter was supplemented by Gimperlein's talk for the analytic case.
Questions from classical harmonic analysis and Fourier transformation were treated in Delorme's presentation of a Plancherel theorem for p–adic reductive groups, in the shorter contributions by van den Ban and Kuit for symmetric spaces as well as by Sayag and Vargas.
The spectral theory of automorphic forms and trace formulas were surveyed by Müller. The number theoretic issues were followed up on by Offen's application of a relative trace formula to period integrals. Aizenbud generalized Jaquet's smooth transfer of Kloosterman integrals to the Archimedean case. Also of relevance to local trace formulas, Opdam employed techniques for affine Hecke algebras to generalize a formula by Arthur for an Euler–Poincaré pairing to fields of arbitrary characteristic.
Among the further topics, Sahi outlined a detailed analysis of Whittaker functionals and associated varieties for irreducible unitary representations of $GL(n, \mathbb{R})$. In a shorter presentation, Möllers exhibited explicit L^2–models for minimal representations. Connections to the infinite–dimensional theory featured in the parallel workshop were present in Neretin's talk on the permutation group S_∞. Hilgert discussed quantum ergodicity on Riemannian symmetric spaces of noncompact type. Finally, the desingularization of deformation spaces associated to discontinuous group actions was considered by Yoshino.

Workshop: Representation Theory and Harmonic Analysis

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Abstracts

Smooth transfer of Kloosterman integrals

Avraham Aizenbud

Fix a local field F and it's quadratic extension E. We consider an integral of a Schwartz function on $GL_n(F)$ along the orbits of the two sided action of the groups of upper unipotent matrices twisted by a non-degenerate character. This gives a smooth function on the torus. We prove that the space of all functions obtained in such a way coincides with the space that is constructed analogously when $GL_n(F)$ is replaced with the variety of non-degenerate hermitian forms.

The non-Archimedean case is done by Jaquet and the Archimedean case is done by Gourevitch and myself. In the talk we discussed the main ingredients of the proof, the difficulties that occur in the Archimedean case and the methods we used to overcome them.

Polynomial estimates for c-functions on semisimple symmetric spaces

Erik van den Ban
(joint work with Henrik Schlichtkrull)

In the talk I reported on estimates for c-functions associated to semisimple symmetric spaces. The c-functions determine the top order asymptotic behavior of Eisenstein integrals for such spaces. The estimates are of importance for the comparison of different formulations of the Paley-Wiener theorem, see [3].

Let G be a real semisimple Lie group with finite center, let σ be an involution of G, and let H be an open subgroup of the group G^σ of fixed points. Then G/H is a semisimple symmetric space.

There exists a maximal compact subgroup K of G which is stable under σ and let θ be the associated Cartan involution. Let (τ, V_τ) be a finite dimensional unitary representation of K and let $C^\infty(G/H : \tau)$ denote the space of smooth functions $G/H \to V_\tau$ that are τ-spherical, i.e.,

$$f(kx) = \tau(k)f(x), \qquad (x \in G/H, k \in K).$$

Then there exists a most continuous Fourier transform on the subspace of $C_c^\infty(G/H : \tau)$ of compactly supported spherical functions. This transform is defined in terms of normalized Eisenstein integrals

$$E^\circ(\lambda) \in C_c^\infty(G/H : \tau) \otimes \mathcal{A}_2$$

which are defined as spherical matrix coefficients of the minimal principal series for G/H. Here $\mathcal{A}_2$ is a certain finite dimensional Hilbert space, and the Eisenstein integral depends meromorphically on the parameter λ, which ranges over the complex dual $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ of $\mathfrak{a}_{\mathrm{q}}$, a maximal abelian subspace of $\ker(I + d\theta(e) \cap \ker(I + d\sigma(e))$.

The generic directions to infinity in the symmetric space G/H are described in terms of a positive Weyl chamber A_{q}^+ and a quotient $\mathcal{W} = W/W_{K\cap H}$, where

$W = N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$ and where $W_{K\cap H} = N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})/Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$. In fact, the following union is disjoint and gives an open dense subset of G/H :

$$\bigcup_{v\in\mathcal{W}} KA_{\mathrm{q}}^+ vH.$$

Let M denote the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in G, let τ_M denote the restriction of τ to $K\cap M$. Then the space $\mathcal{A}_2$ mentioned above equals the following direct sum of Hilbert spaces

$$\mathcal{A}_2 = \bigoplus_{v\in\mathcal{W}} L^2(M/M\cap vHv^{-1} : \tau_M).$$

Moreover, the top order asymptotic behavior of the Eisenstein integral is described by

$$E^\circ(\lambda : mav)\psi \sim \sum_{s\in W} a^{s\lambda-\rho}[\mathrm{pr}_v C^\circ(s:\lambda)\psi](m), \qquad (A_{\mathrm{q}}^+ \ni a \to \infty),$$

for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$, $v \in \mathcal{W}$ and $m \in M/M\cap vHv^{-1}$. Here the $C^\circ(s:\lambda)$ are $\mathrm{End}(\mathcal{A}_2)$-valued meromorphic functions of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. The estimates mentioned in the title take the following form.

Theorem. *For every $R > 0$ there exists a polynomial functions $q : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$ and constants $C > 0, N \in \mathbb{N}$ such that*

$$\|q(\lambda)\, C^\circ(s:\lambda)\| \leq C(1+|\lambda|)^N,$$

for all $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with $|\mathrm{Re}\lambda| \leq R$.

Moreover, for the above polynomial q one may take a product of linear factors of the form $\langle\, \lambda\,,\, \alpha\,\rangle - c$, with α an $\mathfrak{a}_{\mathrm{q}}$-root, and $c \in \mathbb{R}$.

The proof of this result is based on application of the following results:

(a) the Vogan-Wallach functional equation for standard intertwining operators (cf. [1]);
(b) the functional equation for H-fixed distribution vectors of the minimal principal series of G/H (cf. [2]);
(c) estimates for the evaluation of the above mentioned H-fixed distribution vectors at points of $\mathcal{W}$.

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Understanding Casselman–Wallach's globalization theorem

JOSEPH BERNSTEIN
(joint work with Bernhard Krötz)

In this talk I describe a new approach to the proof of the Casselman–Wallach theorem, which shows that the category of Harish-Chandra modules for a real reductive group G is naturally equivalent to the category of smooth admissible F–representations (i.e. Fréchet representations of moderate growth). This approach was developed by B. Krötz and myself. It describes the globalization of Harish-Chandra modules uniformly in their parameters. This fact has important applications in the theory of Eisenstein series.

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Plancherel formula for Whittaker functions on a p-adic reductive group

PATRICK DELORME

Let G be the group of F- points of a connected reductive group defined over F, where F is a non archimedian local field.
Let (P_0, P_0^-) be a pair of opposite parabolic subgroups of G and let M_0 be their common Levi subgroup.
Let U_0 be the unipotent radical of P_0 and A_0 be the maximal split torus M_0. Let K be an A_0-good maximal compact subgroup of G.
Let ψ be non degenerate smooth unitary character of U_0.
Let $C^\infty(U_0\backslash G, \psi)$ be the space of smooth Whittaker functions on G i.e. functions f on G which are right invariant by a compact open subgroup of G and such that:

$$f(u_0 g) = \psi(u_0) f(g), g \in G, u_0 \in U_0.$$

One defines a Schwartz space $\mathcal{C}(U_0\backslash G, \psi) \subset C^\infty(U_0\backslash G, \psi)$ (analogue of Harish-Chandra Schwartz space for functions on G), endowed with a natural topology.
One defines a Fourier transform for $\mathcal{C}(U_0\backslash G, \psi)$, called Whittaker-Fourier (WFT) transform.
The main result is the determination of the image of this transform and an inversion formula. Moreover $\mathcal{C}(U_0\backslash G, \psi)$ is a subspace of $L^2(U_0\backslash G, \psi)$ and one determines also how the scalar product behaves under WFT.
This is an unpublished work of Harish-Chandra, as said by Clozel.
For the real case there is a proof by Wallach in his book.
Here, it is a different type of proof. It follows closely Waldspurger redaction of the Harish-Chandra Plancherel formula for functions on the group.
Apart the results, this work introduce a setting with B-matrices and wave packets which should work also for p-adic symmetric spaces.

Analytic globalizations of Harish-Chandra modules

Heiko Gimperlein
(joint work with Bernhard Krötz, Henrik Schlichtkrull)

Let G be a real reductive Lie group and $K < G$ a maximal compact subgroup. We consider representations (π, E) of G on a locally convex Hausdorff space E. A vector $v \in E$ is said to be *analytic* provided that the orbit map $\gamma_v : G \to E$, $x \mapsto \pi(x)v$, extends to a holomorphic function on a left–G–invariant neighborhood in $G_{\mathbb{C}}$. The space E^ω of analytic vectors is endowed with a natural inductive limit topology $E^\omega = \lim_{n\to\infty} E_n$,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\} ,$$

indexed by a neighborhood basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity in $G_{\mathbb{C}}$, and (π, E) is said to be *analytic* provided that $E = E^\omega$ as topological vector spaces. No completeness assumptions on E are imposed, so that the quotient of an analytic representation by a closed invariant subspace is again analytic.

Moderately growing analytic representations allow for an additional action by an algebra of superexponentially decaying functions. To be specific, consider a Banach representation (π, E). Fix a left–invariant Riemannian metric on G and let d be the associated distance function. The continuous functions on G decaying faster than $e^{-n\mathrm{d}(\cdot,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a convolution algebra $\mathcal{R}(G)$, which is a G–module under the left regular representation. If we denote the space of analytic vectors of $\mathcal{R}(G)$ by $\mathcal{A}(G)$, the map

$$\Pi : \mathcal{A}(G) \to \mathrm{End}(E^\omega),\ \Pi(f)v = \int_G f(x)\ \pi(x)v\ dx \ , \tag{1}$$

gives rise to a continuous algebra action on E^ω. More general representations on a sequentially complete space will be called $\mathcal{A}(G)$*–tempered* provided that the integral in (1) converges strongly and defines a continuous action of $\mathcal{A}(G)$.

Recall that to an admissible G-representation (π, E) of finite length one can associate the Harish-Chandra module E_K of its K-finite vectors and that $E_K \subset E^\omega$ e.g. if E is a Banach space. Conversely, a *globalization* of a given Harish–Chandra module V is an admissible representation of G with $V = E_K$. Our main result is an independent approach to aspects of Schmid's and Kashiwara's theory of globalizations, which asserts in particular that every Harish-Chandra module admits a unique *minimal globalization*, equivalent to E^ω for all Banach globalizations E.

Theorem ([3]). *Let G be a real reductive group. Then every Harish-Chandra module V for G admits a unique $\mathcal{A}(G)$–tempered analytic globalization V^{mg}. Moreover, V^{mg} has the property $V^{mg} = \Pi(\mathcal{A}(G))V$.*

Corollary. *a) $E^\omega \simeq V^{mg}$ for every $\mathcal{A}(G)$-tempered globalization E of V.*
b) For an irreducible admissible Banach representation, E^ω is an algebraically simple $\mathcal{A}(G)$-module.

Schmid constructs his minimal globalization from a realization of V as smooth functions on G using matrix coefficients. His and Kashiwara's results employ the convolution algebra of test functions instead of $\mathcal{A}(G)$ and apply to any complete globalization without temperedness assumptions. In our language they obtain a factorization $E^\omega = \Pi(C_c^\infty(G))E_K$ for every complete globalization E.

Our more explicit approach defines V^{mg} as the quotient of $\mathcal{A}(G)^k$ with its diagonal G-action by the kernel of the G-equivariant continuous map

$$\mathcal{A}(G)^k \to E, \quad (f_1, \dots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j)v_j \ ,$$

where V is considered as a subspace generated by $v_1, \dots, v_k$ in an arbitrary $\mathcal{A}(G)$–tempered globalization. V^{mg} is independent of the choice of the globalization and (up to equivalence) of the set of generators and satisfies $V^{\mathrm{mg}} = \Pi(\mathcal{A}(G))V$. It is minimal in the sense that it injects into the space of analytic vectors of any other $\mathcal{A}(G)$–tempered globalization, and its functorial properties are readily deduced from the construction. Topologically V^{mg} turns out to be a reflexive DNF space and is, in particular, complete.

General arguments and the functorial properties reduce the proof of the theorem to the case where the Harish-Chandra module V is the space of K–finite vectors of a spherical principal series representation. The main step is to show that in this situation V^{mg} coincides with the analytic functions on $M\backslash K$ as analytic representations, where $M = Z_K(A)$ for some Iwasawa decomposition $G = KAN$.

The key ingredient in the proof are some recent lower bounds for matrix coefficients obtained in [1]. The techniques developed by Bernstein and Krötz in their article in the context of smooth globalizations can be adapted to yield an explicit factorization of $v \in C^\omega(M\backslash K)$ as $v = \Pi(F)\xi$ for a suitable K-finite $\xi \in V$. The main new difficulty is to show $F \in \mathcal{A}(G)$. To do so, the coincidence of analytic and Δ–analytic vectors for Fréchet representations of moderate growth [2] turns out to be convenient. Both the theorem and the assertion in part b) of the Corollary follow from the thus obtained factorization $C^\omega(M\backslash K) = \Pi(\mathcal{A}(G))V = V^{\mathrm{mg}}$.

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Patterson–Sullivan distributions on locally symmetric spaces

Joachim Hilgert
(joint work with Sönke Hansen, Michael Schröder)

Let $X = G/K$ denote a Riemannian symmetric space of noncompact type, where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $G = KAN$ be a corresponding Iwasawa decomposition of G and let M denote the centralizer of A in K. The Furstenberg boundary of X can be identified with the flag manifold $B := K/M$. Let $o := K \in G/K$ denote the *origin* of the symmetric space X. Further, let Δ, resp. Δ_Γ, denote the Laplace operator of X, resp. X_Γ.

For each joint eigenfunction φ (with exponential growth) of the invariant differential operators there is a unique distribution boundary value $T_\varphi \in \mathcal{D}'(B)$ such that

$$\varphi(x) = \int_B e^{(i\lambda+\rho)\langle x,b\rangle} T_\varphi(db),$$

where $\lambda \in \mathfrak{a}^*$ is a suitable spectral parameter. Here $\langle x, b\rangle$ denotes the horocycle bracket. Given $a \in C_c^\infty(T^*X)$, the *Wigner distribution* associated with two such eigenfunctions φ and ψ is defined by

$$W_{\varphi,\psi}(a) := \langle \mathrm{Op}(a)\varphi, \psi\rangle_{L^2(X_\Gamma)}.$$

The goal is to describe weak limits of Wigner distributions as the spectral parameters of the eigenfunctions tend to infinity. To this end one introduces new distributions, called *Patterson–Sullivan distributions* as a weighted Radon transform of the boundary values $T_\varphi(db) \otimes T_\psi(db')$. The weight function is basically defined by its invariance properties. The construction generalizes a construction of Anantharaman and Zelditch for hyperbolic surfaces (cf. [1]). The two types of distributions can be related by an integral operator and analyzing this operator by stationary phase methods one proves that for functions a supported in an open Weyl chamber Wigner and Patterson-Sullivan distributions asymptotically agree.

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Radon transformation on reductive symmetric spaces: support theorems

Job Kuit

Let G be a connected semisimple Lie group with finite center, let $G = KAN$ be an Iwasawa decomposition of G and let M be the centralizer of A in K. A horosphere in the Riemannian symmetric space $X = G/K$ is a submanifold of the form $gN\cdot K$, with $g \in G$. The set of horospheres is in bijection with the manifold $\Xi = G/MN$ via the map

$$E : g \cdot MN \mapsto gN \cdot K.$$

The *horospherical transform* on X is the Radon transform $\mathcal{R}$ mapping a compactly supported smooth function ϕ on X to the function

$$\mathcal{R}\phi : g \cdot MN \mapsto \int_N \phi(gn \cdot K)$$

on Ξ. In [1, Lemma 8.1] S. Helgason proved the following support theorem for this transform.

Let ϕ be a compactly supported smooth function and let V be a closed ball in X. Assume that

$$\mathcal{R}\phi(\xi) = 0 \quad \textit{whenever} \quad E(\xi) \cap V = \emptyset.$$

Then

$$\phi(x) = 0 \quad \textit{for} \quad x \notin V.$$

Note that this theorem implies that the horospherical transform is injective on the space of compactly supported smooth functions.

In this talk we present a generalization of Helgason's result to a support theorem for a class of Radon transforms (including the horospherical transforms) on a reductive symmetric space $X = G/H$ with G a real reductive Lie group of the Harish-Chandra class and H an essentially connected open subgroup of the fixed-point subgroup G^σ of an involution σ on G.

Let θ be a Cartan involution of G commuting with σ. For each $\sigma \circ \theta$-stable parabolic subgroup P with Langlands decomposition $M_P A_P N_P$ we consider the Radon transform $\mathcal{R}_P$ mapping a function ϕ on X to the function on $\Xi_P = G/(M_P A_P \cap H)N_P$ given by

$$\mathcal{R}_P\phi(g \cdot \xi_P) = \int_{N_P} \phi(gn \cdot H)\, dn.$$

Here ξ_P denotes the coset $e \cdot (M_P A_P \cap H)N_P$ containing the unit element e. This Radon transform, which is initially defined for compactly supported smooth functions, can be extended to a large class of distributions on X.

If P_0 is a minimal $\sigma \circ \theta$-stable parabolic subgroup of G contained in P, then A_P is contained in A_{P_0}. The Lie algebra $\mathfrak{a}_{P_0}$ of A_{P_0} is σ-stable and decomposes as the direct sum of the $+1$ and -1 eigenspace for σ. The latter space is denoted by $\mathfrak{a}_{\mathfrak{q}}$. The connected abelian Lie subgroup of G with Lie algebra $\mathfrak{a}_{\mathfrak{q}}$ is denoted by $A_{\mathfrak{q}}$.

The maps

$$K \times A_{\mathfrak{q}} \to X; \qquad (k,a) \mapsto ka \cdot H$$

and

$$K \times A_{\mathfrak{q}} \to \Xi_P; \qquad (k,a) \mapsto ka \cdot \xi_P$$

are surjective, just as in the Riemannian case. For a subset B of $\mathfrak{a}_{\mathfrak{q}}$, we define

$$X(B) = K\exp(B) \cdot H \quad \text{and} \quad \Xi_P(B) = K\exp(B) \cdot \xi_P.$$

The support of a transformed function or distribution turns out to be non-compact in general. In fact, if the support of a distribution μ is contained in $X(B)$ for some compact convex subset B of $\mathfrak{a}_{\mathfrak{q}}$ that is invariant under the action of the quotient $W_{K\cap H} = \mathcal{N}_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})/\mathcal{Z}_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})$ of the normalizer $\mathcal{N}_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})$ and the centralizer $\mathcal{Z}_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})$ of $\mathfrak{a}_{\mathfrak{q}}$ in $K \cap H$, then the support of the Radon transformed function $\mathcal{R}_P\mu$ is contained in $\Xi_P(B + \Gamma_P)$, where Γ_P is the cone in $\mathfrak{a}_{\mathfrak{q}}$ spanned by the root vectors corresponding to roots that are positive with respect to P. The support theorem that we prove in this article is a partial converse to this statement for distributions μ in a suitable class of distributions, containing the compactly supported ones:

Theorem *Let B be a $W_{K\cap H}$-invariant convex compact subset of $\mathfrak{a}_{\mathfrak{q}}$. If*

$$\operatorname{supp}(\mathcal{R}_P\mu) \subseteq \Xi_P(B + \Gamma_P),$$

then

$$\operatorname{supp}(\mu) \subseteq X(C),$$

where C is the maximal subset of $B + \Gamma_P$ that is invariant under the action of $W_{K\cap H}$.

If $K = H$ and P is a minimal parabolic subgroup of G, then C equals B. Our theorem reduces then to the support theorem of Helgason for the horospherical transform on a Riemannian symmetric space. Just as in the Riemannian case, the support theorem implies injectivity of the Radon transform.

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Minimal representations and special functions

Jan Möllers
(joint work with Joachim Hilgert, Toshiyuki Kobayashi)

We establish a relation between solutions of a certain fourth order ordinary differential operator and K-finite vectors in L^2-models of minimal representations. The L^2-theory of the fourth order operator is studied using unitarity of the minimal representations.

1. Special functions associated to a certain fourth order differential operator

For $\mu,\nu\in\mathbb{C}$ consider the fourth order differential operator

$$\mathcal{D}_{\mu,\nu} := \frac{1}{x^2}\left((\theta+\mu+\nu)(\theta+\mu)-x^2\right)\left(\theta(\theta+\nu)-x^2\right), \qquad \theta = x\frac{\mathrm{d}}{\mathrm{d}x}.$$

It is easy to see that for $\mu,\nu\in\mathbb{R}$ the operator $\mathcal{D}_{\mu,\nu}$ is symmetric on $L^2(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$. For certain parameters μ and ν representation theory can be used to prove the following result:

Theorem. *Suppose $\mu,\nu\in\mathbb{R}$ such that*

$$(*)\qquad \begin{cases} \nu=-1 \text{ and } \mu\in\frac{1}{2}\mathbb{N}_0 & \text{or}\\ \nu=0 \text{ and } \mu\in\mathbb{N}_0 & \text{or}\\ \mu,\nu\in\mathbb{N}_0, \mu\geq\nu \text{ and } \mu+\nu \text{ is even.} \end{cases}$$

Then $\mathcal{D}_{\mu,\nu}$ extends to a self-adjoint operator on $L^2(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$ with discrete spectrum. Moreover, the L^2-eigenvalues are given by

$$\lambda_j^{\mu,\nu} := 4j(j+\mu+1), \qquad j\in\mathbb{N}_0,$$

and the corresponding eigenspaces are one-dimensional.

We then ask the following question:

Question. *What are the L^2-eigenfunctions of $\mathcal{D}_{\mu,\nu}$?*

To obtain explicit L^2-eigenfunctions, we define a generating function $G^{\mu,\nu}(t,x)$ by

$$G_2^{\mu,\nu}(t,x) := \frac{1}{(1-t)^{\frac{\mu+\nu+2}{2}}}\widetilde{I}_{\frac{\mu}{2}}\left(\frac{tx}{1-t}\right)\widetilde{K}_{\frac{\nu}{2}}\left(\frac{x}{1-t}\right),$$

where $\widetilde{I}_\alpha(z) = (\frac{z}{2})^{-\alpha}I_\alpha(z)$ and $\widetilde{K}_\alpha(z) = (\frac{z}{2})^{-\alpha}K_\alpha(z)$ denote the normalized I- and K-Bessel functions. $G^{\mu,\nu}(t,x)$ is analytic near $t=0$ and we define a series $(\Lambda_j^{\mu,\nu}(x))_{j\in\mathbb{N}_0}$ of functions on $\mathbb{R}_+$ by

$$G^{\mu,\nu}(t,x) = \sum_{j=0}^\infty \Lambda_j^{\mu,\nu}(x)t^j.$$

Theorem. *For $\mu+\nu,\mu-\nu>-2$ the function $\Lambda_j^{\mu,\nu}(x)$ is an L^2-eigenfunction of $\mathcal{D}_{\mu,\nu}$ for the eigenvalue $\lambda_j^{\mu,\nu}$. In particular, if μ and ν satisfy (*), then the system $(\Lambda_j^{\mu,\nu}(x))_{j\in\mathbb{N}_0}$ forms an orthogonal basis of $L^2(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$ with norms*

$$\|\Lambda_j^{\mu,\nu}\|^2_{L^2(\mathbb{R}_+,x^{\mu+\nu+1}\,\mathrm{d}x)} = \frac{2^{\mu+\nu-1}\Gamma(j+\frac{\mu+\nu+2}{2})\Gamma(j+\frac{\mu-\nu+2}{2})}{j!(2j+\mu+1)\Gamma(j+\mu+1)}.$$

A detailed study of the operator $\mathcal{D}_{\mu,\nu}$ and the functions $\Lambda_j^{\mu,\nu}(x)$ can be found in [2, 3, 5].

2. L^2-models for minimal representations

Minimal representations are the smallest infinite-dimensional unitary representations. They are thought to correspond to the minimal non-zero nilpotent coadjoint orbit. We explain how the operator $\mathcal{D}_{\mu,\nu}$ and the functions $\Lambda_j^{\mu,\nu}(x)$ arise in minimal representations.

Let V be a simple real reduced Jordan algebra V and denote by $\mathrm{Co}(V)$ its conformal group which is a simple real Lie group. There is a unified way to construct all minimal representations of a finite cover G of $\mathrm{Co}(V)$ on an explicit Hilbert space $L^2(\mathcal{O})$ (excluding $G = \mathrm{SO}(3,q)$, q even). (For a detailed outline of this construction see [6].) Here $\mathcal{O} \subseteq (\mathcal{O}_{\mathbb{C}}^{\min} \cap \mathfrak{g}^*) \neq \emptyset$ is a certain Lagrangian submanifold, where $\mathcal{O}_{\mathbb{C}}^{\min} \subseteq \mathfrak{g}_{\mathbb{C}}^*$ denotes the minimal non-zero nilpotent coadjoint orbit and $\mathfrak{g}$ the Lie algebra of G.

Example. (1) For the group $G = \mathrm{Mp}(n,\mathbb{R})$ we have $L^2(\mathcal{O}) \cong L^2_{\mathrm{even}}(\mathbb{R}^n)$ and one minimal representations is isomorphic to the even part of the metaplectic representation.

(2) All cases except $G = \mathrm{SO}(p,q)_0$ were first treated by A. Dvorsky and S. Sahi in [1, 7].

(3) For $G = \mathrm{SO}(p,q)_0$, $p,q \geq 3$, $p+q$ even, the Lagrangian $\mathcal{O}$ is isomorphic to the isotropic cone

$$\mathcal{O} \cong \{x \in \mathbb{R}^{p+q-2} \setminus \{0\} : x_1^2 + \ldots + x_{p-1}^2 - x_p^2 - \ldots - x_{p+q-2}^2 = 0\}$$

and the L^2-model of the minimal representation was first constructed by T. Kobayashi and B. Ørsted in [4].

Properties.

- $\mathcal{O}$ is too small to carry a non-trivial G-action. Therefore, $\mathfrak{g}$ does not act by vectorfields, but by differential operators of degree ≤ 2. Hence, the Casimir operator of a maximal compact subgroup $K \subseteq G$ is a differential operator of order 4.
- The uniform description of the Lie algebra action in terms of Jordan algebras allows to calculate the Casimir action explicitly (at least on the space $L^2(\mathcal{O})_{\mathrm{rad}} \subseteq L^2(\mathcal{O})$ of 'radial' functions). This leads to the differential operator $\mathcal{D}_{\mu,\nu}$ with constants μ and ν written in terms of the Jordan algebra.
- Minimal representations are ladder representations, i.e. restriction to K yields the multiplicity-free decomposition

$$L^2(\mathcal{O}) \cong \widehat{\bigoplus_{j=0}^{\infty}} W^j,$$

into K-types W^j of highest weight $\alpha_0 + j\gamma$. The Casimir operator of K acts as a scalar on each K-type W^j and the subspace of radial functions in each K-type W^j is one-dimensional:

$$W^j \cap L^2(\mathcal{O})_{\mathrm{rad}} = \mathbb{C}\psi_j.$$

Theorem. $\psi_j(x) = \Lambda_j^{\mu,\nu}(|x|)$, $x \in \mathcal{O}$.

This observation implies all L^2-statements for the operator $\mathcal{D}_{\mu,\nu}$.

3. Open questions

The proof of the L^2-statements for the operator $\mathcal{D}_{\mu,\nu}$ uses representation theory in a crucial way. This puts the restriction (*) on the parameters μ and ν. However, it is likely, that the L^2-statements still hold for a more general set of parameters.

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Spectral theory of automorphic forms and the Arthur-Selberg trace formula

Werner Müller

The purpose of this talk is to discuss the trace formulas of Selberg and Arthur and some of their applications to spectral theory of automorphic forms.

The basic set up is as follows. Let G be a real semi-simple Lie group with finitely many connected components, with finite center and of the non-compact type. We fix a maximal compact subgroup K of G. Let $\Gamma \subset G$ be a lattice, i.e., a discrete subgroup such that $\mathrm{vol}(\Gamma\backslash G) < \infty$, where the volume is taken with any Haar measure on G. Of particular importance are arithmetic subgroups Γ. In this case we consider a semi-simple (or reductive) algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and G is the group of real points of $\mathbf{G}$. We fix an embedding $\rho\colon \mathbf{G} \hookrightarrow \mathrm{GL}(n)$ for some n. Then $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup, if $\rho(\Gamma)$ is commensurable with $\mathrm{GL}(n,\mathbb{Z})$.

Let

$$R_\Gamma\colon G \to \mathrm{Aut}(L^2(\Gamma\backslash G))$$

be the right regular representation which is defined by

$$\left(R_\Gamma(g)f\right)(g') = f(g'g).$$

One of the *fundamental problems in the theory of automorphic forms* is to describe explicitely the spectral resolution of $(R_\Gamma, L^2(\Gamma\backslash G))$. By Langlands' theory of Eisenstein series [9] there is a decomposition

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$$

into invariant subspaces of R_Γ. $L^2_d(\Gamma\backslash G)$ is the subspace spanned by all irreducible subspaces of $L^2(\Gamma\backslash G)$ and $L^2_c(\Gamma\backslash G)$ is described in terms of Eisenstein series. The restriction R^d_Γ of R_Γ to $L^2_d(\Gamma\backslash G)$ decomposes discretely

$$R^d_\Gamma = \bigoplus_{\pi\in\widehat{G}} m_\Gamma(\pi)\,\pi \tag{1}$$

with finite multiplicities $m_\Gamma(\pi)$. $L^2_d(\Gamma\backslash G)$ contains the subspace of cusp forms which is defined as follows. Let $P \subset G$ be a parabolic subgroup with Levi decomposition $P = L_P N_P$. P is called Γ-cuspidal if $(\Gamma \cap N_P)\backslash N_P$ is compact. A function $f \in L^2(\Gamma\backslash G)$ is called cusp form, if f is right K-finite, $\mathcal{Z}(\mathfrak{g})$-finite, and

$$\int_{(\Gamma\cap N_P)\backslash N_P} f(ng)\,dn = 0 \tag{2}$$

for all proper Γ-cuspidal parabolic subgroups P. Let $L^2_{cusp}(\Gamma\backslash G)$ be the subspace of $L^2(\Gamma\backslash G)$, spanned by all cusp forms. Then $L^2_{cusp}(\Gamma\backslash G)$ is contained in $L^2_d(\Gamma\backslash G)$. Let $L^2_{res}(\Gamma\backslash G)$ be the orthogonal complement of $L^2_{cusp}(\Gamma\backslash G)$ in $L^2_d(\Gamma\backslash G)$. Thus

$$L^2_d(\Gamma\backslash G) = L^2_{cusp}(\Gamma\backslash G) \oplus L^2_{res}(\Gamma\backslash G).$$

It follows from [9] that $L^2_{res}(\Gamma\backslash G)$ is spanned by iterated residues of Eisenstein series. By the theory of Eisenstein series it follows that cusp forms are the building blocks of the theory of automorphic forms and one of the main issues is to study the multiplicities $m_\Gamma(\pi)$ with which $\pi \in \widehat{G}$ occurs in the space of cusps forms $L^2_{cusp}(\Gamma\backslash G)$.

For applications to number theory arithmetic groups Γ are significant. To use the arithmetic structure one has to pass to the adelic framework, i.e., replace $\Gamma\backslash G$ by the adelic quotient $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ where $\mathbf{G}$ is the algebraic group such that $\Gamma \subset \mathbf{G}(\mathbb{A})$. The corresponding problem is then to describe the spectral decomposition of the regular representation of $\mathbf{G}(\mathbb{A})$ in $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$. The cuspidal subspace $L^2_{cusp}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ is defined in a similar manner. An irreducible unitary representation of $\mathbf{G}(\mathbb{A})$ that occurs in the space of cusp forms is called an automorphic cuspidal representation of $\mathbf{G}(\mathbb{A})$.

Finally we note that the original approach by Selberg [18], [19] was not group theoretic. Selberg studied the spectrum of the Laplace operator on hyperbolic surfaces of finite area. In general, let $S = G/K$ be the Riemannian symmetric space associated to G and K. Assume that Γ is torsion free. Then $X = \Gamma\backslash S$ is a locally symmetric manifold of finite volume. Let $\mathcal{D}(S)$ be the algebra of invariant differential operators on S. Then $\mathcal{D}(S)$ is a finitely generated, commutative algebra. The problem studied by Selberg is the spectral resolution of $\mathcal{D}(S)$ acting in $L^2(\Gamma\backslash S)$. Since $\Delta \in \mathcal{D}(S)$, this concerns in particular the spectral resolution of the Laplace operator. The connection with the group theoretic point of view is given by $L^2(\Gamma\backslash G)^K \cong L^2(\Gamma\backslash S)$. Especially we have

$$L^2_d(\Gamma\backslash G)^K \cong \bigoplus_{\chi} \mathbb{C}\phi_\chi,$$

where χ runs over the additive characters of $\mathcal{D}(S)$ and ϕ_χ is square integrable and satisfies $D\phi_\chi = \chi(D)\phi_\chi$, $D \in \mathcal{D}(S)$. A joint eigenfunction of $\mathcal{D}(S)$ which satisfies the cuspidal condition (2) is called a Maass cusp form. In the case of a non-compact hyperbolic surface $\Gamma\backslash\mathbb{H}$ of finite area, a Maass cusp form is just a square integrable eigenfunction f of the hyperbolic Laplace operator Δ with vanishing zeroth Fourier coefficients in all cusps. The continuous spectrum of Δ equals $[1/4,\infty)$. Therefore all eigenvalues $\lambda \geq 1/4$ of Maass cusp forms are embedded into the continuous spectrum. It is known from mathematical physics that embedded eigenvalues are unstable with respect to perturbations. This is why the study of cups forms is so difficult and their existence highly non-trivial. In this respect there is the conjecture of Phillips and Sarnak which states that for a generic non-compact hyperbolic surface of finite area (where generic refers to a point in the moduli space) the Laplacian has no embedded eigenvalues.

The trace formula is one of the most important tools to study the cuspidal spectrum. It was introduced by Selberg in [18], who developed the trace formula for quotients of the hyperbolic plane by Fuchsian groups of the first kind. His original motivation and application was to show the existence of Maass forms for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. The trace formula was then vastly generalized by Arthur in the context of adelic quotients $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$. Arthur's main motivation was to attack the Langlands functoriality conjectures.

For a hyperbolic surface $\Gamma\backslash\mathbb{H}$ of finite area, the trace formula is the following statement. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the point spectrum, i.e., the eigenvalues of Δ. Each eigenvalue occurs with finite multiplicity. Write the eigenvalues as $\lambda_j = 1/4 + r_j^2$, where $r_j \in \mathbb{R} \cup i[-1/2,1/2]$. Let $f \in C_c^\infty(\mathbb{R})$ and let h be the Fourier transform of f. Let $C(s)$ be the "scattering matrix" derived from the constant terms of Eisenstein series and put $\phi(s) = \det C(s)$. Then the following equality holds.

$$\begin{aligned}\sum_{j\geq 0} h(r_j) - \frac{1}{4}\int_{\mathbb{R}} h(r)\frac{\phi'}{\phi}(1/2+ir)\,dr - \frac{m-\phi(1/2)}{2}h(0)\\ = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{2\pi}\int_{\mathbb{R}} h(r) r\tanh(\pi r)\,dr\\ + \sum_{k=1}^{\infty}\sum_{\substack{[\gamma]\neq e\\ prime}} \frac{\ell(\gamma)}{\sinh\left(\frac{k\ell(\gamma)}{2}\right)} f(k\ell(\gamma))\\ - \frac{m}{2}\int_{\mathbb{R}} h(r)\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1+ir)\,dr - 2m\ln(2)f(0),\end{aligned} \tag{3}$$

where $[\gamma]$ runs over the hyperbolic conjugacy classes of Γ, $\ell(\gamma)$ is the length of the corresponding closed geodesic, $\mathbf{\Gamma}(s)$ is the Gamma function, and m is number of cusps of $\Gamma\backslash\mathbb{H}$. Let

$$N_\Gamma(\lambda) := \#\{i\colon \lambda_i \leq \lambda^2\} \tag{4}$$

be counting function of the eigenvalues. Selberg used the trace formula to prove the following Weyl law

$$N_\Gamma(\lambda) - \frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}(1/2+it)\,dt \sim \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda^2 \tag{5}$$

as $\lambda \to \infty$. To prove (4) one can proceed as in the compact case. First one applies the trace formula to the heat operator $e^{-t\Delta}$ and determines the asymptotic behavior of the spectral side as $t \to \infty$. Then a Tauberian theorem gives (4). If one uses the trace formula combined with Hörmander's method to estimate the spectral function on a compact manifold, one can estimate the remainder term by $O(\lambda \log \lambda)$ (see [14]). In general one can not separate the eigenvalue counting function and the winding number. However, for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ one can express the determinant of the scattering matrix in terms of known functions of analytic number theory. For example, for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ one has

$$\phi(s) = \sqrt{\pi}\,\frac{\mathbf{\Gamma}(s-1/2)\zeta(2s-1)}{\mathbf{\Gamma}(s)\zeta(2s)} \tag{6}$$

where $\zeta(s)$ is the Riemann zeta function. Using standard estimates for the logarithmic derivatives of $\zeta(s)$ and $\mathbf{\Gamma}(s)$, it follows that the winding number in (5) is of order $O(\lambda \log \lambda)$. A similar result holds for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. Hence for congruence subgroups Γ, the eigenvalue counting function $N_\Gamma(\lambda)$ satisfies Weyl's law which implies that for such lattices Maass cups forms exist in abundance. Reznikov has extended this to other rank one cases [16].

The trace formula has been generalized by Arthur to arbitrary reductive groups. For this however one has to pass to the adelic setting. Let $\mathbf{G}$ be a semi-simple algebraic group over $\mathbb{Q}$. Then Arthur's trace formula in its original form, developed in the 70' and 80', is an identity of sums of distributions on $\mathbf{G}(\mathbb{A})$ of the form

$$\sum_\chi J_\chi(f) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(\mathbf{G}(\mathbb{A})), \tag{7}$$

where χ ranges over spectral data and $\mathfrak{o}$ ranges over semi-simple conjugacy classes of $\mathbf{G}(\mathbb{Q})$ (see [1] for more details). The left hand side is the spectral side. One contribution is the sum over the cuspidal spectrum. The other terms are distributions derived from Eisenstein series. The main ingredients are multidimensional logarithmic derivatives of Intertwining operators, generalizing the corresponding terms on the left hand side of (3). The sum-integral in Arthur's formula was known to converge conditionally. The absolute convergence has recently been established in [7], [8]. This is important for the potential applications of the trace formula to spectral theory. The right hand side of (7) is the geometric side. The distributions

$J_{\mathfrak{o}}$ are given in terms of *weighted orbital integrals.* At the real place they are of the form

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)w(g)\,dg \tag{8}$$

with some weight function $w(g)$. The trace formula has been used to prove Weyl's law for the cuspidal spectrum of the Laplacian in various higher rank cases. S. Miller [12] proved it for Maass forms for $\mathrm{SL}(3,\mathbb{Z})$. In [15] it was proved for vector valued forms for congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$. Lindenstrauss and Venkatesh [11] proved it for all quotients $\Gamma\backslash G/K$, where Γ is a congruence subgroup of G and G is of the adjoint type.

In the higher rank case, we are not only dealing with the Laplace operator, but with the whole algebra $\mathcal{D}(S)$ of invariant differential operators, whose spectrum is multidimensional. Let A be maximal split torus of $G = \mathbf{G}(\mathbb{R})$ and let $\mathfrak{a}$ be its Lie algebra. Let W be the Weyl group. Then the spectrum of $\mathcal{D}(S)$ is a W-invariant subset of $\mathfrak{a}^*_{\mathbb{C}}$. Given $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let $m(\lambda)$ be the multiplicity of the λ-eigenspace of $\mathcal{D}(S)$ in $L^2(\Gamma\backslash G)$. The tempered spectrum is contained in $i\mathfrak{a}^*$. Let $\beta(\lambda)$ denote the Plancherel measure. In [10] we have proved the following theorem.

Theorem *Let $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $d = \dim S$. Let $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ be the principal congruence subgroup of level $N \in \mathbb{N}$. Assume that $N \geq 3$. Suppose that $\Omega \subset i\mathfrak{a}^*$ is a W-invariant bounded domain with piecewise C^2-boundary. Then*

$$\sum_{\lambda\in t\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{|W|}\int_{t\Omega}\beta(\lambda)\,d\lambda + O(t^{d-1}(\log t)^n).$$

For a co-compact lattice this is due to Duistermaat, Kolk, and Varadarajan [6]. The proof of the theorem relies on various results which are currently only known for $\mathrm{GL}(n)$. One important fact that enters the proof is the description of the intertwining operators in terms of Rankin-Selberg L-functions. This is the generalization of (6). Another result that is used is the description of the residual spectrum by Moeglin and Waldspurger [13]. It implies that, compared to the cuspidal spectrum, the residual spectrum is of lower order growth. For classical groups these results are only partially known.

The Weyl law is only the first example of the possible applications of the trace formula to spectral theory. Further applications will include, for example, the study of the cuspidal spectrum, if Γ varies. This is the problem of limit multiplicities first studied by De George and Wallach [3], [4]. It means that we consider a tower of normal subgroups of finite index $\Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots$ such that $\cap\Gamma_n = \{e\}$. We define measures μ_n on the tempered dual $\widehat{G}_{temp}$ as follows. For every bounded Jordan measurable set $\Omega \subset \widehat{G}_{temp}$ put

$$\mu_n(\Omega) = \frac{\sum_{\pi\in\Omega} m_\Gamma(\pi)}{\mathrm{vol}(\Gamma_n\backslash G)}.$$

Then the conjecture is that for regular Ω (which means that the Plancherel measure $\mu_{pl}(\partial\Omega)$ of the boundary $\partial\Omega$ vanishes) we have

$$\lim_{n\to\infty} \mu_n(\Omega) = \mu_{pl}(\Omega).$$

For a co-compact lattice this problem was studied in [3], [4], [5]. For the discrete series the answer is affirmative [17]. Another goal is to include Hecke operators.

One of the most important applications of the trace formula is the comparison of the traces on different groups aiming at the functoriality conjectures of Langlands [2]. This was the driving force for Arthur to develop the trace formula.

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Infinite symmetric group and categories of simplicial bordisms

Yury Neretin

We consider a product of three copies of infinite symmetric group and its representations spherical with respect to the diagonal subgroup. We show that such representations generate functors from a certain category of simplicial two-dimensional surfaces (bordisms) to the category of Hilbert spaces and bounded linear operators.

1. Infinite symmetric groups. By $S_\infty^{fin} \subset S_\infty$ we denote the group of finite permutations of $\mathbb{N}$ (i.e., substitutions g satisfying the condition $gj = j$ for sufficiently large $j \in \mathbb{N}$). Denote by S_∞ the group of all permutations of the set $\mathbb{N} = \{1, 2, 3, \dots\}$. By $S_\infty(\alpha) \subset S_\infty$ we denote the stabilizer of points 1, 2, ..., $\alpha \in \mathbb{N}$. We assume $S_\infty(0) := S_\infty$. The standard topology on the group S_∞ is defined by the condition: subgroups $S_\infty(\alpha)$ are open and cosets $gS_\infty(\alpha) \subset S_\infty$ form a basis of topology.

2. Thoma theorem. In 1964 E. Thoma obtained the classification of all characters of S_∞^{fin}. According his definition, a *character* is an extreme point of the set of all functions F on S_∞^{fin} satisfying the conditions:

(1) F is central, i.e., $F(hgh^{-1}) = F(g)$;
(2) F is positive definite;
(3) $F(1) = 1$.

Theorem. **(Thoma,** [4]**)** *All characters of S_∞^{fin} have the form*

$$\chi(g) = \prod_{k\geq 2} \Bigl(\sum_j \alpha_j^k - \sum_j (-\beta_j)^k\Bigr)^{r_k(g)}, \tag{1}$$

where $r_k(g)$ is the number of cycles of length k in g, and the parameters α, β satisfy

$$\alpha_1 \geq \alpha_2 \geq \dots \geq 0, \qquad \beta_1 \geq \beta_2 \geq \dots \geq 0, \qquad \sum \alpha_j + \sum \beta_j \leq 1. \tag{2}$$

3. Double of symmetric group. First, let G be a group, $K \subset G$ a subgroup. Let ρ be an irreducible unitary representation of G. We say that a *representation* ρ *is K-spherical* if there exists a unique vector v such that $\rho(h)v = v$ for all $h \in K$. A *spherical function* is a function on G given by

$$\phi(g) = \langle \rho(g)v, v\rangle.$$

We say that a *pair* (G, K) *is spherical* if each unitary irreducible unitary representation of G has ≤ 1 fixed vectors.

Now set $G = S_\infty^{fin} \times S_\infty^{fin}$ and $K = S_\infty^{fin}$ is the diagonal of G.

Theorem. ([3]). a) *The pair $(G, K) = (S_\infty^{fin} \times S_\infty^{fin}, S_\infty^{fin})$ is spherical.*
b) *There is a canonical one-to-one correspondence between Thoma characters and spherical representations of the (G, K).*

c) *Moreover, for a character $\chi(g)$ the corresponding spherical function is given by*

$$\Phi(g_1, g_2) = \chi(g_1 g_2^{-1}).$$

4. Sphericity.

Theorem. *Let $G = S_\infty^{fin} \times \cdots \times S_\infty^{fin}$ be the product of n copies of S_∞^{fin}. Let $K = S_\infty^{fin}$ be the diagonal subgroup. Then the pair (G, K) is spherical.*

5. n-symmetric groups. Consider the product $G^{[n]} := S_\infty \times \cdots \times S_\infty$ of n copies of the complete symmetric group. We define the *n-symmetric group* $\mathbb{G} = \mathbb{G}^{[n]}$ as the subgroup of $G^{[n]}$ consisting of collections $(g_1, \dots, g_n)$ such that

$$g_i g_j^{-1} \in S_\infty^{fin} \qquad \text{for all } i,\, j \le n.$$

In other words, $\mathbb{G}^{[n]}$ consists of all collections

$$(gh_1, gh_2, \dots, gh_n) \qquad \text{such that } g \in S_\infty,\ h_j \in S_\infty^{fin}.$$

Denote by $K(\alpha)$ the image of $S_\infty(\alpha)$ under the diagonal embedding $K \mapsto \mathbb{G}^{[n]}$. We define the topology on $\mathbb{G}^{[n]}$ from the condition: the subgroups $K(\alpha)$ are open in $\mathbb{G}^{[n]}$ and form a fundamental system of neighborhoods of 1. In other words, the topology of $K \simeq S_\infty$ is the same as above, the quotient-space $\mathbb{G}^{[n]}/K$ is countable and equipped with the discrete topology.

The existing representation theory of infinite symmetric groups is mainly the representation theory of the bi-symmetric group $\mathbb{G}^{[2]}$, see Thoma [4], Vershik, Kerov [5], Olshanski [3], Okounkov [2], Kerov, Olshanski, Vershik [1]. The situation was explained by Olshanski in [3].

Proposition. *Any S_∞^{fin}-spherical representation of $S_\infty^{fin} \times \cdots \times S_\infty^{fin}$ admits a continuous extension to the group $\mathbb{G}^{[n]}$.*

6. Construction of simplicial complexes. Take 3 copies of the set $\mathbb{N}$, say *red, yellow, and blue.* An element of $\mathbf{g} \in \mathbb{G}^{[3]}$ is a triple of permutations of $\mathbb{N}$, denote it by

$$\mathbf{g} = (g_{red},\ g_{yellow},\ g_{blue}).$$

We draw a collection of disjoint oriented *black* triangles A_j, where j ranges in $\mathbb{N}$, and paint their sides in red, yellow, and blue *anti-clockwise.* We assign labels 1, 2, $\dots n$ to black (resp. white) triangles. We also draw a collection of oriented *white* triangles B_j and paint sides in red, yellow, and blue *clockwise.*

Next, we glue a simplicial complex from these triangles. If g_{red} sends i to j, then we identify the red side of the black triangle A_i with the red side of the white triangle B_j. We repeat the same operation for g_{yellow} and g_{blue}. In this way, we get a disjoint countable union of 2-dimensional compact closed triangulated surfaces.

All components except finite number consist of two triangles (black and white, glued along the corresponding sides).

Next, we forget white labels $> \alpha$ and black labels $> \beta$ and black labels $> \alpha$.

Thus for aech double coset $\in K(\alpha) \setminus \mathbb{G}^{[3]}/K(\beta)$ we have assigned a simplicial two-dimensional surface with labels and coloring.

We define product of double cosets

$$K(\alpha)\setminus \mathbb{G}^{[3]}/K(\beta)\times K(\beta)\setminus \mathbb{G}^{[3]}/K(\gamma)\to K(\alpha)\setminus \mathbb{G}^{[3]}/K(\gamma)$$

as a product of simplicial bordisms. Thus we get a category whose objects are 0, 1, 2, ... and morphisms are double cosets.

7. Main construction Let ρ be a unitary representation of $\mathbb{G}^{[3]}$ in a Hilbert space H. Denote by $H(\alpha)$ the space of $K(\alpha)$-fixed vectors. Denote by $P(\alpha)$ the operator of orthogonal projection to the space $H(\alpha)$.

Fix α, β. For $\mathbf{p}\in\mathbb{G}^{[3]}$, consider the operator

$$H(\beta)\to H(\alpha)$$

given by

$$\overline{\rho}(g):=P(\alpha)\rho(\mathbf{p})=P(\alpha)\rho(\mathbf{p})P(\beta).$$

For $r_1\in K(\alpha)$, $r_2\in K(\beta)$ we have

$$\overline{\rho}(r_1 g r_2)=\overline{\rho}(g). \tag{3}$$

Thus $\overline{\rho}$ is a function on double cosets $K(\alpha)\setminus\mathbb{G}^{[3]}/K(\beta)$.

Theorem. *For any unitary representation ρ of $\mathbb{G}^{[3]}$ for each α, β, $\gamma\in\mathbb{Z}_+$, for each*

$$\mathfrak{a}\in K(\alpha)\setminus\mathbb{G}^{[3]}/K(\beta),\quad \mathfrak{b}\in K(\beta)\setminus\mathbb{G}^{[3]}/K(\gamma), \tag{4}$$

we have

$$\overline{\rho}(\mathfrak{a})\overline{\rho}(\mathfrak{b})=\overline{\rho}(\mathfrak{a}\circ\mathfrak{b}).$$

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Unitary periods

Omer Offen
(joint work with Brooke Feigon, Erez Lapid)

Let E/F be a quadratic extension of number fields. For a cusp form ϕ on $GL_n(\mathbb{A}_E)$ and for a unitary subgroup H associated with E/F we consider the unitary period integral

$$P^H(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)\ dh.$$

A cuspidal automorphic representation π of $GL_n(\mathbb{A}_E)$ is called H-*distinguished* if the restriction of P^H to π is non-zero.

Based on Jacquet's relative trace formula we show that the unitary period is factorizable on any cuspidal π. Furthermore, we provide a criteria for distinction of π by any unitary group H.

To obtain these global results we apply the relative trace formula to carefully study the local components of unitary periods. This study further yields local results on multiplicities of linear forms invariant by a unitary group. In particular, locally we prove that a generic irreducible representation is in the image of quadratic base change if and only if it is distinguished by a quasi-split unitary group. This is a local analog of a similar result of Jacquet for cuspidal automorphic representations.

A formula of Arthur for the Euler-Poincaré pairing

Eric Opdam
(joint work with Maarten Solleveld)

1. The Euler-Poincaré pairing

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category with finite homological dimension. The *Euler-Poincaré pairing* [17] of two finite length objects π, π' of $\mathcal{C}$ is defined by

$$EP_{\mathcal{C}}(\pi, \pi') = \sum_{i\geq 0} (-1)^i \mathrm{dimExt}^i_{\mathcal{C}}(\pi, \pi'). \quad (1)$$

Let $K_{\mathbb{C}}(\mathcal{C})$ denote the Grothendieck group (tensored by $\mathbb{C}$) of finite length objects in $\mathcal{C}$. The Euler-Poincaré pairing extends to a sesquilinear form (conjugate linear in the first variable) on $K_{\mathbb{C}}(\mathcal{C})$, also denoted by $EP_{\mathcal{C}}$. The theme of this talk is the comparison of three instances of the Euler-Poincaré pairing:

(i) The Euler-Poincaré pairing is well defined for the category $\mathcal{C}_L$ of smooth representations of the group $L = \mathbf{L}(F)$ of rational points of a reductive group $\mathbf{L}$ defined over a non-archimedean local field F [17]. The Euler-Poincaré pairing on $G_{\mathbb{C}}(L) := K_{\mathbb{C}}(\mathcal{C}_L)$ is denoted by EP_L. In this case EP_L is Hermitian [17] and plays a fundamental role in the local trace formula and in the study of orbital integrals on the regular elliptic set of L [1, 17, 3, 16].

(ii) The Euler-Poincaré pairing is also naturally defined for the category $\mathcal{C}_{\mathcal{H}}$ of finitely generated modules of an affine Hecke algebra $\mathcal{H}$ with positive parameters, since $\mathcal{C}_{\mathcal{H}}$ has finite cohomological dimension [12]. The form $EP_{\mathcal{H}}$ thus obtained on $G_{\mathbb{C}}(\mathcal{H}) := K_{\mathbb{C}}(\mathcal{C}_{\mathcal{H}})$ is Hermitian ([12, Theorem 3.5 a)]).

(iii) Let Γ be a finite group acting linearly on a lattice X. Put $S = \Gamma \ltimes X$, and $G_{\mathbb{C}}(S) := K_{\mathbb{C}}(\mathcal{C}_S)$ with $\mathcal{C}_S$ the category of finitely generated S-modules. We denote the Euler-Poincaré pairing in this case by EP_S. Again EP_S can be shown to be Hermitian [12].

In all three cases, properly parabolically induced representations (in the sense of [16] in case (iii)) are in the radical of the Euler-Poincaré pairing.

The goal of this talk is the explicit computation of $EP_{\mathcal{C}}$ in these three cases.

Some preliminary remarks are in order. It is clear that $EP_{\mathcal{C}}$ behaves well with respect to block decompositions: If $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ then $K_{\mathbb{C}}(\mathcal{C}) = K_{\mathbb{C}}(\mathcal{C}_1) \oplus K_{\mathbb{C}}(\mathcal{C}_2)$ and $EP_{\mathcal{C}}$ is the direct sum of the sesquilinear forms $EP_{\mathcal{C}_i}$. Second, if $\lambda : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories then λ yields an isometry with respect to the respective Euler-Poincaré pairings. Combining these two general remarks, we see that if a Bernstein block [2] $\mathcal{B}$ of $\mathcal{C}_L$ (case (i)) is Morita equivalent to the module category of an affine Hecke algebra $\mathcal{H}$, then EP_L restricted to $G_{\mathbb{C}}(\mathcal{B}) \times G_{\mathbb{C}}(\mathcal{B})$ is computed by $EP_{\mathcal{H}}$ (case (ii)). This is known in many cases [6], [10], [15]. Next we remark that case (iii) is in some sense trivial; here EP_S can be computed completely explicitly [16], [12] in terms of the *elliptic pairings* defined by Reeder [16] on the Grothendieck groups of the representations of the finite isotropy groups of Γ acting on the torus T of complex characters of X. Finally we note that in case (ii) there is a natural limiting procedure sending the base $\mathbf{q}$ of the Hecke parameters to 1. It is easy to see that this induces a linear isometry with respect to the Euler-Poincaré pairings $EP_{\mathcal{H}}$ (case (ii)) and EP_W (case (iii)) where $W = W_0 \ltimes X$ denotes the (extended) affine Weyl group underlying $\mathcal{H}$.

This last remark is useful to some extend, and reduces the computation of $EP_{\mathcal{H}}$ to the problem of computing the limit for $\mathbf{q} \to 1$ of an irreducible representation of $\mathcal{H}$. In general this is an open problem, but in the special case of equal Hecke parameters this was used to compute $EP_{\mathcal{H}}$ explicitly by Reeder [16, Main Theorem, Theorem 5.10.1] for the standard modules in the Kazhdan-Lusztig classification [8], using properties of the Springer correspondence, a formula of Arthur [1] for case (i), and the comparison of geometric and analytic R-groups. We will generalize these results of Reeder to arbitrary positive Hecke parameters, using analytic R-groups and structure theory of the Schwartz completion $\mathcal{S}$ of $\mathcal{H}$ instead. More precisely, for *tempered* irreducible characters we will prove an analog of Arthur's formula by reducing the computation of Euler-Poincaré pairings in the cases (i) and (ii) to the trivial case (iii) in a completely different fashion.

In case (i) the original formula of Arthur [1] is a formula expressing the so-called *elliptic pairing* of tempered characters of L in terms of the elliptic pairing of the corresponding (twisted) characters of an analytic R-group. The formula of Arthur for the Euler-Poincaré pairing to which we referred above is in fact an aggregation of this result for elliptic pairings with another deep result in harmonic analysis on L, namely Kazhdan's conjecture [7] (proved in [17] and in [3]). This result states that, provided F has characteristic 0, the pairing EP_L for tempered representations of L is equal to the elliptic pairing of their characters. Arthur's formula for Euler-Poincaré pairings makes sense in the context of case (ii) as well, using the analytic R-groups introduced in [5]. In the next section we will discuss some ingredients of the proof of this formula in the case (ii).

2. Arthur's formula for the Euler-Poincaré pairing

Let $\mathcal{H}$ be an affine Hecke algebra in the sense of [11], [4], and let π, π' be finite dimensional tempered modules for $\mathcal{H}$. Let $\mathcal{S}$ be the Schwartz completion [4] of $\mathcal{H}$ (a Fréchet algebra). The first main step in the computation of $EP_{\mathcal{H}}(\pi, \pi')$ is based on the following theorem:

Theorem. *(cf. [13]) For all $i \in \mathbb{Z}$ we have $\mathrm{Ext}^i_{\mathcal{H}}(\pi, \pi') \simeq \mathrm{Ext}^i_{\mathcal{S}}(\pi, \pi')$.*

This is the main result of [13], and it is the analog in case (ii) of a deep result of Meyer [9] for reductive groups (case (i)). It shows immediately that

$$EP_H(\pi, \pi') = EP_{\mathcal{S}}(\pi, \pi') \tag{2}$$

This result is already quite powerful. For example, let π be an irreducible discrete series module for $\mathcal{H}$. The Fourier isomorphism [4] for the Schwartz completion $\mathcal{S}$ shows that π is a projective $\mathcal{S}$ module. Hence it follows that $EP_{\mathcal{H}}(\pi, \pi') = 1$ if π' is equivalent to π, and $EP_{\mathcal{H}}(\pi, \pi') = 0$ if π' is a tempered irreducible $\mathcal{H}$-module inequivalent to π. In particular, the isomorphism classes of the irreducible discrete series characters of $\mathcal{H}$ form an orthonormal set with respect to $EP_{\mathcal{H}}$.

The formula of Arthur extends this result for discrete series characters to arbitrary irreducible tempered characters. By [4] the category of tempered $\mathcal{H}$-modules of finite length falls into blocks which are parameterized by the set $\mathcal{W}\backslash\Xi_u$ of orbits of tempered standard induction data $\xi \in \Xi_u$ for $\mathcal{H}$ under the action of the Weyl groupoid $\mathcal{W}$ for $\mathcal{H}$. The block $\mathcal{B}_{\mathcal{W}\xi}$ of tempered modules associated with $\mathcal{W}\xi$ is the block generated by the standard induced module $\pi(\xi)$. By [5] we have the analog in case (ii) of the Knapp-Stein Dimension Theorem:

Theorem. *A tempered standard induced module $\pi(\xi)$ is a unitary tempered $\mathcal{H}$-module and the commutant of $\pi(\xi)(\mathcal{H})$ is isomorphic to the twisted group algebra ${}^{\gamma_\xi}\mathbb{C}[\mathfrak{R}_\xi]$, where $\mathfrak{R}_\xi$ is the so-called analytic R-group associated to ξ, and γ_ξ is a certain 2-cocycle of $\mathfrak{R}_\xi$.*

The group $\mathfrak{R}_\xi$ is a finite group acting faithfully on the the tangent space $\mathfrak{a}_\xi$ of the space of tempered standard induction data. It follows that the set of equivalence classes of irreducible objects in $\mathcal{B}_{\mathcal{W}\xi}$ is in bijection with the set of irreducible γ_ξ-twisted characters of $\mathfrak{R}_\xi$. Given an irreducible γ_ξ-twisted character χ of $\mathfrak{R}_\xi$ let $\pi(\mathcal{W}\xi, \chi)$ be an irreducible tempered $\mathcal{H}$-module in the corresponding equivalence class. We summarize the above results of [4], [5] by:

Theorem. *The $\mathcal{H}$-modules $\pi(\mathcal{W}\xi, \chi)$ (with $\mathcal{W}\xi \in \mathcal{W}\backslash\Xi_u$ and χ running over the set of irreducible γ_ξ-twisted characters of $\mathfrak{R}_\xi$) form a complete set of representatives of the equivalence classes of tempered irreducible characters of $\mathcal{H}$.*

Now we have everything in place to formulate the formula of Arthur:

Theorem.

$$EP_{\mathcal{H}}(\pi(\mathcal{W}\xi,\chi),\pi(\mathcal{W}\xi',\chi')) = \delta_{\mathcal{W}\xi,\mathcal{W}\xi'}|\mathfrak{R}_\xi|^{-1}\sum_{r\in\mathfrak{R}_\xi}\det{}_{\mathfrak{a}_\xi}(1-r)\overline{\chi(r)}\chi'(r)$$

By the Langlands parameterization of irreducible characters of $\mathcal{H}$ (which is available in this generality, see e.g. [20]) it follows that $G_{\mathbb{C}}(\mathcal{H})$ has a linear bases provided by the classes of the standard induced modules $\pi(\lambda)$ where $\lambda = (\mathcal{H}^P, \sigma, t)$ with $\mathcal{H}^P$ a standard "Levi-subalgebra" of $\mathcal{H}$, σ a tempered irreducible module of $\mathcal{H}^P$, and t a positive induction parameter (in the face of the positive Weyl chamber defined by the simple roots of $\mathcal{H}^P$). Theorem 2.4 therefore yields the computation of $EP_{\mathcal{H}}$ on $G_{\mathbb{C}}(\mathcal{H}) \times G_{\mathbb{C}}(\mathcal{H})$ in terms of the bases provided by the classes of the standard modules $\pi(\lambda)$. The computation of $EP_{\mathcal{H}}(\pi,\pi')$ for arbitrary *irreducible* $\mathcal{H}$-modules π,π' is clearly a problem of a different nature on which our methods do not provide any information.

The proof (reported on in [14]) is based on a direct computation of $\mathrm{Ext}^i_{\mathcal{S}}(\pi,\pi')$ using the Fourier isomorphism [4] for $\mathcal{S}$. The proof is thus based on the Fourier isomorphism for $\mathcal{S}$, Theorem 2.1, and Theorem 2.2. In particular the proof goes through for case (i) for arbitrary F (cf. [9], [18], [21]). In particular this remark proves Arthur's formula for EP_L for F of arbitrary characteristic and also yields another proof of Kazhdan's conjecture when combined with Arthur's original formula for elliptic pairing [1].

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Associated varieties, derivatives, Whittaker functionals, and rank for GL$(n, \mathbb{R})$

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(joint work with Dmitry Gourevitch)

Let $G = G_n = GL(n, \mathbb{R})$, let N be the subgroup of strictly upper triangular matrices, and let $\mathfrak{g}, \mathfrak{n}$ denote their respective complexified Lie algebras. Characters of N correspond bijectively to elements of $\mathfrak{n}^*$ vanishing on $[\mathfrak{n}, \mathfrak{n}]$, *i.e.* to elements of $\mathfrak{X} = (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. By the Killing form on $\mathfrak{g}$, we can idenitfy $\mathfrak{g}^*$ with $\mathfrak{g}$ and obtain thereby an imbedding $\mathfrak{n}^* \subset \mathfrak{g}^*$, allowing us to regard $\mathfrak{X}$ as a subset of $\mathfrak{g}^*$.

First suppose π is an irreducible admissible representation of G on a Banach space, and let π_K and π^∞ denote the spaces of K-finite and smooth vectors, respectively. For $\psi \in \mathfrak{X}$ we define the spaces of Whittaker functionals:

$$Wh_\psi^*(\pi_K) = Hom_{\mathfrak{n}}(\pi_K, \psi), \; Wh_\psi^*(\pi^\infty) = Hom_N(\pi^\infty, \psi)$$

where in the latter space we require the morphism to be continuous with repect to the Frechet topology of π^∞. We also denote by $\mathcal{V}_\pi \subset \mathfrak{g}^*$ the associated variety of the annhilator of π.

Our first main result is as follows:

Theorem. *Let π be an irreducible admissible representation of $GL(n, \mathbb{R})$, and let $\psi \in \mathfrak{X}$ be as above, then the following are equivalent:*

(1) $\psi \in \mathcal{V}_\pi$
(2) $Wh_\psi^*(\pi_K) \neq 0$
(3) $Wh_\psi^*(\pi^\infty) \neq 0$

Now suppose π is an irreducible unitary representation of $G_n = GL(n,\mathbb{R})$, i.e. $\pi \in \widehat{G_n}$ then we define the depth d of π, and the highest derivative $A\pi \in \widehat{G_{n-d}}$ as follows: First of all we know by a result of Baruch that π restricts irreducibly to the *mirabolic* subgroup $P_n \subset G_n$ consisting of invertible matrices with last row $(0,\cdots,0,1)$. The group P_n is a semidirect product $G_n \ltimes \mathbb{R}^{n-1}$, and analysing its unitary dual iteratively by Mackey theory we obtain

$$\widehat{P_n} \approx \widehat{G_{n-1}} \coprod \widehat{P_{n-1}} \approx \coprod_{d=1}^{n} \widehat{G_{n-d}}$$

We define d and $A\pi$ by restricting π to P_n and using the above isomorphism. We also define iterated highest derivatives by $A^0\pi = \pi$ and $A^i\pi = A\left(A^{i-1}\pi\right)$, and define the depth sequence of π to be

$$\delta(\pi) = (d_1, d_2, \ldots) \text{ where } d_i = \text{depth}\left(A^{i-1}\pi\right)$$

We recall next the notion of Howe rank for $\pi \in \widehat{G_n}$. Let $m = \lfloor n/2 \rfloor$ and let $P_{m,n-m} = (G_m \times G_{n-m}) \ltimes N_{m,n-m}$ be the "middle" maximal parabolic subgroup of G_n, consisting of matrices with 0's in the lower left $(n-m)\times m$ block. $N_{m,n-m}$ is an abelian group isomorphic to the space $\text{Mat}_{m\times(n-m)}$ of $m \times (n-m)$ real matrices. By Stone's theorem the restriction $\pi|_{N_{m,n-m}}$ is given by a projection-valued Borel measure μ_π on the unitary dual $\hat{N}_{m,n-m}$. The latter space can also be identified with $\text{Mat}_{m\times(n-m)}$ and Scarmuzzi has shown that $\mu_\pi$is of pure rank, i.e. there exists some integer k, called the Howe rank of π, such that

$$\mu_\pi(E) = \mu_\pi(E \cap \mathcal{R}_k) \text{ for all Borel subsets } E \subset \text{Mat}_{m\times(n-m)}$$

where $\mathcal{R}_k \subset \text{Mat}_{m\times(n-m)}$ denotes the subset of matrices of rank k.

Our second main result is as follows:

Theorem. *Let π be an irreducible unitary representation of $G_n = GL(n,\mathbb{R})$. Then*

1. *The depth sequence $\delta(\pi)$ is decreasing, and hence defines a partition $\lambda = \lambda_\pi$*
2. *The associated variety $\mathcal{V}_\pi$ is the closure of the nilpotent coadjoint orbit $O(\lambda)$, where the latter consists of all nilpotent matrices of Jordan type λ.*
3. *The Howe rank of π is $\min(\lfloor n/2 \rfloor, n - l(\lambda))$, where $l(\lambda)$ is the number of nonzero parts of λ.*

Vanishing at infinity on homogeneous spaces and applications to lattice counting

Eitan Sayag
(joint work with Bernhard Krötz, Henrik Schlichtkrull)

Let G be a real Lie group and $H \subset G$ be a closed subgroup. Consider the homogenous space $Z = G/H$ and assume that it is unimodular, that is, it carries a G-invariant measure μ_Z. Note that such a measure is unique up to a scalar multiple.

For a Banach representation (π, E) of G let us denote by E^∞ the space of smooth vectors which is naturally a Fréchet module for G. In the special case for the regular representation L of G on $E = L^p(Z)$ with $1 \leq p < \infty$, it follows from the local Sobolev lemma that $E^\infty \subset C^\infty(Z)$. Let $C_0^\infty(Z)$ be the space of smooth functions on Z that vanish at infinity. Motivated by the decay of eigenfunctions on symmetric spaces ([9]), the following definition was taken in [5]:

Definition. *We say Z has the property* VAI *(vanishing at infinity) if for all $1 \leq p < \infty$ we have*

$$L^p(Z)^\infty \subset C_0^\infty(Z).$$

By a result of [7] $Z = G$ has the VAI property for G unimodular and $H = \{\mathbf{1}\}$. Furthermore, by [5] all semisimple symmetric spaces admit the VAI property. On the other hand, if H is a non-cocompact lattice in G then $Z = G/H$ is not VAI.

Assume that H is connected. We say that Z is of *reductive type* in case H is a reductive subgroup of G, that is, if the adjoint representation of H in the Lie algebra $\mathfrak{g}$ of G is completely reducible. In this article we prove the following.

Theorem. *Let G be a real reductive group, $H \subset G$ a closed connected subgroup such that $Z = G/H$ is unimodular. Then* VAI *holds for Z if and only if it is of reductive type.*

In fact, we show that if Z is as above and not of reductive type, then there exist unbounded functions in $L^p(Z)^\infty$ for all $1 \leq p < \infty$.

If Z is of reductive type and $B \subset G$ is a compact ball we provide essentially sharp lower and upper bounds for $\mathrm{vol}_Z(Bgz_0)$ where $z_0 \in Z$ is a base point and $g \in G$ is such that gz_0 moves off to infinity. We found simple and short arguments for these bounds. Our results generalize and simplify previous approaches in [6] and [4]. The lower bounds in particular imply that Z has VAI.

In case Z is not of reductive type we essentially show that there is a compact ball $B \subset G$ and a sequence $(g_n)_{n\in\mathbb{N}}$ such that

- $Bg_nz_0 \cap Bg_mz_0 = \emptyset$ for $n \neq m$.
- $\mathrm{vol}_Z(Bg_nz_0) \leq e^{-n}$ for all $n \in \mathbb{N}$.

Out of these data it is straightforward to construct a smooth L^p-function which does not vanish at infinity.

We did not address here the cases where H is not connected or G is not reductive. Without any further assumption let us assume that G is a connected Lie group and $H \subset G$ is a closed subgroup such that $Z = G/H$ is unimodular. In case G is infinitesimally simple and Z is not compact one might suspect that Z has VAI if and only if the Zariski closure of H is a proper reductive subgroup. For G and H algebraic and G reductive, H is reductive in G if and only if it is reductive. One might then suspect for G and H algebraic and G general, that Z has VAI if and only if the nilradical of H is contained in the nilradical of G.

Initially we wanted to prove the converse implication in Theorem via a temperedness result for invariant measures. To be more specific, assume G and $H < G$ to be algebraic groups and $Z = G/H$ to be unimodular and quasi-affine. Under these assumptions we conjecture that there is a rational G-module V, and an embedding $Z \to V$ such that the invariant measure μ_Z, via pull-back, defines a tempered distribution on V. Note that if Z is of reductive type, then there exists a V such that the image of $Z \to V$ is closed, and hence μ_Z defines a tempered distribution on V. If Z is not of reductive type, then all images $Z \to V$ are non-closed and our conjecture would imply that VAI does not hold. Our conjecture is supported by a result of Deligne, established in [8], which asserts that for a reductive group G and $X \in \mathfrak{g} := \mathrm{Lie}(G)$ the invariant measure on the adjoint orbit $Z := \mathrm{Ad}(G)(X) \subset \mathfrak{g}$ defines a tempered distribution on $\mathfrak{g}$. Various particular results in the theory of prehomogeneous vector spaces provide additional support for our conjecture (see [1]).

Finally we wish to put VAI in the context of ergodic theory. We recall that the main ergodic theorem is implied by the fact that matrix coefficients of unitary representations which do not contain the trivial representation vanish at infinity. Now VAI may be interpreted as analogous to this fact. Thus it is natural expect that VAI is related to counting problems on Z's of reductive type. In fact this is the case: we show that soft harmonic analysis puts the counting results of [2] and [3] in the realm of VAI.

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Hodge theory and unitary representations of reductive Lie groups

Wilfried Schmid
(joint work with Kari Vilonen)

Let $G_\mathbb{R}$ be a linear, reductive matrix group, $K_\mathbb{R} \subset G_\mathbb{R}$ a maximal compact subgroup, G and K the complexifications of the two groups, and $U_\mathbb{R} \subset G$ a compact real form which contains $K_\mathbb{R}$. The Lie algebras will be denoted by the corresponding lower case German letters $\mathfrak{g}_\mathbb{R}$, $\mathfrak{k}_\mathbb{R}$, $\mathfrak{g}$, etc.

Recall the notion of a Harish Chandra module for $G_\mathbb{R}$: a finitely generated module V over the universal enveloping algebra $U(\mathfrak{g})$, equipped with an algebraic action of K with finite multiplicities, such that the two actions are compatible. It has been understood for a long time that the problem of understanding the irreducible unitary representations of $G_\mathbb{R}$ can be reduced to the following three seemingly algebraic problems:

a) classify the irreducible Harish Chandra modules for $G_\mathbb{R}$;
b) determine which of these carry a non-zero $\mathfrak{g}_\mathbb{R}$-invariant hermitian form (which is necessarily unique up to scaling);
c) when a nonzero $\mathfrak{g}_\mathbb{R}$-invariant hermitian form exists, determine whether it is (positive or negative) definite.

Of these, a) has three equivalent, though different solutions, due to Langlands, Vogan-Zuckerman, and Beilinson-Bernstein, and b) can be answered by an essentially formal argument. Problem c) is still open. It should be mentioned that the irreducible unitary representations of certain groups $G_\mathbb{R}$, and some classes of irreducible unitary representations of any $G_\mathbb{R}$, have been determined; however, these results do not suggest a general pattern.

More than twenty years ago, Vogan argued that the problem c) can be reduced to the case when the Harish Chandra module V has a real infinitesimal character – in other words, an infinitesimal character χ_λ in Harish Chandra's notation, with parameter λ in the $\mathbb{R}$-linear span of the weight lattice. More recently [1], Vogan and his coworkers observed that any irreducible Harish Chandra module V with real infinitesimal character carries a non-zero $\mathfrak{u}_\mathbb{R}$-invariant hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$. Moreover, if V also carries a nonzero $\mathfrak{g}_\mathbb{R}$-invariant hermitian form, the two hermitian forms are related in a direct, explicitly describable manner. Thus, if one could solve the problem

c') explicitly describe the non-zero $\mathfrak{u}_\mathbb{R}$-invariant hermitian form on an irreducible Harish Chandra module V with real infinitesimal character,

one could treat the problem c).

The $\mathfrak{u}_\mathbb{R}$-invariant hermitian form on V can be constructed geometrically, on the Beilinson-Bernstein realization of V, in terms of Sabbah's notion of polarization of a $\mathcal{D}$-module. Using Morihiko Saito's theory of mixed Hodge modules – or more precisely, an extension of his theory – Vilonen and I have constructed two canonical filtrations on any Harish Chandra module V, irreducible or not, with real infinitesimal character: an increasing "Hodge filtration"

$$0 \subset F_a V \subset F_{a+1} V \subset \ldots \subset F_p V \subset \ldots \subset \cup_{p=a}^{\infty} F_p V = V$$

by finite dimensional K-invariant subspaces $F_p V$, and a finite increasing "weight filtration"

$$0 \subset W_0 V \subset \ldots \subset W_k V \subset W_{k+1} V \subset \ldots \subset V$$

by Harish Chandra submodules $W_k V$. These filtrations are induced by analogous filtrations of the Beilinson-Bernstein realization of V, and are functorial on that level, with respect to all the standard $\mathcal{D}$-module morphisms.

Conjecture. *If V is an irreducible Harish Chandra module with real infinitesimal character, the non-zero $\mathfrak{u}_{\mathbb{R}}$-invariant hermitian form is nondegenerate on each $F_p V$, and the induced hermitian form on the quotients*

$$F_p V / F_{p-1} V \cong F_p V \cap (F_{p-1} V)^{\perp}$$

is alternatingly positive and negative definite, depending on the parity of p.

The conjecture, if proved, would not answer the question c') directly, but it would put c'), and thereby also c), into a functorial context, and would make c) approachable by geometric arguments.

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Restriction of discrete series representations

Jorge Vargas

Let H denote a closed reductive subgroup of a reductive Lie group G. Assume there exists (π, V) an irreducible square integrable representation for G. The work I have done in the last few years is aimed to understand the structure of the restriction $res_H(\pi)$ of (π, V) to the subgroup H.

In joint work with Michel Duflo, we found necessary and sufficient conditions to assure that the restriction of (π, V) to H is an admissible restriction, extending work of Toshiyuki Kobayashi. We also have obtained, under certain conditions, formula for the multiplicity of each irreducible factor of $res_H(\pi)$ quite similar the one obtained by Kostant for the multiplicity of a weight and as the formula of Blattner proved by Hecht and Schmid. Some of conditions to assure admissibility are algebraic, others are formulated in the language of coadjoint orbits and projection maps.

I have also collaborated with Bent Orsted.

A new result is the following, $G = SO(2n,1) \times SO(2n,1)$ and H equal to the diagonal subgroup, $K = SO(2n)$, (τ, W) lowest $K \times K$ type of π. Then, the lowest

$K-$type of any discrete factor of π restricted to H is contained in $res_K(W) \otimes Ind_M^K(Trivial)$.

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Topological blow-up and discontinuous groups

Taro Yoshino

Let X be a non-Hausdorff space with a crack S. We introduce the concept of "Topological blow-up" as a 'repair' of the crack. In general, the 'repaired' space $\widetilde{X}$ is a sequential Hausdorff space containing $X \setminus S$ as its open subset. Moreover, in many cases, $\widetilde{X}$ is Hausdorff. The original space X can be recovered from the pair of $(\widetilde{X}, S)$.

1. Example

Let $\mathbb{R}$ act on $\mathbb{R}^4$ linearly and unipotently by

$$\mathbb{R} \to GL(4, \mathbb{R}), \qquad t \mapsto \begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & t \\ & & & 1 \end{pmatrix}.$$

We consider the quotient space $X := \mathbb{R} \backslash \mathbb{R}^4$. This space naturally divides into two parts:

$$R := \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in X \mid (y, w) \neq (0,0) \right\}, \qquad S := \left\{ \begin{bmatrix} x \\ 0 \\ z \\ 0 \end{bmatrix} \in X \right\}.$$

The subset R (*regular part*) is the set of all one-dimensional orbits. On the other hand, S (*singular part*) is the set of all zero-dimensional orbits. In the induced topology, we have $R \simeq (\mathbb{R}^3 \setminus \mathbb{R})$ and $S \simeq \mathbb{R}^2$. Roughly speaking two-dimensional space S is in the one dimensional 'hole' of R. So, X is not Hausdorff. This space, however, is not so 'bad'. In fact, the 'generic part' is Hausdorff. Our method works well for such a space.

2. Setting

Let X be a (non Hausdorff) topological space. We say a closed subset S is a *crack*, if for any distinct elements $x, y \in X$,

$$\{x, y\} \not\subset S \Longrightarrow x \text{ and } y \text{ have disjoint neighbourhoods.}$$

By definition, the complementary part of S is Hausdorff. We may think of S as the 'lack of Hausdorffness' of X. Our construct of *topological blow-up* 'repairs' this crack by patching another topological space $\mathfrak{L}$.

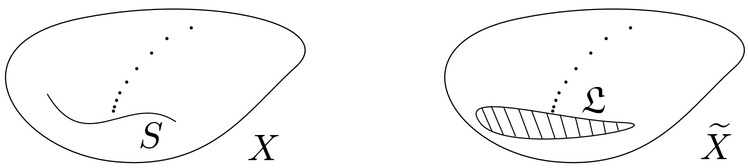
To be precise, the 'repaired' space $\widetilde{X}$ will be defined as:

$$\widetilde{X} := \mathfrak{L} \sqcup R,$$

where $R := X \setminus S$ and $\mathfrak{L} \subset 2^S$ (the power set of S). Here, we want to ask: What is $\mathfrak{L}$? How to define a topology on $\widetilde{X}$? But, before that, let us see a naive idea to define topological blow-up.

3. Naive idea

Suppose that a sequence in R approaches to the singular part S. Note that the limit points of this sequence are not necessarily unique. So, we consider the set of all limit points, and denote it by l.



Then, l is a subset of S, and l should be a point in $\mathfrak{L}$. Here, recall that the 'regular parts' of X and $\widetilde{X}$ are the same space R. So we can 'copy' the sequence to the right picture. Then, our naive idea is:

It is desirable if the 'copied' sequence convergences to the point l in the topology on $\widetilde{X}$.

Unfortunately this idea does not work well. We need to modify our idea. There are three steps.

4. Filter

We need to replace the word 'sequence' by 'filter', because 'convergence of sequence' does not have enough information on the topological space which is not first countable.

5. Prime filter

For a filter $\mathcal{F}$, we define

$$\lim \mathcal{F} := \{s \in S \mid \mathcal{F} \to s\}.$$

And we put

$$\mathfrak{L}^{\mathrm{all}} := \{\lim \mathcal{F} \mid \mathcal{F} \text{ is a filter of } R\}.$$

In light of the naive idea, the patching space $\mathfrak{L}$ should be $\mathfrak{L}^{\mathrm{all}}$. However, $\mathfrak{L}^{\mathrm{all}}$ is too large as the patching space.

Observation. *Take two elements $l_1, l_2 \in \mathfrak{L}^{all}$ such that $l_1 \cap l_2 \neq \emptyset$. For simplicity, in what follows, we use sequence instead of filter. There are sequences $\{x_n\}$, $\{y_n\}$ whose limit sets are l_1 and l_2, respectively. Define the third sequence by*

$$z_n := \begin{cases} x_n & (n \text{ is odd}) \\ y_n & (n \text{ is even}). \end{cases}$$

Then, the limit set of $\{z_n\}$ is $l_3 := l_1 \cap l_2 \in \mathfrak{L}^{all}$. In the topology on $\widetilde{X}$, while $\{z_n\}$ convergences to a point l_3, the subsequence $\{z_{2n}\}$ convergences to a point l_1. This must not happen. In other word, l_3 should not be a point of the patching space $\mathfrak{L}$.

To remove such elements, we introduce:

Definition. *A filter $\mathcal{F}$ is* **prime**, *if any finer filter $\mathcal{F}'$ has the same limit set* ($\lim \mathcal{F}' = \lim \mathcal{F}$).

Here, we have used the terminology "finer" which corresponds to "subsequence" in terms of sequences. We put

$$\mathfrak{L}^{\text{prime}} := \{\lim \mathcal{F} \mid \mathcal{F} \text{ is a primer filter of } R\}.$$

It turns out that there are enough prime filters to understand all convergent filters of R in X. To be precise, we have:

Lemma. *For any filter $\mathcal{F}$, there exists a finer prime filter $\mathcal{F}'$.*

Then, in fact, $\mathfrak{L}^{\text{prime}}$ is good as patching space. However, it is not so easy to tell whether a given limit set is prime or not. So we introduce

$$\mathfrak{L}^{\text{max}} := \left\{ l \in \mathfrak{L}^{\text{all}} \mid l \text{ is maximal in } \mathfrak{L}^{\text{all}} \right\}.$$

Then we have:

Observation. *Maximal limit is prime. ($\mathfrak{L}^{max} \subset \mathfrak{L}^{prime}$).*

6. Analogy to algebraic geometry

There are some similarities with the elementary theory of algebraic geometry.

$\mathfrak{L}^{\text{all}}$	limit set	$\longleftrightarrow$	ideal
$\mathfrak{L}^{\text{prime}}$	prime limit	$\longleftrightarrow$	prime ideal
$\mathfrak{L}^{\text{max}}$	maximal limit	$\longleftrightarrow$	maximal ideal

Moreover, in the algebraic geometry, Spec is important. It is the set of all prime ideals with topology. In what follows, we define a topology on $\mathfrak{L} := \mathfrak{L}^{\text{prime}}$, which is the set of all prime limits.

7. Patching map

To define a topology on $\widetilde{X}$, we introduce a *patching map*. Apart from our setting for a while, we consider a topological space $X = A \sqcup B$ in general such that B is open in X.

¿From the topology on X, we obtain the induced topology on A and B. On the other hand, if we forget the topology on X, we cannot recover the original topology from only the topology on A and B. In fact, we do not know how A and B are connected. In other words, we need '$+\alpha$ information' to recover.

$$\text{(topology on } X) \longleftrightarrow \text{(topology on } A) + \text{(topology on } B) + \alpha$$

Patching map is just such information.

Definition. *A* **patching map** *is a map $\mu : \mathcal{O}_B \to \mathcal{O}_A$ having the following properties:* $(U, V \in \mathcal{O}_B)$

(1) $\mu(B) = A$, (2) $\mu(U \cap V) = \mu(U) \cap \mu(V)$, (3) $U \subset V \Rightarrow \mu(U) \subset \mu(V)$.

Here, we denote by $\mathcal{O}_A$ (resp. $\mathcal{O}_B$) the set of all open subsets in A (resp. B). Then, we have

Lemma. *Fix topologies on A and B. Then there is a one-to-one correspondence:*

$$\textit{(topology on } X \textit{ such that } B \textit{ is open)} \longleftrightarrow \textit{(patching map } \mu : \mathcal{O}_B \to \mathcal{O}_A\textit{)}.$$

In the above lemma, the correspondence is given by:
[left to right] $\mu(U) := A \cap \mathrm{Int}_X(A \sqcup U)$ for $U \in \mathcal{O}_B$.
[right to left] T is open in $X \overset{\text{def}}{\Longleftrightarrow} A \cap T \in \mathcal{O}_A$, $B \cap T \in \mathcal{O}_B$, and $A \cap T \subset \mu(B \cap T)$.

Now, return to our setting. For a prime limit $l \in \mathfrak{L} := \mathfrak{L}^{\text{prime}}$, we set

$$\Omega_l := \{\mathcal{F} : \text{ prime filter of } R \mid \lim \mathcal{F} = l\}.$$

Define a map $\mu : \mathcal{O}_R \to 2^{\mathfrak{L}}$ by

$$\mu(U) := \{l \in \mathfrak{L} \mid U \in \mathcal{F} \text{ for any } \mathcal{F} \in \Omega_l\}.$$

We define a topology on $\mathfrak{L}$ by the system of open sets generated from the image of μ. Then μ satisfies the properties of patching map. So we obtain a topology on $\widetilde{X} = \mathfrak{L} \sqcup R$.

8. Example

We apply our method to the first example. The patching space $\mathfrak{L}$ is given by

$$\mathfrak{L} := \left\{ l \subset S \mid l \text{ is a line in } S \, (\simeq \mathbb{R}^2) \right\}.$$

As a topological space, $\mathfrak{L}$ is homeomorphic to the Möbius band. Let us consider how $\mathfrak{L}$ and R are connected. Recall that R is homeomorphic to $\mathbb{R}^3 \setminus \mathbb{R}$, thus also homeomorphic to $\mathbb{R}^3 \setminus T$, where T is a solid tube in $\mathbb{R}^3$. We consider an equivalence relation on the boundary of T by:

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} x = \pm y \qquad (\text{for } x, y \in \partial T).$$

Since the quotient space $\partial T / \sim$ is homeomorphic to the Möbius band, there is a two-to-one map $f : \partial T \to \mathfrak{L}$. This map gives a topology on $\widetilde{X} = \mathfrak{L} \sqcup R$. In particular, the 'repaired space' $\widetilde{X}$ is Hausdorff in this case.

9. Motivation

The deformation space $\mathcal{T}(\Gamma, G, H)$ of discontinuous groups Γ on a non-Riemannian homogeneous space G/H is not a Hausdorff space in general. This work was originally motivated from the trial of understanding of such deformation space (see [K93, KN06, BKY08]).

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