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Mini-Workshop: Dynamics of Trace Maps and Applications to Spectral Theory

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Abstract. Recently some exciting results for the spectrum of Schrödinger operators with self-similar potentials and the dynamics of the associated Schrödinger equation have been established using input from complex and smooth dynamics. The workshop allowed us to bring together experts in dynamical systems, spectral theory, and quasi-crystals to study these deep and promising relations.

Mathematics Subject Classification (2000): 37D20, 47B36, 81Q10.

Introduction by the Organisers

In the early 1980's, two groups independently laid the groundwork for much activity in (solid state) physics and mathematics (spectral theory) in the decades since then. Shechtman et al. discovered new structures, nowadays called quasi-crystals, that have unexpected and intriguing behavior. Namely, these structures have a diffraction pattern resembling that of crystals but also displaying rotational symmetries that are impossible for crystals. Kohmoto et al. on the other hand proposed a simple quasi-periodic Schrödinger operator, the Fibonacci Hamiltonian, with critical behavior for all non-zero values of the coupling parameter. That is, the eigenfunctions are neither localized nor extended, the spectral measures are purely singular continuous, and quantum transport is anomalous. All these properties are by now rigorously established.

The mathematical models of quasi-crystals have a number of features, such as being constructed by a cut-and-project method (projection of a part of a higher-dimensional lattice along incommensurate directions), displaying self-similarity (resulting from a construction based on inflation), and of course the desired diffraction behavior. The model proposed by Kohmoto et al. has all these features and it has consequently become the standard one-dimensional quasi-crystal model.

The mathematics involved in the analysis of such models has many facets. Naturally, it involves discrete geometry, spectral theory, and harmonic analysis. In addition, dynamical systems play an important role. As shown by Kohmoto et al., the self-similarity gives rise in a natural way to a renormalization procedure which results in a direct correspondence between the desired spectral properties and the dynamics of a three-dimensional polynomial map, $T(x, y, z) = (2xy-z, x, y)$. Since the variables correspond to traces of transfer matrices, the map T is called the *trace map*. In fact, any self-similar model gives rise to a corresponding polynomial map and hence there is a rich class of trace maps whose dynamics are directly related to a certain class of Schrödinger operators. Moreover, since the spectra of these Schrödinger operators are zero-measure Cantor sets, methods from geometric measure theory have also been applied successfully to study the scaling properties of these Cantor sets.

Trace maps have been studied with a view towards spectral theory in many works. Most of them regard T as a real dynamical system, that is, as $T : \mathbb{R}^3 \to \mathbb{R}^3$. This is in some sense natural since due to self-adjointness of the Schrödinger operators, spectra and spectral measure live on $\mathbb{R}$ and for real energies, all traces are real. However, some recent works have challenged this picture and instead studied $T : \mathbb{C}^3 \to \mathbb{C}^3$. This innocent-looking change of perspective has opened up a new tool-box, that of complex analysis and complex dynamics, and in fact allowed one to prove spectral results that had been completely out of reach.

Another promising new fusion of ideas involves the application of uniformly and normally hyperbolic dynamics to trace maps and has led to a multitude of new results for the weakly coupled Fibonacci Hamiltonian. Among these results are estimates of the fractal dimension of the spectrum and complete gap labeling in the sense of Bellissard.

The aim of the workshop was to pursue these new points of view vigorously.

The structure of the workshop was the following. The meeting was attended by 17 participants, with each presenting their results and/or a historical overview of the subject. We started off with several overview lectures (by Kohmoto, Sütő, Bellissard, Grimm, Damanik), and continued with presentations of particular results. One of the talks (by Lifshitz) provided an exposition of physical point of view, which was quite inspiring for mathematicians. In the very last talk Embree presented numerical results on spectral properties of Fibonacci Hamiltonian, which also suggest numerous new conjectures. One night Grimm presented a beautiful general-audience talk, entitled "A hexagonal monotile for the Euclidean plane," which many participants from the other mini-workshops also attended.

We have the feeling that bringing together people from dynamical systems, spectral theory, mathematical physics, and physics of quasicrystals turned out to be amazingly productive, provided deeper understanding of the subject by all the participants, and will eventually lead to new exciting results.

Mini-Workshop: Dynamics of Trace Maps and Applications to Spectral Theory

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Abstracts

TKNN and KKT: topological theory of QHE and the dynamical system for the quasiperiodic operator

Mahito Kohmoto

2D Bolch electrons in a magnetic field are considered, and it is shown that Hall conductance is given by integers, Chern number. Then it is related to 1D quasiperiodic problem. Matrix iteration is considered as a renormalization group procedure, and the trace map is derived for the Fibonacci lattice. Several properties of the systems are presented, i.e., Cantor set spectrum, critical wave functions etc.

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The Fibonacci Hamiltonian

András Sütő

This is an account of my two papers [1, 2] on the difference equation

$$(H\psi)_n = \psi_{n-1} + \psi_{n+1} + \lambda(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)\psi_n \tag{1}$$

where $\alpha = (\sqrt{5}-1)/2$. Earlier nonrigorous work by physicists [3, 4, 5] arrived at the conclusion that the spectrum of H is a Cantor set of zero Lebesgue measure and the spectral measures are singular continuous. The mathematical study of the problem started with Casdagli's paper [6] in which it was proven that the so-called dynamical spectrum is a Cantor set of zero Lebesgue measure for $|\lambda| \geq 16$. In the proof and in all subsequent papers the best approximants of α, $\alpha_n = F_{n-1}/F_n$, (F_n is the sequence of Fibonacci numbers with initial conditions $F_0 = F_1 = 1$) and the associated F_n-periodic approximations H_n of H play a crucial role. The well-known method of solution of (1) is the use of transfer matrices. If $\Psi_N = (\psi_{N+1}\ \psi_N)^T$ and M_n is the transfer matrix over the sites $1, 2, \ldots, F_n$, then $\Psi_{F_n} = M_n\Psi_0$. Due to the quasi-periodicity of the sequence $\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$, both M_n and its trace $2x_n$ satisfy recurrence relations. There is also an invariant I associated with the sequence x_n. The spectrum of H_n is $\sigma_n = \{E \in \mathbb{R} : |x_n(E)| \leq 1\}$. The following theorem can be proven using basic spectral theory.

Theorem *$\sigma_n \cup \sigma_{n+1}$ is a decreasing sequence, and* $\sigma(H) = \bigcap_{n=1}^{\infty}(\sigma_n \cup \sigma_{n+1})$.

The spectrum of H is purely continuous.

In the context of trace maps the Lyapunov exponent is defined as

$$\gamma(E) = \lim_{N\to\pm\infty} (1/N) \ln \|T_N\|,$$

provided that the limit exists. Here T_N is the transfer matrix over the sites $1, 2, \ldots, N$.

Proposition *For $E \in \sigma(H)$, $\gamma(E) = 0$.*

Confronting two theorems of Kotani [7, 8] on the set $\{E : \gamma(E) = 0\}$ and using the continuity of the spectrum one arrives at the the following result.

Theorem *For any nonzero λ, H has a purely singular continuous spectrum on a Cantor set of zero Lebesgue measure.*

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1D-Quantum systems, 1983-1990: a review.

Jean V. Bellissard

This talk presents a review of the problems and results obtained during the eighties for one dimensional Schrödinger operators with aperiodic potentials, either on the continuum or on the lattice.

The first part starts with a description of the formalism of transfer matrix. Then the examples of potentials that were already under scrutiny in 1979, the Anderson model in one-dimension, quasi periodic potentials, and a model with singular continuous spectrum. The Harper model is part of this program, and the known (non-rigorous) results on this model were provided.

The second part concerns the formalism used by the author and his collaborators in the eighties [4, 3] to describe such systems: (i) the tight binding representation (called at the time the *French connection*), (ii) the Hull, its transversal and the corresponding groupoids, (iii) the C^*-algebra of the Hull, (iv) the use of this algebra to compute the Integrated Density of States (IDS) and the *Gap labeling Theorem*.

The third part concerns the results obtained by various authors concerning the existence of a Cantor spectrum. The first existence theorem by Moser [5], the results obtained on the almost Mathieu and Harper models between 1982 and 2009, the study of one-dimensional quasicrystals using the trace map [1, 2], the case of substitution sequences like the Thue-Morse one.

The last part concludes with a series of open problems, such as the gap opening, the properties of potentials with positive configurational entropy, the study of transport exponents.

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Trace maps, invariants and applications

Uwe Grimm

Trace maps emerged in physics in 1983 [2, 4] in the study of non-periodic 1D quantum systems; they appeared before in a mathematical context in [5]. As trace maps apply to any system of (2×2)-matrices derived from a two-letter substitution rule, they have numerous applications [1].

In the most general setting, a two-letter subsitution can be regarded as an endomorphism ϱ of the free group $F_2 = \langle a, b \rangle$ generated by two letters a and b. The invertible substitutions are the automorphisms of F_2, which form the group $\mathrm{Aut}(F_2)$. Let $A, B \in \mathrm{SL}(2, \mathbb{C})$ denote unimodular 2×2 matrices, and consider the corresponding matrix system, where a sequence of letters a and b is replaced by a product of matrices A and B. Due to the Cayley-Hamilton theorem, the trace of the product can be expressed in terms of the traces of A, B and AB. In particular, if $x = \frac{1}{2}\mathrm{tr}(A)$, $y = \frac{1}{2}\mathrm{tr}(B)$ and $z = \frac{1}{2}\mathrm{tr}(AB)$, then any substitution ϱ gives rise to an associated trace map $F_\varrho \in \mathbb{Z}[x, y, z]^3$, with fixed point $(1, 1, 1)$.

Any trace map that stems from an automorphism leaves the Fricke-Vogt character $I(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1$ invariant, while for a general $\varrho \in \mathrm{End}(F_2)$ it transforms as $F_\varrho(I(x,y,z)) = P_\varrho(x,y,z)I(x,y,z)$, with $P_\varrho \in \mathbb{Z}[x,y,z]$. The level set $\{(x,y,z) \in \mathbb{C}^3 \mid I(x,y,z) = 0\}$ is then invariant for any such trace map; compare the other contributions to this workshop.

The Fricke-Vogt character is not the only possible invariant. For instance, the trace map for the generalised Fibonacci substitution $\varrho^{(k)}$ which map $a \mapsto b$ and $b \mapsto b^{k-1}a^k$, with $k \in \mathbb{Z}$, preserves the function

$$H(x,y,z) = yU_k(x) - zU_{k-1}(x)\,,$$

where U_k, with $U_{-k} = -U_{k-2}$, denote the Chebyshev polynomials of the second kind [1].

Trace maps can also be applied to systems with non-unimodular matrices. An example for one-dimensional classical Ising chains with modulated interactions and fields is discussed in [1]. Here, one considers the determinants separately, which are easily computed from the number of letters a and b in the substitution sequence. Another possible application of trace maps is to kicked two-level systems, where the sequence corresponds to a periodic kicking of a system with two different strength and direction of kicks [1].

For some applications, the trace alone may not contain the information that is required. An example is transmission through aperiodic multilayers [3], where the transmission coefficient is the sum of the four entries of the transfer matrix. An approach using an 'anti-trace' map was proposed in [6]. As an example, for the Fibonacci case $T_{n+1} = T_{n-1}T_n$, the traces $x_n = \frac{1}{2}\mathrm{tr}(T_n)$ and the anti-traces a_n (here defined as half the difference between the off-diagonal elements of T_n) satisfy the coupled set of equations

$$x_{n+1} = 2x_n x_{n-1} - x_{n-2} \quad \text{and} \quad a_{n+1} = 2x_n a_{n-1} + a_{n-2}\,.$$

Some generalisations and applications are discussed in [6].

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Some developments in 1D quantum systems (1998–2010)

David Damanik

In this talk we surveyed some results for Schrödinger operators in $\ell^2(\mathbb{Z})$ with low-complexity potentials obtained between 1998 and 2010. The presentation was centered around the class of Schrödinger operators with Sturmian potentials, that is,

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \theta \mod 1)\psi(n)$$

with $\lambda > 0$, $\alpha \in [0,1) \setminus \mathbb{Q}$, and $\theta \in [0,1)$.

The results were grouped as follows:

(i) Using partitions and Gordon-type criteria, absence of eigenvalues was shown for all Sturmian potentials and many related models (see, e.g., [7, 8]).

(ii) Using a quantitative version of subordinacy theory and a mass-reproduction technique to prove the required solution estimates, absolute continuity with respect to suitable Hausdorff measures was shown for the spectral measures associated with some Sturmian potentials (see, e.g., [1, 7, 17, 18]).

(iii) Uniform convergence to the Lyapunov exponents for all energies was shown for all Sturmian models and many related ones. This provides an elegant and simple proof of zero-measure spectrum; compare [9, 10, 20, 21].

(iv) Time-averaged quantum dynamics was studied with the help of the Plancherel theorem (resp., the Dunford functional calculus) and Green function estimates via transfer matrix bounds; compare [2, 11, 12, 13, 14, 15, 19, 24].

(v) The fractal dimension of the spectrum was studied in the papers [3, 4, 5, 6, 16, 22, 23].

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Dynamical properties of the trace map and spectrum of the weakly coupled Fibonacci Hamiltonian

Anton Gorodetski
(joint work with David Damanik)

We consider the spectrum of the Fibonacci Hamiltonian for small values of the coupling constant, and study the limit, as the value of the coupling constant approaches zero, of its thickness and its Hausdorff dimension. We prove that the thickness tends to infinity and, consequently, the Hausdorff dimension of the spectrum tends to one. We also show that at small coupling, all gaps allowed by the gap labeling theorem are open and the length of every gap tends to zero linearly. Moreover, for sufficiently small coupling, the sum of the spectrum with itself is an interval. This last result provides a rigorous explanation of a phenomenon for the Fibonacci square lattice discovered numerically by Even-Dar Mandel and Lifshitz [6, 7]. Finally, we show that the density of states is exact-dimensional, and its dimension also tends to one as coupling constant tends to zero. The proofs of these results [4, 5] are based on hyperbolicity of the trace map associated with Fibonacci Hamiltonian [1, 2, 3].

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(Reversing) symmetries of trace maps

Michael Baake

The trace maps discovered in the study of 1D discrete Schrödinger operators (see [1] and the other contributions of this workshop for references) occurred earlier in the context of algebraic geometry and group theory; compare [10, 3, 11] and references therein. Indeed, given the Fricke-Vogt invariant [6]

$$I(x,y,z) \,=\, x^2+y^2+z^2-2xyz-1\,,$$

one can define two sets of polynomial mappings, namely

$$\mathcal{A}=\{A\in\mathbb{C}[x,y,z]^3 \mid I\circ A = I\,\} \quad \text{and}$$
$$\mathcal{G}=\{G\in\mathbb{Z}[x,y,z]^3 \mid G\in\mathcal{A} \text{ with } G(1,1,1)=(1,1,1)\}\,.$$

It is a surprising (but well-known) fact [7, 11, 1] that $\mathcal{A}$ and $\mathcal{G}$ are groups, with $\mathcal{A}=\Sigma\rtimes\mathcal{G}$ (where $\Sigma\simeq C_2\times C_2$) and $\mathcal{G}\simeq \mathrm{PGL}(2,\mathbb{Z})$. The latter are the invertible trace maps that also emerge from elements of the automorphism group $\mathrm{Aut}(F_2)$ of the free group F_2 with two generators; see [1] and references therein for more. Note that this also implies $\mathcal{A}\subset\mathbb{Z}[x,y,z]^3$.

This algebraic setting allows the classification of (reversing) symmetries of (invertible) trace maps in a group theoretic setting; see [5] for general results and [9, 2, 3] for the case of trace maps. Given $F\in\mathcal{G}$, one defines

$$\mathcal{S}(F)=\mathrm{cent}_{\mathcal{G}}(F) \quad \text{and} \quad \mathcal{R}(F)=\{H\in\mathcal{G}\mid HFH^{-1}=F^{\pm1}\}\,,$$

which are subgroups of $\mathcal{G}$. Note that $\mathcal{S}(F)$ is a normal subgroup of $\mathcal{R}(F)$, with the corresponding factor group being either trivial or C_2. The possibilities for trace maps are classified in [3, Thms. 1 and 2], while [3, Thm. 3] gives the extension to the larger group $\mathcal{A}$. The proof can either be based on the isomorphism $\mathrm{PSL}(2,\mathbb{Z})\simeq C_2 * C_3$ and standard results from combinatorial group theory [8, 6], or on the known structure of $\mathrm{GL}(2,\mathbb{Z})$ and the symmetries of 'cat maps' [2, 4]. More detailed consequences for the dynamics of trace maps are discussed in [9, 8]; see also the other contributions of this workshop.

Various generalisations are possible from here, such as mappings of a similar structure (with an invariant) in higher dimensions [9, Sec. 7], or (non-invertible) trace maps derived from the monoid $\mathrm{Hom}(F_2)$. They lead to $I \circ F = P_F \cdot I$ with a polynomial $P_F \in \mathbb{Z}[x, y, z]$; see [11, 1] for details. None of them seem to have been pursued systematically so far.

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Some observations on the trace map related to period doubling potential

Yanhui Qu

In this talk we study the trace map related to the Period doubling potential. Base on the work of [1], we give some observations on it.

Consider the substitution σ over two letter alphabet $\{a, b\}$ defined as

$$\sigma(a) = ab, \quad \sigma(b) = aa.$$

The fixed point of σ is

$$\sigma^\infty(a) = abaaabababaaabaa \cdots := \zeta_1\zeta_2\zeta_3\cdots$$

Define a two-sided sequence β as

$$\beta_n = \beta_{-n} = \zeta_n, \quad (n \geq 1); \quad \beta_0 = a \text{ or } b.$$

β is called the period doubling sequence.

For any $E \in \mathbb{R}$ define

$$A_a := \begin{bmatrix} E-V & -1 \\ 1 & 0 \end{bmatrix} \quad A_b := \begin{bmatrix} E+V & -1 \\ 1 & 0 \end{bmatrix}$$

For $w = w_1 \cdots w_n \in \{a,b\}^n$ define $A_w := A_{w_n} \cdots A_{w_1}$. Define

$$x_n := \mathrm{tr}(A_{\sigma^n(a)}) \quad \text{and} \quad y_n := \mathrm{tr}(A_{\sigma^n(b)}).$$

It is known (see for example [1]) that

$$x_{n+1} = x_n y_n - 2; \qquad y_{n+1} = x_n^2 - 2.$$

Thus the computation of these quantities is realized by the following dynamic $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = (xy-2, x^2-2).$$

F is called the trace map related to the period doubling potential.

The two observations are that: at first we give sufficient condition for the points which have bounded orbits; then we give the following asymptotic expansion for the function g (whose graph $\Gamma(g)$ form the upper boundary of the stable set of F):

$$g(x) = \frac{2+\sqrt{2}}{x} + \frac{2+\sqrt{2}}{4x^3} + O(\frac{1}{x^5}), \quad (x \to \infty).$$

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On the spectrum of the 1D quantum quasiperiodic Ising model

William Yessen

We consider the Hamiltonian, the *1D quantum quasiperiodic Ising model,* given by

$$H = -\sum_n \sigma_n^{(z)} - \sum_n J_n \sigma_n^{(x)} \sigma_{n+1}^{(x)}$$

acting on the one-dimensional array of spins, $\otimes_n \mathbb{C}^2$, where $\sigma_n^{(x),(z)}$ are the spin-1/2 operators and $h > 0$ is the magnetic field in the direction transversal to the spin lattice, and $\{J_n\}$ is generated from $J_0, J_1 > 0$ by Fibonacci substitution [1]. The spin variables can be transformed into Fermi variables and diagonalized [1], resulting in

$$P = \sum_k \Lambda_k \eta_k^\dagger \eta_k$$

where $\eta_k, \eta_k^\dagger$ are noninteracting Fermi fields. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the Fibonacci trace map: $f(x,y,z) = (2xy - z, x, y)$. It was shown in [1] that the eigenvalues Λ_k lie on the curve

$$\gamma(\Lambda) = \left(\frac{\Lambda^2 - (h^2 + J_0^2/4)}{2hJ_0}, \frac{\Lambda^2 - (h^2 + J_1^2/4)}{2hJ_1}, \left(\frac{J_0}{J_1} + \frac{J_1}{J_0} \right) /2 \right)$$

and $\{f^n(\gamma(\Lambda_k))\}_{n\in\mathbb{N}}$ is bounded. This leads to the definition of the dynamical spectrum: $\mathbf{B}_\infty = \{\Lambda \in \mathbb{R} : \{f^n(\gamma(\Lambda))\}_{n\in\mathbb{N}}$ is bounded$\}$. We prove

Theorem [2] *For any $J_0, h > 0$ there exists $r_0 \in (0,1)$ such that for all J_1 satisfying $J_0/J_1 \in (1-r_0, 1+r_0)$, the dynamical spectrum $\mathbf{B}_\infty$ is a Cantor set of zero Lebesgue measure, whose local Hausdorff dimension is nonconstant, continuous, and lies strictly between zero and one.*

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Separable models of Fibonacci quasicrystals - numerical results

Shahar Even-Dar Mandel
(joint work with Ron Lifshitz)

This works uses two-dimensional and three-dimensional models of quasicrystals based on the one-dimensional Fibonacci quasicrystal[1]. The process by which these models are constructed yields a separable Schrödinger equation and hence allows us to use the one-dimensional results to calculate the physical properties of the higher-dimensional models[3].

The electronic energy spectra in these quasicrystals are studied. The one-dimensional spectrum is known to be a Cantor-like set of zero Lebesgue measure for any choice of physical parameters. The higher-dimensional models have spectra which can consist of continuous energy intervals, similar to the spectra of periodic crystals, for weak quasiperiodicity. For strong quasiperiodicity the higher-dimensional models yield nowhere-dense spectra similar to the one-dimensional model. In the intermediate range we encounter spectra with mixed structure. Estimates on the critical values of the physical parameters of the model in which the transitions between regimes occur are obtained numerically[2, 4, 5]. Analytical results by Damanik and Gorodetski, motivated by these numerical results confirmed their vailidity[6, 7].

The dynamics of electronic wave-packets that are initially localized at a single site of the crystals is also studied. Transitions between regimes in the dynamics of wave-packets are associated with the transitions between regimes in the structure of the spectra. Estimates for the transitional values are obtained numerically. The power-law decay of the survival probability and the inverse participation ratio is observed to be also modulated by log-periodic oscillations[8].

Hypothesizing a relation between the spectra and the nature of the electronic eigenfunctions of these quasicrystals, we use linear combinations of degenerate (or nearly-degenerate) eigenfunctions, and by utilizing an efficient search algorithm we succeed in generating extended eigenfunctions. This is in contrast to the known nature of the one-dimensional eigenfunctions which are characterized by polynomial spatial decay. The extended eigenfunctions obtained by the search algorithm display a long-range order reflecting the quasiperiodicity of the underlying potential, and indicate the possible existence of a quasiperiodic version of the Bloch theorem [9].

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Scaling properties of Sturmian potential based Schrödinger operator: some useful tools

Laurent Raymond

We consider a class of one-dimensional discrete quasiperiodic Schrödinger operator H_α, with two valued potential arranged along a sturmian sequence associated to the slope $\alpha \in [0,1)$. These operators act on $\mathbb{C}^\mathbb{Z}$ in the following way:

$$(H\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + ([(n+1)\alpha] - [n\alpha])V\Psi_n \, \forall n \in \mathbb{Z}\,.$$

The spectrum of such an operator is the set of energies E such that $H\Psi = E\Psi$ for some non exponentially growing Ψ. Given an energy value E, the *pseudo-eigenvector* Ψ is entirely determined by the *transfer matrix* and two successive values.

When α is a rational number p/q, the potential is q-periodic and by the Bloch-Floquet theorem, the spectrum is the set of energies E for which $M_E(q)$ the transfer matrix over a period is a unitary matrix. Namely, $Tr(M_E(q)) \in [-2,2]$. When α is not rational, one can use the sequence $\frac{p_k}{q_k}$ of best periodic approximations constructed using the continued fraction expansion $\alpha = [a_1, a_2, \ldots, a_k, \ldots]$. The sequence of transfer matrices over q_k sites is $M_k(E)$.The evolution of their trace values for a given energy E is the *trace map*. The spectrum is view as the set of bounded orbits. As far as the spectrum as a set is concerned, the trace map is the main tool to be used. By the characterization of escaping orbits, it has been shown that it is a Cantor set of zero Lebesgue measure [1, 2, 3].

In order to get more information on the generalized eigenvectors, the traces are not sufficient, and the transfer matrix evolution has to be studied. The matrices $M_{(k,p)}(E) = M_{k-1}M_k^p$ are computed recursively by

$$M_0 = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix},\ M_{-1} = \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix},\ M_{k+1} = M_{(k,a_{k+1}+1)}\,.$$

This evolution leaves invariant the $[M_{k-1}, M_{(k,p)}]^2$ whose value can be computed for the initial conditions and is given by $V^2\mathbb{I}_2$ for this particular model.

Using the Chebyshev polynomials of the second kind and the Cayleigh-Hamilton theorem, it is possible to compute the evolution of the transfer matrices in a linearized form. The two matrices $M_{k+1} = M_{(k+2,0)}$ and $Z_{k+1} = M_{(k+1,1)}$ can be expressed as a linear combination of $M_k = M_{(k+1,0)}$, $M_{k-1} = M_{(k,0)}$, $Z_k = M_{(k,1)}$ and $\mathbb{I}_2$. The coefficients are polynomial expressions in the trace of M_k. This can be used to prove a polynomial upper bound for the matrix norm [4, 5]. This could also be useful to characterize the time evolution of a distribution under the time evolution associated to this Schrödinger operator. It is related to the asymptotic behavior of the generalized eigenvectors.

When the on-site potential strength is greater than a critical value ($V > 4$), we can describe the spectrum by putting it in one-to-one correspondence with a set of sequences of symbols, this construction is related to the symbolic dynamics of the trace-map [6] that is confined to an hyperbolic invariant set.

In the present case, a decreasing sequence of covering of the spectrum is constructed. It is composed by disjoint intervals, which are bands of the periodic approximations of H. Namely, for given k and p, we call $\sigma_{(k,p)}$ the spectrum of a periodic operator described by the transfer matrix $M_{(k,p)}$ over a period. We know that a covering is obtained by $\sigma_{(k,0)} \bigcup \sigma_{(k+1,0)}$. We can identify 3 different types of intervals in this covering:

type I gap: an interval in a a gap of $\sigma_{(k+1,0)}$, which is a band of $\sigma_{(k,1)}$ included in a band of $\sigma_{(k,0)}$,

type II band: a band of $\sigma_{(k+1,0)}$ included in a band of $\sigma_{(k,-1)}$ and in a gap of $\sigma_{(k,0)}$,

type III band: a band of $\sigma_{(k+1,0)}$ included in a band of $\sigma_{(k,0)}$ and in a gap of $\sigma_{(k,1)}$.

The covering is hierarchically arranged in the following way: at a given level k, a type I gap contains a unique band of $\sigma_{(k+2,0)}$ denoted by II. It is a type II band at level $k+1$. A type II band contains $(a_{k+1}+1)$ bands of $\sigma_{(k+1,1)}$ denoted by $\left(I^{(j)}\right)_{j=1\cdots a_{k+1}+1}$, they are all type I gaps at level $k+1$. They are alternated with a_{k+1} bands of $\sigma_{(k+2,0)}$ denoted by $\left(III^{(j)}\right)_{j=1\cdots a_{k+1}}$, they are all type III bands at level $k+1$.

$$I^{(1)} < III^{(1)} < I^{(2)} \cdots < I^{(a_{k+1})} < III^{(a_{k+1})} < I^{(a_{k+1}+1)}.$$

A type III band contains (a_{k+1}) bands of $\sigma_{(k+1,1)}$ denoted by $\left(I^{(j)}\right)_{j=1\cdots a_{k+1}}$, they are all type I gaps at level $k+1$. They are alternated with $(a_{k+1}-1)$ bands of $\sigma_{(k+2,0)}$ denoted by $\left(III^{(j)}\right)_{j=1\cdots a_{k+1}-1}$, they are all type III bands at level $k+1$.

$$I^{(1)} < III^{(1)} < I^{(2)} \cdots < I^{(a_{k+1}-1)} < III^{(a_{k+1}-1)} < I^{(a_{k+1})}.$$

We can show that at level 0, $[-2, 2]$ is of type III, and $[V-2, V+2]$ of type I, and that this covers the spectrum.

We now have a set of rules to construct an infinite length code which is associated to one and only one energy of the spectrum. This coding is an increasing function if the set of codes is ordered lexicographically. This allows a computation of the integrated density of states $\mathcal{N}$ for all energies of the spectrum by counting the number of codes less than the considered energy code. This allows a constructive version of the gap labeling of the operator (a labeling of the allowed gaps can be obtained in a more general setting using the Shubin formula and K-theory of C^*-algebras). Namely, the set of gaps of the spectrum is in one-to-one correspondence with the set of relative integers, by the relation:

$$\mathcal{N}(E) \in \{(l\alpha) \bmod 1, l \in \mathbb{Z}\} \bigcup \{1\}$$

The description of the spectrum is sufficiently detailed to allow an estimation of the scaling of a decreasing sequence of its covering [7, 8], by the control of the derivatives of the traces with respect to the energy E.

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Gibbs-like measure for spectrum of 1D Schrödinger operator with Sturm potentials

Zhi Ying Wen

(joint work with Fan Shen, Liu Qinghui)

The discrete Schrödinger operator acting on $l^2(\mathbb{Z})$ is defined as follows: for any $\psi = \{\psi_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$,

$$(H\psi)_n := \psi_{n-1} + \psi_{n+1} + v_n\psi_n, \ \forall n \in \mathbb{Z}. \tag{1}$$

with the Sturm potential

$$v_n = V\chi_{[1-\alpha,1)}(n\alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z}, \tag{2}$$

where $\alpha \in (0,1)$ is an irrational. We denote the Schrödinger operator with Sturm potential as $H_{\alpha,V}$. For some related previous results, see [1, 2, 5, 4] and references therein. Based on dimensional theory of Cookie-cutter sets and Cookie-cutter-like sets (see [6]), we proved in [3] the following result.

Let $V > 20$ and $\alpha = [0; a_1, a_2, a_3, \cdots]$ with $(a_n)_{n\geq 1}$ bounded. Then

$$\dim_H \sigma(H_{\alpha,V}) = s_*, \quad \overline{\dim}_B \sigma(H_{\alpha,V}) = s^*.$$

And in the case of $(a_n)_{n\geq 1}$ be ultimate periodic, $s_* = s^*$.

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Uniform convergence of Schrödinger cocycles over simple Toeplitz subshift

Qing Hui Liu
(joint work with Qu Yan Hui)

For locally constant cocycles defined on an aperiodic subshift, Lenz ([1]) proved that if the subshift satisfies positive weight, then the cocycle is uniform. Damanik and Lenz([2]) proved that if the subshift satisfies a certain condition (B), then the cocycle is uniform. In [3], we study simple Toeplitz subshifts. We prove the following results.

Let (Ω_β, T) be a simple Toeplitz subshift. Let M^E be defined as above. Then the function M^E is uniform for every $E \in \mathbb{R}$..

Let (Ω_β, T) be a simple Toeplitz subshift. Then there exists a compact set $\Sigma \subset \mathbb{R}$ of Lebesgue measure 0 such that $\sigma(H_\omega) = \Sigma$ for any $\omega \in \Omega_\beta$.

Let β be a simple Toeplitz word with coding $(a_k, n_k, l_k)_{k\geq 1}$. If $n_k \geq 4$ for any $k > 0$, then for any $\omega \in \Omega_\beta$, H_ω has purely singular continuous spectrum.

Given $\{(a_k, n_k)\}_{k\geq 1}$ with $\#\tilde{\mathcal{A}} \geq 3$

- Case 1. $\lim_{k\to\infty} n_k = \infty$.

- Case 2. $(n_k)_{k\geq 1}$ bounded, but

$$\lim_{k\to\infty} \max_{c\in\tilde{\mathcal{A}}\setminus\{a_k\}} p_{c,k} - k = \infty,$$

where $p_{c,k} := \min\{i > k \mid a_i = c\}$. For example, we can take

$$a_1 a_2 a_3 \cdots = (ab)c(ab)^2 d(ab)^3 c(ab)^4 d \cdots .$$

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Trace map hyperbolicity for a large coupling Schrödinger operator

Laurent Marin

(joint work with Emiliano De Simone)

We consider the trace map associated with the silver ratio Schrdinger operator as a diffeomorphism on the invariant surface associated with a given coupling constant and prove that the non-wandering set of this map is hyperbolic if the coupling is sufficiently large. As a consequence, for this values of the coupling constant, the local and global Hausdorff dimension and the local and global box counting dimension of the spectrum of this operator all coincide and are smooth functions of the coupling constant.

We also derive dynamical upper bound for the propagation of the wavepacket. The method is to bound the outside probabilties with the inverse power of transfer matrix norms using complex analysis and Weyl Theory. Our bound is valid for a set of parameter of Lebegue measure 1.

We study the fractal dimension of the spectrum of a quasiperiodical Schrodinger operator associated to a sturmian potential. We consider potential defined with irrationnal number verifying a generic diophantine condition. We recall how shape and box dimension of the spectrum is linked to the irrational number properties. We give general lower bound of the box dimension of the spectrum, true for all irrational numbers. Finally we recall dynamical implication of the bound on spectra dimension.

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Mesoscopic quasicrystals: can we observe the dynamics directly?

Ron Lifshitz

In recent years we have witnessed a surge of scientific interest in experimental systems, exhibiting quasiperiodic long-range order on a scale much greater than that of atomic quasicrystals—typically from tens of nanometers to tens of microns—collectively referred to as *mesoscopic quasicrystals*. Depending on the particular physical realization, mesoscopic quasicrystals might be artificially fabricated at the level of individual structural elements [1, 2]; they may form dynamically as a result of trapping or manipulation by external forces or fields [4, 5, 3]; or may self-assemble spontaneously [6, 7, 8, 9, 10], as observed to date in close to half a dozen soft matter systems. These newly-realized mesoscopic quasicrystals not only provide exciting platforms for the fundamental study of the physics of quasicrystals, with the experimental ability of tracking the trajectories of individual particles, and viewing the dynamics of spreading wave-packets [4, 11]. They also hold the promise for new applications based on artificial or self-assembled nanomaterials with unique physical properties that take advantage of the quasiperiodicity, such as novel photonic metamaterials [1, 2, 12, 13].

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Dynamical analysis of one-dimensional quantum systems

Serguei Tcheremchantsev

Let $\mathcal{H}$ be a complex separable Hilbert space, H a bounded self-adjoint operator in $\mathcal{H}$ and $\psi \in \mathcal{H}, \|\psi\| = 1$. The time evolution of the state ψ is given by $\psi(t) = \exp(-itH)$. Let $\mathcal{B} = \{e_n\}$ be some orthonormal basis of $\mathcal{H}$, labeled by $n \in \mathbb{Z}_+$ or by $n \in \mathbb{Z}$. We are interested in the spreading in time of $\psi(t)$ over the basis $\mathcal{B}$. To describe it, consider the moments of the position operator associated to $\mathcal{B}$:

$$|X|^p(t) = \sum_n (|n|^p + 1)| < \psi(t), e_n > |^2, \quad p > 0.$$

Define also the time-averaged moments

$$< |X|^p > (T) = \frac{1}{T} \int_0^T |X|^p(t)dt.$$

Define the growing exponents

$$\beta^+(p) = \limsup_{t \to +\infty} \frac{\log|X|^p(t)}{p\log t}, \beta^-(p) = \liminf_{t \to +\infty} \frac{\log|X|^p(t)}{p\log t},$$

and similar quantities $\tilde{\beta}^\pm(p)$ for the time-averaged moments.

Let μ_ψ be the spectral measure associated to the state ψ and the operator H. We discuss the links between various dimensions of μ_ψ and the growth exponents $\tilde{\beta}^\pm(p)$. In particular, under some condition on μ_ψ,

$$\lim_{p \to +\infty} \tilde{\beta}^\pm(p) \geq \dim_B^\pm(\mathrm{supp}\mu_\psi),$$

where $\dim_B$ denotes the box-counting dimension of a set. We discuss applications of these general results to the case of the one-dimensional Fibonacci hamiltonian.

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Hofstadter butterfly, critical wave functions, phase transition, and multifractal

Mahito Kohmoto

2D Bloch electrons in a magnetic field is again considered. 1D Harper equation describing this 2D problem is derived. The energy spectrum as a function of flux per plaquette shows the intricate Hofstadter. The existence of the phase transition is shown from the total band widths computations following the scaling-system-size. Hofstadter for both subcritical and supercritical are shown and the Aubry duality is indicated. In fact, it is shown that the Aubry duality is a consequence of the gauge invariance of the original 2D problem.

The multifractal method is applied to both the energy spectra and the wave functions. The critical behavior is well described, and the exact multifractal spectrum is presented for the wave function at the center of the energy spectrum of the Fibonacci model.

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A hexagonal monotile for the Euclidean plane

Uwe Grimm

(joint work with Michael Baake)

In his seminal paper [2], Berger proved the undecidability of the domino problem by constructing an aperiodic set of 20,426 Wang tiles (marked squares). An aperiodic set of prototiles can tile space without gaps or overlaps, but does not admit any periodic tilings, which means tilings that are invariant under any nontrivial translation. The number of tiles that are required to enforce aperiodicity has subsequently been reduced, with the Penrose tiling [6] requiring just two different prototiles (up to Euclidean motions). The smallest set of Wang tiles (up to translations) known at present contains 13 tiles [5, 3].

It has been a long-standing problem whether a single prototile (which is usually meant to be a topological disk) exists such that it tiles the plane only aperiodically. In three dimensions, the Schmitt-Conway-Danzer (SCD) 'einstein' is an example of a single tile that does not admit tilings with translational symmetry; however, it does allow for a screw symmetry (a combination of a translation and a rotation by an irrational multiple of π). Until recently, the closest contender in the plane was Penrose's $1 + \varepsilon + \varepsilon^2$ tiling [7], which comprises a hexagonal prototile and two 'key' prototiles which encode the local matching rules. While the corner key tile (ε^2) can be made arbitrarily small, the edge key tile (ε) transports information along an edge of a hexagon, and hence can only be made arbitrarily thin in one direction, which means that the information cannot be encoded in the shape of the corresponding edge of the hexagon alone.

In a recent preprint [9], a single hexagonal prototile with local matching rules was announced. Its discovery is due to Joan Taylor from Burnie, Tasmania, an amateur mathematician who has been fascinated by tilings for many years; see [11] for her original argument based on the composition-decomposition method

[4]. While the matching rules of the marked hexagon (which occurs in rotated and reflected versions as well) are local, they act on next-nearest neighbours, and thus cannot be encoded in the shape of the prototile, unless one allows for a disconnected tile [9] or 'thickens' the tile into the third dimension [10]; see also [1] for an artistic variant, and [12] for the pseudo-inflation rule based on 7 hexagons.

The resulting inflation tilings are limit periodic. They have a 2-adic structure which is apparent from a hierarchy of triangular structures, and thus resemble Robinson's tilings [8]. It turns out that there is a close relation to Penrose's $1 + \varepsilon + \varepsilon^2$ tiling; the ensemble of $1 + \varepsilon + \varepsilon^2$ tilings forms a 3-fold cover of the (minimal) hull of the Taylor inflation tiling, where the elements of the latter can be obtained from the former by a local derivation rule.

There is a subtle difference between the Taylor tilings of [11] and the Socolar-Taylor tilings of [9]. Taylor's matching rules [11] determine a single local isomorphism (LI) class of the corresponding inflation tiling, so they constitute perfect matching rules. In contrast, the matching rules given in [9] allow for non-repetitive tilings, which contain a singular vertex.

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Spectral calculations for Fibonacci Hamiltonians

Mark Embree

(joint work with David Damanik, Anton Gorodetski, Serguei Tcheremchantsev)

What insights can careful computations give into the spectral structure of Fibonacci Hamiltonians? This talk addressed that question in three parts.

First, we described various strategies for computing the spectrum $\sigma_{\lambda,k}$ of Sütő's periodic approximations $H_{\lambda,k}$ [4] to the Fibonacci Hamiltonian H_λ with coupling constant λ: here H_λ operates on $\psi \in \ell^2(\mathbf{Z})$ as

$$(H_\lambda \psi)(n) = \psi(n+1) + \psi(n-1) + V_\lambda(n)\psi(n)$$

for

$$V_\lambda(n) = \begin{cases} \lambda, & (n/\phi \bmod 1) \geq 1 - 1/\phi; \\ 0, & \text{otherwise.} \end{cases}$$

The spectrum of $\sigma_{k,\lambda}$ comprises F_k intervals whose calculation becomes sensitive as k increases. (F_k denotes the kth Fibonacci number.) We illustrated the pitfalls of computing these bands via polynomial root finding, and argued that iterations of the trace map also fail to give a robust procedure for computing the entire spectrum. Most of calculations derive the bands via two symmetric eigenvalue problems involving $F_k \times F_k$ matrices (see, e.g., [5, Ch. 7]). With this approach we routinely compute $\sigma_{k,\lambda}$ for values of k up to around $k = 17$. (The same computational approach is used by Even-Dar Mandel and Lifshitz.)

Next, we outlined the determination of the rate at which the dimension of the spectrum scales in the large-coupling limit, as reported in [1]. We explained how combinatorial computations counting band types between successive periodic approximations allowed us to compute that the dimension of the spectrum Σ_λ of the Fibonacci Hamiltonian scales like $\log(1+\sqrt{2})\log\lambda$ as $\lambda \to \infty$.

Finally, we consider the spectrum of the Fibonacci Hamiltonian on a two-dimensional lattice, described by the set $\Sigma_\lambda + \Sigma_\lambda$. (For similar calculations involving the off-diagonal Fibonacci model, see [3].) Using accurate computations of the periodic approximation $\Sigma_{\lambda,k} = \sigma_{\lambda,k} \cup \sigma_{\lambda,k+1}$ and inspired by recent work of Damanik and Gorodetski [2], we presented estimates of the thickness of Σ_λ. From these computations we conjecture that the thickness is a monotone decreasing function of λ that behaves like $1/\lambda$ as $\lambda \to 0$.

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