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Operator Algebras and Representation Theory: Frames, Wavelets and Fractals

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Abstract. The central focus of the workshop was Kadison-Singer conjecture and its connection to operator algebras, harmonic analysis, representation theory and the theory of fractals. The program was intrinsically interdisciplinary and represented areas with much recent progress. The workshop includes talks on operator theory, wavelets, shearlets, frames, fractals, representations theory and compressed sensing.

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Introduction by the Organisers

Operator algebras, representation theory, and harmonic analysis have always been closely related to physics, in particular quantum theory. Indeed, quantum theory was a driving force behind the early history of operator algebras and representation theory. Early pioneers, such as J. von Neumann, viewed those subjects not as separate fields, but two sides of a general theory. Very recently, other connections to more applied sciences, in particular to engineering, have emerged and stimulated research in mathematics which in turn has led to interdisciplinary work. These connections include wavelet theory, frame theory, fractals, function spaces related to representations, analysis on loop groups, and the geometry of tilings. New connections to approximation theory, numerical mathematics, and microlocal analysis have also influenced today's research atmosphere. Advances made in any one of these subfields will have direct implications for the others. The central topic of our workshop is an excellent example of recent changes in paradigms and creative interactions. It was built around a single conjecture, the Kadison-Singer conjecture, and yet it involved four separate areas, spanning a wide range from pure to applied mathematics.

Each area was represented by several experts. The meeting therefore was especially fruitful in facilitating discussions across fields, and involving researchers each having unique perspectives, but perhaps not insight all the four main themes.

The Kadison-Singer conjecture was stated in a pioneering 1959-paper by R. V. Kadison, who was a participant at this meeting, and I. Singer [5]. The two authors were inspired in turn by P. A. M. Dirac's use of operators in Hilbert space $\mathcal{H}$: quantum mechanical observables and states. In the theory, pure states are the building blocks. Further Dirac was interested in the role of maximal abelian algebras of bounded operators. The Kadison-Singer conjecture asks if every pure state on the infinite by infinite diagonal matrices (as they are represented by operators on $\mathcal{H}$) extends "uniquely" to a pure state on the algebra of all bounded operators on $\mathcal{H}$. Existence is clear, so the open problem is the uniqueness.

For nearly two decades the Kadison-Singer conjecture (KS) represented only a specialized corner of operator algebra theory. This has all changed. Recently KS has blossomed into a vibrant interdisciplinary research endeavor, with research teams coming from disparate corners of pure and applied mathematics. Our workshop aimed at bringing all of this into focus.

Initially, like with other parts of operator algebras, the KS problem had its roots in quantum mechanics. More recently, two things happened to bring KS back front and center, as a common thread in four fields, some of them often thought of as disparate: (i) operator algebras, representation theory and harmonic analysis, (ii) signal processing, and the use of frames (over-complete "bases"), (iii) combinatorics of pavings, and (iv) Banach space geometry, see e.g., [3]. The four themes in turn overlap with other research trends, one being multi-scale theory. This lies behind powerful tools used both in frames and in the analysis of fractals. An important class of frames constitutes wavelet families built from scale similarity (hence multi-scale). By their very definition, fractals are understood from the similarity of data or geometries at different scales.

One breakthrough was papers [1, 2] by Joel Anderson which established the equivalence between KS and what is now called the paving conjecture; and the other was work [4] initiated by P. Casazza, who also attended the workshop, showing that KS is equivalent to key conjectures in signal and image processing, one of them known as the Feightinger conjecture. These conjectures are, on the face of it, from quite disparate areas of mathematics. It is therefore especially intriguing that, quite recently, their equivalence has been established. This also means that the resolution of the conjecture is now more likely, and that the answer will have implications going far beyond just finding out if the answer is "yes" or "no", The implications of a no-answer will be as important as those deriving from an affirmative resolution.

This diversity of topics closely connected to the Kadison-Singer conjecture is illustrated by the listed topics below, and by speakers at the meeting. Each talk was interdisciplinary in the sense of giving rise to a lively discussion between participants from diverse areas. The topics covered included:

(1) Operator Algebras, including various aspects of the Kadison-Singer conjecture and, related to this, extensions of the Schur-Horn theorem and the Bourgain-Tzafriri restricted invertibility theorem.
(2) Representation Theory, relating Feichtinger's conjecture to duality principles.
(3) Frame Theory, with topics ranging from spanning and linear independence properties over semiframes to duality principles.
(4) Applied Harmonic Analysis, focusing in particular on the novel anisotropic system of shearlets and on various aspects of uncertainty principles.
(5) Fractal Theory, in particular, analysis on Cantor sets.
(6) Classical Harmonic Analysis and Sampling Theory, covering diverse aspects of sampling from the Zak transform to extensions of Shannon's sampling theorem.
(7) Compressed Sensing and Sparse Approximation, including the theory related to the Johnson-Lindenstrauss lemma.

During the meeting 21 expert talks were presented. To also accommodate younger researchers, we had one session of shorter talks.

In addition to the Oberwolfach Reports with Abstracts of talks (edited by J. Lemvig in collaboration with the organizers), we have arranged a journal special issue in the journal *Numerical Functional Analysis and Optimization* to help disseminate results from our workshop. This will include refereed papers also from participants who did not get a chance to deliver formal talks at the workshop. The editors are Pete Casazza, Palle Jorgensen, Keri Kornelson, Gitta Kutyniok, David Larson, Peter Massopust, Gestur Ólafsson, Judith Packer, Sergei Silvestrov, and Qiyu Sun.

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Workshop: Operator Algebras and Representation Theory: Frames, Wavelets and Fractals

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Abstracts

The Heisenberg relation – mathematical formulations

Richard Kadison
(joint work with Zhe Liu)

We study the Heisenberg relation, in its original (pure) form, $QP - PQ = i\hbar I$, where $h = 2\pi\hbar$ is Planck's experimentally determined quantum of action $\sim 6.625 \cdot 10^{-27}$ *erg s*, with a focus on the ways it can be realized ("represented") mathematically. We show, among other things, that this cannot be accomplished with (unbounded) operators affiliated with a factor of type II. This results from a special case of our conjecture: If p is a non-commutative polynomial in n variables such that $\tau(p(A_1, \ldots, A_n)) = 0$ for all $A_1, \ldots, A_n$ in a II_1 factor $\mathcal{M}$, then $\tau(p(\tau_1, \ldots, \tau_n)) = 0$ whenever $\tau_1, \ldots, \tau_n$ are operators affiliated with $\mathcal{M}$ for which $p(\tau_1, \ldots, \tau_n)$ is bounded (τ is the trace on $\mathcal{M}$). We have been able to prove the special case, where $p(x_1, x_2)$ is $x_1x_2 - x_2x_1$, provided at least one of x_1, x_2 is replaced by a self-adjoint operator affiliated with $\mathcal{M}$.

Operators and frames

Peter G. Casazza
(joint work with Jameson Cahill and Gitta Kutyniok)

We address several fundamental problems in Hilbert space frame theory which will appear in three papers - two of Cahill and Casazza and one by Cahill, Casazza and Kutyniok.

Definition 1. For any vectors $\{f_i\}_{i=1}^M$ in a Hilbert space $\mathbb{H}_N$:

- The **synthesis operator** is the $N \times M$ matrix $F := \begin{bmatrix} f_1 & \cdots & f_M \end{bmatrix}$.
- The **analysis operator** is the $M \times N$ matrix $F^* = \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix}$.
- The **frame operator** is the $N \times N$ matrix $S = FF^* = \sum_{i=1}^{M} f_i f_i^*$.
- $\{f_i\}_{i=1}^M$ is a **frame** if there are $0 < A \leq B$ so that
$$A\ I \leq S \leq B\ I$$
- $\{f_m\}_{m=1}^M$ is a **tight frame** if $S = A\ I$ for some $A > 0$, and a **Parseval frame** if $A = 1$.
- $\{f_m\}_{m=1}^M$ is a **equal norm frame** and $\|f_m\| = c$ for all $m = 1, 2, \ldots, M$, and a **unit norm frame** if $c = 1$.

- Two frames $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ for $\mathbb{H}_N$ are **isomorphic** if there is an invertible operator $T : \mathbb{H}_N \to \mathbb{H}_N$ with

$$Tf_i = g_i, \text{ for all } i = 1, 2, \ldots, M.$$

- Two frames are **frame equivalent** if they are isomorphic and have the same frame operator.

Main Question: To understand how invertible operators change a frame. I.e. Given a frame, when is it isomorphic to a frame with another (preferably better) set of properties?

Some special cases are:

- Given a frame $\{f_i\}_{i\in I}$, find the invertible operator T so that $\{Tf_i\}_{i\in I}$ is the **closest** to being an equal norm Parseval frame.
- Which frames are isomorphic to unit norm frames?
- Which frames have the same frame operator?
- Given a frame $\{f_i\}_{i\in I}$, find the numbers $\{a_i\}_{i\in I}$ so that the frame $\{a_i f_i\}_{i\in I}$ is the **closest** to being Parseval.
- Given a frame $\{f_i\}_{i\in I}$ with frame operator S, classify the invertible operators T so that $\{Tf_i\}_{i\in I}$ has frame operator S. This problem will be answered in these papers.
- Given a frame $\{f_i\}_{i\in I}$, classify the set

$$\{\{\|Tf_i\|\}_{i=1}^M : T \text{ is an invertible operator}\}.$$

Closest will need to be defined above. The reason these problems are important is that often in practice we are given a frame which arises in an application and we have to work with it. This frame may have very bad frame properties such as a too small lower frame bound or too big upper frame bound as well as the norms of the vectors being very spread out. It is possible that this was a very good frame - such as an equal norm Parseval frame - but an invertible operator has been applied to it ruining its good properties. So our goal long term is to take a given frame and apply an invertible operator to it to turn it into the *best* frame possible. At this time, equal norm Parseval frames are the best frames in that we have some strong results here [1] which show that such frames can be partitioned into a number of linearly independent spanning sets with a possibly non-spanning linearly independent set left over.

A fundamental lemma for our work is:

Lemma 2. *Let $T : \mathbb{H}_N \to \mathbb{H}_N$ be an invertible operator on $\mathbb{H}_N$ and let $\{e_i\}_{i=1}^N$ be an orthonormal basis for $\mathbb{H}_N$. The following are equivalent:*

(1) $\{Te_i\}_{i=1}^N$ is an orthogonal set,

(2) $\{e_i\}_{i=1}^N$ is an eigenbasis for T^*T with respective eigenvalues $\|Te_i\|^2$.

In particular, T must map some orthonormal basis to an orthogonal set.

This lemma clearly fails for infinite dimensions since in this case the frame operator may not have any eigenvectors.

Some Sample Results

A unitary operator clearly takes a frame to a frame equivalent frame. But as we will see, these are not the only operators doing this.

Fix an orthonormal basis $\mathcal{E} = \{e_i\}_{i=1}^N$ for $\mathbb{H}_N$ and fix positive constants $\mathcal{U} = \{\mu_i\}_{i=1}^N$ and let $\{g_i\}_{i=1}^N$ be any orthonormal basis for $\mathbb{H}_N$, with

$$g_j = \sum_{i=1}^{N} \langle f_j, e_i \rangle e_i.$$

Define vectors $\{h_i\}_{i=1}^N$ by:

$$h_i = \sum_{i=1}^{N} \sqrt{\frac{\mu_i}{\mu_j}} \langle g_i, e_j \rangle e_j.$$

An operator $T : \mathbb{H}_N \to \mathbb{H}_N$ is called **admissible** for $(\mathcal{U}, \mathcal{E})$ if

$$T^* e_i = h_i, \text{ for all } i = 1, 2, \ldots, N.$$

Let S be a positive operator on a Hilbert space $\mathbb{H}_N$ with eigenvectors $\mathcal{E} = \{e_i\}_{i=1}^N$ and respective eigenvalues $\Lambda = \{\lambda_i\}_{i=1}^N$, and let T be an invertible operator on $\mathbb{H}_N$.

We now have:

Theorem 3. *The following are equivalent:*

(1) $S = TST^*$.

(2) T *is an admissible operator for* $(\Lambda, \mathcal{E})$.

In another direction we ask:

Problem: Given a frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$ and a constant c, can we classify the operators T so that $\|Tf_i\| = c\|f_i\|$?

Note: Clearly a multiple of a unitary operator does this. Can any other operator do this?

Definition 4. Let $\mathcal{F} = \{f_i\}_{i=1}^M$ be a frame for $\mathbb{H}_N$ and let $\mathcal{E} = \{e_j\}_{j=1}^N$ be an orthonormal basis for $\mathbb{H}_N$. We define:

$$\mathcal{H}(\mathcal{F}, \mathcal{E}) = span \ \ \{|\langle f_i, e_1 \rangle|^2, |\langle f_i, e_2 \rangle|^2, \ldots, |\langle f_i, e_N \rangle|^2) : 1 \le i \le M\}.$$

Now we can answer our problem.

Theorem 5. *Let* $\mathcal{F} = \{f_i\}_{i=1}^M$ *be a frame for* $\mathbb{H}_N$*. Let* T *be an invertible operator on* $\mathbb{H}_N$ *and let* T^*T *have the orthonormal basis* $\mathcal{E} = \{e_j\}_{j=1}^N$ *as eigenvectors with respective eigenvalues* $\{\lambda_j\}_{j=1}^N$.

The following are equivalent:

(1) We have $\|Tf_i\| = c\|f_i\|$, *for all* $i = 1, 2, \dots, M$.

(2) We have

$$(\lambda_1 - c^2, \lambda_2 - c^2, \dots, \lambda_N - c^2) \perp \mathcal{H}(\mathcal{F}, \mathcal{E}).$$

Hence, if $\mathcal{H}(\mathcal{F}, \mathcal{E}) = \mathbb{H}_N$, *then* $\lambda_i = c^2$, *for all* $i = 1, 2, \dots, M$ *and so* T *must be a multiple of a unitary operator.*

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The Zak transforms

Guido Weiss

(joint work with Eugenio Hernández, Hrvoje Šikić, and Edward Wilson)

We introduce two (associated) Zak transforms:

$$(\mathcal{Z}f)(x,\xi) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i k\xi}, \qquad (\tilde{\mathcal{Z}}g)(x,\xi) = \sum_{\ell\in\mathbb{Z}} g(\xi+\ell)e^{2\pi i\ell x}.$$

where $f, g \in L^2(\mathbb{R})$. They map $L^2(\mathbb{R})$ isometrically onto the Hilbert spaces $\mathcal{M} = \{\varphi(x,\xi), (x,\xi) \in \mathbb{R}^2 : \varphi$ is 1-periodic in x and satisfies

$$(1) \qquad \varphi(x+\ell,\xi) = e^{2\pi i\ell\xi}\varphi(x,\xi), \quad \|\varphi\|^2_{\mathcal{M}} = \int_0^1\int_0^1 |\varphi(x,\xi)|^2 d\xi dx < \infty\}$$

and

$\tilde{\mathcal{M}} = \{\tilde\varphi(x,\xi), (x,\xi) \in \mathbb{R}^2 : \tilde\varphi$ is 1-periodic in x and satisfies

$$(2) \qquad \tilde\varphi(x+\ell,\xi) = e^{2\pi i\ell\xi}\tilde\varphi(x,\xi), \quad \|\tilde\varphi\|^2_{\tilde{\mathcal{M}}} = \int_0^1\int_0^1 |\tilde\varphi(x,\xi)|^2 dx d\xi < \infty\} \ .$$

The isometric property is easily seen from the property $\sum_{k\in\mathbb{Z}} |f(x+k)|^2 < \infty$ a.e. (since $\int_0^1 \sum |f(x+k)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ from which we deduce $\int_0^1\int_0^1 |\varphi(x,\xi)|^2 dx d\xi = \int_0^1 \sum_k |f(x+k)|^2 dx = \|f\|^2_{L^2(\mathbb{R})}$). That $\mathcal{Z}$ maps onto $\mathcal{M}$ follows from the observation that $\varphi(x,\xi) = \sum_{k\in\mathbb{Z}} c_k(x)e^{-2\pi i k\xi}$ in $\mathcal{M}$ is $\mathcal{Z}f$,where $f(x+k) = c_k(x)$ for $x \in [0,1)$. The proof that $\tilde{\mathcal{Z}}$ on $L^2(\mathbb{R})$ is an isometry onto $\tilde{\mathcal{M}}$ is completely analogous. It is also easy to see that $\mathcal{Z}^{-1}$ is the operator $\int_0^1 d\xi$ (If

$\varphi(x,\xi) = \sum_k f(x+k)e^{-2\pi i k\xi}$, then, clearly,

$$\int_0^1 \varphi(x,\xi)d\xi = f(x),$$

the 0 coefficient of the Fourier series $\varphi(x,\xi) = \sum_k f(x+k)e^{-2\pi i k\xi}$.) Similarly, $\tilde{\mathcal{Z}}^{-1} = \int_0^1 dx$. It is also easy to check that the operator $U : \mathcal{M} \to \tilde{\mathcal{M}}$ defined by $(U\varphi)(x,\xi) = e^{-2\pi i x\xi}\varphi(x,\xi) \equiv \tilde{\varphi}(x,\xi)$ is a unitary operator. In addition,

$$(U\mathcal{Z}f)(x,\xi) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i(x+k)\xi} \equiv \tilde{\varphi}(x,\xi).$$

Thus,

$$(\tilde{\mathcal{Z}}^{-1}U\mathcal{Z})f(x,\xi) = \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i(x+k)\xi}dx = \hat{f}(\xi).$$

The justification of these equalities is an easy exercize. This gives us the Plancherel theorem.

Theorem 1. *The Fourier transform $\mathcal{F}$ is equal to $\mathcal{F} = \tilde{\mathcal{Z}}^{-1}U\mathcal{Z}$ and $\mathcal{F}^{-1} = \mathcal{Z}^{-1}U^*\tilde{\mathcal{Z}}$, and both operators are unitary.*

This proof depends on the elementary properties of Fourier series in $L^2([-1,1))$: $\sum_k c_k e^{-2\pi i k\xi} \sim h(\xi) \in L^2([-1,1))$ and $\|h\|^2_{L^1([-1,1))} = \sum_{k\in\mathbb{Z}} |c_k|^2$.

The material in this lecture represents a collaboration with E. Hernandez, H. Sikic and E. Wilson [1].

Some simple applications of this material were given in this lecture and it was indicated that all this can be extended to higher dimensions as well as in the setting of locally compact, Abelian groups G and their duals $\hat{G}$. The space $L^2(\mathbb{R}^n)$ corresponds to a separable Hilbert space $\mathbb{H}$ on which acts a unitary representation of G.

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Norm-controlled inversion

Karlheinz Gröchenig
(joint work with Andreas Klotz)

Let $\mathcal{A} \subseteq \mathcal{B}$ be pair of nested Banach $*$-algebras with a common unit element. We say that $\mathcal{A}$ is inverse-closed in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \Rightarrow a^{-1}\mathcal{A}.$$

The classical example of this situation is Wiener's Lemma for absolutely convergent Fourier series. It states that the algebra of absolutely convergent Fourier series in inverse-closed in the (C*-) algebra of continuos periodic functions.

If $\mathcal{A}$ is inverse-closed in $\mathcal{B}$, then a natural next question is whether there is some form of norm control for the inverse a^{-1} in the smaller algebra $\mathcal{A}$. Precisely, we say that $\mathcal{A}$ admits norm control in $\mathcal{B}$, if there exists a function $h : R_+^2 \to R_+$, such that

$$\|a^{-1}\|_{\mathcal{A}} \leq h(\|a\|_{\mathcal{A}}, \|a^{-1}\|_{\mathcal{B}}).$$

Clearly, norm control is stronger than inverse-closedness. In view of many applications of inverse-closed Banach algebras, norm control would help to turn qualitative results into quantitative results.

At this time it is a complete mystery when an inverse-closed subalgebra admits norm control.

Consider the following examples.

Example 1. Let $C(\mathbb{T})$ consist of all continuous functions on the torus $\mathbb{T}$ and $C^1(\mathbb{T})$ consists of all continuously differentiable functions on $\mathbb{T}$ with norm $\|f\|_{C^1} = \|f'\|_\infty + \|f\|_\infty$. The quotient rule $(f^{-1})' = -f'/f^2$ implies that $C^1(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$. In addition,

$$\begin{aligned} \|f^{-1}\|_{C^1} &\leq \|f'\|_\infty \|f^{-1}\|_\infty^2 + \|f^{-1}\|_\infty \\ &\leq \|f\|_{C^1} \|f^{-1}\|_\infty^2 + \|f^{-1}\|_\infty \leq 2\|f\|_{C^1}\|f^{-1}\|_\infty^2 . \end{aligned}$$

As the controlling function h one may take $h(u,v) = uv^2 + v$ or $h(u,v) = 2uv^2$.

Example 2. Let $\mathcal{A}(\mathbb{T})$ be the algebra of absolutely convergent Fourier series with norm $\|f\|_{\mathcal{A}} = \|a\|_1$ for $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikt}$. By Wiener's Lemma $\mathcal{A}(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$. A deep result of Nikolski [7] shows that $\mathcal{A}$ does not admit norm control in $C(\mathbb{T})$.

Example 3. Let $\mathcal{B}$ the C^*-algebra of bounded operators on $\ell^2(\mathbb{Z})$ and $\mathcal{J}_s$ be the Banach algebra of matrices with polynomial off-diagonal decay with norm

$$\|A\|_{\mathcal{J}_s} = \sup_{k,l\in\mathbb{Z}} |A_{kl}|(1+|k-l|)^s .$$

By a result of Jaffard [5] and Baskakov [1] off-diagonal decay is preserved and $\mathcal{J}_s$ is inverse-closed in $\mathcal{B}(\ell^2)$ for $s > 1$. It is implicit in [8] that $\mathcal{J}_s$ possesses norm control in $\mathcal{B}(\ell^2)$, but the controlling function h does not seem to be known yet.

The three examples offer many puzzles. When is a subalgebra of a Banach algebra inverse-closed? Under which additional conditions does a subalgebra admit norm-control? The examples of absolutely convergent series shows that even in concrete situations these questions may be difficult to answer.

As a generic result we may formulate the following insight:

Theorem 4. *Assume that $\mathcal{A}$ is a smooth subalgebra of $\mathcal{B}$.*
(i) Then $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.
(ii) $\mathcal{A}$ admits norm-control in $\mathcal{B}$.

This theorem is of course meaningless, unless we give a precise definition of smooth subalgebras. Some recipes are known in the theory of operator algebras [2]. A systematic construction of smooth subalgebras can be based on classical approximation theory [4]. The standard smoothness spaces in analysis, for instance spaces of n-times differentiable functions, Hölder-Lipschitz spaces, Besov spaces, and Bessel potential spaces, possess an analogue in the world of Banach algebras. This discovery enabled us to give a systematic construction of inverse-closed subalgebras of a given Banach algebra [4].

It turns out that all these subalgebras admit also norm-control.

We give two examples of how to fill the above meta-theorem with content.

(1) By replacing the derivative of a function by an unbounded (closed) derivation on a Banach algebra, one can repeat the proof of the quotient rule and obtain results on inverse-closedness and norm-control.

A derivation δ on a Banach algebra is a linear mapping that satisfies the product rule $\delta(ab) = a\delta(b) + \delta(a)b$ for $a, b \in \mathcal{A}$. Under certain conditions one can show that

$$0 = \delta(e) = \delta(a^{-1}a) = a\delta(a^{-1}) + \delta(a)a^{-1}$$

and thus obtains the quotient rule $\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}$.

If the derivation is unbounded and closed, then the domain $\operatorname{dom}\delta$ is inverse-closed in $\mathcal{A}$ and the quotient rule implies norm-control in precisely the same manner as for $C^1(\mathbb{T})$. This observation goes back to Bratteli and Robinson [2] and is the main construction of smooth subalgebras in operator theory.

(2) If $\mathcal{A}$ possesses a bounded group of automorphisms parametrized by $\mathbb{R}^d$, then one can imitate difference operators. Precisely, let $t \in \mathbb{R}^d \to \psi_t \in \operatorname{Aut}(\mathcal{A})$ be a commutative subgroup of the automorphism group, satisfying $\psi_{s+t} = \psi_s\psi_t$ for $s, t \in \mathbb{R}^d$ and $\|\psi_t(a)\|_{\mathcal{A}} \leq M\|a\|_{\mathcal{A}}$ for all $t \in \mathbb{R}^d$ and $a \in \mathcal{A}$. Then one can define the difference operator $\Delta_t(a) = \psi_t(a) - a$ and higher order difference operators. Furthermore, one can define Besov spaces in complete analogy to the definition on $\mathbb{R}^d$. Let $s > 0$ be a smoothness parameter, k an integer $> s$, and $1 \leq p \leq \infty$. Then the Besov space $B_s^p(\mathcal{A})$ consists of all $a \in \mathcal{A}$, such that the norm

$$\|a\|_{B_s^p(\mathcal{A})} = \|a\|_{\mathcal{A}} + \Big(\int_{\mathbb{R}^d} \big(|t|^{-s}\|\Delta_t^k(a)\|_{\mathcal{A}}\big)^p \frac{d}{|t|^d}\Big)^{1/p}$$

is finite. Definitions of this type go back to [3], but so far Besov spaces have been studied exclusively with respect to Banach space properties. If $\mathcal{A}$ is a Banach algebra with a uniformly bounded automorphism group $\mathbb{R}^d$, then $B^p_s(\mathcal{A})$ is a Banach algebra.

As a first theorem to give precise mathematical substance to the meta-theorem 4 we formulate the following result.

Theorem 5. *Let $\mathcal{A}$ be a Banach $*$-algebra with a bounded automorphism group $\{\psi_t : t \in \mathbb{R}^d\}$.*

(i) Then $B^p_s(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ [6].

(ii) $B^p_s(\mathcal{A})$ admits norm-control in $\mathcal{A}$. Explicitly, for $n-1 \le s < n$ we have

$$\|a^{-1}\|_{B^p_s(\mathcal{A})} \le (2M)^{2^n-1} \|a\|^{2^n-1}_{B^p_s(\mathcal{A})} \|a^{-1}\|^{2^n}_{\mathcal{A}} .$$

One may now develop a systematic correspondence between smoothness spaces in approximation theory and smooth subalgebras of a Banach algebra with an automorphism group. In all known cases the smooth subalgebra is inverse-closed and admits norm-control. The inverse-closedness was already investigated in [4] and [6].

The investigation of norm-control is work in progress with Andreas Klotz, University of Vienna.

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Operator-valued measures, dilations, and the theory of frames

David R. Larson

(joint work with Deguang Han, Bei Liu, and Rui Liu)

This talk represents some ongoing joint work with Deguang Han, Bei Liu and Rui Liu. We develop elements of a dilation theory for operator-valued measures from a σ-algebra of sets into $B(X)$, where X is a Banach space. Our main results are apparently new even for the special case where X is a Hilbert space, and in fact it is the Hilbertian case that provides the prime motivation for this work. Hilbertian operator-valued measures are closely related to bounded linear maps on abelian von Neumann algebras, and some of our results in this setting include new dilation results for bounded linear maps that are not necessarily completely bounded, and from domain algebras that are not necessarily abelian. There are applications to both the discrete and the continuous frame theory. We investigate some natural associations between the theory of frames (including continuous frames and framings), the theory of operator-valued measures on sigma-algebras of sets, and the theory of continuous linear maps between C^*-algebras. In this connection frame theory itself is identified with the special case in which the domain algebra for the maps is an abelian von Neumann algebra and the map is ultraweakly (i.e., σ-weakly) continuous. Some of results for maps extend to the case where the domain algebra is non-commutative. It has been known for a long time that a necessary and sufficient condition for a bounded linear map from a unital C*-algebra into $B(H)$ to have a Hilbert space dilation to a $*$-homomorphism is that the mapping needs to be completely bounded. Our theory shows that if the domain algebra is commutative, then even if it is not completely bounded it still has a Banach space dilation to a homomorphism. We view this as a generalization of the known result of Cazzaza, Han and Larson that arbitrary framings have Banach dilations, and also the known result that completely bounded maps have Hilbertian dilations. Our methods extend to some cases where the domain algebra need not be commutative, leading to new dilation results for maps of general von Neumann algebras. This paper was motivated by some recent results in frame theory and the observation that there is a close connection between the analysis of dual pairs of frames (both the discrete and the continuous theory) and the theory of operator-valued measures.

Framings are the natural generalization of discrete frame theory (more specifically: dual-frame pairs) to a non-Hilbertian setting, and even if the underlying space is a Hilbert space the dilation space can fail to be Hilbertian. This theory was originally developed by Casazza, Han and Larson in [1] as an attempt to introduce *frame theory with dilations* into a Banach space context. The initial motivation for the present manuscript was to completely understand the dilation theory of framings. In the context of Hilbert space, we realized that the dilation theory for discrete framings from [1] induces a dilation theory for discrete operator valued measures that fail to be to be completely bounded in the sense of (c.f. [3]). This gives a generalization of Naimarks's Dilation Theorem for the special case. This result led us to consider general (non-necessarily-discrete) operator valued measures. Our main results give a dilation theory for general (non-necessarily-cb) OVM's that completely generalizes Naimark's Dilation theorem in a Banach space setting, and which is new even for Hilbert space.

There is a well-known theory establishing a connection between general bounded linear mappings from the C^*-algebra $C(X)$ of continuous functions on a compact Hausdorf space X into $B(H)$ and operator valued measures on the sigma algebra of

Borel subsets of X (c.f. [3]). If A is an abelian C^*-algebra then A can be identified with $C(X)$ for a topological space X and can also be identified with $C(\beta X)$ where βX is the Stone-Cech compactification of X. Then the support σ-algebra for the OVM is the sigma algebra of Borel subsets of βX which is enormous. However in our generalized (commutative) framing theory A will always be an abelian von-Neumann algebra presented up front as $L^\infty(\Omega, \Sigma, \mu)$, with Ω a topological space and Σ its algebra of Borel sets, and the maps on A into $B(H)$ are normal. In particular, to model the discrete frame and framing theory Ω is a countable index set with the discrete topology (most often $\mathbb{N}$), so Σ is its power set, and μ is counting measure. So in this setting it is more natural to work directly with this presentation in developing dilation theory rather than passing to $\beta\Omega$, and we take this approach in this paper. We feel that the connection we make with established discrete frame and framing theory in the literature is transparent, and then the OVM dilation theory for the continuous case becomes a natural but nontrivial generalization of the theory for the discrete case that was inspired by framings. After doing this we attempted to apply our techniques to the case where the domain algebra for a map is non-commutative. We obtained some results which we discussed in our workshop talk. We show that all bounded maps have Banach dilations. However, additional hypotheses are needed if dilations of maps are to have strong continuity and structural properties. If the range space is a Hilbert space then it is well-known that there is a Hilbertian dilation if the map is completely bounded. The part of this theory that is apparently new is that even if a map is not cb it has a Banach dilation. In the discrete abelian case we show that the dilation of a normal map can be taken to be normal and the dilation space can be taken to be separable. We show that under suitable hypotheses this type of result can be generalized to the noncommutative setting.

A point on terminology: As in the past, we will call a framing for a Hilbert space it Hilbertian if it has a Hilbert space dilation to a pair consisting of a basis and its dual basis, or equivalently if the framing is a dual-pair of frames, and to call it a *non-Hilbertian* framing if it is a framing for a Hilbert space which has only a Banach dilation. There is no ambiguity here with terminology for framings for Banach spaces because it goes without saying that they are not Hilbertian. For consistency we adopt the same convention for operator-valued measures on Hilbert space and linear maps of operator algebras on Hilbert space, calling them it Hilbertian if they admit a dilation where the dilation space is a Hilbert space, and calling them non-Hilbertian otherwise.

This theory is really a symbiosis between aspects of Hilbert space operator algebra theory and aspects of Banach space theory, so we try to present Banach versions of Hilbertian results when we can obtain them. Some of the essential Hilbertian results we use are proven more naturally in a wider Banach context. Operator valued measures have many different dilations to idempotent valued measures on larger Banach spaces (even if the measure to be dilated is a cb measure on a Hilbert space) and a part of this theory necessarily deals with classification issues. In the workshop talk we discussed some additional results and exposition for Hilbert space operator-valued framings and measures, including the non-cb measures and their Banach dilations. In particular, we discussed some examples on the manner in which frames and framings on a Hilbert space induce natural operator valued measures on that Hilbert space.

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Wavelet techniques in multifractal analysis: recent advances and open problems

Stéphane Jaffard

(joint work with Patrice Abry, Stéphane Roux and Herwig Wendt)

The starting point of multifractal analysis is the analysis and classification of irregular signals. A seminal idea lies in the notion of self-similarity of fractal sets: The triadic Cantor set, or the Van Koch curve are made of pieces similar to the whole. In such cases, one adopts the following rule: If a set A is composed of N pieces similar to A with ratio r, then $\dim(A) = -\frac{\log N}{\log r}$, which yields a dimension of $\frac{\log 2}{\log 3}$ for the Cantor set, and of $\frac{\log 4}{\log 3}$ for the Van Koch curve.

Consider now a random function, such as Fractional Brownian Motion (FBM): sample paths do not satisfy such an exact selfsimilarity relationship, which only holds *in law*:

$$B_H(ax) \stackrel{\mathcal{L}}{=} a^H B_H(x). \tag{1}$$

The exponent H is the selfsimilarity exponent and is related with the dimension of the graph $\mathcal{G}_H$ of B_H by: $\dim(\mathcal{G}_H) = 2 - H$; (1) cannot be checked on one sample path, because it requires the knowledge of all sample paths. However, one can make the following considerations on one sample path: $B_H(t)$ satisfies $\forall s, t \geq 0$, $\mathbb{E}(|B_H(t) - B_H(s)|^2) = |t-s|^{2H}$; It follows that $|B_H(t+\delta) - B_H(t)| \sim |\delta|^H$; integrating on a whole interval, we get

$$\int |B_H(x+\delta) - B_H(x)|^p dx \sim |\delta|^{Hp}.$$

Therefore, when $\delta \to 0$, the left-hand side gets arbitrarily close to a deterministic quantity from which one can recover the exponent H. Following this intuition, Kolmogorov, in 1942, proposed to associate to an irregular signal f its *scaling function* $\zeta_f(p)$ defined for $p > 0$ by

$$\int |f(x+\delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}.$$

Note that, numerically, it can be easily recovered by a regression on a log-log plot.

Kolmogorov expected a linear scaling function for the velocity of fully developed turbulence: $\zeta_v(p) = p/3$, which is compatible with a FBM modeling. However, in the 1960s, experimental evidence showed that the scaling function of fully developed turbulence is not linear. Cascade models were proposed in order to take this behavior into account, by Kolmogorov, Obukhov, Novikov, Stewart, Yaglom, and culminated with the famous "Mandelbrot cascades", which had an important impact inside mathematics and in modeling. Understanding their properties opened the way to the theory of multiplicative martingales and multiplicative chaos, started by J.-P. Kahane and J. Peyrière, [4]. More and more general models of random cascades were constructed afterwards; these constructions also had implications outside of turbulence models, e.g., in *fragmentation*, and for the study of the *harmonic measure* on a fractal boundary, see [1].

An important step was taken when G. Parisi and U. Frisch proposed to interpret the nonlinearity of the scaling function as revealing the presence of Hölder singularities of different strengths, [5]. Informally, the Hölder exponent $h_f(x_0)$ of a function f at x_0 is defined by $|f(x) - f(x_0)| \sim |x - x_0|^{h_f(x_0)}$. The *spectrum of singularities* of f is

$$d_f(H) = \dim \left(\{x_0 : \quad h_f(x_0) = H\}\right)$$

where dim stands for the Hausdorff dimension. The *Multifractal formalism* asserts that the scaling function and the spectrum of singularities are related through a Legendre transform:

$$d_f(H) = \inf_p \left(1 + Hp - \zeta_f(p)\right). \tag{2}$$

However, (2) usually fails for Hs such that the infimum is attained for $p < 0$; therefore, one should not take it as a formula to be checked, but rather as a research program. Alternative scaling functions based on wavelet techniques were proposed by A. Arneodo et al.. The idea is to replace increments by integrals against wavelets, i.e. smooth, localized, oscillating functions at different scales. A *wavelet basis* on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the $2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbf{Z}$ form an orthonormal basis of $L^2(\mathbb{R})$.

We use the following notations: Dyadic intervals are $\lambda = [k2^{-j}, (k+1)2^{-j}[$, wavelet coefficients: $c_\lambda = 2^j \int f(x)\psi(2^j x - k)dx$ and $\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$. A first possibility is to consider the wavelet scaling function $\eta_f(p)$, defined by

$$2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\eta_f(p)j}.$$

Recall that f belongs to the Besov space $B_p^{s,\infty}(\mathbb{R})$ if

$$\exists C, \forall j : \qquad 2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \leq C \cdot 2^{-spj}.$$

Therefore, $\forall p > 0$, $\zeta_f(p) = \sup\{s : \quad f \in B_p^{s/p,p}\}$. Thus, the wavelet scaling function allows to determine simply (by a regression on log-log plot) which function spaces data belong to. In 2D, this has implications in image processing, where models often make the assumption that, indeed, data belong to certain function spaces; for instance, a standard assumption in the "$u + v$" models is that the "cartoon part" of the image is in BV and and "texture part" in L^2, [3]. The talk shows examples taken from natural textures, that either satisfy or do not satisfy such assumptions. However, the wavelet scaling function does not solve the initial problem we mentioned of supplying a scaling function for which the multifractal formalism would be valid for $p < 0$. This requirement calls for a new scaling function defined as follows. Let λ be a dyadic interval; 3λ is the interval of same center and three times wider. Let f be a bounded function; the *wavelet leaders* of f are the quantities $d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$. If

$$2^j \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j},$$

then $\eta_f(p)$ is the *leader scaling function.* The *Legendre spectrum* of f is

$$L_f(H) = \inf_{p \in \mathbb{R}} (1 + Hp - \eta_f(p)).$$

The *wavelet leaders multifractal formalism* holds if $d_f(H) = L_f(H)$. *Oscillation spaces* are associated with the leader scaling function: Let $p > 0$; $f \in O^s_p\ (\mathbb{R})$ if

$$\exists C, \forall j: \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \leq 2^{-spj}.$$

As in the case of Besov spaces,

$$\forall p > 0, \qquad \eta_f(p) = \sup \{s : p \in O^{s/p}_p\}.$$

One can show that $O^s_p = B^{s,\infty}_p$ if $s > 1/p$. However, If $s < 1/p$, they are different families of function spaces, [2]. Comparing the spaces O^s_p and B^s_p which contain f yields an information on the *clustering* of the large wavelet coefficients of f, [2]. The leader scaling function has the following robustness properties : It is independent of the wavelet basis $\forall p \in \mathbb{R}$ if the $\psi^{(i)}$ belong to $\mathcal{S}$ (however, the question is still open for wavelets with finite smoothness, such as compactly supported wavelets); it is invariant under the addition of a C^∞ function, or under a C^∞ change of variable. If f belongs to $C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then $\forall H$, $d_f(H) \leq L_f(H)$, and one can show that equality holds for "many" functions and stochastic processes such as FBM, Random wavelets series, Random wavelet cascades, ... and equality also holds *generically* (in the Baire or Prevalence sense). The talk shows several examples taken form various fields of applications where the wavelet leader scaling function is used for classification and model validation.

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Probability measures on solenoids and induced representations

Judith A. Packer
(joint work with Lawrence W. Baggett, Kathy D. Merrill, and Arlan Ramsay)

We discuss a construction, first due to D. Dutkay and P. Jorgensen ([4], [3]), that creates probability measures on solenoids from filter functions defined on the unit circle related to generalized wavelets. Using this construction, a representation of the Baumslag-Solitar group is obtained, and properties of the representation are related to properties of the original wavelet and filter systems. In particular, the existence of what is called a wavelet set is contingent on the corresponding representation of the Baumslag-Solitar group $\mathbb{Q}_d \ltimes \mathbb{Z}$ being induced from a representation of the normal abelian subgroup $\mathbb{Q}_d$. Here $\mathbb{Q}_d = \cup_{n=0}^{\infty} d^{-n}(\mathbb{Z})$, where d is an integer greater than 1. This work is joint with L. Baggett, K. Merrill, and A. Ramsay ([2]).

Let N and d be positive integers, with $N > 1$. Let $\mathbf{Q_{N,d}} = \cup_{j=0}^{\infty}(N^{-j}(\mathbb{Z}^d)) \subset [\mathbb{Q}]^d$. Recall that one can form the generalized Baumslag-Solitar group $BS_{N,d}$, as a semidirect product, with elements in $\mathbf{Q_{N,d}} \times \mathbb{Z}$ and with group operation given by

$$(\beta_1, m_1) \cdot (\beta_2, m_2) \;=\; (\beta_1 + N^{-m_1}(\beta_2), m_1 + m_2),$$

$\beta_1,\ \beta_2\ \in\ \mathbf{Q_{N,d}},\ m_1,\ m_2\ \in\ \mathbb{Z}$. The Pontryagin dual of $\mathbf{Q_{N,d}}$ is the (N, d)-solenoid, denoted by $\mathcal{S}_{N,d}$. It is the compact inverse limit abelian group $\{(z_i)_{i=0}^{\infty} : z_i \in \mathbb{T}^d,\ (z_{i+1})^N = z_i,\ \forall i\}$. For every $j \in \mathbb{N} \cup \{0\}$, there is a map $\pi_j : \mathcal{S}_{A,d} \to \mathbb{T}^d$ given by $\pi_j((z_i)_{i=0}^{\infty}) = z_j$.

For $d = 1$, the dual pairing between the two groups is given by:

$$\langle N^{-j}(k), (z_i)_{i=0}^{\infty} \rangle \;=\; [\pi_j((z_i))]^k \;=\; (z_j)^k.$$

In this report, a review of the construction, originally to Dutkay and Jorgensen, on constructing probability measures on $\mathcal{S}_{N,d}$ from filters from wavelet theory was given. These measures on the solenoid were then used to construct a different version of the wavelet representation of the Baumslag-Solitar group. It was shown that analyzing a certain decomposition of these measures gives further information about the representation involved.

Definition 1. Let N and d be as before, and let $h : \mathbb{T}^d \to \mathbb{C}$. We call a non-constant function h a low-pass filter for dilation by N if it is non-zero in a neighborhood of 1 and satisfies

$$\sum_{\{w:w^N=z\}} |h(w)|^2 = N,\ z \in \mathbb{T}^d.$$

With this definition in hand, we can describe the probability measure construction of Dutkay and Jorgensen:

Theorem 2 ([3], [1]). *Let $h : \mathbb{T}^d \to \mathbb{C}$ be a "low-pass" filter for dilation $N > 1$ such that the Haar measure of $h^{-1}(\{0\})$ is equal to 0. Then there is a unique probability measure τ_h on $\mathcal{S}_{N,d}$ such that for every $f \in C(\mathbb{T}^d)$,*

$$\int_{\mathcal{S}_{N,d}} f \circ \pi_j((z_i))d\tau_h \ = \int_{\mathbb{T}^d} f(z) \prod_{k=0}^{j-1} |h((z)^{N^k})|^2 dz.$$

One way that filters naturally arise is as follows: let $X \subset \mathbb{R}^d$, and suppose X is invariant under multiplication by N, and under translations by $\{v : v \in \mathbb{Z}^d\}$. Let μ be a Borel measure on X. Suppose there is a constant $K > 0$ such that $\mu(N(S)) = K\mu(S)$ for S a Borel subset of X. Since X is invariant under dilation by N, and since X is invariant under translations $\{v : v \in \mathbb{Z}^d\}$, X is also invariant under all compositions of translations from $\mathbb{Z}^d$ and powers of N, so (X, μ) will be invariant under translation by every element in $\mathbf{Q_{N,d}}$.

One defines dilation and translation operators D and $\{T_\beta : \beta \in \mathbf{Q_{N,d}} \subset \mathbb{Q}^d\}$ on $L^2(X, \mu)$ by

$$D(f)(x) = \sqrt{K} f(Nx),$$
$$T_\beta(f)(x) = f(x - \beta),\ f \ \in \ L^2(X, \mu).$$

A calculation shows that $T_\beta D \ = \ DT_{N(\beta)},\ \beta \in \mathbf{Q_{N,d}}$. The **wavelet representation** of $BS_{N,d}$ on $L^2(X, \mu)$ is obtained from the unitary operators above. Our ultimate aim is to give conditions under which this representation is induced from a representation of the normal subgroup $\mathbf{Q_{N,d}}$.

In examples we study, there is at least one scaling function ϕ in $L^2(X, \mu)$ satisfying:

(1) $\{T_v(\phi) : v \in \mathbb{Z}^d\}$ form an orthonormal set
(2) There exists $\{a_v : v \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$ such that

$$\phi = \sum_{v\in\mathbb{Z}^n} a_v DT_v(\phi).$$

In our examples, only finitely many of the $\{a_v\}$ will be non-zero.

(3) Setting $V_0 \ = \ \overline{\text{span}}\{T_v(\phi) : v \in \mathbb{Z}^n\}$ and $V_j = D^j(V_0)$, $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(X, \mu)$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

The 'low-pass' filter h in this case is defined by $h(z) = \sum_{v\in\mathbb{Z}^d} a_v \ < z, v > ,\ z \in \mathbb{T}^d$.

D. Dutkay in [3] has constructed a generalized Fourier transform

$$\mathcal{F} : L^2(X, \mu) \to L^2(\mathcal{S}_{N,d}, \tau_h)$$

satisfying

$$\mathcal{F}T_\beta\mathcal{F}^{-1}f((z_i)) \;=\; \langle\beta,(z_i)\rangle f((z_i)),\ \beta\in\mathbf{Q_{N,d}}\cong\ \widehat{\mathcal{S}_{N,d}},$$

$$\mathcal{F}D^{-1}\mathcal{F}^{-1}f((z_i)) = h(z_0)f(\sigma^{-1}((z_i)_{i=0}^{\infty}),(z_n)_{n=0}^{\infty}\in\mathcal{S}_{N,d}.$$

In this situation, $\mathcal{F}(\phi) \;=\; 1_{\mathcal{S}_{A,d}}$, and σ^{-1} is the inverse shift

$$\sigma^{-1}((z_n)) = (\zeta_n)_{n=0}^{\infty},\ \zeta_0 = (z_0)^N,\ \zeta_n = z_{n-1},\ n\geq 2.$$

Definition 3. ([2]) Let $h : \mathbb{T}^d \to \mathbb{C}$ be a polynomial filter for dilation by N corresponding to a multiresolution analysis on $L^2(X,\mu)$. Let τ_h be the probability measure on $\mathcal{S}_{N,d}$ associated to h, and $\mathcal{F} : L^2(X,\mu)\to L^2(\mathcal{S}_{N,d},\tau_h)$ the generalized Fourier transform of Dutkay. Let $\psi\in L^2(X,\mu)$ be a single wavelet for dilation by N and translation by $\mathbb{Z}^d$, so that $\{D^jT_k(\psi) : j,k\in\mathbb{Z}\}$ is an orthonormal basis for $L^2(X,\mu)$. We say that ψ is a generalized M.S.F. wavelet if $\mathcal{F}(\psi) = \lambda((z_n)_{n=0}^{\infty})1_E$, where $E\subset\mathcal{S}_{N,d}$ is a measurable set such that $\{\sigma^j(E) : j\in\mathbb{Z}\}$ tile $\mathcal{S}_{N,d}$ up to sets of τ-measure 0.

We outline here the main results, first noting that it is possible to define a Borel isomorphism Θ between the Cartesian product $\mathbb{T}^d\times\Sigma_{N,d}$, and the solenoid space $\mathcal{S}_{N,d}$ and where $\Sigma_{N,d}$ is the 0-dimensional space $\Sigma_{N,d} = \prod_{j=0}^{\infty}[\{0,1,\cdots,N-1\}^d]$. Using the map Θ, we obtain the following result concerning the decomposition of the measure τ_h. To simplify the statement, we let $d = 1$.

Theorem 4 ([2])**.** *Let $h : \mathbb{T}\to\mathbb{C}$ be a low-pass filter for dilation by $N > 1$, let τ_h be the Borel measure on $\Sigma_{N,1}$ constructed using the filter h. Using the Borel isomorphism $\Theta : \mathbb{T}\times\prod_{j=0}^{\infty}\{0,1,\cdots,N-1\}\to\mathcal{S}_{N,1}$ defined above, the measure $\widetilde{\tau} = \tau_h\circ\Theta$ corresponds to a direct integral $\widetilde{\tau} = \int_{\mathbb{T}} d\nu_z dz$,*
where the measure ν_z for $z = e(t)$ on $\pi_0^{-1}(\{z\})\cong\Sigma_{N,1} = \prod_{j=0}^{\infty}[\{0,1,\cdots,N-1\}]$ is defined on cylinder sets by
$\nu_z(\{a_0\}\times\cdots\times\{a_{k-1}\}\times\prod_{j=k}^{\infty}[\{0,1,\cdots,N-1\}])$ *]*
$=\ \frac{1}{N^k}\prod_{j=1}^{k}|h(e(N^{-j}(t)\cdot e(A^{-j}(\sum_{i=0}^{j-1}N^i(a_i)))|^2.$

Using the above result, we can analyze whether or not the fiber measures ν_z described above are atomic, which obviously depends on h. This allows us to deduce the following result, relating the existence of generalized MSF wavelets to induced representations:

Theorem 5 ([2])**.** *Let $\psi\in L^2(X,\mu)$ be a single wavelet for dilation by N, i.e. suppose that $\{D^jT_v(\psi) :\ j\in\ \mathbb{Z},\ v\ \in\ \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(X,\mu)$. Then, the following are equivalent:*

i) *ψ is a generalized M.S.F. wavelet in the sense defined earlier.*
ii) *W_0 is invariant under every translation in $\{T_\beta :\ \beta\in\mathbf{Q_{N,d}}\}$.*
iii) *The wavelet subspaces $W_j = \overline{span}\{D^jT_v(\psi)\}$ are the closed subspace corresponding to a system of imprimitivity $\{P_j :\ [j]\in BS_{N,d}/\mathbf{Q}_{N,d}\}$.*

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Frames and semi-frames

Jean-Pierre Antoine

(joint work with Peter Balazs)

Given a separable Hilbert space $\mathcal{H}$, it is often convenient to expand an arbitrary element $f \in \mathcal{H}$ in a sequence of simple, basic elements (atoms) $\Psi = (\psi_k)$, $k \in \Gamma$, $f = \sum_{k\in\Gamma} c_k\psi_k$, where the sum converges in an adequate fashion (e.g. strongly and unconditionally) and the coefficients c_k are (preferably) unique and easy to compute. There are several possibilities for obtaining that result. In order of increasing generality, we can require that Ψ be: (i) an orthonormal basis; (ii) a Riesz basis; (iii) a frame. Uniqueness is achieved in the first two cases, but lost in the third one. However, even a frame may be too restrictive, in the sense that it may impossible to satisfy the two frame bounds simultaneously. Accordingly, we define Ψ to be an *upper (resp. lower) semi-frame* if it is a total set and satisfies the upper (resp. lower) frame inequality. Then the question is to find whether the signal can still be reconstructed from its expansion coefficients.

We start with the so-called continuous generalized frames, introduced some time ago [1,2], and studied further by a number of authors (see [3] for references).

Let $\mathcal{H}$ be a Hilbert space and X a locally compact space with measure ν. Then a *generalized frame* for $\mathcal{H}$ is a family of vectors $\Psi := \{\psi_x,\, x \in X\}$, $\psi_x \in \mathcal{H}$, indexed by points of X, such that the map $x \mapsto \langle f, \psi_x\rangle$ is measurable, $\forall f \in \mathcal{H}$, and

$$\int_X \langle f, \psi_x\rangle\langle \psi_x, f'\rangle \,\mathrm{d}\nu(x) = \langle f, Sf'\rangle,\ \forall f, f' \in \mathcal{H},$$

where S is a bounded, positive, self-adjoint, invertible operator on $\mathcal{H}$, called the *frame operator*.

The operator S is invertible, but its inverse S^{-1}, while self-adjoint and positive, need not be bounded. We say that Ψ is a *frame* if S^{-1} is bounded or, equivalently, if there exist constants $\mathsf{m} > 0$ and $\mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m}\,\|f\|^2 \le \langle f, Sf\rangle = \int_X |\langle \psi_x, f\rangle|^2 \,\mathrm{d}\nu(x) \le \mathsf{M}\,\|f\|^2\,, \forall f \in \mathcal{H}. \tag{1}$$

First, Ψ is a total set in $\mathcal{H}$. Next define the *analysis operator*. $C_\Psi : \mathcal{H} \to L^2(X, \mathrm{d}\nu)$ by $(C_\Psi f)(x) = \langle \psi_x, f \rangle$, $f \in \mathcal{H}$, with range by $R_C := \mathsf{Ran}(C_\Psi)$. Its adjoint $C_\Psi^* : L^2(X, \mathrm{d}\nu) \to \mathcal{H}$ is called the *synthesis operator*. Then $C_\Psi^* C_\Psi = S$ and $\|C_\Psi f\|^2_{L^2(X)} = \|S^{1/2} f\|^2_{\mathcal{H}} = \langle f, Sf \rangle$. Furthermore, C_Ψ is injective, since $S > 0$, so that $C_\Psi^{-1} : R_C \to \mathcal{H}$ is well-defined.

Next, the lower frame bound implies that R_C is a *closed* subspace of $L^2(X, \mathrm{d}\nu)$. The corresponding projection is an integral operator with (reproducing) kernel $K(x, y) = \langle \psi_x, S^{-1} \psi_y \rangle$, thus R_C is a reproducing kernel Hilbert space. In addition, the subspace R_C is also complete in the norm $\|\cdot\|_\Psi$, associated to the inner product

$$\langle F, F' \rangle_\Psi := \langle F, C_\Psi \, S^{-1} \, C_\Psi^{-1} F' \rangle_{L^2(X)}, \text{ for } F, F' \in R_C. \tag{2}$$

Hence $(R_C, \|\cdot\|_\Psi)$ is a Hilbert space, denoted by $\mathcal{H}_\Psi$, and the map $C_\Psi : \mathcal{H} \to \mathcal{H}_\Psi$ is unitary. Therefore, it can be inverted on its range by the adjoint operator $C_\Psi^{*(\Psi)} : \mathcal{H}_\Psi \to \mathcal{H}$. Thus one gets, for every $f \in \mathcal{H}$, a *reconstruction formula*:

$$f = C_\Psi^{*(\Psi)} = \int_X F(x) \, S^{-1} \, \psi_x \; \mathrm{d}\nu(x), \text{ for } F = C_\Psi f \in \mathcal{H}_\Psi \quad \text{(weak integral)}. \tag{3}$$

Let now Ψ be a (continuous) *upper semi-frame*, that is, there exists $\mathsf{M} < \infty$ s.t.

$$0 < \int_X |\langle \psi_x, f \rangle|^2 \; \mathrm{d}\nu(x) \leq \mathsf{M} \, \|f\|^2, \; \forall f \in \mathcal{H}, \; f \neq 0. \tag{4}$$

In this case, Ψ is a total set in $\mathcal{H}$, the operators C_Ψ and S are bounded, S is injective and self-adjoint. Therefore $R_S := \mathsf{Ran}(S)$ is dense in $\mathcal{H}$ and S^{-1} is also self-adjoint. S^{-1} is unbounded, with dense domain $\mathsf{Dom}(S^{-1}) = R_S$.

Define the Hilbert space $\mathcal{H}_\Psi := \overline{C_\Psi(R_S)}^{\Psi}$, where the completion is taken with respect to the norm $\|\cdot\|_\Psi$. defined in (2). Then, the map C_Ψ is an isometry from $\mathsf{Dom}(S^{-1}) = R_S$ onto $C_\Psi(R_S) \subset \mathcal{H}_\Psi$, thus it extends by continuity to a *unitary* map from $\mathcal{H}$ onto $\mathcal{H}_\Psi$. Therefore, $\mathcal{H}_\Psi$ and R_C coincide as sets, so that $\mathcal{H}_\Psi$ is a vector subspace (though not necessarily closed) of $L^2(X, \mathrm{d}\nu)$.

Consider now, in the Hilbert space $\overline{R_C}$, the operators $G = \overline{C_\Psi \, S \, C_\Psi^{-1}}$ and G^{-1}, the self-adjoint extension of $G_S^{-1} = C_\Psi \, S^{-1} \, C_\Psi^{-1}$, the latter being essentially self-adjoint. Both operators are self-adjoint and positive, G is bounded and G^{-1} is densely defined in $\overline{R_C}$. Furthermore, they are inverse of each other on appropriate domains. Moreover, the norm $\|\cdot\|_\Psi$ is equivalent to the graph norm of $G^{-1/2}$.

We will say that the upper semi-frame $\Psi = \{\psi_x, \, x \in X\}$ is *regular* if all the vectors ψ_x, $x \in X$, belong to $\mathsf{Dom}(S^{-1})$. In that case, the discussion proceeds exactly as in the bounded case. In particular, the reproducing kernel $K(x, y) = \langle \psi_x, S^{-1} \psi_y \rangle$ is a *bona fide* function on $X \times X$. One obtains the same weak reconstruction formula, but restricted to the subspace $R_S = \mathsf{Dom}(S^{-1})$:

$$f = C_\Psi^{*(\Psi)} F = \int_X F(x) \, S^{-1} \, \psi_x \; \mathrm{d}\nu(x), \forall \, f \in R_S, \; F = C_\Psi f \in \mathcal{H}_\Psi. \tag{5}$$

On the other hand, if Ψ is not regular, one has to treat the kernel $K(x, y)$ as a bounded sesquilinear form over $\mathcal{H}_\Psi$ and use the language of distributions, for instance, in terms of a Gel'fand triplet [3].

Given a frame $\Psi = \{\psi_x\}$, one says that a frame $\{\chi_x\}$ is *dual* to the frame $\{\psi_x\}$ if one has, in the weak sense, $f = \int_X \langle \chi_x, f\rangle\, \psi_x \; \mathrm{d}\nu(x)$, $\forall\, f \in \mathcal{H}$. Then the frame $\{\psi_x\}$ is dual to the frame $\{\chi_x\}$. We want to extend this notion to semi-frames.

Let first $\Psi = \{\psi_x\}$ be an arbitrary total family in $\mathcal{H}$. As usual, we define the analysis operator as $C_\Psi f(x) = \langle \psi_x, f\rangle$ and the synthesis operator $D_\Psi F = \int_X F(x)\, \psi_x \; \mathrm{d}\nu(x)$, on natural domains. They are both unbounded.

Lemma 1. *(i) Given any total family Ψ, the analysis operator C_Ψ is closed. Then Ψ satisfies the lower frame condition iff C_Ψ has closed range and is injective.*

(ii) If the function $x \mapsto \langle \psi_x, f\rangle$ is locally integrable for all $f \in \mathcal{H}$, then the operator D_Ψ is densely defined and one has $C_\Psi = D_\Psi^*$.

The condition of local integrability is satisfied for all $f \in \mathsf{Dom}(C_\Psi)$, but not necessarily for all $f \in \mathcal{H}$, unless Ψ is an upper semi-frame, since $\mathsf{Dom}(C_\Psi) = \mathcal{H}$.

Finally, one defines the frame operator as $S = D_\Psi C_\Psi$ on the obvious domain $\mathsf{Dom}(S) \subset \mathsf{Dom}(C_\Psi)$. However, if the upper frame inequality is not satisfied, S and C_Ψ could have nondense domains, in which case one cannot define a unique adjoint C_Ψ^* and S may not be self-adjoint. However, if $\psi_y \in \mathsf{Dom}(C_\Psi)$, $\forall\, y \in X$, then C_Ψ is densely defined, $D_\Psi \subseteq C_\Psi^*$ and D_Ψ is closable. Finally, D_Ψ is closed iff $D_\Psi = C_\Psi^*$. Then $S = C_\Psi^* C_\Psi$ is self-adjoint.

Next, we say that a family $\Phi = \{\phi_x\}$ is a *lower semi-frame* if it satisfies the lower frame condition, that is, there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\, \|f\|^2 \leq \int_X |\langle \phi_x, f\rangle|^2 \; \mathrm{d}\nu(x), \quad \forall\, f \in \mathcal{H}. \tag{6}$$

Clearly, (6) implies that the family Φ is total in $\mathcal{H}$. With these definitions, we obtain a nice duality property between upper and lower semi-frames.

Proposition 2. *(i) Let $\Psi = \{\psi_x\}$ be an upper semi-frame, with upper frame bound M and let $\Phi = \{\phi_x\}$ be a total family dual to Ψ. Then Φ is a lower semi-frame, with lower frame bound M^{-1}.*

(ii) Conversely, if $\Phi = \{\phi_x\}$ is a lower semi-frame, there exists an upper semi-frame $\Psi = \{\psi_x\}$ dual to Φ, that is, one has, in the weak sense,

$$f = \int_X \langle \phi_x, f\rangle\, \psi_x \; \mathrm{d}\nu(x), \quad \forall\, f \in \mathsf{Dom}(C_\Phi).$$

We can give concrete examples [3] of a non-regular upper semi-frame (from affine coherent states) and of a lower semi-frame (from wavelets on the 2-sphere).

If X is a discrete set and ν a counting measure, we go back to the familiar (discrete) frame. All the previous results hold true, the only difference being that, instead of weakly convergent integrals, we are interested in expansions with norm convergence. The Hilbert space $L^2(X, \mathrm{d}\nu)$ becomes ℓ^2 and the analysis operator $C : \mathcal{H} \to \ell^2$, the synthesis operator $D : \ell^2 \to \mathcal{H}$, and the frame operator

$S : \mathcal{H} \to \mathcal{H}$ take their usual form. In the case of a Bessel sequence (i.e. when the upper frame inequality is satisfied), all three operators are bounded and one has $D = C^*,\ C = D^*$ and $S = C^*C$.

For lack of space, we will not go into details, but simply refer to [3].

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On the duality principle by Casazza, Kutyniok, and Lammers

Ole Christensen

(joint work with Hong Oh Kim and Rae Young Kim)

Let $\{f_i\}_{i\in I}$ denote a frame for a separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. In [1], Casazza, Kutyniok, and Lammers introduced the *R-dual sequence* of $\{f_i\}_{i\in I}$ with respect to a choice of orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ as the sequence $\{w_j\}_{j\in I}$ given by

$$w_j = \sum_{i\in I}\langle f_i, e_j\rangle h_i,\ j \in I. \qquad (1)$$

The paper [1] demonstrates that there is a strong relationship between the frame-theoretic properties of $\{w_j\}_{j\in I}$ and $\{f_i\}_{i\in I}$. In the talk we analyze the concept of R-dual sequence from another angle than it was done in [1]. Technically this is done by considering a dual formulation of (1), namely, for a given frame $\{f_i\}_{i\in I}$ and a (Riesz) sequence $\{w_j\}_{j\in I}$ to search for orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ such that

$$f_i = \sum_{j\in I}\langle w_j, h_i\rangle e_j,\ i \in I.$$

Using this approach we state a number of equivalent conditions for $\{w_j\}_{j\in I}$ to be an R-dual of $\{f_i\}_{i\in I}$. In particular we introduce a sequence $\{n_i\}_{i\in I}$ that can be used to check whether $\{w_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ or not; in fact, the answer is yes if and only if $\{n_i\}_{i\in I}$ is a tight frame sequence with frame bound $E = 1$.

One of the key properties of the R-duals is a certain duality relation that resembles the duality principle in Gabor analysis. The driving force in the article [1] was the question whether the duality principle in Gabor analysis actually can be derived from the theory of the R-duals. The question remains unsolved, but in [1] a positive conclusion is derived in the special case of a tight Gabor frame. The results presented here shed new light on this issue: in fact, the partial result in [1] turns out to be a consequence of a general result about R-duals, valid for any tight frame in any Hilbert space.

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Directional tight framelets with OEP filter banks

Bin Han

The main theme of this talk is to study wavelets and framelets within the framework of nonhomogeneous wavelet systems. Let us first introduce some notation and definitions.

For a function $f:\mathbb{R}^d \to \mathbb{C}$ and a $d\times d$ invertible matrix U, we define

$$f_{U;k,n}(x) := |\det U|^{1/2} e^{-in\cdot Ux} f(Ux-k), \quad x,k,n \in \mathbb{R}^d.$$

In particular, $f_{U;k} := f_{U;k,0} = |\det U|^{1/2} f(Ux-k)$.

For a subset Ψ of functions/distributions and a $d\times d$ invertible matrix M, a homogeneous M-wavelet system is defined to be

$$\mathrm{WS}(\Psi) := \{\psi_{M^j;k} \ | \ j\in\mathbb{Z}^d, k\in\mathbb{Z}, \psi\in\Psi\}.$$

Similarly, for sets Φ,Ψ of functions/distributions, a nonhomogeneous M-wavelet system is defined to be

$$\mathrm{WS}_J(\Phi;\Psi) := \{\phi_{M^J;k} \ | \ k\in\mathbb{Z}^d, \phi\in\Phi\} \cup \{\psi_{M^j;k} \ | \ j\geq J, k\in\mathbb{Z}^d, \psi\in\Psi\}.$$

Though nonhomogeneous wavelet systems are much less studied than homogeneous wavelet systems, in this talk we show that nonhomogeneous wavelet systems play a fundamental role in wavelet analysis and link together many aspects of wavelet analysis, for example, multiresolution analysis, refinable functions, filter banks, homogeneous wavelet systems, and etc.. This talk concentrates on the investigation of several aspects of nonhomogeneous wavelet systems.

First, we show that a nonhomogeneous orthonormal wavelet basis, a nonhomogeneous Riesz wavelet basis, or a nonhomogeneous tight or dual wavelet frame naturally leads to a homogeneous orthonormal wavelet basis, a homogeneous Riesz wavelet basis, or a homogeneous tight or dual wavelet frame, respectively. In other words, a nonhomogeneous wavelet system induces a sequence of nonhomogeneous wavelet systems at all the scale levels J preserving almost all its properties; a homogeneous wavelet system is the limiting system of this sequence of nonhomogeneous wavelet systems and preserves almost all the properties of the given nonhomogeneous wavelet system. More details are given in [1] (also see [2] for the case of dimension one).

Then we shall study nonhomogeneous wavelet systems in the frequency domain. Let $\mathcal{D}(\mathbb{R}^d)$ denote the linear space of all C^∞ compactly supported functions. For a tempered distribution f, we have

$$\widehat{f_{U;k}}(\xi) = e^{-i(U^T)^{-1}k\cdot\xi}\widehat{f}((U^T)^{-1}\xi) =: \widehat{f}_{(U^T)^{-1};0,k}.$$

Let Φ, Ψ be subsets of tempered distributions and $\mathrm{WS}_J(\Phi;\Psi)$ be an M-wavelet system. Define $\mathbf{\Phi} := \{\widehat{\phi} : \phi \in \Phi\}$ and $\mathbf{\Psi} := \{\widehat{\psi} : \psi \in \Psi\}$. Then the image of $\mathrm{WS}_J(\Phi;\Psi)$ under the Fourier transform becomes $\mathrm{FWS}_J(\mathbf{\Phi};\mathbf{\Psi})$:

$$\mathrm{FWS}_J(\mathbf{\Phi};\mathbf{\Psi}) = \{\phi_{N^J;0,k} \mid k \in \mathbb{Z}^d, \phi \in \mathbf{\Phi}\} \cup \cup_{j=J}^\infty\{\psi_{N^j;0,k} \mid k \in \mathbb{Z}^d, \psi \in \mathbf{\Psi}\},$$

where $N := (M^T)^{-1}$. Then we introduce a notion of a pair of frequency-based dual N-wavelet frames in the distribution space. Such a notion has been introduced in dimension one in [2] and generalized to high dimensions in [1]. Roughly speaking, the notion of a pair of frequency-based dual N-wavelet frames in the distribution space allows one to have the perfect reconstruction for any smooth function from the test function space $\mathcal{D}(\mathbb{R}^d)$. We present a complete characterization of a pair of frequency-based dual N-wavelet frames in the distributions (see [1, 2] for more detail). The same technique can be used to provide a complete characterization of frequency-based fully nonstationary dual wavelet frames in the distribution space. Here the word *fully nonstationary* means that at the scale level j, one is not only be able to change to the set of wavelet generators but also be able to change the associated dilation matrix.

Next, we present several applications of the notion of a pair of frequency-based dual wavelet frames in the distribution space. The oblique extension principle (OEP) has been introduced in [5–7] for the construction of tight or dual wavelet frames with high vanishing moments. See [5–9] and many references therein for the study and construction of tight or dual wavelet frames in the square integrable space. We show that without any extra condition such as vanishing moments and smoothness, every OEP-based filter bank with perfect reconstruction is always naturally linked to a pair of frequency-based dual wavelet frames in the distribution space. This provides a precise connection between the theory of filter bank and the theory of wavelet analysis.

Using the characterization of a pair of frequency-based dual wavelet frames in the distribution space, for any $d \times d$ real-valued expansive matrix M, we construct a smooth tight M-wavelet frame in $L_2(\mathbb{R}^d)$ such that the tight wavelet frame is generated by two Schwarz functions which are radial functions. Then we show that by a simple modification, directional tight wavelet frames can be easily constructed. Both types of tight wavelet frames have an underlying OEP-based filter bank with the perfect reconstruction property. Currently, the application of the directional tight wavelet frames to image denoising is undergoing.

Finally, in this talk, we introduce the notion of wavelets and framelets in function spaces. For the study of wavelets and framelets in the Sobolev spaces, see [4] for more detail. For the notion of wavelets and framelets in a general function space, see [3] for more detail. Then we show that a wavelet or framelet in a general function space can be completely characterization by two properties: the stability of the nonhomogeneous wavelet systems in associated function spaces and a pair of frequency-based dual wavelet frames in the distribution space.

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The digital shearlet transform on pseudo-polar grids

Xiaosheng Zhuang

(joint work with David L. Donoho, Gitta Kutyniok, and Morteza Shahram)

Directional representative systems ([2–4]) provide sparse approximation of anisotropic features are highly desired in both theory and application. The shearlet system is a novel system which provides a unified treatment of both the continumn and digital realms. Our main goal is to develop a digital shearlet theory which is rationally designed in the sense that it is the natural digitalization of the existing shearlet theory for continuum data. More precisely, let

$$\{\phi_n := \phi(\cdot - n)\}_{n\in\mathbb{Z}^2} \cup \{\psi^\iota_{jsm} := 2^{\frac{3}{4}j}\psi^\iota(S^\iota_s A_j \cdot - m)\}_{j,s\in\mathbb{Z},m\in\mathbb{Z}^2;\iota=1,2}$$

be a shearlet system, where $A_j = \begin{pmatrix} 4^j & 0 \\ 0 & 2^j \end{pmatrix}$ is the parabolic scaling matrix and $S^1_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $S^2_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ are the shearing matrices with respect to vertical and horizontal cones ([4]). The digitization of the continumn shearlet transform:

$$f \mapsto \{\langle f, \phi_n\rangle\}_n \cup \{\langle f, \psi^\iota_{jsm}\rangle\}_{j,s,m;\iota}$$

cascades the following three main steps for an $N \times N$ image:

1) Pseudo-polar Fourier transform with oversampling factor of R ([1]).
2) Multiplication by ‘density-compensation-style’ weights.

3) Decomposing the pseudo-polar grids into rectangular subbands with additional 2D-iFFT.

Weighted Pseudo-Polar Fourier Transform. Since $\langle f, \psi^{\iota}_{jsm}\rangle = \langle \hat{f}, \hat{\psi}^{\iota}_{jsm}\rangle$, we first apply a weighted pseudo-polar Fourier transform to an $N \times N$ image I so that the image is tranformed from the cartesian grid in time domain to a pseudo-polar grid in the frequency domain. This is the contents of 1) and 2), for which we need to find a weight function $w : \Omega_R \to \mathbb{R}^+$ so that

$$\sum_{u,v=-N/2}^{N/2-1} |I(u,v)|^2 = \sum_{(\omega_x,\omega_y)\in\Omega_R} w(\omega_x,\omega_y)\cdot|\hat{I}(\omega_x,\omega_y)|^2, \tag{1}$$

where $\hat{I}(\omega_x,\omega_y)$ is the pseudo-polar Fourier transform given by

$$\hat{I}(\omega_x,\omega_y) = \sum_{u,v=-N/2}^{N/2-1} I(u,v)e^{-\frac{2\pi i}{m_0}(u\omega_x+v\omega_y)}, \tag{2}$$

and $\Omega_R = \Omega_R^1 \cup \Omega_R^2$ is the pseudo-polar grid with

$$\begin{aligned} \Omega_R^1 = &\quad \{(-\tfrac{4\ell k}{RN}, \tfrac{2k}{R}) : -\tfrac{N}{2} \le \ell \le \tfrac{N}{2}, -\tfrac{RN}{2} \le k \le \tfrac{RN}{2}\},\\ \Omega_R^2 = &\quad \{(\tfrac{2k}{R}, -\tfrac{4\ell k}{RN}) : -\tfrac{N}{2} \le \ell \le \tfrac{N}{2}, -\tfrac{RN}{2} \le k \le \tfrac{RN}{2}\}, \end{aligned}$$

and $R \ge 2$ is the oversampling factor. Notice that the center $\mathcal{C} = \{(0,0)\}$ appears $N+1$ times in Ω_R^1 and Ω_R^2, and the points on the seam lines $\mathcal{S}_R^1 = \{(-\frac{2k}{R}, \frac{2k}{R}) : -\frac{RN}{2} \le k \le \frac{RN}{2}, k \ne 0\}$ and $\mathcal{S}_R^2 = \{(\frac{2k}{R}, -\frac{2k}{R}) : -\frac{RN}{2} \le k \le \frac{RN}{2}, k \ne 0\}$ appear in both Ω_R^1 and Ω_R^2. It has been shown that (2) can be fastly computed with order $O(N^2 \log N)$ (see [1]). Choosing the weights carefully, the following 'Plancherel theorem' – similar to the one for the discrete Fourier transform – can be proved for the pseudo-polar grid $\Omega_R = \Omega_R^1 \cup \Omega_R^2$.

Theorem 1. *Let N be even, and let $w : \Omega_R \to \mathbb{R}^+$ be a weight function satisfies the symmetry conditions: $w(\omega_x,\omega_y) = w(\omega_y,\omega_x)$, $w(\omega_x,\omega_y) = w(-\omega_x,\omega_y)$, and $w(\omega_x,\omega_y) = w(\omega_x,-\omega_y)$ for all $(\omega_x,\omega_y) \in \Omega_R$. Then (1) holds if and only if, for all $-N+1 \le u,v \le N-1$, the weight function w satisfies*

$$\begin{aligned} \delta(u,v) = w(0,0) &+ 4\cdot \sum_{\ell=0,N/2}\sum_{k=1}^{RN/2} w(\tfrac{2k}{R}, \tfrac{2k}{R}\cdot\tfrac{-2\ell}{N})\cdot\cos(2\pi u\cdot\tfrac{2k}{m_0R})\cdot\cos(2\pi v\cdot\tfrac{2k}{m_0R}\cdot\tfrac{2\ell}{N})\\ &+ 8\cdot\sum_{\ell=1}^{N/2-1}\sum_{k=1}^{RN/2} w(\tfrac{2k}{R}, \tfrac{2k}{R}\cdot\tfrac{-2\ell}{N})\cdot\cos(2\pi u\cdot\tfrac{2k}{m_0R})\cdot\cos(2\pi v\cdot\tfrac{2k}{m_0R}\cdot\tfrac{2\ell}{N}). \end{aligned}$$

Digital Shearlets on the Pseudo-Polar Grid. For 3), we need to construct a sequence of subband window functions on the pseudo-polar grid. In summary, we construct a tight frame $\{\varphi_n^{\iota_0}, \sigma_{jsm}^{\iota} : \Omega_R \to \mathbb{C}\}_{j,s,m;\iota_0,\iota}$ on the psuedo-polar grid Ω_R, which we call *digital shearlets.*

let W_0 be the Fourier transform of a Meyer scaling function with supp$W_0 \subseteq [-4^{j_R}, 4^{j_R}]$ and $j_R := -\lceil\log_4(R/2)\rceil$, and let V_0 be a 'bump function' satisfying

$\text{supp}V_0 \subseteq [-4^{j_R} - 1/2, 4^{j_R} + 1/2]$ with $V_0(\xi) \equiv 1$ for $|\xi| \leq 4^{j_R}$. Then we define the scaling function ϕ for the digital shearlet system to be

$$\hat{\phi}(\xi_1, \xi_2) = W_0(\xi_1)V_0(\xi_2), \quad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

We further choose W to be the Fourier transform of the Meyer wavelet function satisfying $\text{supp}W \subseteq [-4^{j_R+1}, -4^{j_R-1}] \cup [4^{j_R-1}, 4^{j_R+1}]$, and V to be a 'bump' function satisfying $\text{supp}V \subseteq [-1, 1]$ and $|V(\xi-1)|^2 + |V(\xi)|^2 + |V(\xi+1)|^2 = 1$ for all $|\xi| \leq 1$ and $\xi \in \mathbb{R}$. Then the generating shearlet ψ for the digital shearlet system on Ω_R^2 is defined as

$$\hat{\psi}(\xi_1, \xi_2) = W(\xi_1)V(\tfrac{\xi_2}{\xi_1}), \quad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Before stating the definition of digital shearlets, we first partition the set Ω_R beyond the already defined partitioning into Ω_R^1 and Ω_R^2 by setting $\Omega_R^1 = \Omega_R^{11} \cup \mathcal{C} \cup \Omega_R^{12}$ and $\Omega_R^2 = \Omega_R^{21} \cup \mathcal{C} \cup \Omega_R^{22}$, where $\Omega_R^{11} := \{(\omega_x, \omega_y) \in \Omega_R^1 : \omega_y \geq 1\}$, $\Omega_R^{12} := \{(\omega_x, \omega_y) \in \Omega_R^1 : \omega_y \leq -1\}$, and so on. The number of sampling points in radial and angular direction affected by a window at scale j and shear s is now given by

$$\mathcal{L}_j^1 = \begin{cases} 4^{j+j_R-1}\frac{R}{2}15 + 1 & : 0 \leq j < \lceil \log_4 N \rceil - j_R, \\ \lfloor \frac{R}{2}(N - 4^{j+j_R-1}) \rfloor + 1 & : j = \lceil \log_4 N \rceil - j_R, \end{cases}$$

and

$$\mathcal{L}_{j,s}^2 = \begin{cases} 2^{-j}N + 1 & : -2^j < s < 2^j, \\ 2^{-j}\frac{N}{2} + 1 & : s \in \{-2^j, 2^j\}. \end{cases}$$

We define $\mathcal{R}_{j,s}$ to be a rectangle given by

$$\mathcal{R}_{j,s} = \{((\mathcal{L}_j^1)^{-1}4^j(R/2)r_1, -(\mathcal{L}_{j,s}^2)^{-1}(N/2^{j+1})r_2)\}_{0\leq r_1 \leq L_j^1 - 1; 0 \leq r_2 \leq L_{j,s}^2 - 1}$$

and set the low frequency rectangle to be $\mathcal{R} = \{(r_1, r_2)\}_{-1 \leq r_1 \leq 1; -N/2 \leq r_2 \leq \frac{N}{2}}$.

We are now ready to define digital shearlets.

Definition 2. At scale $j \in \{0, \dots, \lceil \log_4 N \rceil - j_R\}$, shear $s = \{-2^j, \dots, 2^j\}$, and spatial position $m \in \mathcal{R}_{j,s}$, the *digital shearlets* on the cone Ω_R^{21} are defined by

$$\sigma_{j,s,m}^{21}(\omega_x, \omega_y) = \frac{C(\omega_x,\omega_y)}{\sqrt{|\mathcal{R}_{j,s}|}} W(4^{-j}\omega_x)\, V(s + 2^j \tfrac{\omega_y}{\omega_x}) \cdot \chi_{\Omega_R^{21}}(\omega_x, \omega_y)\, e^{-2\pi i m'(4^{-j}\frac{2k}{R}, -2^{j+1}\frac{\ell}{N})},$$

where $C(\omega_x, \omega_y) = 1$ if $(\omega_x, \omega_y) \notin \mathcal{S}_R^1 \cup \mathcal{S}_R^2$, $C(\omega_x, \omega_y) = \frac{1}{\sqrt{2}}$ if $(\omega_x, \omega_y) \in (\mathcal{S}_R^1 \cup \mathcal{S}_R^2) \setminus \mathcal{C}$, and $C(\omega_x, \omega_y) = \frac{1}{\sqrt{2(N+1)}}$ if $(\omega_x, \omega_y) \in \mathcal{C}$. The shearlets $\sigma_{j,s,m}^{11}, \sigma_{j,s,m}^{12}, \sigma_{j,s,m}^{22}$ on the remaining cones are defined accordingly by symmetry with equal indexing sets. For $n \in \mathcal{R}$, we further define the functions

$$\varphi_n^{\iota_0}(\omega_x, \omega_y) = \frac{C(\omega_x,\omega_y)}{\sqrt{|\mathcal{R}|}} W_0(\omega_x)\, V_0(\omega_y) \cdot \chi_{\Omega_R^{21}}(\omega_x, \omega_y)\, e^{-in'(\frac{k}{3}, \frac{\ell}{N+1})}, \ \iota_0 = 1, 2.$$

Summarizing, we call the system

$$\{\varphi_n^{\iota_0}\}_{\iota_0 = 1,2; n \in \mathcal{R}} \cup \{\sigma_{j,s,m}^{\iota}\}_{\iota = 11,12,21,22; 0 \leq j \leq \lceil \log_4 N \rceil - j_R; -2^j \leq s \leq 2^j, m \in \mathcal{R}_{j,s}}$$

the *digital shearlet system*, and denote it by $\mathcal{DSH}$.

This system has the following desirable property:

Theorem 3. *The digital shearlet system $\mathcal{DSH}$ defined in Definition 2 forms a tight frame for functions $J : \Omega_R \to \mathbb{C}$.*

Digital Shearlet Transform. Now, given an image I of size $N \times N$, the digital shearlet transform of I produces a sequnce of digital shearlet coefficients $\{c_n^{\iota_0}, c_{jsm}^{\iota}\}_{\iota_0,\iota;j,s,m}$, where $c_n^{\iota_0} := \langle \hat{J}_w, \varphi_n^{\iota_0}\rangle$ and $c_{jsm}^{\iota} := \langle \hat{J}_w, \sigma_{jsm}^{\iota}\rangle$, and $\hat{J}_w$ is the weighted pseudo-polar Fourier tranform of I defined to be $J_w(\omega_x,\omega_y) = \sqrt{w(\omega_x,\omega_y)}\hat{I}(\omega_x,\omega_y)$, $(\omega_x,\omega_y) \in \Omega_R$ with $\hat{I}$ being given in (2). The digital shearlet transform, together with many quantitative measures for quantifying and comparing performances of different directional systems, has been implemented in the ShearLab package, which can be accessed through the website: www.shearlab.org.

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Operator representation: from time-frequency multipliers... to sound design

Bruno Torrésani

(joint work with Monika Dörfler, Anaïk Olivero, and Richard Kronland-Martinet)

Gabor multipliers. This talk is concerned with the representation of linear operators on, say $L^2(\mathbb{R})$, as superpositions of elementary time-frequency building blocks, namely rank one projection operators associated with fonctions (called atoms) labelled by time-frequency index. Such operators, called Gabor multipliers, can be written as

$$\mathbb{M}_{\mathbf{m}\ g,h}x(t) = \sum_{m,n=-\infty}^{\infty} \mathbf{m}(m,n)\langle x, g_{mn}\rangle h_{mn}\ ,$$

where $g_{mn}(t) = e^{2i\pi m\nu_0 t} g(t - mb_0)$ are copies of a reference function (window), obtained by time and frequency shifts on a lattice $\Lambda = \mathbb{Z}b_0 \times \mathbb{Z}\nu_0$, and $\mathbf{m}$, called *mask* or *upper symbol* is a bounded sequence on Λ. Such multipliers are of frequent use in signal processing applications.

Such operators and their approximation properties, have been studied by various authors (see for example [2, 4, 7] and references therein). A main result states that when the windows g, h and the sampling lattice Λ are suitably chosen, then the optimal Hilbert-Schmidt approximation by Gabor multipliers is well-defined. The result relies on the so-called spreading function representation (see [8] for example).

Theorem 1. (1) *Let $H \in \mathcal{H}$ be a Hilbert-Schmidt operator on $L^2(\mathbb{R})$. Then there exists a function $\eta = \eta_H \in L^2(\mathbb{R}^2)$, called the* spreading function, *such that*

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_H(t,\xi)\pi(t,\xi)\, dtd\xi \ .$$

the integral being interpreted in the weak operator sense.

(2) *The relation $\eta_H \in L^2(\mathbb{R}) \longleftrightarrow H \in \mathcal{H}$ extends to a Gel'fand triple isomorphism $(S_0(\mathbb{R}), L^2(\mathbb{R}), S_0'(\mathbb{R})) \longleftrightarrow (\mathcal{B}, \mathcal{H}, \mathcal{B}')$, where $S_0(\mathbb{R})$ is the Feichtinger algebra, $S_0'(\mathbb{R})$ is its dual, and $\mathcal{B}$ and $\mathcal{B}'$ denote respectively the space of bounded operators $S_0' \to S_0$ and its dual.*

Then, denoting by $\Lambda^\circ = \mathbb{Z}/\nu_0 \times \mathbb{Z}/b_0$ the adjoint lattice, and by $\square^\circ$ the corresponding fundamental domain, introduce the (ν_0^{-1}, b_0^{-1}) periodic function

$$\mathcal{U}(t,\xi) = \sum_{k,\ell=-\infty}^{\infty} \left|\mathcal{V}_g h\left(t + k/\nu_0, \xi + \ell/b_0\right)\right|^2 \ .$$

One then proves

Theorem 2 ([2,4])**.** *Assume g, h, b_0 and ν_0 are chosen so that*

$$A \le \mathcal{U} \le B \quad \text{a. e. on} \quad \square^\circ \ ,$$

for some constants $0 < A \le B < \infty$. Then the best GM approximation (in Hilbert-Schmidt sense) of $H \in \mathcal{H}$ is defined by the mask

$$\mathbf{m}(m,n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{M}(t,\xi) e^{2i\pi(n\nu_0 t - m b_0 \xi)}\, dtd\xi$$

whose discrete symplectic Fourier transform reads

$$\mathcal{M}(t,\xi) = \frac{\sum_{k,\ell} \overline{\mathcal{V}_g h}\left(t + k/\nu_0, \xi + \ell/b_0\right) \eta_H\left(t + k/\nu_0, \xi + \ell/b_0\right)}{\mathcal{U}(t,\xi)}$$

The approximation error can be computed explicitly. This yields for example [5]

Corollary 3. *Assume that η is supported inside $\square^\circ$. Then*

$$\|H - \mathbb{M}_{\mathbf{m};g,h}\|_{\mathcal{H}}^2 \le \|H\|_{\mathcal{H}}^2 \left[1 - \inf_{(t,\xi)\in\square^\circ} \frac{|\mathcal{V}_g h(t,\xi)|^2}{\mathcal{U}(t,\xi)}\right] .$$

This means that the approximation quality depends heavily on the concentration of $\mathcal{V}_g h$ inside the fundamental domain $\square^\circ$.

Multiple Gabor Multipliers (MGM). When the spreading function η_H of H is not well enough localized, it is still possible to seek approximations as sums of Gabor multipliers. Multiple Gabor multipliers are linear combinations of Gabor multipliers, with a fixed analysis window g, several different synthesis windows (with different time-frequency localizations) $h^{(j)}$, $j \in \mathcal{J}$, and corresponding masks $\mathbf{m}_j$. They can be written as

$$\mathbb{M}_{\{\mathbf{m}_j;g,h^{(j)}\}} = \sum_{j\in\mathcal{J}} \mathbb{M}_{\mathbf{m}_j;g,h^{(j)}} .$$

Optimal MGM approximations for Hilbert-Schmidt operators can be obtained as in the GM case. For $(t,\xi) \in \Box^\circ$, set

$$\mathcal{U}(t,\xi)_{jj'} = \sum_{k,\ell} \overline{\mathcal{V}_g h^{(j)}}(t + k/\nu_0, \xi + \ell/b_0)\mathcal{V}_g h^{(j')}(t + k/\nu_0, \xi + \ell/b_0).$$

Theorem 4. *Let $g \in S_0(\mathbb{R})$ and $h^{(j)} \in S_0(\mathbb{R})$, $j \in \mathcal{J}$ be such that the matrix $\mathcal{U}(t,\xi)$ is invertible a.e. on $\Box^\circ$. Let $H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$ be an operator with spreading function $\eta \in (S_0, \mathbf{L}^2, S_0')$. Then the functions $\underline{\mathcal{M}} = \{\mathcal{M}_j, j \in \mathcal{J}\}$ yielding the optimal MGM approximation of may be obtained as*

$$\underline{\mathcal{M}}(t,\xi) = \mathcal{U}(t,\xi)^{-1} \cdot \mathbf{V}(t,\xi) \ ,$$

where $\mathbf{V}$ is the vector whose entries read

$$\mathbf{V}_{j_0}(t,\xi) = \sum_{k,\ell} \eta(t + k/\nu_0, \xi + \ell/b_0)\overline{\mathcal{V}_g h^{j_0}}(t + k/\nu_0, \xi + \ell/b_0).$$

As in the GM case, explicit error estimates can be obtained [5].

Application: Gabor multiplier estimation. We now address the following more concrete problem, in view of signal processing applications: given two functions $x_0, x_1 \in L^2(\mathbb{R})$, under which assumptions can one find a mask $\mathbf{m}$ such that $x_1 \approx \mathbb{M}_{\mathbf{m}} x_0$?

In signal processing applications, x_0 would be some input signal, x_1 the output signal of some linear system, which one would like to estimate. Potential applications include among others channel identification for mobile phone signal transmission, room estimation for acoustic de-reverberation, or musical instrument classification and sound morphing, which is our domain of interest.

In such a context, it is natural to turn to variational formulations and seek a solution by minimizing in $\ell^2(\mathbb{Z}^2)$ a quantity such as

$$\Phi[\mathbf{m}] = \frac{1}{2}\|x_1 - \mathbb{M}_{\mathbf{m};g,g} x_0\|_2^2 + \frac{\lambda}{p}\|\mathbf{m} - 1\|_p^p \ ,$$

with $\lambda \in \mathbb{R}^+$ a Lagrange parameter introduced to control the norm of the mask $\mathbf{m}$.

The minimization problem can be solved for $p = 2$, and yields a nice, closed form... but huge matrix equation, not suitable for numerical purpose. However one can reformulate the problem as an inverse problem [10]

$$\min_{\mathbf{m} \in \ell^2} \left[\frac{1}{2}\|x_1 - T\mathbf{m}\|_2^2 + \frac{\lambda}{p}\|\mathbf{m} - 1\|_p^p\right]$$

where the linear operator T represents pointwise multiplication with $\mathcal{V}_g x_0$ followed by Gabor synthesis $\mathcal{V}_h^*$, and resort to efficient numerical algorithms, for instance Landweber type iterations in the spirit of [3], or accelerated versions [1].

The same approach is extended to multiple input-output signal pairs

$$x_0^{(k)} \approx \mathbb{M}_{\mathbf{m}} x_0^{(k)} \ , \qquad k = 1, \dots K$$

or to multiple Gabor multipliers estimation. The same algorithmic structure can be used in these two contexts, with increased complexity.

Application to sound design. One motivation for studying such questions was the sound design problem, which can be roughly stated as follows: how can one construct methods that would produce sound signals that *make sense*, i.e. that would sound natural to a listener, and convey some specific information ?

We are very far from being able to answer such a question, and a first step is to better understand the nature and structure of sound signals. It turns out that Gabor multipliers provide a sensible way to perform pairwise comparisons between sounds in a given database, and that the estimated masks yield interpretable information. During the talk, we presented two applications, limiting to simple sounds (namely, single note signals from different wind instruments):

- Sound categorization: starting from estimated multipliers, use the information contained in the masks $\mathbf{m}$ to generate divergence measures, further exploited via classification algorithms.
- Sound morphing: given a pair of sound signals, estimate a Gabor multiplier, and use it to synthesize families of sound signals that interpolate between the input and the output.

The results presented during the workshop are part of the PhD project of Anaïk Olivero. Practical implementations (in the Ltfat package software, see [11]) and sound examples will be made available soon.

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Lipschitz equivalence of Cantor sets

Yang Wang

(joint work with Hui Rao and Huo-Jun Ruan)

Let E, F be compact sets in $\mathbb{R}^d$. We say that E and F are *Lipschitz equivalent*, and denote it by $E \sim F$, if there exists a bijection $\psi : E \longrightarrow F$ which is *bi-Lipschitz*, i.e. there exists a constant $C > 0$ such that

$$C^{-1}|x - y| \leq |\psi(x) - \psi(y)| \leq C|x - y|$$

for all $x, y \in E$.

An area of interest in the study of self-similar sets is the Lipschitz equivalence property. With Lipschitz equivalence many important properties of self-similar sets are preserved. There is an extensive literature on Lipschitz equivalence of self-similar sets, see e.g. [2] for a comprehensive discussion.

This talk concerns with the Lipschitz equivalence of dust-like self-similar sets in $\mathbb{R}^d$. Recall that in general we characterize a self-similar set as the attractor of an *iterated functions system (IFS)*. Let $\{\phi_j\}_{j=1}^m$ be an IFS on $\mathbb{R}^d$ where each ϕ_j is a contractive similarity with contraction ratio $0 < \rho_j < 1$. The attractor of the IFS is the unique nonempty compact set F satisfying $F = \bigcup_{j=1}^m \phi_j(F)$, see [3]. We say that the attractor F is *dust-like*, or alternatively, the IFS $\{\phi_j\}$ satisfies the *strong separation condition (SSC)*, if the sets $\{\phi_j(F)\}$ are disjoint. It is well known that if F is dust-like then the Hausdorff dimension $s = \dim_H(F)$ of F satisfies $\sum_{j=1}^m \rho_j^s = 1$. It is well known that two dust-like self-similar sets with the same contraction ratios are Lipschitz equivalent. For any $\rho_1, \ldots, \rho_m \in (0, 1)$ with $\sum_{j=1}^m \rho_j^d < 1$, we will call $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ a *contraction vector*, and use the notation $\mathcal{D}(\boldsymbol{\rho}) = \mathcal{D}(\rho_1, \ldots, \rho_m)$ to denote the set of all dust-like self-similar sets that are the attractor of some IFS with contraction ratios $\rho_j, j = 1, \ldots, m$ on $\mathbb{R}^d$. We use $s = \dim_H \mathcal{D}(\boldsymbol{\rho})$ to denote the Hausdorff dimension of sets in $\mathcal{D}(\boldsymbol{\rho})$. Let $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be two contraction vectors. We say $\mathcal{D}(\boldsymbol{\rho})$ and $\mathcal{D}(\boldsymbol{\tau})$ are Lipschitz equivalent, and denote it by $\mathcal{D}(\boldsymbol{\rho}) \sim \mathcal{D}(\boldsymbol{\tau})$, if $E \sim F$ for some (and thus for all) $E \in \mathcal{D}(\boldsymbol{\rho})$ and $F \in \mathcal{D}(\boldsymbol{\tau})$. Note that if $\boldsymbol{\tau}$ is a permutation of $\boldsymbol{\rho}$ then we clearly have $\mathcal{D}(\boldsymbol{\tau}) = \mathcal{D}(\boldsymbol{\rho})$. One of the most fundamental results in the study of Lipschitz equivalence is the following theorem, proved by Falconer and Marsh [2], that establishes a connection to the algebraic properties of the contraction ratios:

Theorem 1 ([2], Theorem 3.3)**.** *Let $\mathcal{D}(\boldsymbol{\rho})$ and $\mathcal{D}(\boldsymbol{\tau})$ be Lipschitz equivalent, where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ are two contraction vectors. Let $s = \dim_H \mathcal{D}(\boldsymbol{\rho}) = \dim_H \mathcal{D}(\boldsymbol{\tau})$. Then*

1. *$\mathbb{Q}(\rho_1^s, \ldots, \rho_m^s) = \mathbb{Q}(\tau_1^s, \ldots, \tau_n^s)$, where $\mathbb{Q}(a_1, \ldots, a_m)$ denotes the subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and $a_1, \ldots, a_m$.*
2. *There exist positive integers p, q such that*

$$\mathrm{sgp}(\rho_1^p, \ldots, \rho_m^p) \subseteq \mathrm{sgp}(\tau_1, \ldots, \tau_n),$$
$$\mathrm{sgp}(\tau_1^q, \ldots, \tau_n^q) \subseteq \mathrm{sgp}(\rho_1, \ldots, \rho_m),$$

where $\mathrm{sgp}(a_1, \ldots, a_m)$ *denotes the subsemigroup of* $(\mathbb{R}^+, \times)$ *generated by* $a_1, \ldots, a_m$*.*

Using this theorem, it was shown in [1,2] that there exist dust-like self-similar sets E and F such that $\dim_H E = \dim_H F$ but E and F are not Lipschitz equivalent. Also, from this theorem, the following question arises naturally:

Question 1. Can we present nontrivial sufficient conditions and necessary conditions on $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ such that $\mathcal{D}(\boldsymbol{\rho}) \sim \mathcal{D}(\boldsymbol{\tau})$?

Since the above work by Falconer and Marsh, there have been little progress in this direction as we know of. In this talk we present some recent progresses in this direction. To answer the question in general is likely to be extremely hard. We present several important special cases that should allow us to gain some deep insight into the problem. Our results establish further connections between algebraic properties of contraction ratios and Lipschitz equivalence of dust-like self-similar sets.

We introduce the notion of *rank* for a contraction vector $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$, which is the rank (the number of generators) of the free subgroup of $(\mathbb{R}^+, \times)$ generated by $\rho_1, \ldots, \rho_m$. According to Theorem 1 (2), if $\mathcal{D}(\boldsymbol{\rho}) \sim \mathcal{D}(\boldsymbol{\tau})$, then $\mathrm{rank}\langle\boldsymbol{\rho}\rangle = \mathrm{rank}\langle\boldsymbol{\tau}\rangle = \mathrm{rank}\langle\boldsymbol{\rho}, \boldsymbol{\tau}\rangle$, where $\langle\boldsymbol{\rho}, \boldsymbol{\tau}\rangle := \langle\rho_1, \ldots, \rho_m, \tau_1, \ldots, \tau_n\rangle$ for $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$. One of our main theorems is:

Theorem 2. *Let* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ *and* $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$ *be two contraction vectors such that* $\mathrm{rank}\langle\boldsymbol{\rho}\rangle = m$*. Then* $\mathcal{D}(\boldsymbol{\rho})$ *and* $\mathcal{D}(\boldsymbol{\tau})$ *are Lipschitz equivalent if and only if* $\boldsymbol{\tau}$ *is a permutation of* $\boldsymbol{\rho}$*.*

A special case we study involve self-similar sets with two branches. This seemingly simple case turns out to be rather challenging. We show that Theorem 2 and a result on the irreducibility of certain trinomials by Ljunggren [4] allows us to completely characterize the Lipschitz equivalence of dust-like self-similar sets with two branches. We prove:

Theorem 3. *Let* (ρ_1, ρ_2) *and* (τ_1, τ_2) *be two contraction vectors with* $\rho_1 \le \rho_2$, $\tau_1 \le \tau_2$*. Assume that* $\rho_1 \le \tau_1$*. Then* $\mathcal{D}(\boldsymbol{\rho}) \sim \mathcal{D}(\boldsymbol{\tau})$ *if and only if one of the two conditions holds:*

(1) $\rho_1 = \tau_1$ *and* $\rho_2 = \tau_2$*.*

(2) *There exists a real number* $0 < \lambda < 1$*, such that*

$$(\rho_1, \rho_2) = (\lambda^5, \lambda) \quad \textit{and} \quad (\tau_1, \tau_2) = (\lambda^3, \lambda^2).$$

Another case where the Lipschitz equivalence of dust-like self-similar sets can be characterized completely is when one of them has uniform contraction ratio.

Theorem 4. *Let* $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_m) = (\rho, \ldots, \rho)$ *and* $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$*. Then* $\mathcal{D}(\boldsymbol{\rho})$ *and* $\mathcal{D}(\boldsymbol{\tau})$ *are Lipschitz equivalent if and only if the following conditions hold:*

(1) $\dim_H \mathcal{D}(\boldsymbol{\tau}) = \dim_H \mathcal{D}(\boldsymbol{\rho}) = \log m / \log \rho^{-1}$.

(2) *There exists a* $q \in \mathbb{Z}^+$ *such that* $m^{1/q} \in \mathbb{Z}$ *and*

$$\frac{\log \tau_j}{\log \rho} \in \frac{1}{q}\mathbb{Z} \quad \textit{for all} \quad j = 1, 2, \ldots, n.$$

The results of this talk along with many other results can be found in the paper [5].

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Unions of subspaces models: theory and applications

Akram Aldroubi

Let $\mathcal{H}$ be a Hilbert space, $\mathbf{F} = \{f_1, \ldots, f_m\}$ a finite set of vectors in $\mathcal{H}$, $\mathcal{C}$ a family of closed subspaces of $\mathcal{H}$, $\mathcal{V}$ the set of all sequences of elements in $\mathcal{C}$ of length l (i.e., $\mathcal{V} = \mathcal{V}(l) = \big\{\{V_1, \ldots, V_l\} : V_i \in \mathcal{C}, 1 \leq i \leq l\big\}$). The following problem has several applications in mathematics, engineering, and computer science:

Problem 1 (Non-Linear Least Squares Subspace Approximation)**.**

(1) Given a finite set $\mathbf{F} \subset \mathcal{H}$ and a fixed integer $l \geq 1$, find the infimum of the expression
$$e(\mathbf{F}, \mathbf{V}) := \sum_{f \in \mathbf{F}} \min_{1 \leq j \leq l} d^2(f, V_j),$$
over $\mathbf{V} = \{V_1, \ldots, V_l\} \in \mathcal{V}$, and $d(x, y) := \|x - y\|_{\mathcal{H}}$.

(2) Find a sequence of l-subspaces $\mathbf{V}^o = \{V_1^o, \ldots, V_l^o\} \in \mathcal{V}$ (if it exists) such that

$$e(\mathbf{F}, \mathbf{V}^o) = \inf\{e(\mathbf{F}, \mathbf{V}) : \mathbf{V} \in \mathcal{V}\}. \tag{1}$$

This is a nonlinear version of the least squares problem ($l = 1$), and it has many applications in mathematics and engineering. For example, in finite dimensions, the subspace segmentation problem in computer vision (see e.g., [18] and the references therein), the problems of face recognition, and motion tracking in videos (see e.g., [6,10,12,18,19]), and the problem of segmentation and data clustering in Hybrid Linear Models (see e.g., [8,14] and the reference therein). Compressed sensing is another related area where s-sparse signals in $\mathbb{C}^d$ can be viewed as belonging to a union of subspaces $\mathcal{M} = \cup_{i \in I} V_i$, with $\dim V_i \leq s$ [9].

Examples where $\mathcal{H}$ is infinite-dimensional occurs in signal modeling. The typical situation is $\mathcal{H} = L^2(\mathbb{R}^d)$, $\mathbf{F} \subset L^2(\mathbb{R}^d)$ is a set of finite signals, and $\mathcal{C}$ is the set of all finitely-generated shift-invariant spaces of $L^2(\mathbb{R}^d)$ [5, 15, 21]. Subspaces in $\mathcal{C}$ are infinite-dimensional as well but they are shift-invariant, i.e., for $V \in \mathcal{C}$, $f \in V$ implies that $f(\cdot - k) \in V$ for all $k \in \mathbb{Z}$ (see e.g., [7]). The class of signals with finite rate of innovation is another situation where the space $\mathcal{H}$ is infinite-dimensional [16]. Applications where a union of subspaces underly the signal model in infinite dimensions can be found in [4, 15, 16].

The Minimal Subspace Approximation Property. It has been shown that, given a family of closed subspaces $\mathcal{C}$, the existence of a minimizing sequence of subspaces $\mathbf{V}^o = \{V_1^o, \ldots, V_l^o\}$ that solves Problem 1 is equivalent to the existence of a solution to the same problem but for $l = 1$ [4]:

Theorem 2. *Problem 1 has a minimizing set of subspaces for any $l \geq 1$ if and only if it has a minimizing subspace for $l = 1$.*

This suggests the following definition:

Definition 3. A set of closed subspaces $\mathcal{C}$ of a separable Hilbert space $\mathcal{H}$ has the Minimum Subspace Approximation Property (MSAP) if for every finite subset $\mathbf{F} \subset \mathcal{H}$ there exists an element $V \in \mathcal{C}$ that minimizes the expression $e(\mathbf{F}, V) = \sum_{f \in \mathbf{F}} d^2(f, V)$ over all $V \in \mathcal{C}$. We will say that $\mathcal{C}$ has $MSAP(k)$ for some $k \in \mathbb{N}$ if the previous property holds for all subsets $\mathbf{F}$ of cardinality $m \leq k$.

Using this terminology, Problem 1 has a minimizing sequence of subspaces if and only if $\mathcal{C}$ satisfies the MSAP.

Remark 4. We will see that, in general, $MSAP(k+1)$ is strictly stronger than $MSAP(k)$. Obviously, MSAP is stronger than $MSAP(k)$ for any $k \in \mathbb{N}$.

There are some cases for which it is known that the MSAP is satisfied. For example, if $\mathcal{H} = \mathbb{C}^d$ and $\mathcal{C} = \{V \subset \mathcal{H} : \dim V \leq s\}$, the Eckhard-Young theorem [11] implies that $\mathcal{C}$ satisfies MSAP. Another example is when $\mathcal{H} = L^2(\mathbb{R}^d)$ and $\mathcal{C} = \overline{\text{span}\{\phi_1, \ldots, \phi_r\}}$ is the set of all shift-invariant spaces of length at most r. For this last example, a result in [3] implies that $\mathcal{C}$ satisfies the MSAP.

The general approach for the existence of a minimizer has been recently considered in [1]. The family $\mathcal{C}$ is viewed as a set of projectors and the characterization is in terms of the augmented set $\mathcal{C}^+$ consisting of $\mathcal{C}$ together with the positive operators added to it. In finite dimensions, the necessary and sufficient condition is that $\mathcal{C}^+$ is closed [1] . An equivalent characterization is that the convex hull $\text{co}(\mathcal{C}^+)$ of $\mathcal{C}^+$ is equal to the convex hull $\text{co}(\overline{\mathcal{C}^+})$ of its closure [1] . A third characterization for a finite d-dimensional space $\mathcal{H}$ is that $\mathcal{C}$ satisfies the $MSAP(d-1)$ [1] (see Definition 3).

The infinite dimensions is different. For this case, the necessary and sufficient conditions are in terms of the set of contact half-spaces $\tau(\mathcal{C}^+)$, and $\tau(\overline{\mathcal{C}^+})$ of $\mathcal{C}^+$ and $\overline{\mathcal{C}^+}$, respectively. Specifically, Problem 1 has a minimizer if and only if $\tau(\mathcal{C}^+) = \tau(\overline{\mathcal{C}^+})$ where the closure is in the weak operator topology [1].

Algorithms and Dimensionality Reduction. Search algorithms for finding solutions to Problem 1 are often iterative. A general abstract algorithm of this kind is described in [4]. Iterative algorithms often need a good initial approximation to a solution. Thus there are several important related problems including

1) Finding a good initial approximate solution; 2) Dimensionality reduction for efficient computations.

Often, the search for the optimal model from observed data **F** involves heavy computations that dramatically increase with the dimensionality of $\mathcal{H}$. However, the effective dimension of the space $S = V_1 + \cdots + V_l$ associated with the model $\mathcal{M} = \cup_i V_i$ is much smaller than the dimension of the ambient space $\mathcal{H}$. Thus one important feature is to map the data into a lower dimensional space, and solve the transformed problem in this lower dimensional space. If the mapping is chosen appropriately, the original problem can be solved exactly or approximately using the solution of the transformed data. For example, when the model $\mathcal{M} = \cup_i V_i$ inside $\mathcal{H} = \mathbb{R}^N$ is of dimension $\dim \mathcal{M} = k << N$ then it is possible to transform Problem 1 into a problem in which the ambient space $\mathcal{H}'$ has dimension $k+1$ and obtain an exact solution to the original problem provided no noise is present [2].

However, when noise is present, the mapping to low dimensional subspace are more constrained and the solution in the reduced space can only be used to approximate the real solution. The type of mapping that needs to be used for the noisy case and an estimate of the error can be found in [2].

Application to motion segmentation. Consider a moving affine camera that captures N frames of a scene that contains multiple moving objects. Let p be a point of one of these objects and let $x_i(p), y_i(p)$ be the coordinates of p in frame i. Define the *trajectory vector* of p in $\mathbb{R}^{2N}$ as $w(p) = (x_1(p), y_1(p) \ldots, x_N(p), y_N(p))^t$. It can be shown that the trajectory vectors of all points of an object in a video belong to a vector subspace in $\mathbb{R}^{2N}$ of dimension no larger than 4 [13]. Thus, trajectory vectors in videos can be modeled by a union $\mathcal{M} = \cup_{i \in I} V_i$ of l subspaces where l is the number of moving objects (background is itself a motion).

A precise description of the motion tracking in video can be found in [6]. Finding the nearest unions of subspaces to a set of trajectory vectors as in Problem 1 allows for segmenting and tracking the moving objects. Techniques for motion tracking can be compared to state of the art methods on the Hopkins 155 Data set [VMS10] . The Hopkins 155 Dataset was created as a benchmark database to evaluate motion segmentation algorithms. The ground truth segmentations are also provided for comparison. Our algorithm's recognition rates for two and three motion video sequences are 99.15% and 98.85%, respectively [6] which are best to date.

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Shearlet coorbit spaces I: General setting (in arbitrary space dimensions)

Stephan Dahlke
(joint work with Gabriele Steidl and Gerd Teschke)

Multivariate Continuous Shearlet Transform. Let us start by introducing the continuous shearlet transform on $L_2(\mathbb{R}^n)$. This requires the generalization of the parabolic dilation matrix and of the shear matrix. Let I_n denote the (n,n)-identity matrix and 0_n, resp. 1_n the vectors with n entries 0, resp. 1. For

$a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{n-1}$, we set

$$A_a := \begin{pmatrix} a & 0_{n-1}^{\mathrm{T}} \\ 0_{n-1} & \operatorname{sgn}(a)|a|^{\frac{1}{n}} I_{n-1} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s^{\mathrm{T}} \\ 0_{n-1} & I_{n-1} \end{pmatrix}.$$

Lemma 1. *The set $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ endowed with the operation*

$$(a, s, t) \circ (a', s', t') = (aa', s + |a|^{1-1/n} s', t + S_s A_a t')$$

is a locally compact group $\mathbb{S}$ which we call full shearlet group*. The left and right Haar measures on $\mathbb{S}$ are given by*

$$d\mu_l(a, s, t) = \frac{1}{|a|^{n+1}}\, da\, ds\, dt \quad \text{and} \quad d\mu_r(a, s, t) = \frac{1}{|a|}\, da\, ds\, dt.$$

For $f \in L_2(\mathbb{R}^n)$ we define

(1) $$\pi(a, s, t) f(x) = f_{a,s,t}(x) := |a|^{\frac{1}{2n}-1} f(A_a^{-1} S_s^{-1}(x - t)).$$

Theorem 2. *The mapping π defined by* (1) *is a unitary representation of $\mathbb{S}$. Moreover, a function $\psi \in L_2(\mathbb{R}^n)$ is admissible if and only if*

(2) $$C_\psi := \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^n}\, d\omega < \infty.$$

Then, for any $f \in L_2(\mathbb{R}^n)$, the following equality holds true:

(3) $$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2\, d\mu_l(a, s, t) = C_\psi\, \|f\|^2_{L_2(\mathbb{R}^n)}.$$

Multivariate Shearlet Coorbit Theory. We consider weight functions $w(a, s, t) = w(a, s)$ that are locally integrable with respect to a and s, i.e., $w \in L_1^{loc}(\mathbb{R}^n)$ and fulfill $w((a, s, t) \circ (a', s', t')) \leq w(a, s, t) w(a', s', t')$ and $w(a, s, t) \geq 1$. Let

$$L_{p,w}(\mathbb{S}) := \{F \ : \|F\|_{L_{p,w}(\mathbb{S})} := \left(\int_{\mathbb{S}} |F(g)|^p\, w(a, s, t)^p d\mu(a, s, t) \right)^{1/p} < \infty\}.$$

In order to construct the coorbit spaces related to the shearlet group we have to ensure that there exists a function $\psi \in L_2(\mathbb{R}^n)$ such that

(4) $$\mathcal{SH}_\psi(\psi) = \langle \psi, \pi(a, s, t)\psi \rangle \in L_{1,w}(\mathbb{S}).$$

Theorem 3. *Let ψ be a Schwartz function such that* $\operatorname{supp} \hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times [-b_1, b_1] \times \cdots \times [-b_{n-1}, b_{n-1}]$*. Then we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$.*

For ψ satisfying (4) we can consider the space

(5) $$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^n) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi \rangle \in L_{1,w}(\mathbb{S})\},$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}_{1,w}^\sim$. The spaces $\mathcal{H}_{1,w}$ and $\mathcal{H}_{1,w}^\sim$ are π-invariant Banach spaces with continuous embeddings $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{1,w}^\sim$, and their definition is independent of the shearlet ψ. Then the inner product on $L_2(\mathbb{R}^n) \times L_2(\mathbb{R}^n)$ extends to a sesquilinear form on $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$, therefore for $\psi \in \mathcal{H}_{1,w}$ and $f \in \mathcal{H}_{1,w}^\sim$ the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a, s, t) := \langle f, \pi(a, s, t)\psi \rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}$$

are well-defined. The next step is to consider an additional weight function m which is *moderate* with respect to w, i.e., $m((a,s,t)\circ(a',s',t')\circ(a'',s'',t'')) \leq w(a,s,t)m(a',s',t')w(a'',s'',t'')$. Then, with respect to the new weight m, we define the *shearlet coorbit spaces*

$$\mathcal{SC}_{p,m} := \{f \in \mathcal{H}_{1,w}^{\sim} : \mathcal{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\} \tag{6}$$

with norms $\|f\|_{\mathcal{SC}_{p,m}} := \|\mathcal{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}$.

The Feichtinger-Gröchenig theory provides us with a machinery to construct atomic decompositions and Banach frames for our shearlet coorbit spaces $\mathcal{SC}_{p,w}$. A (countable) family $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{S}$ is said to be *U-dense* if $\cup_{\lambda\in\Lambda}(a,s,t)_\lambda U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of e we have $(a_i,s_i,t_i)Q \cap (a_j,s_j,t_j)Q = \emptyset$, $i \neq j$, and *relatively separated* if X is a finite union of separated sets.

Lemma 4. *Let U be a neighborhood of the identity in $\mathbb{S}$, and let $\alpha > 1$ and $\beta, \gamma > 0$ be defined such that $[\alpha^{\frac{1}{n}-1}, \alpha^{\frac{1}{n}}) \times [-\frac{\beta}{2}, \frac{\beta}{2})^{n-1} \times [-\frac{\gamma}{2}, \frac{\gamma}{2})^n \subseteq U$. Then the sequence*

$$\{(\epsilon\alpha^j, \beta\alpha^{j(1-\frac{1}{n})}k, S_{\beta\alpha^{j(1-\frac{1}{n})}k}A_{\alpha^j}\gamma m) : j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n, \ \epsilon \in \{-1,1\}\}$$

is U-dense and relatively separated.

Next we define the *U–oscillation* as

$$\operatorname{osc}_U(a,s,t) := \sup_{u\in U} |\mathcal{SH}_\psi(\psi)(u\circ(a,s,t)) - \mathcal{SH}_\psi(\psi)(a,s,t)|. \tag{7}$$

Then, the following decomposition theorem, which was proved in a general setting in [3–5], says that discretizing the representation by means of an U-dense set produces an atomic decomposition for $\mathcal{SC}_{p,w}$.

Theorem 5. *Assume that the irreducible, unitary representation π is w-integrable and let an appropriately normalized $\psi \in L_2(\mathbb{R}^n)$ which fulfills*

$$M\langle\psi, \pi(a,s,t)\rangle := \sup_{u\in(a,s,t)U} |\langle\psi, \pi(u)\psi\rangle| \in L_{1,w}(\mathbb{S}) \tag{8}$$

be given. Choose a neighborhood U of e so small that $\|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})} < 1$. Then for any U-dense and relatively separated set $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ the space $\mathcal{SC}_{p,m}$ has the following atomic decomposition: *If $f \in \mathcal{SC}_{p,m}$, then*

$$f = \sum_{\lambda\in\Lambda} c_\lambda(f)\pi((a,s,t)_\lambda)\psi \tag{9}$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}} \sim \|f\|_{\mathcal{SC}_{p,m}}. \tag{10}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by its *moments* $\langle f, \pi((a,s,t)_\lambda)\psi\rangle$ and how f can be reconstructed from these moments.

Theorem 6. *Impose the same assumptions as in Theorem 5. Choose a neighborhood U of e such that $\|\operatorname{osc}_U\|_{L_{1,w}(\mathbb{S})} < 1/\|\mathcal{SH}_\psi(\psi)\|_{L_{1,w}(\mathbb{S})}$. Then $\{\pi((a,s,t)_\lambda)\psi : \lambda \in \Lambda\}$ is a Banach frame for $\mathcal{SC}_{p,m}$. This means that*

i) *$f \in \mathcal{SC}_{p,m}$ if and only if $((\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda} \in \ell_{p,m}$;*

ii)

$$\|f\|_{\mathcal{SC}_{p,m}} \sim \|((\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\|_{\ell_{p,m}};$$

iii) *there exists a bounded, linear operator $\mathcal{S}$ from $\ell_{p,m}$ to $\mathcal{SC}_{p,m}$ such that $\mathcal{S}\left(((\langle f, \psi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\right) = f.$*

To apply the whole machinery it remains to prove that $\|\mathrm{osc}_U\|_{L_{1,w}(\mathbb{S})}$ becomes arbitrarily small for a sufficiently small neighborhood U of e.

Theorem 7. *Let ψ be a function contained in the Schwartz space $\mathcal{S}$ with $\operatorname{supp}\hat{\psi} \subseteq ([-a_1,-a_0]\cup[a_0,a_1]) \times_b [-b_1,b_1]\times\cdots\times[-b_{n-1},b_{n-1}]$. Then, for every $\varepsilon > 0$, there exists a sufficiently small neighborhood U of e so that*

$$\|\mathrm{osc}_U\|_{L_{1,w}(\mathbb{S})} \le \varepsilon. \tag{11}$$

Further information concerning the coorbit and group theory related with the continuous shearlet transform can be found in [1, 2].

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Shearlet coorbit spaces II: compactly supported shearlets, traces and embeddings

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(joint work with Stephan Dahlke and Gabriele Steidl)

We show that compactly supported functions with sufficient smoothness and enough vanishing moments can serve as analyzing vectors for shearlet coorbit spaces. We use this approach to prove embedding theorems for subspaces of shearlet coorbit spaces resembling shearlets on the cone into Besov spaces. Furthermore, we show embedding relations of traces of these subspaces with respect to the real axes.

The Shearlet group and the continuous Shearlet transform. The (full) shearlet group $\mathbb{S}$ is defined to be the set $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ endowed with the group operation $(a,s,t)\,(a',s',t') = (aa', s+s'\sqrt{|a|}, t+S_sA_at')$. A right–invariant and left-invariant

Haar measures of $\mathbb{S}$ is given by $\mu_{\mathbb{S},r} = da/|a|\, ds\, dt$ and $\mu_{\mathbb{S},l} = da/|a|^3\, ds\, dt$, respectively and the modular function of $\mathbb{S}$ by $\triangle(a,s,t) = 1/|a|^2$. For the shearlet group the mapping $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ defined by

$$\pi(a,s,t)\,\psi(x) := |a|^{-\frac{3}{4}}\,\psi\left(\frac{1}{a}\,(x_1 - t_1 - s(x_2 - t_2))\,, \frac{\operatorname{sgn} a}{\sqrt{|a|}}\,(x_2 - t_2)\right) \tag{1}$$

is a unitary representation of $\mathbb{S}$, see [1,2]. With the help of [2] it follows that the unitary representation π defined in (1) is a square-integrable representation of $\mathbb{S}$. The transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^2) \to L_2(\mathbb{S})$ defined by $\mathcal{SH}_\psi f(a,s,t) := \langle f, \psi_{a,s,t}\rangle$ is called Continuous Shearlet Transform.

Shearlet coorbit spaces from Shearlets with compact support. Let w be a positive, real-valued, continuous submultiplicative weight on $\mathbb{S}$. To define our coorbit spaces we need the set $\mathcal{A}_w := \{\psi \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(\psi) = \langle \psi, \pi(\cdot)\psi\rangle \in L_{1,w}\}$ of analyzing vectors, see [3–5, 7]. In the following, we assume that our weight is symmetric with respect to the modular function, i.e., $w(g) = w(g^{-1})\triangle(g^{-1})$. Let $Q_D := [-D,D] \times [-D,D]$. The following theorem shows that $\mathcal{A}_w$ contains shearlets with compact support.

Theorem 1. *Let $\psi(x) \in L_2(\mathbb{R}^2)$ fulfill $\operatorname{supp}\psi \in Q_D$. Suppose that the weight function satisfies $w(a,s,t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ for $\rho_1, \rho_2 > 0$ and that*

$$|\hat{\psi}(\omega_1,\omega_2)| \le C\frac{|\omega_1|^n}{(1+|\omega_1|)^r}\,\frac{1}{(1+|\omega_2|)^r} \tag{2}$$

with $n \ge \max(\frac{1}{4}+\rho_2, \frac{9}{4}+\rho_1)$ and $r > n + \max(\frac{7}{4}+\rho_2, \frac{9}{4}+\rho_1)$. Then we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$.

For an analyzing ψ we can consider

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}_{1,w}^\sim$. As the inner product on $L_2(\mathbb{R}^2) \times L_2(\mathbb{R}^2)$ extends to a sesquilinear form on $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$, the extended representation coefficients $\mathcal{SH}_\psi(f)(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}$ are well-defined. Let m be a w-moderate weight on $\mathbb{S}$, i.e. $m(xyz) \le w(x)m(y)w(z)$ for all $x,y,z \in \mathbb{S}$. Then we can define the called shearlet coorbit spaces

$$\mathcal{SC}_{p,m} := \{f \in \mathcal{H}_{1,w}^\sim : \mathcal{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\}, \quad \|f\|_{\mathcal{SC}_{p,m}} := \|\mathcal{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}. \tag{3}$$

Atomic decompositions and Shearlet Banach frames. To construct atomic decompositions and Banach frames the subset $\mathcal{B}_w$ of $\mathcal{A}_w$,

$$\mathcal{B}_w := \{\psi \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(\psi) \in \mathcal{W}(C_0, L_{1,w})\}$$

has to be non-empty. Here $\mathcal{W}(C_0, L_{1,w}) := \{F : \|(L_x\chi_{\mathcal{Q}})F\|_\infty \in L_{1,w}\}$ and $\mathcal{Q}$ is a relatively compact neighborhood of the identity element in $\mathbb{S}$, see [7]. It can be shown, for a classes of weights w sufficiently smooth and compactly supported $\psi(x) \in L_2(\mathbb{R}^2)$ belong to $\mathcal{B}_w$. As it was shown in [2] that for $\alpha > 1$ and $\sigma, \tau > 0$ the set $X := \{(\epsilon\alpha^{-j}, \sigma\alpha^{-j/2}k, S_{\sigma\alpha^{-j/2}k}A_{\alpha^{-j}}\tau l) : j \in \mathbb{Z}, k \in \mathbb{Z}, l \in \mathbb{Z}^2, \epsilon \in \{-1,1\}\}$ forms a U-dense and relatively separated family, we can deduce by Theorem 3.1

and 3.2 in [2] that we can establish atomic decompositions and Banach frames for the shearlet coorbit spaces.

Structure of Shearlet Coorbit Spaces. We now establish relations between scales of Shearlet coorbit spaces and relations to Besov spaces. To establish relations to Besov spaces we apply the characterization of homogeneous Besov spaces $B^{\sigma}_{p,q}$ from [6], see also [8,10]. For inhomogeneous Besov spaces we refer to [9]. The full analysis is restricted to weights $m(a,s,t) = m(a) := |a|^{-r}, r \geq 0$, suggesting to use the abbreviation $\mathcal{SC}_{p,r} := \mathcal{SC}_{p,m}$. For simplicity, we further assume that we can use $\sigma = \tau = 1$ in the U-dense, relatively separated set X and restrict ourselves to the case $\epsilon = 1$. Therefore, we assume that $f \in \mathcal{SC}_{p,r}$ can be written as

$$(4) \quad f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2).$$

To derive reasonable trace and embedding theorems, it is necessary to introduce the following subspaces of $\mathcal{SC}_{p,r}$. For fixed $\psi \in B_w$ we denote by $\mathcal{SCC}_{p,r}$ be the closed subspace of $\mathcal{SC}_{p,r}$ consisting of those functions which are representable as in (4) but with integers $|k| \leq \alpha^{j/2}$. As we shall see in the sequel for each of these ψ the resulting spaces $\mathcal{SCC}_{p,r}$ embed in the same scale of Besov spaces, and the same holds true for the trace theorems.

In most of the classical smoothness spaces like Sobolev and Besov spaces dense subsets of 'nice' functions can be identified.

Theorem 2. *Let* $\mathcal{S}_0 := \{f \in \mathcal{S} : |\hat{f}(\omega)| \leq \omega_1^{2\alpha}(1+\|\omega\|^2)^{-2\alpha}\ \forall\ \alpha > 0\}$ *and* $m(a,s,t) = m(a,s) := |a|^r(1/|a| + |a| + |s|)^n$ *for some* $r \in \mathbb{R}, n \geq 0$. *Then the set of Schwartz functions forms a dense subset of the shearlet coorbit space* $\mathcal{SC}_{p,m}$.

We now investigate the traces of functions lying in $\mathcal{SCC}_{p,r}$ with respect to the horizontal and vertical axes, respectively.

Theorem 3. *Let* $Tr_h f$ *denote the restriction of* f *to the (horizontal)* x_1*-axis, i.e.,* $(Tr_h f)(x_1) := f(x_1, 0)$. *Then* $Tr_h(\mathcal{SCC}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$, *where*

$$B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R}) := \{h \mid h = h_1 + h_2, h_1 \in B^{\sigma_1}_{p,p}(\mathbb{R}), h_2 \in B^{\sigma_2}_{p,p}(\mathbb{R})\}$$

and the parameters σ_1 *and* σ_2 *satisfy the conditions* $\sigma_1 = r - \frac{5}{4} + \frac{3}{2p}$, $\sigma_2 = r - \frac{3}{4} + \frac{1}{p}$.

Corollary 4. *For* $p = 1$, *the embedding* $Tr_h(\mathcal{SC}_{1,r}) \subset B^{\sigma}_{1,1}(\mathbb{R})$ *with* $\sigma = r - \frac{3}{4} + \frac{1}{p}$ *holds true.*

Theorem 5. *Let* $Tr_v f$ *denote the restriction of* f *to the (vertical)* x_2*-axis, i.e.,* $(Tr_v f)(x_2) := f(0, x_2)$. *Then the embedding* $Tr_v(\mathcal{SCC}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$, *holds true, where* σ_1 *is the largest number such that* $\sigma_1 + \lfloor\sigma_1\rfloor \leq 2r - \frac{9}{2} + \frac{3}{p}$, *and* $\sigma_2 = 2r - \frac{3}{2} + \frac{1}{p}$.

We turn now to embedding results.

Corollary 6. *For* $1 \leq p_1 \leq p_2 \leq \infty$ *the embedding* $\mathcal{SC}_{p_1,r} \subset \mathcal{SC}_{p_2,r}$ *holds true. Introducing the 'smoothness spaces'* $\mathcal{G}^r_p := \mathcal{SC}_{p,r+d(\frac{1}{2}-\frac{1}{p})}$. *This implies the continuous embedding* $\mathcal{G}^{r_1}_{p_1} \subset \mathcal{G}^{r_2}_{p_2}$, *if* $r_1 - \frac{d}{p_1} = r_2 - \frac{d}{p_2}$.

Theorem 7. *The embedding $\mathcal{SCC}_{p,r} \subset B^{\sigma_1}_{p,p}(\mathbb{R}^2) + B^{\sigma_2}_{p,p}(\mathbb{R}^2)$, holds true, where σ_1 is the largest number such that $\sigma_1 + \lfloor \sigma_1 \rfloor \leq 2r - \frac{9}{2} + \frac{4}{p}$, and $\sigma_2 - \frac{\lfloor \sigma_2 \rfloor}{2} = r + \frac{3}{2p} + \frac{1}{4}$.*

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Generalized sampling and infinite-dimensional compressed sensing

Anders C. Hansen

(joint work with Ben Adcock)

We will discuss a generalization of the Shannon Sampling Theorem that allows for reconstruction of signals in arbitrary bases in a completely stable way. When extra information is available, such as sparsity or compressibility of the signal in a particular bases, one may reduce the number of samples dramatically. This is done via Compressed Sensing techniques, however, the usual finite-dimensional framework is not sufficient. To overcome this obstacle I'll introduce the concept of Infinite-Dimensional Compressed Sensing.

The well known Shannon Sampling Theorem states that if

$$f = \mathcal{F}g, \qquad g \in L^2(\mathbb{R}),$$

(note that $\mathcal{F}$ is the Fourier Transform) and $\mathrm{supp}(g) \subset [-T, T]$ for some $T > 0$, then both f and g can be reconstructed from point samples of f. In particular, if $\epsilon \leq \frac{1}{2T}$ (the Nyquist rate) then

$$f(t) = \sum_{k=-\infty}^{\infty} f(k\epsilon)\mathrm{sinc}\left(\frac{t + k\epsilon}{\epsilon}\right), \quad L^2 \text{ and unif. conv.,} \tag{1}$$

$$g = \epsilon \sum_{k=-\infty}^{\infty} f(k\epsilon) e^{2\pi \mathrm{i}\epsilon k \cdot}, \qquad L^2 \text{ convergence.} \tag{2}$$

In practice, one cannot process nor acquire the infinite amount of information $\{f(k\epsilon)\}_{k\in\mathbb{Z}}$ that is needed to fully reconstruct f and g and thus one must resort to forming, for some $N \in \mathbb{N}$, the approximations

$$f_N = \sum_{k=-N}^{N} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \qquad g_N = \epsilon \sum_{k=-N}^{N} f(k\epsilon) e^{2\pi \mathrm{i}\epsilon k \cdot}.$$

The question on how well these functions approximate f and g is related to the speed of convergence of the series in (1) and (2). Which again is related to how suitable the functions $\{\mathrm{sinc}\,((\cdot + k\epsilon)/(\epsilon))\}_{k\in\mathbb{Z}}$ and $\{e^{2\pi \mathrm{i}\epsilon k \cdot}\}_{k\in\mathbb{Z}}$ are in series expansions of f and g. In particular, there may be L^2 functions $\{\varphi_k\}_{k\in\mathbb{N}}$ and coefficients $\{\beta_k\}_{k\in\mathbb{N}}$ such that the series

$$f = \sum_{k\in\mathbb{N}} \beta_k \mathcal{F}\varphi_k, \qquad g = \sum_{k\in\mathbb{N}} \beta_k \varphi_k$$

converge faster than the series in (1). There are therefore two important questions to ask:

(i) Can one obtain the coefficients $\{\beta_k\}_{k\in\mathbb{N}}$ (or at least approximations to them) in a stable manner, based on the same sampling information $\{f(k\epsilon)\}_{k\in\mathbb{N}}$, and will this yield better approximations to f and g?

(ii) Can one subsample from $\{f(\epsilon k)\}_{k\in\mathbb{N}}$ (e.g. not sampling at the Nyquist rate) and still get recovery of $\{\beta_k\}_{k\in\mathbb{N}}$ and hence f and g?

The final answer to the first question YES! and can be summarized in the following generalization of the Shannon Sampling Theorem below.

The answer to the second question is also YES! (given some extra requirements on the signals f and g). This is done via the concept of Infinite-Dimensional Compressed Sensing.

Theorem 1. *Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R}^d)$. Suppose that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal set in $L^2(\mathbb{R}^d)$ such that there exists a $T > 0$ with $\mathrm{supp}(\varphi_j) \subset [-T,T]^d$ for all $j \in \mathbb{N}$. For $\epsilon > 0$, let $\rho : \mathbb{N} \to (\epsilon\mathbb{Z})^d$ be a bijection. Define the infinite matrix*

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = (\mathcal{F}\varphi_j)(\rho(i)). \tag{3}$$

Then, for $\epsilon \leq \frac{1}{2T}$, we have that $\epsilon^{d/2}U$ is an isometry. Also, set

$$f = \mathcal{F}g, \quad g = \sum_{j=1}^{\infty} \beta_j \varphi_j \in L^2(\mathbb{R}^N),$$

and let (for $l \in \mathbb{N}$)P_l denote the projection onto $\mathrm{span}\{e_1, \ldots, e_l\}$*. Then, for every $K \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for all $N \geq n$, the solution to*

$$
(4) \qquad A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \vdots \\ \tilde{\beta}_K \end{pmatrix} = P_K U^* P_N \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ f(\rho(3)) \\ \vdots \end{pmatrix}, \; A = P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})},
$$

is unique. If

$$
\tilde{g}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \varphi_j, \qquad \tilde{f}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \mathcal{F}\varphi_j,
$$

then

$$
\|g - \tilde{g}_{K,N}\|_{L^2(\mathbb{R}^d)} \leq (1 + C_{K,N}) \|P_K^{\perp}\beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \ldots\},
$$

and

$$
\|f - \tilde{f}_{K,N}\|_{L^\infty(\mathbb{R}^d)} \leq (2T)^{d/2}(1 + C_{K,N}) \|P_K^{\perp}\beta\|_{l^2(\mathbb{N})},
$$

where, for fixed K, the constant $C_{K,N} \to 0$ as $N \to \infty$.

The results can be found in [1–3], and the ideas stem from [4].

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An infinite dimensional extension of the Schur-Horn theorem for operators with finite spectrum

Marcin Bownik

(joint work with John Jasper)

The purpose of this talk is to outline a recent progress in extending the Schur-Horn theorem for operators on an infinite dimensional Hilbert space with finite spectrum. That is, we are interested in giving necessary and sufficient conditions for a sequence $\{d_i\}$ to be the diagonal of a self-adjoint operator with specified list of eigenvalues. The classical Schur-Horn theorem [12, 19] can be formulated as follows.

Theorem 1. *Let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ be real sequences with nonincreasing order. There exists $N \times N$ hermitian matrix with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$ $\Longleftrightarrow$*

$$\forall\, n = 1, \ldots, N, \ \sum_{i=1}^{n} d_i \le \sum_{i=1}^{n} \lambda_i \qquad \textit{and} \qquad \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} d_i.$$

This line of research is motivated partly by the frame theory, where the problem of characterizing norms of frames with prescribed frame operator attracted a significant number of researchers, see [6–9, 17]. Antezana, Massey, Ruiz, and Stojanoff [1] established the connection of this problem with infinite dimensional Schur-Horn problem and gave refined necessary conditions and sufficient conditions. Indeed, we have the following extension of the well-known dilation theorem for Parseval frames due to Han and Larson [11].

Theorem 2. *$\{f_i\}_{i\in I}$ is a frame on a Hilbert space $\mathcal{H}$ with a frame operator S $\Longleftrightarrow$ there exists an orthonormal basis $\{e_i\}_{i\in I}$ of some larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $f_i = Ee_i$, where $E = S^{1/2} \bigoplus \mathbf{0}$ and the zero operator $\mathbf{0}$ acts on $\mathcal{K} \ominus \mathcal{H}$.*

In the special case of Parseval frames $\{f_i\}$, the frame operator $S = Id$. Thus, a sequence of frame norms $\{||f_i||^2\}$ corresponds to a diagonal of a certain orthogonal projection. In this case, Kadison [14, 15] gave the complete characterization of sequences which are diagonals of projections.

Theorem 3. *Let $\{d_i\}_{i\in I}$ be a sequence in $[0,1]$ and $\alpha \in (0,1)$. Define*

$$a = \sum_{d_i < \alpha} d_i, \qquad b = \sum_{d_i \ge \alpha} (1 - d_i).$$

There is a projection with diagonal $\{d_i\}_{i\in I}$ if and only if $a = \infty$, or $b = \infty$, or both $a, b < \infty$ and $a - b \in \mathbb{Z}$.

Kadison's theorem can be considered as an infinite dimensional extension of the Schur-Horn Theorem for operators with two points in the spectrum. It falls into a broader category of research that aims at finding an analogue of the Schur-Horn theorem for operators on an infinite dimensional Hilbert space. Recently there has been a great deal of progress by a number of authors. The work of Gohberg and Markus [10] and Arveson and Kadison [4] extended the Schur-Horn theorem to positive trace class operators. More recently Kaftal and Weiss [16] have extended this to all positive compact operators. Other work in this area includes the study of II_1 factors by Argerami and Massey [2] and normal operators by Arveson [3]. Neumann [18] proved what may be considered an approximate Schur-Horn theorem since it is given in terms of the ℓ^∞-closure of the set of diagonal sequences. Bownik and Jasper [5] established a variant of the Schur-Horn theorem for the set of locally invertible positive operators.

Theorem 4. *Let $0 < A < B < \infty$ and $\{d_i\}_{i\in I}$ be a nonsummable sequence in $[0, B]$. Define*

$$C = \sum_{d_i < A} d_i \quad \textit{and} \quad D = \sum_{d_i \ge A} \big(B - d_i\big). \tag{1}$$

Then, there is a positive operator E on a Hilbert space $\mathcal{H}$ with $\{A,B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A,B]$ and diagonal $\{d_i\}$ $\iff$ one of the following holds: (i) $C=\infty$, (ii) $D=\infty$, (iii) both $C,D<\infty$ and there exists $n \in \mathbb{N} \cup \{0\}$ such that

$$nA \leq C \leq A + B(n-1) + D.$$

The natural next step after Kadison's theorem is to consider operators with three points in the spectrum. Jasper [13] has recently shown the following result.

Theorem 5. *Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a sequence in $[0,B]$ with $\sum d_i = \sum (B-d_i) = \infty$. Define*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a self-adjoint operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$ $\iff$ one of the following holds: (i) $C=\infty$, (ii) $D=\infty$, or (iii) both $C,D<\infty$ and there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$C - D = NA + kB \qquad \text{and} \qquad C \geq (N+k)A.$$

One should add that the assumption that $\sum d_i = \sum (B-d_i) = \infty$ is not a true limitation. Indeed, the summable case $\sum d_i < \infty$ requires more restrictive conditions which can be deduced from a more precise variant of Jasper's theorem [13]. In fact, the main result in [13] gives a complete list of characterization conditions of diagonals of operators with prescribed multiplicities.

In the final part of the talk we describe the current joint work with Jasper on characterizing diagonals of operators with finite spectrum. It turns out that the key role is played by the two extreme eigenvalues with infinite multiplicity. The full characterization result involves 3 ingredients:

(1) the lower exterior majorization of diagonal terms below the smallest eigenvalue with infinite multiplicity,
(2) the upper exterior majorization above the largest eigenvalue with infinite multiplicity, and
(3) the interior majorization for diagonal terms lying between those two extreme infinite multiplicity eigenvalues.

The last condition involves both the trace condition and majorization inequalities similar as in Theorem 5. Finally, an interesting feature of the interior majorization is the fact all of these numerical conditions disappear completely in the case of more than two eigenvalues with infinite multiplicity.

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Spanning and independence properties of frame partitions

Bernhard G. Bodmann and Darren Speegle
(joint work with Pete G. Casazza and Vern I. Paulsen)

Introduction. Over the last decades, frame theory has developed into a vibrant subject including contributions in time-frequency analysis [7–10, 12, 18] and applications in engineering such as wireless communications or other types of signal and image processing techniques, see the survey papers [14, 15] and the many references therein. In pure mathematics, frame theory has opened up new approaches to one of the significant open problems in analysis today - the notoriously intractable 1959 Kadison-Singer Problem [2, 5, 6].

Formally, a *frame* is a family of vectors $\{f_i\}_{i\in I}$ in a real or complex Hilbert space $\mathbb{H}$ so that there are constants $0 < A \leq B < \infty$ (called the lower and upper frame bounds, respectively) satisfying

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{H}. \tag{1}$$

We refer the reader to [7] for the requisite background in frame theory.

What makes frames so useful in practice is their *redundancy.* That is, in general a frame might have smaller subsets, each with a dense linear span in the space $\mathbb{H}$. The flexibility in choosing representations is key to many applications [14, 15]. One of the most important problems in frame theory today is therefore to understand redundancy and its role in these applications. This includes a long list of fundamental questions concerning the behavior of subsets of a frame such as how many (disjoint) spanning sets does a frame contain? Or, how many (disjoint) linearly independent subsets does it contain? Or, putting these together, how many disjoint linearly independent spanning sets does our frame contain? Can we partition any unit norm frame into a finite number of subsets with specialized properties such as each subset being *nearly tight* for its span, i.e. with $\frac{B}{A} \approx 1$? This innocent looking problem in frame theory is now known to be equivalent to the 1959 Kadison-Singer Problem [3, 6].

One of the first results concerning the decomposition of frames into linearly independent sets is in [2] where it is shown that a frame can be partitioned into $\lceil B\rceil$-linearly independent sets. In [6] it is shown that a unit norm tight frame $\{f_i\}_{i=1}^{KM}$ in an M-dimensional space can be partitioned into K linearly independent spanning sets. It has been an open question since [6] appeared whether there is a similar result for unit norm frames $\{f_i\}_{i=1}^{KM+r}$ for $0 < r < M$. This is one of the questions we will answer in this paper. Recently, an intuitive, quantitative measure for redundancy was given for finite frames [1]. It is interesting to note that it relates to independence and spanning properties of frame partitions which we explore here. We expect that such a systematic refinement of our understanding of redundancy will have an important impact on applications.

We will concentrate on Parseval frames here since with respect to independence and spanning properties, this is really the general case in the sense that every frame $\{f_i\}_{i\in I}$ is isomorphic to the Parseval frame $\{S^{-1/2}f_i\}_{i\in I}$, where S is the frame operator. That is, $S^{-1/2}$ is an invertible operator mapping our frame to a Parseval frame and hence it maintains linearly independence properties, spanning properties and Riesz basic sequences. Many of our results rely on the assumption that the norms of the Parseval frame vectors are uniformly bounded away from 1. This is a necessary assumption since for a Parseval frame $\{f_i\}_{i\in I}$, Equation 1 quickly yields that if $\|f_j\| = 1$, then $f_j \perp \mathit{span}\ \{f_i\}_{i\neq j}$.

Results. First, we establish a dichotomy between independence properties of subsets of a finite Parseval frame and spanning properties of complementary subsets of its Naimark complement. This gives a new approach to the Kadison-Singer Problem which is complementary to the standard equivalences of the problem.

Proposition 1. *Let* $\mathbb{H}$ *be a Hilbert space with orthonormal basis* $\{e_j\}_{j\in S}$, *let* P *be the orthogonal projection onto a closed subspace of* $\mathbb{H}$, *and let* $B \subseteq S$. *Then* *the linear span of* $\{Pe_j\}_{j\in B}$ *is dense in* $P(\mathbb{H})$ *if and only if the operator* $(\langle (I-P)e_j,(I-P)e_i\rangle)_{i,j\in B^c}$ *on* $\ell^2(B^c)$ *is one-to-one.*

Next, we apply this result to show that given a Parseval frame $\{f_n\}_{n\in\mathbb{N}}$ with $\|f_n\|^2 \le 1-\delta$ we can partition $\mathbb{N}$ into r-subsets (r only depending upon δ) $\{A_j\}_{j=1}^r$ so that span $\{f_n\}_{n\in A_k^c} = \mathbb{H}$, for all $1\le k\le r$.

Theorem 2. *Let* $0<\delta<1$, *and set* $r = 2\lceil\frac{1}{\delta}\rceil$. *If* $\{f_n\}_{n\in\mathbb{N}}$ *is a Parseval frame for a Hilbert space* $\mathbb{H}$, *with* $\|f_n\|^2\le 1-\delta$ *for all* $n\in\mathbb{N}$, *then there exists a partition of* $\mathbb{N}$ *into* r *disjoint sets,* $A_1\cup\cdots\cup A_r = \mathbb{N}$, *such that* $\mathbb{H}_{A_k^c} = \mathbb{H}$, *for* $k=1,\ldots,r$.

Until now, we have considered the problem of partitioning Parseval frames into spanning sets or linearly independent sets. Now we will examine the much deeper problem of partitioning a Parseval frame into linearly independent spanning sets. This is a fundamental problem in the field which until now has had only one case that has been answered. Namely, in [6] it is shown that a equal norm Parseval frame $\{f_i\}_{i=1}^{KM}$ for $\mathbb{H}_M$ can be partitioned into K-linearly independent spanning sets. This proof relies on the Rado-Horn Theorem and to prove our result, we have to first strengthen the Rado-Horn Theorem itself.

Theorem 3. *Let* $\{f_i\}_{i\in I}$ *be an equal norm Parseval frame for an* N *dimensional Hilbert space* $\mathbb{H}_N$ *with* $|I| = rN+k$ *with* $0\le k<N$. *Then there is a partition* $\{I_i\}_{i=1}^{r+1}$ *of* I *so that for* $i\in\{2,\ldots,r+1\}$, $\{f_j\}_{j\in I_i}$ *is a linearly independent spanning set and* $\{f_j\}_{j\in I_1}$ *is linearly independent.*

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The Bourgain Tzafriri restricted invertibility theorem in infinite dimensions

Götz E. Pfander
(joint work with Peter G. Casazza)

We give a generalization of the Bourgain Tzafriri restricted invertibility theorem [3] to infinite dimensional Hilbert spaces. Our approach is based on notions of *localized frames* and *density* similar to those used in the work of Balan, Casazza, Heil, and Landau [1, 2], respectively Gröchenig [5]. Here, we streamline the notion of density to avoid having to specify an index map for a frame localized with respect to a (possibly overcomplete) reference system. Our work is available in detail as preprint [4].

Restricted invertibility refers to restricting a non injective real or complex valued matrix to a subspace of its domain so that the restricted map becomes injective. Clearly, we can always restrict to a subspace of dimension being the rank of the matrix. The strength of the restricted invertibility theorem is that it allows to select a (generally smaller) subspace while controlling the operator norm of the left inverse of the matrix. In this extended abstract, we will use an equivalent formulation of restricted invertibility based on terms from frame theory. Then, given a family of vectors in a finite / infinite dimensional Hilbert space, the goal is to choose a large subset from that family so that the subset forms a Riesz basis with sufficiently large lower Riesz bound.

Recall that $\mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{H}$ is *Bessel* with *Bessel bound* $0 < B < \infty$ if

$$\|\sum_{i\in I} a_i f_i\|^2 \leq B \sum_{i\in I} |a_i|^2, \quad \{a_i\}_{i\in I},$$

it is a *Riesz sequence* with *Riesz bounds* $0<A\leq B<\infty$ if

$$A \sum_{i\in I} |a_i|^2 \leq \|\sum_{i\in I} a_i f_i\|^2 \leq B \sum_{i\in I} |a_i|^2, \quad \{a_i\}_{i\in I}$$

The restricted invertibility theorem in finite dimensions has recently been reproven (with improved constants) by Spielman, Srivastava [6].

Theorem 1. *For every $n \in \mathbb{N}$ and set of n vectors f_i in $\mathbb{C}^n$ with Bessel bound B and $\|f_i\| = 1$ for all i, exist subsets $J_\alpha \subseteq \{1, 2, \ldots, n\}$, $\alpha \in (0, 1)$, satisfying*

(1) $|J_\alpha| \geq \alpha^2 \dfrac{n}{B}$, *and*

(2) For all $\{b_j\}_{j \in J_\alpha}$ we have $(1-\alpha)^2 \sum_{j \in J_\alpha} |b_j|^2 \leq \| \sum_{j \in J_\alpha} b_j \, f_j \|^2.$

Simple examples illustrating the appearance of n/B in (1) include the cases (i) $\{f_i\}_{i=1,\ldots,n}$ is an orthonormal basis, (ii) all f_i are identical, or (iii) $\{f_i\}_{i=1,\ldots,n}$ is the union of two identical orthonormal bases. Also consider (iv) $\{f_i\}_{i=1,\ldots,n-1}$ is an orthonormal set and $f_n = f_1$.

To generalize the finite dimensional restricted invertibility theorem we must establish a notion of density which is based on a coordinate / reference system. We aim to replace (1) by

$$\text{(1')} \qquad \frac{\text{Density of } J_\alpha \text{ with respect to a coordinate system } \mathcal{G}}{\text{Density of } I \text{ with respect to a coordinate system } \mathcal{G}} \geq \frac{\alpha^2}{\|T\|^2}.$$

The herein proposed notions of localization and density are stated in loose terms. Similarly to the concept of Beurling density, a precise statement of the proposed notion of density must employ lim sup and lim inf. The definition below coincides with the precise definition in canonical examples.

Definition 2. The *density* of $\mathcal{F}$ with respect to $\mathcal{G}$ is given by

$$D(\mathcal{F}; \mathcal{G}) \quad = \quad \lim_{R \to \infty} \frac{\sum_{f \in \mathcal{F}} a_f \sum_{n \in B_R} |\langle f, g_n \rangle|^2}{|B_R|},$$

where $a_f = \left(\sum_{n \in G} |\langle f, g_n \rangle|^2 \right)^{-1}$, $f \in \mathcal{F}$.
(If $\mathcal{G}$ is a Parseval frame, then all $a_f = 1$.)

Definition 3. For $0 < D(\mathcal{F}; \mathcal{G}) < \infty$, the *relative density* of $\mathcal{F}' \subseteq \mathcal{F}$ with respect to $\mathcal{G}$ is

$$R(\mathcal{F}', \mathcal{F}; \mathcal{G}) = D(\mathcal{F}'; \mathcal{G}) \; / \; D(\mathcal{F}; \mathcal{G}).$$

Definition 4. $\mathcal{F}$ is ℓ_1-*localized* with respect to $\mathcal{G} = \{g_k\}_{k \in G}$ if there exists a sequence $r \in \ell_1(G)$ so that for all $f \in \mathcal{F}$ there is $k \in G$ with $|\langle f, g_n \rangle| \leq r(n-k)$, $n \in G$.

These concepts lead to the following infinite dimensional version of the Bourgain-Tzafriri restricted invertibility theorem.

Theorem 5. *Let $\mathcal{F}$ be ℓ_1-localized with respect to the frame $\mathcal{G}$ and assume that $\|f\| \geq u$, $f \in \mathcal{F}$, and $\mathcal{F}$ Bessel with bound $B_{\mathcal{F}}$. Assume*
$\mathcal{G} = \{g_k : k \in G\}$ is a frame for $\mathcal{H}$ with ℓ_1- self-localized dual frame
If $(\mathcal{F}; \mathcal{G})$ is ℓ_1-localized, then for every $\alpha \in (0, 1)$ and $\delta > 0$ there is a subset $\mathcal{F}_{\alpha\delta} \subseteq \mathcal{F}$ with

(1) $R(\mathcal{F}_{\alpha\delta}, \mathcal{F}; \mathcal{G}) \geq \dfrac{\alpha^2 u^2}{B_{\mathcal{F}}}$,

(2) *$\mathcal{F}_{\alpha\delta}$ is a Riesz sequence with Riesz bounds* $(1-\alpha)^2(1-\delta)\, u^2$, $B_{\mathcal{F}}$.

We will now present an application of the infinite dimensional restricted invertibility theorem to Gabor analysis.

For $\lambda{=}(y,\omega){\in}\mathbb{R}^{2d}$, $\varphi{\in}L^2(\mathbb{R}^d)$, set $\pi(\lambda)\varphi(x){=}\pi(y,\omega)\varphi{=}e^{2\pi i x\omega}\varphi(x{-}y)$ and consider the *Gabor system* $(\varphi,\Lambda) = \{\pi(\lambda)\varphi\}_{\lambda\in\Lambda}$. In this case, relative density of $\Lambda_{\alpha\delta}$ as subset of $\Lambda \subseteq \mathbb{R}^2$ can be expressed by $D(\Lambda_{\alpha\delta})/D(\Lambda)$ where D denotes the classical Beurling density. Note that in order to obtain ℓ^1-localization of the Gabor system with respect to a reference system, we must choose the reference system to be overcomplete. In this sense, we are using a redundant coordinate system, as permitted by the infinite dimensional version of the restricted invertibility theorem.

Theorem 6. *Let $\alpha \in (0,1)$, $\delta > 0$. Let $\varphi \in S_0(\mathbb{R})$ and let the Gabor system (φ,Λ) have Bessel bound $B < \infty$. Then exists $\Lambda_{\alpha\delta} \subseteq \Lambda$ with uniform density and*

(1) $\dfrac{D(\Lambda_{\alpha\delta})}{D(\Lambda)} \geq \dfrac{\alpha^2}{B}\|\varphi\|^2$,

(2) For all $\{b_\lambda\}_{\lambda\in\Lambda} \in \ell_2(\Lambda)$,

$$(1-\alpha)^2(1-\delta)\|\varphi\| \sum_{\lambda\in\Lambda_{\alpha\delta}} |b_\lambda|^2 \leq \| \sum_{\lambda\in\Lambda_{\alpha\delta}} b_\lambda \pi(\lambda)\varphi\|^2 \leq B \sum_{\lambda\in\Lambda_{\alpha\delta}} |b_\lambda|^2$$

If (φ,Λ) is a tight frame, then $D(\Lambda) = \frac{B}{\|\varphi\|^2}$, and for $\alpha < 1$ exists $\Lambda_{\alpha\delta}$ with

(1) $D(\Lambda_{\alpha\delta}) \geq \alpha^2$.

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Sampling of band-limited functions for Gelfand pairs

Jens Gerlach Christensen
(joint work with Azita Mayeli and Gestur Ólafsson)

We recall the definition of a Gelfand pairs (G, K) and the basic facts about the Fourier transform on the associated commutative space G/K. Details and more discussion can be found in [5]. We then derive sampling results for band-limited functions on G/K.

Let G be a Lie group and $K \subset G$ a compact subgroup. Denote by ℓ the left regular representation $\ell(a)f(x) = f(a^{-1}x)$ Since K is compact we identify functions on G/K with right invariant functions on G such that $L^p(G/K) \subseteq L^p(G)$. Define convolution of functions by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dx$$

whenever it makes sense. The left K-invariant and integrable functions are denoted by $L^1(G/K)^K$ and forms an algebra under convolution. (G, K) is a Gelfand pair if $L^1(G/K)^K$ is commutative. In this case we also say that G/K is a commutative space. Equivalently (G, K) is a Gelfand pair if the algebra of left invariant differential operators $\mathbb{D}(G/K)$ is commutative. If G/K is a commutative space, then there exists a measurable set $\Lambda \subseteq \widehat{G}$, where $\widehat{G}$ is the unitary dual of G, such that

$$(\ell, L^2(G/K)) \simeq \int_\Lambda^\oplus (\pi_\lambda, \mathcal{H}_\lambda)\, d\mu(\lambda)\,. \tag{1}$$

From this can be defined the *vector valued Fourier transform* for functions $f \in L^1(G/K)$

$$\widehat{f}(\lambda) = \mathcal{F}(f)(\lambda) := \pi_\lambda(f)p_\lambda \tag{2}$$

where $p_\lambda \in \mathcal{H}_\lambda^K$ is a unit vector. Note, if $f \in L^1(G)$ and $g \in L^2(G/K)$ then

$$\mathcal{F}(f * g)(\lambda) = \pi_\lambda(f)\widehat{g}(\lambda)\,.$$

For $f \in C_c^\infty(G/K)$ it follows that

$$\|f\|_2^2 = \int \|\widehat{f}(\lambda)\|_{\mathcal{H}_\lambda}^2\, d\mu(\lambda)\,.$$

and

$$f(x) = \int (\widehat{f}(\lambda), \pi_\lambda(x)p_\lambda)_{\mathcal{H}_\lambda}\, d\mu(\lambda)$$

Hence, the vector valued Fourier transform extends to an unitary isomorphism

$$L^2(G/K) = \int^\oplus (\pi_\lambda, \mathcal{H}_\lambda)\, d\mu(\lambda)\,.$$

The vectors p_λ are eigenvectors for $\pi_\lambda(D)$ when $D \in \mathbb{D}(G/K)$. By [2] the topology on the spectrum Λ is equivalent to the Euclidean topology on the eigenvalues of

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left invariant differential operators. Let $\Omega \subseteq \Lambda$ be compact. A continuous function $f \in L^2(G/K)$ is Ω-band-limited if $\text{supp}(\widehat{f}) \subseteq \Omega$ and the space of such functions is denoted $PW(\Omega)$. The eigenvalues of $\pi_\lambda(\Delta)$, where Δ is the Laplacian on G/K, are uniformly bounded for $\lambda \in \Omega$ and thus there is a constant $C(\Omega)$ such that

$$\|(I-\Delta)^n f\|_{L^2} \leq C(\Omega)^n \|f\|_L^2 \tag{3}$$

for functions $f \in PW(\Omega)$.

Let (G, K) be a Gelfand pair and fix an $Ad(K)$-invariant Riemannian metric on G. Fix an orthonormal basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. We investigate how functions in Paley-Wiener space vary inside neighborhoods of the form

$$U_\epsilon = \{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_n X_n} \mid -\epsilon \leq t_k \leq \epsilon \text{ for all } 1 \leq k \leq n\}$$

and find that the local oscillations

$$\text{osc}_\epsilon(f)(x) = \sup_{u \in U_\epsilon} |f(x) - f(xu^{-1})|$$

have norm estimated by

$$\|\text{osc}_\epsilon(f)\|_{L^2(G)}^2 \leq C_\epsilon \sum_{k=0}^{n} \|(I-\Delta)^k f\|_{L^2(G)} \|f\|_{L^2(G)} \leq C_\epsilon \|f\|_{L^2(G)}^2$$

for $f \in PW(\Omega)$. The constant C_ϵ tends to zero as $\epsilon \to 0$ and in the last step we used the Bernstein inequality (3).

Let ϕ be the continuous representative in $L^2(G/K)^K$ for which

$$\widehat{\phi}(\lambda) = 1_\Omega(\lambda) p_\lambda.$$

Along the same lines as [1, 3, 4] we obtain the following sampling results

Main theorem. *Let $\Omega \subseteq \Lambda$ be a compact set, then it is possible to choose ϵ small enough that for any U_ϵ-relatively separated family x_i*

- *the functions $\ell_{x_i}\phi$ form a frame for L^2_Ω,*
- *the operator $T_1 f = \sum_i f(x_i)\psi_i * \phi$ is invertible on $PW(\Omega)$*
- *$T_1^{-1}(\psi_i * \phi)$ is a dual frame for $\ell_{x_i}\phi$.*

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Fourier series on Cantor sets

Dorin Ervin Dutkay

Everybody knows about Fourier series on the interval: the set of functions

$$\{e^{2\pi inx} : n \in \mathbb{Z}\}$$

forms an orthonormal basis for $L^2[0,1]$, where the interval $[0,1]$ is endowed with Lebesgue measure. But how about some other measures? Can one have orthonormal bases of exponentials on some singular measure? In 1998 Jorgensen and Pedersen [3] discovered a surprising example: a wonderful Cantor type set which, when endowed with the appropriate Hausdorff measure, possesses an orthonormal basis of exponential functions. It is not the usual middle-third Cantor set; Jorgensen and Pedersen's example is obtained from the unit interval, dividing it into 4 equal pieces and keeping the first and the third and then repeating the procedure *ad infinitum.* The resulting Cantor set X_4 has Hausdorff dimension $\frac{1}{2}$. Consider μ_4 the Hausdorff measure of dimension $\frac{1}{2}$ restricted to the set X_4. Jorgensen and Pedersen proved that for the following very sparse set

$$\Lambda := \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0,1\}, n \in \mathbb{N} \right\},$$

the set of exponential functions $\{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthonormal basis. Such a measure is called *spectral* and Λ is called a *spectrum.*

Spectral measures appear also in the Fuglede conjecture which states that a the normalized Lebesgue measure on some Borel subset Ω of $\mathbb{R}^d$ is a spectral measure if and only if Ω tiles $\mathbb{R}^n$ by translations. The conjecture was disproved in one direction first by Tao for dimensions $d \geq 5$, then down to dimension $d \geq 3$ in both directions by Matolcsi et. al.

Jorgensen and Pedersen's example was further generalized for various affine iterated function systems or infinite convolution measures, see e.g [1, 2, 4, 5]. For example, consider the affine iterated function system $\tau_b(x) = R^{-1}(x+b)$, $b \in B$, on $\mathbb{R}^d$, where R is an expansive integer matrix and B is a subset of $\mathbb{R}^d$. Let μ_B be the invariant measure associated to this IFS in the sense of Hutchinson. In [2] we proposed the following conjecture:

Suppose there exist a set $L \subset \mathbb{Z}^d$ such that (R, B, L) forms a Hadamard system, i.e. $\#B = \#L$ and the matrix

$$\frac{1}{\sqrt{\#B}} \left(e^{2\pi i R^{-1} b \cdot l} \right)_{b \in B, l \in L}$$

is unitary. Then the measure μ_B is spectral.

The conjecture was proved in dimension 1 [1], for many cases in dimension 2 by Bengt Alrud in his thesis and under a certain "reducibility assumption" for higher dimensions [2].

In their paper [3], Jorgensen and Pedersen also proved that the more familiar middle-third Cantor set with its Hausdorff measure μ_3 of dimension $\frac{\ln 2}{\ln 3}$ is far from being spectral. Actually, no three exponential functions are mutually orthogonal in $L^2(\mu_3)$. Following this Strichartz [5] raised the following question:

Is there a frame of exponentials on the middle-third Cantor set?

The problem turned out to be quite difficult and it still open. Some researchers believe that, because of this complete lack of orthogonality for exponentials on the middle-third Cantor set, this could be a good ground to look for a counterexample to the Feichtinger conjecture.

We were able to construct Bessel sequences and Riesz basis sequences of positive Beurling dimension, but no frames yet.

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Shearlets and representation theory (and the Wick calculus)

Emily J. King

(joint work with Wojtek Czaja)

In what follows a new multi-dimensional transformation which contains directional components and which generalize the concepts of the $L^2(\mathbb{R}^2)$ shearlet transform with isotropic dilations is presented. Some of the properties of this new family of isotropic shearlet transformations and of their associated Lie groups are shown. Under the wavelet representation, these groups are square-integrable over generalized Hardy Spaces. Some new structural results using Calderón-Toeplitz $(\mathrm{TDS})_k$ multiplication operators and the Wick Calculus will be presented.

We begin by fixing our notation. For $f : L^2(\mathbb{R}^k)$, $y \in \mathbb{R}^k$, and $A \in \mathrm{GL}(\mathbb{R}, d)$ define the following (unitary) operators $D_A f(x) = |\det A|^{-1/2} f(A^{-1}x)$ and $T_y f(x) = f(x - y)$. Shearlets were created in order to analyze directional information in 2-dimensional images [9] [13]. The *continuous shearlet group* is the group $\{(y, (S_\ell A_a)^{-1}) : a > 0, \ell \in \mathbb{R}, y \in \mathbb{R}^2\}$, where A_a is the parabolic diagonal dilation matrix with diagonal values a and $\sqrt{a}$ and S_ℓ is the upper triangular matrix which shears by ℓ. The representation ν which maps each $(y, (S_\ell A_a)^{-1})$ to $T_y D_{(S_\ell A_a)^{-1}}$ is the *wavelet representation.* As real-life data sets are more and more commonly 3 dimensions or more, we would like to generalize the concept of shearlets to higher dimensions. We would like this generalization to have an underlying group structure that is *reproducing*; that is, for all $f \in L^2(\mathbb{R}^d)$, there exists a $\phi \in L^2(\mathbb{R}^d)$ such that $f = \int_G \langle f, \nu(g)\phi\rangle \nu(g)\phi dg$ holds weakly where ν is the wavelet representation and dg is a Haar measure for G. We would also like the representation to be *square-integrable*. A unitary representation π mapping a locally compact group G with Haar measure to a Hilbert space $\mathcal{H}$ is called *square-integrable* if it is irreducible and there exists an $f \in \mathcal{H}$ such that $\int_G |\langle f, \pi(g)f\rangle|^2 \, dg < \infty$.

There exist a few generalizations of shearlets to $L^2(\mathbb{R}^k)$ which include traditional shearlets as the case $k = 2$ [3] [5] [6] [11] [12] [10]. One may also generalize shearlets using isotropic dilations, which include traditional wavelets as the case $k = 1$.

Definition 1. Consider the matrix group $M = \{t^{-1}(S_\ell^t) : \ell \in \mathbb{R}^{k-1}, t > 0\}$ where S_ℓ is formally defined to be 1 when $k = 1$ and for $k \geq 2$ is the shearing matrix $\{(a_{ij})_{i,j} : a_{ii} = 1, 1 \leq i \leq k; a_{ik} = \ell_i, 1 \leq i \leq k-1; 0, \text{else}\}$. Set $(\mathrm{TDS})_k = \mathbb{R}^k \ltimes M$.

Theorem 2. $(\mathrm{TDS})_k$ *is a reproducing group under the wavelet representation and square integrable over* $\mathcal{H}_\pm(\mathbb{R}^k) = \{f \in L^2(\mathbb{R}^k) : \operatorname{supp} \hat{f} \subseteq \dot{\mathbb{R}}^k_\pm\}$, *where* $\dot{\mathbb{R}}^k_\pm = \{(x_1, \ldots, x_k) : x_1, \ldots, x_{k-1} \neq 0, \pm x_k > 0\}$.

The only other isotropic generalization in the literature uses a Toeplitz shearing matrix [7] or is only for the case $k = 2$ [1] [2].

In analogy with the structural results found in [8] [14], we have the following results concerning Calderón-Toeplitz $(\mathrm{TDS})_k$ operators and their corresponding Wick Calculus [3]. In what follows, let a be a bounded function on $\dot{\mathbb{R}}^k_+$ and ψ be a reproducing function for the wavelet representation of $(\mathrm{TDS})_k$.

Definition 3. The *Calderón-Toeplitz* $(\mathrm{TDS})_k$ *operator* $\mathcal{T}_a$ acting on $L^2(\mathbb{R}^k)$ is given weakly by

$$\mathcal{T}_a f = \int_{(\mathrm{TDS})_k} a(\ell, t)\langle f, T_y D_{t^{-1}(S_\ell^t)}\psi\rangle T_y D_{t^{-1}(S_\ell^t)}\psi \frac{dt}{t^{k+1}} dy d\ell.$$

Its *Wick symbol* $\tilde{a}$ is defined as

$$\tilde{a}(y, \ell, t) = \frac{\langle \mathcal{T}_a(T_y D_{t^{-1}(S_\ell^t)}\psi), T_y D_{t^{-1}(S_\ell^t)}\psi\rangle}{\langle T_y D_{t^{-1}(S_\ell^t)}\psi, T_y D_{t^{-1}(S_\ell^t)}\psi\rangle}.$$

Theorem 4. *Calderón-Toeplitz* $(\mathrm{TDS})_k$ *operator* $\mathcal{T}_a$ *is unitarily equivalent to a multiplication operator acting on* $L^2(\mathbb{R}^k)$. *Furthermore, the operator* $T_a T_b$ *has Wick symbol*

$$\tilde{a} \star \tilde{b}(t, \ell) = \tilde{c}(t, \ell) = t^k \int_{\mathbb{R}^k} \gamma_a(\xi)\gamma_b(\xi)|\hat{\psi}(tS_{-\ell}\xi)|^2 d\xi.$$

Further work can be done to relate Calderón-Toeplitz $(\mathrm{TDS})_k$ operators to results known for Fourier multipliers. The methods used to prove these results concerning the $(\mathrm{TDS})_k$ operators do not work for $(\mathrm{CSG})_k$ operators.

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New and improved Johnson-Lindenstrauss embeddings via the restricted isometry property

Felix Krahmer

(joint work with Rachel Ward)

The Johnson-Lindenstrauss (JL) Lemma [1] states that any set E of p points in high dimensional Euclidean space can be embedded into $O(\varepsilon^{-2}\log(p))$ dimensions, without distorting the distance between any two points by more than a factor between $1-\varepsilon$ and $1+\varepsilon$. We establish a new connection between the JL Lemma and the Restricted Isometry Property (RIP), a well-known concept in the theory of sparse recovery often used for showing the success of ℓ_1-minimization.

More precisely (see for example [2]), a matrix $\Phi \in \mathbb{R}^{m\times N}$ is said to have the Restricted Isometry Property of order k and level $\delta \in (0,1)$ (equivalently, (k,δ)-RIP) if

$$(1-\delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta)\|x\|_2^2 \qquad \text{for all } k\text{-sparse } x \in \mathbb{R}^N.$$

Here a vector is k-sparse if it has only k non-zero entries.

Our main result is the following.

Theorem 1 ([3]). *Fix $\eta > 0$ and $\varepsilon \in (0,1)$, and consider a finite set $E \subset \mathbb{R}^N$ of cardinality $|E| = p$. Set $k \geq 40 \log \frac{4p}{\eta}$, and suppose that $\Phi \in \mathbb{R}^{m\times N}$ satisfies the Restricted Isometry Property of order k and level $\delta \leq \frac{\varepsilon}{4}$. Let $\xi \in \mathbb{R}^N$ be a Rademacher sequence, i.e., uniformly distributed on $\{-1,1\}^N$. Then with probability exceeding $1-\eta$,*

$$(1-\varepsilon)\|x\|_2^2 \leq \|\Phi D_\xi x\|_2^2 \leq (1+\varepsilon)\|x\|_2^2$$

uniformly for all $x \in E$.

Applying this result to the set of distances $E' = \{x - y : x, y \in E\}$, we obtain that RIP matrices with randomized column signs satisfy the Johnson-Lindenstrauss Lemma. In a sense, this is a converse to a recent result by Baraniuk et al. [4], who show that the Restricted Isometry Property is implied by a concentration inequality closely related to the JL Lemma.

Consequently, matrices satisfying the Restricted Isometry Property of optimal order provide optimal Johnson-Lindenstrauss embeddings up to a logarithmic factor in N. In particular, combining Theorem 1 with results on RIP matrices (see for example [5]) yields the best known bounds on the necessary embedding dimension m for matrices with fast multiplication properties. More specifically, for partial Fourier and partial Hadamard matrices, our method optimizes the dependence of m on the distortion ε: We improve the recent bound $m = O(\varepsilon^{-4}\log(p)\log^4(N))$ of Ailon and Liberty [6] to $m = O(\varepsilon^{-2}\log(p)\log^4(N))$, which is optimal up to the logarithmic factors in N.

The proof of the theorem is based on concentration inequalities for Rademacher sums and Rademacher chaos combined with matrix norm estimates as they commonly appear in the compressed sensing literature.

Applications of our results include compressed sensing for redundant dictionaries and matrix recovery.

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Shearlets and sparse approximations in $L^2(\mathbb{R}^3)$

Jakob Lemvig

(joint work with Gitta Kutyniok and Wang-Q Lim)

In this report, we introduce generalized three-dimensional cartoon-like images, i.e., three-dimensional functions which are C^β except for discontinuities along C^α surfaces for $\alpha, \beta \in (1, 2]$, and consider sparse approximations of such for $\beta \geq \alpha$. We first derive the optimal rate of approximation which is achievable by exploiting information theoretic arguments. Then we introduce three-dimensional pyramid-adapted shearlet systems with compactly supported generators and prove that such shearlet systems indeed deliver essentially optimal sparse approximations of three-dimensional cartoon-like images. Finally, we even extend this result to the situation of surfaces which are C^α except for zero- and one-dimensional singularities, and again derive essential optimal sparsity of the constructed shearlet frames.

We start by defining the 3D cartoon-like image class. Fix $\mu, \nu > 0$. By $\mathcal{E}_2^2(\mathbb{R}^3)$ we denote the set of functions $f : \mathbb{R}^3 \to \mathbb{C}$ of the form

$$f = f_0 + f_1 \chi_B,$$

where $B \subset [0, 1]^3$ with ∂B a closed C^2-surface for which the principal curvatures are bounded by ν and $f_i \in C^2(\mathbb{R}^3)$ with $\mathrm{supp} f_i \subset [0, 1]^3$ and $\|f_i\|_{C^2} \leq \mu$ for each $i = 0, 1$. We enlarge this cartoon-like image model class to allow less regular images as follows. Let $1 < \alpha \leq \beta \leq 2$. We then only require that ∂B is a C^α-surface and that $f_i \in C^\beta(\mathbb{R}^3)$ with $\|f_i\|_{C^\beta} \leq \mu$ for each $i = 0, 1$. We speak of $\mathcal{E}_\alpha^\beta(\mathbb{R}^3)$ as consisting of generalized cartoon-like 3D images having C^β smoothness apart from a C^α discontinuity surface.

In [3], it was shown that the optimal approximation rate for generalized 3D cartoon-like images $f \in \mathcal{E}_\alpha^\beta(\mathbb{R}^3)$ which can be achieved for almost any frame representation systems is

$$\|f - f_n\|_2^2 = O(n^{-\alpha/2}), \quad n \to \infty,$$

where f_n is the nonlinear n-term approximation obtained by choosing the n largest coefficients of f in the canonical frame expansion. In the case of a basis, f_n will correspond to the best n-term approximation.In the following paragraphs we introduce shearlet systems for $L^2(\mathbb{R}^3)$ which nearly deliver this approximation rate for all $1 < \alpha \leq \beta \leq 2$.

Let $\alpha \in (1,2]$. We scale according to the *scaling matrix* A_{2^j}, $j \in \mathbb{Z}$, and represent directionality by the *shear matrix* S_k, $k = (k_1, k_2) \in \mathbb{Z}^2$, defined by

$$A_{2^j} = \begin{pmatrix} 2^{j\alpha/2} & 0 & 0 \\ 0 & 2^{j/2} & 0 \\ 0 & 0 & 2^{j/2} \end{pmatrix}, \qquad S_k = \begin{pmatrix} 1 & k_1 & k_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. The translation lattices will be generated by the matrix $M_c = \mathrm{diag}(c_1, c_2, c_2)$, where $c_1 > 0$ and $c_2 > 0$.

With the notation in place, we are ready to introduce our 3D shearlet system. For fixed $\alpha \in (1,2]$ and $c = (c_1, c_2) \in (\mathbb{R}_+)^2$, the *pyramid-adapted 3D shearlet system* generated by $\phi, \psi, \tilde{\psi}, \breve{\psi} \in L^2(\mathbb{R}^3)$ is defined by

$$SH(\phi, \psi, \tilde{\psi}, \breve{\psi}) = \Phi(\phi) \cup \Psi(\psi) \cup \tilde{\Psi}(\tilde{\psi}) \cup \breve{\Psi}(\breve{\psi}),$$

where

$$\Psi(\psi) = \{2^{j(2+\alpha)/4}\psi(S_k A_{2^j} \cdot - m) \,:\, j \geq 0, |k_1|, |k_2| \leq \lceil 2^{j(\alpha-1)/2} \rceil, m \in M_c\mathbb{Z}^3\},$$

and similar for $\tilde{\Psi}(\tilde{\psi}; c)$ and $\breve{\Psi}(\breve{\psi}; c)$ (switching the role of the variables), and

$$\Phi(\phi; c_1) = \{\phi(\cdot - m) \,:\, m \in c_1\mathbb{Z}^3\}.$$

The functions $\phi, \psi, \tilde{\psi}, \breve{\psi} \in L^2(\mathbb{R}^3)$ are called shearlets, and the function ϕ is a scaling function. The case $\alpha = 2$ correspond to paraboloidal scaling; allowing $\alpha = 1$ would yield isotropic scaling. Hence as α approaches 1, the shearlet system becomes more and more wavelet-like (and less and less 'directional').

In [3], it is shown that one can construct frames of the form $SH(\phi, \psi, \tilde{\psi}, \breve{\psi})$, where the generators $\phi, \psi, \tilde{\psi}, \breve{\psi} \in L^2(\mathbb{R}^3)$ are compactly supported. The following result tells us that compactly supported pyramid-adapted shearlets provide nearly optimal approximation rate for the class of generalized 3D cartoon-like images.

Theorem 1 ([3])**.** *Let $\alpha \in (1,2]$ and $c \in (\mathbb{R}_+)^2$, and let $\phi, \psi, \tilde{\psi}, \breve{\psi} \in L^2(\mathbb{R}^3)$ be compactly supported. Suppose that, for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, the function ψ satisfies:*

(i) $|\hat{\psi}(\xi)| \leq C \cdot \min\{1, |\xi_1|^{\delta}\} \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\} \cdot \min\{1, |\xi_3|^{-\gamma}\}$,

(ii) $\left|\frac{\partial}{\partial \xi_i}\hat{\psi}(\xi)\right| \leq |h(\xi_1)| \cdot \left(1 + \frac{|\xi_2|}{|\xi_1|}\right)^{-\gamma} \cdot \left(1 + \frac{|\xi_3|}{|\xi_1|}\right)^{-\gamma}$, *for* $i = 1, 2$,

where $\delta > 8$, $\gamma \geq 4$, $h \in L^1(\mathbb{R})$, and C a constant, and suppose that $\tilde{\psi}$ and $\breve{\psi}$ satisfy analogous conditions with the obvious change of coordinates. Further, suppose that the shearlet system $SH(\phi, \psi, \tilde{\psi}, \breve{\psi})$ forms a frame for $L^2(\mathbb{R}^3)$.

Then, for any $\beta \geq \alpha$ and $\mu, \nu > 0$, the frame $SH(\phi, \psi, \tilde{\psi}, \breve{\psi})$ provides nearly optimally sparse approximations of functions $f \in \mathcal{E}_{\alpha}^{\beta}(\mathbb{R}^3)$ in the sense that:

$$\|f - f_n\|_2^2 = O(n^{-\alpha/2+\tau}) \qquad \text{as } n \to \infty,$$

where $\tau = \tau(\alpha)$ satisfies $\tau < 0.04$.

For $\alpha = 2$, we even have $\|f - f_n\|_2^2 = O(n^{-\alpha/2}\,(\log n)^2)$ in Theorem 1 which is optimal (up to a log-factor). As α approaches 1 we have less and less directional information of the C^{α} discontinuity surface. Theorem 1 tells us that, as $\alpha \to 1$, the 'optimal' shearlet system should become more and more wavelet-like. We also remark that a large class of generators $\psi, \tilde{\psi}$, and $\breve{\psi}$ satisfy the conditions (i) and (ii) in Theorem 1.

Theorem 1 deals with discontinuity *surfaces*, but, as opposed to the two-dimensional setting, anisotropic structures in three-dimensional data comprise of *two* morphologically different types of structure, namely surfaces *and* curves. It would therefore be desirable to allow our 3D image class to also contain cartoon-like images with certain *curve* singularities. To achieve this we allow our discontinuity surface ∂B to be a closed, continuous, *piecewise* C^α smooth surface. We denote this function class $\mathcal{E}^{\beta}_{\alpha,L}(\mathbb{R}^3)$, where $L \in \mathbb{N}$ is the maximal number of C^α pieces. The pyramid-adapted shearlet systems still deliver the same nearly optimal rate for this extended image class $\mathcal{E}^{\beta}_{\alpha,L}(\mathbb{R}^3)$ [3].

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Uncertainty principles and localization measures

Peter Maass and Nir Sochen

(joint work with Chen Sagiv and Hans-Georg Stark)

The construction of bases $\{\varphi_i | i \in N\}$, which allow an efficient representation of signals or images f, such that the coefficients $\{c_i\}$ in

$$f = \sum_{i \in N} c_i \varphi_i$$

determine the phase space characteristics of f is one of the most basic tasks in signal and image processing. The basis functions φ_i should be somehow optimal in terms of their localization properties in a given phase space such as time-frequency or location-scale phase spaces. A classical approach links this optimality concept to so-called uncertainty principles:

For the classical time-frequency phase space, the uncertainty principle reads as ($\|\varphi\| = 1$)

$$\min_{\varphi} \int \xi^2 \, |\varphi(\xi)|^2 \, d\xi \int \omega^2 \, |\hat{\varphi}(\omega)|^2 \, d\omega \ \geq 1/4 \ .$$

If we measure the localization properties of f in the time-frequency phase space by the quantity on the left, then optimal localization in this case is obtained by functions, which attain equality in this uncertainty principle. These are the Gaussian functions.

This concept has been generalized in the 1960's, see e.g. [3], which leads to general uncertainty principles related to Lie algebras with non-vanishing commutators: Let A_k, $k = 1, 2$ be selfadjoint operators and define $e_A(f) = \langle Af, f\rangle$ and $var_A(f) = \langle (A - e_A)^2 f, f\rangle$ If $\|f\|^2 = 1$ then

$$var_{A_1}(f)\ var_{A_2}(f)\ \geq\ 4\ e_{[A_1-e_1,A_2-e_2]}(f)^2\ .$$

Again, bases functions, which are 'optimal' with respect to the underlying phase space were usually constructed by determining functions, which obtain an equality in this uncertainty principle. This typically leads to some system of ordinary differential equations, see e.g. [1, 2] for such constructions related to the wavelet and shearlet transforms.

However, the right hand side of the general uncertainty principle depends on f itself. It is no longer obvious that functions with optimal localization properties in terms of minimizing the joint product of variances, are obtained by this procedure. In fact it was shown in [4], that in case of the wavelet transform and the underlying affine group, there exists a sequence of functions f_n, such that ($\|f_n\| = 1$)

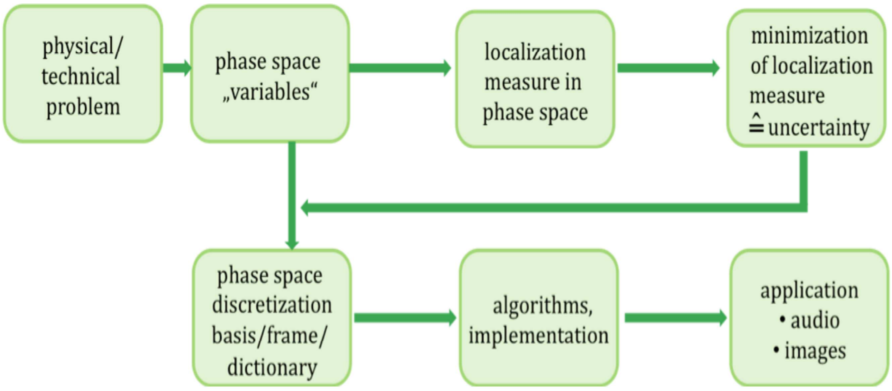
$$\lim_{n\to\infty}\ var_{A_1}(f_n)\ var_{A_2}(f_n) = 0\ \ .$$

Hence, the 'uncertainty minimizers' obtained by the classical approach, which do have a non-zero value of the variance product, do not have optimal localization properties. Conversely, neither of these functions f_n obtain an equality in the uncertainty principle.

This observation results in two problems:

(1) Can we characterize Lie algebras, where the 'product of variances' have a vanishing infimum?
(2) Which localization concepts should be used as a replacment of the uncertainty principle?

The first question was answered in [5], where a complete characterization was obtained in terms of the structure of the commutator $[A_1, A_2]$. The second question is being presently investigated in the UnLocX project. This project aims at developing a general concept of optimal localization concepts in phase space and at the development of related signal and image processing algorithms.



Ideally, we will start with a phase space definition and a localization concept, which arises from some specific application. We then determine reated optimal function systems as well as efficient decomposition and synthesis operators. his project started last September and lasts unil August 2013. It involves researchers from Vienna, Marseille, Tel Aviv, Lausanne, Aschaffenburg and Bremen.

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Frames for group representations: A duality principle and Feichtinger's frame conjecture

Deguang Han

We examine the duality principle and the Feichtinger's frame conjecture for frames induced by (projective) unitary representations of infinite countable groups. Our focus is on the infinite conjugate class (ICC) groups with special interests in the free groups.

A general duality principle: One of the well studied classes of frames is the class of Gabor (or Weyl-Heisenberg) frames: Let $\mathcal{K} = A\mathbb{Z}^d$ and $\mathcal{L} = B\mathbb{Z}^d$ be two full-rank lattices in $\mathbb{R}^d$, and let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \mathcal{L} \times \mathcal{K}$. Then the *Gabor (or Weyl-Heisenberg) family* is the following collection of functions in $L^2(\mathbb{R}^d)$:

$$\mathbf{G}(g, \Lambda) = \mathbf{G}(g, \mathcal{L}, \mathcal{K}) := \{e^{2\pi i<\ell,x>}g(x-\kappa) \Big| \ell \in \mathcal{L}, \kappa \in \mathcal{K}\}.$$

For convenience, we write $g_\lambda = g_{\kappa,\ell} = e^{2\pi i<\ell,x>}g(x-\kappa)$, where $\lambda = (\kappa, \ell)$. If E_ℓ and T_κ are the modulation and translation unitary operators defined by

$$E_\ell f(x) = e^{2\pi i<\ell,x>}f(x)$$

and

$$T_\kappa f(x) = f(x-\kappa)$$

for all $f \in L^2(\mathbb{R}^d)$. Then we have $g_{\kappa,\ell} = E_\ell T_\kappa g$. The well-known Ron-Shen duality principle states that a Gabor sequence $\mathbf{G}(g, \Lambda)$ is a frame (respectively, Parseval frame) for $L^2(\mathbb{R}^d)$ if and only if the adjoint Gabor sequence $\mathbf{G}(g, \Lambda^o)$ is a Riesz sequence (respectively, orthonormal sequence), where $\Lambda^o = (B^t)^{-1}\mathbb{Z}^d \times (A^t)^{-1}\mathbb{Z}^d$ is the adjoint lattice of Λ.

The duality principle [2, 7, 10] for Gabor frames states that a Gabor sequence $\mathbf{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if the associated adjoint Gabor sequence $\mathbf{G}(g, \Lambda^o)$ is a Riesz sequence. We prove that this duality principle extends to any dual pairs of projective unitary representations of countable groups. We examine the existence problem of dual pairs and establish some connection with classification problems for II_1 factors. While in general such a pair may not exist for some groups, we show that such a dual pair always exists for every subrepresentation of the left regular unitary representation when G is an abelian infinite countable group or an amenable ICC group. For free groups with finitely many generators, the existence problem of such a dual pair is equivalent to the well known problem about the classification of free group von Neumann algebras.

Two (projective) unitary representationsπ and σ) of G on a Hilbert space H are called to form a dual pair if they satisfy the following three conditions:

(i) $\pi(G)' = \sigma(G)''$

(ii) π and σ share the same Bessel vectors

(iii) π admits a frame vector for H and σ admits a Riesz sequence vector

Theorem 1 ([3]) *Let (π, σ) be a dual pair. Then $\{\pi(g)\xi\}_{g\in G}$ is a frame for H if and only if $\{\sigma(g)\xi\}_{g\in G}$ is a Riesz sequence.*

It has been a longstanding unsolved problem to decide whether the factors obtained from the free groups with n and m generators respectively are isomorphic if n is not equal to m with both $n, m > 1$. This problem can be rephrased as: *Let $\mathcal{F}_n$ $(n > 1)$ be the free group of n-generators and $P \in \lambda(\mathcal{F}_n)'$ is a nontrivial projection. Is $\lambda(\mathcal{F}_n)'$ $*$-isomorphic to $P\lambda(\mathcal{F}_n)'P$?* The following result shows that this problem is equivalent to the existence problem of dual pairs for free groups:

Theorem 2 ([3]) *Let $\pi = \lambda|_P$ be a subrepresentation of the left regular representation of an ICC (infinite conjugate class) group G and $P \in \lambda(G)'$ be a projection. Then the following are equivalent:*

(i) $\lambda(G)'$ and $P\lambda(G)'P$ are isomorphic von Neumann algebras;
(ii) there exists a group representation σ such that (π, σ) form a dual pair.

Corollary 3([3]) *Let G be a countable group and λ be its left regular unitary representation (i.e. $\mu \equiv 1$). Then we have*

(i) If G is either an abelian group, then for every projection $0 \neq P \in \lambda(G)'$, there exists a unitary representation σ of G such that $(\lambda|_P, \sigma)$ is a dual pair ([3]).

(ii) If G is an amenable ICC group, then for every projection $0 \neq P \in \lambda(G)'$, there exists a unitary representation σ of G such that $(\lambda|_P, \sigma)$ is a dual pair ([1]).

(iii) There exist ICC groups such that none of the nontrivial subrepresentations $\lambda|_P$ admits a dual pair ([8]).

(iv)If $G = \mathcal{F}_\infty \Rightarrow \lambda|_P$ admits a dual pair for every nontrivial projection $P \in \lambda(G)'$ ([9]).

Feichtinger's frame conjecture: Feichtinger's frame conjecture states that every norm bounded (from below) frame is a finite union of Riesz sequences. This conjecture turns out to be equivalent to several well-known open problems in mathematics including Kadison-Singer pure state extension problem, Bourgain-Tzafriri restricted invertibility conjecture and Anderson's paving conjecture. For group representation frames we parametrize the set of all such Parseval frames by operators in the commutant of the corresponding representation. We characterize when two such frames are strongly disjoint. We prove an undersampling result showing that if the representation has a Parseval frame of norm $\frac{1}{\sqrt{N}}$, the Hilbert space is spanned by an orthonormal basis generated by a subgroup. As applications we obtain some sufficient conditions under which a unitary representation admits a Parseval frame which is spanned by an Riesz sequences generated by a subgroup. In particular, every subrepresentation of the left regular representation of a free group has this property.

Theorem 4 ([4]) *Let G be a countable ICC group and let π be a unitary representation of the group G on the Hilbert space H. Suppose there exists a Parseval frame vector $\xi \in H$ with $\|\xi\|^2 = \frac{1}{N}$, $N \in \mathbb{Z}$. Assume in addition that H is a normal ICC subgroup of G with index $[G : H] = N$, and H contains elements of infinite order. Then there exist a strongly disjoint N-tuple $\eta_1, \ldots, \eta_N$ of Parseval frame vectors for H such that for all $i = 1, \ldots, N$, the family $\sqrt{N}\{\pi(h)\eta_i \mid h \in H\}$ is an orthonormal basis for H.*

As a consequence of Theorem 1 we obtain

Theorem 5 ([4]) *Let $G = \mathcal{F}_n$ be a free group with more than one generator. Then every frame representation admits a frame satisfying the Feichtinger's frame conjecture.*

Question. *For free group $G = \mathcal{F}_n$ $(n > 1)$, does every frame vector satisfy the Feichtinger's frame conjecture.*

Remark. An affirmative answer to the above question will yield that every exponential frame $\{e^{2\pi int}|_E\}_{n\in\mathbb{Z}}$ satisfies the Feichtinger's frame conjecture, where E is any subset of $[0,1]$ with positive Lebesgue measure.

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Sampling for time- and bandlimited signals

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(joint work with Jeffrey A. Hogan and Scott Izu)

The classical sampling theorem provides a formula by which bandlimited signals can be recovered by *sinc-interpolating* their integer samples. Usually, signals of interest have finite duration. The "Bell Labs" theory of time- and bandlimiting, developed by Landau, Slepian and Pollak starting in the 1960s, identified those bandlimited signals having most of their energies concentrated in a given time interval. However, the theory of time- and bandlimiting was never really reconciled with the classical sampling theorem until 2003 when Walter and Shen [1] as well as Khare and George [2] observed, independently, a sense in which approximately time- and bandlimited signals might be recovered, locally, from sinc series built from samples within the time interval of interest. We discuss here a quantitative estimate of the error between a member of a suitably defined class of approximately time- and bandlimited signals and a corresponding sampling series.

To fix ideas, for $f \in L^1 \cap L^2(\mathbb{R})$, let $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi i t\xi}\, dt$, and let $Pf = (\widehat{f}\,\mathbb{1}_{[-1/2,1/2]})^\vee$ denote the orthogonal projection onto the Paley-Wiener space $\mathrm{PW}(\mathbb{R})$ of L^2-signals bandlimited to $[-1/2, 1/2]$. The sampling theorem says that if $f \in \mathrm{PW}$ then $f(t) = \sum_{k=-\infty}^{\infty} f(k)\,\mathrm{sinc}\,(t-k)$. Let $Q_T f = f\,\mathbb{1}_{[-T,T]}$

denote the orthogonal projection onto the subspace of $L^2(\mathbb{R})$ consisting of signals supported in $[-T, T]$. The *time- and bandlimiting operator* PQ_T is self-adjoint on $\mathrm{PW}(\mathbb{R})$. Its eigenvalues form a discrete subset of $(0, 1)$. Landau and Widom [3] proved that the number $N(\alpha)$ of its eigenvalues larger than α satisfies $N(\alpha) = 2T + \log((1-\alpha)/\alpha)\,\log(2T)/\pi^2 + o(\log T)$. In particular, if λ_n denotes the nth eigenvalue of PQ_T when listed in decreasing order—PQ_T has a simple spectrum—then λ_n is close to one for $n < [2T] - \log(2T)$.
The eigenfunctions of PQ_T are called *prolate spheroidal wave functions* (PSWFs). For $f \in \mathrm{PW}$ let

$$f_N(t) = \sum_{n=0}^{N} \Big(\sum_k f(k)\,\varphi_n(k)\Big)\,\varphi_n(t)$$

denote the projection onto the span of the first N PSWFs and let

$$f_{N,T} = \sum_{n=0}^{N} \Big(\sum_{|k|\le M(T)} f(k)\,\varphi_n^T(k)\Big); \qquad \varphi_n^T(t) = \sum_{|k|\le M(T)} \varphi_n(k)\,\mathrm{sinc}\,(t-k)$$

for a suitable value of $M(T)$. We show that if $M(T) = T\,(1+\pi^2)(1+\log^\gamma T)$ where $\gamma > 1$ is fixed, then

$$\|Q_T\big(f_N - f_{N,T}\big)\|^2 \le \sum_{n=0}^{N} \lambda_n |\langle (f_N - f_{N,T}),\, \varphi_n\rangle|^2 \le C\|f\|^2 \sum_{n=0}^{N} \lambda_n(1-\lambda_n).$$

This estimate shows that if f lies in the span of the first N PSWFs where N is a bit smaller than $2T$ then f can be approximated accurately, over $[-T, T]$, by its classical sampling series. The result improves a bound due to Walter and Shen [1] in which the terms $1-\lambda_n$ on the right hand side are replaced by their square roots. However, in Walter and Shen's estimates $M(T)$ is also replaced by T. Methods for estimating numerically the samples of the PSWFs are also presented here.

Applications of multiband signals are growing. By a multiband signal we mean a signal in $L^2(\mathbb{R})$ whose Fourier transform is supported on a finite union of bounded, pairwise disjoint intervals. Building on the observation that time- and bandlimited signals can be approximated locally by their sinc interpolated samples, we give a result for constructing eigenfunctions for time- and bandlimiting to unions of frequency supports from the separate supports that is consistent with building sampling expansions for unions. For a compact set $\Sigma \subset \mathbb{R}$, let $P_\Sigma f = (f\,\mathbb{1}_\Sigma)^\vee$ and, for $S \subset \mathbb{R}$, set $Q_S f = f\,\mathbb{1}_S$. Now let Σ_1 and Σ_2 be disjoint, compact sets and let $\lambda_n^{\Sigma_i} \sim \varphi_n^{\Sigma_i}$, $i = 1, 2$ denote the nondegenerate eigen–pairs of $P_{\Sigma_i} Q_S P_{\Sigma_i}$. Denote by $\Lambda_{\Sigma_i} = \mathrm{diag}\,\lambda_n^{\Sigma_i}$ the diagonal matrix whose diagonal entries are the eigenvalues of $P_{\Sigma_i} Q_S P_{\Sigma_i}$ expressed in decreasing order of magnitude and by $\Gamma = \{\gamma_{nm}\}$ the matrix whose entries are the inner products of the time-localizations of $\varphi_n^{\Sigma_1}$ with $\varphi_m^{\Sigma_2}$, that is, $\gamma_{nm} = \langle Q_S \varphi_n^{\Sigma_1},\, \varphi_m^{\Sigma_2}\rangle$. We show then that any eigen–pair $\psi \sim \lambda$ for $P_{\Sigma_1\cup\Sigma_2} Q_S$ can be written $\psi = \sum_{n=0}^{\infty} (\alpha_n \varphi_n^{\Sigma_1} + \beta_n \varphi_n^{\Sigma_2})$ where the vectors $\boldsymbol{\alpha} = \{\alpha_n\}$ and $\boldsymbol{\beta} = \{\beta_n\}$ together form a discrete eigenvector for the block matrix eigenvalue problem

$$\lambda \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \Lambda_{\Sigma_1} & \bar{\Gamma} \\ \Gamma^T & \Lambda_{\Sigma_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} .$$

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