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Real Analysis, Harmonic Analysis and Applications

Organised by
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Abstract. The workshop has focused on important developments within the last few years in the point of view and methods of real and harmonic Analysis as well as significant concurrent progress in the application of these to various other fields.

Mathematics Subject Classification (2000): 42xx, 43xx, 44xx, 22xx, 35xx.

Introduction by the Organisers

This workshop, which continued the triennial series at Oberwolfach on Real and Harmonic Analysis that started in 1986, has brought together experts and young scientists working in harmonic analysis and its applications (such as arithmetic combinatorics, quasiconformal mappings, nonlinear dispersive and elliptic PDE, and ergodic theory) with the objective of furthering the important interactions between these fields.

Major areas and results represented at the workshop are:

- The use of the polynomial method in harmonic analysis has recently led to advancement on several classical problems such as the multilinear Kakeya problem, the Kakeya problem in finite fields, and the related joints problem. One highlight presented at the workshop was the solution of the Erdös distance problem: Given a set of N points in the plane, the number of distinct distances between these points is at least $C\sqrt{N}/\log N$.
- Harmonic analysis questions motivated by several complex variables, such as the Cauchy integral and related operators on higher dimensional domains with minimally smooth boundary, and multiplier estimates for the Kohn Laplacian on forms on the sphere of $\mathbb{C}^n$.

- The interplay between martingale methods and harmonic analysis, for example to obtain sharp weighted estimates on singular integrals, including the recent solution of the long standing A_2 conjecture, that A_2-weighted bounds for Calderón Zygmund singular integrals depend in first order on the A_2 constant of the weight. Progress has been achieved by advances on martingale based operators and sharpened transfer principles between martingale estimates and classical Calderón Zygmund theory. The methods are also applicable in the study of questions in geometric measure theory, which require understanding of singular integral theory in very hostile environments such as spaces not of homogeneous type.
- Improved understanding of invariants of higher degree analytic and smooth surfaces and their singularities as they play a role in estimating analytic expressions such as oscillatory integrals, Fourier restriction maps, or measure of sublevel sets. Typically invariants are derived from the Newton polyhedron, the convex hull of points in the integer lattice representing non-vanishing Taylor coefficients of smooth functions in question, relative to appropriate coordinates.
- Progress on multilinear estimates in recent years has been due to the application of a range of novel techniques such as time frequency analysis, additive combinatorics in the form of Gowers uniformity norms and related topics and the polynomial method. These methods also bear fruit on more classical problems such as the Hilbert transform along vector fields.
- Sharp invariance properties on quasiconformal mappings have led to new understanding of function spaces in harmonic analysis.
- New applications of real and harmonic analysis to elliptic PDE's had implications ranging from understanding of divergence form operators to nonlinear elliptic operators such as the k-Hessian.
- New localization techniques in frequency and space have led to progress on linear and non-linear dispersive PDE's.
- Harmonic analysis questions in the continuous setting have turned out to be an important guide line for the understanding of discrete analogues of these questions, and have led to interesting insights about questions from analytic number theory, such as decay estimates for exponential sums, and the interplay between number theory and real and harmonic analysis.

The meeting took place in a lively and active atmosphere, and greatly benefited from the ideal environment at Oberwolfach. It was attended by 48 participants. The program consisted of 28 lectures of 40 minutes. Long afternoon breaks have been intensively used by the participants for mathematical discussions and collaborations. The organisers made an effort to include young mathematicians, and greatly appreciate the support through the Oberwolfach Leibniz Graduate Students Program, which allowed to invite several outstanding young scientists.

Workshop: Real Analysis, Harmonic Analysis and Applications

Table of Contents

Abstracts

Cauchy integrals for minimally smooth domains in $\mathbb{C}^n$

Elias M. Stein

This is a report on joint work with Loredana Lanzani. Our goal is the study of the Cauchy integral, and the Szegö and Bergman projections for domains D in $\mathbb{C}^n$ whose boundaries have minimal smoothness.

In the case $n = 1$ this has been carried out in the work of Calderón, Coifman-McIntosh-Meyer, and David for the Cauchy integral and a number of authors for the Szegö and Bergman projections. When $n = 1$ and the boundary curve is of class $C^{1+\varepsilon}$, the results are amenable to classical methods. The limiting situation, when the boundary is of class C^1, or Lipschitz, (or another geometric variant that essentially requires one derivative of smoothness) needed further ideas and techniques, such as those related to the T(1) theorem.

As soon as $n > 1$, the situation changes radically. Among the reasons are the necessary role of pseudo-convexity, (which essentially requires two derivatives of the defining function) and the very nature of the Cauchy-Fantappiè integrals that also require two such derivatives.

We state our results for domains whose defining functions are either of class $C^{1,1}$ (that is, its first derivatives are Lipschitz) or of class C^2. For a domain of the first type we say it is strongly $\mathbb{C}$-linearly convex if $d(z, T^{\mathbb{C}}_\omega) \geq c\,|z-\omega|^2$, where $z \in D$, $\omega \in bD$, and $T^{\mathbb{C}}_\omega$ is the complex sub-space of the tangent space at ω.

Theorem 1. *If D is of class $C^{1,1}$ and is strongly $\mathbb{C}$-linearly convex, then the Cauchy-Leray-Fantappiè integral is bounded on $L^p(bD)$, for $1 < p < \infty$.*

Theorem 2. *If D is of class C^2 and is strongly pseudo-convex, then the Bergman projection is bounded on $L^p(D)$, $1 < p < \infty$.*

Corollary 1. *The same L^p boundedness holds for the operator whose kernel is the absolute value of the Bergman kernel.*

There is also an analogue to Theorem 2 for the Szegö projection, if D is of class C^2 and strongly pseudo-convex.

The proof of Theorem 2 requires that we first construct a family $\{B^1_\varepsilon\}$ of (non-orthogonal) projections of $L^2(D)$ to $O(D) \wedge L^2(D)$. Here we use ideas of Kerzman-Stein, Ligocka, and Range. A crucial difference however is that in these works one needed that the domain was smooth and it sufficed to consider a single such B^1. Here the B^1_ε may be viewed as "regularizations" of a single B^1. However, the B^1_ε do *not* approximate the true Bergman projection, (their norms may be unbounded as $\varepsilon \to 0$). Their key role is captured in the following.

Lemma 1. *For each $\varepsilon > 0$ we can write $B^1_\varepsilon = A_\varepsilon + C_\varepsilon$, where the norm of $A_\varepsilon - A^*_\varepsilon$ (as an operator on L^p, $1 < p < \infty$) is less than εc_p. Also C_ε maps L^1 to L^∞, (but its norm may be unbounded as $\varepsilon \to 0$).*

Variation for Riesz transforms and uniform rectifiability

Xavier Tolsa
(joint work with Albert Mas)

For $1 \leq n < d$ integers and $\rho > 2$, we have proved in [13] that an n-dimensional Ahlfors-David regular measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if the ρ-variation for the Riesz transform with respect to μ is a bounded operator in $L^2(\mu)$. This result can be considered as a partial solution to a well known open problem posed by G. David and S. Semmes which relates the $L^2(\mu)$ boundedness of the Riesz transforms to the uniform rectifiability of μ.

Recall that the n-dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

We will use the notation $\mathcal{R}^\mu f(x) := \{R^\mu_\epsilon f(x)\}_{\epsilon>0}$. When $d = 2$ (i.e., μ is a Borel measure in $\mathbb{C}$), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where $C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, d\mu(y)$, for $x \in \mathbb{C}$. To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R^\mu_* f(x) = \sup_{\epsilon>0} |R^\mu_\epsilon f(x)|$ and $C^\mu_* f(x) = \sup_{\epsilon>0} |C^\mu_\epsilon f(x)|$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. Given $d \geq 2$, the kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that we consider in this paper satisfy

$$(1) \qquad |K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x^i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x^i}\partial_{x^j} K(x)| \leq \frac{C}{|x|^{n+2}},$$

for all $1 \leq i, j \leq d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $1 \leq n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). For $f \in L^1(\mu)$ and $x \in \mathbb{R}^d$, we set

$$T^\mu_\epsilon f(x) \equiv T_\epsilon(f\mu)(x) := \int_{|x-y|>\epsilon} K(x-y) f(y)\, d\mu(y),$$

and we denote $\mathcal{T}^\mu f(x) = \{T^\mu_\epsilon f(x)\}_{\epsilon>0}$.

Let $\mathcal{F} := \{F_\epsilon\}_{\epsilon>0}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho > 0$, the *ρ-variation* of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{V}_\rho(\mathcal{F})(x) := \sup_{\{\epsilon_m\}} \left(\sum_{m \in \mathbb{Z}} |F_{\epsilon_{m+1}}(x) - F_{\epsilon_m}(x)|^\rho \right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$. Fix a decreasing sequence $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$. On the other hand, the *oscillation* of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by $\mathcal{O}(\mathcal{F})(x) := \sup_{\{\epsilon_m\},\{\delta_m\}} \Big(\sum_{m\in\mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \Big)^{1/2}$, where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ and $\{\delta_m\}_{m\in\mathbb{Z}}$ such that $r_{m+1} \le \epsilon_m \le \delta_m \le r_m$ for all $m \in \mathbb{Z}$.

The ρ-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [10], [1], [7], [2], [8], [9], and [17], for example). We are interested in the ρ-variation and oscillation of the family $\mathcal{T}^\mu f$. That is, given a Borel measure μ in $\mathbb{R}^d$ and $f \in L^1(\mu)$ we will deal with $(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x) := \mathcal{V}_\rho(\mathcal{T}^\mu f)(x)$ and $(\mathcal{O}\circ\mathcal{T}^\mu)f(x) := \mathcal{O}(\mathcal{T}^\mu f)(x)$. Notice that $T^\mu_* f(x) \le \mathcal{V}_\rho(\mathcal{T}^\mu f)(x)$ for any compactly supported function $f \in L^1(\mu)$.

When μ coincides with the Lebesgue measure in the real line and $K(x) = 1/x$ is the kernel of the Hilbert transform, in [2] it was shown that $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ and $\mathcal{O} \circ \mathcal{T}^\mu$ are bounded in $L^p(\mu)$, for $1 < p < \infty$, and of weak type $(1,1)$. This result was extended to other singular integrals in higher dimensions in [3]. The case of the Cauchy transform and other odd Calderón-Zygmund operators on Lipschitz graphs was studied recently in [12].

Let us turn our attention to uniform rectifiability now. Recall that a Borel measure μ in $\mathbb{R}^d$ is called n-rectifiable if there exists a countable family of n-dimensional C^1 submanifolds $\{M_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ such that $\mu(E\setminus\bigcup_{i\in\mathbb{N}} M_i) = 0$. Moreover, μ is said to be n-dimensional Ahlfors-David (AD) regular if there exists some constant $C > 0$ such that $C^{-1}r^n \le \mu(B(x,r)) \le Cr^n$ for all $x \in \operatorname{supp}\mu$ and $0 < r \le \operatorname{diam}(\operatorname{supp}\mu)$. One also says that μ is uniformly n-rectifiable if there exist $\theta, M > 0$ so that, for each $x \in \operatorname{supp}\mu$ and $r > 0$, there is a Lipschitz mapping g from the n-dimensional ball $B^n(0,r) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that $\operatorname{Lip}(g) \le M$ and $\mu\big(B(x,r)\cap g(B^n(0,r))\big) \ge \theta r^n$, where $\operatorname{Lip}(g)$ stands for the Lipschitz constant of g. In particular, uniform rectifiability implies rectifiability.

David and Semmes asked more than twenty years ago the following question, still open:

Question 1. *Is it true that an n-dimensional AD regular measure μ is uniformly n-rectifiable if and only if R^μ_* is bounded in $L^2(\mu)$?*

By the results in [5], the "only if" implication of the question above is already known to hold. Also in [5], G. David and S. Semmes gave a positive answer to Question 1 if one replaces the L^2 boundedness of R^μ_* by the L^2 boundedness of T^μ_* for a wide class of odd kernels K. In the case $n = 1$ (in particular, for the Cauchy transform), the "if" implication was proved by P. Mattila, M. Melnikov and J. Verdera in [15] using the notion of curvature of measures. Later on, G. David and J. C. Léger [11] proved that the L^2 boundedness C^μ_* implies that μ is rectifiable, even without the AD regularity assumption.

When μ is the n-dimensional Hausdorff measure on a set $E \subset \mathbb{R}^d$ such that $\mu(E) < \infty$, the rectifiability of μ is also related with the existence μ-a.e. of the principal value of the Riesz transform of μ, that is, the existence of $R^\mu 1(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon 1(x)$ for μ-a.e. $x \in E$. For some results in this direction, see [14], [16], and [18].

Our main new result is the following:

Theorem 2. *Let $1 \leq n < d$ and $\rho > 2$. An n-dimensional AD regular Borel measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$. Moreover, if μ is n-uniformly rectifiable, then for any kernel K satisfying* (1)*, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.*

Notice that the preceding theorem asserts that if we replace the $L^2(\mu)$ boundedness of R^μ_* by the stronger assumption that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then μ must be uniformly rectifiable. On the other hand, the theorem claims that the variation for the n-dimensional Riesz transforms is bounded in $L^2(\mu)$.

A natural question then arises. Given an arbitrary measure μ on $\mathbb{R}^d$, without atoms say, does the $L^2(\mu)$ boundedness of R^μ_* implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$, for $\rho > 2$? By the results of [15] and Theorem 2, this is true in the case $n = 1$ if μ is AD regular 1-dimensional.

Concerning the proof of Theorem 2, in our previous paper [12] we showed that, if μ stands for the n-dimensional Hausdorff-measure on an n-dimensional Lipschitz graph, then the ρ-variation for Riesz transforms and odd Calderón-Zygmund operators with smooth truncations are bounded in $L^2(\mu)$. This is a fundamental step to prove that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ and, more generally, $\mathcal{V}_\rho \circ \mathcal{T}^\mu$, are bounded in $L^2(\mu)$ if μ is uniformly n-rectifiable. Another basic tool in our arguments is the geometric corona decomposition of uniformly rectifiable measures introduced by David and Semmes in [5].

The proof of the fact that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies the uniform rectifiability of μ is not so laborious as the one of the converse implication. The arguments are partially inspired by some of the techniques in [19].

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Hardy spaces associated with certain Schrödinger operators

Jacek Dziubański
(joint work with Marcin Preisner, Jacek Zienkiewicz)

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$ with a nonnegative locally integrable potential V and let $K_t(x, y)$ be the integral kernels of the semigroup $\{T_t\}_{t>0}$ generated by $-L$. We say that an $L^1(\mathbb{R}^d)$-function f belongs to H^1_L if $\mathcal{M}_L f \in L^1(\mathbb{R}^d)$, where

$$\mathcal{M}_L f(x) = \sup_{t>0} |T_t f(x)|.$$

Then we set $\|f\|_{H^1_L} = \|f\|_{H^1_{L,\max}} = \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}$.

Hardy spaces associated with certain operators attracted attention of many authors. We refer the reader to [1], [2], [4], [5], [9], [10], [11], [15], [16], [17] and references therein.

In [15] the authors provide a general approach to the theory of H^1-spaces associated with semigroups satisfying the Davies-Gaffney estimates, and in particular Schrödinger semigroups, proving the following atomic decomposition of the elements of H^1_L. *A function $a \in L^2(\mathbb{R}^d)$ is a (1,2,1)-atom associated with the operator L if there exist a function $b \in \mathcal{D}(L)$ and a ball B such that*

(i) $a = Lb$,
(ii) $\operatorname{supp} b \subset B$,
(iii) $\|L^k b\|_{L^2(\mathbb{R}^d)} \leq r_B^{2-2k} |B|^{-1/2}, \quad k = 0, 1.$

Let us recall that $H^1_{\rm atom} = \big\{ f : f = \sum_j \lambda_j a_j ,\ \sum_j |\lambda_j| < \infty \big\}$, where $\lambda_j \in \mathbb{C}$, a_j are H^1-atoms. We set $\|f\|_{H^1_{\rm atom}} = \inf \big\{ \sum_j |\lambda_j| \big\}$, where the infimum is taken over all representations of f as above.

One of the results of [15] asserts that $\|f\|_{H^1_{L,{\rm atom}}} \sim \|f\|_{H^1_{L,\max}}$.

The purpose of this talk is to present that certain classes of Schrödinger operators admit other atomic decompositions. Atoms that occur in these atomic decompositions are similar to those of the classical theory of real Hardy spaces, local or global, and they can be defined by means of behavior of the potential V.

Hardy spaces for Schrödinger operators with local atoms. Let $\mathcal{Q} = \{Q_j\}_{j=1}^\infty$ be a family of closed cubes in $\mathbb{R}^d$ with disjoint interiors such that $\mathbb{R}^d \setminus \bigcup_{j=1}^\infty Q_j$ is of Lebesgue measure zero. We shall assume that there exist constants $C, \beta > 0$ such that if $Q_i^{****} \cap Q_j^{****} \neq \emptyset$ then $d(Q_i) \leq C d(Q_j)$, where $d(Q)$ denotes the diameter of Q, and $Q^* = (1+\beta)Q$.

We say that a function a is an $H^1_{\mathcal{Q}}$-atom if there exists $Q \in \mathcal{Q}$ such that either $a = |Q|^{-1}\mathbf{1}_Q$ or a is the classical atom with support contained in Q^* (that is, there is a cube $Q' \subset Q^*$ such that $\operatorname{supp} a \subset Q'$, $\int a = 0$, $|a| \leq |Q'|^{-1}$).

Following [10] we impose two additional assumptions on V and $\mathcal{Q}$, mainly:

$$\text{(D)} \qquad (\exists C, \varepsilon > 0) \qquad \sup_{y \in Q^*} \int T_{2^n d(Q)^2}(x,y)\,dx \leq C n^{-1-\varepsilon} \quad \text{for } Q \in \mathcal{Q}, n \in \mathbb{N};$$

$$\text{(K)} \quad (\exists C, \delta > 0) \quad \int_0^{2t} (\mathbf{1}_{Q^{***}} V) * P_s(x)\,ds \leq C \left(\frac{t}{d(Q)^2} \right)^{\delta} \quad \text{for } x \in \mathbb{R}^d,\ Q \in \mathcal{Q},\ t \leq d(Q)^2.$$

Results of [10] state that if we assume (D) and (K) then we have the following atomic characterization of the Hardy space H^1_L:

$$\text{(1)} \quad f \in H^1_L \iff f \in H^1_{\mathcal{Q}} \quad \text{and} \quad C^{-1}\|f\|_{H^1_{\mathcal{Q},{\rm atom}}} \leq \|f\|_{H^1_{L,\max}} \leq C\|f\|_{H^1_{\mathcal{Q},{\rm atom}}}.$$

Moreover, it was proved in [7] that in the case of V satisfying (D) and (K) the space H^1_L is characterized by the Riesz transforms $R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$.

Summarizing: if (D) and (K) hold, then the atoms of H^1_L are local atoms in the sense of [14], where the scale and localization are adapted to the behavior of V.

`Examples.`

- The Hardy space H^1_L associated with one-dimensional Schrödinger operator L was studied in [4]. It was proved there that for any nonnegative $V \in L^1_{loc}(\mathbb{R})$ the collection $\mathcal{Q}$ of maximal dyadic intervals Q of $\mathbb{R}$ that are defined by the stopping time condition

$$\text{(2)} \qquad |Q| \int_{16Q} V(y)\,dy \leq 1,$$

fulfils (D) (see [4, Lemma 2.2]). One can proved that also (K) is satisfied.

- $V(x) = \gamma |x|^{-2}$, $d \geq 3$, $\gamma > 0$. Then for $\mathcal{Q}$ being the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$ that consists of dyadic cubes the conditions (D) and (K) hold (see [10]).

• $d \geq 3$, V satisfies the reverse Hölder inequality with exponent $q > d/2$, that is,

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(y)\,dy \quad \text{for every ball } B.$$

Define $\mathcal{Q}$ by: $Q \in \mathcal{Q}$ if and only if Q is the maximal dyadic cube for which $d(Q)^2|Q|^{-1}\int_Q V(y)\,dy \leq 1$. Then the conditions (D) and (K) are true (see [10]).

Hardy spaces H^1_L with weighted cancellation condition. Different type of cancellation conditions may occur when, in dimensions $d \geq 3$, one considers potentials which are in some sense small. This happens if e.g. V is compactly supported and belongs to $L^p(\mathbb{R}^d)$ for certain $p > d/2$ (see [11]). Then, in this case, the limit $\lim_{t\to\infty}\int_{\mathbb{R}^d} K_t(x,y)\,dy$ exists and defines an L-harmonic function $w(x)$. The function w satisfies: $0 < \delta \leq w(x) \leq 1$, $|w(x) - w(y)| \leq C|x-y|^\alpha$, and $\lim_{|x|\to\infty} w(x) = 1$. A function a is an atom for H^1_L if there exists a ball B such that

$$\operatorname{supp} a \subset B, \quad \|a\|_\infty \leq |B|^{-1}, \quad \int a(x)w(x)\,dx = 0. \tag{3}$$

It was proved in [11] that the space H^1_L admits the atomic decomposition with these atoms. Moreover, H^1_L is also characterized by the Riesz transforms (see [6]).

It turns out that the condition about the support of V could be relaxed. If we assume that $V(x) = \sum_{i=1}^n V_i(x)$, where each $V_i(x)$ is a nonnegative potential that depends only on certain number of variables (possibly different for every i) and belongs to L^p classes of these variables for certain range of p. The space H^1_L has an atomic decomposition, which is similar to that of compactly supported potentials (see [8]). Also the characterization by the Riesz transforms $\partial_{x_j}L^{-1/2}$ is true.

The case of compactly supported potentials in dimension 2. Assume that V is a nonnegative nonzero compactly supported C^2-function. It was proved in [12] that there exists a regular L-harmonic weight w such that

$$C^{-1}\ln(2+|x|) \leq w(x) \leq C\ln(2+|x|), \quad |\nabla w(x)| \leq C(1+|x|)^{-1},$$

and the space H^1_L admits atomic decomposition with atoms satisfying (3) with the new weight w.

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Spectral multipliers for commuting differential operators on Lie groups

Alessio Martini

This is a summary of results contained in the author's PhD thesis [11], written under the supervision of Fulvio Ricci.

The classical Mihlin-Hörmander and Marcinkiewicz theorems for Fourier multipliers on $\mathbb{R}^n$ give sufficient conditions for the L^p-boundedness ($1 < p < \infty$) of operators belonging to the joint functional calculus of the partial derivatives on $\mathbb{R}^n$, i.e., operators of the form

$$m(L_1, \dots, L_n) = \int_{\mathbb{R}^n} m(\lambda)\, dE(\lambda) \tag{1}$$

where $L_j = -i\frac{\partial}{\partial x_j}$ and E is the joint spectral resolution of $L_1, \dots, L_n$, in terms of smoothness properties of the multiplier $m : \mathbb{R}^n \to \mathbb{C}$. Namely, a Mihlin-Hörmander condition

$$\sup_{t>0} \|(m \circ \epsilon_t)\, \eta_n\|_{W^q_s} < \infty \tag{2}$$

(where η_n is a smooth cutoff function which is nonzero on an annulus centered at the origin on $\mathbb{R}^n$, $(\epsilon_t)_{t>0}$ is a family of dilations on $\mathbb{R}^n$, and W^q_s is an L^q Sobolev norm of fractional order s) for $q = 2$ and $s > n/2$, or a Marcinkiewicz condition

$$\sup_{t_1,\dots,t_n>0} \|m(t_1\cdot, \dots, t_n\cdot)\, \eta_1 \otimes \cdots \otimes \eta_1\|_{S^q_s W} < \infty \tag{3}$$

(where η_1 is a cutoff as before but on $\mathbb{R}^1$, and $S^q_{\vec{s}}W$ is an L^q Sobolev norm with dominating mixed smoothness [17] of order $\vec{s} = (s_1, \dots, s_n)$) for $q = 2$ and $s_1, \dots, s_n > 1/2$ are sufficient conditions on m for the L^p-boundedness of (1).

In contexts other than $\mathbb{R}^n$, this kind of problem has been extensively studied for the functional calculus of a single operator $L = L_1$. In particular, for a positive Rockland operator L on a homogeneous Lie group [9], condition (2) for $n = 1$ with $q = 2$ and $s > Q_\infty/2$ (where Q_∞ is the dimension at infinity of the group) is sufficient for the L^p-boundedness of $m(L)$ [14, 2, 8]. This result, on one hand, has been extended to operators with Gaussian-type heat kernel bounds (or finite propagation speed) on doubling metric-measure spaces, although requiring a stronger condition on m — i.e., (2) with $q = \infty$ and $s > Q/2$, where Q is the "homogeneous dimension" associated with the doubling condition [1, 8, 5]; on the other hand, it has been sharpened in particular cases (positive Rockland operators on Heisenberg-type groups, and distinguished sublaplacians on SU_2 and on the complex spheres), where condition (2) with $q = 2$ and s greater than half the topological dimension of the manifold is proved to be sufficient [7, 16, 10, 4, 3].

The problem for multiple operators ($n > 1$), instead, has been considered only in quite special cases: particularly for sublaplacians and central derivatives on Heisenberg-type groups (for which sharp results have been obtained [15, 6, 19]), and for systems of operators acting on different factors of a direct product [18].

In the context of a homogeneous Lie group G, we introduce the notion of *Rockland system*: a system $L_1, \dots, L_n$ of pairwise commuting, formally self-adjoint, homogeneous left-invariant differential operators which are jointly injective in every nontrivial irreducible unitary representation of G. This notion is sufficiently wide to include all the previously considered systems of operators (in the context of homogeneous Lie groups) together with several other examples, and sufficiently strong to ensure the existence of a joint functional calculus on $L^2(G)$ [13]. Under this condition, we prove analogues of the Mihlin-Hörmander and Marcinkiewicz multiplier theorems [12].

Namely, for a Rockland system $L_1, \dots, L_n$ on a homogeneous Lie group G of dimension at infinity Q_∞, a system of dilations $(\epsilon_t)_{t>0}$ on $\mathbb{R}^n$ is naturally defined by means of the homogeneity degrees of the operators. We then obtain

Theorem 1. *If $m : \mathbb{R}^n \to \mathbb{C}$ satisfies condition* (2) *with $q = \infty$ and $s > Q_\infty/2$, then $m(L_1, \dots, L_n)$ is bounded on $L^p(G)$ for $1 < p < \infty$.*

This general result can be sharpened in particular cases, by extending the techniques of [10] to our setting; in fact, if G is a direct product of Euclidean and Heisenberg-type (or Métivier) groups, then condition (2) with $q = \infty$ and s greater than half the topological dimension of G is sufficient.

A sort of product theory can then be developed, by considering several homogeneous Lie groups G_j, each endowed with a Rockland system; for simplicity, here we restrict to the case of a single self-adjoint Rockland operator L_j on each G_j. These operators can be considered as a system on the direct product $G_1 \times \cdots \times G_n$, or more generally on a connected Lie group G endowed with homomorphisms $\kappa_j : G_j \to G$

such that the pushforwards $L_j^\natural = \kappa_j(L_j)$ commute and admit a joint functional calculus on $L^2(G)$. If Q_j denotes the dimension at infinity of G_j, then we have

Theorem 2. *If $m : \mathbb{R}^n \to \mathbb{C}$ satisfies condition (3) with $q = 2$ and $s_j > Q_j/2$ for $j = 1, \dots, n$, then $m(L_1^\natural, \dots, L_n^\natural)$ is bounded on $L^p(G)$ for $1 < p < \infty$.*

As before, this general result can be sharpened; for instance, if some of the G_j are Heisenberg-type, then the corresponding s_j can be taken greater than half the topological dimension of these G_j. An interesting feature of this result is that the environment group G needs not be homogeneous, nor nilpotent. In fact, as a corollary, for a distinguished sublaplacian Δ on a diamond group $G = \mathrm{H}_k \rtimes \mathbb{T}^d$ (semidirect product of a Heisenberg group H_k and a torus $\mathbb{T}^d$), we obtain that condition (2) with $q = 2$ and s greater than half the topological dimension of G is sufficient for the L^p-boundedness of $m(\Delta)$.

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Davis and Garsia inequalties for Hardy Martingales

Paul F. X. Müller

Let $\mathbb{T}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty}\}$ denote the countable product of the torus $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$,equipped with its normalized Haar measure $\mathbb{P}$. A natural filtration of $\sigma-$ algebras on $\mathbb{T}^{\mathbb{N}}$ is given by the coordinate projections

$$P_k : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^k, \quad (x_i)_{i=1}^{\infty} \to (x_i)_{i=1}^{k}.$$

Define $\mathcal{F}_k$ to be the $\sigma-$ algebra on $\mathbb{T}^{\mathbb{N}}$ generated by P_k.

Let $F = (F_k)$ be an $L^1(\mathbb{T}^{\mathbb{N}})-$bounded martingale on the filtered probability space $(\mathbb{T}^{\mathbb{N}}, (\mathcal{F}_k), \mathbb{P})$. Conditioned on $\mathcal{F}_{k-1}$ the martingale differnce $\Delta F_k = F_k - F_{k-1}$ defines an element in the Lebesque space of integrable, function of vanishing mean $L^1_0(\mathbb{T})$. By definition the martingale $F = (F_k)$ belongs to the class of Hardy martingles, if, conditioned on $\mathcal{F}_{k-1}$,

$$\Delta F_k = F_k - F_{k-1} \quad \text{defines an element in the Hardy space } H^1_0(\mathbb{T}).$$

Hardy martingales arose in Complex and Functional Analysis. Originally with embedding problems [1], isomorphic classification [2, 9], factorization problems [11], similarity problems [12], boundary convergence of analytic functions [7, 6], Jensen measures [3], renorming for Banach spaces [4], vector valued Littlewood Paley Theory [14] .

Two robustness properties of Hardy martingales are particularily important for their use in Analysis.

(1) The class of Hardy martingales is closed under martingale transforms.
(2) For Hardy martingales, their L^1 norm is determined by square functions. There exist $c, C > 0$ so that for any Hardy martingale $F = (F_k)$,

$$c\mathbb{E}|F_n| \leq \mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} \leq C\mathbb{E}|F_n|. \tag{1}$$

In the talk we gave a strengthening of the square function characterization (1) for Hardy martingales from [10]. We prove that every Hardy martingale $F = (F_k)_{k=1}^{n}$ can be written as

$$F = G + B$$

where $G = (G_k)_{k=1}^{n}$ and $B = (B_k)_{k=1}^{n}$ are again Hardy martingales so that

$$\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|) \leq C\mathbb{E}|F_n|. \tag{2}$$

and

$$(3) \qquad |\Delta G_k| \le A_0 |F_{k-1}|, \qquad k \le n.$$

The estimate (2) implies of course the right hand side of the square function estimate (1) since the triangle inequality and the Burkholder-Gundy martingale inequality [5] give

$$\begin{aligned}\mathbb{E}(\sum_{k=1}^n |\Delta F_k|^2)^{1/2} &\le \mathbb{E}(\sum_{k=1}^n |\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B_k|)\\ &\le 2\mathbb{E}(\sum_{k=1}^n \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B_k|).\end{aligned}$$

The uniform previsible estimate (3) should be compared with uniform previsible estimates appearing in the classical Davis and Garsia inequality [5, Chapters III and IV]. For general martingales the Davis decomposition [5, Chapter III] guarantees only uniform estimates by previsible *and increasing* functionals such as $\max_{m \le k-1} |F_m|$. Hence a routine application of the Davis decomposition could yield only

$$|\Delta G_k| \le A_0 \max_{m \le k-1} |F_m|.$$

The presented paper [10] exploits stochastic holomorphy [13, 8] to prove that if $h \in H_0^1(\mathbb{T})$ and $z \in \mathbb{C}$, then

$$\rho = \inf\{t < \tau : |h(B_t)| > 2\alpha_0^{-1}|z|\}, \quad \text{and} \quad g(e^{i\theta}) = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}).$$

satisfy: $g \in H_0^\infty(\mathbb{T})$ with $|g| \le A_0|z|$ and

$$(|z|^2 + A_0^{-2}\int_{\mathbb{T}} |g|^2 dm)^{1/2} + A_0^{-1}\int_{\mathbb{T}} |h-g| dm \le \int_{\mathbb{T}} |z+h| dm.$$

These estimates are used together with a general iteration principle extracted from the work of J. Bourgain [1]. In its simplest form the iteration principle yields a comparison theorem between square functions as follows: Assume that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are non- negative, integrable functions so that the following set of estimates hold true,

$$\mathbb{E}(\sum_{m=1}^{k-1} u_m^2 + v_k^2)^{1/2} \le \mathbb{E}(\sum_{m=1}^{k} u_m^2)^{1/2}, \qquad k \le n.$$

Then we have

$$\mathbb{E}(\sum_{m=1}^{n} v_m^2)^{1/2} \le 2\mathbb{E}(\sum_{m=1}^{n} u_m^2)^{1/2}.$$

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Progress on the HRT conjecture

Ciprian Demeter

The following conjecture, known as the HRT conjecture was proposed in [4]. See also [5] for a nice discussion on the subject.

Conjecture 1. *Let $(t_j, \xi_j)_{j=1}^n$ be $n \geq 2$ distinct points in the plane (call them time-frequency shifts). Then there is no nontrivial L^2 function $f : \mathbb{R} \to \mathbb{C}$ satisfying a nontrivial linear dependence*

$$\sum_{j=1}^{n} d_i f(x + t_j) e^{2\pi i \xi_j x} = 0, \tag{1}$$

for a.e. $x \in \mathbb{R}$.

The Conjecture was proved when $(t_i, \xi_i)_{i=1}^n$ sit on a lattice by Linnell [6], using von Neumann algebra techniques. In [2], together with Gautam we produced a simpler argument which is inspired by the theory of random Schrödinger operators.

The following weaker conjecture has also been circulated (see for example [5]).

Conjecture 2. *Let $(t_j,\xi_j)_{j=1}^n$ be $n\geq 2$ distinct points in the plane. Then there is no nontrivial Schwartz function $f:\mathbb{R}\to\mathbb{C}$ satisfying a nontrivial linear dependence*

$$\sum_{j=1}^n d_i f(x+t_j)e^{2\pi i\xi_j x}=0,$$

for a.e. $x\in\mathbb{R}$.

In light of the discussion above, both conjectures are true if one considers only three time-frequency shifts (t_i,ξ_i). The case involving four shifts is open. We proposed to investigate special configurations of four shifts, and were less to interesting number theoretical considerations.

Call an (n,m) configuration any collection of $n+m$ distinct points in the plane such that there exist two distinct parallel lines such that one of them contains exactly n of the points, and the other one contains exactly m of the points. Our main results are:

Theorem 3 ([1]). *Conjecture 2 holds for all $(1,3)$ and $(2,2)$ configurations.*

Let $\|x\|$ denote the distance of x to the nearest integer.

Theorem 4 ([1]). *Conjecture 1 holds for special $(1,3)$ configurations*

$$(0,0),(1,0),(1,\alpha),(1,\beta)$$

(a) if

$$\liminf_{n\to\infty} n\log n\|n\frac{\beta}{\alpha}\|<\infty$$

(b) if at least one of α,β is rational
In either case, no nontrivial solution f can exist satisfying minimal decay

$$\lim_{\substack{|n|\to\infty\\ n\in\mathbb{Z}}}|f(x+n)|=0,\quad a.e.\ x$$

As a consequence, the Conjecture 1 holds for almost all $(1,3)$ configurations.

In the $(2,2)$ case, we obtained the strongest possible result with Zaharescu

Theorem 5 ([3]). *Consider any special $(2,2)$ configuration $(0,0),(1,0),(0,\alpha),(1,\beta)$. Then the equation (1) has no nontrivial solution f satisfying minimal decay*

$$\lim_{\substack{|n|\to\infty\\ n\in\mathbb{Z}}}|f(x+n)|=0,\quad a.e.\ x$$

As a consequence, the Conjecture 1 holds true for all $(2,2)$ configurations.

One of the key ingredients in our arguments is the existence of simultaneous approximants $P\in\mathbb{R}$, $P\to\infty$ such that

$$P\max\{\|P\alpha\|,\|P\beta\|\}\lesssim 1.$$

The key feature of any special (n,m) configuration of points is the fact that any linear dependence between the corresponding time-frequency translates of a hypothetical solution forces a (scalar) recurrence along $\mathbb{Z}$ orbits $x+\mathbb{Z}$. We use

Diophantine approximation to identify appropriate scales. For each fixed scale, we investigate the recurrence along finite portions of two carefully chosen distinct orbits, with length comparable to the scale.

1. Questions

1. Our approach fails for $(1,4)$ configurations such as $(0,0),(1,0),(1,\alpha),(1,\beta),(1,\gamma)$. This is because the best one can guarantee in general is the existence of arbitrarily large P such that $\max\{\|P\alpha\|,\|P\beta\|,\|P\gamma\|\} \lesssim \frac{1}{\sqrt{P}}$. It is not clear whether working with 3 or more orbits would have more to say about this case.
2. How to deal with four point configurations sitting on three, rather than two lines? Here, matrix (rather than scalar) recurrences occur.

For example, does

$$f(x+1) = C_1 f(x-1) + C_2 f(x)e(\alpha x) + C_3 f(x)e(\beta x)$$

have a Schwartz solution? This amounts to the recurrence

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} C_2 e(\alpha x) + C_3 e(\beta x) & C_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$$

admitting a rapidly decaying (at $\pm\infty$) solution $u_n = u_n(x)$ for a set of x in $[0,1)$ of positive measure.

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On Erdös' distinct distances problem

Nets Katz

(joint work with Larry Guth)

In joint work with Larry Guth, we obtain nearly sharp bounds in the Erdös distance problem. That is, we show that if P is a set of N points in the Euclidean plane and $d(P)$ is the set of distances between pairs of points in P, we have the inequality:

$$d(P) \geq \frac{N}{C \log N}.$$

To show this, we apply the reduction of Elekes and Sharir which serves as the application of the Erlangen program to the study of the distance problem. We pass to the group of rigid motions of the plane. For any pair (p, p') of points from P, we define the line $l_{pp'}$ of rigid motions taking p to p'. It turns out that obtaining our bound amounts to studying intersections between these lines. We let P_k denote the set of rigid motions contained in at least k of the lines. We must prove for $2 < k < N$, the bound

$$|P_k| \leq C \frac{N^3}{k^2}.$$

For $k = 2$, the proof is an exercise in algebra. We tie the problem to the classical theory of ruled surfaces through the use of the polynomial method of Dvir and the flecnode polynomial of Salmon.

For k large, the proof is necessarily topological. We apply the polynomial ham sandwich theorem to obtain a cell decomposition and then prove our bound mimicking Clarkson et. al.'s proof of the Szemeredi Trotter theorem. Our cell decomposition admits an error set, living in the zeroes of a polynomial of low degree. For this error set, we simply apply the polynomial method following our proof of the Joints conjecture and Bourgain's conjecture for the discrete Kakeya set.

The nonlinear Schrödinger equation below L^2

Herbert Koch
(joint work with Daniel Tataru)

We consider the cubic Nonlinear Schrödinger equation (NLS)

$$iu_t - u_{xx} \pm u|u|^2 = 0, \qquad u(0) = u_0, \tag{1}$$

in one space dimension, either focusing or defocusing. This problem is invariant with respect to the scaling

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$$

as is the Sobolev space $\dot{H}^{-\frac{1}{2}}$, which one may view as the critical Sobolev space. The NLS equation (1) is also invariant under the Galilean transformation

$$u(x,t) \to e^{icx - ic^2 t} u(x + 2ct, t)$$

which corresponds to a shift in the frequency space. However the space $\dot{H}^{-\frac{1}{2}}$ is not Galilean invariant.

This problem is globally well-posed for initial data $u_0 \in L^2$, and the L^2 norm of the solution is conserved along the flow. Furthermore, the solution has a Lipschitz dependence on the initial data, uniformly for time in a compact set and data in bounded sets in L^2. Precisely, if u and v are two solutions for (1) with initial data u_0, respectively v_0 then we have

$$\|u(t) - v(t)\|_{L^2} \lesssim \|u_0 - v_0\|_{L^2}, \qquad |t| < 1, \quad \|u_0\|_{L^2}, \|v_0\|_{L^2} \leq 1$$

By scaling and reiteration this implies a global in time bound

$$\|u(t) - v(t)\|_{L^2} \lesssim e^{C|t|(\|u_0\|_{L^2}+\|v_0\|_{L^2})^4}\|u_0 - v_0\|_{L^2} \tag{2}$$

As a consequence of the Galilean invariance, the map from initial data to the solution at time 1 cannot be uniformly continuous in the unit ball in H^s with $s < 0$. However, one may hope for continuous dependence on the initial data, a question which is outside the scope of our work.

The problem of obtaining apriori estimates in negative Sobolev spaces was previously considered by Christ-Colliander-Tao [1] ($s \geq -1/12$) and by the authors [2]($s \geq -1/6$). One key idea was that one can bootstrap suitable Strichartz type norms of the solution but only on frequency dependent time-scales. Another idea was to use the I-method to construct better almost conserved H^s type norms for the problem.

Local energy bounds, a new ingredient here, gives apriori estimates in H^s for $s \geq \frac{1}{4}$. It is likely that $-1/4$ is not optimal. Our main result is as follows:

Theorem 1. *There exists* $\varepsilon > 0$ *such that the following is true. Let*

$$-\frac{1}{4} \leq s < 0, \qquad \Lambda \geq 1$$

and assume that the initial data $u_0 \in L^2$ *satisfies*

$$\|u_0\|^2_{H^s_\Lambda} := \int (\Lambda^2 + \xi^2)^s |\hat{u}_0|^2 d\xi < \varepsilon^2.$$

Then the solution u *to* (1) *satisfies*

$$\sup_{0\leq t\leq 1} \|u(t)\|_{H^s_\Lambda} \leq 2\|u_0\|_{H^s_\Lambda}. \tag{3}$$

Applying the above theorem to a rescaled solution for $s = \frac{1}{4}$ with increasing values of Λ yields global in time bounds.

Corollary 2. *Let* $-\frac{1}{4} < s < 0$ *and* $M \geq 1$. *Let* u *be a solution to* (1) *with initial data* $u_0 \in L^2$ *so that*

$$\|u_0\|_{H^s} \leq M$$

Then for all $T > 0$ *the function* u *satisfies*

$$\sup_{|t|\leq T} \|u(t)\|_{H^s_{\Lambda(T)}} \lesssim M, \qquad \Lambda(T) = \max\{T^{\frac{1}{8s+2}} M^{\frac{4}{4s+1}}, M^{\frac{2}{2s+1}}\} \tag{4}$$

The apriori estimates suffice to construct global weak solutions. Using the uniform bounds one may prove the following statement.

Theorem 3. *Suppose that* $u_0 \in H^s$, $s \geq -\frac{1}{4}$. *Then there exists a weak solution* u *in a function space which imbeds into* $C(\mathbb{R}, H^s)$.

Some heuristic considerations. The nonlinear Schrödinger equation is completely integrable. Depending on whether we look at the focusing or the defocusing problem, we expect two possible types of behavior for frequency localized data.

In the defocusing case, we expect the solutions to disperse spatially. However, in frequency there should only be a limited spreading, to a range below the dyadic scale, which depends only on the L^2 size of the data. Energy estimates show that for frequency localized data with L^2 norm λ, frequency spreading occurs at most up to scale λ.

Examples of solutions studied either by inverse scattering or by a nonlinear semiclassical analysis indicate that energy may spread over a large frequency interval even if the energy is concentrated at frequencies $\lesssim 1$ initially, and there are solutions with energy distributed over a large frequency interval with velocity zero.

For the proof of our main result we use localization in frequency and space. Thereby we limit the spreading in frequency, and we connect velocity and frequency.

An overview of the proof. We begin with a dyadic Littlewood-Paley frequency decomposition of the solution u,

$$u = \sum_{\lambda \geq \Lambda} u_\lambda, \qquad u_\lambda = P_\lambda u$$

where λ takes dyadic values not smaller than Λ, and u_Λ contains all frequencies up to size Λ. Here the multipliers P_λ are standard Littlewood-Paley projectors. For each such λ we also use a spatial partition of unity on the λ^{1+4s} scale,

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x), \qquad \chi_j^\lambda(x) = \chi(\lambda^{-1-4s}x - j) \tag{5}$$

with $\chi \in C_0^\infty(-1,1)$. To prove the theorem we will use

(i) Two energy spaces, namely a standard energy norm

$$\|u\|^2_{l^2 L^\infty H^s_\Lambda} = \sum_{\lambda \geq \Lambda} \lambda^{2s} \|u_\lambda\|^2_{L^\infty L^2} \tag{6}$$

and a local energy norm adapted to the λ^{1+4s} spatial scale,

$$\|u\|^2_{l^2 l^\infty L^2 H^{-s}_\Lambda} = \sum_{\lambda \geq \Lambda} \lambda^{-2s-2} \sup_{j \in \mathbb{Z}} \|\chi_j^\lambda \partial_x u_\lambda\|^2_{L^2} \tag{7}$$

(ii) Two Banach spaces X^s_Λ and $X^s_{\Lambda,le}$ measuring the space-time regularity of the solution u.

(iii) Two corresponding Banach spaces Y^s_Λ and $Y^s_{\Lambda,le}$ measuring the regularity of the nonlinear term $|u|^2 u$.

The proof relies on

$$\|u\|_{X^s_\Lambda} \lesssim \|u\|_{l^2 L^\infty H^s_\Lambda} + \|(i\partial_t - \Delta)u\|_{Y^s_\Lambda}, \quad \|u\|_{X^s_{\Lambda,le}} \lesssim \|u\|_{l^2 l^\infty L^2 H^{-s}_\Lambda} + \|(i\partial_t - \Delta)u\|_{Y^s_{\Lambda,le}}$$

a cubic bound,

$$\||u|^2 u\|_{Y^s_\Lambda \cap Y^s_{\Lambda,le}} \lesssim \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}}$$

and an energy estimate and a local energy estimate for a solution u to (1) with $\|u\|_{l^2 L^\infty H^s_\Lambda} \ll 1$:

$$\|u\|_{l^2L^\infty H^s_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda} + \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}}, \qquad \|u\|_{l^2 l^\infty L^2 H^{-s}_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda} + \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}}.$$

The bootstrap argument leads to Theorem 1.

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The sharp L^p-L^2 Fourier restriction estimates for hypersurfaces in $\mathbb{R}^3$.

Isroil A. Ikromov

(joint work with Detlef Müller)

Let S be a smooth, finite type hypersurface in $\mathbb{R}^3$ with Riemannian surface measure $d\sigma$, and consider the compactly supported measure $d\mu := \rho d\sigma$ on S, where $0 \le \rho \in C_0^\infty(S)$. The goal of this talk is to determine, the sharp range of exponents p for which the Fourier restriction estimate

$$\Big(\int_S |\widehat{f}|^2 \, d\mu\Big)^{1/2} \le C_p \|f\|_{L^p(\mathbb{R}^3)}, \qquad f \in \S(\mathbb{R}^3), \tag{1}$$

holds true. To this end, we may localize to a sufficiently small neighborhoods of a given point x^0 on S. By applying a suitable Euclidean motion of $\mathbb{R}^3$ we may then assume that $x^0 = (0,0,0)$, and that S is the graph

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

of a smooth function ϕ defined on a sufficiently small neighborhood Ω of the origin, such that $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$.

In our article [3], this problem had been solved, in terms of Newton diagrams associated to ϕ, under the assumption that there exists a linear coordinate system which is adapted to the function ϕ, in the sense of Varchenko.

We had proved the following result:

Theorem 1. *Assume that, after applying a suitable linear change of coordinates, the coordinates (x_1, x_2) are adapted to ϕ e.g $h_{lin}(\phi) = h(\phi)$. We then define the critical exponent p_c by*

$$p'_c := 2h(\phi) + 2 = 2h_{lin}(\phi) + 2, \tag{2}$$

where p' denotes the exponent conjugate to p, i.e., $1/p + 1/p' = 1$.

Then there exists a neighborhood $U \subset S$ of the point x^0 such that for every non-negative density $\rho \in C_0^\infty(U)$ the Fourier restriction estimate(1) holds true for every p such that

$$1 \le p \le p_c. \tag{3}$$

Moreover, if $\rho(x^0) \neq 0$, then the condition (3) *on p is also necessary for the validity of* (1)*.*

From now on, we shall therefore always make the following

Assumption 2. *There is no linear coordinate system which is adapted to ϕ e.g. $h_{lin}(\phi) < h(\phi)$.*

In order to formulate our main result, we need more notation. Also we use notations from our papers [2] and [3].

A given coordinate system x is said to be *adapted* to ϕ if $h(\phi) = d_x$.

In the case where the coordinates (x_1, x_2) are not adapted to ϕ, we see that the principal face $\pi(\phi)$ is a compact edge lying on a unique line

$$L = \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

and that $m_1 := \kappa_2/\kappa_1 \in \mathbb{N}$. Moreover, one can prove that under the assumption 2 if the coordinates are linearly adapted to ϕ then

$$m := m_1 = \kappa_2/\kappa_1 \in \mathbb{N} \qquad \text{and } m \geq 2. \tag{4}$$

In addition if we have two linearly adapted coordinates systems with $\kappa_2 \geq \kappa_1$ and $\kappa'_2 \geq \kappa'_2$ then $\kappa_2/\kappa_1 = \kappa'_2/\kappa'_1 := m$. Thus the number m is wel-defined.

Then, by Theorem 5.1 in [2], there exists a smooth real-valued function ψ (which we may choose as the so-called principal root jet of ϕ) of the form

$$\psi(x_1) = c x_1^m + O(x_1^{m+1}) \tag{5}$$

with $c \neq 0$ defined on a neighborhood of the origin such that an adapted coordinate system (y_1, y_2) for ϕ is given locally near the origin by means of the (in general non-linear) shear

$$y_1 := x_1, \ y_2 := x_2 - \psi(x_1). \tag{6}$$

In these coordinates, ϕ is given by

$$\phi^a(y) := \phi(y_1, y_2 + \psi(y_1)). \tag{7}$$

Let us then denote the vertices of the Newton polyhedron $\mathcal{N}(\phi^a)$ by (A_ℓ, B_ℓ), $\ell = 0, \ldots, n$, where we assume that they are ordered so that $A_{\ell-1} < A_\ell$, $\ell = 1, \ldots, n$, with associated compact edges given by the intervals $\gamma_\ell := [(A_{\ell-1}, B_{\ell-1}), (A_\ell, B_\ell)]$, $\ell = 1, \ldots .n$ The unbounded horizontal edge with left endpoint (A_n, B_n) will be denoted by γ_∞. To each of these edges γ_ℓ,, we associate the weight $\kappa^\ell = (\kappa_1^\ell, \kappa_2^\ell)$, so that γ_ℓ is contained in the line

$$L_\ell := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^\ell t_1 + \kappa_2^\ell t_2 = 1\}.$$

For $\ell = \infty$, we have $\kappa_1^\infty := 0$. We denote by

$$a_\ell := \frac{\kappa_2^\ell}{\kappa_1^\ell}$$

the reciprocal of the slope of the line L_ℓ.

Then the κ^ℓ-principal part ϕ_{κ^ℓ} of ϕ corresponding to the supporting line L_ℓ is of the form

$$\phi_{\kappa^\ell}(x) = c_\ell\, x_1^{A_{\ell-1}} x_2^{B_\ell} \prod_\alpha \Big(x_2 - c_\ell^\alpha x_1^{a_\ell}\Big)^{N_\alpha}$$

(cf. [3]). In view of this identity, we shall say that the edge

$$\gamma_\ell := [(A_{\ell-1}, B_{\ell-1}), (A_\ell, B_\ell)]$$

is associated to the cluster of roots $[\ell]$.

Consider the line parallel to the bi-sectrix

$$\Delta^{(m)} := \{(t, t+m+1) : t \in \mathbb{R}\}.$$

For any edge $\gamma_\ell \subset L_\ell := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^\ell t_1 + \kappa_2^\ell t_2 = 1\}$ define h_ℓ by

$$\Delta^{(m)} \cap L_\ell = \{(h_\ell - m, h_\ell + 1)\},$$

i.e.,

$$h_\ell = \frac{1 + m\kappa_1^\ell - \kappa_2^\ell}{\kappa_1^\ell + \kappa_2^\ell}, \tag{8}$$

and define the *m-height* of ϕ by

$$h^{(m)}(\phi) := \max(d, \max_{\ell : a_\ell > m} h_\ell).$$

Remarks 3. (a) *For L in place of L_ℓ and κ in place of κ^l, one has $m = \kappa_2/\kappa_1$ and $d = 1/(\kappa_1 + \kappa_2)$, so that one gets d in place of h_ℓ in* (8)

(b) *Since $m < a_\ell$, we have $h_\ell < 1/(\kappa_1^\ell + \kappa_2^\ell)$, hence $h^{(m)}(\phi) < h(\phi)$.*

Theorem 4. *Assume that there is no linear coordinate system adapted to the non-zero real analytic function ϕ, and that $m \geq 2$ in* (5). *Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every non-negative density $\rho \in C_0^\infty(U)$, the Fourier restriction estimate* (1) *holds true for every $p \geq 1$ such that $p' \geq p'_c := 2h^{(m)}(\phi) + 2$.*

Remarks 5. (a) *The condition $p' \geq 2p'_c + 2$ is weaker than the condition $p' \geq 2h(\phi) + 2$, which would follow from Greenleaf's result* [1] *unless $\nu(\phi) = 1$.*

(b) *Again, Knapp type examples show that our result is sharp.*

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A Multidimensional Resolution of Singularities with Applications

Allan Greenleaf
(joint work with Tristan Collins, Malabika Pramanik)

Let $F \in C^\omega(\mathbb{R}^n)$, $F(0) = 0$. Some basic numerical invariants associated to F are

(i) the *critical integrability index* (or *log canonical threshold*), $\mu(F)$;
(ii) the *sublevel set growth rate*, $\nu(F)$; and
(iii) the *oscillatory index*, $\rho(F)$.

It is known that $\mu(F) = \nu(F)$, and Greenblatt showed that these equal $\rho(F)$ (if $\notin 2\mathbb{Z}+1$). In $\mathbb{R}^2$, Phong, Stein and Sturm showed that $\nu(F)$ equals the reciprocal of the *height* of F, $d(F)$, defined by Varchenko as follows. Let $N(F) \subset \mathbb{R}^n_+$ denote the Newton polygon of F, and $d_0(F) = \inf\{d > 0 | (d, d, \dots, d) \in N(F)\}$ the Newton distance, which is coordinate-dependent. Then,

$$d(F) := \sup\{d_0(F \circ \Phi) | \Phi : \mathbb{R}^n \xrightarrow{C^\omega} \mathbb{R}^n,\ \Phi(0) = 0\},$$

and a coordinate system Φ is said to be *adapted* if $d_0(F \circ \Phi) = d(F)$. Varchenko showed that adapted coordinates always exist in two dimensions, and can be taken to be of the form

$$\Phi(x_1, x_2) = (x_1, x_2 - r(x_1)) \text{ or } (x_1 - r(x_2), x_2) \text{ for some } r \in C^\omega.$$

Building on work of Parusiński [1994,200], we describe an algorithm, inductive on the dimension, for constructing a class $\mathcal{C}$ of *local* coordinate systems $\Phi(\phi, V, r)$, given by fractional power series on domains (called *towers of horns*) V such that $0 \in \overline{V}$. The Newton distance $d_0(F \circ \Phi(\phi, V, r))$ is defined for each such Φ, and we can then define a new height function,

$$d(F) := \sup\{d_0(F \circ \Phi(\phi, V, r)) | \Phi \in \mathcal{C}\}.$$

Among other things, we prove that this is computable:

Theorem 1. There is an algorithmically constructible finite set, $\mathcal{C}^* \subset \mathcal{C}$, such that $d(F) = \sup\{d(F \circ \Phi) | \Phi \in \mathcal{C}^*\}$.

Define the resulting Newton exponent or decay rate, $\delta(F) := d(F)^{-1}$; we prove

Theorem 2. For $F \in C^\omega(\mathbb{R}^n)$, the Newton exponent $\delta(F)$ equals the critical integrability index $\mu(F)$, sublevel growth index $\nu(F)$, and oscillatory index $\rho(F)$ (if $\notin 2\mathbb{Z} + 1$).

A dyadic model for Toeplitz products on Bergman space

Sandra Pott
(joint work with Alexandru Aleman)

In the early 90's, D. Sarason posed conjectures on the characterization of the boundedness of Toeplitz products on Hardy and Bergman spaces. The Hardy space case attracted much attention because of its close relation to the joint A_2 conjecture for the famous two-weight problem for the Hilbert transform in Real Analysis, pointed out by Cruz-Uribe in [1], but both conjectures, the Sarason conjecture for Toeplitz products on Hardy space and the joint A_2 conjecture, were shown to be false by F. Nazarov around 2000 [2].

The Bergman space case of Sarason's conjecture is still open, and is likewise connected to two-weighted inequalities on Bergman space.

In the talk, I will present a dyadic model for Toeplitz products on Bergman space. We will prove a test function criterion for boundedness of such dyadic model operators, in the style of the test function criteria of Nazarov, Treil and Volberg for boundedness of two-weighted dyadic shifts in [4].

This is joint work with A. Aleman.

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Dyadic shifts—new building blocks of singular integral operators

Tuomas Hytönen

In my talk, I discussed the following dyadic model of singular integral operators:

Definition (Dyadic shifts, [6])**.** An operator S is called a dyadic shift of type $(i,j)\in\mathbb{N}\times\mathbb{N}$, associated to a system of dyadic cubes $\mathscr{D}$ on $\mathbb{R}^d$, if it has the form

$$S=\sum_{K\in\mathscr{D}}A_K,\qquad A_Kf=\sum_{\substack{I,J\in\mathscr{D},I,J\subseteq K\\ \ell(I)=2^{-i}\ell(K)\\ \ell(J)=2^{-j}\ell(K)}}a_{IJK}h_I\langle h_J,f\rangle,$$

where h_I is a Haar function on the cube I (constant on the dyadic subcubes of I and normalized in L^2), similarly h_J on J, and the a_{IJK} are constants normalized by $|a_{IJK}|\leq\sqrt{|I||J|}/|K|$. It is also required that all subshifts $S_{\mathscr{Q}}:=\sum_{K\in\mathscr{Q}}A_K$ are bounded on L^2 with operator norm at most 1, for all $\mathscr{Q}\subseteq\mathscr{D}$.

A shift is called cancellative if all appearing Haar functions satisfy $\int h_I = \int h_J = 0$. In this case the boundedness of the subshifts is automatic from the uniform boundedness of all the A_K (which is easy to check) and orthogonality. Examples include the Haar multipliers (of type $(0,0)$, also known as dyadic martingale transforms) with $A_K f = \lambda_K h_K \langle h_K, f\rangle$, and Petermichl's dyadic shift (of type $(1,0)$, in dimension $d=1$) with $A_K f = 2^{-1/2}(h_{K_{\text{left}}} - h_{K_{\text{right}}})\langle h_K, f\rangle$, which was used in her celebrated representation of the Hilbert transform [10], and gave the name to the whole family of shifts as defined above. The general definition is due to Lacey, Petermichl and Reguera [6].

The main example of non-cancellative shifts is the dyadic paraproduct (of type $(0,0)$) with $A_K f = \lambda_K h_K \langle h_K^0, f\rangle$, where $h_K^0 := |K|^{-1/2} 1_K$, and the coefficients are required to satisfy the Carleson condition.

The importance of this dyadic model comes from the fact that it is rich enough to represent all classical Calderón–Zygmund operators:

Theorem (The dyadic representation theorem, [1]). *Let $T \in \mathrm{CZO}_\alpha$—a singular integral operator with a Calderón–Zygmund kernel of Hölder exponent $\alpha \in (0,1]$—, and satisfy the $T1$ conditions*

$$|\langle 1_Q, T1_Q\rangle| \leq c_T, \qquad T1 \in \mathrm{BMO}, \qquad T^*1 \in \mathrm{BMO}.$$

Then it has the following representation valid for all $f, g \in C_c^1(\mathbb{R}^d)$:

$$\langle g, Tf\rangle = c_{T,\alpha}\mathbb{E}_{\mathscr{D}} \sum_{i,j=0}^{\infty} 2^{-\max\{i,j\}\alpha/2} \langle g, S_{\mathscr{D}}^{ij} f\rangle,$$

where $c_{T,\alpha}$ is a constant, $S_{\mathscr{D}}^{ij}$ is a dyadic shift of type (i,j) associated to the dyadic system $\mathscr{D}$, and $\mathbb{E}_{\mathscr{D}}$ is the expectation over a certain random choice of $\mathscr{D}$. The shift $S_{\mathscr{D}}^{00}$ is the sum of a Haar multiplier, a paraproduct, and an adjoint paraproduct, while all other shifts are cancellative.

Since all the shifts $S_{\mathscr{D}}^{ij}$ are uniformly bounded operators on L^2, this representation contains, in particular, the $T1$ theorem of David and Journé—the L^2 boundedness of T under the $T1$ conditions. This is not a coincidence, since my proof of the dyadic representation was in fact inspired by an extension of the $T1$ theorem to non-doubling measures by Nazarov, Treil and Volberg [8], which implicitly involves a predecessor of such a decomposition. Their important notions of random dyadic systems, good and bad cubes, and estimates for their probabilities, again appear in the proof of the dyadic representation, although with some new twist.

Besides reproving the $T1$ theorem, the dyadic representation has the following main consequence for sharp weighted norm inequalities, which have attracted some attention in the last few years:

Corollary (The A_2 theorem, [1]). *Let $T \in \mathrm{CZO}_\alpha$, $\alpha \in (0,1]$, and $w \in A_2$—Muckenhoupt's weight class with*

$$[w]_{A_2} := \sup_Q \fint_Q w \cdot \fint_Q \frac{1}{w} < \infty,$$

where the supremum is over all axes-parallel cubes in $\mathbb{R}^d$*. Then* $T : L^2(w) \to L^2(w)$*—this qualitative mapping property is classical!—satisfies the quantitative bound*

$$\|Tf\|_{L^2(w)} \leq c_{T,\alpha}[w]_{A_2}\|f\|_{L^2(w)},$$

and the linear bound in terms of $[w]_{A_2}$ *is best possible.*

The dyadic representation effectively reduces the proof of the A_2 theorem to obtaining a similar bound for the shifts S^{ij} in place of T, with subexponential growth of the constant $c_{S^{ij}}$ in terms of the shift parameters (i, j) to be able to sum up the series. In the past one year after my first proof of this result, a number of different proofs by several authors have appeared [2, 3, 4, 5, 7, 9, 11]. However, they are all based on the dyadic representation theorem, which seems like a strong illustration of the usefulness of such an expansion: As soon as one gets one's hands on the dyadic shifts, there are many possible ways to carry out the further analysis, while no method for getting the sharp weighted estimate is yet known without going through this dyadic model.

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New methods for boundary value problems of elliptic equations

Pascal Auscher

(joint work with Andreas Rosén)

We report on new representations and new solvability methods for boundary value problems (BVPs) for divergence form second order, real and complex, elliptic systems. We look here at BVPs in domains Lipschitz diffeomorphic to the upper half space $\mathbb{R}^{1+n}_+ := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \ ; \ t > 0\}$, $n \geq 1$. Details are to be found in [1] joint with Andreas Rosén (formerly Axelsson). For bounded domains, see [2].

The strategy applies to systems of equations $Lu = -\mathrm{div} A \nabla u = 0$ where

$$(1) \qquad Lu^\alpha(t,x) = \sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i \Big(A^{\alpha,\beta}_{i,j}(t,x) \partial_j u^\beta(t,x) \Big) = 0, \qquad \alpha = 1, \dots, m$$

in $\mathbb{R}^{1+n}_+$, where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$. We assume

$$(2) \qquad A = (A^{\alpha,\beta}_{i,j}(t,x))^{\alpha,\beta=1,\dots,m}_{i,j=0,\dots,n} \in L_\infty(\mathbb{R}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m})),$$

and that A is strictly accretive on $\mathcal{H}$, meaning that there exists $\kappa > 0$ such that

$$(3) \qquad \sum_{i,j=0}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathbb{R}^n} \mathrm{Re}(A^{\alpha,\beta}_{i,j}(t,x) f^\beta_j(x) \overline{f^\alpha_i(x)}) dx \geq \kappa \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbb{R}^n} |f^\alpha_i(x)|^2 dx,$$

for all $f \in \mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^m) \oplus \overline{\mathsf{R}(\nabla_x)} = \mathsf{N}(\mathrm{curl}_x)$ seen as a subspace of $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ and a.e. $t > 0$. For equations ($m = 1$) it is equivalent to the usual pointwise accretivity. For our results to hold, a natural Carleson type assumption is made on the coefficients. This includes all systems with t-independent coefficients.

We seek to prove well-posedness for (1), *i.e.* unique solvability in appropriate spaces given Dirichlet data $u|_{t=0}$, Neumann data $\partial_{\nu_A} u|_{t=0}$ or Dirichlet regularity data $\nabla_x u|_{t=0}$, assumed to satisfy an L_2 condition. Note that the continuity estimate required for well-posedness in the sense of Hadamard is not included in our notion of well-posedness, but is shown to hold. For the Neumann and Dirichlet regularity problems, we work in the class of weak solutions whose gradient $\nabla_{t,x} u$ has L_2 modified non-tangential maximal function $\widetilde{N}_*(\nabla_{t,x} u)$ in L_2. Under our assumptions, we obtain a representation, describe the limiting behaviour of $\nabla_{t,x} u$ at $t = 0$ and ∞ and obtain a perturbation result for well-posedness. For the Dirichlet problem, it is more natural given our method to work in the class of weak solutions with square function estimate $\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} u|^2 t dt dx < \infty$. Under our assumptions, we prove a rigidity theorem (up to a constant, such solutions are continuous in $t \geq 0$ with values in $L_2(\mathbb{R}^n, \mathbb{C}^m)$ and vanish at ∞ with $\sup_{t>0} \|u_t\|_2 \lesssim \|\nabla_{t,x} u\|_{L_2(tdt; L_2)}$), show new *a priori* non-tangential maximal estimates (fixing the constant to be 0, $\|\widetilde{N}_*(u)\|_2 \lesssim \|\nabla_{t,x} u\|_{L_2(tdt;L_2)}$) and obtain a perturbation result for well-posedness.

We refer to [1] for a comprehensive historical background. We only mention several things for the purpose of this report. Not so much was known for systems. Next, for t-independent equations, the theory is relatively complete for the real symmetric ones from the works of Dahlberg, Jerison-Kenig, Kenig-Pipher [9, 12, 13] using the technology of harmonic measure based on the maximum principle, regularity theory and Rellich identities. For the real non symmetric ones, only some specific situations are known so far. Eventually, for t-dependent equations, Caffarelli, Fabes and Kenig [7] show the necessity of a square Dini condition in the transverse direction and Fabes, Jerison, Kenig [11] showed solvability of the Dirichlet problem assuming continuity on $A(t,x)$ (We notice that as an outcome, we solve the regularity problem under this condition in [2]). Dahlberg formulated a variant which is scale-invariant: a condition of Carleson type for the coefficients $A(t,x) - A(0,x)$, the smallness of which guarantees perturbation results for the Dirichlet problem [10]. Kenig-Pipher [13] developed the corresponding perturbation theory from real symmetric t-independent equations for the Neumann and regularity problems using variational solutions.

At our level of generality, we must give up regularity theory and maximum principle, hence harmonic measure. Instead we develop a Hardy space method, going back to the origins of the development of the Hardy spaces on Euclidean space by Stein and Weiss [14].

We limit ourselves here to a rough description of the method for Neumann and regularity problems. Our basic idea for constructing solutions u to the divergence form equation (1) in $\mathbb{R}^{1+n}_+$ is to consider it as a first order div − curl system with the gradient $\nabla_{t,x}u$ as the unknown function. In fact, as first done in [3], solving for the t-derivatives in the equation, the divergence form equation for u becomes a vector-valued ODE

$$\partial_t(\nabla_{t,x}u) + T_A(\nabla_{t,x}u) = 0,$$

where T_A is an operator only involving the first order derivatives along $\mathbb{R}^n$ and operators of pointwise multiplication. It turns out that if one instead of $\nabla_{t,x}u$ takes as the unknown the **conormal gradient**

$$\nabla_A u := \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_x u \end{bmatrix}, \tag{4}$$

with inward (for convenience) conormal derivative $\partial_{\nu_A} u = (A\nabla_{t,x}u, e_0)$, then the corresponding operator T_A has a simpler structure; the ODE reads

$$\partial_t f + DBf = 0, \qquad \text{with } f := \nabla_A u, \tag{5}$$

where $D := \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1}$ writing $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the splitting $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m}) = L_2(\mathbb{R}^n; \mathbb{C}^m) \oplus L_2(\mathbb{R}^n; \mathbb{C}^{nm})$ is also a strictly accretive matrix on $\mathcal{H}$.

The first order approach is most natural for solving the Neumann and regularity BVPs, since these boundary conditions are conditions on the conormal gradient f, not on the potential u. Indeed, the Neumann BVP means that the normal part of f_0 (i.e. first component) $(f_0)_\perp = \partial_{\nu_A} u|_{\mathbb{R}^n}$ is given at the boundary $t = 0$, whereas the regularity condition is that the tangential part of f_0 (i.e. second component) $(f_0)_\parallel = \nabla_x u|_{\mathbb{R}^n}$ is given. Note that for both BVPs, only "one half" of the function f_0 is prescribed.

This algebraic reduction was the key discovery in [4] when coefficients A do not depend on t and the ODE is autonomous as DB does not depend on t. It carries in extenso to all coefficients. However, the ODE becomes non-autonomous. The main results in [1] are to integrate this ODE (a Duhamel type formula) in the appropriate and most general classes of weak solutions to obtain representation and the boundary maps between initial values f_0 and boundary datas that allow to formulate solvability. We develop for that new maximal regularity estimates using the tent space theory of Coifman-Meyer-Stein [8]. For the Dirichlet problem, one must integrate once more as one seeks estimates on the potential u rather than its gradient.

Let us finish by explaining the case of t-independent coefficients $B(t,x) = B_0(x)$. In this case, we view DB_0 as an unbounded operator in $L_2(\mathbb{R}^n;\mathbb{C}^{(1+n)m})$, and at a first glance the solution to (5) with initial datum f_0 seems to be $f_t = e^{-tDB_0}f_0$. However, the problem is that DB_0 is not a sectorial operator, but instead bisectorial, *i.e.* its spectrum is contained in a double sector around the real axis. This indefiniteness means that the operators e^{-tDB_0} are not well defined on $L_2(\mathbb{R}^n;\mathbb{C}^{(1+n)m})$ for any $t \neq 0$. Another technical problem is that DB_0 has an infinite dimensional null space. The fact is that there are topological splittings

$$L_2 = \mathcal{H} \oplus \mathsf{N}(DB_0) = \left(E_0^+\mathcal{H} \oplus E_0^-\mathcal{H}\right) \oplus \mathsf{N}(DB_0),$$

noticing that $\overline{\mathsf{R}(DB_0)} = \overline{\mathsf{R}(D)} = \mathcal{H}$. The splitting of $\mathcal{H}$ into the spectral subspace $E_0^+\mathcal{H}$ for the sector in the right half plane and the spectral subspace $E_0^-\mathcal{H}$ for the sector in the left half plane is a deep result, and builds as was done in [6] on the Kato square root problem solved in [5]. This proof also shows that DB_0 has square function estimates, which in particular shows that $-DB_0$ generates a bounded holomorphic semigroup in $E_0^+\mathcal{H}$, and that DB_0 generates a bounded holomorphic semigroup in $E_0^-\mathcal{H}$. The spectral subspace $E_0^+\mathcal{H}$ (resp. $E_0^-\mathcal{H}$) looks like the holomorphic Hardy space on the upper (resp. lower) half-space, but associated to (1) instead of the Laplace equation (if $B_0 = I$, then the elements exactly are the Riesz systems of Stein and Weiss [14]: in boundary dimension $n = 1$, this comparison is exact). We note that $B_0^{-1}E_0^{\pm}\mathcal{H}$ are the spectral spaces in the splitting associated to B_0D. Let us isolate one of the results in [1] to conclude.

Theorem 1. *(i) All weak solutions u of* (1) *with L_2 modified non-tangential maximal function $\widetilde{N}_*(\nabla_{t,x}u) \in L_2$ are of the form $\nabla_A u(t,x) = (e^{-tDB_0}f_0)(x)$ for some $f_0 \in E_0^+\mathcal{H}$ and, moreover, $\|\widetilde{N}_*(\nabla_{t,x}u)\|_2 \sim \|f_0\|_2$.*

(ii) All weak solutions u of (1) *with square function $\|\nabla_{t,x}u\|_{L_2(tdt;L_2)} < \infty$ are of the form $u(t,x) = (e^{-tB_0D}\tilde{f}_0)_\perp(x)$ for some $\tilde{f}_0 \in B_0^{-1}E_0^+\mathcal{H}$ and, moreover, $\|\widetilde{N}_*(u)\|_2 \lesssim \|\nabla_{t,x}u\|_{L_2(tdt;L_2)} \sim \|\tilde{f}_0\|_2$.*

The non-tangential maximal and square function estimates for solutions of this form were already proved in [4]. The novelty is that they all are of that form. As corollaries, (i) allows right away to formulate the Neumann problem in the optimal class and to see that well-posedness is equivalent to the invertibility of the map $f_0 \in E_0^+\mathcal{H} \mapsto (f_0)_\perp \in L_2(\mathbb{R}^n, \mathbb{C}^m)$, while (ii) tells that the Dirichlet problem can always be formulated in the square function class (while it is not the case in the class $\|\widetilde{N}_*(u)\|_2 < \infty$) and that well-posedness is equivalent to the invertibility of the map $\tilde{f}_0 \in B_0^{-1}E_0^+\mathcal{H} \mapsto (\tilde{f}_0)_\perp \in L_2(\mathbb{R}^n, \mathbb{C}^m)$.

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Hessian Sobolev and Poincaré inequalities

Igor Verbitsky

(joint work with Fausto Ferrari, Bruno Franchi)

The fully nonlinear k-Hessian operator $F_k[u]$ $(k = 1, \dots, n)$ is defined as the sum of all the $k \times k$ principal minors of the Hessian matrix D^2u, where $u \in C^2(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$. For $k = 1$, $F_k[u]$ coincides with the Laplacian $\triangle u$, and for $k = n$, with the Monge–Ampère operator.

The k-Hessian is elliptic when restricted to the cone of k-convex functions such that $F_j[u] \geq 0$ for $j = 1, 2, \dots, k$. This definition has been extended to upper semicontinuous functions $u : \Omega \to [-\infty, +\infty)$ by Trudinger and Wang [8]. In this case $F_k[u]$ is a nonnegative measure defined in terms of viscosity solutions: u is *k-convex* if $F_k[q] \geq 0$ for any quadratic polynomial q such that $u - q$ has a local finite maximum in Ω.

Denote by $\Phi^k(\Omega)$ the class of all k-convex functions in Ω. Then

$$\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega).$$

Here $\Phi^1(\Omega)$ is the class of all subharmonic functions, and $\Phi^n(\Omega)$ is the class of all convex functions in Ω. The domain Ω is assumed to be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$, $H_j(\partial\Omega) > 0$, $j = 1, ..., k-1$; $H_j(\partial\Omega)$ denotes the j-mean curvature of the boundary $\partial\Omega$.

The following Hessian Sobolev inequality is due to X.-J. Wang (1994):

$$\left(\int_\Omega |u|^q\,dx\right)^{\frac{1}{q}} \leq C\left(\int_\Omega -u\,F_k[u]\,dx\right)^{\frac{1}{k+1}}.$$

$q = \frac{(k+1)n}{n-2k}$, $1 \leq k \leq \frac{n}{2}$. Here u is k-convex, $u = 0$ on $\partial\Omega$, and Ω is $(k-1)$-convex.

The Hessian–Poincaré inequality:

$$\left(\int_\Omega |Du|^2\,dx\right)^{\frac{1}{2}} \leq C\left(\int_\Omega -u\,F_k[u]\,dx\right)^{\frac{1}{k+1}},$$

as well as the higher–order Poincaré inequalities:

$$\left(\int_\Omega -u\,F_l[u]\,dx\right)^{\frac{1}{l+1}} \leq C\left(\int_\Omega -u\,F_k[u]\,dx\right)^{\frac{1}{k+1}},$$

for $0 \leq l < k \leq n$, were established by Trudinger and Wang in [7]. New simpler proofs were given in [9] together with some extensions.

The following two theorems [3] establish the relation between the k-Hessian energy $\mathcal{E}_k[u] = \int_\Omega -u\,F_k[u]\,dx$, and the more conventional fractional Laplacian energy $E_k[u] = \int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{k}{k+1}}u\right|^{k+1}dx$, for k-convex functions in the case $\Omega = \mathbb{R}^n$. Denote by $\Phi_0^k(\mathbb{R}^n)$ the class of k-convex functions vanishing at infinity.

Theorem 1 [3]. *Let $u \in \Phi_0^k(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$. Then there exists a positive constant $c_{k,n}$ such that*

$$\int_{\mathbb{R}^n} -u\,F_k[u]\,dx \leq c_{k,n}\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{k}{k+1}}u\right|^{k+1}dx.$$

If $k \geq \frac{n}{2}$ then the Hessian energy on $\mathbb{R}^n$ is infinite unless $u = 0$.

Theorem 2 [3]. *Let $u \in \Phi_0^k(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$. If $(-\Delta)^{\frac{k}{k+1}}[-(-\Delta)^{\frac{k}{k+1}}u]^k \geq 0$, then there exists a positive constant $C_{k,n}$ such that*

$$\int_{\mathbb{R}^n}|(-\Delta)^{\frac{k}{k+1}}u|^{k+1}dx \leq C_{k,n}\int_{\mathbb{R}^n} -u\,F_k[u]\,dx.$$

Conjecture. *The extra assumption$(-\Delta)^{\frac{k}{k+1}}[-(-\Delta)^{\frac{k}{k+1}}u]^k \geq 0$ in Theorem 2 can be dropped. In other words, $\mathcal{E}_k[u]$ is equivalent to $E_k[u]$.*

The following corollary implies that the k-Hessian capacity introduced in [8] is equivalent to the classical Bessel capacity associated with the Sobolev space $W^{\frac{2k}{k+1},k+1}$ (see [5]).

Corollary [3]. *Let $u \in \Phi_0^k(\mathbb{R}^n)$, where $1 \leq k < \frac{n}{2}$. Then there exists $\tilde{u}$ such that $c_1 \leq |u|/|\tilde{u}| \leq c_2$, and*

$$C_1\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\alpha}{2}}\tilde{u}\right|^{k+1}dx \leq \int_{\mathbb{R}^n} -u\,F_k[u]\,dx \leq C_2\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\alpha}{2}}\tilde{u}\right|^{k+1}dx,$$

where c_i, C_i $(i = 1, 2)$ are positive constants depending only on k, n.

Applications to fully nonlinear equations involving k-Hessian operators can be found in [4], [5], [6].

The following inequality for fractional Laplacians, together with some applications, is deduced as a corollary in [3], Sec. 4: for a convex function $\phi : \mathbb{R} \to \mathbb{R}_+$,

$$(-\Delta)^{\alpha/2}\,(\phi(u)) \le \phi'(u)\cdot(-\Delta)^{\alpha/2}u, \quad \text{if } 0 < \alpha \le 2.$$

I would like to thank Elias Stein for pointing out that such inequalities appeared earlier in the literature. (See [2] for $\phi(x) = x^2$, and [1] in the general case.)

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Exponential sums in two variables: the quasi-homogeneous case

James Wright

Let $f \in \mathbb{Z}[X, Y]$ be a quasi-homogeneous polynomial in two variables by which we mean there exists two positive numbers $\kappa_1, \kappa_2 > 0$ so that $f(r^{\kappa_1}x, r^{\kappa_2}y) = rf(x, y)$ for every $r \ge 0$. Our goal is to give sharp bounds on the exponential sums

$$\mathcal{S}(f;p^s) \;=\; \frac{1}{p^{2s}} \sum_{x \bmod p^s}\; \sum_{y \bmod p^s} e^{2\pi i f(x,y)/p^s}$$

where the parameter p^s is a power of a prime number p.

We will be particularly interested when the estimates are uniform in p and s with the underlying constants in the estimates depending only on the degree of f, say. Uniform estimates of the form

$$|\mathcal{S}(f;p^s)| \;\le\; C s^{i(f)} p^{-s/h(f)} \tag{1}$$

will be proved for almost every prime p and where C is an absolute constant depending only on the degree of f; that is, there is an exceptional finite set of primes $\mathcal{P}(f)$ and a constant $C = C_{\deg(f)}$ such that (1) holds for every $p \notin \mathcal{P}(f)$. In fact in almost every case the exponents $h(f)$ and $i(f)$ will be the same as those arising in the best uniform estimates for the corresponding euclidean oscillatory integrals

$$(2) \qquad \Big| \iint_{\mathbb{R}^2} e^{2\pi i \lambda f(x,y)} \phi(x,y) dx dy \Big| \ \le\ C \left[\log(|\lambda|)\right]^{i(f)} |\lambda|^{-1/h(f)}$$

where the *height* of f is defined as $h(f) := \sup_z\{d_z(f)\}$, the supremum being taken over all smooth local coordinate systems $z = (x, y)$ of the origin and d_z denotes the *Newton distance* of f in the coordinates z. In [6] an intrinsic description of the height $h(f)$ is given when f is a quasi-homogeneous polynomial; in fact in this case $h(f)$ can be described explicitly in terms of the homogeneity dilation parameters κ_1, κ_2 and the maximum multiplicity of the real roots of f. The exponent $i(f)$ is sometimes referred to as Varchenko's exponent or the *multiplicity of oscillation* of f and takes only the values 0 or 1; it is always equal to 0 except when $h(f) \ge 2$ and the *principal face* of f in *adapted coordinates*[1] is a vertex of the Newton diagram in which case we set $i(f) = 1$. Again an explicit, intrinsic description of the exponent $i(f)$ is given in [6]. The estimate (2) is sharp in the sense that

$$(3) \qquad \lim_{\lambda \to +\infty} \frac{\lambda^{1/h(f)}}{\log^{i(f)}(\lambda)} \iint_{\mathbb{R}^2} e^{2\pi i \lambda f(x,y)} \phi(x,y) dx dy \ =\ c\, \phi(0,0)$$

for some nonzero constant c if the support of ϕ is sufficiently small and if the principal face of f in adapted coordinates is a compact set. For proofs of (2) and (3), see for example [7] where these results are established for any smooth real-valued phase f of finite-type.

It turns out that the uniform estimates in (1), discrete analogues of (2), hold for every quasi-homogeneous polynomial $f \in \mathbb{Z}[X, Y]$ *except* for a single family of degenerate f of the form

$$(4) \qquad f(x, y) \ =\ a(by^2 + cxy + dx^2)^m$$

where the quadratic polynomial $by^2 + cxy + dx^2$ is irreducible over the rationals $\mathbb{Q}$. In this case (1) holds with the same decay parameter $h(f)$ but now the 0-1 valued exponent $i(f) = i_p(f)$ depends on the prime p. For example when $f(x, y) = a(y^2 - 2x^2)^m$, it turns out that $i_p(f) = 1$ when $p \equiv 1$ or 7 mod 8 and $i_p(f) = 0$ when $p \equiv 3$ or 5 mod 8.

We also mention a trivial exception to (1); the linear factor s in (1) appears for $f(x, y) = axy$ (one can simply evaluate the sum $\mathcal{S}(f; p^s)$ in this case) but the exponent $i(f)$ in (2) is 0. The difference is easily accounted for by the smooth cut-off function ϕ. To avoid mentioning this trivial exception in the statement of our main result, we reset the value $i(f)$ to be equal to 1 for $f(x, y) = axy$. In this way the only truly exceptional case to the euclidean estimate (2) will be those f of the form (4). We also obtain a version of (3) for $\mathcal{S}(f; p^s)$ in the following theorem.

Theorem 0.1. *For any quasi-homogeneous polynomial* $f \in \mathbb{Z}[X, Y]$ *not of the form* (4)*, there is a finite collection* $\mathcal{P}(f)$ *of prime numbers and constant* $C > 0$*,*

[1] a local coordinate system z where the supremum defining the height is achieved; that is, $h(f) = d_z$

depending only on the degree of f, so that

$$|\mathcal{S}(f;p^s)| \;\leq\; C s^{i(f)} p^{-s/h(f)}$$

holds for every prime $p \notin \mathcal{P}(f)$.

When f is of the form (4), the above estimate still holds but now $i(f) = i_p(f)$ depends on p; more precisely $i_p(f) = 1$ or 0 depending on whether the roots of f (a conjugate pair of algebraic numbers of degree 2 over $\mathbb{Q}$) lie in the p-adic field $\mathbb{Q}_p$ or not, respectively.

Furthermore when f is homogeneous but not in the exceptional class (4) and $h(f) > 2$, there is a constant $c > 0$, depending only on the degree of f, so that for $p \notin \mathcal{P}(f)$,

$$c s^{i(f)} p^{-s/h(f)} \;\leq\; |\mathcal{S}(f;p^s)| \tag{5}$$

holds for infinitely many $s \geq 1$. When f is as in (4), the estimate (5) still holds but with $i(f) = i_p(f)$ defined above.

For quasi-homogeneous polynomials $f \in \mathbb{Z}[X_1, \ldots, X_n]$ in arbitrary number of variables, Denef and Sperber [4] and Cluckers [1], [2] have established the estimate (1) when f is nondegenerate with respect to its Newton diagram which is related to certain conjectures of Igusa found in [5]. The estimates in Theorem 0.1 extend their work in the two variable setting to arbitrary quasi-homogeneous polynomials. In fact in [4], Denef and Sperber make a conjecture for general homogeneous polynomials (extended to quasi-homogeneous polynomials by Cluckers) and Theorem 0.1 verifies this conjecture in the two variable setting. The lower bound (5) shows the general sharpness of the estimate with respect to p and s. Sharp estimates for arbitrary quasi-homogeneous polynomials have been obtained previously by Cluckers [3] in the case when $s = 1$ or $s = 2$, again for polynomials in any number of variables.

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Hilbert transforms along one-variable vector fields

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(joint work with Christoph Thiele)

We consider the Hilbert transform

$$H_v f(x) = p.v. \int \frac{f(x - tv(x))}{t} dt. \tag{1}$$

defined for $x \in \mathbf{R}^2$, where $v\colon \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{0\}$ is a nonvanishing vector field. We can also define the analogous maximal operator

$$M_v f(x) = \sup_{\epsilon>0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x + v(x)t) dt, \tag{2}$$

again defined for $x \in \mathbf{R}^2$. Well-known conjectures of Stein and Zygmund concern the L^p boundedness properties of H_v and M_v, respectively, when the vector field v is Lipschitz.

Negative results are known to hold when v is only $C^{1-\epsilon}$ for any ϵ. This follows from rather standard examples involving Besicovitch sets in $\mathbf{R}^2$. Best known positive results are due to Stein/Stree (for H_v) and Bourgain (for M_v)– in both situations some L^p estimates are obtained for real-analytic vector fields. General results for vector fields v in smoothness classes between Hölder and real-analytic are unknown.

Recent results have been obtained by the speaker with Christoph Thiele on the L^p boundedness of H_v when the vector field v is assumed to depend only on one variable; i.e., $v(x, y) = v(x)$. The dependence on the x-coordinate can be arbitrary. In this case, we obtain

- If $p \in (\frac{3}{2}, \infty)$, then

$$||H_v f||_p \leq C||f||_p. \tag{3}$$

The most obvious open questions have to do with extending this theorem to the maximal operator M_v, and extending it to a broader class of vector fields. The result above relies heavily on estimates obtained en route to stronger results obtained by the speaker when the function in question has restricted frequency support. Specifically,

- Assume the support of $\hat{f}$ lies in an annulus. If $p \in (1, \infty)$, then

$$||H_v f||_p \leq C||f||_p. \tag{4}$$

  (Here the implicit constant depends only on p and the ratio of the outside and inside radii of the annulus.)

This second theorem relies heavily on earlier work of Lacey and Li, who established the single-annulus theorem for $p > 2$ for arbitrary v. Lacey and Li also established an approach for proving L^p results when $p < 2$ provided certain maximal estimates are known. Again, extensions to M_v and more general vector fields are of interst.

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Observations on Multilinear Oscillatory Integral Operator and Multilinear Sublevel Set Inequalities

Michael Christ

Consider a scalar-valued multilinear form

$$\mathcal{I}_\lambda(f_1, \cdots, f_n) = \int_{\mathbb{R}^d} e^{i\lambda P(y)} \prod_{j=1}^{n} \big(f_j \circ \ell_j\big)(y)\, \eta(y)\, dy$$

with $\ell_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ surjective, linear; $P : \mathbb{R}^d \to \mathbb{R}$ a real-valued polynomial phase; η a compactly supported smooth cutoff function, and $\lambda \in \mathbb{R}$ a large parameter. Is there an operator decay bound of the form

$$|\mathcal{I}_\lambda(f_1, \cdots, f_n)| \le C|\lambda|^{-\delta} \prod_j \|f_j\|_L^\infty \text{ as } |\lambda| \to \infty?$$

One can alternatively ask for a weaker bound of the form $|\mathcal{I}_\lambda(f_1, \cdots, f_n)| \le C\varepsilon(\lambda) \prod_j \|f_j\|_{L^\infty}$ where $\varepsilon(\lambda) \to 0$ as $|\lambda| \to \infty$.

A necessary condition is that P cannot be expressed as $\sum_j \big(\phi_j \circ \ell_j\big)$ for any functions $\{\phi_j\}$. We say that P is nondegenerate relative to $\{\ell_j\}$ if it cannot be so expressed. A natural conjecture is that for any nondegenerate polynomial phase P, there is a decay bound of the above type.

The bilinear case has been extensively studied, with detailed and often optimal results by many authors. Much less is known in the singular multilinear case. The subject is still at a preliminary stage, and it is not yet appropriate to ask for optimal decay exponents or L^p classes. Therefore we put the strongest relevant norm, L^∞, on f_j, and ask whether there is any positive decay exponent δ.

Christ-Li-Tao-Thiele established some relevant results in 2005. (i) $P = \sum_j \phi_j \circ \ell_j$ for distributions ϕ_j if and only if this holds with polynomials ϕ_j with degree$(\phi_j) \le$ degree(P). (ii) P is nondegenerate if and only if there exists a linear partial differential operator with constant coefficients such that $L(P) \neq 0$, but $L(f \circ \ell_j) = 0$

for all functions f and all j. (iii) Nondegeneracy implies power law decay in codimension 1 case ($d_j = d-1$ for all j). (iv) Nondegeneracy implies power law decay if all $d_j = 1$ and the number of factors f_j is $< 2d$.

Theorem. Nondegeneracy implies power law decay if $2\max_i d_i + \sum_j d_j \le 2d$, provided that $\{\ell_j\}$ lies in general position. □

If all $d_j = 1$, this just barely misses the result of Christ-Li-Tao-Thiele.

By a multilinear sublevel set associated to P we mean a set of the form

$$E_\varepsilon(P, g_1, \cdots, g_n) = \big\{y \in B : |P(y) - \sum_{j=1}^{n} g_j(\ell_j(y))| < \varepsilon\big\}.$$

It is natural to conjecture that if P is nondegenerate then $\big|E_\varepsilon(P, g_1, \cdots, g_n)\big| \le C\varepsilon^\delta$ uniformly for all functions g_j; this conjecture would be a direct consequence of the corresponding conjecture for multilinear oscillatory integrals, formulated above.

We say that $\{\ell_j\}$ is rationally commensurate if it is possible to make $\mathbb{R}$–linear changes of coordinates so that all ℓ_j are simultaneously represented by matrices with rational entries.

Theorem. Let a polynomial P be nondegenerate with respect to a finite *rationally commensurate* family $\{\ell_j\}$. Then for any measurable $\{f_j\}$, $|E_\varepsilon(P, f_1, \cdots, f_n)| \le \Theta(\varepsilon)$ where $\Theta(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$. Θ depends on P and on $\{\ell_j\}$, but not on $\{f_j\}$. If one modifies the definition of the sublevel set E_ε by requiring only that the distance from $|P(y) - \sum_{j=1}^{n} g_j(\ell_j(y))|$ to $\mathbb{Z}$ be less than ε, then the same bound holds. □

Let $P, \{\ell_j\}$ as above. We say that P is nondegenerate with a finite witness relative to $\{\ell_j\}$ if there exists a finite set $S \subset \mathbb{R}^d$ such that the restriction $P\big|_S$ of P to S does not belong to the span of the set of all functions $(f_j \circ \ell_j)\big|_S$. It is a tautology that nondegeneracy with a finite witness implies nondegeneracy.

The second theorem stated above follows by combining the following two subtheorems.

Theorem. Nondegeneracy implies nondegeneracy with a finite witness provided that $\{\ell_j\}$ is rationally commensurate. □

Theorem. If P is nondegenerate with a finite witness, then sublevel set bounds hold with some rate function $\Theta(\varepsilon)$. □

The following property of sublevel sets is used to prove the second subtheorem. Let P be a polynomial, with finite witness set S. Then there exist coefficients c_s such that $\sum_{s\in S} c_s P(s) = 1$ but $\sum_{s\in S} c_s (f_j \circ \ell_j)(s) = 0$ for all j and f_j. Define $q(r, a) = \sum_{s\in a+rS} c_s \Big(P(s) - \sum_j (f_j \circ \ell_j)(s)\Big)$. Then whenever $a + rS \subset E_\varepsilon$, necessarily $|q(r, a)| \le C\varepsilon$. Now q is a polynomial independent of $\{f_j\}$, which does not vanish identically. So $|q(r, a)| \ll 1$ occurs rarely. The thrust of Szemerédi's theorem (as extended by Furstenberg and Katznelson) is that a set which contains no images $rS + a$ must be small. A simple discretization and lifting argument allows one to conclude that if $E \subset \mathbb{R}^d$ contains $rS + a$ for few (a, r), then $|E|$ must be small.

The first subtheorem asserts the existence of finite witness sets. Its purely algebraic proof is based on finite difference equations for functions whose domains are sufficiently large finite lattices.

The first theorem formulated above, which concerns oscillatory integrals, is proved by an induction on dimensions. The analysis consists primarily of linear algebra, resting ultimately on the 2005 result of Christ-Li-Tao-Thiele for the special case in which each of the linear mappings ℓ_j has one-dimensional codomain. The proof is best illustrated via an example.

Consider

$$\iint_{\mathbb{R}^4} e^{iP(x_1,x_2,y_1,y_2)} f_0(x_1,y_1) f_1(x_2,y_2) f_2(x_1+x_2, y_1+y_2)\, dx_1\, dx_2\, dy_1\, dy_2.$$

Rewrite this as

$$\iint \Big(\iint e^{iP(s,u,t,-t+v)} f_0(s,t) f_1(u,-t+v) f_2(s+u,v)\, ds\, dt \Big)\, du\, dv,$$

which has the advantage that in the inner integral, f_1, f_2 each depend on only one of the two variables of integration.

Fix u, v. The inner integral equals $\Big\langle f_0,\, e^{iQ_{u,v}} (F_{1,u,v} \circ L_1)(F_{2,u,v} \circ L_2) \Big\rangle$ where $F_{1,u,v}(t) = f_1(u, -t+v)$, $F_{2,u,v}$ has a similar expression in terms of f_2, $Q_{u,v}$ is a certain polynomial in (s,t), $L_1(s,t) = t$, and $L_2(s,t) = s+t$.

If the original integral is not suitably small, then there exists (u,v) for which the inner integral is not small. Therefore f_0 has a nonsmall inner product with some $G = e^{iQ}(F_1 \circ L_1)(F_2 \circ L_2)$. Decompose $f_0 = \alpha G + \tilde{f}_0$ where $\alpha \in \mathbb{C}$, $|\alpha|$ is not small, and $\|\tilde{f}_0\|_2 \le (1 - c|\alpha|^2)\|f_0\|_2$. The contribution of $\tilde{f}_0$ is controlled by "induction on norm". Substitute αG for f_0 in original integral, and absorb e^{iQ} into e^{iP}. This represents progress, for one factor f_0 which depended on two variables, has been replaced by two factors which each depend on only one variable. Apply the same reasoning to f_1, f_2. In the end, we have reduced matters to the case where all of the projections ℓ_j have one-dimensional codomains.

Christ and Oliveira e Silva have found a different proof for trilinear oscillatory integral forms. Rather than reducing to product functions, one first reduces to functions of the form $e^{i\Phi(x_1,x_2)}$ where Φ is a real polynomial of controlled degree with respect to x_2, whose coefficients are unknown measurable functions of x_1. A second step then reduces further to case where these coefficients are likewise polynomials. Facts about sublevel sets are used.

Final remark. I should have tried to bypass Szemerédi's theorem by using instead theorems of Balog-Szemerédi-Gowers and Freiman (which are central ingredients in Gowers' proof of Szemerédi's theorem). This might lead to power law bounds for sublevel sets. For oscillatory integrals, repeated applications of Cauchy-Schwarz lead to a bound $\big|I_\lambda(f_1, \cdots f_n)\big| \le C\|f_i\|_{U^N} \prod_{j \ne i} \|f_j\|_{L^\infty}$ where $\|\cdot\|_{U^N}$ is the Gowers norm of sufficiently high order N. If $\|f_i\|_{U^N} \le C|\lambda|^{-\delta}\|f_i\|_\infty$ for some index i, then the proof is complete. If not, then the recent inverse theorem of Green-Tao-Ziegler for the Gowers norms gives certain information about each f_i. Does this information suffice? I hope to answer this in a future talk.

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Global Smoothing for the Periodic KdV Evolution

M. Burak Erdoğan
(joint work with Nikos Tzirakis)

In this talk we consider the Korteweg de Vries (KdV) equation on the torus

$$\begin{cases} u_t + u_{xxx} + 2uu_x = 0, & x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}), \end{cases} \tag{1}$$

The main result is the following smoothing theorem from [3]

Theorem 1. *Fix $s > -1/2$ and $a < \min(2s+1, 1)$. Consider the real valued solution of KdV (1) on $\mathbb{T} \times \mathbb{R}$ with initial data $u(x,0) = g(x) \in H^s$. Assume that we have a growth bound $\|u(t)\|_{H^s} \leq C(\|g\|_{H^s})(1+|t|)^{\alpha(s)}$. Then $u(t) - e^{tL}g \in C_t^0 H_x^{s+a}$ and*

$$\|u(t) - e^{tL}g\|_{H^{s+a}} \leq C(s, a, \|g\|_{H^s})(1+|t|)^{1+6\alpha(s)},$$

where $L = -\partial_x^3 + \langle g \rangle \partial_x$.

The proof of this theorem utilizes normal form transforms as it was used in [1], Bourgain's restricted norm method and L^6 Strichartz estimate [2].

Remarks
1) A similar result is valid for the KdV equatioin with a smooth space time potential in the case $s \geq 0$.
2) A similar result follows for the modified KdV equation using Miura transform and the observation that the commutator of the linear evolution and the Miura transform is smoother.
3) For L^2 initial data g, Theorem 1 implies that

$$u - e^{tL}g \in C_t^0 H_x^{1-},$$

and hence is a continuous function of x and t.

We have the following corollaries

Corollary 1. *Let u be the real valued solution of (1) with initial data $g \in BV \subset L^2$. Then, u is a continuous function of x if $t/2\pi$ is an irrational number. For rational values of $t/2\pi$, it is a bounded function with at most countably many discontinuities. Moreover, if g is also continuous then $u \in C_t^0 C_x^0$.*

This corollary follows by Remark 3 and a similar theorem by Oskolkov [5] which is valid for the linear evolution.

Corollary 2. *Let u be the real valued solution of* (1) *with initial data $g \in H^s$, $s > 3/7$. Then, for almost every $x \in \mathbb{T}$,*

$$\lim_{t\to 0} u(x,t) = g(x).$$

Once again this corollary follows from a corresponding theorem for the linear evolution which in turn follows from the following Strichartz estimate by Hu and Li [4]:

$$\|e^{Lt}g\|_{L^{14}_{x,t}} \lesssim \|g\|_{H^{\frac{3}{14}+}}.$$

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Quasiconformal mappings and function spaces

Pekka Koskela

The invariance of the homogeneous Sobolev space $\dot{W}^{1,n}$ under quasiconformal mappings of $\mathbb{R}^n$, $n \geq 2$, is essentially simply the definition of quasiconformality and the chain rule. Using the fact that $J_f \in A_\infty$, one further observes that BMO is invariant. Further invariant function spaces can be obtained via interpolation. It turns out that the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{n/s,q}$ are invariant for $0 < s < 1$, $\frac{n}{n+s} < q \leq \infty$. This is proven via a new pointwise characterization for these spaces.

A $T(1)$-Theorem for non-integral operators

Dorothee Frey
(joint work with Peer Christian Kunstmann)

We consider new types of paraproducts constructed via H^∞-functional calculus and develop a $T(1)$-Theorem for non-integral operators by combining methods used in the study of L^p theory for non-integral operators and the Kato square root problem with the recently developed theory of Hardy and BMO spaces associated to sectorial operators.

The underlying space (X, d, μ) is a space of homogeneous type as introduced by Coifman and Weiss. We consider a **sectorial operator** L of order $2m$ on $L^2(X)$ with the following properties:

- L has a bounded holomorphic functional calculus on $L^2(X)$;
- The semigroup e^{-tL} generated by L satisfies Davies-Gaffney estimates, also called L^2 off-diagonal estimates;
- The semigroup e^{-tL} satisfies an $L^p - L^2$ off-diagonal estimate for some $1 < p < 2$ and an $L^2 - L^q$ off-diagonal estimate for some $2 < q < \infty$.

Under the first two assumptions on L, there was recently developed a theory of Hardy spaces $H^p_L(X)$ and of a corresponding space $BMO_L(X)$ associated to the operator L (cf. [4] and the literature cited there). These newly defined spaces generalize the usual Lebesgue spaces and the BMO space of John and Nirenberg. Various properties of those still remain true. In particular, there exists a generalization of a criterion of Fefferman and Stein, describing the connection of Carleson measures and elements of $BMO_L(X)$. This connection sets the stage for a definition of **paraproducts** constructed via holomorphic functional calculus.

We show in [2] that, under the above three assumptions on L, for every $b \in BMO_L(X)$ the paraproduct operator

$$\Pi_b : f \mapsto \int_0^\infty \tilde{\psi}(t^{2m}L)[\psi(t^{2m}L)b \cdot A_t(e^{-t^{2m}L}f)]\,\frac{dt}{t} \tag{1}$$

is bounded on $L^2(X)$, where ψ, $\tilde{\psi}$ are taken from the set Ψ consisting of bounded holomorphic functions on a sector with decay at zero and infinity, and A_t denotes some averaging operator. The appearance of the operator A_t might seem surprising, but this is due to the fact that we do not impose any kernel estimates on the semigroup e^{-tL}.

Besides, we show that Π_b extends to a bounded operator from $L^p(X)$ to $H^p_L(X)$ for $p \in (2, \infty)$ and from $L^\infty(X)$ to $BMO_L(X)$.

The paraproduct Π_b defined in (1) is one of the main tools in the examination of the L^2-boundedness of so-called **non-integral operators**. This type of operators generalizes Calderón-Zygmund operators in the sense that kernel estimates are substituted by certain off-diagonal estimates. In detail, we consider operators $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$ such that for functions $\psi_1, \psi_2 \in \Psi$ with sufficient decay at zero the following **off-diagonal estimates** are valid:

$$\|T\psi_1(tL)f\|_{L^2(B_2)} \leq C\left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t}\right)^{-\gamma}\|f\|_{L^2(B_1)} \tag{2}$$

$$\|T^*\psi_2(tL^*)f\|_{L^2(B_2)} \leq C\left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t}\right)^{-\gamma}\|f\|_{L^2(B_1)} \tag{3}$$

for some $\gamma > 0$, for all $t > 0$, all balls B_1, B_2 with radius $r = t^{1/2m}$ and all $f \in L^2(X)$ supported in B_1.

On the Euclidean space $\mathbb{R}^n$ let us denote by G_L the vertical Littlewood-Paley-Stein square function associated to L, i.e. let

$$G_L(f)(x) := \left(\int_0^\infty \left|t\nabla e^{-t^{2m}L}f(x)\right|^2 \frac{dt}{t}\right)^{1/2}$$

for all $x \in \mathbb{R}^n$ and all $f \in L^2(\mathbb{R}^n)$. Then the main result, a $T(1)$**-Theorem for non-integral operators**, reads as follows:

Theorem 1 ([3])**.** *Let L be the sectorial operator of order $2m$ as specified above such that G_L and G_{L^*} are bounded on $L^2(\mathbb{R}^n)$. Let T be a non-integral operator satisfying* (2) *and* (3) *for sufficiently large $\gamma > 0$. Then T is bounded on $L^2(\mathbb{R}^n)$ if and only if*

$$T(1) \in BMO_L(\mathbb{R}^n) \qquad \textit{and} \qquad T^*(1) \in BMO_{L^*}(\mathbb{R}^n).$$

Here, $T(1)$ and $T^*(1)$ are appropriately defined linear functionals on a subspace of $H^1_L(\mathbb{R}^n)$ and $H^1_{L^*}(\mathbb{R}^n)$, respectively.
If the space $\mathbb{R}^n$ is replaced by an arbitrary space X of homogeneous type, we require in addition the validity of some Poincaré inequality and have to reformulate the boundedness of the Littlewood-Paley-Stein square functions.
The assumptions on the non-integral operator T are chosen in such a way that the boundedness on Hardy spaces $H^p_L(X)$ is an immediate consequence of the boundedness on $L^2(X)$.
With the same methods used in the proof of this $T(1)$-Theorem, we moreover show a second version of a $T(1)$-Theorem with weaker assumptions in the case that the conservation properties $e^{-tL}(1) = 1$ and $e^{-tL^*}(1) = 1$ hold. This generalizes a $T(1)$-Theorem due to Bernicot [1] who assumed Poisson kernel bounds for the semigroup.
Under the additional assumption that e^{-tL} is bounded on $L^\infty(X)$ uniformly in $t > 0$, we then apply this second version to prove the boundedness of the paraproduct operator $\tilde{\Pi}_f$ on $L^2(X)$, where $\tilde{\Pi}_f$ is defined by

$$\tilde{\Pi}_f(g) := \int_0^\infty \tilde{\psi}(t^{2m}L)[e^{-t^{2m}L}g \cdot e^{-t^{2m}L}f]\,\frac{dt}{t}$$

for $f \in L^\infty(X)$, $g \in L^2(X)$ and $\tilde{\psi} \in \Psi$ with sufficient decay at zero and infinity.
We moreover study conditions for a $T(b)$-Theorem to be valid.

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A sharp multiplier theorem for the Kohn Laplacian on forms of the sphere in $\mathbb{C}^n$

Michael Cowling

(joint work with Alessio Martini)

Suppose that M is a compact manifold with a smooth measure, and $\mathcal{E}$ and $\mathcal{F}$ are vector bundles over M, both of which are equipped with inner products. For notational consistency, we write $\mathcal{T}$ for the trivial bundle $M \times \mathbb{C}$ over M. Suppose also that $d : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ is a first order differential operator from the space of smooth sections of $\mathcal{E}$ to the space of smooth sections of $\mathcal{F}$. Using the inner products, we may construct the adjoint $d^* : C^\infty(\mathcal{F}) \to C^\infty(\mathcal{E})$ and hence consider the self-adjoint operator $d + d^*$, on the sum of the bundles:

$$d + d^* : C^\infty(\mathcal{E} \oplus \mathcal{F}) \to C^\infty(\mathcal{E} \oplus \mathcal{F}).$$

Define $D = d + d^*$ and $\Delta := D^2$. Then Δ is a positive self-adjoint operator on $L^2(\mathcal{E} \oplus \mathcal{F}^*)$. This construction may be applied to the de Rham operator d and the Kohn boundary operator $\bar{\partial}_b$ to produce the Hodge and Kohn Laplacians; in particular, since $d^2 = 0$, then $\Delta = dd^* + d^*d$. For convenience, we will henceforth consider a first-order self-adjoint operator D acting on sections of a bundle $\mathcal{G}$.

It is a question of some interest to consider a bounded Borel-measurable function $G : [0, +\infty) \to \mathbb{C}$ and to ask whether the operator $G(\Delta)$, which is defined and bounded on $L^2(\mathcal{G})$ by spectral theory, is also bounded on some L^p-spaces. Actually, to do this, it is easier to work with the even function $F : \mathbb{R} \to \mathbb{C}$, given by $F(x) = G(x^2)$ for all $x \in \mathbb{R}$, and to consider $F(\sqrt{D^2})$, or better, $F(D)$, since the homogeneity of F allows Fourier transform techniques to be applied more easily. Note that the operator $F(D)$ is well-defined and equal to $G(\Delta)$ on $L^2(\mathcal{G})$. We look for results that show that a condition of the form

$$\sup_{t>0} \|F(t\cdot)\, \eta\|_{H^s} < +\infty$$

implies that $F(D)$ is bounded on $L^p(\mathcal{G})$ for all $p \in (1, +\infty)$; here η is a smooth (nonzero) function with compact support in $\mathbb{R}^+$ and $H^s(\mathbb{R})$ is the Sobolev space of functions in $L^2(\mathbb{R})$ with s derivatives in $L^2(\mathbb{R})$. For example, when D is elliptic, it suffices to take s equal to half the dimension n of M; when D is subelliptic, it suffices to take s equal to half the so-called homogeneous dimension Q of M.

Sometimes, M is subelliptic, it is possible to improve the critical index from $Q/2$ to $n/2$. For example, this is known for some step 2 nilpotent Lie groups ([4] or [6]) and for some scalar operators on the sphere in $\mathbb{C}^n$ (see [2] or [3]). We announce the following result.

Theorem 1. *Suppose that D is the operator arising from the Kohn operator $\bar{\partial}_b$, acting on spaces of forms on the sphere S in $\mathbb{C}^n$. If*

$$\sup_{t>0} \|F(t\cdot)\,\eta\|_{H^{n-1/2}} < +\infty,$$

then $F(D)$ is bounded on $L^p(\mathcal{G})$. Here the sections of $\mathcal{G}$ are a D-invariant space of forms.

1. The distance associated to a first-order differential operator

The first step in dealing with the operator D is to associate to it a distance ϱ. For $u \in C^\infty(\mathcal{T})$ and $U \in C^\infty(\mathcal{G})$, Leibniz' rule shows that

$$D(uU) = (D^\sigma u)U + u(DU).$$

Note that $D^\sigma u \in \operatorname{End}(\mathcal{G})$. The operator D^σ is essentially the so-called symbol $\sigma(D)$ of D; more precisely, $D^\sigma u = \sigma(D)(du)$, where d now denotes the usual differential. We define

$$\varrho(x,y) = \sup\{|u(x) - u(y)| : u \in C^\infty(\mathcal{T}),\ \|D^\sigma u\| \le 1\}.$$

The norm $\|D^\sigma u\|$ is a supremum of pointwise operator norms. This gives rise to a *sub-Finsler geometry.* A certain amount of work is required to show that this is the appropriate distance for D, but here is one result that indicates that this is so. For convenience, given a compact subset K of M and $t \in \mathbb{R}$, we write $N(K,t)$ for the set of all $x \in M$ such that $\varrho(x,K) \le |t|$; we also write I for the interval $(-\delta,\delta)$, where $\delta > 0$.

Theorem 2. *Suppose that $u(\cdot,t) \in C^\infty(\mathcal{G})$ for each $t \in I$, that u is also smooth as a function of t, that*

$$\frac{\partial}{\partial t} u(x,t) + iDu(x,t) = 0,$$

for all $(x,t) \in M \times I$, and that $\|u(\cdot,t)\|_2$ is independent of t in I. Then, for all $t \in I$, $\operatorname{supp} u(\cdot,t) \subseteq N(\operatorname{supp} u(\cdot,0),t)$.

The proof of this theorem is not new, but we have not found this precise version in the literature. We combine ideas from analysis on "metric measure spaces" with old ideas on wave propagation.

2. The technique of the sharp theorem

Since D is self-adjoint and M is compact, there exist an orthonormal basis for $L^2(\mathcal{G})$ of sections $\{e_j : j \in \mathbb{N}\}$ and real scalars λ_j such that

$$De_j = \lambda_j e_j.$$

Then the operator $F(D)$ is, at least formally, a kernel operator, and its kernel K_F is given by

$$K_F(x,y) = \sum_{j\in\mathbb{N}} F(\lambda_j)\, e_j(x)\, e_j(y)^* \quad \forall x,y \in M.$$

We show that $F(D)$ maps $L^1(\mathcal{G})$ to the weak L^1-space $L^{1,\infty}(\mathcal{G})$. Loosely speaking, we do this by using Hölder's inequality to estimate L^1 norms of "components" of $K_F(\cdot,y)$ with support in a neighbourhood $N(y,\delta)$ of y from their L^2 norms, which can be found using the Plancherel theorem; finite propagation speed enables us to find these components. This technique goes back at least as far as Coifman and Weiss's work on "spaces of homogeneous type" [1], and without a little extra information, can only be used to prove the multiplier theorem with a critical index of $Q/2$. To improve it, we need a little extra information, namely an estimate of the form

$$\int_M \varrho^\alpha(x,y)\,|K_F(x,y)|^2\,dx \le C\,l^{(Q-\alpha)} \sum_{k=1}^{l} \max\{|F(\lambda)| : k-1 < \lambda \le l\}^2$$

for all functions F with support in $[-l,l]$, and all positive integers l. This is proved with a careful study of the tensor product of representations of $\mathrm{SU}(n)$, and is an extension of the ideas in [2] and [3]. An additional complication is that in some cases, the kernel of $\bar{\partial}_b$ (or its adjoint) is infinite-dimensional; fortunately, we can subtract off a multiple of the Szegő projection, which was shown to be bounded on all the L^p spaces by Korányi and Vagi [5], to deal with this.

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Higher order Riesz transforms on Heisenberg groups

Sundaram Thangavelu

We characterize higher order Riesz transforms on Heisenberg groups H^n and also show that they satisfy dimension free bounds under some assumptions on their multipliers.

Given a bigraded solid harmonic P on $\mathbb{C}^n$ we define the operator R_P as a Fourier multiplier on the Heisenberg group H^n corresponding to the multiplier

$G_\lambda(P)H(\lambda)^{-1/2(p+q)}$. Here $H(\lambda) = -\Delta + \lambda^2|x|^2$ is the rescaled Hermite operator and $G_\lambda(P)$ is the operator associated to P via Weyl correspondence. Thus

$$\pi_\lambda(R_P f) = \pi_\lambda(f)G_\lambda(P)H(\lambda)^{-1/2(p+q)}$$

where $\pi_\lambda, \lambda \in \mathbb{R} \setminus 0$ are the Schrodinger representations. When $P(z) = z_j$ or $\bar{z}_j$ we get back the first order Riesz transforms studied by Coulhon-Muller-Zienkiewicz [1] and others.

Let $R(\sigma)$ stand for the representation of the unitary group $U(n)$ on the space $\mathcal{H}_{p,q}$ of bigraded solid harmonics of bidegree (p, q). Let $\rho(\sigma)$ stand for the action of $U(n)$ on functions $f(z, t)$ defined on the Heisenberg group. The following result justifies why R_P are the most natural candidates for higher order Riesz transforms.

Theorem 3. *Let T be a translation invariant operator taking $L^2(H^n)$ into $L^2(H^n, \mathcal{H}_{p,q})$ and let $M(\lambda)$ be the corresponding Fourier multiplier. Assume that*

1. *(i) $R(\sigma)Tf(z, t) = \rho(\sigma)T\rho(\sigma^*)f(z, t)$ for every $\sigma \in U(n)$,*
2. *$T\delta_r f(z, t) = \delta_r Tf(z, t)$ for every $r > 0$ where δ_r are the non-isotropic dilations of the Heisenberg group and*
3. *$M(\lambda)P_k(\lambda) = ((2k+n)|\lambda|)^{-\frac{1}{2}(p+q)}S(\lambda)$ for some unbounded operator $S(\lambda)$ where $P_k(\lambda)$ are the spectral projections associated to $H(\lambda)$.*

Then for any linear functional β of $\mathcal{H}_{p,q}$ the operator $\beta(T)f = \beta(Tf)$ is a linear combination of R_P as P runs through an orthonormal basis of $\mathcal{H}_{p,q}$.

Let $R_j, \bar{R}_j, j = 1, 2, \cdots n$ stand for the first order Riesz transforms on H^n. Then a result of Coulton et al [1] says that

$$\| (\sum_{j=1}^{n} |R_j f|^2 + |\bar{R}_j f|^2)^{1/2} \|_p \leq C \| f \|_p$$

where C is independent of the dimension n. We are interested in proving such dimension-free estimates for higher order Riesz transforms R_P. In this direction we have the following partial result. Let $P_0(z) = z_1^p \bar{z}_2{}^q$ and let $O(P_0)$ stand for the orbit of P_0 under the action of $U(n)$.

Theorem 4. *For every $P \in O(P_0)$ the Riesz transform R_P satisfies dimension-free bounds on $L^p(H^n), 1 < p < \infty$.*

We can view the Riesz transforms R_P as operator valued Fourier multipliers for the Euclidean Fourier transform on $\mathbb{R}$. Then using an analogue of de Leeuw's theorem for operator valued Fourier multipliers we can deduce an analogue of Theorem 0.2 for Riesz transforms on the reduced Heisenberg group. By considering functions of the form $f(z)e^{it}$ we obtain boundedness of higher order Riesz transforms for special Hermite expansions. An application of Mauceri's transference gives us boundedness of higher order Riesz tansforms for the Hermite operator.

By considering functions on $\mathbb{C}^n$ which are homogeneous of degree $m \in \mathbb{N}^n$ we can obtain weighted norm estimates for multiple Laguerre expansions. Indeed, let $R_{j,m}$ stand for the Riesz transforms associated to Laguerre expansions of type m on $\mathbb{R}^n_+$. Then we have

Theorem 5. *For every $m \in \mathbb{N}^n$ we have the weighted norm inequality*

$$\int_{\mathbb{R}_+^n} |R_{j,m}f(r)|^p \Pi_{j=1}^n r_j^{m_j(p-2)} d\mu_m$$

$$\leq C_p \int_{\mathbb{R}_+^n} |f(r)|^p \Pi_{j=1}^n r_j^{m_j(p-2)} d\mu_m$$

for all $f \in L^p(\mathbb{R}_+^n, d\mu_m), 1 < p < \infty$ where C_p is independent of n and m.

In the above $d\mu_m(r) = \Pi_{j=1}^n r_j^{2m_j+1} dr_j$. We conjecture that the above result is true for Laguerre expansions of arbitrary type studied by Nowak and Stempak [2].

The results mentioned here are from my joint work with my student P.K.Sanjay [3].

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Multilinear Kakeya, Factorisation and Algebraic Topology

Anthony Carbery

(joint work with Stefán Ingi Valdimarsson)

In this talk we examine the recent proof by Guth [1] of the endpoint multilinear Kakeya inequality. In a certain simplified version of this result, let $\mathcal{T}_j$ be the set of doubly-infinite tubes in $\mathbb{R}^n$ of cross-sectional diameter 1 whose direction is within 10° of the unit vector e_j.

Theorem[Multilinear Kakeya] If $p \geq 1/(n-1)$ then

$$\int \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \chi_{T_j}(x) \right)^p dx \leq C_{p,n} \prod_{j=1}^n \left(\sum_{T_j \in \mathcal{T}_j} c_{T_j} \right)^p .$$

This is obvious when $n = 2$ and $p = 1$ and is due to Jon Bennett, Terry Tao and AC in the case $p > 1/(n-1)$, and to Guth in the endpoint case $p = 1/(n-1)$. The two methods of proof are very different.

We provide an abstract interpretation of Guth's method of proof and this results in an easy boundedness criterion as follows:

Proposition Let X, Y_j, $1 \leq j \leq n$ be measure spaces, let $U_j : \mathcal{M}(Y_j) \to \mathcal{M}(X)$ be linear. Let $r_j > 0$, $r = \sum_j r_j \geq 1$ and $p_j \geq 1$. Suppose that for every $M \in L^{r'}(X)$, $M \geq 0$, there exist $S_1(x), \ldots, S_n(x)$ such that

$$M(x) \leq \prod_{j=1}^{n} S_j(x)^{r_j/r} \text{a.e.},$$

and, for all $\tilde{S}_j$ with $|\tilde{S}_j(x)| = |S_j(x)|$ a.e.,

$$\|U_j^* \tilde{S}_j\|_{p_j'} \leq A\|M\|_{r'}.$$

Then

$$\int_X \prod_{j=1}^{n} |U_j f_j(x)|^{r_j} \, d\mu(x) \leq A^r \prod_{j=1}^{n} \|f_j\|_{L^{p_j}(Y_j)}^{r_j}.$$

Interestingly, this result admits a dual formulation:

Theorem[AC and S. Valdimarsson] Suppose that X and Y_j are measure spaces satisfying reasonable conditions, U_j is linear taking positive functions on Y_j to positive functions on X, and that $r_j > 0$, $r = \sum_j r_j \geq 1$ and $p_j \geq 1$. If

$$\int_X \prod_{j=1}^{n} U_j f_j(x)^{r_j} \, d\mu(x) \leq A^r \prod_{j=1}^{n} \|f_j\|_{L^{p_j}(Y_j)}^{r_j}$$

then, for all nonnegative $M \in L^{r'}(X)$, there exist $S_1(x), \ldots, S_n(x)$ such that

$$M(x) \leq \prod_{j=1}^{n} S_j(x)^{r_j/r} \quad a.e.$$

and

$$\|U_j^* S_j\|_{p_j'} \leq A\|M\|_{r'}.$$

This does not seem to be a standard result of functional analysis, but instead comes about as a result of duality methods in the theory of convex optimisation. These ultimately rely upon some form of the Ky Fan minimax Theorem, which itself is closely related to the Brouwer fixed point theorem. The proof is highly non-constructive.

Example. Let $(T_1 f)(x_1, x_2) = f(x_1)$ and$(T_2 g)(x_1, x_2) = g(x_2)$; then

$$\int_{\mathbb{R}^2} T_1 f(x) T_2 g(x) dx = \int_{\mathbb{R}^2} f(x_1) g(x_2) dx = \int_{\mathbb{R}} f \int_{\mathbb{R}} g.$$

So with $n = 2$, $r_1 = r_2 = p_1 = p_2 = 1$ the duality principle gives that for every $M \geq 0$ in $L^2(\mathbb{R}^2)$ we can factorise it as $M = (GH)^{1/2}$ such that

$$\sup_x \int G(x, y) dy \leq \|M\|_2$$

and

$$\sup_y \int H(x,y)dx \le \|M\|_2.$$

Of course this latter example may be also established by elementary means as pointed out to the authors by M. Christ.

Guth's proof of the endpoint multilinear Kakeya inequality proceeds by constructing a factorisation suitable for use in the Proposition above, and in addition to the polynomial method, employs a variety of techniques from algebraic topology, in particular $\mathbb{Z}_2$-cohomology. We give an argument which avoids this use of heavy-duty algebraic topology and instead uses only the Borsuk–Ulam theorem.

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Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure

Andreas Seeger
(joint work with Jong-Guk Bak, Daniel Oberlin)

For a smooth curve $t \mapsto \gamma(t)$ in $\mathbb{R}^d$, $d \ge 3$, consider the Fourier restriction operator,

$$\mathcal{R}f(t) = \widehat{f}(\gamma(t)),$$

defined on an interval I. We are interested in the mapping properties of $\mathcal{R}$ in Lebesgue spaces $L^q(I; d\lambda)$ where $d\lambda = w(t)dt$ and the weight is given by *affine arclength measure*,

$$w(t) = |\tau(t)|^{\frac{2}{d^2+d}}, \qquad \text{with } \tau(t) = \det(\gamma'(t), \dots, \gamma^{(d)}(t)) .$$

For nondegenerate curves (with $\tau(t) \ne 0$) affine arclength measure is comparable to Lebesgue measure on any compact interval. For this case the sharp $L^p \to L^q$ estimates in dimensions $d \ge 3$ are due to Drury [8], $\mathcal{R} : L^p(\mathbb{R}^d) \to L^q(I)$ for $1 < p < p_d := \frac{d^2+d+2}{d^2+d}$, $p' = q\frac{d(d+1)}{2}$. More recently, a sharp endpoint result has been proven in [1], namely $\mathcal{R}f \in L^{p_d}$ when f belongs to the Lorentz space $L^{p_d,1}(\mathbb{R}^d)$. The analogue of this result is false in two dimensions, by an argument [4] based on the Kakeya set. We are now concerned with the extension of this result to more general classes of curves where Lebesgue measure is replaced by affine arclength measure. We note that affine arclength measure is essentially optimal in this respect, namely if $\mathcal{R} : L^{p,1}(\mathbb{R}^d) \to L^q(I; d\mu)$ for the critical $p' = q\frac{d(d+1)}{2}$ then $d\mu = v(t)dt$ where $v(t) \le C_d \|\mathcal{R}\|^q_{p\to q} w(t)$ a.e., with w as above. The following theorem ([3]) is about the model class of 'monomial' curves.

Theorem. *Let $d \geq 3$ and let $w_a dt$ denote the affine arclength measure for the curve $t \mapsto \gamma_a(t) = (t^{a_1}, t^{a_2}, \dots, t^{a_d})$, $\quad 0 < t < \infty$. Then there is $C(d) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^d)$*

$$\Big(\int_0^\infty |\widehat{f}(\gamma_a(t))|^{p_d} w_a(t) dt \Big)^{1/p_d} \leq C(d) \|f\|_{L^{p_d,1}(\mathbb{R}^d)}.$$

The constant is universal in the sense that it does not depend on $a_1, \dots, a_d$.

We prove a similar result for 'simple' polynomial curves which are of the form $(t, t^2, \dots, t^d, P(t))$ where P is a polynomial of degree n and the constant depends only on n. One might conjecture that a corresponding result holds for general polynomial curves as considered recently in [7] (with general polynomial entries in all coordinates) but even the optimal range of the $L^p \to L^q$ estimates is currently open for this class.

Two geometric inequalities are crucial in our proof. The first one was introduced in the works by Drury and Marshall [9], [10] (and also appears in [1], [2], [7], [6]). Consider the Jacobian $\mathcal{J}_{\Phi_\gamma}$ of the map $\Phi_\gamma(t_1, \dots, t_d) = \sum_{j=1}^d \gamma(t_j)$. The relevant inequality is

$$|\mathcal{J}_{\Phi_\gamma}(t_1, \dots, t_d)| \geq c \Big(\prod_{i=1}^d \tau_\gamma(t_i) \Big)^{1/d} \prod_{1 \leq j < k \leq d} (t_k - t_j).$$

We rely on a result by Drury and Marshall [10] who verified this inequality for the exponential reparametrizations $\Gamma(t) = (e^{a_1 t}, \dots, e^{a_d t})$ of the monomial curves (see also [6] for some extensions).

Another inequality concerns the so called offspring curves. If $\kappa = (\kappa_1, \dots, \kappa_d)$ so that $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_d$, and one of the coordinates κ_i is 0, then a κ-offspring curve γ_κ is defined (on a suitable interval) by $\gamma_\kappa(t) = \sum_{j=1}^d \gamma(t + \kappa_j)$. If Γ is an exponential parametrization of a monomial curve then the offspring curves of Γ are affine images of Γ. Moreover they satisfy the following inequality relating the torsion of the offspring curves Γ_κ to the torsion of the original curves,

$$|\tau_{\Gamma_\kappa}(t)| \geq c \max_{j=1,\dots,d} |\tau_\Gamma(t + \kappa_j)| \,.$$

This is an easy to check strengthening of a weaker inequality which appeared in [9], [10], [1], [2], where the max on the right hand side is replaced by a geometric mean. The stronger inequality allows for increased flexibility so that various powers of the weights can be used.

As a technical tool we use an interpolation procedure introduced in [5], for multilinear operators which exhibit many symmetries. The relevant examples here are expressions $\prod_{i=1}^n \mathcal{R}^* g_i$ where $\mathcal{R}^*$ is the adjoint of the Fourier restriction operator. Unfortunately this procedure does not readily apply in our setting since real interpolation spaces of weighted Lorentz spaces (with change of weight) may not be weighted Lorentz spaces. Given a weight w on an interval I and a Lorentz space X of measurable functions on I one is then led to consider block spaces: Set $\Omega_k = \{t \in I : 2^k \leq w(x) < 2^{k+1}\}$ for $k \in \mathbb{Z}$. Define the block Lorentz space

$b^q_s(w;X)$ as the class of measurable functions for which

$$\|f\|_{b^q_s(w;X)} := \Big(\sum_{k\in\mathbb{Z}} \big[2^{ks}\|\chi_{\Omega_k} f\|_X\big]^q\Big)^{1/q}$$

is finite. Note that $b^q_{1/q}(w;L^q)$ is just the weighted $L^q(w)$ space. In our setting we have to deal with inequalities that involve a weak type space on the left hand side and the space $b^1_s(w;L^{q,1})$ (or even $b^r_s(w;L^{q,r})$ for $r<1$) on the right hand side. Such bounds are weaker then the usual restricted weak type estimates with weights. The following interpolation theorem turns out to be helpful in order to deal with this difficulty.

We use the notation $DS^\circ(n)$ for the interior of the Birkhoff polytop of doubly stochastic matrices, i.e. $A=(a_{ij}$ belongs to $DS^\circ(n)$ if $\sum_{j=1}^n a_{ij}=1$, $i=1,\dots,n$ and $\sum_{i=1}^n a_{ij}=1$, $j=1,\dots,n$, and all entries lie in $(0,1)$.

Theorem. *Suppose we are given $m\in\{1,\dots,n\}$ and $\delta_1,\dots,\delta_n\in\mathbb{R}$ so that the numbers δ_i with $i\neq m$ are not all equal. Let $0<r\le 1$, and let $q_1,\dots,q_n\in[r,\infty]$ such that $\sum_{i=1}^n q_i^{-1}=r^{-1}$. Let V be an r-convex Lorentz space (e.g. a normed Lorentz space when $r=1$ or the space $L^{r,\infty}$ when $r<1$). Let $\overline{X}=(X_0,X_1)$ be a couple of compatible complete quasi-normed spaces of measurable functions on an interval I, and let w be a weight on I.*

Let T be a multilinear operator defined on n-tuples of X_0+X_1 valued sequences and suppose that for every permutation π on n letters we have the inequality

$$\|T(f_{\pi(1)},\dots,f_{\pi(n)})\|_V \le \|f_m\|_{b^r_{\delta_m}(w;X_1)} \prod_{i\neq m} \|f_i\|_{b^r_{\delta_i}(w;X_0)}\,.$$

Then for every $A\in DS^\circ(n)$ and every $B\in DS(n)$ such that $b_{im}=r/q_i$, $i=1,\dots,d$, there is a constant $C(A,B,r)$ so that for $\vec{s}=BA\vec{\delta}$ and $\vec{\theta}=BA\vec{e}_m$

$$\|T(f_1,\dots,f_n)\|_V \le C(A,B,\vec{\delta},r)\prod_{i=1}^n \|f_i\|_{b^{q_i}_{s_i}(w;\overline{X}_{\theta_i,q_i})}$$

for all $(f_1,\dots,f_n)\in\prod_{i=1}^n b^{q_i}_{s_i}(w;\overline{X}_{\theta_i,q_i})$.

In particular one may choose the spaces on the right hand side to be $b^{nr}_\nu(w;\overline{X}_{\frac{1}{n},nr})$ with $\nu=\frac{1}{n}\sum_{i=1}^n\delta_i$, which in the relevant application (the proof of the weak type (p'_d,p'_d) inequality for the adjoint restriction operator) will become a weighted $L^{p'_d}$ space.

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On some new divergence-curl inequalities

Po-Lam Yung

(joint work with Sagun Chanillo and Yi Wang)

Our subject of interest involves some kind of compensation phenomena that has to do with divergence, curl and the space L^1 of Lebesgue integrable functions or differential forms. In the forms stated below they were discovered by Bourgain-Brezis, Lanzani-Stein and van Schaftingen around 2004. Also lying beneath these results is the failure of the critical Sobolev embedding of $\dot{W}^{1,n}$ into L^∞. We mention here that these results seem to be quite different from the more classical theory of compensated compactness; no connection between them is known so far. Towards the end we mention some extensions of these known results to subelliptic and hyperbolic settings.

For simplicity, for the moment we work on $\mathbb{R}^n$, $n \geq 2$. We denote by d the Hodge-de Rham exterior derivative, and d^* its (formal) adjoint. The theory we are going to describe consists of three major pillars, each best illustrated by a separate theorem. The first involves the solution of d^*:

Theorem 1 (Bourgain-Brezis [1]). *Suppose $l \neq n-1$. Then for any l-form $f \in L^n$ that is in the image[1] of d^*, there exists a $(l+1)$-form Y with coefficients in L^∞ such that*

$$d^*Y = f$$

in the sense of distributions, and $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.

In particular, we have

Corollary 1 (Bourgain-Brezis [2]). *For any function $f \in L^n$, there exists a vector field $Y \in L^\infty$ such that*

$$\operatorname{div} Y = f$$

and $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.

The second pillar is a Gagliardo-Nirenberg inequality for differential forms:

[1] By this we mean f is the d^* of some forms with coefficients in $\dot{W}^{1,n}$.

Theorem 2 (Lanzani-Stein [5]). *Suppose $u \in C_c^\infty$ is an l-form. We have*

$$\|u\|_{L^{n/(n-1)}} \leq C(\|du\|_{L^1} + \|d^*u\|_{L^1})$$

unless d^*u is a function or du is a top form. If d^*u is a function, the result remains true if $d^*u = 0$. If du is a top form, the result remains true if $du = 0$.

Since d of a 1-form is its curl and d^* of a 1-form is its divergence, this is sometimes called a divergence-curl inequality.

The third theorem is the following compensation phenomenon:

Theorem 3 (van Schaftingen [6]). *If u is a 1-form with $d^*u = 0$, then for any 1-form $\phi \in C_c^\infty$, we have*

$$\int_{\mathbb{R}^n} u \cdot \phi dx \leq C\|u\|_{L^1}\|\phi\|_{\dot{W}^{1,n}}.$$

If $\dot{W}^{1,n}$ were embedded into L^∞, the first theorem would be trivial by Hodge decomposition, and so will be the third by Holder's inequality. It is remarkable that these theorems remain true even though the desired Sobolev embedding fails.

It turns out all three theorems are equivalent by duality. van Schaftingen gave a beautiful elementary proof of the third theorem, thereby proving all of them.

What is more remarkable is the following observation of Bourgain-Brezis [1]. They proved that in all the above results, the space L^∞ can be replaced by the smaller Banach space $L^\infty \cap \dot{W}^{1,n}$, and the space L^1 can be replaced by the bigger Banach space $L^1 + (\dot{W}^{1,n})^*$. (Here X^* denotes the dual of a Banach space X.) They proved this by giving a direct constructive proof of the analog of Theorem 1, where the space L^∞ is replaced by $L^\infty \cap \dot{W}^{1,n}$, and then deducing the rest by duality. In the former they used the following approximation lemma, which is another remedy of the failure of the critical Sobolev embedding, and which is of independent interest:

Lemma 1 (Bourgain-Brezis [1]). *Given any $\delta > 0$, there exists $C_\delta > 0$ such that the following holds: For any function $f \in \dot{W}^{1,n}$, there exists a function $F \in L^\infty \cap \dot{W}^{1,n}$ such that*

$$\sum_{i=2}^{n} \|\partial_i f - \partial_i F\|_{L^n} \leq \delta\|\nabla f\|_{L^n}$$

and

$$\|F\|_{L^\infty} + \|\nabla F\|_{L^n} \leq C_\delta\|\nabla f\|_{L^n}.$$

Here one should think of F as an $L^\infty \cap \dot{W}^{1,n}$ function whose derivatives approximates those of the given f in all but one direction.

In joint work with Yi Wang, we proved an analog of this approximation lemma on the Heisenberg group $\mathbb{H}^n$:

Lemma 2. *Given any $\delta > 0$, there exists $C_\delta > 0$ such that the following holds: For any function f on $\mathbb{H}^n$ with $\|\nabla_b f\|_{L^Q} < \infty$, there exists a function $F \in L^\infty$*

with $\nabla_b F \in L^Q$ *such that*

$$\sum_{i=2}^{2n} \|X_i f - X_i F\|_{L^Q} \leq \delta \|\nabla_b f\|_{L^Q}$$

and

$$\|F\|_{L^\infty} + \|\nabla_b F\|_{L^Q} \leq C_\delta \|\nabla_b f\|_{L^Q}.$$

Here $Q = 2n+2$ *is the homogeneous dimension of* $\mathbb{H}^n$*,* $X_1, \dots, X_{2n}$ *is a basis of left-invariant vector fields of degree 1, and* $\nabla_b f = (X_1 f, \dots, X_{2n} f)$*.*

With this we deduce, for instance, a Gagliardo-Nirenberg inequality for $\overline{\partial}_b$ on $\mathbb{H}^n$:

Theorem 4. *If* u *is a* $(0,q)$ *form on* $\mathbb{H}^n$ *with* $2 \leq q \leq n-2$*, then*

$$\|u\|_{L^{Q/(Q-1)}} \leq C(\|\overline{\partial}_b u\|_{L^1 + (\dot{NL}^{1,Q})^*} + \|\overline{\partial}_b^* u\|_{L^1 + (\dot{NL}^{1,Q})^*})$$

where $\dot{NL}^{1,Q}$ *is the space of functions whose* ∇_b *is in* L^Q*. Also, if* $n \geq 2$ *and* u *is a function on* $\mathbb{H}^n$ *that is orthogonal to the kernel of* $\overline{\partial}_b$*, then*

$$\|u\|_{L^{Q/(Q-1)}} \leq C \|\overline{\partial}_b u\|_{L^1 + (\dot{NL}^{1,Q})^*}.$$

A weaker version of this result, namely what one has by replacing $L^1 + (\dot{NL}^{1,Q})^*$ above by L^1, can also be deduced easily from the work of Chanillo-van Schaftingen [3] (c.f. also [7]).

Finally we adapt the result of van Schaftingen to the study of wave equations. In joint work with Sagun Chanillo [4], we proved the following improved Strichartz inequality for systems of wave equations where the inhomogeneous term is a divergence-free vector field. For simplicity we state it in $2+1$ dimensions.

Theorem 5 ([4])**.** *Suppose* $u\colon \mathbb{R}^{1+2} \to \mathbb{R}^2$ *is a (weak) solution of the following system of wave equations* $\Box u = f$ *with* $u|_{t=0} = 0$ *and* $\partial_t u|_{t=0} = 0$*, where* $f = (f_1, f_2)\colon \mathbb{R}^{1+2} \to \mathbb{R}^2$ *is a divergence free vector field at each given time* t*, i.e.*

$$\partial_{x_1} f_1 + \partial_{x_2} f_2 = 0$$

for each t*. Then*

$$\|u\|_{C^0_t L^2_x} + \|\partial_t u\|_{C^0_t \dot{H}^{-1}_x} \leq C \|f\|_{L^1_t L^1_x}.$$

The remarkable phenomenon here is that we have L^1_x norm of f on the right hand side, and this was made possible only because f is a vector field and is divergence free at each time.

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Multiparameter singular integrals

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We present a theory of singular integrals based on multiparameter Carnot-Carathéodory balls; i.e., balls defined by vector fields. Let $\nu \in \mathbb{N}$ and suppose we are given ν lists of C^∞ vector fields X_j^μ ($1 \le \mu \le \nu$, $1 \le j \le q_\mu$) defined on some open set $\Omega \subset \mathbb{R}^n$, and suppose we pair each vector field with a positive real number $d_j^\mu > 0$. We define ν parameter Carnot-Carathéodory balls by

$$B(x,(\delta_1,\dots,\delta_\nu)) = \Bigg\{ y \in \Omega \Bigg| \exists \gamma : [0,1] \to \Omega, \gamma(0) = x, \gamma(1) = y, \\ \gamma'(t) = \sum_{\mu=1}^{\nu} \sum_{j=1}^{q_\mu} a_j^\mu(t) \delta_\mu^{d_j^\mu} X_j^\mu(\gamma(t)), |a_j^\mu(t)| < 1 \Bigg\}.$$

We assume

$$\left[\delta_{\mu_1}^{d_j^{\mu_1}} X_j^{\mu_1}, \delta_{\mu_2}^{d_k^{\mu_2}} X_k^{\mu_2} \right] = \sum_\mu \sum_l c_{j,k,l}^{\mu,\delta} \delta_\mu^{d_l^\mu} X_l^\mu,$$

where $c_{j,k,l}^{\mu,\delta} \in C^\infty$ is "bounded uniformly in δ." These balls were studied when $\nu = 1$ by Nagel, Stein, and Wainger [4]. For higher ν, they were studied under more restrictions by Tao and Wright [6], and this was later extended to the above conditions in [5]. Under the above assumption we define a class of singular integrals based on these balls. These singular integrals form an algebra and are bounded on natural non-isotropic Sobolev spaces ($NL^p_{(s_1,\dots,s_\nu)}$, $1 < p < \infty$, $s_\mu \in \mathbb{R}$). In particular, they are bounded on L^p ($1 < p < \infty$).

When $\nu = 1$, these balls form a space of homogeneous type ([4]) and the corresponding singular integrals are standard Calderón-Zygmund singular integrals. More precisely, they are the NIS operators developed by [2, 1]). When the ambient space is a product space, $\Omega = \Omega_1 \times \cdots \times \Omega_\nu$, each X_j^μ is a vector field on d_j^μ, the singular integrals are standard product type Calerón-Zygmud singular integrals. More precisely, they are the product NIS operators developed in [3]. When the balls are not of product type, this theory introduces a new type of singular integral. These new singular integrals contain parametricies for certain partial differential operators defined by vector fields.

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Singular integrals survival in bad neighborhoods and related topics

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Singular integrals in very bad behaving measure spaces are coming into the focus of attention because of problems of Geometric Measure Theory (GMT), where a priori measures have no smoothness. Another source is the sharp weighted estimates of singular operators often appearing from PDE. The sharpness makes necessary the decoupling of measure and kernel, sometimes a quite radical one. The typical setting would be a metric space of homogeneous type (so it has a doubling measure), but the singular operator on it is considered with respect to *another* non-doubling and very badly behaving measure. Examples, where this happens (or may happen), are Painlevé, Denjoy, Vitushkin's problems; David–Semmes problem; analysis on the boundaries of pseudoconvex domains that goes beyond the scope of Carnot–Carathéodory spaces; two weight Hilbert transform; one weight sharp estimates of Calderón–Zygmund operators, et cetera...

1. Introduction. David–Semmes' problem, its variants

The starting point is the book [5] relating Calderón–Zygmund theory and GMT. The interesting thing is that the book itself has nothing to do with "non-homogeneous" harmonic analysis. It is completely within the realm of homogeneous Calderón–Zygmund theory as extended by Michael Christ in [1]. In other words, the underlying measure μ, with respect to which the singular integrals are considered is always doubling in [5]. In fact, $\mu = H^s|E$, where E is assumed to be regular in the sense of Ahlfors:

$$c\, r^s \leq H^s(B(x,r)\cap E) \leq C\, r^s\,,\ x\in E, r\leq \mathrm{diam} E$$

uniformly. However, the feeling is that this regularity is not really needed. It feels like it can be assumed "without the loss of generality". And this is indeed the fact. It is not a simple fact, but it is true that non-homogeneous $T1$ theorems as in [17], [29] (see also some discussion in [6] and below) allow us to reduce (often) the general case to the case of Ahlfors regularity.

A disclaimer: all purely mathematical content of this part of this note is joint with Vladimir Eiderman and Fedja Nazarov. However, all mistakes are mine.

Let E be a compact set in $\mathbb{R}^d$ such that $0 < H^s(E) < \infty$, $0 < s \leq d$. Let $R^s = (R^{1,s}, \dots, R^{d,s})$ be vector Riesz kernel of singularity s: namely, $R^{i,s} = \frac{x_i}{|x|^{1+s}}$, $i = 1, \dots, d$. The question of David–Semmes posed in the book [5] the following question:
let E be Ahlfors regular, namely,

$$c\, r^s \leq H^s(B(x,r) \cap E) \leq C\, r^s\,,\ \forall x \in E, 0 < r < \operatorname{diam} E\,, \tag{1}$$

let $R^s : L^2(E, H^s|E) \to L^2(E, H^s|E)$ be bounded, and also require that $s = d-1$. Is is true that E is uniformly rectifiable?

Uniformly rectifiable means here that

for all $x \in E, 0 < r < \operatorname{diam} E$, there exists a Lipschitz image $\Gamma_{x,r}$ of R^{d-1} into R^d with Lipschitz constant independent of x, r such that $H^{d-1}(B(x,r) \cap E \cap \Gamma_{x,r}) \geq c\, r^{d-1}$.

David–Semmes proved this "analysis-to-geometry" result under a stricter assumption, namely, the boundedness is required for **all** Calderón–Zygmund operators, not only for Riesz transforms.

Mattila–Melnikov–Verdera proved this result for $d = 2$, [12]. But it is known that in the plane case there is a miracle of Melnikov's formula, which introduces Menger's curvature tool into analysis, see [13], [14], [24].

The higher dimensional version seems to be one of the leading question now. Let us now elaborate on David–Semmes question and consider several variants of it and their reformulations.

2. Variants of David–Semmes question

2.1. **Integer $s = n$.** The first natural thing to do is to consider $s = n \neq d-1$, where $n \in \mathbb{Z}_+$ is an integer, $0 < n < d$. Then the question is exactly the same as before, only the uniform rectifiability becomes n-uniform rectifiability, namely, now $\Gamma_{x,r}$ is a Lipschitz image of R^n into R^d with Lipschitz constant independent of x, r such that $H^n(B(x,r) \cap E \cap \Gamma_{x,r}) \geq c\, r^n$. It is widely believed that the answer is correct: if the set E, $0 < H^n(E) < \infty$ is n-Ahlfors regular (see (1) with $s = n$) and all Riesz transforms of singularity n are bounded in $L^2(E, H^n|E)$, then E is n-uniformly rectifiable. Again, if one assumes that **all** Calderón–Zygmund operators of singularity n are bounded, then the conclusion follows: [5].

2.2. **Integer $s = n$, but Ahlfors regularity** (1) **is dropped.** Let $s = n \leq d-1$ be integer, but the assumption of Ahlfors regularity be dropped. The conclusion must be obviously altered. Instead of the existence of big piece of Lipschitz image in all scales one should hope for just one such Lipschitz image. So the assumption of the boundedness of all Riesz transforms in $L^2(E, H^n|E)$ stays the same, but the conclusion must be changed to

(2) There exists a Lipschitz image Γ of $\mathbb{R}^n$ into $\mathbb{R}^d$ such that $H^n(E \cap \Gamma) > 0$.

This is done by Tolsa [24] for $d = 2, n = 1$. The proofs use the ubiquitous Melnikov's formula and Menger's curvature, the tool, which by the words of Guy David "is cruelly missing" in $d > 2, s \geq 1; d = 2, s > 1$.

2.3. **What if s is not integer?** For non-integer s, $0 < s < d$, there is no "Lipschitz images of R^s into R^d", because there is no R^s, and there is no good way to express the structural condition on E saying that E has good "lipshitz smooth" pieces. Therefore, it is natural to think that the boundedness of Riesz transforms of singularity $s \notin \mathbb{Z}_+$ in $L^2(H^s|E)$ does not happen at all on E such that $0 < H^s(E) < \infty$. This is actually proved in the case of s-Ahlfors regularity (1) of E by Vihtila [28]. In fact, more is proved in [28]. Instead of imposing a strong estimate from below as in (1), Vihtila in [28] could have required only that for H^s-a.e. point $x \in E$ the lower density be strictly positive

$$\liminf_{r \to 0} \frac{H^s(B(x,r) \cap E)}{r^s} > 0\,. \tag{3}$$

The technique of tangent measures then allows her to prove the non-existence of such sets having bounded Riesz transforms on them.

However, dropping (1) and (3) completely seems to represent huge difficulty. Even the case $d = 2, 1 < s < 2$ is difficult and was open till recently, see Eiderman–Nazarov–Volberg's [6], where it has been settled.

On the other hand, dropping (1) and (3) for $d = 2, s = 1$ was achieved by Tolsa [24] (with combination with Léger's [9]), and we want to mention that these are **very difficult papers**.

For $d = 2, s < 1$ one can use Prat's paper [21], and again the problem gets solved: no such E exists. Here one uses the same Melnikov's approach but for Riesz kernels of singularity $s < 1$. A small miracle happens–a miracle known to the experts–that the symmetrization trick works and gives a positive kernel. As we already mentioned this is "cruelly" false for $s > 1, d = 2$ and $s \geq 1, d > 2$.

However, looking at our approach in [6], one can see that it leads to the following claim:

Theorem 1. *Let $s \in (d-1, d)$.*

$$E \text{ is a compact in } \mathbb{R}^d,\ 0 < H^s(E) < \infty,\ \textit{such that} \tag{4}$$

$$R^s : L^2(E, H^s|E) \to L^2(E, H^s|E) \textit{ is bounded},$$

$$\textit{and } \liminf_{r \to 0} \frac{H^s(B(x,r) \cap E)}{r^s} = 0\ H^s - \textit{a.e. on } E \Rightarrow \textit{ contradiction}.$$

The proof in [6] replaces the "the cruelly missing" Melnikov's formula and Menger's curvature by Maximum principle for fractional Laplacian and variational method of estimating singular integrals from below. Combining with Vihtila's result (or rather its extension to the case (3), the extension being valid because of NTV's [18], [17]) we get that for $s \in (d-1, d)$ there are no sufficiently symmetric sets E, $0 < H^s(E) < \infty$, to have $R^s : L^2(E, H^s|E) \to L^2(E, H^s|E)$ is bounded, and there are no s-dimensional measure $\mu \neq 0$ such that $R^s(\mu)$ is bounded in $\mathbb{R}^d$ almost everywhere with respect to Lebesgue measure in $\mathbb{R}^d$.

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