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Homotopy Theory

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Abstract. This workshop was a forum to present and discuss the latest result and ideas in homotopy theory and the connections to other branches of mathematics, such as algebraic geometry, representation theory and group theory.

Mathematics Subject Classification (2000): 55xx.

Introduction by the Organisers

Algebraic topology in general and homotopy theory in particular is in an exciting period of growth and transformation, driven in part by strong interactions with algebraic geometry, mathematical physics, and representation theory. This has led to new approaches to our classical problems and the emergence of entirely new areas of study, such as derived algebraic geometry and the 'homotopy theory of homotopy theories'. In this workshop we had presentations from across the full range of new developments.

In 2009, Hill, Hopkins and Ravenel announced a proof that there do not exist manifolds with Kervaire invariant 1 in dimensions $2^k - 2$ for any $k \geq 8$. This solves the long standing open Kervaire invariant problem (except for dimension 126, which is the only dimension which now remains open). The Kervaire invariant is of a geometric nature, work of Browder had reformulated the question about the existence of a Kervaire manifold in terms of pure homotopy theory. Hill, Hopkins and Ravenel solved this homotopy theoretic formulation of the problem with the help of equivariant stable homotopy theory and structured ring spectra. The solution to the Kervaire invariant problem boosted new activities in these areas.

Two example were described in the talks by Hill and Strickland. Michael Hill (reporting on joint work with Michael Hopkins) explained a subtle and previously misunderstood feature of equivariant commutative ring spectra. In this equivariant context, the slogan that 'E_∞ is commutative' is no longer true in a simple way; there are different kinds of equivariant E_∞-operads, giving rise to successively more structure on homotopy groups, in the form of norm maps. In his talk, Neil Strickland presented a new approach towards Tambara functors, the bookkeeping device for the natural structure that exists on the homotopy ring of an equivariant commutative ring spectrum.

The talk of Brooke Shipley (reporting on joint work with John Greenlees) discussed equivariant cohomology theories taking rational values, in a particularly accessible case. Rational equivariant cohomology theories on free G-spaces, for a compact Lie group G are represented by free rational G-spectra. The category of these is Quillen equivalent to the category of torsion modules over the twisted group ring $H^*(BN)[W]$ where N is the identity component of G and $W = G/N$ acts on N (and hence on $H^*(BN)$) by conjugation. The talk described how the proof proceeds by a change of rings, an Eilenberg-Moore fixed point argument and rigidity.

Steffen Sagave presented new insight into the concept of 'units' of a structured ring spectrum. The classical theory sees only the connective part of a ring spectrum and does not distinguish, for example, between a periodic ring spectrum and its connective cover. Sagave introduced a version of the 'spectrum of units' of a commutative structured ring spectrum that remembers if (and how) units in non-zero degrees exist.

Three talks were devoted to applications of Tom Goodwillie's *calculus of functors*. In the early 90's Tom Goodwillie devised a 'calculus' designed to analyze highly nonlinear functors in non-semisimple contexts. Calculus provides a systematic theory of approximations to such functors, encoded in a sequence of spaces (or rather spectra) with properties analogous to the coefficients in a Taylor series. The analogy with classical calculus breaks down at certain points where the topological theory is richer. For example, the identity functor of topological spaces has non-trivial higher derivatives and the derivatives of a functor are not independent of each other, i.e., a functor is not simply a sum of the Taylor terms. Mark Behrens (reporting on joint work with Charles Rezk) and Nick Kuhn (reporting on joint work with Jason McCarty) exploited the Taylor towers of specific functors for purposes of mod-2 homology calculations. Gregory Arone reported on joint work with Michael Ching about the additional structure that is needed to assemble the Taylor tower from the derivatives of a functor; the data is in terms of actions of an operad formed by the derivatives of the identity functor. Operads were also a key tool in Kathryn Hess' talk (reporting on joint work with Bill Dwyer); they give an new interpretation, in terms of spaces of operad morphisms, of a certain double delooping first constructed by McClure and Smith. A consequence is a new interpretation of the space of long knots as the double loop space of the space of operad morphisms into the Kontsevich operad.

An important point of contact between homotopy theory and algebra is in the theory of group actions on spaces and the closely-related study of the classifying spaces of groups. These connections manifested themselves in the talks by Adem, Grodal and Benson. Alejandro Adem reported on joint work with José Manuel Gómez, resulting in new calculations of equivariant K-groups for certain compact connected Lie groups, acting on compact spaces with 'maximal rank isotropy', i.e., in a way that every stabilizer group contains a maximal torus. Jesper Grodal presented a new description of the Grothendieck group of homotopy classes of maps between p–completed classifying spaces, out of an arbitrary finite group G, and into symmetric groups. The answer is as the Grothendieck group of G-stable finite S-sets, where S is a p-Sylow subgroup of G. The talk of Dave Benson (reporting on joint work with Srikanth Iyengar and Henning Krause) described the classification of localizing subcategories of $D(C^*(BG;k))$ for a finite group G. There is a one to one correspondence with subsets of $\mathrm{Spec}(H^*(BG))$. The proof starts by converting the problem to algebra by noting that $D(C^*(BG))$ is equivalent to the localizing subcategory of $K(\mathrm{Inj}(kG))$ generated by the tensor identity. After this it applies the authors' general stratification machinery. The talk went on to make a similar conjecture for compact Lie groups G, and included a number of illuminating examples where the collection of localizing subcategories is more complicated.

John Rognes gave a survey talk on the phenomenon of redshift in algebraic K-theory. The term 'redshift', coined by Rognes, refers to the observation that algebraic K-theory and topological cyclic homology have then tendency to increase the chromatic level (in the sense of stable homotopy theory) when applied to commutative structure ring spectra.

Niko Naumann spoke on joint work with Tyler Lawson giving a criterion for certain complex oriented cohomology theories to be represented by an E_∞-ring spectrum. Complex oriented theories (those with Chern classes) are basic to the chromatic viewpoint on stable homotopy theory, while an E_∞ ring spectrum has power operations akin to Adams operations. The interplay between Chern classes and power operations can be quite subtle, and the Naumann-Lawson criterion addresses this issue using the algebraic geometry of p-divisible groups.

Hans-Werner Henn (reporting on joint work with Paul Goerss) gave a calculation of the Brown-Comenetz dual of the $K(2)$-local sphere at the prime 3. In a celebrated paper, Gross and Hopkins calculated the E_2-homology of this object using the rigid analytic geometry of the Lubin-Tate deformation space; however, this is not enough to determine the homotopy type in examples such as this. In fact, there is a twist by an element of Hopkins's Picard group.

John Francis's talk began with a basic question: what would a homology theory for manifolds look like? Specifically, with any homology theory, global data can be assembled from local data using a Mayer-Vietoris sequence; so we want some sort of local-to-global calculating tool for manifolds. After giving an axiomatic characterization, he then gave an elegant construction of topological chiral homology using factorization algebras and a new and intuitive proof of the nonabelian Poincaré duality theorem of Salvatore and Lurie.

Julie Bergner (reporting on joint work with Charles Rezk) gave an introduction into the homotopy theoretic approach to higher category theory, advertising Rezk's category of Θ_n-spaces as a convenient model for the concept of (∞, n)-categories.

A new feature in this workshop was the 'gong show' on Wednesday morning. Here the following nine junior participants took the opportunity to present their work in 10-minute presentations.

Nora Seeliger: *Group models for fusion systems and cohomology*

George Raptis: *Presheaves of coalgebras*

Martin Palmer: *Homological stability for oriented configuration spaces*

Justin Noel: *Maps of homotopy T-algebras*

Lennart Meier: *Modules over TMF*

Martin Langer: *Cohomology of certain crystallographic groups*

Geoffroy Horel: *Higher topological Hochschild cohomology*

Dustin Clausen: *p-adic analogs of the real J-homomorphism*

Tarje Bargher: *An operadic E_{n+1}-construction that acts on $Map(S^n, M)$ stably*

Workshop: Homotopy Theory

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Abstracts

Equivariant K-theory of group actions with maximal rank isotropy

Alejandro Adem
(joint work with José Manuel Gómez)

Let G denote a compact connected Lie group with torsion–free fundamental group. Suppose that G acts on a compact space X so that each isotropy subgroup is a connected subgroup of maximal rank; that is, G_x contains a maximal torus $T \subset G$ for every $x \in X$. We study the problem of computing $K_G^*(X)$, the complex G–equivariant K–theory of X. Our work is primarily motivated by the examples given by spaces of ordered commuting n-tuples in compact matrix groups such as $U(m)$, $SU(m)$ and $Sp(m)$ with the conjugation action. These examples can also be described as spaces of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G) \subset G^n$ with the conjugation action. However there are other interesting types of examples for which these techniques will apply: let G act on its Lie algebra $\mathfrak{g}$ via the adjoint representation; then the G-sphere $S^{\mathfrak{g}}$ is an action with connected maximal rank isotropy.

The starting point for the computation of $K_G^*(X)$ for such actions is the observation that if $T \subset G$ is a maximal torus, then by restriction there is an associated action of $N_G(T)$ on X^T. This in turn yields a corresponding action of the Weyl group W on the fixed-point set X^T. The associated action of W on X^T determines, in some way, the behaviour of the original action of G on X. We derive some conditions on the action of W on X^T, assuming that X^T has the homotopy type of a W-CW complex, so that $K_G^*(X)$ or $K_G^*(X) \otimes \mathbb{Q}$ can be computed from the strong collapse of a spectral sequence associated to the skeletal filtration which can be expressed in terms of Bredon cohomology. Before stating our main results we briefly recall a definition from the theory of reflection groups.

Suppose that $W_i \subset W$ is a reflection subgroup. Let Φ_i be the corresponding root system and Φ_i^+ the corresponding positive roots. Define

$$W_i^\ell := \{w \in W \mid w(\Phi_i^+) \subset \Phi^+\}.$$

The set W_i^ℓ forms a system of representatives for the different cosets in W/W_i. Let $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ be a family of reflection subgroups of W. We say that $\mathcal{W}$ satisfies the *coset intersection property* if given $i, j \in \mathcal{I}$ we can find some $k \in \mathcal{I}$ such that $W_i \cup W_j \subset W_k$ and $W_k^\ell = W_i^\ell \cap W_j^\ell$. We are now ready to state our first result:

Theorem 1 ([1]). *Let G be a compact connected Lie group with torsion-free fundamental group and $T \subset G$ a maximal torus. Suppose that G acts on a compact space X with connected maximal rank isotropy such that X^T has the homotopy type of a W-CW complex. Assume that there is a CW-subcomplex K of X^T containing a unique representative under the action of W for all the cells in X^T and the family $\{W_\sigma \mid \sigma$ is a cell in $K\}$ is contained in a family $\mathcal{W}$ of reflection subgroups of W satisfying the coset intersection property. If $H^*(X^T;\mathbb{Z})$ is torsion-free, then $K_G^*(X)$ is a free module over $R(G)$ of rank equal to $\sum_{i\geq 0} \mathrm{rank}_{\mathbb{Z}} H^i(X^T;\mathbb{Z})$.*

The previous theorem has the following applications.

Corollary 2. *Let G be a simply connected compact Lie group acting on itself by conjugation. Then $K^*_G(G)$ is a free module over $R(G)$ of rank 2^r, where r denotes the rank of G.*

This yields a new proof for a result first obtained in Brylinski and Zhang ([2]) where the structure of $K^*_G(G)$ was determined as an algebra over $R(G)$ for compact connected Lie groups with $\pi_1(G)$ torsion-free, and which is a precursor to our work. On the level of Lie algebras this corollary has the following analogue.

Corollary 3. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Let G act on its Lie algebra $\mathfrak{g}$ by the adjoint representation. If r is the rank of G, then*

$$\tilde{K}^q_G(S^{\mathfrak{g}}) \cong \begin{cases} R(G) & \textit{if} \quad q \equiv r \ (\textit{mod}\ 2), \\ 0 & \textit{if} \quad q+1 \equiv r \ (\textit{mod}\ 2). \end{cases}$$

We also obtain representation–theoretic conditions which imply that $K^*_G(X)\otimes\mathbb{Q}$ is a free $R(G)\otimes\mathbb{Q}$–module and which in many cases are not hard to verify. Let $\mathcal{M}_W$ be the family of W-representations A over $\mathbb{Q}$ for which there is an isomorphism of $R(G)\otimes\mathbb{Q}$ modules

$$(A \otimes R(T))^W \cong A \otimes R(G).$$

Theorem 4 ([1])**.** *Let G be a compact connected Lie group with torsion-free fundamental group and $T \subset G$ a maximal torus. Suppose that G acts on a compact space X with connected maximal rank isotropy subgroups in such a way that X^T has the homotopy type of a W-CW complex. If $H^*(X^T, \mathbb{Q})$ belongs to the family $\mathcal{M}_W$, then $K^*_G(X)\otimes\mathbb{Q}$ is a free module over $R(G)\otimes\mathbb{Q}$ of rank equal to $\sum_{i\geq 0} \mathrm{rank}_{\mathbb{Q}} H^i(X^T;\mathbb{Q})$.*

Here are some applications of our second theorem. Let $C_n(\mathfrak{g})$ denote the algebraic variety of ordered commuting n-tuples in $\mathfrak{g}$. As before, this variety is endowed with an action of G via the adjoint representation. Consider the collection $\mathcal{P}$ of all compact Lie groups arising as finite products of the classical groups S^1, $SU(r)$, $U(q)$ and $Sp(k)$.

Corollary 5. *Suppose that $G \in \mathcal{P}$ is of rank r. Then there is an isomorphism of modules over $R(G)\otimes\mathbb{Q}$*

$$\tilde{K}^q_G(C_n(\mathfrak{g})^+) \otimes \mathbb{Q} \cong \begin{cases} R(G)\otimes\mathbb{Q} & \textit{if} \quad q \equiv rn \ (\textit{mod}\ 2), \\ 0 & \textit{if} \quad q+1 \equiv rn \ (\textit{mod}\ 2). \end{cases}$$

Recall that if G acts on a topological space X then its inertia space is defined as $\Lambda X := \{(g,x) \in G \times X \mid gx = x\}$. Note that ΛX inherits an action of G and the basic observation is that if G acts on X with connected maximal rank isotropy subgroups and $\pi_1(G_x)$ is torsion-free, then ΛX has connected maximal rank isotropy groups.

Theorem 6 ([1])**.** *Let X denote a compact G–CW complex with connected maximal rank isotropy subgroups all of which have torsion–free fundamental group. Assume furthermore that $H^k(T \times X^T;\mathbb{Q})$ is in $\mathcal{M}_W$ for every $k \geq 0$. Then*

$K^*_G(\Lambda X) \otimes \mathbb{Q}$ *(as an ungraded module) is a free* $R(G) \otimes \mathbb{Q}$ *module of rank equal to* $2^r \cdot \left(\sum_{i\geq 0} \operatorname{rank}_{\mathbb{Q}}(H^i(X^T;\mathbb{Q}))\right)$*, where* r *is the rank of* G*.*

The construction of inertia spaces can be iterated. In this way we obtain a sequence of spaces $\{\Lambda^n(X)\}_{n\geq 0}$. If we further require that $G_x \in \mathcal{P}$ for all $x \in X$, then the action of G on $\Lambda^n(X)$ has connected maximal rank isotropy subgroups for every $n \geq 0$.

Theorem 7 ([1])**.** *Let* X *denote a compact* G*–*CW *complex such that all of its isotropy subgroups lie in* $\mathcal{P}$ *and are of maximal rank. Assume furthermore that* $H^k(T^n \times X^T;\mathbb{Q})$ *is in* $\mathcal{M}_W$ *for every* $k \geq 0$*. Then* $K^*_G(\Lambda^n(X))\otimes\mathbb{Q}$ *(as an ungraded module) is a free* $R(G)\otimes\mathbb{Q}$ *module of rank equal to* $2^{nr} \cdot \left(\sum_{i\geq 0} \operatorname{rank}_{\mathbb{Q}}(H^i(X^T;\mathbb{Q}))\right)$ *where* r *is the rank of* G*.*

Taking X to be a single point with the trivial G–action yields $\Lambda^n(X) = \operatorname{Hom}(\mathbb{Z}^n, G)$ (the space of ordered commuting n-tuples in G) with the conjugation action, and our result can be applied.

Corollary 8. *Suppose that* $G \in \mathcal{P}$ *is of rank* r*. Then* $K^*_G(\operatorname{Hom}(\mathbb{Z}^n, G)) \otimes \mathbb{Q}$ *is free of rank* 2^{nr} *as an ungraded* $R(G) \otimes \mathbb{Q}$ *module.*

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Introduction to Redshift

John Rognes

1. Homotopy fixed points of smash powers

Consider a compact Lie group G and a commutative S-algebra B. The tensor product $G \otimes B = \bigwedge_G B$ is a commutative B-algebra with G-action. Consider the G-homotopy fixed points $(G \otimes B)^{hG} = F(EG_+, G \otimes B)^G$. Experience has shown that if $\pi_*(B)$ contains v_n-periodic families then $\pi_*(G \otimes B)^{hG}$ often contains v_{n+k}-periodic families, where k is the rank of G. Since the v_{n+k}-periodic families have longer periods, or longer wavelength, than the v_n-periodic families, we refer to this as a *redshift* phenomenon.

The case $G = \mathbb{T}$ was investigated in the context of topological cyclic homology by Madsen and coauthors [2], [5]. The tensor product $\mathbb{T} \otimes B = THH(B)$ is the *topological Hochschild homology* of B, and $THH(B)^{h\mathbb{T}}$ is closely related to the topological Frobenius homology $TF(B;p) = \operatorname{holim}_{n,F} THH(B)^{C_{p^n}}$. The higher abelian cases, with $G = \mathbb{T}^k$ for $k \geq 2$, have been emphasized by Carlsson, Dundas and coauthors [3], [4].

2. Examples of redshift

We write H for $H\mathbb{F}_p$, and focus on $p = 2$, but the results extend to odd primes.

2.1. **The case $G = \mathbb{T}$ and $B = H$.** In this case $H_*(H) = \mathscr{A}_* = P(\bar{\xi}_1, \bar{\xi}_2, \dots)$ is dual to the Steenrod algebra $\mathscr{A}$. Computation shows that $H_*(THH(\mathbb{F}_2)) = \mathscr{A}_* \otimes P(x)$ where x is represented by $\sigma\bar{\xi}_1$, so $\pi_* THH(\mathbb{F}_2) = P(x)$ with $\deg(x) = 2$. The odd spheres filtration on $E\mathbb{T} = S(\mathbb{C}^\infty)$ induces a tower of fibrations with limit $THH(B)^{h\mathbb{T}} = F(E\mathbb{T}_+, THH(B))^{\mathbb{T}}$. The *continuous cohomology* $H^*_c(THH(B)^{h\mathbb{T}})$ is the colimit of the cohomology groups in the tower. Computations show that $H^*_c(THH(\mathbb{F}_2)^{h\mathbb{T}}) \cong \bigoplus_{j\in\mathbb{Z}} \Sigma^{2j} H^*(H\mathbb{Z})$, so $\pi_* THH(\mathbb{F}_2)^{h\mathbb{T}} \cong \bigoplus_{j\in\mathbb{Z}} \Sigma^{2j}\mathbb{Z}_2$. This is a first example of redshift, where v_0-periodicity arises in $\pi_* THH(B)^{h\mathbb{T}}$, even if $\pi_*(B)$ is v_0-torsion.

2.2. **The case $G = \mathbb{T}^2$ and $B = H$.** Computation shows that $H_*(\mathbb{T}^2 \otimes H) = \mathscr{A}_* \otimes P(x_1, x_2) \otimes E(y)$, where x_1, x_2 and y are represented by $\sigma_1\bar{\xi}_1$, $\sigma_2\bar{\xi}_1$ and $\sigma_1\sigma_2\bar{\xi}_1$, in degrees 2, 2 and 3, respectively. Here σ_1 and σ_2 are the differentials and derivations induced by the standard generators of $H_1(\mathbb{T}^2)$. A full computation of $H^*_c((\mathbb{T}^2 \otimes H)^{h\mathbb{T}^2})$ is complicated, but the image of the edge homomorphism $H^*(\mathbb{T}^2 \otimes H) \to H^*_c((\mathbb{T}^2 \otimes H)^{h\mathbb{T}^2})$ contains many copies of $H^*(ku)$, some of which produce copies of $\pi_*(ku^\wedge_2)$ in homotopy. This is an example of higher redshift, where v_0- and v_1-periodicity arises in $\pi_*(\mathbb{T}^2 \otimes B)^{h\mathbb{T}^2}$, even if $\pi_*(B)$ is v_0-torsion.

2.3. **The case $G = \mathbb{T}$, $B = tmf$.** Inspired by the redshift phenomenon, Bruner and Rognes are investigating $THH(tmf)^{h\mathbb{T}}$ as a possible example of a v_3-periodic theory with interesting maps $S \to K(tmf) \to THH(tmf)^{h\mathbb{T}}$. This has the potential of detecting γ-family elements in $\pi_*(S)$, which until now have not been observed at $p = 2$.

2.4. **The case $G = \mathbb{T}$, $B = MU$.** We are also interested in $K(MU)$ and $THH(MU)^{h\mathbb{T}}$, as a half-way house between $K(S) = A(\star)$ and $K(\mathbb{Z})$. Is $K(MU)$ simpler to describe then $K(S)$, in the way that $\pi_*(MU)$ is simpler than $\pi_*(S)$? Can $K(S)$ be recovered from $K(MU)$ by descent along the Hopf–Galois extension $S \to MU$, in the sense of [9]?

3. Algebraic and topological Singer constructions

The prime order case $G = C_p$ is well understood.

3.1. **The algebraic Singer construction.** For each $\mathscr{A}$-module M the *algebraic Singer construction* is the tensor product $R_+(M) = \Sigma P(x^{\pm 1}) \otimes M$, with $\deg(x) = 1$, with an explicitly given action by the Steenrod algebra [10]. There is a natural $\mathscr{A}$-module homomorphism $\epsilon\colon R_+(M) \to M$, taking $\Sigma x^r \otimes a$ to $Sq^{r+1}(a)$, which turns out to be an Ext-equivalence [1].

3.2. **The topological Singer construction.** Let B be bounded below with $H_*(B)$ of finite type. Lunøe-Nielsen and Rognes [7] define the *topological Singer construction* $R_+(B) = (B \wedge B)^{tC_2}$ to be the Tate construction on the spectrum $B \wedge B$, with the C_2-action given by transposition. There is a natural isomorphism of $\mathscr{A}$-modules $H^*_c(R_+(B)) \cong R_+(H^*(B))$.

When $B \wedge B$ is realized with the C_2-equivariant structure given by the *Bökstedt smash product*, there is a natural stable map $\epsilon_B \colon B \to R_+(B)$, from the geometric fixed points of $B \wedge B$ to the Tate construction, which induces Singer's homomorphism in continuous cohomology. Hence $\epsilon_B \colon B \to R_+(B)$ and the related map $\Gamma_1 \colon (B \wedge B)^{C_2} \to (B \wedge B)^{hC_2}$ are both 2-adic equivalences. The case $B = S$ is Lin's theorem [6], being a special case of Segal's Burnside ring conjecture.

4. Cyclic fixed points

The following theorem tells us that when the Segal conjecture for the C_p-action on $THH(B)$ holds "in high degrees", then we are free to replace the homotopy fixed points $THH(B)^{h\mathbb{T}}$ with the Tate construction $THH(B)^{t\mathbb{T}}$, and either one of these is a good approximation to the topological Frobenius theory $TF(B;p)$.

Theorem 1 (Tsalidis [11])**.** *Let B be a connective S-algebra with $H_*(B)$ of finite type. If the map $\Gamma_1 \colon THH(B)^{C_p} \to THH(B)^{hC_p}$ becomes k-coconnected after p-adic completion, then so do all of the maps*

$$\hat{\Gamma}_1 \colon THH(B) \to THH(B)^{tC_p},$$
$$\Gamma \colon TF(B;p) \to THH(B)^{h\mathbb{T}} \text{ and}$$
$$\hat{\Gamma} \colon TF(B;p) \to THH(B)^{t\mathbb{T}}.$$

Calculations are often easier for the Tate constructions, since the Tate cohomology of C_p is (almost) a graded field, while the group cohomology has a more complicated module theory.

5. Additive approximations

The unit map $\eta \colon B \to THH(B)$ extends to an $\mathbb{T}$-equivariant map $\omega \colon \mathbb{T} \ltimes B \longrightarrow THH(B)$. Given $\alpha \in H_q(B)$ we write $\sigma\alpha \in H_{q+1}(THH(B))$ for the image of $\sigma \ltimes \alpha$, where $\sigma \in H_1(\mathbb{T})$ is the generator.

The inclusion $C_2 \subset \mathbb{T}$ induces a C_2-equivariant map $\eta_2 \colon B \wedge B \to THH(B)$, which extends to an $\mathbb{T}$-equivariant map $\omega_2 \colon \mathbb{T} \ltimes_{C_2} B \wedge B \longrightarrow THH(B)$. Applying $(-)^{tC_2}$ we get a $\mathbb{T}/C_2$-equivariant map $\omega^t \colon \mathbb{T}/C_2 \ltimes R_+(B) \simeq (\mathbb{T} \ltimes_{C_2} B \wedge B)^{tC_2} \longrightarrow THH(B)^{tC_2}$.

Theorem 2 (Lunøe-Nielsen, Rognes [8])**.** *There is a homotopy commutative square*

$$\begin{array}{ccc}
\mathbb{T} \ltimes B & \xrightarrow{\ \omega\ } & THH(B) \\
{\scriptstyle \rho \wedge \epsilon_B}\downarrow & & \downarrow{\scriptstyle \hat{\Gamma}_1} \\
\mathbb{T}/C_2 \ltimes R_+(B) & \xrightarrow{\ \omega^t\ } & THH(B)^{tC_2}
\end{array}$$

where $\rho\colon \mathbb{T} \to \mathbb{T}/C_2$ *is the square root isomorphism, and* $\epsilon_B\colon B \to R_+(B)$ *is the map inducing the* Ext*-equivalence in cohomology.*

Using this diagram we can compute the effect of $\hat{\Gamma}_1$ on classes in $H_*(THH(B))$ that are in the image of ω_*, i.e., the classes α and $\sigma\alpha$ for $\alpha \in H_*(B)$. This is made possible by explicit formulas for $(\epsilon_B)_* = \epsilon_*$ and $(\omega^t)_*$ in homology.

6. THH of complex bordism

In view of the examples $B = H\mathbb{F}_p$, $H\mathbb{Z}$, ℓ, ku and tmf, where Γ_1 and $\hat{\Gamma}_1$ only become equivalences with suitably finite coefficients and in high degrees, the following theorem is a little surprising. It asserts that the C_p-equivariant Segal conjecture holds for $THH(MU)$, in just as strong a form as it holds for the C_p-equivariant sphere spectrum $THH(S) = S$.

Theorem 3 (Lunøe-Nielsen, Rognes [8])**.** *The map*

$$\hat{\Gamma}_1\colon THH(MU) \longrightarrow THH(MU)^{tC_p}$$

is a p-adic equivalence.

Outline of proof. We prove that $\hat{\Gamma}_1$ is a p-adic equivalence by showing that there is an isomorphism of $\mathscr{A}$-modules $\Phi^*\colon H^*_c(THH(MU)^{tC_p}) \xrightarrow{\cong} H^*_c(R_+(THH(MU)))$ such that $(\hat{\Gamma}_1)^* = \epsilon \circ \Phi^*$. To achieve this we show that there is a bicontinuous isomorphism of complete $\mathscr{A}_*$-comodules

$$\Phi\colon H^c_*(R_+(THH(MU))) \xrightarrow{\cong} H^c_*(THH(MU)^{tC_p})$$

such that $(\hat{\Gamma}_1)_* = \Phi \circ \epsilon_*$.

This is done by two careful calculations, showing that there are pro-isomorphisms of $\mathscr{A}_*$-comodules

$$f\colon H^c_*(R_+(MU)) \otimes_{H_*(MU)} H_*(THH(MU)) \longrightarrow H^c_*(R_+(THH(MU)))$$

and

$$g\colon H^c_*(R_+(MU)) \otimes_{H_*(MU)} H_*(THH(MU)) \longrightarrow H^c_*(THH(MU)^{tC_p})$$

under $H_*(THH(MU))$. □

Corollary 4. *The maps* $\Gamma_1\colon THH(MU)^{C_p} \to THH(MU)^{hC_p}$, $\Gamma\colon TF(MU;p) \to THH(MU)^{h\mathbb{T}}$ *and* $\hat{\Gamma}\colon TF(MU;p) \to THH(MU)^{t\mathbb{T}}$ *are p-adic equivalences.*

This result tells us that we have a good chance at determining $K(MU)$ by way of $TC(MU;p)$, since $TF(MU;p)$, $THH(MU)^{h\mathbb{T}}$ and $THH(MU)^{t\mathbb{T}}$ are all p-adically equivalent.

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Operads, modules and Goodwillie towers

Gregory Arone

(joint work with Michael Ching)

Let $\mathcal{C}$ and $\mathcal{D}$ each be either the category of pointed spaces or the category of spectra. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pointed homotopy functor. By the work of Goodwillie [3], the derivatives of F form a symmetric sequence of spectra $\partial_* F$. This symmetric sequence determines the homogeneous layers in the Taylor tower of F, but not the extensions in the tower. We explore the following question: what natural structure does $\partial_* F$ possess, beyond that of a symmetric sequence? Our ultimate goal is to describe a structure that is sufficient to recover the Taylor tower of F from the derivatives. Such a description could be considered an extension of Goodwillie's classification of homogeneous functors to a classification of Taylor towers.

Our starting point is a theorem of Ching [2], which says that *the derivatives of the identity functor form an operad.* Our first theorem says that *the derivatives of a general functor F form a bimodule* (or a right/left module, depending on the source and target categories of the functor) *over this operad* [1]. Koszul duality for operads plays an interesting role in the proof. As an application we show that the module structure on derivatives is exactly what one needs to write down a chain rule for the calculus of functors.

However, this module structure on $\partial_* F$ is not sufficient to recover the Taylor tower of F and therefore it does not tell the whole story. We are led to seek a refinement of the module structure. We may now consider Goodwillie differentiation as a functor from a category of functors to a suitable category of bimodules over an operad

$$\partial_*\colon [\mathcal{C},\mathcal{D}] \longrightarrow \text{Bimodules}.$$

We make the following key observation: The differentiation functor functor ∂_*, considered as a functor into the category of bimodules, has a right adjoint. We will denote this right adjoint by Φ.

$$\Phi\colon \text{Bimodules} \longrightarrow [\mathcal{C},\mathcal{D}].$$

The composite functor $\Phi\partial_*$ defines a comonad in the category of bimodules, and the derivatives of a functor from $\mathcal{C}$ to $\mathcal{D}$ form a coalgebra over this comonad. From this coalgebra structure one can, in principle, reconstruct the Taylor tower of a functor. This leads to our main theorem:

Theorem. *Goodwillie differentiation induces an equivalence of homotopy categories between the category of homotopy functors $[\mathcal{C},\mathcal{D}]$, where a weak equivalence is a natural transformation that induces an equivalence of Taylor towers, and the category of coalgebras over the comonad $\Phi\partial_*$.*

Our next task is to give a more concrete characterization of what it means to be a coalgebra over $\Phi\partial_*$ in specific cases. To this end we prove the following results: Taylor towers of functors from Spaces to Spectra are classified by right modules over something that may be called the "Lie pro-operad". This structure is a refinement of that of a right module over the Lie operad (which is what one expects). Alternatively, one can say that (Taylor towers of) functors from Spaces to Spectra are classified by "restricted right modules" over the Lie operad. In the same vein we show that Taylor towers of functors from Spectra to Spectra are classified by right modules over the "sphere pro-operad". This structure is a refinement of that of a symmetric sequence, again in accordance with expectation.

An interesting example to test our theory on is the functor $X \mapsto \Sigma^\infty\Omega^\infty(E\wedge X)$. Here E is a fixed spectrum, and the functor can be thought as a functor from either the category of Spaces or Spectra to the category of Spectra. The derivatives of this functor are given by the sequence $E, E^{\wedge 2}, \ldots, E^{\wedge n}, \ldots$. The fact that this sequence is the sequence of derivatives of a functor tells us something interesting about the structure possessed by spectra in general. In particular it tells us that spectra possess a natural structure of a *restricted coalgebra* over the Lie operad.

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Homological behavior of the Goodwillie tower

Mark Behrens
(joint work with Charles Rezk)

1. Calculus background

Let $F : \mathrm{Top}_* \to \mathrm{Top}_*$ be a functor from spaces to spaces, which preserves the zero object, weak equivalences, and filtered homotopy colimits. Goodwillie [8] associates to such a functor a tower of functors under F

$$\cdots \to P_3(F) \to P_2(F) \to P_1(F).$$

The fibers $D_n(F) = \mathrm{fiber}(P_n(F) \to P_{n-1}(F))$ satisfy

$$D_n(F)(X) = \Omega^\infty \mathbb{D}_n(F)(X), \quad \mathbb{D}_n(F)(X) = \partial_n(F) \wedge_{h\Sigma_n} X^{\wedge n}.$$

Here, $\partial_n(F)$ is a Σ_n-spectrum. Under favorable circumstances, the tower converges, in the sense that $F(X) \simeq \mathrm{holim}_n P_n(F)(X)$. We then get a Goodwillie spectral sequence (GSS)

$$\pi_* \mathbb{D}_n(F)(X) \Rightarrow \pi_* F(X).$$

The fiber sequences that give the layers deloop to give "k invariants" $k_n : P_n(F) \to BD_{n+1}(F)$. The "attaching map" between consecutive layers of the tower is given by the composite

$$\alpha_n : D_n(F) \to P_n(F) \xrightarrow{k_n} BD_{n+1}(F).$$

These attaching maps are the d_1-differentials in the GSS. We are interested in studying the homology of the layers, and the homological behavior of these attaching maps, where F is the identity functor Id, and X is a sphere S^q. In this case, the GSS converges to the unstable homotopy groups of spheres. The homology theories we will be discussing will be either (1) mod 2 homology, or (2) Morava E-theory. All of the work in case (2) is joint with Charles Rezk.

2. Facts about $P_n(\mathrm{Id})$

We recall the following:

- The derivatives $\partial_*(\mathrm{Id})$ form an operad [7] which is topologically Koszul dual to the commutative operad. As such, it should be regarded as a homotopical Lie operad.
- For any functor F, the derivatives $\partial_*(F)$ are a bimodule over $\partial_*(\mathrm{Id})$ [1].
- For $n \neq p^k$ (and q odd if $p > 2$) the layer $\mathbb{D}_n(S^q)$ is p-locally contractible [4]. Therefore, p-locally one actually studies the attaching maps α_k : $D_{p^k}(S^q) \to BD_{p^{k+1}}(S^q)$.
- The p-local attaching maps above deloop k-times: $\alpha_k = \Omega^k \beta_k$ [3].

3. Mod 2 homology

Just as the E_∞ operad has associated to it Dyer-Lashof operations, the Lie operad $\partial_*(\mathrm{Id})$ has Dyer-Lashof-Lie operations [5]. Using the left module structure on $\partial_*(F)$ induces (mod 2) homology operations

$$\bar{Q}^i : H_*\mathbb{D}_n(F)(X) \to H_{*+i-1}\mathbb{D}_{2n}(F)(X)$$

such that

$$\bar{Q}^r\bar{Q}^s = \sum_t \left[\binom{s-r+t}{s-1} + \binom{s-r+t}{2t-r}\right] \bar{Q}^{r+s-t}\bar{Q}^t$$

$$\bar{Q}^i x = 0 \quad \text{if: } i < |x|.$$

These relations give the Lie-Dyer-Lashof algebra a basis of admissibles $\bar{Q}^{i_1}\cdots\bar{Q}^{i_s}$ with an admissibility criterion $i_j \geq 2i_{j+1}+1$. In the language of these operations, we get the following computation (compare with [4]).

Theorem 1 ([5])**.** *We have*

$$H_*\mathbb{D}_{2^k}(\mathrm{Id})(S^q) = \mathbb{F}_2\{\bar{Q}^{i_1}\cdots\bar{Q}^{i_k}\iota_q :\ :: i_j \geq 2i_{j+1}+1, i_k \geq q\}.$$

Our next task is to describe the effect on homology of the (delooped) attaching maps

$$\beta_k : B^k D_{2^k}(\mathrm{Id})(S^q) \to B^{k+1}D_{2^{k+1}}(\mathrm{Id})(S^q).$$

Arone and Dwyer showed that there are equivalences $\Sigma^k\mathbb{D}_{2^k}(\mathrm{Id})(S^q) \simeq \Sigma^q L(k)_q$, where $L(k)_q$ is the Steinberg summand of the Thom spectrum $B(\mathbb{F}_2^k)^{q\bar{\rho}}$ ($\bar{\rho}$ is the reduced regular representation). In particular, $\Sigma^k\mathbb{D}_{2^k}(\mathrm{Id})(S^q)$ is a retract of a suspension spectrum, and hence $H_*B^kD_{2^k}(\mathrm{Id})(S^q)$ is the free allowable algebra over the Dyer-Lashof algeba generated by $H_*\Sigma^k\mathbb{D}_{2^k}(\mathrm{Id})(S^q)$.

Theorem 2 ([6])**.** *The induced map on mod 2 homology $(\beta_k)_*$ is described on algebra generators by*

$$\beta_k : Q^{j_1}\cdots Q^{j_\ell}\sigma^k\bar{Q}^{i_1}\cdots\bar{Q}^{i_k}\iota_q = \sum_{s=1}^{\ell} Q^{j_1}\cdots\bar{Q}^{j_s}\cdots Q^{j_\ell}\sigma^k\bar{Q}^{i_1}\cdots\bar{Q}^{i_k}\iota_q$$

Up to a weight filtration, the map is multiplicative.

In the formula in the above theorem, the Dyer-Lashof operations and the Lie-Dyer-Lashof operations commute with the mixed Adem relations:

$$\bar{Q}^r Q^s = \sum_t \left[\binom{s-r+t}{s-1} + \binom{s-r+t}{2t-r}\right] Q^{r+s-t}\bar{Q}^t.$$

In the case of $q=1$, these formulas coincide with the formulas of N. Kuhn of the homology of the James-Hopf maps used in his proof of the Whitehead conjecture. We therefore extract the following corollary (proved independently by Arone-Dwyer-Lesh).

Corollary 3 ([6], Arone-Dwyer-Lesh)**.** *The GSS for S^1 collapses at the E_2 page.*

4. Morava E-theory

Let E denote height n Morava E-theory. Let Δ_q be the algebra of natural additive operations which act on the module of indecomposables $[QE_*^\wedge(R)]_q$ for R an augmented E_∞ ring spectrum. Rezk recently showed these algebras are *Koszul*. What does this mean? These algebras decompose additively as

$$\Delta_q = \bigoplus_k \Delta_q[k]$$

where $\Delta_q[k]$ is the E_0-module spanned by "length k sequences of operations". Let $\bar{E}_0$ denote the Δ_q-module structure on E_0 induced from the augmentation $\Delta_q \to \Delta_q[0] = E_0$. Then the bar complex breaks up as

$$B_\bullet(\bar{E}_0, \Delta_q, \bar{E}_0) = \bigoplus_k B_\bullet(\bar{E}_0, \Delta_q, \bar{E}_0)[k].$$

Being Koszul means that

$$H_* B_\bullet(\bar{E}_0, \Delta_q, \bar{E}_0)[k] = \begin{cases} 0, & * \neq k, \\ C[k]_q, & * = k. \end{cases}$$

The modules $C[k]_q$ give the *Koszul resolution* — for M a Δ_q-module, there is an associated cochain complex

$$C[0]_q^\vee \otimes_{E_0} M \xrightarrow{d_0} C[1]_q^\vee \otimes_{E_0} M \xrightarrow{d_1} C[2]_q^\vee \otimes_{E_0} M \xrightarrow{d_2} \cdots$$

whose cohomology is $\mathrm{Ext}_{\Delta_q}(\bar{E}_0, M)$.

Theorem 4. *For odd q there are canonical isomorphisms*

$$E_q^\wedge(\Sigma^k \mathbb{D}_{2^k}(\mathrm{Id})(S^q)) \cong C[k]_{-q}^\vee.$$

We are making progress on the following (Φ_n is the Bousfield-Kuhn functor):

Conjecture 5. *Under the isomorphism above, the induced map $(\Phi_n \beta_k)_*$ on E-homology is given by the differential in the Koszul resolution for $\tilde{E}^q(S^q)$.*

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Localizations of Equivariant Commutative Rings

Michael A. Hill
(joint work with Michael J. Hopkins)

In this talk, I discussed joint work with Hopkins which addresses the question: "When is the localization of a commutative G-equivariant ring a commutative G-equivariant ring?". In all that follows, let G be a finite group. The talk sketches a proof of the following theorem.

Theorem 1. *If for all acyclics Z for a localization L and for all subgroups H, $N_H^G Z$ is acylic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

The proof is modeled on the standard non-equivariant proof in EKMM [1]. The essential twist is understanding the interplay between the G-action on the E_∞ operad and the norm.

Already in the statement of the theorem we have used the norm (as described in Hill-Hopkins-Ravenel [2]). This is a symmetric monoidal functor $N_H^G\colon \mathcal{S}_H \to \mathcal{S}_G$ from the category of H-spectra (with its smash product) to the category of G-spectra. This has the distinguished feature of also refining to the left adjoint to the forgetful functor from commutative G-ring spectra to commutative H-ring spectra. Thus there is for any commutative G-ring spectrum R a canonical map of commutative G-ring spectra

$$N_H^G Res_H^G(R) \to R.$$

These satisfy axioms analogous to the norm maps in Tambara functors, making commutative G-rings into spectral Tambara functors (analogous to Guillou-May's description of equivariant spectra).

1. Localizations Need Not Be Commutative

We first sketch a counterexample to the obvious conjecture. Let $\mathcal{P}$ denote the family of proper subgroups of G, and let $\tilde{E}\mathcal{P}$ denote the cofiber of the natural map from the classifying space $E\mathcal{P}_+$ to S^0. The spectrum $\tilde{E}\mathcal{P}$ is a localization of S^0: we kill all maps from induced cells. Since G is finite, we can also realize $\tilde{E}\mathcal{P}$ as $S^0[a_{\bar{\rho}}^{-1}]$, where $a_{\bar{\rho}}$ is the inclusion of $\{0,\infty\}$ into the representation sphere associated to $\bar{\rho}$, the quotient of the real regular representation by its trivial summand.

This spectrum does not admit maps from the norms of its restrictions. For any proper subgroup H, the commutative H-ring spectrum $Res_H^G(\tilde{E}\mathcal{P})$ is contractible. This is the terminal commutative H-ring, and since $\tilde{E}\mathcal{P}$ is not contractible, we cannot have a commutative ring map

$$* \simeq N_H^G Res_H^G(\tilde{E}\mathcal{P}) \to \tilde{E}\mathcal{P}.$$

The example already underscores the role the norm will play. Here there is an obstruction to being a commutative ring spectrum. One way to interpret our theorem is that this is the obstruction; if localization "plays nicely with the norm", then it takes commutative ring objects to commutative ring objects. On the other hand, the spectrum $\tilde{E}\mathcal{P}$ is a the result of inverting a map from an invertible element ($S^{-\bar{\rho}}$) to the symmetric monoidal unit S^0. For formal reasons, this is guaranteed to be "infinitely coherently commutative". Thus we need to understand the different ways a G-commutative ring spectrum can be commutative.

2. Flavors of E_∞

In the non-equivariant context, the model of the E_∞ operad used for commutative rings is the linear isometries operad on an separable Hilbert space U. In the equivariant context, we have an additional choice: how does the group act on U? We consider only universes (so if an irreducible representation occurs, it does so infinitely often) that contain a trivial summand. This gives a hierarchy of operads, all of which are underlain by the ordinary linear isometries operad. In all cases, the key determination is which subgroups H are such that G/H embeds in U (just as with transfers in the additive context). The extremal cases are where U is a trivial universe (so only G/G embeds within), giving the "naive E_∞ operad" and U a complete universe (so all G/H embed within), giving the commutative operad.

There is a huge difference between the algebras over these operads. Operads over the naive E_∞ operad are "coherently homotopy commutative", but for a free algebra over this operad on Z, the geometric fixed points is the free algebra on the geometric fixed points of Z. In particular, for $Z = G_+$, the geometric fixed points are S^0. In stark contrast, for the commutative operad, the geometric fixed points can be much more complicated.

The spaces in these linear isometries operads are universal spaces for families of subgroups of $G\times\Sigma_n$. For the naive operad, the family is those subgroups contained in G. For the commutative operad, the family is those subgroups $H \subset G\times\Sigma_n$ such that $H\cap\Sigma_n = \{e\}$. As universal families, they have nice simplicial decompositions, and the only cells which appear are those with stabilizer in the families associated to the operad.

In all cases, Σ_n acts freely. Since $G \times \Sigma_n$ need not, there can be fixed points produced. The easiest way to describe how these interact, and to see the result, is to utilize the natural enrichment of the symmetric monoidal structure on G-spectra.

2.1. **Tensoring over G-spaces.** It is well known that the category of commutative G-ring spectra is tensored over G-spaces (see for instance Mandell-May or the appendix to Hill-Hopkins-Ravenel [3]). The universal property defining the tensor structure establishes canonical equivalences

$$G/H \otimes R \simeq N_H^G Res_H^G(R).$$

This tells us how to tensor any commutative G-ring spectrum with any finite G-set (and therefore by the usual tricks with any G-space, though this will not directly be needed). What is perhaps more exciting is that while the left-hand side is not defined for R an arbitrary element of $\mathcal{S}_G$, the right-hand side is. This means that via the norm, we can define $X \otimes Z$ for a finite G-set X and a G-spectrum Z, and the underlying spectrum is just the $|X|$-fold smash power of Z.

The key step in proving the theorem is to identify

$$(G \times \Sigma_n/H)_+ \wedge_{\Sigma_n} Z^{\wedge n} = (G \times \Sigma_n/H)_+ \wedge_{\Sigma_n} (\underline{n} \otimes Z),$$

where $\underline{n}$ denotes the set with n-elements and a trivial G-action. Depending on H, smashing over Σ_n converts $\underline{n}$ into a different G-set (possibly with a non-trivial G-action). This means that smashing $G \times \Sigma_n/H$ over Σ_n with $\underline{n} \otimes Z$ yields a wedge of spectra of the form $X \otimes Z$ for various G-sets (determined by H).

The key example is as follows. Let $n = 2$, and let $G = C_2$. There is a subgroup, $\mathbb{Z}/2$ of $G \times \Sigma_2$ given by the diagonal (it's ludicrous, since all the groups are the same). Now consider

$$G \times \Sigma_2/H_+ \wedge_{\Sigma_2} (\underline{2} \otimes Z).$$

The quotient by H shows that we identify the canonical Σ_2-action on $\underline{2}$ with a C_2-action. This converts $\underline{2}$ into C_2 as a C_2-space, and hence

$$G \times \Sigma_2/H_+ \wedge_{\Sigma_2} (\underline{2} \otimes Z) = C_2 \otimes Z = N_e^{C_2} Res_e^{C_2}(Z).$$

A similar analysis holds in the general case.

3. Sketch of the proof

We need to show that if Z is acyclic, then $Z^{\wedge n}/\Sigma_n$ is acyclic. Knowing this will allow us to simply copy EKMM. We can replace $Z^{\wedge n}/\Sigma_n$ with $\mathcal{L}_n(U)_+ \wedge_{\Sigma_n} Z^{\wedge n}$, where $\mathcal{L}_n(U)$. Using the cell decomposition and the described analysis of the possible stabilizer subgroups, we see that this has a filtration with associated graded suspensions of wedges of smash products of norms. EKMM arguments handle the wedges and smash products, and by assumption, the norms are also acyclic.

We can do something better. We described above a hierarchy of commutative operads. The stronger statement, showed in essentially the same way, is as follows.

Theorem 2. *If for all L-acyclic spectra Z and for all G/H embedding in U the spectrum $N_H^G Res_H^G(Z)$ is L-acyclic, then for all commutative G-ring spectra R, $L(R)$ is an algebra over $\mathcal{L}(U)$.*

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Commutative multiplications on $BP\langle n\rangle$

Niko Naumann
(joint work with Tyler Lawson)

We reported on joint work with T. Lawson using derived algebraic geometry to approach $\mathcal{E}_\infty$-structures on truncated Brown Peterson spectra $BP\langle n\rangle$. The main result of [1] gives a purely algebraic criterion in case $n = 2$. This criterion roughly demands the existence of a coordinate on Lubin Tate deformation space of a 1-dimensional formal group of height 2 with strong rationality properties. The existence or otherwise of such a coordinate is one problem at every prime p and at present we are only able to check this for $p = 2, 3$ using algebraization via elliptic curves. We also reported on some applications [2] including an $\mathcal{E}_\infty$-orientation $tmf_{(2)} \to BP\langle 2\rangle$. To conclude, we explained why one might hope to extend these results to $n \geq 3$ using [3] as a starting point.

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Spectra of units for periodic ring spectra

Steffen Sagave

One can associate a spectrum of units $\mathrm{gl}_1 E$ to a commutative structured ring spectrum E. This is analogous to forming the abelian group of units $\mathrm{GL}_1 R$ in the underlying multiplicative monoid of an ordinary commutative ring R. The spectrum $\mathrm{gl}_1 E$ is useful because it controls the orientation theory of E.

The spectrum of graded units. The various equivalent definitions for $\mathrm{gl}_1 E$ in the literature have the disadvantage that they do not see the difference between a periodic ring spectrum and its connective cover. Our aim is to define a spectrum of *graded* units which detects periodicity. It is a functor

$$\mathrm{gl}_1^{\mathcal{J}} \colon \mathcal{C}\mathrm{Sp}^{\Sigma} \to \Gamma^{\mathrm{op}}\text{-}\mathcal{S}/b\mathcal{J}, \qquad E \mapsto \mathrm{gl}_1^{\mathcal{J}} E$$

from the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^{\Sigma}$ to the category of Γ-spaces augmented over a certain Γ-space $b\mathcal{J}$.

Commutative symmetric ring spectra are one possible incarnation of a category of commutative structured ring spectra. They are strictly commutative monoids with respect to the smash product of symmetric spectra. Segal's category of Γ-spaces Γ^{op}-$\mathcal{S}$ is a convenient way to encode connective spectra. The Γ-space $b\mathcal{J}$ arises from a symmetric monoidal category $\mathcal{J}$ and represents the sphere spectrum. So $\mathrm{gl}_1^{\mathcal{J}} E$ may be viewed as a connective spectrum over the sphere spectrum.

Below we outline some aspects of the definition of $\mathrm{gl}_1^{\mathcal{J}} E$ and explain why we think of it as *graded* units. Before that, we discuss how it relates to the ordinary units $\mathrm{gl}_1 E$.

Graded units and ordinary units. Let $\mathrm{bgl}_1^* E$ be the spectrum associated with the homotopy cofiber of the augmentation $\mathrm{gl}_1^{\mathcal{J}} E \to b\mathcal{J}$.

Theorem 1. [2] *Let E be a positive fibrant commutative symmetric ring spectrum. The ordinary spectrum of units $\mathrm{gl}_1 E$ is the connective cover of $\Omega \mathrm{bgl}_1^* E$, and $\mathrm{bgl}_1^* E$ is connective.*

The bottom homotopy group $\pi_0(\mathrm{bgl}_1^* E)$ is isomorphic to $\mathbb{Z}/n_E\mathbb{Z}$ where $n_E \in \mathbb{N}_0$ is the periodicity *of E. By definition, $n_E = 0$ (and $\mathbb{Z}/n_E\mathbb{Z} \cong \mathbb{Z}$) if all units of the multiplicative graded monoid $\pi_*(E)$ have degree 0, and n_E is the smallest positive degree of a unit in $\pi_*(E)$ otherwise.*

In other words, $\mathrm{bgl}_1^* E$ is a not necessarily connected delooping of the ordinary spectrum of units whose bottom homotopy group detects periodicity. The periodic and connective complex K-theory spectra KU and ku illustrate this: The map $ku \to KU$ that exhibits ku as the connective cover of KU induces the surjection

$$\mathbb{Z} \cong \pi_0(\mathrm{bgl}_1^*(ku)) \to \pi_0(\mathrm{bgl}_1^*(KU)) \cong \mathbb{Z}/2.$$

In contrast, the induced map of ordinary units $\mathrm{gl}_1(ku) \to \mathrm{gl}_1(KU)$ is a stable equivalence.

Since $b\mathcal{J}$ represents the sphere spectrum, the definition of $\mathrm{bgl}_1^* E$ provides a map $\mathbb{S} \to \mathrm{bgl}_1^* E$. The induced map

$$\mathbb{Z}/2 \cong \pi_1(\mathbb{S}) \to \pi_1(\mathrm{bgl}_1^* E) \cong \pi_0(\mathrm{gl}_1 E) \cong (\pi_0(E))^{\times}$$

is the sign action of the additive group structure on $\pi_0(E)$. This implies that the first k-invariant of $\mathrm{bgl}_1^* E$ is non-trivial as soon as $\{\pm 1\}$ acts non-trivially on $\pi_0(E)$.

Graded E_∞ spaces. The definition $\mathrm{gl}_1^{\mathcal{J}} E$ builds on the diagram space of graded units $\mathrm{GL}_1^{\mathcal{J}} E$ that we introduced in joint work with Schlichtkrull [3], and we will now summarize the relevant material from [3].

In the same way as commutative symmetric ring spectra model E_∞ spectra, one can give strictly commutative models for E_∞ spaces: The category $\mathcal{S}^{\mathcal{I}}$ of space valued functors on the category of finite sets and injections $\mathcal{I}$ has a symmetric monoidal product $\boxtimes$ such that all homotopy types of E_∞ spaces are represented by commutative monoids in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$ [3, Theorem 1.2]. We call these commutative monoids in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$ *commutative $\mathcal{I}$-space monoids.* For example, the underlying multiplicative E_∞ space of a commutative symmetric ring spectrum E arises as commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}} E$ in a natural way. The value of $\Omega^{\mathcal{I}} E$ at the finite set $\mathbf{m} = \{1, \ldots, m\}$ is the space $\Omega^m E_m$.

In this terminology, the construction of the ordinary units $\mathrm{gl}_1 E$ goes as follows: The commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}} E$ has a sub commutative $\mathcal{I}$-space monoid $\mathrm{GL}_1^{\mathcal{I}} E$ of invertible path components, and Schlichtkrull [4] showed how to build a Γ-space $\mathrm{gl}_1 E$ from $\mathrm{GL}_1^{\mathcal{I}} E$. Defining $\mathrm{gl}_1 E$ using $\mathrm{GL}_1^{\mathcal{I}} E$ explains why $\mathrm{gl}_1 E$ does not detect periodicity: The inclusion $\mathrm{GL}_1^{\mathcal{I}} E \to \Omega^{\mathcal{I}} E$ corresponds to the inclusion

$\pi_0(E)^\times \to \pi_0(E)$, and both $\mathrm{GL}_1^{\mathcal{I}} E$ and $\Omega^{\mathcal{I}} E$ do not detect multiplicative units of $\pi_*(E)$ in non-zero degrees because they are build from the spaces $\Omega^m E_m$ which do not carry information about the negative dimensional homotopy groups of E.

The key idea is now to pass to a more elaborate indexing category in order to include information about units in all degrees of $\pi_*(E)$. Let $\mathcal{J}$ be the Quillen localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ. This is a symmetric monoidal category whose objects are pairs of finite sets $(\mathbf{m_1}, \mathbf{m_2})$. Its classifying space has the homotopy type of QS^0. As for $\mathcal{I}$, we obtain a symmetric monoidal category of space valued functors $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$. We call the commutative monoids in $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$ *commutative $\mathcal{J}$-space monoids* and write $\mathcal{CS}^{\mathcal{J}}$ for the resulting category.

In [3] we develop a homotopy theory for commutative $\mathcal{J}$-space monoids that clarifies their relationship to ordinary E_∞-spaces:

Theorem 2. [3, Theorem 1.7] *The category of commutative $\mathcal{J}$-space monoids admits a model structure such that it is Quillen equivalent to the category of E_∞ spaces over $B\mathcal{J}$.*

Thinking of E_∞ spaces as a homotopical generalization of commutative monoids and of commutative symmetric ring spectra as a generalization of commutative rings, the following analogy explains this statement: An ordinary $\mathbb{Z}$-graded commutative monoid may be defined as a commutative monoid with a map to the additive monoid of the initial commutative ring $\mathbb{Z}$. By the theorem, a commutative $\mathcal{J}$-space monoid is up to homotopy an E_∞ space with a map to the underlying additive E_∞ space $B\mathcal{J} \simeq QS^0$ of the initial commutative ring spectrum $\mathbb{S}$. Therefore we think of commutative $\mathcal{J}$-space monoids as graded E_∞ spaces.

Exploiting a close relationship between $\mathcal{J}$ and the combinatorics of symmetric spectra, a commutative symmetric ring spectrum E has an associated commutative $\mathcal{J}$-space monoid $\Omega^{\mathcal{J}} E$ that is defined by $(\Omega^{\mathcal{J}} E)(\mathbf{m_1}, \mathbf{m_2}) = \Omega^{m_2} E_{m_1}$ on the objects of $\mathcal{J}$. This description indicates that $\Omega^{\mathcal{J}}(E)$ also captures information about negative dimensional homotopy groups of E. In view of the above discussion, $\Omega^{\mathcal{J}} E$ is a model for the graded multiplicative E_∞-space of E.

Grouplike graded E_∞ spaces. A classical theorem in stable homotopy theory states that grouplike E_∞ spaces are equivalent to connective spectra. This has a counterpart for graded E_∞ spaces:

Theorem 3. [2] *There is a chain of Quillen equivalences that induces an equivalence between the homotopy category of grouplike commutative $\mathcal{J}$-space monoids and the homotopy category of connective spectra over the sphere spectrum.*

If E is a commutative symmetric ring spectrum, one can form a grouplike sub commutative $\mathcal{J}$-space monoid $\mathrm{GL}_1^{\mathcal{J}} E$ of *graded units* in $\Omega^{\mathcal{J}} E$ that corresponds to the inclusion of the graded commutative group of units $\pi_*(E)^\times$ into the underlying graded multiplicative monoid of $\pi_*(E)$. The augmented Γ-space of units $\mathrm{gl}_1^{\mathcal{J}} E$ discussed above is constructed from $\mathrm{GL}_1^{\mathcal{J}} E$, and the theorem makes clear why a spectrum associated with $\mathrm{GL}_1^{\mathcal{J}} E$ should be a connective spectrum over the sphere spectrum.

The last theorem is also a key ingredient for

Theorem 4. [2] *The functor* $\mathrm{Ho}(\mathcal{C}\mathrm{Sp}^{\Sigma}) \to \mathrm{Ho}(\Gamma^{\mathrm{op}}\text{-}\mathcal{S}/b\mathcal{J})$ *induced by* $\mathrm{gl}_1^{\mathcal{J}}$ *is a right adjoint.*

A similar statement about the ordinary units is proven by Ando, Blumberg, Gepner, Hopkins, and Rezk [1, Theorem 3.2].

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An algebraic model for free rational equivariant stable homotopy theories

Brooke Shipley

(joint work with John Greenlees)

In previous work we gave a small and concrete model of free rational G-spectra when G is a connected compact Lie group [1]. This talk discussed the extension to general compact Lie groups, and used a new method of proof (following [2]).

Assume G is a compact Lie group with identity component N and component group $W = G/N$. Here W acts on the polynomial ring $H^*(BN)$ by ring isomorphisms. We write $H^*(\tilde{B}N)$ to make the W action explicit. We then consider the twisted group ring $H^*(\tilde{B}N)[W]$. The algebraic model for free rational G-spectra is given by differential graded torsion modules over $H^*(\tilde{B}N)[W]$. Here, a module is torsion if it is torsion as a module over the polynomial ring $H^*(\tilde{B}N)$.

Theorem 1. [3] *For any compact Lie group G, with identity component N and component group $W = G/N$, there is a Quillen equivalence*

$$\textit{free-}G\textit{-spectra}/\mathbb{Q} \simeq \textit{tors-}H^*(\tilde{B}N)[W]\textit{-mod}$$

of model categories. In particular their derived categories are equivalent

$$Ho(\textit{free-}G\textit{-spectra}/\mathbb{Q}) \simeq D(\textit{tors-}H^*(\tilde{B}N)[W]\textit{-mod})$$

as triangulated categories.

Note that this algebraic model does not use the full information given by the extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow W \longrightarrow 1.$$

For example, the relevant twisted group ring for both $O(2)$ and $Pin(2)$ is the twisted polynomial ring $\mathbb{Q}[c][W]$ where W is the group of order 2 whose nontrivial element negates c. In fact, this shows that free rational $O(2)$-spectra and free rational $Pin(2)$-spectra are Quillen equivalent homotopy theories. The 2 to 1 map $Pin(2) \longrightarrow O(2)$ in fact induces this rational equivalence.

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Group actions on sets, at a prime p

Jesper Grodal

In this talk I presented an identification of the Grothendieck group of maps between p–completed classifying spaces $Gr([BG\hat{_p}, \coprod_n B\Sigma_n\hat{_p}])$, for G an arbitrary finite group, as the Burnside ring $A(\mathcal{F}_p(G))$ of the p–fusion system $\mathcal{F}_p(G)$ of G.

Let S be a Sylow p–subgroup of G. A finite S–set X is said to be G–stable if for all subgroups $Q \leq S$ and all $g \in G$ such that $gQg^{-1} \leq S$, the Q–set obtained by restricting the S–action on X to Q is isomorphic, as a Q–set, to the Q–set obtained by restricting the S-action to gQg^{-1}, and then viewing X as a Q–set via the conjugation map c_g. The Burnside ring $A(\mathcal{F}_p(G))$ is defined as the Grothendieck group of the monoid of G–stable finite S–sets, under disjoint union — this is easily seen only to depend on the p–fusion system $\mathcal{F}_p(G)$ of G, explaining the notation.

Our main theorem can now be stated as follows:

Theorem 1. *Let G be a finite group with Sylow p-subgroup S, and let $A(\mathcal{F}_p(G))$ be the Burnside ring of G–stable finite S–sets, as defined above. Then*

$$Gr([BG\hat{_p}, \coprod_n (B\Sigma_n)\hat{_p}]) \xrightarrow{\cong} A(\mathcal{F}_p(G))$$

It is easy to see that $A(\mathcal{F}_p(G))$ is a free abelian group of rank the number of conjugacy classes of p-subgroups in G, and it is naturally as a subring of the (ordinary) Burnside ring $A(S)$ of the Sylow p–subgroup S. The algebraic properties of $A(\mathcal{F}_p(G))$ has been studied by several authors, see e.g., [2]. It was proven in [1] that the map in Theorem 1 had finite kernel and cokernel, by observing that the relevant obstructions took values in finite groups; we here show that the obstruction groups in fact vanish.

Applying Theorem 1 for the different primes p dividing the order of G, one obtains an 'integral' version, where the p–completion is replaced by the Quillen plus construction $(-)^+$.

Theorem 2. *The following diagram is a pull-back of rings*

$$\begin{array}{ccc} Gr([BG, \coprod_n (B\Sigma_n)^+]) & \longrightarrow & \prod_{p||G|} A(\mathcal{F}_p(G)) \\ \downarrow{\scriptstyle aug} & & \downarrow{\scriptstyle aug} \\ \mathbb{Z} & \xrightarrow{diag} & \prod_{p||G|} \mathbb{Z} \end{array}$$

Said differently, $Gr([BG, \coprod_n (B\Sigma_n)^+])$ is just the product of the $A(\mathcal{F}_p(G))$, for $p \mid |G|$, in the category of $\mathbb{Z}$–augmented rings.

Theorem 1 can be viewed as an 'uncompleted' version of the Segal conjecture via the following commutative diagram

$$\begin{array}{ccc} Gr([BG\hat{_p}, \coprod_n (B\Sigma_n)\hat{_p}]) & \xrightarrow{\cong} & A(\mathcal{F}_p(G)) \\ \downarrow & & \downarrow{\scriptstyle (-)\hat{_I}} \\ [BG\hat{_p}, Q(S^0)\hat{_p}] & \xrightarrow{\cong} & A(\mathcal{F}_p(G))\hat{_I} \end{array} \tag{3}$$

where I is the augmentation ideal. Here the top isomorphism is Theorem 1 and bottom isomorphism is the Segal conjecture, Ragnarsson style [5, 2].

Our proof follows the general outline of a celebrated result of Jackowski–Oliver [4] on vector bundles over classifying spaces, but requires as new input an 'equivariant stability theorem' for the symmetric groups. The results of this talk will appear in [3].

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Factorization homology

John Francis

We offer an axiomatic description of the factorization homology (a.k.a. topological chiral homology) of topological manifolds, in a sense analogous to (and generalizing) the Eilenberg-Steenrod axioms for usual homology. This point of view provides a new proof of the nonabelian Poincare duality of Salvatore and Lurie, that factorization homology with coefficients in an n-fold loop space is homotopy equivalent to a space of compactly supported maps. The method of proof generalizes to manifolds with boundary, in joint work with David Ayala and Hiro Tanaka, and stratified manifolds.

Tambara functors

Neil Strickland

Tambara functors were introduced to encode the relationship between restriction maps, transfers and norm maps in the homotopy groups of strictly commutative equivariant ring spectra. They can be defined as follows: we introduce a bicategory $\mathcal{U}$ whose 1-truncation is the Lawvere theory for commutative semirings, then form the evident analogous bicategory $\mathcal{U}_G$ of finite G-sets (as 0-cells), bispan diagrams of finite G-sets (as 1-cells) and equivariant isomorphisms (as 2-cells). We then truncate to form an ordinary category $\overline{\mathcal{U}}_G$, and define Tambara functors to be product-preserving functors from $\overline{\mathcal{U}}_G$ to sets. There is a rich algebraic theory of Tambara functors and their relationships with semirings, Mackey functors, Green rings, functors on orbit categories and equivariant spectra. There is also a globally equivariant theory defined using bispan diagrams of groupoids, which relates to coherent systems of G-equivariant ring spectra defined for all groups G simultaneously. In this context we obtain an elegant description of the generalized Witt rings of Dress and Siebeneicher as components of the left adjoint to an evaluation functor.

Homotopy-theoretic approaches to (∞, n)-categories

Julie Bergner
(joint work with Charles Rezk)

The goal of this talk is to look at some of the details behind the machinery of $(\infty, 1)$-categories and more general (∞, n)-categories currently being used in a number of applications throughout homotopy theory and other areas of mathematics.

We begin with the basic idea of higher categories. An n-category should have objects, together with 1-morphisms between objects, 2-morphisms between 1-morphisms, and so forth up to n-morphisms between $(n-1)$-morphisms. Continuing for all n, we can also obtain the notion of ∞-category. If units and associativity at all levels are defined strictly, there is no problem defining these structures rigorously; an n-category is defined to be a category enriched in $(n-1)$-categories.

However, most examples that one finds within mathematics are not defined so strictly. In homotopy-theoretic examples, often associativity only holds up to homotopy, or more generally up to some kind of equivalence. While there are many definitions of weak n-categories, comparing these definitions has proved to be a very difficult task.

On the other hand, so-called $(\infty, 1)$-categories, or weak ∞-categories with k-morphisms weakly invertible for $k > 1$ have proved to be much more tractable, in that a number of approaches to them have been shown to be equivalent to one another. The comparisons between more general (∞, n)-categories, where k-morphisms are invertible for $k > n$, are still being developed and are the subject of this talk.

To understand the development of (∞, n)-categories, we begin with the simplest example, that of $(\infty, 0)$-categories, or weak ∞-groupoids. It is often taken as a definition that $(\infty, 0)$-categories are just topological spaces or simplicial sets. To see that this definition is sensible, think of the points of a topological space as objects, paths between points (which are invertible up to homotopy) as 1-morphisms, homotopies between paths as 2-morphisms, homotopies between homotopies as 3-morphisms, and so forth. Furthermore, the homotopy theory of $(\infty, 0)$-categories is well-developed, via the classical model structures on the categories of topological spaces and simplicial sets.

Moving up a categorical level, we can use the idea that an $(\infty, 1)$-category should be a category enriched in $(\infty, 0)$-categories. In other words we can consider topological or simplicial categories as models for $(\infty, 1)$-categories. Since there is a model structure on the category of small simplicial categories, we can still do homotopy theory here [1].

However, if we try to continue this process of enrichment, we get less manageable models. First, the iterated enrichments are still going to be too rigid for many examples. Second, the model structure on the category of simplicial categories is not cartesian, so it is not expected that one can obtain a suitable model structure for categories enriched in them. Therefore, we would like to find another model for $(\infty, 1)$-categories which has a cartesian model structure so that we have a good homotopy theory of $(\infty, 2)$-categories.

There are in fact several models for $(\infty, 1)$-categories, each of which weakens the definition of a simplicial category in some way. Quasi-categories, Segal categories, and complete Segal spaces all have respective model structures which have been shown to be Quillen equivalent to one another, thus establishing them as alternative approaches to $(\infty, 1)$-categories [2], [3], [4], [5], [6]. In this talk we focus on complete Segal spaces, which were developed by Rezk [9].

These objects are simplicial objects in the category of simplicial sets, satisfying a Segal condition and a completeness condition. The Segal condition gives an up-to-homotopy-composition, while the completeness condition compensates for the fact that the objects of a simplicial category form a discrete space, where as the 0-space of a complete Segal space may not. One way to think of the completeness condition is that it requires the 0-space to be a moduli space for equivalences in the 1-space. There is a model structure on the category of simplicial spaces such that the complete Segal spaces are the fibrant objects. Furthermore, this model structure is cartesian, and there is in fact a model structure on the category of categories enriched over it, giving a model for $(\infty, 2)$-categories.

However, complete Segal spaces have only solved our problem at one level, as we still have difficulty if we want to continue enriching to obtain (∞, n)-categories for still larger n. What we really need are higher analogues of complete Segal spaces. There are multiple ways to define such objects, and here we look at Rezk's Θ_n-spaces [8].

The categories Θ_n are defined iteratively so that $\Theta_1 = \Delta$, and the objects of Θ_n look like simple models for strict n-categories, just as the objects of Δ can be thought of a simple models for ordinary categories. Considering functors $\Theta_n^{op} \to s\mathcal{S}ets$, we can require Segal and completeness conditions, but now at multiple levels, and call the resulting objects Θ_n-spaces. These objects are the fibrant objects in a cartesian model structure on the category of all functors $\Theta_n^{op} \to s\mathcal{S}ets$.

While it seems that Θ_n-spaces give a good model for (∞, n)-categories, it would be ideal to show that their model structure is Quillen equivalent to one on the category of small categories enriched in $(\infty, n-1)$-categories, since enrichment is a desired approach to higher categories. Finding a chain of Quillen equivalences between these model structures is the subject of work in progress. One consequence that will follow from this result will be a chain of Quillen equivalences between the model structure for Θ_n-spaces and the one for the multi-simplicial model of Barwick and Lurie, as used in [7].

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Long knots and maps of operads

Kathryn Hess

(joint work with William G. Dwyer)

1. Operads with multiplication

Let $(\mathbf{M}, \otimes, I)$ be a monoidal category, and let $\mathbf{Op}$ denote the category of non-symmetric operads in $\mathbf{M}$. Let $\mathscr{A}$ and $\mathscr{D}_2$ denote the associative operad and the little 2-disks operad, respectively.

In [3] McClure and Smith constructed a functor from the category of operads under $\mathscr{A}$ to the category of cosimplicial objects in $\mathbf{M}$,

$$\mathscr{A}\downarrow\mathbf{Op}\to\mathbf{M}^{\mathbf{\Delta}} : (\mathscr{A}\xrightarrow{\omega}\mathscr{O})\mapsto\mathscr{O}^{\bullet}_{\omega},$$

where the cofaces and codegeneracies of $\mathscr{O}^{\bullet}_{\omega}$ are defined in terms of ω and the multiplication of operad. They then proved that if $\mathbf{M}$ is the category of simplicial sets, then $\operatorname{holim}_{\mathbf{\Delta}}\mathscr{O}^{\bullet}_{\omega}$ is a $\mathscr{D}_2$-space and therefore a double loop space, if it is connected.

Our main contribution is a determination of the homotopy type of a double delooping of $\operatorname{holim}_{\mathbf{\Delta}}\mathscr{O}^{\bullet}_{\omega}$, in terms of a derived mapping space of operad maps.

Theorem 1. [2] *If $\mathscr{O}$ is a simplicial operad such that $\mathscr{O}(0)\sim *\sim\mathscr{O}(1)$, then for every operad morphism $\omega:\mathscr{A}\to\mathscr{O}$, there is a natural weak homotopy equivalence*

$$\operatorname{holim}_{\mathbf{\Delta}}\mathscr{O}^{\bullet}_{\omega}\sim\Omega^2\operatorname{Map}_{\mathbf{Op}}(\mathscr{A},\mathscr{O})_{\omega},$$

where $\operatorname{Map}_{\mathbf{Op}}(\mathscr{A},\mathscr{O})_{\omega}$ denotes the derived mapping space of operad maps from $\mathscr{A}$ to $\mathscr{O}$, based at ω.

Let $\mathscr{K}_m$ denote the m^{th}-Kontsevich operad. Sinha proved in [5] that for all $m\geq 4$, there is an operad morphism $\omega_m:\mathscr{A}\to\mathscr{K}_m$ such that $\operatorname{holim}_{\mathbf{\Delta}}(\mathscr{K}_m)^{\bullet}_{\omega_m}$ has the homotopy type of the space L_m of tangentially straightened long knots in $\mathbb{R}^m$. Our theorem thus implies that $\Omega^2\operatorname{Map}_{\mathbf{Op}}(\mathscr{A},\mathscr{K}_m)_{\omega_m}\sim L_m$, providing an intriguing new description of the space of long knots.

2. The fiber sequence theorem

To prove Theorem 1, we apply the following result twice.

Theorem 2. [2] *Let $(\mathbf{M},\otimes,I)$ be a monoidal category with an "appropriately compatible" model category structure. For every "nice" monoid morphism $\omega:R\to S$ in $\mathbf{M}$, there is a fibration sequence*

$$\Omega\operatorname{Map}_{\mathbf{Mon}}(R,S)_{\omega}\to\operatorname{Map}_{R,R}(R,S_{\omega})\xrightarrow{\eta_R^*}\operatorname{Map}_{\mathbf{M}}(I,S),$$

where $\operatorname{Map}_{\mathbf{Mon}}$, $\operatorname{Map}_{R,R}$ and $\operatorname{Map}_{\mathbf{M}}$ denote the derived mapping spaces of monoid maps, (R,R)-bimodule maps and maps in $\mathbf{M}$, respectively. Moreover, S_{ω} denotes S endowed with R-bimodule structure induced by ω, and the fibre is taken over the unit map $\eta_S:I\to S$ of the monoid S.

In the next section we give some indication of the meaning of "appropriate compatibility" and of "niceness", when we explain the first step in the proof of Theorem 2. In both of the applications below, the hard work consists in verifying these conditions in each particular context.

Applying Theorem 2 to the category $\mathbf{sSet}^{\mathbb{N}}$ of sequences of simplicial sets, endowed with the composition monoidal product $\circ$ and unit object $\mathscr{J}=(\emptyset,*,\emptyset,...)$, we obtain the following corollary.

Corollary 3. *For any map of simplicial operads $\omega : \mathscr{A} \to \mathscr{O}$, there is a fibration sequence*

$$\Omega \operatorname{Map}_{\mathbf{Op}}(\mathscr{A}, \mathscr{O})_\omega \to \operatorname{Map}_{\mathscr{A},\mathscr{A}}(\mathscr{A}, \mathscr{O}_\omega) \to \operatorname{Map}_{\mathbf{sSet}^{\mathbb{N}}}(\mathscr{J}, \mathscr{O}).$$

In particular, if $\mathscr{O}(1) \sim *$, then

$$\Omega \operatorname{Map}_{\mathbf{Op}}(\mathscr{A}, \mathscr{O})_\omega \sim \operatorname{Map}_{\mathscr{A},\mathscr{A}}(\mathscr{A}, \mathscr{O}_\omega).$$

The category of $(\mathscr{A}, \mathscr{A})$-bimodules in $\mathbf{sSet}^{\mathbb{N}}$ can be identified with the category of graded monoids in the category of right $\mathscr{A}$-modules. To analyze the middle term of the fibration sequence in Corollary 3 , we can therefore apply Theorem 2 to the category $\mathbf{Mod}_{\mathscr{A}}$ endowed with the graded monoidal product $\odot$ and unit object $\mathscr{I} = (*, \emptyset, \emptyset ...)$.

Corollary 4. *Let $\mathscr{A}_\odot$ denote the symmetric sequence $\mathscr{A}$, considered as an $(\mathscr{A}, \mathscr{A})$-bimodule (or, equivalently, as a graded monoid in $\mathbf{Mod}_{\mathscr{A}}$). For any morphism $\varphi : \mathscr{A}_\odot \to \mathscr{X}$ of $(\mathscr{A}, \mathscr{A})$-bimodules, there is a fibration sequence*

$$\Omega \operatorname{Map}_{\mathscr{A},\mathscr{A}}(\mathscr{A}_\odot, \mathscr{X})_\varphi \to \operatorname{Map}_{\mathscr{A}_\odot,\mathscr{A}_\odot}(\mathscr{A}_\odot, \mathscr{X}_\varphi) \to \operatorname{Map}_{\mathbf{Mod}_{\mathscr{A}}}(\mathscr{I}, \mathscr{X}).$$

In particular, if $\mathscr{X}(0) \sim *$, then

$$\Omega \operatorname{Map}_{\mathscr{A},\mathscr{A}}(\mathscr{A}, \mathscr{X})_\varphi \sim \operatorname{Map}_{\mathscr{A}_\odot,\mathscr{A}_\odot}(\mathscr{A}, \mathscr{X}_\varphi).$$

To complete the proof of Theorem 1, we observe that Corollaries 3 and 4 together imply that if $\mathscr{O}$ is a simplicial operad such that $\mathscr{O}(0) \sim * \sim \mathscr{O}(1)$, then for any morphism of simplicial operads $\omega : \mathscr{A} \to \mathscr{O}$

$$\Omega^2 \operatorname{Map}_{\mathbf{Op}}(\mathscr{A}, \mathscr{O})_\omega \sim \operatorname{Map}_{\mathscr{A}_\odot,\mathscr{A}_\odot}(\mathscr{A}_\odot, \mathscr{O}_\omega).$$

Using the two-sided simplicial bar resolution of $\mathscr{A}_\odot$ as a graded bimodule over itself and applying mapping space level adjunctions, we obtain that

$$\operatorname{Map}_{\mathscr{A}_\odot,\mathscr{A}_\odot}(\mathscr{A}_\odot, \mathscr{O}_\omega) \sim \operatorname{holim}_{\mathbf{\Delta}} \mathscr{O}_\omega^\bullet$$

and thus conclude the proof.

3. From monoid maps to bimodule maps

The key to proving Theorem 2 is a particularly simple description of the derived mapping space, which follows from work of Dugger [1] and Rezk [4]. Let N denote the nerve functor.

Lemma 5. *Let $\mathbf{M}$ be a left proper model category. Let $X, Y \in \operatorname{Ob} \mathbf{M}$, and let $X^c \xrightarrow{\sim} X$ be a cofibrant replacement. If the induced map $X^c \coprod Y \to X \coprod Y$ is a weak equivalence, then*

$$\operatorname{Map}_{\mathbf{M}}(X, Y) \sim \mathsf{N}(\mathbf{M}^d_{X,Y}),$$

where $\mathbf{M}^d_{X,Y}$ denotes the full subcategory of $(X \coprod Y) \downarrow \mathbf{M}$ of distinguished objects, i.e., morphisms $X \coprod Y \to Z$ such that the composite $Y \to X \coprod Y \to Z$ is a weak equivalence.

Lemma 5 enables us to relate spaces of monoid maps to spaces of bimodule maps, as specified in the next result.

Theorem 6. *Let* $(\mathbf{M}, \otimes, I)$ *be a monoidal category that is also endowed with a model category structure. Let* R *and* S *be monoids in* $\mathbf{M}$*.*

If

(1) *all categories of monoids, right modules and bimodules in* $\mathbf{M}$ *inherit model structures from* $\mathbf{M}$ *in which a morphism is a weak equivalence (respectively, fibration) if the underlying morphism in* $\mathbf{M}$ *is;*

(2) *the forgetful functor* $(R \coprod S) \downarrow \mathbf{Mon} \to (R \otimes S) \downarrow {}_R\mathbf{Mod}_S$ *admits a left adjoint, denoted* E*;*

(3) *the functor* E *restricts to a functor from the category* $\mathbf{Mod}_{R,S}^{d,cof}$ *of distinguished, cofibrant* (R, S)*-bimodules under* $R \otimes S$ *to the category* $\mathbf{Mon}_{R,S}^{d}$ *of distinguished monoids under* $R \coprod S$*; and*

(4) (a) R *is cofibrant in* $\mathbf{Mon}$*, or*
(b) $\mathbf{Mon}$ *is left proper, and* $R^c \coprod S \xrightarrow{\sim} R \coprod S$*;*

then

$$\mathrm{Map}_{\mathbf{Mon}}(R, S) \sim \mathsf{N}(\mathbf{Mod}_{R,S}^{d}).$$

Hypotheses (1)-(4) above form the core of the definition of "appropriately compatible" model structure and of "nice" monoid morphisms. To complete the proof of Theorem 2, we need two further hypotheses, one of which is the analogue of (4) for bimodules, while the other says that cofibrant replacements of the unit I and of the monoid R must act as units, up to homotopy, in $\mathbf{M}$ and in the category of left R-modules, respectively.

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The Brown-Comenetz dual of the $K(2)$-local sphere at the prime 3

Hans-Werner Henn
(joint work with Paul Goerss)

1. Background

1.1. **Brown-Comenetz duality on spectra.** By Brown representability there is a spectrum I such that

$$\mathrm{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z}) \cong [X, I]\ .$$

If we define $IX = F(X, I)$ (in particular $IS^0 = F(S^0, I) = I$) we get a duality on the category of spectra. In particular $\pi_k(I) \cong \mathrm{Hom}(\pi_{-k}S^0, \mathbb{Q}/\mathbb{Z})$.

1.2. **Chromatic tower and chromatic square.** For finite p-local spectra X there is a convergent tower of Bousfield localizations

$$\ldots \to L_n X = L_{E(n)}X \to L_{n-1}X \to \ldots$$

where $E(n)$ is the n-th Johnson-Wilson spectrum at p. Moreover, for every X there is a homotopy pull back square (a "chromatic square")

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array} .$$

where $K(n)$ demotes the n-th Morava K-theory at p. The (commun) vertical fibres are denoted $M_n X$.

1.3. **Brown-Comenetz duality in the $K(n)$-local category.** Define I_n via

$$\mathrm{Hom}(\pi_0(M_n X), \mathbb{Q}/\mathbb{Z}) \cong [X, I_n]$$

If we define $I_n X = F(M_n X, I) = IM_n X$ (in particular $I_n S^0 = F(M_n S^0, I) \simeq F(S^0, I_n) = I_n$) we get a duality on the category $\mathcal{K}_n$ of $K(n)$-local spectra, in particular $I_n X$ and I_n are $K(n)$-local. Furthermore we have

$$\pi_k(I_n) \cong \mathrm{Hom}(\pi_{-k}(M_n S^0), \mathbb{Q}/\mathbb{Z}) .$$

2. The $K(n)$-local Picard group and the Hopkins-Gross formula

2.1. **Morava modules.** Let E_n be the n-th Lubin Tate spectrum at p. This is a complex oriented periodic spectrum whose π_0 classifies deformations of the p-typical formal group law F_n of height n over $\mathbb{F}_{p^n}$ whose p-series is given by x^{p^n}. If $\mathbb{W}_{\mathbb{F}_{p^n}}$ denotes the ring of Witt vectors of $\mathbb{F}_{p^n}$ there is a non-canonical isomorphism

$$(E_n)_* \cong \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1, \ldots, u_{-1}]][u^{\pm 1}] .$$

This is acted on by the automorphism group $Aut(F_n, \mathbb{F}_{p^n}) =: \mathbb{G}_n$. Furthermore

$$\mathbb{G}_n \cong \mathbb{S}_n \rtimes Gal(\mathbb{F}_{p^n} : \mathbb{F}_p)$$

where $\mathbb{S}_n$ is the usual Morava stabilizer group which can be identified with the group of units in the maximal order of the division algebra over $\mathbb{Q}_n$ of dimension n^2 and Hasse invariant $1/n$.

The Morava module of X is given by $(E_n)_*X = \pi_*(L_{K(n)}(E_n \wedge X))$. Under suitable assumptions on X (which will be satisfied in the sequel) this is a continuous twisted $(E_n)_*[[\mathbb{G}_n]]$-module.

2.2. **Picard groups.** The categories $\mathcal{K}_n$ resp. $\mathcal{EG}_n$ of $K(n)$-local spectra resp. continuous twisted $(E_n)_*[[\mathbb{G}_n]]$-modules are symmetric monoidal with respect to

$$X \wedge_{\mathcal{K}(n)} Y := L_{K(n)}(X \wedge Y) \text{ resp. } M \otimes_{E_{n*}} N \ .$$

The invertible objects with respect to the monoidal structure form a group $Pic_n := Pic_{\mathcal{K}(n)}$ resp. $(Pic_n)_{alg} := Pic_{\mathcal{EG}_n}$ and the functor $X \mapsto (E_n)_*X$ induces a homomorphism $Pic_n \to (Pic_n)_{alg}$ whose kernel is denoted by κ_n, i.e. we have a short exact sequence

$$0 \to \kappa_n \to Pic_n \to (Pic_n)_{alg} \ .$$

If n is sufficiently large with respect to p then κ_n is known to be trivial.

2.3. **Some elements in Pic_n and $(Pic_n)_{alg}$.** The spectra S^1 and I_n are invertible in $\mathcal{K}_n$. There is another important invertible spectrum denoted $S^0[det]$. It can be obtained as follows.

We note that the group $\mathbb{G}_n$ admits a surjective homomorphism $det : \mathbb{G}_n \to \mathbb{Z}_p^\times$ with kernel $S\mathbb{G}_n$. Let $\mathbb{G}_n^1$ be the kernel of the composition $\mathbb{G}_n \to \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times/\mu$ where μ is the subgroup of roots of unity of $\mathbb{Z}_p^\times$. Let ψ denote the self map of E_n given by the action of a fixed chosen topological generator of $\mathbb{Z}_p^\times/\mu$. Then there is a fibre sequence

$$L_{K(n)}S^0 \to E_n^{h\mathbb{G}_n^1} \stackrel{\psi - id}{\longrightarrow} E_n^{h\mathbb{G}_n^1}$$

which for $n = 1$ and the obvious suitable choice for ψ gives the well known fibration (because of $E_1 = K\mathbb{Z}_p$)

$$L_{K(1)}S^0 \to (K\mathbb{Z}_p)^{h\mu} \stackrel{\psi^{p+1} - id}{\longrightarrow} (K\mathbb{Z}_p)^{h\mu} \ .$$

The spectrum $S^0[det]$ is given as the fibre in the following exact triangle

$$S^0[det] \to (E_n^{hS\mathbb{G}_n})_\chi \stackrel{\psi - det(\psi).id}{\longrightarrow} (E_n^{hS\mathbb{G}_n})_\chi$$

where $(E_n^{hS\mathbb{G}_n})_\chi$ is the eigenspectrum of $E_n^{h\mathbb{G}_n^1}$ with respect to the action of μ via its fundamental character $\chi : \mu \to \mathbb{Z}_p^\times$. Then we have

$$(E_n)_*(S^0[det]) \cong (E_n)_*(S^0)[det]$$

where det on the right hand side means twisting the action by the determinant.

2.4. **The Hopkins Gross formula.** The following result which describes I_n modulo κ_n is known as the Hopkins-Gross formula (cf. [3], [5]).

Theorem 1. *There is an isomorphism of continuous twisted $(E_n)_*[[\mathbb{G}_n]]$-modules*

$$(E_n)_*I_n \cong \Sigma^{n^2-n}(E_n)_*[det] \ .$$

3. The case $n = 2$ and $p = 3$

The case $n = 1$ is well understood [4]. For $n = 2$ it is known that $\kappa_2 = 0$ if $p > 3$. For $n = 2$ and $p = 3$ the structure of κ_2 is given by the following result [2].

Theorem 2. *Let $p = 3$. There is an isomorphism*

$$H^5(\mathbb{G}_2, (E_2)_4) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$$

and the d_5-differential in the Adams Novikov spectral sequence induces an isomorphism

$$\varphi : \kappa_2 \to H^5(\mathbb{G}_2, (E_2)_4) \ .$$

The next result characterizes a particularly important element in κ_2 at $p = 3$. For this we recall that $V(1)$ denotes the cofibre of the Adams self map of the mod-p Moore spectrum and G_{24} is a certain subgroup of $\mathbb{G}_2$ of order 24 which is unique up to conjugacy and for which the homotopy fixed point spectrum $E_2^{hG_{24}}$ is a form of the $K(2)$-localization of the spectrum of topological modular forms TMF.

Theorem 3. *Let $p = 3$. There is a unique spectrum $P \in \kappa_2$ satisfying*
- $E_2^{hG_{24}} \wedge P \cong \Sigma^{48} E_2^{hG_{24}}$
- $P \wedge V(1) \cong \Sigma^{48} L_{K(2)} V(1)$.

Finally we get the following refinement of the Hopkins-Gross formula for $n = 2$ and $p = 3$.

Theorem 4. *Let $p = 3$. Then there is an equivalence*

$$I_2 \simeq L_{K(2)} S^2 \wedge S^0[det] \wedge P \ .$$

The main tools for proving these results are taken from [1]. In particular we make heavy use of a certain finite length algebraic resolution of the trivial $\mathbb{Z}_3[[\mathbb{G}_2]]$-module $\mathbb{Z}_3$ in terms of permutation modules on cosets of finite subgroups, as well as its "realization" in terms of a "resolution of $L_{K(2)} S^0$ by spectra" whose iterated cofibres are the homotopy fixed point spectra of the action of these finite subgroups on the spectrum E_2.

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Localising subcategories for cochains on BG

David J. Benson
(joint work with Srikanth Iyengar, Henning Krause)

This is a report on joint work with Srikanth Iyengar and Henning Krause in which we classified the localising subcategories of the derived category of cochains on the classifying space of a finite group. The proof is given in [3], and depends heavily on the paper [6]

Let G be a finite group and k a field of characteristic p. We write $\operatorname{Spec} H^*(G,k)$ for the set of homogeneous prime ideals in the cohomology ring $H^*(G,k) = H^*(BG;k)$, including the maximal ideal. The main theorem is as follows.

Theorem 1. *There is a natural one to one correspondence between the localising subcategories of $D(C^*(BG;k))$ and the subsets of the set $\operatorname{Spec} H^*(G,k)$.*

Here, we are regarding the cochains $C^*(BG;k)$ as a commutative S-algebra in the sense of [8], and $D(C^*(BG;k))$ is the derived module category. This is a tensor triangulated category with $X \otimes^{\mathbb{L}}_{C^*(BG;k)} Y$ as the tensor product. A *localising subcategory* of a triangulated category is a triangulated subcategory that is closed under direct sums. Our methods say nothing about the following possible generalisation of the main theorem.

Conjecture 2. *The theorem remains true if we replace finite groups with compact Lie groups (and $H^*(G,k)$ with $H^*(BG;k)$).*

Algebraisation. The main theorem was proved by first proving an algebraic theorem about modular representations of kG. We write $\mathsf{K}(\mathsf{Inj}\, kG)$ for the category whose objects are the chain complexes of injective = projective = flat kG-module and whose arrows are the homotopy classes of degree preserving chain maps. This is a tensor triangulated category in which the tensor product is $X \otimes_k Y$ with diagonal action of G. The following theorem is proved in [7].

Theorem 3. *If G is a p-group then there is an equivalence of tensor triangulated categories $D(C^*(BG;k)) \simeq \mathsf{K}(\mathsf{Inj}\, kG)$.*

More generally, for any finite group $D(C^*(BG;k))$ is equivalent to the localising subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by the tensor identity ik, namely the injective resolution of the field k. We write $\operatorname{Loc}(ik)$ for this.

What does $\mathsf{K}(\mathsf{Inj}()kG)$ look like? As described in [7], there is a recollement of triangulated categories

$$\mathsf{StMod}(kG) \simeq \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG) \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathsf{K}(\mathsf{Inj}\, kG) \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; D(kG)$$

where $\mathsf{StMod}(kG)$ is the stable category of kG-modules, which is equivalent via Tate resolutions to the homotopy category $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, kG)$ of acyclic complexes of injective modules. The compact objects are only preserved by the left adjoints in the recollement, and give the more familiar sequence

$$\mathsf{stmod}(kG) \leftarrow D^b(kG) \leftarrow \mathsf{Perf}(kG)$$

where $\mathsf{Perf}(kG)$ is the category of perfect complexes in the bounded derived category $D^b(kG)$.

Stratification. The setup is as follows. Let T be a compactly generated triangulated category with coproducts. We write Σ for the shift in T. We denote by $Z^*(\mathsf{T})$ the centre of T, namely the graded ring whose degree n elements are the natural transformations from the identity to Σ^n which commute with Σ up to the appropriate sign. This is a graded commutative ring, but may be unmanageably large.

If R is a graded commutative Noetherian ring together with a homomorphism $R \to Z^*(\mathsf{T})$, we say that R *acts* on T.

For each $\mathfrak{p} \in \operatorname{Spec} R$ we have a *local cohomology functor* $\Gamma_\mathfrak{p} \colon \mathsf{T} \to \mathsf{T}$ picking out the layer of T corresponding to $\mathfrak{p}$. It is a composite of a Bousfield localisation picking out the primes containing $\mathfrak{p}$ and another deleting the primes properly containing $\mathfrak{p}$. Then for X an object in T we define

$$\operatorname{supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_\mathfrak{p} X \ncong 0\}.$$

We say that the *local-global principle* holds for the action of R on T if for all X in T we have

$$\operatorname{Loc}_\mathsf{T}(X) = \operatorname{Loc}_\mathsf{T}\{\Gamma_\mathfrak{p} X \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

(1) If R has finite Krull dimension then the local-global principle automatically holds.
(2) In a tensor triangulated category we always have the corresponding tensor ideal statement. Namely, define $\operatorname{Loc}^\otimes_\mathsf{T}(X)$ to be the smallest tensor ideal (i.e., closed under tensor product with an arbitrary object in T or equivalently with a set of compact generators) localising subcategory containing X. It is easy to check that this is the same as $\operatorname{Loc}_\mathsf{T}(\{X \otimes Y \mid Y \in \mathsf{T}\})$. If the tensor identity $\mathbb{1}$ generates T then we have $\operatorname{Loc}^\otimes_\mathsf{T}(X) = \operatorname{Loc}_\mathsf{T}(X)$. Then the tensor ideal version that always holds is the statement that

$$\operatorname{Loc}^\otimes_\mathsf{T}(X) = \operatorname{Loc}^\otimes_\mathsf{T}\{\Gamma_\mathfrak{p} X \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Definition 4. If the local-global principle holds and each $\Gamma_\mathfrak{p}\mathsf{T}$ is either zero or a minimal localising subcategory, we say that T is *stratified* by the action of R.

Theorem 5. [4]. *If T is stratified by the action of R then there is a one to one correspondence between the localising subcategories of T and the subsets of the set*

$$\operatorname{supp}_R(\mathsf{T}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \Gamma_\mathfrak{p}\mathsf{T} \neq 0\}.$$

The map giving this correspondence is given by $\mathcal{C} \mapsto \{\mathfrak{p} \mid \Gamma_\mathfrak{p}\mathcal{C} \neq 0\}$. Its inverse sends a subset S to the full subcategory consisting of objects X such that $\operatorname{supp}_R(X) \subseteq S$.

In this language, Theorem 1 may be restated to say that the triangulated category $D(C^*(BG;k))$ is stratified by the action of $H^*(G,k)$. This is deduced from the following theorem, proved in [6].

Theorem 6. $\mathsf{K}(\mathsf{Inj}\, kG)$ *is tensor stratified by $H^*(G,k)$.*

The meaning of this is that each $\Gamma_{\mathfrak{p}}\mathsf{K}(\mathsf{Inj}\, kG)$ is a minimal *tensor ideal* localising subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$. This implies that the maps described in Theorem 5 give a one to one correspondence between the tensor ideal localising subcategories of $\mathsf{K}(\mathsf{Inj}\, kG)$ and the subsets of $\operatorname{Spec} H^*(G,k)$.

This theorem was proved by means of a long series of reductions until methods of Mike Hopkins and Amnon Neeman finished the problem.

If G is a finite p-group then Theorems 3 and 6 immediately imply Theorem 1, and we are done. For a more general finite group we need the following.

Theorem 7. *Suppose that R has finite Krull dimension and that T is a tensor stratified by R. Then $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is stratified by R. Thus there is a one to one correspondence between the tensor ideal localising subcategories of T and localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. The map giving this correspondence sends $\mathcal{C}$ to $\mathcal{C} \cap \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, and its inverse sends $\mathcal{D}$ to $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathcal{D})$.*

Note that without stratification, these maps need not give a one to one correspondence. The map sending $\mathcal{C}$ to $\mathcal{C}\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is always surjective, but need not be injective. For example if $\mathsf{T} = D(\mathsf{QCoh}(\mathbb{P}^1))$, the derived category of quasi-coherent sheaves on the projective line, then the tensor identity is $\mathcal{O}$. In this case $\operatorname{Loc}_{\mathsf{T}}(\mathcal{O})$ has no proper localising subcategories, while there are many tensor ideal localising subcategories of T.

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The mod 2 homology of infinite loopspaces

Nicholas J. Kuhn

(joint work with Jason B. McCarty)

This is a report on work in [6]. An infinite loopspace is a space of the form $\Omega^\infty X$, the 0th space of a spectrum X. All homology is with mod 2 coefficients.

Problem How can one compute $H_*(\Omega^\infty X)$ from knowledge of $H_*(X)$?

The graded vector space $H_*(X)$ has a minimum of extra structure: it is a locally finite right module over the mod 2 Steenrod algebra $\mathscr{A}$. By contrast, the structure of $H_*(\Omega^\infty X)$ is much richer: it is a restricted Hopf algebra in the abelian category of left modules over the Dyer–Lashof algebra with compatible unstable right $\mathscr{A}$–module structure.

We study the problem via the spectral sequence obtained by applying mod 2 homology to the Goodwillie tower of the functor sending X to $\Sigma^\infty\Omega^\infty X$. Goodwillie's general theory [4] implies that this left half plane spectral sequence strongly converges to $H_*(\Omega^\infty X)$ if X is 0–connected, and has $E^1_{-d,d+*}(X) = H_*(D_dX)$, where $D_dX = X^{\wedge d}_{h\Sigma_d}$. We use the further geometric structure on the tower that was explored in [1] using a model for the tower derived from Arone's model for the suspension spectra of function spaces [2].

1. Lots of categories and a description of E^1

We introduce various algebraic categories.

- $\mathcal{M}$ is the category of locally finite right $\mathscr{A}$–modules. The Steenrod squares go down in degree: given $x \in M \in \mathcal{M}$, $|xSq^i| = |x| - i$.
- $\mathcal{U}$ is the full subcategory of $\mathcal{M}$ consisting of modules satisfying the unstable condition: $xSq^i = 0$ whenever $2i > |x|$.
- $\mathcal{Q}$ is the category of graded vector spaces M acted on by Dyer–Lashof operations $Q^i : M_n \to M_{n+i}$, for $i \in \mathbb{Z}$, satisfying the Adem relations and the unstable relation: $Q^ix = 0$ whenever $i < |x|$.
- $\mathcal{QM}$ is the full subcategory of $\mathcal{M} \cap \mathcal{Q}$ consisting of objects whose Dyer–Lashof structure is intertwined with the Steenrod structure via the Nishida relations. $\mathcal{QU} = \mathcal{QM} \cap \mathcal{U}$.

All these categories admit tensor products, via the Cartan formula for both Steenrod and Dyer–Lashof operations. We define various categories of Hopf algebras.

- $\mathcal{HM}$ is the category of bicommutative Hopf algebras in $\mathcal{M}$.
- $\mathcal{HQM}$ is the category of bicommutative Hopf algebras in $\mathcal{QM}$ satisfying the Dyer–Lashof restriction axiom: $Q^{|x|}x = x^2$. $\mathcal{HQU} = \mathcal{HQM} \cap \mathcal{U}$.

We also need two 'free' functors.

- $\mathcal{R}_* : \mathcal{M} \to \mathcal{QM}$ is left adjoint to the forgetful functor. Explicitly, $\mathcal{R}_*M = \bigoplus_{s=0}^\infty \mathcal{R}_sM$ where $\mathcal{R}_s : \mathcal{M} \to \mathcal{M}$ is given by

  $$\mathcal{R}_sM = \langle Q^Ix \mid l(I) = s, x \in M\rangle/(\text{unstable and Adem relations}).$$

  Here, if $I = (i_1, \ldots, i_s)$, $Q^Ix = Q^{i_1}\cdots Q^{i_s}x$, and $l(I) = s$.
- $U_{\mathcal{Q}} : \mathcal{QM} \to \mathcal{HQM}$ is left adjoint to the functor taking an object $H \in \mathcal{HQM}$ to its module PH of primitives. Explicitly,

  $$U_{\mathcal{Q}}(M) = S^*(M)/(Q^{|x|}x - x^2).$$

We begin our study of the spectral sequence knowing the following.

- $H_*(\Omega^\infty X)$ is an object in $\mathcal{HQU}$.

- $E^1_{*,*}(X) = U_{\mathcal{Q}}(\mathcal{R}_*(H_*(X)))$ as an object in $\mathcal{HQM}$. Here, if $x \in H_*(X)$ and $I = (i_1, \ldots, i_s)$, then $Q^I x$ has bidegree $(-2^s, 2^s + |x| + |I|)$, where $|I| = i_1 + \cdots + i_s$. Steenrod operations act vertically, while Dyer–Lashof operations double the horizontal degree.
- Each $E^r_{*,*}(X)$ is an object in $\mathcal{HM}$, and each d^r is $\mathscr{A}$–linear and both a derivation and coderivation.

2. Universal differentials

Our first theorem identifies universal structure. Parts (b) and (c) are proved by using properties of the $\mathbb{Z}/2$–Tate construction combined with an action of the little cubes operad $\mathcal{C}_\infty$ on our tower.

Theorem 1. *For all spectra X, the following hold in $\{E^r_{*,*}(X)\}$.*

(a) *For all $x \in H_*(X)$, $d^1(x) = \sum_{i \geq 0} Q^{i-1}(xSq^i)$.*

(b) *If $y \in H_*(D_d X)$ lives to E^r, and $d^r(y)$ is represented by $z \in H_*(D_{d+r}X)$, then $Q^i y \in H_*(D_{2d}X)$ lives to E^{2r}, and $d^{2r}(Q^i y)$ is represented by $Q^i(z) \in H_*(D_{2d+2r}X)$.*

(c) *$y \in H_*(D_d X)$ represents $z \in H_*(\Omega^\infty X)$ in $E^\infty_{-d,*}(X)$, then $Q^i y \in H_*(D_{2d}X)$ represents $Q^i z \in H_*(\Omega^\infty X)$ in $E^\infty_{-2d,*}(X)$.*

Corollary 2. *$E^\infty_{*,*}(X) \in \mathcal{HQM}$ with structure induced from $E^1_{*,*}(X)$, and compatible with the structure on $H_*(\Omega^\infty X)$.*

Corollary 3. *For all spectra X, $x \in H_*(X)$, and I of length s, $Q^I x$ lives to $E^{2^s}_{-2^s,*}(X)$ and $d^{2^s}(Q^I x) = \sum_{i \geq 0} Q^I Q^{i-1}(xSq^i)$.*

3. An algebraic spectral sequence

We now build an algebraic spectral sequence using only differentials as in the last corollary. Our discovery is that this spectral sequence can be completely described, with an interesting E^∞ term.

We need some notation related to the category $\mathcal{U}$ of unstable right $\mathscr{A}$–modules.

- Let $\Omega^\infty : \mathcal{M} \to \mathcal{U}$ be right adjoint to the inclusion. Explicitly, $\Omega^\infty M$ is the largest unstable submodule of M.
- Let $\Omega : \mathcal{U} \to \mathcal{U}$ be right adjoint to the suspension $\Sigma : \mathcal{U} \to \mathcal{U}$. Explicitly, ΩM is the largest unstable submodule of $\Sigma^{-1} M$.
- The functor Ω^∞ is left exact, and we let $\Omega^\infty_s : \mathcal{M} \to \mathcal{U}$ denote the associated right derived functors.

It is convenient to let $L_s M = \Omega \Omega^\infty_s \Sigma^{1-s} M$. A reworked development of the 'Singer complex' [5, 3] for computing the derived functors Ω^∞_s reveals that the functors L_s have extra structure.

Proposition 4. *There are natural operations $Q^i : L_s M \to L_{s+1}$, raising degree by i, giving $L_* M$ the structure of an object in $\mathcal{QM}$.*

Our second theorem now goes as follows.

Theorem 5. *For all $M \in \mathcal{M}$, there is a spectral sequence $\{E^{alg,r}_{*,*}(M)\}$ with the following properties.*

(a) *The spectral sequence is a functor of M taking values $\mathcal{HM}$, with each d^r both a derivation and coderivation.*

(b) $E^{alg,1}_{*,*}(M) = U_{\mathcal{Q}}(\mathcal{R}_* M)$ *as an object in $\mathcal{HQM}$.*

(c) *d^r is not zero only when $r = 2^s$. For $x \in M$ and I of length s, $Q^I x$ lives to $E^{alg,2^s}_{-2^s,*}(M)$, and $d^{2^s}(Q^I x) = \sum_{i\geq 0} Q^I Q^{i-1}(xSq^i)$.*

(d) $E^{alg,\infty}_{*,*}(M) \simeq U_{\mathcal{Q}}(L_* M)$ *as an object in $\mathcal{HQU}$.*

4. Examples

By playing off the two theorems, one can often prove that the algebraic spectral sequence for $H_*(X)$ agrees with the topological spectral sequence for X.

This happens, for example, in the following cases.

- $X =$ a generalized Eilenberg–MacLane spectrum, unless there is 2–torsion of order at least 4 in $\pi_0(X)$ or $\pi_{-1}(X)$.
- $X =$ a suspension spectrum.
- $X =$ the cofiber of $S \to H\mathbb{Z}$.

Even when the spectral sequences differ, the two theorems severely constrain how this can happen.

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