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Nonlinear Evolution Problems

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Abstract. In this workshop geometric evolution equations of parabolic type, nonlinear hyperbolic equations, and dispersive equations and their interrelations were the subject of 21 talks and several shorter special presentations.

Mathematics Subject Classification (2000): 47J35, 35Q30, 74J30, 35L70, 53C44, 35Q55.

Introduction by the Organisers

In this workshop again we focussed on mainly three types of nonlinear evolution problems and their interrelations: geometric evolution equations (essentially all of parabolic type), nonlinear hyperbolic equations, and dispersive equations. As in previous editions of our workshop, this combination turned out to be very fruitful.

Altogether there were 21 talks, presented by international specialists from Australia, Canada, Germany, Great Britain, Italy, France, Switzerland, Russia and the United States. Many of the speakers were only a few years past their Ph.D., some even still working towards their Ph.D.; 6 out of 49 participants and 4 out of the 21 main speakers were women. As a rule, three lectures were delivered in the morning session; two lectures were given in the late afternoon, which left ample time for individual discussions, including some informal seminar style presentations where Ph.D. students and recent postdoctoral researchers were able to present their work. This report also contains abstracts of all informal seminar style presentations.

In geometric evolution equations, the prominent themes were mean curvature and Ricci flow. It became even more apparent that these equations have many features in common, both on the geometric and on the analytical level. Moreover, the talks sparked discussions between researchers specializing in these types of equations and experts in dispersive and hyperbolic equations, in particular, concerning techniques involving matched asymptotic expansions. These techniques in recent years have been used with amazing success not only in Ricci flow and mean curvature flow but also for demonstrating the existence of stable blow-up regimes for wave maps and nonlinear wave or Schrödinger equations. It became apparent that a further link between these different sets of equations is the feature of pseudo-locality, which is very close in spirit to the concept of finite propagation speed for wave equations. While they are false for the linear heat equation, pseudo-locality estimates hold e.g. for the two-dimensional Ricci flow, establishing another surprising connection between geometric evolution equations of parabolic and hyperbolic type.

In the field of nonlinear hyperbolic equations special focus was laid on critical growth and focussing nonlinearities, for which thresholds for concentration behavior and asymptotic profiles were determined, again often using matched asymptotic expansions. Connections between nonlinear hyperbolic equations and dispersive equations arise, for instance concerning the use of vector fields and the treatment of space-time resonances in capillary water waves and relativity. Finally the study of blow up together with a more dynamical system approach to identify central stable manifolds for Schrödinger and wave maps is again related to stationary solutions that are in turn solutions to certain geometric elliptic equations.

Workshop: Nonlinear Evolution Problems

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Abstracts

How to produce a Ricci flow via Cheeger-Gromoll exhaustion

Esther Cabezas-Rivas
(joint work with Burkhard Wilking)

Given a fixed Riemannian metric g on a smooth n-manifold M, we wonder about short time existence of the Ricci flow:

$$\frac{\partial}{\partial t}g(t) = -2\text{Ric}(g(t)), \tag{1}$$

with $g(0) = g$. This was completely settled for closed (i.e. compact and without boundary) manifolds by Hamilton in [7]. But it seems hopeless to expect similar results for the open (i.e. complete and non compact) case; e.g. it is difficult to imagine how to construct a solution to (1) starting at a complete n-manifold ($n \geq 3$) built by attaching in a smooth way long spherical cylinders with radii converging to zero. The natural way to prevent similar situations is to add extra conditions on the curvature; in this spirit, W. X. Shi proved in [9] that the Ricci flow starting on an open manifold with bounded curvature (i.e. $\sup_M |\text{Rm}_g| \leq k_0 < \infty$) admits a solution for a time interval $[0, T(n, k_0)]$ also with bounded curvature.

Later on M. Simon (cf. [11]), assuming further that the manifold has nonnegative curvature operator ($\text{Rm}_g \geq 0$) and is non-collapsing ($\inf_M \text{vol}_g(B_g(\cdot, 1)) \geq v_0 > 0$), was able to extend Shi's solution for a time interval $[0, T(n, v_0)]$, with $|\text{Rm}_{g(t)}| \leq \frac{c(n,v_0)}{t}$ for $t > 0$. Although $T(n, v_0)$ does not depend on an upper curvature bound, such a bound is still needed to guarantee short time existence. Our first result in [3] manages to remove any restriction on upper curvature bounds for open manifolds with $K^{\mathbb{C}} \geq 0$ (which is a condition weaker than $\text{Rm} \geq 0$):

Theorem 1. *Let (M^n, g) be an open manifold with nonnegative (and possibly unbounded) complex sectional curvature. Then there exists a constant $\mathcal{T} = \mathcal{T}(n, g)$ such that (1) has a smooth solution on $[0, \mathcal{T}]$, with $g(0) = g$ and $K^{\mathbb{C}}_{g(t)} \geq 0$.*

Our solution is obtained as a limit of closed solutions with $K^{\mathbb{C}} \geq 0$. Using that by Brendle [1] the trace Harnack inequality holds for the closed case, it follows that the above solution on the open manifold satisfies the trace Harnack estimate as well. This solves an open question posed in [5, Problem 10.45]. Next, we can wonder if the curvature will be instantaneously bounded by our Ricci flow. The negative answer is illustrated by

Theorem 2. *There is an immortal 3-dimensional nonnegatively curved complete Ricci flow $(M, g(t))_{t\in[0,\infty)}$ with unbounded curvature for each t.*

Asking further a non-collapsing condition as in [11], the answer to the above question becomes affirmative:

Corollary 3. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} \geq 0$. If*

$$\inf\{\mathrm{vol}_g(B_g(p,1)) : p \in M\} = v_0 > 0, \tag{2}$$

then the curvature of $(M, g(t))$ is bounded above by $\frac{c(n,v_0)}{t}$ for $t \in (0, \mathcal{T}(n, v_0)]$.

Moreover, even if the initial metric has bounded curvature one can run into metrics with unbounded curvature, as shown by

Theorem 4. *There is an immortal complete Ricci flow $(M^4, g(t))_{t\in[0,\infty)}$ with positive curvature operator such that $\mathrm{Rm}_{g(t)}$ is bounded if and only if $t \in [0, 1)$.*

Our next result gives a precise lower bound on the existence time for (1) in terms of supremum of the volume of balls, instead of infimum as in Corollary 3 and [11]. We stress that this is new even for initial metrics of bounded curvature.

Corollary 5. *In each dimension there is a universal constant $\varepsilon(n) > 0$ such that for each complete manifold (M^n, g) with $K_g^{\mathbb{C}} \geq 0$ the following holds: If we put*

$$\mathcal{T} := \varepsilon(n) \cdot \sup\Big\{ \tfrac{\mathrm{vol}_g(B_g(p,r))}{r^{n-2}} \mid p \in M, r > 0 \Big\} \in (0, \infty],$$

then any complete maximal Ricci flow $(M, g(t))_{t\in[0,T)}$ with $K_{g(t)}^{\mathbb{C}} \geq 0$ and $g(0) = g$ satisfies $\mathcal{T} \leq T$.

If M has a volume growth larger than r^{n-2}, this ensures $T = \infty$. Previously (cf. [10]) long time existence was only known in the case of Euclidean volume growth (EVG) under the stronger assumptions $\mathrm{Rm}_g \geq 0$ and bounded curvature. We highlight that our volume growth condition cannot be further improved: indeed, as the Ricci flow on the metric product $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ exists only for a finite time, the power $n-2$ is optimal. For $n = 3$ we can even determine exactly the extinction time depending on the structure of the manifold:

Corollary 6. *Let (M, g) be an open 3-manifold with $K_g \geq 0$ and soul Σ. Then a maximal complete Ricci flow $(M, g(t))_{t\in[0,T)}$ with $g(0) = g$ and $K_{g(t)} \geq 0$ has*

$$T = \begin{cases} \frac{\mathrm{area}(\Sigma)}{4\pi\chi(\Sigma)} & \text{if} \quad \dim\Sigma = 2 \\ \infty & \text{if} \quad \dim\Sigma = 1 \\ \frac{1}{8\pi}\lim_{r\to\infty} \frac{\mathrm{vol}_g(B_g(p,r))}{r} & \text{if} \quad \Sigma = \{p_0\} \end{cases}.$$

If $\Sigma = \{p_0\}$ and $T < \infty$, then (M, g) is asymptotically cylindrical and $\mathrm{Rm}_{g(t)}$ is bounded for $t > 0$.

By Corollary 5 a finite time singularity T on open manifolds with $K^{\mathbb{C}} \geq 0$ can only occur if the manifold collapses uniformly as $t \to T$. For immortal solutions we also give an analysis of the long time behaviour of (1): In the case of an initial metric with EVG we remark that a result in [10] can be adjusted to see that a suitable rescaled Ricci flow subconverges to an expanding soliton. Furthermore,

Theorem 7. *Let $(M^n, g(t))$ be a non flat immortal Ricci flow with $K^{\mathbb{C}} \geq 0$ satisfying the trace Harnack inequality. If $(M, g(0))$ does not have EVG, then for $p_0 \in M$ there are sequences $t_k \to \infty$ and $Q_k > 0$ such that the rescaled flow*

$(M, Q_k g(t_k + t/Q_k), p_0)$ converges in the Cheeger-Gromov sense to a steady soliton which is not isometric to $\mathbb{R}^n$.

Here is the technical definition of the condition $K^{\mathbb{C}} \geq 0$: Let $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$ be the complexified tangent bundle. We extend Rm and g at $p \in M$ to $\mathbb{C}$-multilinear maps. The complex sectional curvature of a 2-plane σ of $T_p^{\mathbb{C}}M$ is defined by

$$K^{\mathbb{C}}(\sigma) = \mathrm{Rm}(u, v, \bar{v}, \bar{u}) = g(\mathrm{Rm}(u \wedge v), \overline{u \wedge v}),$$

where $\{u, v\}$ is any unitary basis for σ, i.e. $g(u, \bar{u}) = g(v, \bar{v}) = 1$ and $g(u, \bar{v}) = 0$. We say M has nonnegative complex sectional curvature if $K^{\mathbb{C}} \geq 0$. Notice that this is weaker than $\mathrm{Rm} \geq 0$ and implies nonnegative sectional curvature ($K \geq 0$). Unlike $K \geq 0$, the condition $K^{\mathbb{C}} \geq 0$ has the advantage to be invariant under (1).

During the talk, we sketched the proof of Theorem 1 in the case $K_g^{\mathbb{C}} > 0$, which is considerably easier since e.g. then, by Gromoll and Meyer [6], M is diffeomorphic to $\mathbb{R}^n$. We overcome the lack of such a property in the general case by proving the following result (see [8] for a version asking $\mathrm{Rm}_g \geq 0$):

Theorem 8. *Let (M^n, g) be an open, simply connected Riemannian manifold with nonnegative complex sectional curvature. Then M splits isometrically as $\Sigma \times F$, where Σ is the k-dimensional soul of M and F is diffeomorphic to $\mathbb{R}^{n-k}$.*

Thus combining with the knowledge from [2] of the compact case, this extends the same classification of [2] for open manifolds with $K^{\mathbb{C}} \geq 0$.

Finally, we mention that the general case in the proof Theorem 1 includes working directly with the sublevel sets of the Busemann function, which form part of the convex exhaustion used by Cheeger and Gromoll for the soul construction (cf. [4]). The hardest issue is that such sets have non-smooth boundary (for details about how we deal with such a difficulty, see [3]).

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Properties of the Navier-Stokes equation on negatively curved manifolds.

Magdalena Czubak
(joint work with Chi Hin Chan)

The Navier-Stokes equation on $\mathbb{R}^n$ is given by

$$(NS(\mathbb{R}^n)) \qquad \begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla P &= 0, \\ \operatorname{div} u &= 0, \end{aligned}$$

where $u = (u_1, \ldots, u_n)$ is the velocity of the fluid, P is the pressure, and $\operatorname{div} u = 0$ means the fluid is incompressible. Existence of global weak solutions

$$u \in L^\infty(0,\infty; L^2(\mathbb{R}^n)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^n)) \tag{1}$$

for $n = 2, 3$ has been established in the work of Leray [6] and Hopf [4]. In addition, the global weak solutions satisfy the *global energy inequality*

$$\int_{\mathbb{R}^n} |u(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 dx ds \le \int_{\mathbb{R}^n} |u_0|^2 dx, \quad t \ge 0. \tag{2}$$

Solutions satisfying (1)-(2) are referred to as Leray-Hopf solutions, and have historically served as a foundation for further work in the regularity theory for $NS(\mathbb{R}^n)$.

There is a big difference between the regularity of Leray-Hopf solutions in 2 dimensions and in 3 dimensions. Indeed, the Leray-Hopf solutions for $NS(\mathbb{R}^2)$ are smooth and unique [5]. However, as is well-known, the regularity and uniqueness of solutions to the $NS(\mathbb{R}^3)$ equation is a long standing open problem.

When we move from the Euclidean setting to the Riemannian manifold, the first question is how to write the equations. In particular, what is the natural generalization of the Laplacian, Δ? Ebin and Marsden [3] indicated that the ordinary Laplacian should be replaced by the following operator

$$L = 2 \operatorname{Def}^* \operatorname{Def} = (dd^* + d^*d) + dd^* - 2 \operatorname{Ric},$$

where Def and Def* are the deformation tensor and its adjoint respectively, $(dd^* + d^*d) = -\Delta$ is the Hodge Laplacian with d^* as the formal adjoint of the exterior differential operator d, and Ric is the Ricci operator.

First, note that L *is* the ordinary Laplacian on $\mathbb{R}^n$, since then $\operatorname{Ric} \equiv 0$. Second, L as given above sends 1-forms into 1-forms. Hence, it is more convenient to formulate the Navier-Stokes equation on a Riemannian manifold M in terms of 1-forms U^* instead of vector fields U on M. There is a natural correspondence between vector fields U and 1-forms U^*, which allows us to freely move between the two, and rewrite the equation as

$$(NS(M)) \qquad \begin{aligned} \partial_t U^* - \Delta U^* + \overline{\nabla}_U U^* - 2\operatorname{Ric}(U^*) + dP = 0, \\ d^* U^* = 0, \end{aligned}$$

where $\overline{\nabla}$ stands for the induced Levi-Civita connection on the cotangent bundle T^*M. Arguably less natural equation to study is the one without the Ricci operator

$$\begin{aligned} \partial_t U^* - \Delta U^* + \overline{\nabla}_U U^* + dP = 0. \\ d^* U^* = 0. \end{aligned} \tag{3}$$

Before we state the main result, we mention the only other work we are aware of on a non-compact manifold for $NS(M)$. Q.S. Zhang [10] shows the ill-posedness of the *weak solution with finite L^2 norm* on a connected sum of two copies of $\mathbb{R}^3$.

Consider both $NS(M)$ and (3). We show

Theorem 1 (Non-uniqueness of $NS(\mathbb{H}^2(-a^2))$**.** *Let $a > 0$. Let $M = H^2(-a^2)$, the space form with the constant sectional curvature equal to $-a^2$. Then, $NS(M)$ is ill-posed in the following sense: given smooth $u_0^* \in L^2(M)$, there exist infinitely many smooth solutions satisfying*

$$\text{(finite energy)} \quad \int_M |U^*|^2 < \infty, \tag{4}$$

$$\text{(finite dissipation)} \quad \int_0^t \int_M |\operatorname{Def} U^*|^2 < \infty, \tag{5}$$

$$\text{(global energy inequality)} \quad \int_M |U^*|^2 + 4\int_0^t \int_M |\operatorname{Def} U^*|^2 \le \int_M |u_0^*|^2. \tag{6}$$

If we do not include the Ricci term in the equation, we can also have a non-uniqueness result on a more general negatively curved Riemannian manifold.

Theorem 2. *Let $a, b > 0$ be such that $\frac{1}{2}b < a \le b$, and let M be a simply connected, complete 2-dimensional Riemannian manifold with sectional curvature satisfying $-b^2 \le K_M \le -a^2$. Then there exist non-unique solutions to* (3) *satisfying* (4)*-*(6)*.*

There are some very easy to establish corollaries. In particular

Corollary 3 (Lack of the Liouville theorem for space forms)**.** *Let $n \ge 2$, and $a > 0$, then there exist nontrivial bounded solutions of $NS(\mathbb{H}^n(-a^2))$.*

Corollary 4 (Lack of the Liouville theorem in the hyperbolic setting)**.** *Let $n \ge 2$, and $b \ge a > 0$ and let M be a simply connected, complete n-dimensional Riemannian manifold with sectional curvature satisfying $-b^2 \le K_M \le -a^2$. Then there exist nontrivial bounded solutions of* (3)*.*

The non-uniqueness results heavily relies on the existence of nontrivial bounded harmonic functions on negatively curved Riemannian manifolds due to the works of Anderson [1] and Sullivan [8]. The solution pairs (U^*, P) we consider are (for $NS(\mathbb{H}^2(-a^2))$ and similarly for (3))

$$U^* = \psi(t)dF, \quad \text{and} \quad P = -\partial_t\psi(t)F - \frac{1}{2}\psi^2(t)|dF|^2 - 2a^2\psi(t)F,$$

where $\psi(t) = \exp(-\frac{At}{2})$ for any $A \geq 4a^2$, and F is a nontrival bounded harmonic function on $\mathbb{H}^2(-a^2)$. Verifying that (U^*, P) solves $NS(\mathbb{H}^2(-a^2))$ is simple when we use Hodge theory. In fact, taking solutions of the form $\psi(t)\nabla F$ seems to be a well-known convention, and one could set out to try a similar solution on $\mathbb{R}^n$. However, such solutions would not be interesting, because it would not be possible to show that they are even in L^2 since only bounded harmonic functions on $\mathbb{R}^n$ are trivial. In the hyperbolic setting, given the abundance of the bounded harmonic functions, at least we have a hope, but a priori, it is not obvious that our solutions have to satisfy (4)-(6). Hence the main contribution stems from showing (4)-(6).

The flavor of the proofs is very much in the differential geometry framework of the book of Schoen and Yau [7]. There, amongst many things, one can find a simple proof of the existence of the bounded nontrivial harmonic function, which was presented originally in [2]. A careful study of the proof combined with the gradient estimate for harmonic functions due to S.-T. Yau [9] leads to an exponential decay of the gradient of the harmonic function:

$$|\nabla F| \leq C(a, \delta, \phi')e^{-\delta\rho}, \tag{7}$$

where $\delta < a$ is some constant, ρ is the distance function, and ϕ boundary data for F at infinity. This result has nothing to do with Navier-Stokes, and it might be of independent interest. Estimate (7) very easily gives property (4). Showing finite dissipation (5) is more involved, and we omit the details.

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Construction of dynamical vacuum black holes without symmetries

Mihalis Dafermos
(joint work with Gustav Holzegel, Igor Rodnianski)

A fundamental open problem in general relativity is that of the stability of the Kerr black holes, the celebrated 2 parameter family of solutions to the *Einstein vacuum equations*

$$\mathrm{Ric}(g) = 0. \tag{1}$$

According to the *black hole stability conjecture*, small perturbations of Kerr initial data with sub-extremal Kerr parameters $|a_0| < M_0$ would asymptote in time under evolution by (1) to a nearby member of the Kerr family (with parameters $|a| < M$).

At present, not only is this stability conjecture not resolved, but there are no known examples of dynamic black holes with smooth event horizons which asymptote in time to a Kerr solution. It is this more basic question whose resolution is discussed in the present talk:

Theorem ([7])**.** *Given suitable smooth scattering "data" on the horizon* $\mathcal{H}^+$ *and future null infinity* $\mathcal{I}^+$*, asymptoting to the induced Kerr geometry with parameters* $|a| \le M$*, then there exists a corresponding smooth vacuum black hole spacetime* $(\mathcal{M}, g)$ *asymptotically approaching in its exterior region the Kerr solution with parameters* a *and* M*.*

In particular,

Corollary. *There exist black hole spacetimes with smooth horizons which are not exactly Schwarzschild or Kerr.*

As is suggested by the statement of the above theorem, the black hole spacetimes are constructed by prescribing "scattering data" on the event horizon $\mathcal{H}^+$ and on null infinity $\mathcal{I}^+$, and solving *backwards* as a characteristic initial value problem for (1). More precisely, they are constructed by taking the limit of a finite problem where null infinity is replaced by a far-away light cone and the two null hypersurfaces are supplemented with a late time spacelike piece. Note that this problem is well posed in the smooth category by work of Rendall [11]. See also [10].

For global estimates, one needs a formulation of the Einstein equations which captures both the hyperbolicity *per se* and the asymptotics towards null infinity. Moreover, some version of the null condition must be captured, as one cannot construct solutions even just in a neighourhood of a point in null infinity for general non-linear equations with quadratic nonlinearities. We adopt thus a formulation where the hyperbolic aspects of the Einstein equations are captured at the level of the Bianchi equations, but to close the system these must be coupled with transport and elliptic equations for the metric and spin coefficients. This formulation first appeared in [3] and has been much exploited to understand global properties of (1), see for instance [2, 9].

Without getting technical, let us briefly motivate a certain assumption on the scattering data that plays a fundamental role in the proof. To understand this, one should remark first that the simplest scattering problems on *Schwarzschild* can of course be formulated with respect to the degenerate ∂_t-energy. See [8]. Thus, for those problems, the notion of scattering data is defined simply by the finiteness of the corresponding flux at $\mathcal{H}^+$ and $\mathcal{I}^+$. For non-linear problems, this is insufficient near infinity as one must consider weighted energies–this already requires imposing some decay along $\mathcal{I}^+$. But even for the linear problem of the fixed wave equation

$$\Box_g \psi = 0 \tag{2}$$

on *Kerr*, the ∂_t energy would already be inappropriate near the horizon as it does not yield a positive definite quantity. This is the well-known phenomenon of *superradiance*. For non-linear problems, one expects to have to control a non-degenerate energy, analogous to the J^N-flux introduced in [4] (see also [5]), exploiting the celebrated *red-shift effect*. To control such a non-degenerate energy when solving backwards, however **the red-shift effect is seen as a blue-shift effect**. Thus, to counterbalance this, one must impose suitably fast exponential decay on the scattering data.

It is interesting to recall that in the forward problem, understanding (2) on extremal Kerr $|a| = M$ is considerably more difficult than the subextremal case $|a| < M$, in view of the fact that the red-shift degenerates, and in fact, solutions of (2) exhibit a mild instability exactly on the horizon. See recent work of Aretakis [1]. (It is for this reason that we in particular exclude the extremal case from the stability conjecture.) In view of the comments of the previous paragraph, however, it should not be surprising that the extremal case is *not* excluded from the above theorem. In view of the fact that the red-shift now would appear as an obstacle, its degeneration is not such a bad thing, and thus the extremal case is if anything easier!

Details can be found in [7].

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Concentration compactness for the L^2 - critical nonlinear Schrödinger equation

Benjamin Dodson

The nonlinear Schrödinger equation

$$iu_t + \Delta u = \mu|u|^{\frac{4}{d}}u \tag{1}$$

is said to be mass critical since the scaling $u(t,x) \mapsto \frac{1}{\lambda^{d/2}}u(\frac{t}{\lambda^2},\frac{x}{\lambda})$ preserves the L^2 - norm, $\mu = \pm 1$. In this talk we will discuss the concentration compactness method, which is used to prove global well - posedness and scattering for (1) for all initial data $u(0) \in L^2(\mathbf{R}^d)$ when $\mu = +1$, and for $u(0)$ having L^2 norm below the ground state when $\mu = -1$. This result is sharp.

As time permits the talk will also discuss the energy - critical problem in $\mathbf{R}^d \setminus \Omega$,

$$\begin{gathered} iu_t + \Delta u = |u|^{\frac{4}{d-2}}u, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{2}$$

where Ω is a compact, convex obstacle, $d = 4, 5$.

Global existence for capillary water waves

Pierre Germain
(joint work with Nader Masmoudi, Jalal Shatah)

We prove global existence for the capillary water waves problem for small data in weighted Sobolev spaces. The proof combines in a novel way the energy method, the space-time resonance method and commuting vector fields.

We consider the global existence and asymptotic behavior of surface waves for irrotational, incompressible, and inviscid fluid in the presence of surface tension. The fluid velocity is given by Euler's equation in a domain $\mathcal{U}$:

$$\mathscr{D} = \mathscr{D}_t = \{(x, z) = (x_1, x_2, z) \in \mathbb{R}^3, z \leq h(x, t)\}, \quad \mathcal{U} = \bigcup_t \mathscr{D}_t,$$

and the free boundary of the fluid at time t

$$\mathscr{B} = \mathscr{B}_t = \{(x, h(x, t)), x \in \mathbb{R}^2\} = \partial\mathscr{D}$$

moves by the normal velocity of the fluid . The surface tension is assumed to be proportional (by the coefficient c) to the mean curvature κ of $\mathscr{B}$ and we neglect the presence of gravity. In this setting the Euler equation for the fluid velocity v, and the boundary conditions are given by

$$\text{(1a)} \qquad \begin{cases} D_t v \overset{def}{=} \partial_t v + \nabla_v v = -\nabla p & (x, z) \in \mathscr{D}, \\ \nabla \cdot v = 0 & (x, z) \in \mathscr{D}, \end{cases}$$

$$\text{(1b)} \qquad \begin{cases} \partial_t h + \nabla_v (h - z) = 0 & (x, z) \in \mathscr{B}, \\ p = c\kappa, & (x, z) \in \mathscr{B}. \end{cases}$$

Since the flow is assumed to be irrotational, the Euler equation can be reduced to an equation on the boundary and thus the system of equations (E–BC) reduces to a system defined on $\mathscr{B}$. This is achieved by introducing the potential $\psi_{\mathcal{H}}$ where $v = \nabla\psi_{\mathcal{H}}$. Denoting the trace of the potential on the free boundary by $\psi(x, t) = \psi_{\mathcal{H}}(x, h(x, t), t)$, the system of equations for ψ and h are

$$\text{(WW)} \qquad \begin{cases} \partial_t h = G(h)\psi \\ \partial_t \psi = c\kappa - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2(1+|\partial h|^2)} \left(G(h)\psi + \partial h \cdot \nabla\psi\right)^2 \\ (h, \psi)(t = 0) = (h_0, \psi_0). \end{cases}$$

where

$$G(h) = \sqrt{1 + |\partial h|^2}\mathcal{N},$$

$\mathcal{N}$ being the Dirichlet-Neumann operator associated with $\mathscr{D}$, and the mean curvature can be expressed as

$$\kappa = \frac{(1 + (\partial_1 h)^2)\partial_2^2 h + (1 + (\partial_2 h)^2)\partial_1^2 h - 2\partial_1 h\partial_2 h\partial_1\partial_2 h}{2\sqrt{1 + (\partial_1 h)^2 + (\partial_2 h)^2}} \sim \Delta h + (\text{cubic}).$$

In the sequel we take $c = 2$ for simplicity. The dispersive nature of (WW) is revealed by writing the linearization of this system around $(h, \psi) = (0, 0)$:

$$\text{(2)} \qquad \begin{cases} \partial_t h = \Lambda\psi + \mathfrak{R}_1, \\ \partial_t \psi = \Delta h + \mathfrak{R}_2, \end{cases} \qquad \Lambda \overset{def}{=} |(\partial_1, \partial_2)|,$$

where $\mathfrak{R}_i$ are at least quadratic in $(\partial h\partial^2 h, \partial\psi, \mathcal{N}\psi)$. By introducing the variable

$$u \overset{def}{=} \Lambda^{1/2} h + i\psi,$$

the above system can be written as a single equation

$$\partial_t u = -i\Lambda^{3/2} u + \mathfrak{R}, \tag{3}$$

where $\mathfrak{R} = \Lambda^{\frac{1}{2}}\mathfrak{R}_1 + i\mathfrak{R}_2$

Main result and plan of the proof. To state our main result we need to introduce the following notation: let

$$\partial \stackrel{def}{=} (\partial_1, \partial_2), \;\; \Omega \stackrel{def}{=} x^1\partial_2 - x^2\partial_1 = \omega^i\partial_i, \;\text{ and } \mathcal{S} \stackrel{def}{=} \frac{3}{2}t\partial_t + x^i\partial_i,$$

$\Sigma \stackrel{def}{=} x^i\partial_i$ the spatial part of $\mathcal{S}$, and let Γ denote any of these operators $\Gamma = \Sigma, \Omega$, or ∂^3.

Our main theorem reads

Theorem 1. *Assume that the initial data u_0 satisfies*

$$\sum_{|k|\le 2K} \left\|\Gamma^k \Lambda^{1/2} u_0\right\|_{W^{9/2,2}(\mathbb{R}^2)} + \sum_{|k|\le 2K} \left\|\Gamma^k \Lambda^{\alpha} u_0\right\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon, \tag{4}$$

where $K \ge 10$ and $\epsilon > 0$ and $\alpha > 0$ are sufficiently small. Then there exists a global solution (h, ψ) of (3). Furthermore, this solution scatters, i.e., there exists a solution (h_ℓ, ψ_ℓ) of (3) such that

$$\|h(t) - h_\ell(t)\|_2 + \|\psi(t) - \psi_\ell(t)\|_2 \to 0 \qquad \textit{as } t \to \infty.$$

The proof of this result is based on combining the vector field method (which is based on invariances of the equation), and the space time resonance method (which is based on resonant interactions of waves). Below we give a brief description of the proof.

The classical Stefan problem: well-posedness and asymptotic stability

Mahir Hadžić
(joint work with Steve Shkoller)

The Stefan problem is a non-local geometric evolution problem used in the description of solid-liquid interfaces and crystal formation [14]. The unknowns are the phase temperature p and the moving interface Γ and they satisfy the following equations:

$$p_t - \Delta p = 0 \qquad \text{in } \; \Omega(t)\,; \tag{1a}$$

$$\partial_n p = V_{\Gamma(t)} \qquad \text{on } \; \Gamma(t)\,; \tag{1b}$$

$$p = \sigma \kappa_{\Gamma(t)} \qquad \text{on } \; \Gamma(t)\,; \tag{1c}$$

$$p(0, \cdot) = p_0\,,\; \Gamma(0) = \Gamma_0\,. \tag{1d}$$

Here, $\Omega(t)$ is an evolving open subset of $\mathbb{R}^n$, with $\Gamma(t)$ denoting the moving boundary (which may be a connected subset of $\partial\Omega(t)$ if a part of the boundary is fixed).

$\partial_n p = \nabla p \cdot n$ is the normal derivative of p on $\Gamma(t)$ where n stands for the outward pointing unit normal and $V_{\Gamma(t)}$ denotes the normal velocity of the hypersurface

$\Gamma(t)$. When $\sigma = 0$, problem (1) is called the *classical Stefan problem.* In this case, freezing of the liquid occurs at a constant temperature $p = 0$. If however the surface tension coefficient $\sigma > 0$ in (1c) then the problem (1) is termed the *Stefan problem with surface tension*, whereby $\kappa_{\Gamma(t)}$ stands for the mean curvature of the moving boundary $\Gamma(t)$.

In absence of surface tension, the local-in-time existence of classical solutions has been studied by various authors: Meirmanov, Hanzawa, Prüss-Saal-Simonett, Frolova-Solonnikov (see [14, 15] and references therein for a complete overview). Weak solutions were shown to exist by Friedman [4], Friedman and Kinderlehrer [6], Kamenomostskaya [10]. The regularity of such solutions was further studied by Friedman, Kinderlehrer, Caffarelli, Evans, Stampacchia, DiBenedetto and others - for an overview see [5, 15] and references therein. Viscosity solutions were introduced and shown to be smooth in seminal works of Athanasopoulos, Caffarelli, Salsa (for an overview see [1]), and have been also studied in [11]. In the presence of surface tension local-in-time existence (for one and two-phase problem) is shown by Radkevich, Escher-Prüss-Simonett (see [15]), global stability of steady states is investigated in [9, 8, 12] and weak solutions (without uniqueness) are shown to exist by Luckhaus, Almgren-Wang (see [15] for an overview and references).

We are interested in the classical Stefan problem from the point of view of well-posedness in smooth functional spaces and its qualitative dynamic properties. Our first main result is the development of the well-posedness theory in high-order Sobolev-type energy spaces, naturally associated with the problem. Inspired by [2], the problem is first pulled back onto the fixed domain by introducing the so-called *Arbitrary Lagrange-Eulerian* change of variables. As a consequence, the equations take a form analogous to that of the fluid dynamics equations [3]. We identify a weighted energy quantity which controls the regularity of the moving boundary, where the weight is given by the Neumann derivative of the temperature. Under the so-called Taylor sign condition:

$$\partial_n p_0 < 0, \tag{2}$$

this energy quantity is indeed coercive and we prove local-in-time well-posedness.

With surface tension ($\sigma > 0$) (1) is a micro-scale model where the phase transition does *not* occur at a constant temperature. This is opposed to the macro-scale classical Stefan problem where, given suitable sign on the initial and boundary data, maximum principle implies that the phase is *characterized* by the sign of the temperature p. In particular, it is not clear how to obtain uniform-in-σ bounds that allow one to rigorously link the two problems. Our well-posedness treatment applies to the Stefan problem with and without surface tension and as a second result we establish the vanishing surface tension limit as $\sigma \to 0$ [7]. Note that such a limit is a singular limit and it is a-priori not clear whether surface tension always acts as a stabilizing effect [13].

Finally, we study the question of global-in-time stability close to circular steady states. Note that the problem (1) has infinitely many steady states of the form $(p, \Gamma) \equiv (0, \bar{\Gamma})$, where $\bar{\Gamma}$ is some given C^1-hypersurface. It is thus degenerate in the sense that it is a-priori unclear where the solution converges asymptotically. We use the new energy structure mentioned above to address the stability question. Since the weight $\partial_n p$ is expected to decay exponentially fast, the control of the boundary regularity becomes problematic. Relying on Harnack-type bounds and a careful bootstrap argument, we prove global stability of near circular steady states, under the assumption of strictly positive initial temperature.

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On scattering for the quintic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$

Zaher Hani
(joint work with Benoît Pausader)

The purpose of this work [5] is to study the asymptotic behavior of the defocusing quintic nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$ given by:

$$(1) \qquad (i\partial_t + \Delta_{\mathbb{R}\times\mathbb{T}^2})\, u = |u|^4 u, \quad u(t=0) = u_0 \in H^1(\mathbb{R} \times \mathbb{T}^2)$$

Our main motivation is to better understand the broad question of the effect of the geometry of the domain on the asymptotic behavior of large solutions to nonlinear dispersive equations. While scattering holds for the quintic equation on $\mathbb{R}^3$, it is not expected to hold (apart from trivial cases) on $\mathbb{T}^3$. As it turns out, the situation on $\mathbb{R}\times\mathbb{T}^2$ seems to be a borderline case for this question.

To further explain our motivation and our choice of domain, we present the following heuristic relating the domain geometry to the asymptotic behavior of power type nonlinear Schrödinger equations of the form:

$$(i\partial_t+\Delta_{M^d})\,u=|u|^{p-1}u,\quad u(0)\in H^1(M^d) \tag{2}$$

Here M^d stands for a $d-$dimensional Riemannian manifold. From the heuristic that linear solutions with frequency $\sim N$ initially localized around the origin will disperse at time t in the ball of radius $\sim Nt$, one can hope that scattering is partly determined by the asymptotic volume growth of balls with respect to their radius. In fact, if

$$V(r):=\inf_{q\in M^d}\{\mathrm{Vol}_{M^d}(B(q,r))\}\sim_{r\to\infty} r^g,$$

then one would expect that linear solutions decay at a rate $\sim t^{-g/2}$ and based on the Euclidean theory on $\mathbb{R}^g$, the equation (2) would scatter in the range $1+4/g\leq p\leq 1+4/(d-2)$, while one might expect more exotic behavior, at least when $p\leq 1+2/g$.

We don't know whether such a simple picture is completely accurate, but testing this hypothesis motivated us to study the asymptotic behavior for (1) in the case $g=1$ and $d=3$, which seems to be the hardest case that can be addressed in light of the recent developments in [6, 4]. Indeed, as we will argue later, this problem is both *mass-critical* and *energy-critical* $(1+4/g=1+4/(d-2)=5)$.

Our two main results seem to confirm the picture above about scattering, at least in the case of quotients of Euclidean spaces. The first result asserts that small initial data lead to solutions which are global *and scatter*.

Theorem 1 ([5])**.** *There exists $\delta>0$ such that any initial data $u_0\in H^1(\mathbb{R}\times\mathbb{T}^2)$ satisfying*

$$\|u_0\|_{H^1(\mathbb{R}\times\mathbb{T}^2)}\leq\delta$$

leads to a unique global solution $u\in X^1_c(\mathbb{R})$ of (1) *which scatters in the sense that there exists $v^\pm\in H^1(\mathbb{R}\times\mathbb{T}^2)$ such that*

$$\|u(t)-e^{it\Delta_{\mathbb{R}\times\mathbb{T}^2}}v^\pm\|_{H^1(\mathbb{R}\times\mathbb{T}^2)}\to 0\quad as\quad t\to\pm\infty. \tag{3}$$

The uniqueness space $X^1_c\subset C_t(\mathbb{R}:H^1(\mathbb{R}\times\mathbb{T}^2))$ was essentially introduced by Herr-Tataru-Tzvetkov [6]. The main novelty here is the scattering statement on a manifold with such little volume (and so many trapped geodesics[1]). A key fact about Theorem 1 is that it requires only a control provided by the conserved mass and energy of the solution defined respectively by

$$M(u):=\|u(0)\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)},\quad E(u):=\frac{1}{2}\|\nabla u(0)\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}+\frac{1}{6}\|u(0)\|^6_{L^6(\mathbb{R}\times\mathbb{T}^2)}. \tag{4}$$

[1]The presence of trapped geodesics is known to have nontrivial effects on the linear flow and could be expected to also affect the asymptotic behavior of nonlinear solutions.

Having such a small-data scattering result with control in terms of conserved quantities (compare to Tzvetkov-Visciglia [15]) is crucial in extending a small data result to a global result. This is precisely the question we tackle in the main part of this paper.

For this, we follow the Kenig-Merle "concentration-compactness-rigidity" machinery [11] along with later adaptations to deal with inhomogeneous critical equations ([13, 8, 9]). One key ingredient here is a linear and nonlinear profile decomposition for solutions with bounded energy, which identifies sequences of initial data, called "profiles", exhibiting an extreme behavior (in fact a defect of compactness) and possibly "leaving" the geometry. It is there that the "energy-critical" and "mass-critical" nature of our equation become manifest. Each type of profile singles out an effective equation (in general different from (1)) that governs its dynamic, thus linking the asymptotic behavior of (1) to that of the effective equations. We elaborate on this point for the two main types of profiles[2]:

i) Small-scale profiles: These correspond to sequences of initial data that concentrate at a point (typically like $u(0) = N_k^{1/2}\phi(N_k z)$ with $\phi \in C_0^\infty(\mathbb{R}^3)$ and $N_k \to \infty$). These solutions live at very small-scales, so one would expect them not to sense the distinction between $\mathbb{R}\times\mathbb{T}^2$ and $\mathbb{R}^3$ before they scatter. We prove that this is indeed the case, and the dynamic of those profiles is dictated by the quintic nonlinear Schrödinger equation on $\mathbb{R}^3$, which is *energy-critical* and was proven to be globally well-posed and scattering in [2].

ii) Large-scale profiles: These are sequences of initial data typically of the form $u(0) = M_k^{1/2}\psi(M_k x, y)$ with $(x,y) \in \mathbb{R}\times\mathbb{T}^2, \psi \in C_0^\infty(\mathbb{R}\times\mathbb{T}^2)$ and $M_k \to 0$. They are also products of the profile decomposition and their importance can be heuristically anticipated by looking at one of the scaling limit of the manifold $\mathbb{R}\times\mathbb{T}^2$ (namely $\mathbb{R}\times\frac{1}{\lambda}\mathbb{T}^2$ as $\lambda\to\infty$, cf. [5]). One is tempted to guess that the behavior of such profiles is governed by that of *mass-critical* quintic NLS on $\mathbb{R}$ (as is the case if the initial data are independent of the periodic variable y). However, the situation turns out to be more complicated, and the behavior of such profiles is governed by what we call the "quintic resonant system" given by:

$$
\begin{aligned}
&(i\partial_t + \partial_{xx})\, u_j = \sum_{\mathcal{R}(j)} u_{j_1}\overline{u_{j_2}}u_{j_3}\overline{u_{j_4}}u_{j_5} \quad j \in \mathbb{Z}^2\\
&\mathcal{R}(j) = \{(j_1,j_2,j_3,j_4,j_5) \in (\mathbb{Z}^2)^5 : j_1 - j_2 + j_3 - j_4 + j_5 = j \text{ and}\\
&\qquad |j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 + |j_5|^2 = |j|^2\}
\end{aligned} \tag{5}
$$

with unknown $\vec{u} = \{u_j\}_{j\in\mathbb{Z}^2}$, where $u_j : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{C}$. In the special case when $u_j = 0$ for $j \neq 0$, we recover quintic NLS on $\mathbb{R}$, but in general, this is a new equation (see [5] for references on such systems of NLS equations).

[2] These profiles are the product of the profile decomposition, but their appearance can be anticipated by looking at the scaling limits of the manifold $\mathbb{R}$ *times*$\mathbb{T}^2$ (cf. [5]).

It is not very hard to see that (5) is Hamiltonian, has a nice local theory and retains many properties of quintic NLS on $\mathbb{R}$. In view of this and of the result of Dodson [4], it seems reasonable to formulate the following conjecture:

Conjecture 2 ([5])**.** *Let $E \in (0, \infty)$. For any smooth initial data $\vec{u}_0$ satisfying:*

$$E_{ls}(\vec{u}_0) := \frac{1}{2} \sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|u_{0,j}\|^2_{L^2(\mathbb{R})} \leq E$$

there exists a global solution of (5)*, $\vec{u}(t)$, $\vec{u}(t = 0) = \vec{u}_0$ with conserved $E_{ls}(\vec{u}(t)) = E_{ls}(\vec{u}_0)$ that scatters in positive and negative infinite time.*

We can now give the main result of this paper which asserts large data scattering for (1) conditioned on Conjecture 2.

Theorem 3 ([5])**.** *Assume that Conjecture 2 holds for all $E \leq E^{ls}_{max}$, then any initial data $u_0 \in H^1(\mathbb{R} \times \mathbb{T}^2)$ satisfying*

$$L(u_0) := \int_{\mathbb{R} \times \mathbb{T}^2} \left\{ \frac{1}{2}|u_0|^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{6}|u_0|^6 \right\} dx \leq E^{ls}_{max}$$

leads to a solution $u \in X^1_c(\mathbb{R})$ which is global, and scatters in the sense that there exists $v^{\pm} \in H^1(\mathbb{R} \times \mathbb{T}^2)$ such that (3) *holds. In particular, if $E^{ls}_{max} = +\infty$, then all solutions of* (1) *with finite energy and mass scatter.*

A few remarks about this theorem are in order: First, we should point out that the global regularity part holds for all solutions of finite energy, unconditional on Conjecture 2. Second, while Theorem 3 is stated as an implication, it is actually an equivalence (cf. Appendix of [5]). Finally, we note that the full resolution of Conjecture 2 seems to require considerable additional work that is completely independent of the analysis on $\mathbb{R} \times \mathbb{T}^2$, so we choose to leave it for a later work.

To conclude, we point to the main novelties of the proof: i) proving good global Strichartz estimates not only to prove Theorem 1 but also to obtain an L^2-profile decomposition suitable for the large data theory, ii) the analysis of the large-scale profile initial data that appear in the profile decomposition, understanding their "two time-scale" behavior in terms of the quintic system (5) via a normal form transformation, and iii) a final nonlinear profile recomposition similar to that in [9] but with many more cases (cf. [5]).

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A mass-decreasing flow in dimension three

Robert Haslhofer

Let (M, g_{ij}) be an asymptotically flat three-manifold with nonnegative integrable scalar curvature. The ADM-mass [1] from general relativity is defined as

$$m(g) := \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\, dA^i. \tag{1}$$

By the positve mass theorem, the mass is always nonnegative and vanishes only for flat space. Beautiful proofs employing a variety of techniques have been discovered by Schoen-Yau, Witten and Huisken-Ilmanen [2, 3, 4].
The purpose here is to introduce a geometric flow that decreases the mass, for full details please see [5]. Our mass-decreasing flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings.

Definition 1. *Let (M, g_0) be an orientable asymptotically flat three manifold with nonnegative integrable scalar curvature, and fix a parameter $\varepsilon > 0$.*

- *Let $(M(t), g(t))_{t\in[0,\varepsilon]}$ be the surgical Ricci flow solution of* [6] *starting at g_0, with all connected components except the one containing the asymptotically flat end thrown away.*
- *Solve the elliptic equation $(-8\Delta_{g(\varepsilon)} + R_{g(\varepsilon)})w_1 = 0$, $w_1 \to 1$ at ∞, and conformally rescale to the scalar flat metric $g_1 := w_1^4 g(\varepsilon)$.*
- *Let $(M(\varepsilon), g_1)$ be the new initial condition and iterate the above procedure.*

The concatenation 'flow, conformal rescaling, flow, conformal rescaling, ...' gives an evolution $(M(t), g(t))_{t\in[0,\infty)}$ *which we call the* mass-decreasing flow*.*

The point is, that conformal rescalings to scalar flat metrics squeeze out of the manifold as much mass as possible. However, unless the manifold is flat, the scalar curvature becomes strictly positive again under the Ricci flow and thus the mass can be decreased even more by another conformal rescaling. This process can be iterated forever.

Theorem 2. *The mass-decreasing flow exists for all times, and preserves asymptotic flatness and nonnegative integrable scalar curvature. The mass is constant in the time intervals* $t \in ((k-1)\varepsilon, k\varepsilon)$ *and jumps down by*

$$\delta m_k = -\int_M (8|\nabla w_k|^2 + R w_k^2)dV \tag{2}$$

at the conformal rescaling times $t_k = k\varepsilon$*, where* w_k *is the solution of*

$$\left(-8\Delta_{g(t_k)} + R_{g(t_k)}\right) w_k = 0\,, \qquad w_k \to 1 \quad at \quad \infty. \tag{3}$$

The monotonicity of the mass is strict as long as the metric is nonflat.

We remark that the formal limiting equations for $\varepsilon \to 0$ are

$$\partial_t g = -2\mathrm{Ric} + \Delta^{-1}|\mathrm{Ric}|^2 g\,, \qquad \partial_t m = -2\int_M |\mathrm{Ric}|^2 dV. \tag{4}$$

The equations (4) have been discovered independently by Hubert Bray and Lars Andersson. Short-time existence for this nonlocal flow has been proved very recently by Lu-Qing-Zheng [7]. However, we will actually work with the discrete ε-iteration ($\varepsilon > 0$). The point is that our long-time existence result (Theorem 2) relies heavily on the theory of Ricci flow with surgery due to Perelman [8, 9], and the nice variant for noncompact manifolds due to Bessières-Besson-Maillot [6]. Regarding the topological aspects of the long-time behavior, recall that the Ricci flow with surgery on a closed 3-manifold that admits a metric with positive scalar curvature becomes extinct in finite time [10, 11]. In a similar spirit, along the mass-decreasing flow wormholes pinch off and nontrivial spherical space forms bubble off in finite time.

Theorem 3. *There exists a* $T < \infty$*, such that* $M(t) \cong \mathbb{R}^3$ *for* $t > T$*. In particular, the initial manifold had the diffeomorphism type*

$$M \cong \mathbb{R}^3 \# S^3/\Gamma_1 \# \ldots \# S^3/\Gamma_k \# (S^1 \times S^2) \# \ldots \# (S^1 \times S^2). \tag{5}$$

Moreover, one can in fact take $T = \frac{A_0}{4\pi}$*, where* A_0 *is the area of the largest outermost minimal two-sphere in* (M, g_0)*.*

To investigate the geometric-analytic aspects of the long-time behavior we will follow the general principle that monotonicity formulas are a very useful tool. However, Perelman's λ-energy vanishes for all asymptotically flat manifolds with nonnegative scalar curvature. To overcome this difficulty, we consider instead the following variant of Perelman's λ-functional,

$$\lambda_{\mathrm{AF}}(g) := \inf_{w:w\to 1} \int_M \left(4|\nabla w|^2 + Rw^2\right) dV, \tag{6}$$

where the infimum is now taken over all $w \in C^\infty(M)$ such that $w = 1 + O(r^{-1})$ at infinity.

Theorem 4. *Away from the conformal rescaling and surgery times, we have the monotonicity formula*

$$\tfrac{d}{dt}\lambda_{\mathrm{AF}}(g(t)) = 2\int_M |Ric + \nabla^2 f|^2 e^{-f} dV \geq 0, \tag{7}$$

where $(-4\Delta + R)e^{-f/2} = 0$, $f \to 0$ at ∞. At the conformal rescaling times, λ_{AF} jumps down, but the mass jumps down more, i.e. $m - \lambda_{AF}$ is (almost) monotone decreasing at all times (the cumulative error term from the surgeries can be made arbitrarily small by choosing the surgery parameters suitably).

In fact, we had already introduced the energy-functional λ_{AF} in our previous note [12], where we also observed it gives a lower bound for the mass, i.e.

$$m(g) \geq \lambda_{AF}(g). \tag{8}$$

In passing, we remark that the renormalized Perelman-functional also motivates a stability inequality for Ricci-flat cones that we investigated thoroughly in a joint work with Hall and Siepmann [13]. Coming back to the long-time behavior of the mass-decreasing flow we have:

Theorem 5. *Let $(M(t), g(t))_{t\in[0,\infty)}$ be a solution of the mass-decreasing flow and assume a-priori there exist a constant $c > 0$, such that $\lambda_{AF}(g(t_k)) \geq cm(g(t_k))^2$ for all positive integers k. Then there exists a constant $C < \infty$ such that $m(g(t)) \leq C/t$. In particular, the mass-decreasing flow squeezes out all the initial mass, i.e. $\lim_{t\to\infty} m(g(t)) = 0$.*

The a-priori assumption is (partly) motivated by considering the flow on an end close to Schwarzschild. However, we actually have:

Conjecture 6. *The mass-decreasing flow squeezes out all the initial mass even without a-priori assumptions, i.e. $\lim_{t\to\infty} m(g(t)) = 0$.*

Going one step further, one might ask:

Question 7. *Can the mass-decreasing flow be used to give an independent proof of the positive mass theorem?*

The idea is to show that the mass-decreasing flow converges (for $t \to \infty$) to flat space in a sense strong enough to conclude that the mass limits to zero (and hence that the mass was nonnegative at the initial time). Understanding the geometric long time behavior is already very difficul in the case of the Ricci flow with surgery on closed 3-manifolds, see however the recent progress by Bamler [14].

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Ancient solutions to mean curvature flow

Gerhard Huisken
(joint work with Carlo Sinestrari)

We study ancient solutions M_t of the mean curvature flow, with $t \in (-\infty, 0)$, i.e. $F : M \times (-\infty, 0) \to \mathbb{R}^{n+1}$ satisfies

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \qquad p \in M, t < 0, \tag{1}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $M_t = F(\cdot,t)(M)$.

We assume that M is closed and convex for all t. We take $t = 0$ to be the singular time, so that M_t shrinks to a round point as $t \to 0$. We assume that the dimension of M_t is at least two.

In the one-dimensional case it was shown by Daskalopoulos, Hamilton and Sesum [1], that the only such ancient solutions are either circles shrinking by selfsimilarities or special solutions shaped like the gluing of two translating solutions given by the graph of $\log(\cos x)$. For the 2d-Ricci flow a similar result was obtained in [2].

In higher dimensional Ricci flow Brendle, Huisken and Sinestrari [3] have obtained a classification of ancient solutions that are of sufficiently positive pinched curvature while White has given examples of ancient compact convex solutions to mean curvature flow that are obtained from gluing translating solutions. Here we show that convex, compact ancient solution of mean curvature flow are round shrinking spheres if they satisfy natural uniformity assumptions:

Theorem 1 *Let M_t, with $t \in (-\infty, 0)$ be an ancient solution to the mean curvature flow which is closed and convex and satisfies*

$$h_{ij} \geq \epsilon H g_{ij} \tag{2}$$

for some $\epsilon > 0$ independent of t. Then M_t is a shrinking sphere.

Theorem 2 *Let M_t, with $t \in (-\infty, 0)$, be a convex ancient solution of the mean curvature flow satisfying the diameter bound*

$$\mathrm{diam}(M_t) \leq C(1 + \sqrt{-t}) \text{ for all } t < 0. \tag{3}$$

Then M_t is a family of shrinking spheres.

The lecture sketches the arguments of the proof and describes extensions to compact 2-convex ancient solutions of the flow.

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Neckpinching for asymmetric surfaces moving by mean curvature

Dan Knopf
(joint work with Zhou Gang, Israel Michael Sigal)

There is a folklore conjecture in geometric analysis which predicts that finite-time singularities of parabolic geometric PDE asymptotically become as symmetric as their topologies allow. How should this be interpreted?

A first observation is that the statement should be understood in the sense of quasi-isometries. For example, one may deform the standard round metric on $\mathbb{S}^3$ to a 'bumpy' metric g_0, making its isometry group trivial while preserving positive Ricci curvature. Now consider a solution $g(t)$ of Ricci flow with initial data g_0. By Kotschwar's result [14], the isometry group of each $g(t)$ remains trivial. But by Hamilton's seminal Ricci flow result [9], the actions of $\mathrm{O}(4)$ with respect to a fixed atlas become arbitrarily close to isometries as the singularity time is approached.

A second observation is that the statement must be understood locally. For example, asymptotics of Ricci flow neckpinch singularities [2] reveal that metrics possessing rotation and reflection symmetries asymptotically acquire an additional translational symmetry near the singular set by converging to the cylinder soliton.

A third observation is that the conjecture may be broken down into two parts. The first, which can be stated rigorously, is that dilations of finite-time singularities converge (at least modulo subsequences) to self-similar solutions, i.e. solitons.

Indeed, Huisken's monotonicity formula [11] proves this for Type-I singularities of mean curvature flow (MCF). The second part of the conjecture is the heuristic expectation that solitons, as generalized fixed points of geometric heat flows, are in a sense 'maximally diffused' and hence possess symmetry groups that are as large as possible.

How might one investigate the full conjecture? The most powerful methods for studying finite-time singularities of parabolic geometric PDE in greatest generality are surgery programs. These exploit a 'canonical neighborhood' property — the fact that high-curvature regions of a solution have special properties which in some cases allow their classification. Two celebrated examples are Perelman's surgery program for Ricci flow [15, 16] (also see Hamilton's foundational work [10]) and the surgery program of Huisken and Sinestrari [12] for singularities of 2-convex hypersurfaces evolving by MCF. However, even these spectacularly successful surgery programs do not provide independence of subsequence, precisely because of their need to consider quite general solutions. This is an obstacle to showing that a solution asymptotically (locally, quasi-isometrically) approaches a unique singularity model. For example, given a family of hypersurfaces $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ evolving by MCF and becoming singular as $t \nearrow T < \infty$, we call the set of points in the ambient space at which the solution becomes singular its *residue set*. The residue set of a cylinder is a line, and that of a rotationally and reflection symmetric neckpinch is a point [8]. For nonsymmetric neckpinches, however, it is not even known if the residue set is rectifiable (though it is conjecturally a point — see below).

The difficulties in proving independence of subsequence alluded to above are reflected in another conjecture, which we learned of from Ecker: *Do singularities of* MCF *have unique tangent flows?* This has recently been proved by Schulze [17] if one tangent flow consists of a closed, multiplicity-one, smoothly embedded self-similar shrinker, but the general case remains open.

Another approach to studying singularity formation involves matched asymptotic expansions. These generally require much stronger hypotheses than do surgery programs. But in turn, they provide statements that hold uniformly in suitable space-time neighborhoods of a developing singularity. Some examples (certainly not a comprehensive list!) of asymptotics for geometric PDE are work of King [13], Daskalopoulos–del Pino [5], and Daskalopoulos–Šešum [6] for logarithmic fast diffusion, $u_t = \Delta \log u$, which represents the evolution of the conformal factor for a noncompact 2-dimensional solution of Ricci flow encountering a Type-II singularity; work of Angenent–Velázquez [3] for Type-II MCF singularities; work of Angenent and an author [2] for Type-I Ricci flow singularities; work of two authors [8] for Type-I MCF singularities; and work [1] of Angenent–Isenberg and an author for Type-II Ricci flow singularities. Notably, all of these results — except for [6] — require a hypothesis of rotational symmetry, which invites the question: *Do singularities of geometric* PDE *become asymptotically rotationally symmetric?* In other words, is rotational symmetry stable in a suitable geometric sense?

This note is a report on the first step in a program intended to provide an affirmative answer to this question. In the first step [7], we remove the hypothesis of rotational symmetric for surfaces $\mathcal{M}_t^2 \subset \mathbb{R}^3$ evolving by MCF, replacing it by weaker discrete symmetries. We conclude that solutions starting sufficiently close to a standard rotationally-symmmetric neck become asymptotically rotationally symmetric in a precise sense (see below). This result provides another example in which the folklore conjecture outlined above can be made rigorous. In forthcoming work, we plan to remove the dimension restriction as well as the discrete symmetry assumptions. In light of the important result of Colding–Minicozzi [4] that spherical and cylindrical singularities are the only generic MCF singularities, a successful completion of this program will prove that rotationally symmetric neckpinch behavior is 'universal' in a precise sense. As a corollary, it will also prove a version of the conjecture that MCF neckpinch singularities have unique limiting cylinders. What follows is a brief outline of our methods and results in [7].

We study the evolution of graphs over a cylinder $\mathbb{S}^1 \times \mathbb{R}$ embedded in $\mathbb{R}^3$. In coordinates (x, y, z) for $\mathbb{R}^3$, we take as an initial datum a surface $\mathcal{M}_0^2$ around the x-axis, given by a map $\sqrt{y^2 + z^2} = u_0(x, \theta)$, where θ denotes the angle from the ray $y > 0$ in the (y, z)-plane. Then for as long as the flow remains a graph, all $\mathcal{M}_t^2$ are given by $\sqrt{y^2 + z^2} = u(x, \theta, t)$.

Analysis of rotationally symmetric neckpinch formation [8] leads one to expect that perturbations of rotationally symmetric necks should resemble spatially homogeneous cylinders $\sqrt{2(T - t)}$ in a space-time neighborhood of the developing singularity. So we apply adaptive rescaling, transforming the original variables x and t into rescaled blowup variables $y(x, t) := \lambda^{-1}(t)x$ and $\tau(t) := \int_0^t \lambda^{-2}(s)\,\mathrm{d}s$, respectively. (Reflection symmetry fixes the center of the neck at $x = 0$.) What distinguishes this approach from standard parabolic rescaling (e.g. [3] or [2]) is that we do not fix $\lambda(t)$ but instead consider it as a free parameter to be determined from the evolution itself. We study a rescaled radius $v(y, \theta, \tau)$ defined by $v(y(x, t), \theta, \tau(t)) := \lambda^{-1}(t)\, u(x, \theta, t)$. Then in commuting (y, θ, τ) variables, the quantity v evolves by $v_\tau = A_v(v) - (\lambda\lambda_t)v - v^{-1}$, where A_v is a quasilinear elliptic operator. The formal adiabatic approximate solution of this equation is given by $V_{\alpha,\beta}(y) := \sqrt{\alpha(2 + \beta y^2)}$ for positive parameters α and β.

We assume that the initial surface $v_0(y, \theta) = v(y, \theta, 0)$ is sufficiently C^3-close to some $V_{\alpha_0,\beta_0}(y)$. We then prove that for such initial data, the solution $v(y, \theta, \tau)$ of MCF becomes singular at some $T < \infty$ and converges locally to a rotationally symmetric solution $V_{\alpha(\tau),\beta(\tau)}(y)$, where $\alpha \approx 1$ and $\beta \approx (-\log(T - t))^{-1}$ as $t \nearrow T$. The solution's residue set is a point. An interested reader should consult [7] for precise statements of our assumptions, estimates, and convergence results.

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Threshold behavior of solutions for the critical focusing NLW on $\mathbb{R}^{3+1}$

Joachim Krieger
(joint work with Roland Donninger, Kenji Nakanishi, Wilhelm Schlag)

We discuss questions revolving around the behavior of certain large solutions of the energy critical focussing nonlinear wave equation

$$\Box u = -u_{tt} + \Delta u = -u^5$$

on $\mathbb{R}^{3+1}$. More precisely, we consider radial solutions which have energy close to but slightly larger than that of the unique (up to scaling) radial ground state solution

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}} \ .$$

The question we are interested in is the construction of 'bubbling off' solutions of the type

$$u(t,x) = W_{\lambda(t)}(x) + \varepsilon(t,x) \ ,$$

where $W_\lambda = \lambda^{\frac{1}{2}} W(\lambda x)$ and $\lambda(t)$ obeys various types of dynamics, as well as the significance of these types of solutions as a kind of threshold behavior dividing stable blow up/scattering regimes for the equation for data near the ground state.

We discuss finite and infinite time blow up scenarios with a continuum of rates, as well as the existence of a center stable manifold passing through W in a sufficiently strong topology. We also mention quantized blow up rates for smooth data (work in progress), as well as a possible dynamic interpretation of the center stable manifold. This comprises work done in collaboration with R. Donninger, K. Nakanishi and W. Schlag.

Scattering for equivariant wave maps

Andrew Lawrie

(joint work with Raphaël Côte, Carlos Kenig, Wilhelm Schlag)

Wave maps, also known as nonlinear σ-models, constitute a class of nonlinear wave equations defined as critical points (at least formally) of Lagrangians

$$\mathcal{L}(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^{d+1}} \eta^{\alpha\beta} \left\langle \partial_\alpha u \, , \, \partial_\beta u \right\rangle_g \, dtdx$$

where $u : (\mathbb{R}^{d+1}, \eta) \to (M, g)$ is a smooth map from Minkowski space into a Riemannian manifold (M, g). If $M \hookrightarrow \mathbb{R}^N$ is embedded, then critical points are characterized by the property that $\Box u \perp T_u M$ where $\Box$ is the d'Alembertian. In particular, harmonic maps from $\mathbb{R}^d \to M$ are wave maps which do not depend on time. In the presence of symmetries, such as when the target manifold M is rotationally symmetric, one often singles out a special class of such maps called equivariant wave maps. For example, for the sphere $M = S^d$ one requires that $u \circ \rho = \rho^\ell \circ u$ where ℓ is a positive integer and $\rho \in SO(d)$ acts on $\mathbb{R}^d$ on S^d by rotation. These maps themselves have been extensively studied, see for example Shatah [8], Christodoulou, Tahvildar-Zadeh [3], Shatah, Tahvildar-Zadeh [9] for early results. For a summary of these developments, see the book by Shatah, Struwe [10].

1. $2d$ Equivariant Wave Maps

In a forthcoming joint work with Côte, Kenig, and Schlag, we consider energy critical 1-equivariant wave maps $U : \mathbb{R}^{1+2} \to S^2$. In this case, the Cauchy problem reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \end{aligned} \tag{1}$$

where ψ is the azimuth angle measured from the north pole. In this equivariant setting, the conserved energy becomes

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2} \right) r \, dr = \text{const.} \tag{2}$$

Any $\psi(r, t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(t, 0) = m\pi$ and $\psi(t, \infty) = n\pi$ for all $t \in I$ where $m, n \geq 0$ are fixed integers. This requirement splits the energy space into disjoint classes according to this topological condition. The wave map evolution preserves these classes. In light of this, the natural spaces in which to consider Cauchy data for (1) are the energy classes $\mathcal{H}_n := \{(\psi_0, \psi_1) \,|\, \mathcal{E}(\psi_0, \psi_1) < \infty \quad \text{and} \quad \psi_0(0) = 0, \psi_0(\infty) = n\pi\}$.

In the analysis of 1-equivariant wave maps to the sphere, an important role is played by the harmonic map, Q, given by stereographic projection. One can show that the energy $\mathcal{E}(Q, 0)$ is minimal in $\mathcal{H}_1$ and up to a rescaling, Q is the unique 1-equivariant harmonic map to the sphere in $\mathcal{H} = \bigcup_{n \geq 0} \mathcal{H}_n$.

One of our goals is to study the asymptotic behavior of solutions to (1) with data in $\mathcal{H}_0$. In [12], Struwe's work implies that solutions $\vec{\psi}(t)$ to (1) with data in $\vec{\psi}(0) \in \mathcal{H}_0$ are global in time if $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$. Recently, Cote, Kenig and Merle, in [2] extended this result to include scattering to zero in the regime, $\vec{\psi}(0) \in \mathcal{H}_0$ and $\mathcal{E}(\vec{\psi}) < \mathcal{E}(Q) + \delta$ for small $\delta > 0$ and conjecture that scattering should hold for data in $\mathcal{H}_0$ with energy up to $2\mathcal{E}(Q)$. Referred to as the "threshold conjecture," this result is implied by the work of Sterbenz and Tataru in [11] when one restricts their results to the equivariant setting. Here we give an alternative proof of the threshold conjecture in the simpler equivariant setting based on the concentration compactness/rigidity method of Kenig and Merle, [4], [5]. The main new ingredient in the proof is a robust "rigidity" result which states that any solution whose trajectory in the energy space is pre-compact up to the symmetries of the equation must be either identically zero or a rescaled nontrivial harmonic map.

2. 3d Equivariant, Exterior Wave Maps to the Sphere

Recently, together with Wilhelm Schlag in [6], we investigated equivariant wave maps from $3+1$-dimensional Minkowski space exterior to a ball and with S^3 as target. To be specific, let $B \subset \mathbb{R}^3$ be the unit ball. We then consider wave maps $U : \mathbb{R} \times (\mathbb{R}^3 \setminus B) \to S^3$ with a Dirichlet condition on ∂B, i.e., $U(\partial B) = \{N\}$ where N is a fixed point on S^3. In the usual equivariant formulation of this equation, where ψ is the azimuth angle measured from the north pole, the Cauchy problem for 1-equivariant wave maps with Dirichlet boundary condition $\psi(1, t) = 0$ for all $t \geq 0$ reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1 \\ &\psi(1,t) = 0, \quad \forall\, t \geq 0, \quad \psi(r,0) = \psi_0(r), \quad \psi_t(r,0) = \psi_1(r) \end{aligned} \tag{3}$$

$$\mathcal{E}(\psi, \psi_t) = \int_1^\infty \frac{1}{2}\big(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\big) r^2\, dr = \text{const.} \tag{4}$$

Any $\psi(r, t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(\infty, t) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed. Again we denote by $\mathcal{H}$ the energy space.

The advantage of this model lies with the fact that removing the unit ball eliminates the scaling symmetry and also renders the equation subcritical relative to the energy. Both of these features are in stark contrast to the same equation on $3+1$-dimensional Minkowski space, which is known to be super-critical and to develop singularities in finite time, see Shatah [8] and also Shatah, Struwe [10].

Another striking feature of this model, which fails for the $2+1$-dimensional analogue, lies with the fact that it admits infinitely many stationary solutions $Q_n(r)$ which satisfy $Q_n(1) = 0$ and $\lim_{r\to\infty} Q_n(r) = n\pi$, for each $n \geq 1$. These solutions have minimal energy in the class of all functions of finite energy which satisfy the $n\pi$ boundary condition at $r = \infty$, and they are the unique stationary solutions in that class. We denote the latter class by $\mathcal{H}_n$.

The exterior equation (3) was proposed by Bizon, Chmaj, and Maliborski [1] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. Numerical simulations described in [1] indicate that in each equivariance class and topological class given by the boundary value $n\pi$ at $r = \infty$ *every solution* scatters to the unique harmonic map that lies in this class. In this paper we verify this conjecture for $\ell = 1, n = 0$. These solutions start at the north-pole and eventually return there. For $n \geq 1$ we obtain a perturbative result. In particular, we prove:

Theorem 1. *(L, Schlag 2011): Let $(\psi_0, \psi_1) \in \mathcal{H}_0$. Then there exists a unique global evolution to* (3) *scattering to zero in the sense that the energy of the wave map on an arbitrary but fixed compact region vanishes as $t \to \infty$.*

Theorem 2. *(L, Schlag 2011): For any $n \geq 1$ there exists $\varepsilon > 0$ such that for any $(\psi_0, \psi_1) \in \mathcal{H}_n$ such that $\|(\psi_0, \psi_1) - (Q_n, 0)\|_{\mathcal{H}} < \varepsilon$ the solution to* (3) *with data (ψ_0, ψ_1) exists globally, is smooth, and scatters to $(Q_n, 0)$ as $t \to \infty$.*

We prove Theorem 1 by means of the Kenig-Merle concentration compactness/rigidity method [4], [5], [7]. The most novel aspect of our implementation of this method lies with the rigidity argument. Indeed, in order to prove Theorem 1 *without any upper bound on the energy* we demonstrate that the natural virial functional is globally coercive on $\mathcal{H}$. This requires a detailed variational argument, the most delicate part of which consists of a rigorous phase-space analysis of the Euler-Lagrange equation.

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Impulsive gravitational waves

Jonathan Luk
(joint work with Igor Rodnianski)

Explicit solutions of impulsive gravitational waves were first found by Penrose [8], building on earlier works of [1] and [3]. These are spacetimes satisfying the vacuum Einstein equations such that the Riemann curvature tensor has a delta singularity across a null hypersurface. However, the Penrose explicit solution was constructed in plane symmetry. The impulsive gravitational wave thus has plane wavefront and can only be thought of as an idealization that the source of the gravitational wave is at an infinite distance. Moreover, plane symmetry also assumes that the gravitational wave has infinite extent and automatically imposes the assumption of non-asymptotic flatness.

The first study of general spacetimes satisfying the Einstein equations and admitting possible 3-surface delta singularities was first undertaken by Taub [10], who derived a system of consistency relations linking the metric, curvature tensor and the geometry of the singular hypersurface.

We study the dynamical problem for general impulsive gravitational waves by solving the characteristic initial value problem without symmetry assumptions. Characteristic initial data is given on a truncated outgoing cone H_0 and a truncated incoming cone $\underline{H}_0$ intersecting at a two sphere $S_{0,0}$ (Figure 1).

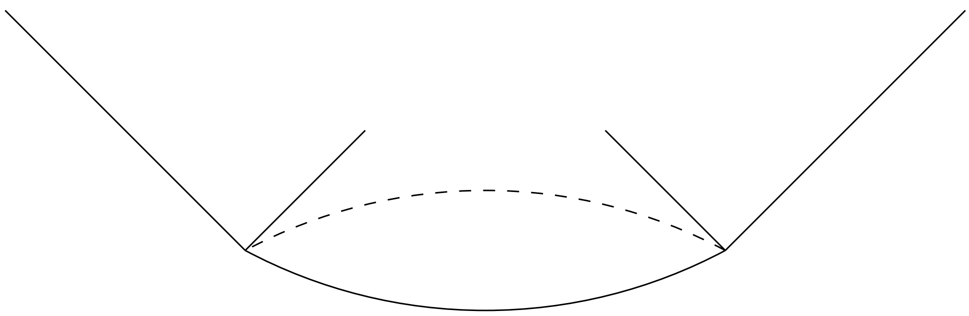

Figure 1. Setup for the characteristic initial value problem

To study the propagation of impulsive gravitational waves, the initial data on the outgoing hypersurface H_0 is prescribed such that the null second fundamental form has a jump discontinuity across an embedded two sphere $S_{0,\underline{u}_s}$ but is smooth otherwise. The curvature tensor for the initial data thus has a delta singularity across $S_{0,\underline{u}_s}$. On the initial incoming hypersurface, the data is smooth but otherwise does not satisfy any smallness assumption. For this class of data, we prove existence and uniqueness of local solutions, as well as a result on the propagation of singularity:

Theorem 1 (L.-Rodnianski [4]). *Given the characteristic initial data as above, there exists a unique local solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. Moreover, the curvature has a delta singularity across the null hypersurface emanating from the initial singularity prescribed on $S_{0,\underline{u}_s}$. The spacetime is smooth away from this null hypersurface.*

The theorem gives a precise description of how the singularity propagates as depicted in Figure 2. This can be thought of as an analog in general relativity of the work of Majda on the propagation of shocks in compressible fluids [6], [7].

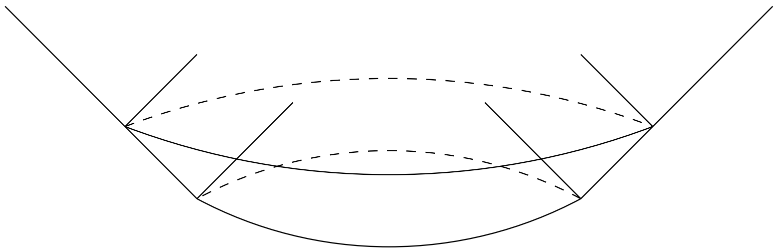

FIGURE 2. Propagation of impulsive gravitational waves

In view of the examples of colliding impulsive gravitational waves found by Khan-Penrose [2] and Szekeres [9], we also studied the collision of these impulsive gravitational waves. More specifically, we prescribe a jump discontinuity across an embedded two sphere in the null second fundamental forms both on the outgoing and the incoming initial hypersurfaces. Locally, by Theorem 1, a unique solution exists and the curvature has a delta singularity across each of the null hypersurfaces emanating from the initial singularity. We show that we can understand the spacetime after the interaction of the two gravitational impulsive waves, which is represented geometrically by the intersection of these two null hypersurfaces. In particular, while the two gravitational impulsive waves interact nonlinearly, the resulting spacetime is smooth except on the union of the two null hypersurfaces emanating from the initial singularities even beyond the interaction:

Theorem 2 (L.-Rodnianski [5]). *Suppose on the initial outgoing hypersurface H_0, the null second fundamental form has a jump discontinuity across the two sphere $S_{0,\underline{u}_s}$ but is smooth otherwise; on the initial incoming hypersurface $\underline{H}_0$, the null second fundamental form has a jump discontinuity across the two sphere $S_{u_s,0}$ but is also smooth otherwise. Then there exists a unique local solution to the vacuum*

Einstein equations $R_{\mu\nu} = 0$*. Moreover, the spacetime is smooth away from the union of incoming null hypersurface* H_{u_s} *emanating from* $S_{u_s,0}$ *and the outgoing null hypersurface* $H_{\underline{u}_s}$ *emanating from* $S_{0,\underline{u}_s}$.

The main difficulty in studying this class of spacetimes is that the Riemann curvature tensor is not in L^2. In this case, the standard energy estimates based on the Bel Robinson tensor do not apply. In the proof, we introduced a new type of energy estimates, which is based on the L^2 norm of only some (renormalized) components of the Riemann curvature tensor. Moreoever, we show that the spacetime geometry can be controlled only with the knowledge of these components of the curvature tensor. In fact this allows us to prove existence and uniqueness of solutions to the vacuum Einstein equations for a more general class of initial data.

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Ricci flow of non-compact warped products with torus fibres

Tobias Marxen

We examine the behaviour of the manifold $M = \mathbf{R} \times N$, where (N, g_N) is a flat, complete, connected Riemannian manifold, with warped product metric $h = f_0^2 dx^2 + g_0^2 g_N$ under Ricci flow, where $f_0, g_0 : \mathbf{R} \to \mathbf{R}$ are C^∞ and positive, such that (M, h) is complete and has bounded curvature.

First we show that the warped product structure is preserved, i. e. the unique maximal solution $h(t), t \in [0, T)$, $0 < T \leq \infty$, of Ricci flow

$$\frac{d}{dt} h(t)(p) = -2\mathbf{Ric}_{h(t)}(p), p \in M, t \in [0, T)$$

with $h(0) = h$ is of the form $h(t) = f^2(\cdot, t)dx^2 + g^2(\cdot, t)g_N$, $t \in [0, T)$ with $f, g : \mathbf{R} \times [0, \infty) \to \mathbf{R}$ C^∞ and positive.

Next we show, in case $\mathrm{Dim}\, N = 2$, long time existence ($T = \infty$) and that the solution is of type III, i. e. $| | \leq \frac{C}{t}$ for some $C > 0$ and for all $t \in (0, \infty)$.

The proof idea is as follows: The product structure $M = \mathbf{R} \times N$ induces a decomposition of each tangent space $T_{(x,q)}M$ of M as a direct sum of the subspace $T_{(x,q)}(\mathbf{R} \times \{q\})$ tangent to the submanifold $\mathbf{R} \times \{q\}$ at (x,q) and of the subspace $T_{(x,q)}(\{x\} \times N)$ tangent to the submanifold $\{x\} \times N$ at (x,q). We call vectors in $T_{(x,q)}(\mathbf{R} \times \{q\})$ horizontal and vectors in $T_{(x,q)}(\{p\} \times N)$ vertical. If we equip $\mathbf{R} \times N$ with the warped product metric $f^2dx^2 + g^2g_N$, where $f, g : \mathbf{R} \to \mathbf{R}$ are C^∞ and positive, we have two main sectional curvatures that can be described via two C^∞ functions $K_V, K_H : \mathbf{R} \times N \to \mathbf{R}$: all two dimensional subspaces of $T_{(x,q)}M$ that are spanned by two vertical vectors have the same sectional curvature, we denote it by $K_V(x,q)$; all two dimensional subspaces of $T_{(x,q)}M$ that are spanned by a horizontal and a vertical vector have the same sectional curvature, we denote it by $K_H(x,q)$. In fact K_V and K_H depend only on $x \in \mathbf{R}$: $K_V(x,q) = \hat{K_V}(x), K_H(x,q) = \hat{K_H}(x)$ with $\hat{K_V}, \hat{K_H} : \mathbf{R} \to \mathbf{R}$.

Now we have $|\ |^2 = a(n)K_V^2 + b(n)K_H^2$, where $a(n), b(n)$ are positive constants depending only on $\operatorname{Dim} N$. We next calculate, in case $\operatorname{Dim} N = 2$, evolution equations for $\hat{K_V}, \hat{K_H}$ and other appropriate geometric quantities. Applying an extension of a maximum principle on noncompact manifolds with time dependent metric then yields

Proposition 1. $\sup_{x \in \boldsymbol{R}} |\hat{K_V}|(x,t) \leq \frac{1}{4t + \frac{1}{\sup_{x\in\boldsymbol{R}} |\hat{K_V}|(x,0)}} = \frac{C}{t+a}$ *for all* $x \in \boldsymbol{R}, t \in [0,\infty)$ *with* $C := \frac{1}{4}$ *and* $a := \frac{1}{4 \sup_{x\in\boldsymbol{R}} |\hat{K_V}|(x,0)}$.

Proposition 2. $|\hat{K_H}|(x,t) \leq \frac{C}{t+a}$ *for all* $x \in \boldsymbol{R}, t \in [0,\infty)$ *with* $a := \frac{1}{4 \sup_{x\in\boldsymbol{R}} |\hat{K_V}|(x,0)}$ *and* $C = C(\sup_{x\in\boldsymbol{R}} |\hat{K_H}|(x,0), \sup_{x\in\boldsymbol{R}} |\hat{K_V}|(x,0))$.

Thereof we get $|\ | \leq \frac{C}{t}$ for some $C > 0$ and for all $t > 0$.

Furthermore we show Gaussian upper bounds for the geometric quantities $|\ |^2, |\nabla\ |^2$ and $|T|^2$ (T denotes the traceless Ricci tensor) on arbitrary complete noncompact manifolds under Ricci flow assuming that the quantities have compact support at time $t = 0$ and appropriate curvature conditions.

The condition that $|\nabla\ |^2$ has compact support at time $t = 0$ could intuitively be called: M has locally symmetric ends at time $t = 0$. And one intuitively starts with flat or Einstein ends, if $|\ |^2$ or $|T|^2$ have compact support at $t = 0$.

Applying the Gaussian estimates to the warped product manifold $\mathbf{R} \times N$ assuming hyperbolic ends at $t = 0$ additionally yields, that the ends have asymptotically constant curvature $C(t)$ for each positive time $t > 0$, and a quantitative estimate of how far the curvatures deviate from this constant. Moreover we determine the exact rate of $C(t)$: $C(t) = -\frac{1}{4t - \frac{1}{C(0)}}$. This is exactly the rate at which the curvatures decay on hyperbolic space H^3 under Ricci flow.

Finally we construct a special class of collapsing (i.e. the injectivity radius goes uniformly to 0 while the curvatures stay bounded) warped product solutions to the Ricci flow.

There are many related results for Ricci flow, we list just a few examples: In 1982 R. Hamilton started Ricci flow by showing that the volume normalized Ricci flow on a closed three dimensional manifold converges to a metric of constant positive sectional curvature, if the initial manifold has strictly positive Ricci curvature ([6]). In [9] M. Simon showed that neckpinching actually happens under Ricci flow by considering a class of warped product metrics on the manifold $\mathbf{R} \times N$, where N is a closed Einstein manifold with positive Einstein constant. S. Angenent and D. Knopf showed neckpinching for a class metrics on S^{n+1}, considering a warped product $\mathbf{R} \times S^n$ [2]. Asymptotics for Ricci flow neckpinches were developed by S. Angenent and D. Knopf in [3] and by S. Angenent, J. Isenberg and D. Knopf in [1]. In [7] R. Hamilton considered the case of a warped product $S^1 \times T^2$, where T^2, the two dimensional torus, carries a flat metric. Furthermore, J. Lott and N. Sesum analysed Ricci flow on warped products with an S^1-fibre over a closed surface and on compact three dimensional manifolds with a free locally isometric T^2 action [8]. Also there is a vast literature on Gaussian estimates for heat type equations, see for example the references in [5]. Our Gaussian estimates are based on [4], p. 355, Theorem 26.25.

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Nonlinear bound states and quasimodes on manifolds

Jeremy L. Marzuola
(joint work with Pierre Albin, Hans Christianson, Jason Metcalfe, Michael Taylor, Laurent Thomann)

This is a collection of observations made in several works with various co-authors, namely [2, 1, 3]. In each of these results, we wish to explore existence of stationary solutions to the nonlinear Schrödinger equation (though other nonlinear dispersive type equations can certainly be considered as well) on a manifold (M, g). Let $-\Delta_g$ be the Laplace-Beltrami operator on M with respect to the metric g. Consider the nonlinear Schrödinger equation $(NLS - g)$ on M:

$$\begin{cases} iu_t + \Delta_g u + |u|^p u = 0, \ x \in M \\ u(0,x) = u_0(x). \end{cases}$$

A nonlinear bound state is a choice of initial condition $R_\lambda(x)$ such that

$$u(t,x) = e^{i\lambda t} R_\lambda(x)$$

satisfies $(NLS - g)$ with initial data $u(0,x) = R_\lambda(x)$.

Plugging in the ansatz yields the following stationary elliptic equation for R_λ:

$$-\Delta_g R_\lambda + \lambda R_\lambda - |R_\lambda|^p R_\lambda = 0. \tag{1}$$

We see a very nice trichotomy in our work:

- $\mathbb{H}^d$: dispersion is so strong that only local nonlinearity dominates,
- $\mathbb{R}^d$: balance of dispersion and nonlinearity globally,
- $(\mathcal{M}, g)$: Locally geometry dominates over nonlinearity.

Recent progress allows us to further explore the existence of global bound states on Weakly Homogeneous Manifolds, [3]. Weakly homogeneous spaces are taken to be spaces such that one can apply compactness techniques locally. Namely, we assume there is a group G of isometries of M and a number $D > 0$ such that for every $x, y \in M$, there exists $g \in G$ such that $\operatorname{dist}(x, g(y)) \leq D$.

We analyze two variational methods of establishing the existence of a solution to (1). One is to mimimize the functional

$$F_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 \tag{2}$$

subject to the constraint

$$J_p(u) = \int_M |u|^{p+1} dV = \beta, \tag{3}$$

with $\beta \in (0, \infty)$ fixed. For this, we will require

$$p \in \Big(1, 1 + \frac{4}{n-2}\Big), \quad \text{i.e.,} \ \ p+1 \in \left(2, \frac{2n}{n-2}\right). \tag{4}$$

From the work of P.L. Lions [5], we have the following means of finding constrained minimizers:

Let $(\rho_n)_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^d)$ satisfying:

$$\rho_n \geq 0 \text{ in } \mathbb{R}^d, \int_{\mathbb{R}^d} \rho_n dx = \lambda$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\geq 1}$ satisfying one of the three following possibilities:

i. (compactness) there exists $y_k \in \mathbb{R}^d$ such that $\rho_{n_k}(\cdot + y_{n_k})$ is tight, i.e.:

$$\forall \epsilon > 0,\ \exists R < \infty,\ \int_{y_k + B_R} \rho_{n_k}(x)dx \geq \lambda - \epsilon;$$

ii. (vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{y+B_R} \rho_{n_k}(x)dx = 0$, for all $R < \infty$;

iii. (dichotomy) there exists $\alpha \in [0, \lambda]$ such that for all $\epsilon > 0$, there exists $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1_+(\mathbb{R}^d)$ satisfying for $k \geq k_0$:

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \epsilon, \\ |\int_{\mathbb{R}^d} \rho_k^1 dx - \alpha| \leq \epsilon, \\ |\int_{\mathbb{R}^d} \rho_k^2 dx - (\lambda - \alpha)| \leq \epsilon, \\ d(\mathrm{Supp}(\rho_k^1), \mathrm{Supp}(\rho_k^2)) \to \infty. \end{cases}$$

This framework can generalized to weakly homogeneous spaces, which include all smooth compact manifolds (including with boundary), universal coverings of compact manifolds and of course standard homogeneous spaces.

1. Nonlinear quasimodes on manifolds with periodic elliptic geodesic orbits

In [1], we construct almost stationary solutions to (1). We do this by separating variables in the t direction, we write

$$\psi(x, t) = e^{-i\lambda t} u(x),$$

from which we get the stationary equation

$$(\lambda - \Delta_g)u = \sigma |u|^p u.$$

The construction in the proof finds a function $u_\lambda(x) = \lambda^{(d-1)/8} g(\lambda^{1/4} x)$ such that g is rapidly decaying away from Γ, $\mathcal{C}^\infty$, g is normalized in L^2, and

$$(\lambda - \Delta_g)u_\lambda = \sigma |u_\lambda|^p u_\lambda + E(u_\lambda),$$

where the error $E(u_\lambda)$ is expressed by the truncation of an asymptotic series similar to that in the work of Thomann and is of lower order in λ.

The result is an improvement over the trivial approximate solution. It is well known that there exist quasimodes for the linear equation localized near Γ of the form

$$v_\lambda(x) = \lambda^{(d-1)/8} e^{is\lambda^{1/2}} f(s, \lambda^{1/4} x),\ (\lambda > 0),$$

with f a function rapidly decaying away from Γ, and s a parametrization around Γ, so that $v_\lambda(x)$ satisfies

$$(\lambda - \Delta_g)v_\lambda = \mathcal{O}(\lambda^{-\infty})\|v_\lambda\|$$

in any seminorm, see Ralston.

Then

$$(\lambda - \Delta_g)v_\lambda = \sigma|v_\lambda|^p v_\lambda + E_2(v_\lambda),$$

where the error $E_2(v_\lambda) = |v_\lambda|^p v_\lambda$ satisfies

$$\|E_2(v_\lambda)\|_{\dot{H}^s} = \mathcal{O}(\lambda^{s/2+p(d-1)/8}).$$

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Partial Regularity for the Harmonic Map Heat Flow

Roger Moser

Many of the questions on the regularity of weakly harmonic maps have been answered by the works of Hélein [5, 6, 7] and subsequent generalisation of Evans [3] and Bethuel [1]. Similar results have been obtained for the harmonic map heat flow as well [4, 2, 8], but only for special target manifolds and for low-dimensional domains. More recently, another approach to regularity for harmonic maps has been found by Rivière [10] and extended by Rivière and Struwe [11]. It turns out that the new method allows a generalisation to variants of the underlying equation, including the harmonic map heat flow.

Suppose that we study maps between the open unit ball B in $\mathbb{R}^m$ and a compact, smooth Riemannian manifold N without boundary. It is convenient to assume that N is embedded isometrically in another Euclidean space $\mathbb{R}^n$. Consider the Dirichlet functional

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2\, dx.$$

Critical points of the functional are called harmonic maps. If A denotes the second fundamental form of N, regarded as a submanifold of $\mathbb{R}^n$, then the corresponding Euler-Lagrange equation is

$$\Delta u + \operatorname{trace} A(u)(\nabla u, \nabla u) = 0.$$

We may study weak solutions of this equation in the Sobolev space

$$W^{1,2}(B;N) = \left\{u \in W^{1,2}(B;\mathbb{R}^n)\colon u(x) \in N \text{ for almost every } x \in B\right\}.$$

By the aforementioned results, weak solutions of the equation are always smooth if $m = 2$. For $m > 2$, there exist examples of weak solutions that are discontinuous everywhere [9]. On the other hand, under a certain stationarity condition, it follows that weak solutions are smooth away from a singular set of codimension 2.

Now consider the L^2 gradient flow of the functional E, called the harmonic map heat flow. The equation for this problem is

$$\frac{\partial u}{\partial t} = \Delta u + \operatorname{trace} A(u)(\nabla u, \nabla u). \tag{1}$$

Obviously, weakly harmonic maps give rise to weak solutions of this equation, thus we cannot expect better regularity results than for the elliptic problem. In particular, in order to obtain any regularity at all, we need to impose additional assumptions. Various conditions have been formulated in this context, all with the aim to use a certain monotonicity formula (found by Struwe [12] for smooth solutions).

We do not state the formula here, but rather one of its consequences. Consider the time interval $(-1, 1)$ for simplicity. Suppose that

$$d^*((t_1, x_1), (t_2, x_2)) = \max\left\{\sqrt{|t_1 - t_2|}, |x_1 - x_2|\right\}$$

is the parabolic metric on the space-time $B^* = (-1, 1) \times B$, and let $B_r^*(z_0)$ be the open ball of radius r about $z_0 = (t_0, x_0)$ with respect to d^*.

Definition 1. *Consider $u \in W^{1,2}(B^*; N)$. Suppose that there exists a constant $c_0 > 0$ such that for all $z_0 = (t_0, x_0) \in B^*$ and $r > 0$ with $B_r^*(z_0) \subset B^*$ and for all $z_1 = (t_1, x_1) \in B^*$ and $s > 0$ with $B_s^*(z_1) \subset B_{r/2}^*(z_0)$, the inequality*

$$s^{2-m}\left(\int_{B_s(x_1)} |\nabla u(t_1, x)|^2 \, dx + \int_{B_s^*(z_1)} \left|\frac{\partial u}{\partial t}\right|^2 dz\right) \leq c_0 r^{-m} \int_{B_r^*(z_0)} |\nabla u|^2 \, dz$$

holds true. Then we say that u satisfies a monotonicity inequality.

Using this inequality as an additional assumption, we can prove a partial regularity result for weak solutions of the harmonic map heat flow.

Theorem 2. *Suppose that $u \in W^{1,2}(B^*; N)$ is a weak solution of* (1) *satisfying a monotonicity inequality. Then there exists a set $S \subset B^*$ that is closed relative to B^*, such that the m-dimensional Hausdorff measure of S with respect to the metric d^* vanishes, and such that $u \in C^\infty(B^* \backslash S)$.*

For the proof of this result, we mostly work on time slices $\{t\} \times B$ and regard $\frac{\partial u}{\partial t}$ as a perturbation of the harmonic map equation. Thus we study an equation of the form

$$\Delta u + \operatorname{trace} A(u)(\nabla u, \nabla u) = f, \tag{2}$$

where we assume that the right-hand side f belongs to $L^p(B; \mathbb{R}^n)$ for some $p > 1$. We want to show that this equation implies regularity under a certain smallness condition, expressed in terms of a Morrey space. More precisely, we require that the norm

$$\|\nabla u\|_{M^{2,2}(B)} = \sup_{x_0 \in B} \sup_{r > 0} \left(r^{2-m} \int_{B \cap B_r(x_0)} |\nabla u|^2 \, dx\right)^{1/2}$$

is sufficiently small.

Theorem 3. *For every $p \in (1, \infty)$ there exists a number $\epsilon > 0$ with the following property. Suppose that $u \in W^{1,2}(B; N)$ and $f \in L^p(B; \mathbb{R}^n)$ satisfy* (2) *weakly in B. If $\|\nabla u\|_{M^{2,2}(B)} \le \epsilon$, then $u \in W^{2,p}_{\mathrm{loc}}(B; \mathbb{R}^n) \cap W^{1,2p}_{\mathrm{loc}}(B; N)$.*

Using this result, it is not too difficult to show that under the assumptions of Theorem 2, an inequality of the form

$$r^{-m} \int_{B_r^*(z_0)} |\nabla u|^2 \, dz \le \epsilon$$

for a sufficiently small $\epsilon > 0$ implies regularity near z_0. Using a standard covering argument, we can thus reduce Theorem 2 to Theorem 3.

The proof of Theorem 3 relies in part on the arguments of Rivière and Struwe [11], but it is not obvious how to control the additional term f with this method. We solve a sequence of auxiliary problems for this purpose, subtracting the solution from u in each step. Passing to the limit, we then obtain an equation that permits the use of known methods.

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Finite time singularities of the Ricci flow

Reto Müller
(joint work with Robert Haslhofer, Carlo Mantegazza)

A smooth compact solution $(M^n, g(t))$ of Hamilton's Ricci flow $\partial_t g(t) = -2\mathrm{Rc}_{g(t)}$ on a time interval $t \in [0,T)$ develops a finite time singularity at $T < \infty$ if and only if the Riemannian curvature tensor Rm satisfies

$$\sup_M |\mathrm{Rm}(\cdot,t)|_{g(t)} \geq \frac{1}{8(T-t)}, \quad \forall t \in [0,T).$$

If there exists a constant C_{I} such that we have in addition

$$\sup_M |\mathrm{Rm}(\cdot,t)|_{g(t)} \leq \frac{C_{\mathrm{I}}}{T-t}, \quad \forall t \in [0,T),$$

then the Ricci flow is said to be of *Type I*, otherwise it is said to be of *Type II*. A natural line to study finite time singularities is to take blow–ups based at a fixed (singular) point $p \in M$. In this talk, we describe how this can be done in the Type I case using Perelman's $\mathcal{W}$–entropy functional

$$\mathcal{W}(g,f,\tau) := \int_M \Big(\tau(\mathrm{R}_g + |\nabla f|_g^2) + f - n\Big)\frac{e^{-f}}{(4\pi\tau)^{n/2}}dV_g.$$

We let $\tau(t) := T - t$ be the remaining time to the finite time singularity and we choose $f(\cdot,t) = f_{p,T}(\cdot,t)$ in such a way that $u_{p,T}(\cdot,t) := \frac{e^{-f_{p,T}(\cdot,t)}}{(4\pi\tau)^{n/2}}$ is an *adjoint heat kernel based at the singular time* (p,T), that is, a locally smooth limit of solutions of the backward parabolic equation $\frac{\partial}{\partial t}u = -\triangle u + \mathrm{R}_g u$, all converging (as distributions) to a Dirac δ–measure at $p \in M$ at times closer and closer to the singular time T (see [3] for a precise definition and an existence proof). From Perelman's entropy formula and Li–Yau–Harnack type inequality [6], we obtain the following monotonicity and nonpositivity result.

Proposition 1 (Monotonicity of $\mathcal{W}(g(t), f_{p,T}(t), \tau(t))$, see [3])**.** *Letting $f_{p,T}(t) : M \to \mathbb{R}$ be as above, we define $\theta_{p,T}(t) := \mathcal{W}(g(t), f_{p,T}(t), \tau(t))$. Then, $\theta_p : [0,T) \to \mathbb{R}$ is nonpositive and non–decreasing along the Ricci flow with derivative*

$$\partial_t \theta_{p,T}(t) = 2\tau \int_M \Big|\mathrm{Rc}_{g(t)} + \nabla^2 f_{p,T}(t) - \frac{g(t)}{2\tau}\Big|^2_{g(t)} \frac{e^{-f_{p,T}(t)}}{(4\pi\tau)^{n/2}} dV_{g(t)} \geq 0.$$

Now, define the pointed flow $(M, \hat{g}(s), p)$ by

$$\hat{g}(s) := \frac{g(t)}{T-t}, \quad s(t) := -\log(T-t) \in [-\log T, +\infty).$$

This so-called *dynamical blow–up* satisfies the evolution equation

$$\partial_s \hat{g}(s) = -2\mathrm{Rc}_{\hat{g}(s)} + \hat{g}(s).$$

A simple rescaling argument shows that

$$\lim_{j\to\infty} \int_j^{j+1} \int_M |\mathrm{Rc}_{\hat{g}(s)} + \nabla^2 \hat{f}(s) - \frac{\hat{g}(s)}{2}|^2_{\hat{g}(s)} e^{-\hat{f}(s)} dV_{\hat{g}(s)}\, ds = 0, \tag{1}$$

where $\hat{f}(s) = f_{p,T}(t(s))$ and $s(t) = -\log(T-t)$ as before. From (1), we obtain the following main result.

Theorem 2 (Blow–ups at Type I singularities, see [3]). *Let $(M, g(t))$ be a compact singular Type I Ricci flow, $p \in M$, and $f_{p,T}(t)$ as above. Let $(M, \hat{g}(s), p)$ be the dynamical blow–up and $\hat{f}(s) = f_{p,T}(t)$ as before. Then there exist $s_j \to \infty$ such that $(M, \hat{g}(s_j), \hat{f}(s_j), p)$ converges smoothly in the pointed Cheeger–Gromov sense to a normalized gradient shrinking Ricci soliton $(M_\infty, g_\infty, f_\infty, p_\infty)$, that is, a complete Riemannian manifold (M_∞, g_∞) satisfying*

$$\mathrm{Rc}_{g_\infty} + \nabla^2 f_\infty = \frac{g_\infty}{2},$$

where $f_\infty : M_\infty \to \mathbb{R}$ is a smooth function with $\int_{M_\infty} \frac{e^{-f_\infty}}{(4\pi)^{n/2}} dV_{g_\infty} = 1$. Moreover, no entropy is lost in the limit process, i.e. we have

$$\begin{aligned}\mathcal{W}(g_\infty, f_\infty) &:= \int_{M_\infty} \big(\mathrm{R}_{g_\infty} + |\nabla f_\infty|^2_{g_\infty} + f_\infty - n\big) \frac{e^{-f_\infty}}{(4\pi)^{n/2}}\, dV_{g_\infty} \\ &= \lim_{j\to\infty} \int_M \big(\mathrm{R}_{\hat{g}(s_j)} + |\nabla \hat{f}(s_j)|^2_{\hat{g}(s_j)} + \hat{f}(s_j) - n\big) \frac{e^{-\hat{f}(s_j)}}{(4\pi)^{n/2}}\, dV_{\hat{g}(s_j)}.\end{aligned}$$

As a consequence of this, in the case where $p \in M$ is a singular point (i.e. there does not exist any neighborhood $U_p \ni p$ on which $|\mathrm{Rm}(\cdot,t)|_{g(t)}$ stays bounded as $t \to T$), the limit soliton is non–flat.

A related blow–up theorem has previously been proved by Naber [5] and Enders–Müller–Topping [1] using a different method. In their works, the soliton potential function is obtained as a limit of a version of Perelman's reduced length *based at the singular time T*.

A key ingredient in the proof of the main theorem are the following upper and lower Gaussian bounds for the adjoint heat kernels based at the singular time. We have

$$\hat{C} e^{-d^2_{g(t)}(p,q)/\hat{C}(T-t)} \leq e^{-f_{p,T}(q,t)} \leq \bar{C} e^{-d^2_{g(t)}(p,q)/\bar{C}(T-t)}$$

for every adjoint heat kernel $f_{p,T}$ and $(q,t) \in M \times [0,T)$, where $\hat{C}$, $\bar{C}$ are positive constants depending only on n and the Type I constant C_{I}. The upper bound follows from a very recent result of Hein and Naber [4].

Finally, we know that the space of singularity models obtained by a blow–up procedure as in the main theorem above is compact. In fact, we can prove a precompactness result (allowing orbifold–singularities) even in the general case without Type I curvature bounds.

Theorem 3 (Orbifold–compactness of singularity models, see [2]). *Let (M_i^n, g_i, f_i) be a sequence of normalized gradient shrinking solitons with entropy uniformly bounded below, $\mathcal{W}(g_i, f_i) \geq \underline{\mu} > -\infty$, and uniform local energy bounds,*

$$\int_{B_r(p_i)} |\mathrm{Rm}_{g_i}|^{n/2}_{g_i} dV_{g_i} \leq E(r) < \infty, \quad \forall i, r. \tag{2}$$

Then a subsequence of (M_i^n, g_i, f_i, p_i) *converges to an orbifold gradient shrinker in the pointed orbifold Cheeger-Gromov sense. Moreover, in dimension* $n = 4$, *the energy bounds (2) follow from an upper bound on the Euler–characteristic* $\chi(M_i) \leq \bar{\chi} < \infty$ *under a technical assumption for the soliton potentials* f_i.

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Almost sure existence of global weak solutions for supercritical Navier-Stokes equations

Andrea R. Nahmod

(joint work with Nataša Pavlović, Gigliola Staffilani)

Consider a viscous, homogenous, incompressible fluid with velocity $\vec{u}$ on $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, d=2, 3 and which is not subject to any external force. Then the initial value problem for the Navier-Stokes equations is given by

$$\text{(NSEp)} \qquad \begin{cases} \vec{u}_t + \vec{v} \cdot \nabla \vec{u} = -\nabla p + \nu \Delta \vec{u}; \quad x \in \Omega \ t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{u}_0(x), \end{cases}$$

where $0 < \nu$ =inverse Reynolds number (non-dimensional viscosity); $\vec{u} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$, $p = p(x,t) \in \mathbb{R}$ and $\vec{u}_0 : \Omega \to \mathbb{R}^d$ is divergence free. For smooth solutions it is well known that the pressure term p can be eliminated via Leray-Hopf projections and view (NSEp) as an evolution equation of $\vec{u}$ alone. The mean of $\vec{u}$ is easily seen to be an invariant of the flow (conservation of momentum) so can reduce to the case of mean zero $\vec{u}_0$. Then the incompressible Navier-Stokes equations (NSEp) (assume $\nu = 1$) can be expressed as

$$\text{(NSE)} \qquad \begin{cases} \vec{u}_t = \Delta \vec{u} - \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{u}); \quad x \in \Omega, \quad t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{u}_0(x), \end{cases}$$

where $\mathbb{P}$ is the Leray-Hopf projection operator into divergence free vector fields. By Duhamel's formula we have

$$\text{(NSEi)} \qquad \vec{u}(t) = e^{t\Delta}\vec{u}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u})\,ds$$

In fact, under suitable general conditions on $\vec{u}$ the three formulations (NSEp), (NSE) and (NSEi) can be shown to be equivalent (weak solutions, mild solutions, integral solutions. Work by Leray, Browder, Kato, Lemarie, Furioli, Lemarié and Terraneo, and others). If the velocity vector field $\vec{u}(x,t)$ solves the Navier-Stokes equations in $\mathbb{R}^d$ or $\mathbb{T}^d$ then $\vec{u}_\lambda(x,t) := \lambda\vec{u}(\lambda x, \lambda^2 t)$, is also a solution for the initial data $\vec{u}_{0\,\lambda} = \lambda\vec{u}_0(\lambda x)$. In particular, $\|\vec{u}_\lambda(x,0)\|_{\dot{H}^{s_c}} = \|\vec{u}(x,0)\|_{\dot{H}^{s_c}}$, $s_c := \frac{d}{2}-1$ The spaces which are invariant under such a scaling are called critical spaces for Navier-Stokes. For example, $\dot{H}^{\frac{d}{2}-1} \hookrightarrow L^d \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{d}{p}} \hookrightarrow BMO^{-1}$ for $1<p<\infty$.

Classical solutions to the (NSE) satisfy the decay of energy which can be expressed as $\|u(x,t)\|_{L^2}^2 + \int_0^t \|\nabla u(x,\tau)\|_{L^2}^2\,d\tau = \|u(x,0)\|_{L^2}^2$. When $d=2$, the energy $\|u(x,t)\|_{L^2}$, which is globally controlled, is exactly the scaling invariant $\dot{H}^{s_c} = L^2$-norm. In this case the equations are said to be *critical*. Classical global solutions have been known to exist (Ladyzhenskaya 69'). When $d=3$, the global well-posedness/regularity problem of (NSE) is a long standing open question.The energy $\|u(x,t)\|_{L^2}$ is at the super-critical level with respect to the scaling invariant $\dot{H}^{\frac{1}{2}}$-norm, and hence the Navier-Stokes equations are said to be *super-critical*. The lack of a known bound for the $\dot{H}^{\frac{1}{2}}$ contributes in keeping the large data global well-posedness question still open.

One way of studying the initial value problem (NSE) is via weak solutions introduced by Leray. Indeed, Leray (1934) and Hopf (1951) showed existence of a global weak solution of the Navier-Stokes equations corresponding to initial data in $L^2(\mathbb{R}^d)$. Lemarié extended this construction and obtained existence of uniformly locally square integrable weak solutions. Questions addressing uniqueness and regularity of these solutions when $d=3$ have not been answered yet. But important contributions in understanding partial regularity and conditional uniqueness of weak solutions by many; see e.g. Caffarelli-Kohn-Nirenberg (82'); Struwe (88'-07'); Lin (98'); P.L. Lions -Masmoudi (98'), Seregin-Šverak (02') Escauriaza-Seregin-Šverak (03'); Vasseur (07'), others. Another approach is to construct solutions to the corresponding integral equation ('mild' solutions) pioneered by Kato and Fujita (1961). Mild solutions to the Navier-Stokes equations for $d\geq 3$ has been studied locally in time and globally for small initial data in various subcritical or critical spaces. Many references; see e.g. T. Kato (84'), Giga-Miyakawa (89'), Cannone (95'), Planchon (96'), H.Koch-Tataru (01'), Gallagher-Planchon (02'), Germain-Pavlovic-Staffilani (07'), Kenig-G. Koch (09'), others.

Periodic Navier-Stokes below L^2. Our goal is to show that after suitable data randomization there exists a large set of super-critical initial data, in $H^{-\alpha}(\mathbb{T}^d)$ for some $\alpha(d)>0$, for both 2d and 3d Navier-Stokes equations for which global energy bounds are proved. As a consequence we then obtain almost sure super-critical global weak solutions. In 2d these global weak solutions are unique.

In the periodic setting similar supercritical randomized well-posedness results were obtained for the 2D cubic NLS by Bourgain (96') and for the 3D cubic NLW by Burq and Tzvetkov (11')[1] This approach was applied in the context of the Navier-Stokes to obtain local in time and small data global solutions to the corresponding integral equation (NSEi) for randomized initial data in $L^2(\mathbb{T}^3)$ by Zhang and Fang (11') and by Deng and Cui (11'). The latter also obtained local in time solutions to (NSEi) for randomized initial data in $H^s(\mathbb{T}^d)$, for $d = 2, 3$ with $-1 < s < 0$.

We are concerned with existence of global in time weak solutions to (NSE) for randomized initial data (without any smallness assumption) in negative Sobolev spaces $H^{-\alpha}(\mathbb{T}^d), d = 2, 3$, for some $\alpha = \alpha(d) > 0$. Roughly, we start with a divergence free and mean zero initial data $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d, d = 2, 3$ and suitably randomize it to obtain $\vec{f}^\omega$ preserving the divergence free condition. The key point is that although the initial data is in $H^{-\alpha}$ for some $\alpha > 0$, the heat flow of the randomized data gives almost surely improved L^p bounds. These bounds yield improved nonlinear estimates arising in the analysis of the difference equation for $\vec{w}$ almost surely. The induced 'nonlinear smoothing' phenomena -akin to the role of Kintchine inequalities in Littlewood-Paley theory- stems from classical results of Rademacher, Kolmogorov, Paley and Zygmund proving that random series on the torus enjoy better L^p bounds than deterministic ones. Indeed, consider the example of Rademacher series $f(\tau) := \sum_{n=0}^{\infty} a_n\, r_n(\tau) \quad \tau \in [0, 1)$, $a_n \in \mathbb{C}$ where $r_n(\tau) := \operatorname{sign} \sin(2^{n+1}\pi\, \tau)$, $n \geq 0$. Then $r_{k,j}(\tau) := r_k(\tau) r_j(\tau)$, $0 \leq k < j < \infty$ is o. n. over $(0, 1)$. It is then a classical theorem (cf. Zygmund Vol I), that if $a_n \in \ell^2$ then the sum $f(\tau)$ converges a.e. and furthermore that the sum $f(\tau)$ belongs to $L^p([0, 1))$ for all p. More precisely, $(\int_0^1 |f|^p \, d\tau)^{1/p} \approx_p \| a_n \|_{\ell^2}$.

These ideas were already exploited in Bourgain's work on NLS, KdV, mKdV, Zakharov system, where global in time existence was obtained almost surely on the statistical ensemble via the existence and invariance of the associated Gibbs measure (after Lebowitz, Rose and Speer's and Zhidkov's works). The starting point of this method is a good local theory on the statistical ensemble (support of the measure) which consists precisely of randomized data of the form $\phi = \phi^\omega(x) = \sum \frac{g_n(\omega)}{|n|^\alpha} e^{i\langle x,n\rangle}$, where $\{g_n(\omega)\}_n$ are independent standard (complex/real) Gaussian random variables on a probability space defining almost surely in ω a function in $H^{\alpha-\frac{d}{2}-\epsilon}$ but not in $H^{\alpha-\frac{3}{2}}$ (α chosen appropriately depending on the Hamiltonian equation). Then almost surely in ω the nonlinear part $(= u - S(t)\phi)$ is showed to be smoother than the linear part. When an invariant measure is available, this is used in lieu of a conserved quantity to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely. Some recent works (after Bourgain's above) include those by Tzvetkov, Burq-Tzvetkov; T. Oh; Nahmod-Oh-Rey Bellet-Staffilani; Nahmod-Rey Bellet-Sheffield-Staffilani; Colliander-Oh, etc. for certain dispersive PDE (NLW,

[1] Burq-Tzvetkov (08') results on compact Riemannian manifolds for 3D radial cubic NLW.

KdV, NLS, etc.) where in the absence of an invariant measure other methods (energy, Bourgain's high-low, etc.) have been adapted.

We seek a solution to the initial value problem (NSE) in the form $\vec{u} = e^{t\Delta}\vec{f}^{\omega} + \vec{w}$ and identify the difference equation that $\vec{w}$ should satisfy. The heat flow of the suitably randomized data gives improved L^p bounds[2] almost surely. These bounds yield improved nonlinear estimates in the analysis of the difference equation for $\vec{w}$ almost surely. We first revisit the proof of equivalence between the initial value problem for the difference equation and the integral formulation of it in our context . We then prove a priori energy estimates for $\vec{w}$. The integral equation formulation is used near time zero and the other one away from zero. A construction of a global weak solution to the difference equation via a Galerkin method is thus possible thanks to the *a priori* energy estimates for $\vec{w}$. We then prove uniqueness of weak solutions when $d = 2$ in the spirit of Ladyzhenskaya-Prodi-Serrin. Finally, we put all ingredients together to conclude. Our main results can be stated as follows:

Theorem [Existence and Uniqueness in 2D]. Fix $T > 0$, $0 < \alpha < \frac{1}{2}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^2))^2$, $\nabla \cdot \vec{f} = 0$ and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^{\omega}$ has a unique global weak solution $\vec{u}$ of the form

$$\vec{u} = \vec{u}_{\vec{f}^{\omega}} + \vec{w}$$

where $\vec{u}_{\vec{f}^{\omega}} = e^{t\Delta}\vec{f}^{\omega}$ and $\vec{w} \in L^{\infty}([0,T];(L^2(\mathbb{T}^2))^2) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^2))^2)$.

Theorem [Existence in 3D]. Fix $T > 0$, $0 < \alpha < \frac{1}{3}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^3))^3$, $\nabla \cdot \vec{f} = 0$, and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^{\omega}$ has a global weak solution $\vec{u}$ of the form

$$\vec{u} = \vec{u}_{\vec{f}^{\omega}} + \vec{w},$$

where $\vec{u}_{\vec{f}^{\omega}} = e^{t\Delta}\vec{f}^{\omega}$ and $\vec{w} \in L^{\infty}([0,T];(L^2(\mathbb{T}^3))^3) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^3))^3)$.

Global existence for models from the Euler-Maxwell equation

Benoît Pausader
(joint work with Alexandru D. Ionescu)

We consider a line of research concerned with global existence for quasilinear dispersive equations, assuming that the initial data are smooth small and localized. In this context, even under such stringent conditions on the initial data, global existence may simply fail.

This study was aparently initiated by F. John [14]. Key early contributions were then made by Klainerman [15] with the introduction of the vector field method and by Shatah [16] with the introduction of the normal form transformation. Many work followed from Ginibre-Velo, Ozawa, Sideris, Tsutsumi among others. Later,

[2] There is no H^s regularization, i.e. $\|\vec{f}^{\omega}\|_{H^s} \sim \|\vec{f}\|_{H^s}$

Germain-Masmoudi-Shatah [6] and Gustafson-Nakanishi-Tsai [11, 12] independently introduced a new point of view sometimes refered to as the "space-time resonance method". This method turned out to be very successful and allowed to prove stability of many physical systems of importance in a situation "close to flat".

The original motivation of our work started from a result of Sideris showing that *there exists solutions of the compressible Euler equations starting from arbitrarily small smooth and localized data which do not remain smooth globally (create a shock in finite time)* [17], which seemed to deem compressible fluids as unstable and not amenable to analysis in a smooth setting. However, Guo [8] showed that this unstable mecanism no longer persists when one considers charged fluids which self-interact through their electrostatic field. Indeed he showed that *3-dimensional small smooth neutral perturbations of a constant equilibrium for the Euler-Poisson systems for the electrons remain smooth globally and scatter back to equilibrium.* The neutral assumption was later removed [5], the result was extended to two dimensions [13].

The Euler-Poisson equation for electron is derived from the full Euler-Maxwell system (see [1]):

$$\begin{aligned}
&\partial_t n_e + \operatorname{div}(n_e v_e) = 0, \quad \partial_t n_i + \operatorname{div}(n_i v_i) = 0,\\
&m_e\left(\partial_t + v_e\cdot\nabla\right)v_e = -\nabla p_e(n_e) - e\left[E + \frac{v_e}{c}\times B\right],\\
&M_i\left(\partial_t + v_i\cdot\nabla\right)v_i = -\nabla p_i(n_i) + e\left[E + \frac{v_i}{c}\times B\right],\\
&\partial_t B + c\nabla\times E = 0, \quad \operatorname{div}(B) = 0\\
&\partial_t E - c\nabla\times B = 4\pi e\left[n_e v_e - n_i v_i\right], \quad \operatorname{div}(E) = 4\pi e\left[n_i - n_e\right]
\end{aligned}$$

which represents the dynamics of a compressible fluid of electrons (resp ions) of mass m_e, density n_e and velocity v_e (resp. M_i, n_i and v_i) subject to the forces created by an electromagnetic field (E, B) originating from the motion of charge and current. Here c denotes the speed of light and p_e and p_i are the respective pressure associated to the fluids. We consider the adiabatic case when this pressure is a function of the density. The goal is to study whether or not the flat equilibrium $(n_e, v_e, n_i, v_i, E, B) = (n_0, 0, n_0, 0, 0, 0)$ is stable in the sense that small perturbations will generate a global dynamics that returns to equilirium.

After scaling, we discover that this system only depends on three dimensionless parameters: the ratio of the inertia of the fluids: $\varepsilon = m_e/M_i$, the ratio of the sound speeds $T = p_e'(n_0)/p_i'(n_0)$ and the ratio of the speed of light with these velocities $C = c/\sqrt{p_e'(n_0)p_i'(n_0)}$. In many situations $C >> 1$ and setting $C = +\infty$ drops the magnetic field and defines the "electrostatic approximation". In practice $\varepsilon << 1$ and setting it to 0 leads to the study of "one fluid" problems.

The electrostatic one fluid problems have been shown to have stable equilibrium [8, 9], while the electron Euler-Maxwell problem has been treated in [4]. Leaving the electrostatic one fluid problem yields many new complications and to understand them, a nice toy model has been singled out in [3]: the system of

Klein-Gordon equations

$$(1)\qquad \left(\partial_{tt} - c_\sigma^2 \Delta + m_\sigma^2\right) u_\sigma = Q_\sigma(\vec{u}, D\vec{u}, D^2\vec{u})$$

for a vector $\vec{u}$, where Q denotes a general quadratic nonlinearity. This system had been studied previously in [2] when the velocities are the same $c_\mu = 1$, $\forall\mu$ and in [3] when the masses are the same $m_\sigma = 1$, $\forall\sigma$, in the semilinear case.

In order to understand better the stability problem for the full Euler-Maxwell system, it is convenient to understand the case of systems of Klein-Gordon equation in a more general case. In this direction, in a joint work with A. Ionescu, we obtain

Theorem 1. *Assume that*

$$m_{\sigma_1} \neq m_{\sigma_2} + m_{\sigma_3}, \quad (c_{\sigma_1} - c_{\sigma_2})(c_{\sigma_1}^2 m_{\sigma_1} - c_{\sigma_2}^2 m_{\sigma_2}) \geq 0, \ \forall \sigma_1, \sigma_2, \sigma_3,$$

and fix any initial data $\vec{u}(0) = \{u_\sigma(0)\}$ smooth and compactly supported, then there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$ the initial data $\delta\vec{u}$ yields a global solution of (1) *which is global and scatters.*

The conditions on the parameters are probbaly not optimal. However some condition is necessary for our method to run.

This result is obtained following a method similar to the "space-time resonance method" as in [4, 6, 12], but with several crucial modifications to handle some of the worst bilinear interactions. The starting point is to conjugate out the linear flow by considering the linear profiles. Then, one tries to control the profiles using energy estimates and dispersion estimate to bound localization norms; we thus focus on control in H^N and in $L^2(x^\alpha dx)$. By dispersion, terms which remain bounded in $L^2(x^{1+\varepsilon}dx)$ can be managed easily. These are the "strong" components. Unfortunately, some of the interactions seem to unavoidably produce terms just outside of these spaces. Taking this into account, we also add a component of our solution with similar properties, except that it only leaves in $L^2(x^{1-\varepsilon}dx)$, which is not enough, but has the redeeming feature of being essentially supported on a two dimensional set. These are the "weak" components.

Besides the energy estimates which are quite standard, our dispersive analysis consists of three different parts. First we quantify position and momentum, and decompose all the interactions with respect to the localization of both intput and the output. This gives a gigantic sum. We then proceed to remove most of the terms where the interaction is inefficient. At the end of this first part, we are left with interactions at a time T, where the output is located at $X \simeq T$ and the inputs are located at $Y \lesssim T$. The second part consists of a careful analysis of the interactions in the case $Y \lesssim T^{1/2}$ when no assumption on the integrability can be of much help (in particular, one may very well have $Y \simeq 1$; on the other hand, in this region, one can integrate by parts efficiently as in stationary phase estimates). This uncovers the "weak" component alluded to above, but also reveals its particular two dimensional structure and the fact that it only barely fails to be "strong". The second part of the analysis deals with the case $T^{1/2} \lesssim Y \lesssim T$ in which case, we verify that the information already obtained is sufficient to output "strong" terms all the time (in this case, the bootstrap information about the norms helps, while the integrations by parts become less and less efficient).

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Flowing to minimal surfaces

Melanie Rupflin
(joint work with Peter Topping)

We introduce a geometric flow of maps from surfaces which has elements in common with both the harmonic map flow and the mean curvature flow, but is more effective at finding minimal surfaces.

Let M be a smooth closed orientable surface of arbitrary genus $\gamma \geq 0$ and let (N, G) be a smooth compact Riemannian manifold of any dimension. We assign to each pair (u, g) consisting of a smooth map $u : M \to N$ and a smooth metric g on the domain M, its Dirichlet energy

$$E(u, g) := \frac{1}{2} \int_M |du|^2 d\mu_g$$

and recall that (u, g) is a critical point of E if and only if u is a weakly conformal harmonic map on (M, g), and thus, if non-constant, a branched minimal immersion.

Taking into account the symmetries of the energy, that is its invariance under conformal variations of the domain as well as under the pull-back by diffeomorphisms (applied to both components), we introduce a natural gradient flow of E of equivalence classes $[(u, g)]$ of pairs of map and domain metric. This flow may be represented by the system

$$\begin{aligned} \partial_t u &= \tau_g(u) \\ \frac{dg}{dt} &= \mathrm{Re}(P_g(\Phi(u, g))) \end{aligned} \tag{1}$$

to be solved in the space of maps $u : M \to N$ and metrics $g \in \mathcal{M}_c$ of constant curvature $c = 1, 0, -1$ for surfaces of genus $\gamma = 0, 1, \geq 2$ (with unit area if $\gamma = 1$). Here $\tau_g(u)$ is the tension field and $\Phi(u, g)$ the Hopf-differential of $u : (M, g) \to (N, G)$. Furthermore P_g denotes the L^2-orthogonal projection from the space of quadratic differentials onto the finite dimensional subspace of *holomorphic* quadratic differentials on (M, g).

We remark that in the special case of M being a sphere the equation (1) is just the harmonic map flow while for M a torus (1) agrees with a flow introduced by Ding-Li-Liu [1] from a somewhat different viewpoint and its metric component reduces to an equation on a 2 dimensional submanifold of $\mathcal{M}_0$.

In general, we obtain

Theorem 1 ([5]). *For any initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_c$ there exists a (weak) solution (u, g) of (1), smooth away from finitely many times T_i and with non-increasing energy, which is defined for all times, unless the metrics $g(t)$ degenerate in moduli space as t approaches a time $T < \infty$, that is unless the length $\ell(g(t))$ of the shortest closed geodesic of $(M, g(t))$ converges to zero as $t \nearrow T$.*

Furthermore, this solution is uniquely determined by its initial data in the class of all weak solutions with non-increasing energy.

The singularities (before a possible time of degeneration) of such a solution are caused by the bubbling off of harmonic spheres at a finite number of points as described by Struwe [6] for the harmonic map flow. Away from these points the map u remains regular while the metric component is Lipschitz-continuous in time on all of M (w.r.t. any C^m metric in space) across the singular time.

The main difficulty in the proof of this result is that for surfaces of genus $\gamma \geq 2$, the metric g moves within the full space $\mathcal{M}_{-1}$ of hyperbolic metrics. This demands that the projection operator P_g which takes the form

$$\mathrm{Re}(P_g(\Psi)) = \mathrm{Re}(\Psi) - \mu(g,\Psi)\cdot g - L_{X(g,\Psi)}g,$$

with X and μ determined as solutions of elliptic PDEs to be solved on the varying Riemannian surface (M,g), needs to be analysed in the full space $\mathcal{M}_{-1}$. In [5] we do this by combining ideas from Teichmüller theory with results on elliptic PDEs.

Given a global solution of the flow whose metric does not degenerate in moduli space even as $t \to \infty$ we obtain asymptotic convergence (up to reparametrisation) to branched minimal immersions.

Theorem 2 ([4])**.** *In the setting of Theorem 1, if the length $\ell(g(t))$ of the shortest closed geodesic of $(M, g(t))$ is uniformly bounded below by a positive constant, then there exist a sequence of times $t_i \to \infty$ and a sequence of orientation-preserving diffeomorphisms $f_i : M \to M$ such that*

$$f_i^* g(t_i) \to \bar{g} \;\text{ and }\; u(t_i) \circ f_i \to \bar{u}$$

converge to a metric $\bar{g} \in \mathcal{M}_c$ and a map $\bar{u}$ with the same action on $\pi_1(M)$ as u_0 which, if non-constant, is a branched minimal immersion.

Here the convergence of metrics is smooth, while the maps converge weakly in $H^1(M,N)$ and strongly in $W^{1,p}_{loc}(M\backslash S)$ for any $p \in [1,\infty)$ away from a finite set of points where energy concentrates.

For incompressible initial maps u_0, that is maps with injective action on π_1, a degeneration of the metric component both at finite and infinite time can be excluded and we recover the results of Schoen-Yau [3] and Sacks-Uhlenbeck [2] concerning the existence of branched minimal immersions with given incompressible action on π_1 with a flow approach. For further details on these results, their proofs and the construction of the flow we refer to [4].

An important tool in the proof of Theorem 2, and a result that is of independent interest, is a Poincaré-type estimate for quadratic differentials of the form

$$\|\Psi - P_g(\Psi)\|_{L^1(M,g)} \leq C \cdot \|\partial_{\bar{z}}\Psi\|_{L^1(M,g)}$$

that bounds the distance of any quadratic differential to its holomorphic part in terms of the antiholomorphic derivative.

Contrary to the Poincaré estimate for functions, we will show in future work that this estimate is *uniform*, valid for all closed hyperbolic surfaces (M,g) and all quadratic differentials Ψ on (M,g) with a constant depending only on the topology of the surface, that is only on the genus.

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Mean curvature flow of complete graphs

Oliver Schnürer
(joint work with Mariel Sáez)

We study the evolution of complete graphs under mean curvature flow. This is illustrated by three examples (Figure 1):

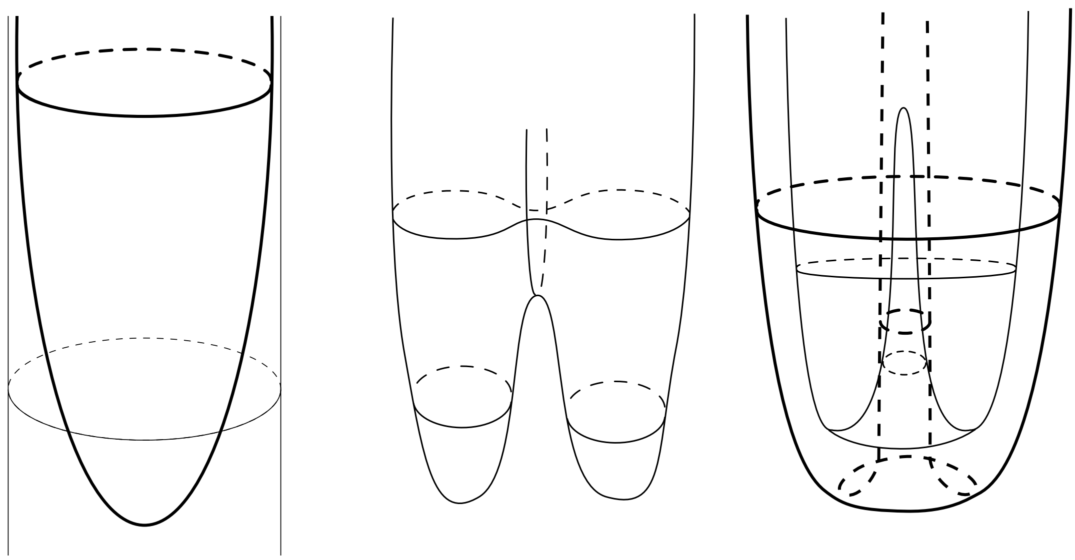

Figure 1. Examples of evolution

(1) Picture on the left: A graph (thick) inside a cylinder (thin) disappears to infinity at the time the cylinder collapses.
(2) Picture in the middle: The middle part of the 4-dimensional graph disappears to infinity and avoids the formation of a neck-pinch.
(3) Picture on the right: Before the cylinder inside the surface (thick) degenerates to a line, a "cap at infinity" is being added to the surface that moves

downwards very quickly. The thin surface depicts the surface shortly after that.

If $u_0 \colon \Omega_0 \to \mathbb{R}$ is locally Lipschitz, defined on a bounded domain $\Omega_0 \subset \mathbb{R}^{n+1}$, $u_0(x) \to \infty$ as $x \to \partial\Omega_0$, and u_0 is bounded below, then there exists a maximal smooth solution u to graphical mean curvature flow with initial value u_0.

The orthogonal projections $\mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ of the evolving graphs yield a level set solution to mean curvature flow.

This allows to define weak solutions as projections of classical graphical solutions.

Uniqueness of compact tangent flows in Mean Curvature Flow

Felix Schulze

In this work we study Mean Curvature Flow (MCF) of n-surfaces of codimension $k \geq 1$ in $\mathbb{R}^{n+k}$, which are close to self-similarly shrinking solutions. In the smooth case we consider a family of embeddings $F : M^n \times (t_1, t_2) \to \mathbb{R}^{n+k}$, for M^n closed, such that

$$\frac{d}{dt} F(p,t) = \vec{H}(p,t) \ ,$$

where $\vec{H}(p,t)$ is the mean curvature vector of $M_t := F(M,t)$ at $F(p,t)$. We denote with $\mathcal{M} = \bigcup_{t \in (t_1,t_2)} (M_t \times \{t\}) \subset \mathbb{R}^{n+k} \times \mathbb{R}$ its space-time track.

In the following, let Σ^n be a smooth, closed, embedded n-surface in $\mathbb{R}^{n+k}$ where the mean curvature vector satisfies

$$\vec{H} = -\frac{x^\perp}{2}.$$

Here x is the position vector at a point on Σ and $^\perp$ the projection to the normal space of Σ at that point. Such a surface gives rise to a self-similarly shrinking solution $\mathcal{M}_\Sigma$, where the evolving surfaces are given by

$$\Sigma_t = \sqrt{-t} \cdot \Sigma, \quad t \in (-\infty, 0).$$

We denote its space-time track by $\mathcal{M}_\Sigma$.

We also want to study the case that the flow is allowed to be non-smooth. Following [5], we say that a family of Radon measures $(\mu_t)_{t \in [t_1,t_2)}$ on $\mathbb{R}^{n+k}$ is an integral n-Brakke flow, if for almost every t the measure μ_t comes from a n-rectifiable varifold with integer densities. Furthermore, we require that given any $\phi \in C^2_c(\mathbb{R}^{n+k}; \mathbb{R}^+)$ the following inequality holds for every $t > 0$

$$\bar{D}_t \mu_t(\phi) \leq \int -\phi |\vec{H}|^2 + \langle \nabla \phi, \vec{H} \rangle \, d\mu_t, \tag{1}$$

where $\bar{D}_t$ denotes the upper derivative at time t and we take the left hand side to be $-\infty$, if μ_t is not n-rectifiable, or does not carry a weak mean curvature. Note that if M_t is moving smoothly by mean curvature flow, then $\bar{D}_t$ is just the usual derivative and we have equality in (1).

We restrict to integral n-Brakke flows which are close to a smooth self-similarly shrinking solution. The assumption that the Brakke flow is close in measure to a smooth solution with multiplicity one actually yields that the Brakke flow has unit density. This implies that for almost all t the corresponding Radon measures can be written as

$$\mu_t = \mathcal{H}^n \llcorner M_t \,.$$

Here M_t is a n-rectifiable subset of $\mathbb{R}^{n+k}$ and $\mathcal{H}^n$ is the n-dimensional Hausdorff-measure on $\mathbb{R}^{n+k}$. If the flow is (locally) smooth, then M_t can be (locally) represented by a smooth n-surface evolving by MCF. Conversely, if M_t moves smoothly by MCF, then $\mu_t := \mathcal{H}^n \llcorner M_t$ defines a unit density n-Brakke flow.

Theorem 1. *Let $\mathcal{M} = (\mu_t)_{t\in(t_1,0)}$ with $t_1 < 0$ be an integral n-Brakke flow such that*

i) *$(\mu_t)_{t\in(t_1,t_2)}$ is sufficiently close in measure to $\mathcal{M}_\Sigma$ for some $t_1 < t_2 < 0$.*

ii) *$\Theta_{(0,0)}(\mathcal{M}) \geq \Theta_{(0,0)}(\mathcal{M}_\Sigma)$, where $\Theta_{(0,0)}(\cdot)$ is the respective Gaussian density at the point $(0,0)$ in space-time.*

Then $\mathcal{M}$ is a smooth flow for $t \in [(t_1+t_2)/2, 0)$, and the rescaled surfaces $\tilde{M}_t := (-t)^{-1/2} \cdot M_t$ can be written as normal graphs over Σ, given by smooth sections $v(t)$ of the normal bundle $T^\perp\Sigma$, with $|v(t)|_{C^m(T^\perp\Sigma)}$ uniformly bounded for all $t \in [(t_1+t_2)/2, 0)$ and all $m \in \mathbb{N}$. Furthermore, there exists a self-similarly shrinking surface Σ' with

$$\Sigma' = graph_\Sigma(v')$$

and

$$|v(t) - v'|_{C^m} \leq c_m (\log(-1/t))^{-\alpha_m}$$

for some constants $c_m > 0$ and exponents $\alpha_m > 0$ for all $m \in \mathbb{N}$.

The above theorem implies uniqueness of compact tangent flows as follows. Let the parabolic rescaling with a factor $\lambda > 0$ be given by

$$\mathcal{D}_\lambda : \mathbb{R}^{n+k} \times \mathbb{R} \to \mathbb{R}^{n+k} \times \mathbb{R}, (x,t) \mapsto (\lambda x, \lambda^2 t) \,.$$

Note that any Brakke flow $\mathcal{M}$ (smooth MCF) is mapped to a Brakke flow (smooth MCF), i.e. $\mathcal{D}_\lambda(\mathcal{M})$ is again a Brakke flow (smooth MCF).

Let (x_0, t_0) be a point in space-time and $(\lambda_i)_{i\in\mathbb{N}}, \lambda_i \to \infty$, be a sequence of positive numbers. If $\mathcal{M}$ is a Brakke flow with bounded area ratios, then the compactness theorem for Brakke flows (see [5, 7.1]) ensures that

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \to \mathcal{M}' \,, \tag{2}$$

where $\mathcal{M}'$ is again a Brakke flow. Such a flow is called a *tangent flow* of $\mathcal{M}$ at (x_0, t_0). Huisken's monotonicity formula ensures that $\mathcal{M}'$ is self-similarly shrinking, i.e. it is invariant under parabolic rescaling.

Corollary 2. *Let $\mathcal{M}$ be an integral n-Brakke flow with bounded area ratios, and assume that at $(x_0, t) \in \mathbb{R}^{m+k} \times \mathbb{R}$ a tangent flow of $\mathcal{M}$ is $\mathcal{M}_\Sigma$. Then this tangent flow is unique, i.e. for any sequence $(\lambda_i)_{i\in\mathbb{N}}$ of positive numbers, $\lambda_i \to \infty$ it holds*

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \to \mathcal{M}_\Sigma \,.$$

Until recently, other than the shrinking sphere and the Angenent torus [2] no further examples of compact self-similarly shrinking solutions in codimension one were known. However, several numerical solutions of D. Chopp [3] suggest that there are a whole variety of such solutions. In a recent preprint [7], N. Møller shows that it is possible to desingularize the intersection lines of a self-similarly shrinking sphere and the Angenent torus to obtain a new, compact, smoothly embedded self-similarly shrinking solution. In higher codimensions this class of solutions should be even bigger.

In a recent work of Kapouleas/Kleene/Møller [6] and X.H. Nguyen [8] non-trivial, non-compact, self-similarly shrinking solutions were constructed. In [4], G. Huisken showed that, under the assumption that the second fundamental form is bounded, the only solutions in the mean convex case are shrinking spheres and cylinders.

The analogous problem for minimal surfaces is the uniqueness of tangent cones. This was studied in [11, 1, 12], and, in the case of multiplicity one tangent cones with isolated singularities, completely settled by L. Simon in [9]. One of the main tools in the analysis therein is the generalisation of an inequality due to Łojasiewicz for real analytic functions to the infinite dimensional setting.

Also in the present work, this Simon-Łojasiewicz inequality for "convex" energy functionals on closed surfaces, plays a central role. We adapt several ideas from [9, 10]. We prove a smooth extension lemma for Brakke flows close to $\mathcal{M}_\Sigma$ and introduce the rescaled flow. Furthermore, we treat the Gaussian integral of Huisken's monotonicity formula for the rescaled flow as an appropriate "energy functional" on Σ and use the Simon-Łojasiewicz inequality to prove a closeness lemma. This lemma and the extension lemma are then applied to prove the main theorem and its corollary.

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A local result for two dimensional Ricci flow

Miles Simon

We consider the Ricci flow of two dimensional Riemannian manifolds without boundary. Assume that the solution is defined on $[0,T)$, is smooth and complete for all times, has no boundary and satisfies the following conditions at time zero: (a) $R(\cdot,0) \geq -1$ on $B_1(x_0,0)$ and (b) $vol(B_1(x_0,0)) \geq v_0 > 0$ ($R(x,t)$ refers to the scalar curvature at $x \in M$ at time t). Then (a_t) $R(\cdot,t) \geq -2$ on $B_{\frac{1}{2}}(x_0,t)$ and (b_t) $vol(B_{\frac{1}{2}}(x_0,t)) \geq \tilde{v}(v_0) > 0$, and (c_t) $R(\cdot,t) \leq \frac{c(v_0)}{t}$ on $B_{\frac{1}{2}}(x_0,t)$ for all $t \in [0, S(v_0)) \cap [0,T)$. This result differs from G.Perelman's Pseudolocality result (Theorem 10.3 of [1]) in that the initial ball of radius one is not necessarily almost euclidean. It allows (for example) initial balls which are cone like.

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Low regularity local wellposedness of Chern-Simons-Schrödinger

Paul Smith
(joint work with Baoping Liu and Daniel Tataru)

We consider the initial value problem for the Chern-Simons-Schrödinger system

$$\begin{cases} D_t\phi &= iD_\ell D_\ell \phi + ig|\phi|^2\phi \\ \partial_t A_1 - \partial_1 A_t &= -\operatorname{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2 - \partial_2 A_t &= \operatorname{Im}(\bar{\phi} D_1 \phi) \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2}|\phi|^2, \end{cases} \tag{1}$$

where $\phi = \phi(t,x)$ is a complex-valued function, the connection coefficients $A_\alpha = A_\alpha(t,x)$ are real-valued functions, the covariant derivatives D_α are defined in terms of the gauge potential A_α via

$$D_\alpha := \partial_\alpha + iA_\alpha,$$

and $(t,x) \in \mathbb{R} \times \mathbb{R}^2$. As the left hand sides of the last three equations of (1) are essentially obtained by commuting various covariant derivatives D_α, these equations are referred to as *curvature constraints*.

The system (1) is a basic model of Chern-Simons dynamics [11, 6, 7, 10]. Two quantities conserved by the flow are *charge*

$$M(\phi) := \int_{\mathbb{R}^2} |\phi|^2 dx$$

and *energy*

$$E(\phi) := \int_{\mathbb{R}^2} |D_x \phi|^2 - \frac{g}{2}|\phi|^4 dx.$$

As the scaling symmetry

$$\phi(t,x) \mapsto \lambda \phi(\lambda^2 t, \lambda x)$$

preserves charge, L^2_x is the critical space for (1).

The Chern-Simons-Schrödinger system exhibits gauge freedom in that (1) is invariant with respect to the transformations

$$\phi \mapsto e^{i\theta}\phi \qquad A_\alpha \mapsto A_\alpha + \partial_\alpha \theta$$

for real-valued functions $\theta(t,x)$. Therefore, in order for (1) to be well-posed, a gauge must be selected.

A classical choice is the *Coulomb gauge*, which is derived by imposing the constraint $\operatorname{div}(A_x) = 0$, where $\operatorname{div}(A_x) = \partial_1 A_1 + \partial_2 A_2$. Using the Coulomb gauge, [1] establishes local wellposedness in H^2 and presents conditions ensuring finite-time blowup. In low dimension, the Coulomb gauge has unfavorable high $\times$ high $\to$ low frequency interactions that are a serious obstacle to proving low-regularity wellposedness results. This motivates searching for a different gauge.

We adopt from [5] a variation of the Coulomb gauge called the *parabolic gauge* or *heat gauge*. The defining condition of the heat gauge is

$$\operatorname{div}(A_x) = A_t. \tag{2}$$

Combining (2) with the curvature constraints of (1) leads to the following set of evolution equations for A_t, A_1, and A_2:

$$\begin{aligned}
(\partial_t - \Delta)A_t &= -\partial_1 \operatorname{Im}(\bar{\phi} D_2 \phi) + \partial_2 (\bar{\phi} D_1 \phi) \\
(\partial_t - \Delta)A_1 &= -\operatorname{Im}(\bar{\phi} D_2 \phi) - \frac{1}{2}\partial_2 |\phi|^2 \\
(\partial_t - \Delta)A_2 &= \operatorname{Im}(\bar{\phi} D_1 \phi) + \frac{1}{2}\partial_1 |\phi|^2.
\end{aligned}$$

We initialize ϕ, A_t, A_1, A_2 at $t_0 = 0$ as follows:

$$\begin{cases}
\phi(0,x) &= \phi_0 \\
A_t(0,x) &= 0 \\
A_1(0,x) &= \frac{1}{2}\Delta^{-1}\partial_2 |\phi_0|^2(x) \\
A_2(0,x) &= -\frac{1}{2}\Delta^{-1}\partial_1 |\phi_0|^2(x).
\end{cases} \tag{3}$$

Our main result is the following.

Theorem 1. *For initial data $\phi_0 \in H^s(\mathbb{R}^2)$, $s > 0$, there is a positive time T, depending only upon $\|\phi_0\|_{H^s}$, such that (1), (2) with initial data (3) has a unique solution $\phi(t,x) \in C([0,T], H^s(\mathbb{R}^2))$. Moreover, $\phi_0 \mapsto \phi$ is Lipschitz continuous from $H^s(\mathbb{R}^2)$ to $C([0,T], H^s(\mathbb{R}^2))$.*

Remark 2. *As (1) is L^2-critical, Theorem 1 establishes local wellposedness over the entire subcritical range.*

Remark 3. *Using the Coulomb gauge,* [4] *establishes local existence in H^s for $s > 1/2$ and uniqueness in H^s for $s \geq 1$. Also, the methods of* [4] *establish an interaction Morawetz estimate for Chern-Simons-Schrödinger.*

We make some remarks regarding the proof of Theorem 1. Deserving special emphasis is the fact that we employ the heat gauge as opposed to one of the standard gauges. Letting $Q_{12}(\cdot,\cdot)$ denote the null form defined by

$$Q_{12}(f,g) = \mathrm{Im}(\partial_1 f \partial_2 g - \partial_2 f \partial_1 g),$$

we have, in the Coulomb gauge, the heuristic that

$$A_t \approx \Delta^{-1} Q_{12}(\bar{\phi}, \phi).$$

On the other hand, the heat gauge obeys the heuristic

$$A_t \approx H^{-1} Q_{12}(\bar{\phi}, \phi),$$

where H^{-1} in this context denotes evolution by the linear heat flow. The symbol H^{-1} provides extra decay at high modulation, which plays a crucial role in controlling the nonlinearities in the ϕ-evolution equation. Our approach also takes advantage of the Q_{12} null form structure, which appears thanks to the sign twists in (1). Refined bilinear Strichartz estimates, in the spirit of [2], for functions with restricted frequency support, also serve as important tools. Finally, we employ modifications of the U^p, V^p spaces [12, 8, 9] to handle various logarithmic losses.

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On the uniqueness of solutions to the periodic 3D Gross-Pitaevskii hierarchy

Vedran Sohinger
(joint work with Gigliola Staffilani)

We present a uniqueness result for solutions to the Gross-Pitaevskii hierarchy on the three-dimensional torus, under the assumption of an a priori spacetime bound. We show that this a priori bound is satisfied for factorized solutions to the hierarchy which come from solutions of the nonlinear Schrödinger equation. This establishes a periodic analogue of the uniqueness result on $\mathbb{R}^3$ previously proved by Klainerman and Machedon.

1. Setup of the problem and statement of our result

We are considering the Gross-Pitaevskii hierarchy, which is a sequence of functions $\Gamma(t) = (\gamma^{(k)}(t;\vec{x}_k;\vec{x}'_k))$ of functions $\gamma^{(k)} : \mathbb{R} \times \Lambda^k \times \Lambda^k \to \mathbb{C}$ satisfying the folllowing symmetry properties:

i) $\gamma^{(k)}(t,\vec{x}_k;\vec{x}'_k) = \overline{\gamma^{(k)}(t,\vec{x}'_k;\vec{x}_k)}$

ii) $\gamma^{(k)}(t,x_{\sigma(1)},\ldots,x_{\sigma(k)};x'_{\sigma(1)},\ldots,x'_{\sigma(k)}) = \gamma^{(k)}(t,x_1,\ldots,x_k;x'_1,\ldots,x'_k)$ for all $\sigma \in S_k$.

which solve the following infinite system of linear equations:

$$(1)\qquad \begin{cases} i\partial_t \gamma^{(k)} + (\Delta_{\vec{x}_k} - \Delta_{\vec{x}'_k})\gamma^{(k)} = \sum_{j=1}^k B_{j,k+1}(\gamma^{(k+1)}) \\ \gamma^{(k)}|_{t=0} = \gamma_0^{(k)}. \end{cases}$$

We are interested in the case of the spatial domain $\Lambda := \mathbb{T}^3$.

Here, $\Delta_{\vec{x}_k} := \sum_{j=1}^k \Delta_{x_j}, \Delta_{\vec{x}'_k} := \sum_{j=1}^k \Delta_{x'_j}$ and the *collision operator* $B_{j,k+1}$ is given by:

$$B_{j,k+1}(\gamma^{(k+1)})(\vec{x}_k;\vec{x}'_k) := Tr_{k+1}[\delta(x_j - x_{k+1}),\gamma^{(k+1)}](\vec{x}_k;\vec{x}'_k)$$

$$= \int_\Lambda dx_{k+1} \big(\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1})\big) \gamma^{(k+1)}(\vec{x}_k, x_{k+1}; \vec{x}'_k, x_{k+1})$$

We suppose that we are working in the class $\mathcal{A}$ of $\Gamma(t) = (\gamma^{(k)}(t))$, satisfying i), ii) as above, such that there exist continuous, positive functions $\sigma, f : \mathbb{R} \to \mathbb{R}$, such that, for all $k \in \mathbb{N}$, $j = 1, \ldots, k$, $t \in \mathbb{R}$, one has:

$$\int_{t-\sigma(t)}^{t+\sigma(t)} \|S^{(k,\alpha)} B_{j,k+1}(\gamma^{(k+1)})(s)\|_{L^2(\Lambda^k \times \Lambda^k)} ds \leq f^{k+1}(t). \tag{2}$$

The result that we prove is:

Theorem 1. *Solutions to the Gross-Pitaevskii hierarchy* (1) *are unique in the class* $\mathcal{A}$ *for* $\alpha > \frac{5}{4}$.

2. Main ideas of the proof

The proof of the uniqueness result is based on the following two results:

This first result resembles a “Strichartz-type estimate”, frequently used in the study of dispersive PDE.

Lemma 2. *(Spacetime estimate) For* $\alpha > \frac{5}{4}$, *there exists* $C_1 = C_1(\alpha) > 0$ *such that, for all* $k \in \mathbb{N}$, *and for all* $j \in \{1, \ldots, k\}$, *one has:*

$$\|S^{(k,\alpha)} B_{j,k+1} \mathcal{U}^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2([0,2\pi] \times \Lambda^k \times \Lambda^k)} \leq C_1 \|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\|_{L^2(\Lambda^{k+1} \times \Lambda^{k+1})}.$$

Here,

$$S^{(k,\alpha)} := \prod_{j=1}^k (1 - \Delta_{x_j})^{\frac{\alpha}{2}} (1 - \Delta_{x'_j})^{\frac{\alpha}{2}}$$

and

$$\mathcal{U}^{(k)}(t) \gamma_0^{(k)} := e^{it \sum_{j=1}^k \Delta_{x_j}} \gamma_0^{(k)} e^{-it \sum_{j=1}^k \Delta_{x'_j}}.$$

The second result states that the class $\mathcal{A}$ is non-empty. In particular, it contains the physically relevant factorized solutions which come from the nonlinear Schrödinger equation. Namely, if ϕ solves.

$$\begin{cases} i\partial_t \phi + \Delta \phi = |\phi|^2 \phi \text{ on } \mathbb{R} \times \Lambda \\ \phi|_{t=0} = \phi_0 \in H^\alpha(\Lambda). \end{cases} \tag{3}$$

then $\Gamma(t) = (|\phi\rangle\langle\phi|^{\otimes k}(t)) = \prod_{j=1}^k \phi(t, x_j) \overline{\phi(t, x'_j)}$ solves (1). Such solutions are called *factorized.*

Lemma 3. *(Verification of the spacetime bound) The factorized solution* $\Gamma(t) = (|\phi\rangle\langle\phi|^{\otimes k}(t))$ *satisfies the spacetime bound* (2) *for* $\alpha \geq 1$.

The proof of Lemma 2 is based on number theoretic techniques used to count lattice points, in the spirit of the work of Bourgain [1]. The two-dimensional analogue of this bound was proved in the work of Kirkpatrick, Schlein, and Staffilani [7] for $\alpha > \frac{3}{4}$. We expect that Lemma 2 can be improved, which would improve the result of the main theorem. This is part of an ongoing joint work of the authors with Philip Gressman.

An analogue of Lemma 3 on $\mathbb{R}^3$ was proved in the work of Klainerman and Machedon by the use of Strichartz estimates on $\mathbb{R}^3$. Due to the loss of derivatives one obtains in the periodic Strichartz estimates [1], the aforementioned argument does not carry through to $\mathbb{T}^3$. We can circumvent this difficulty by using multilinear estimates in atomic spaces, which were first introduced in the work of Koch and Tataru [5]. In particular, we use variants of these spaces adapted to the Schrödinger equation which were used in the context of the energy critical NLS on $\mathbb{T}^3$ by Herr, Tataru, and Tzvetkov [4]. The use of these spaces allows us to prove Lemma 3 when $\alpha \geq 1$.

Since (1) is linear, it suffices to show that, under the assumptions of Theorem 1, $\gamma_0 = 0$ implies that $\Gamma = 0$. We do this by using an iterated Duhamel expansion. Finally, we combine Lemma 2 and Lemma 3 with a Duhamel expansion and the combinatorial argument of Klainerman and Machedon [8] to deduce the main result.

Our uniqueness result is a potential first step in the program of the rigorous derivation of the nonlinear Schrödinger equation from N-body Schrödinger dynamics in the periodic setting. A rigorous derivation was given in the in the non-periodic setting in the work of Erdös, Schlein, and Yau [2].

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Homogenization and asymptotics for small transaction costs

Mete Soner
(joint work with Nizar Touzi)

The problem of investment and consumption in a market with transaction costs was formulated by Magill & Constantinides [2] in 1976. Since then, an impressive understanding of this problem has been achieved. This problem of proportional transaction costs is a special case of a singular stochastic control problem in which the state process can have controlled discontinuities. The related partial differential equation for this class of optimal control problems is a quasi-variational inequality which contains a gradient constraint. Technically, the multi-dimensional setting presents intriguing free boundary problems.

It is well known that in practice the proportional transaction costs are small and in the limiting case of zero costs, one recovers the classical problem of Merton [3]. Then, a natural approach to simplify the problem is to obtain an asymptotic expansion in terms of the small transaction costs. This was initiated in the pioneering paper of Constantinides [1]. The first proof in this direction was obtained in the appendix of [4].

In this talk, we consider this classical problem of small proportional transaction costs and develop a unified approach to the problem of asymptotic analysis. We also relate the first order asymptotic expansion in ϵ to an ergodic singular control problem.

To simplify the presentation, in this abstract we restrict ourselves to a single risky asset with a price process $\{S_t, t \geq 0\}$. We assume S_t is given by a time homogeneous stochastic differential equation together with $S_0 = s$ and volatility function $\sigma(\cdot)$. For an initial capital z, the value function of the Merton infinite horizon optimal consumption-portfolio problem (with zero-transaction costs) is denoted by $v(s,z)$. On the other hand, the value function for the problem with transaction costs is a function of s and the pair (x,y) representing the wealth in the saving accounts and in the stock. Then, the total wealth is simply given by $z = x + y$. For a small proportional transaction cost $\epsilon^3 > 0$, we let $v^\epsilon(s,x,y)$ be the maximum expected discounted utility from consumption. It is clear that $v^\epsilon(s,x,y)$ converges to $v(s,x+y)$ as ϵ tends to zero. Our main analytical objective is to obtain an expansion for v^ϵ in the small parameter ϵ.

To achieve such an expansion, we assume that v is smooth and let

$$\eta(s,z) := -\frac{v_z(s,z)}{v_{zz}(s,z)} \qquad (1)$$

be the corresponding risk tolerance. The solution of the Merton problem also provides us an optimal feedback portfolio strategy $\theta(s,z)$ and an optimal feedback consumption function $\mathbf{c}(s,z)$. Then, the first term in the asymptotic expansion is given through an ergodic singular control problem defined for every fixed point (s,z) by

$$\bar{a}(s,z) := \inf_M J(s,z,M),$$

where M is a control process of bounded variation with variation norm $\|M\|$,

$$J(s,z,M) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{|\sigma(s)\xi_t|^2}{2} + \|M\|_T\right],$$

and the controlled process ξ satisfies the dynamics driven by a Brownian motion B, and parameterized by the fixed data (s,z):

$$d\xi_t = [\theta(s,z)(1-\theta_z(s,z))]dB_t + dM_t.$$

This problem in one space dimension can be solved explicitly.

Let $\{\hat{Z}_t^{s,z}, t \geq 0\}$ be the optimal wealth process using the feedback strategies $\theta, \mathbf{c}$, and starting from the initial conditions $S_0 = s$ and $\hat{Z}_0^{s,z} = z$. Our main result is on the convergence of the function

$$\bar{u}^\epsilon(x,y) := \frac{v(s,x+y) - v^\epsilon(s,x,y)}{\epsilon^2}.$$

Main Theorem. *Let $\bar{a}$ be as above and set $a := \eta v_z \bar{a}$. Then, as ϵ tends to zero,*

$$\bar{u}^\epsilon(x,y) \to u(s,z) := \mathbb{E}\left[\int_0^\infty e^{-\beta t} a(S_t, \hat{Z}_t^{s,z})dt\right], \text{ locally uniformly.} \tag{2}$$

Naturally, the above result requires assumptions and we refer to the original paper for a precise statement. Moreover, the definition and the convergence of u^ϵ is equivalent to the expansion

$$v^\epsilon(s,x,y) = v(z) - \epsilon^2 u(z) + \circ(\epsilon^2), \tag{3}$$

where as before $z = x + y$ and $\circ(\epsilon^k)$ is any function such that $\circ(\epsilon^k)/\epsilon^k$ converges to zero locally uniformly.

The above result provide the connection with homogenization. Indeed, the dynamic programming equation of the ergodic problem described above is the *corrector equation* in the homogenization terminology. This identification allows us to construct a rigorous proof similar to the ones in homogenization. Moreover, the above ergodic problem is a singular one and we show that its continuation region also describes the asymptotic shape of the no-trade region in the transaction cost problem.

The main proof technique is the viscosity approach of Evans to homogenization. This powerful method combined with the relaxed limits of Barles & Perthame provides the necessary tools. As well known, this approach has the advantage of using only a simple L^∞ bound.

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Near soliton dynamics in wave and Schrödinger maps

Daniel Tataru
(joint work with Ioan Bejenaru, Joachim Krieger)

Wave maps into the sphere

$$\phi : \mathbb{R}^{n+1} \to \mathbb{S}^2$$

are solutions to the semilinear wave equation

$$\Box\phi + \phi(|\partial_t\phi|^2 - |\nabla_x\phi|^2) = 0, \qquad \phi(0) = \phi_0, \quad \phi_t(0) = \phi_1$$

This evolution admits a conserved energy

$$E_W(\phi) = \frac{1}{2}\int |\partial_t\phi|^2 + |\nabla_x\phi|^2 dx$$

and dimensionless scaling

$$\phi(t, x) \to \phi(\lambda t, \lambda x)$$

Schrödinger maps into the sphere

$$\phi : \mathbb{R}^{n+1} \to \mathbb{S}^2$$

are solutions to

$$\phi_t + \phi \times \Delta\phi = 0, \qquad \phi(0) = \phi_0$$

This evolution admits a conserved energy

$$E_S(\phi) = \frac{1}{2}\int |\nabla_x\phi|^2 dx$$

and scaling

$$\phi(t, x) \to \phi(\lambda^2 t, \lambda x)$$

Stationary solutions for both equations are called harmonic maps and solve

$$\Delta\phi + \phi|\nabla_x\phi|^2 = 0$$

One can classify dimensions according to the scaling properties of the energy. The case $n = 1$ is energy subcritical. $n = 2$ is the energy critical case, which will be discussed in the sequel. The case $n > 2$ is energy supercritical.

For $n = 2$ all finite energy harmonic maps are smooth, and are critical points for the Lagrangian

$$L(\phi) = \int_{\mathbb{R}^2} |\nabla_x\phi|^2 dx$$

The finite energy maps $\phi : \mathbb{R}^2 \to \mathbb{S}^2$ are classified according to the homotopy class $k \in \mathbb{Z}$. A special role is played by the minimizers of the Lagrangian in homotopy classes. These are unique modulo symmetries. In polar coordinates (r, θ) in $\mathbb{R}^2$ and (ϕ, θ) on $\mathbb{S}^2$ these maps can be represented as

$$Q_k(r, \theta) \to (2\tan^{-1} r^k, \theta), \qquad k \neq 0$$

where the case $k = 1$ is exactly the stereographic projection.

Both equations satisfy globally well-posedness and scattering for small data in the energy space, see [1], [11] and references therein. The **threshold conjecture** asserts that global well-posedness and scattering should still hold for zero homotopy data with energy below $2E(Q_1) = 8\pi$. For wave maps this is a consequence of results in [9], [10]. For Schrödinger maps it is still an open problem; only partial results are available so far. see [2].

The question discussed in the talk concerns the behavior of homotopy one solutions with energy close to the energy of Q_1. Energy considerations show that such solutions must stay close to the family $\mathcal{Q}_1$ of harmonic maps generated from Q via symmetries. However, Q_1 is noncompact so this does not even guarantee global wellposedness.

A slightly simpler problem is to consider a restricted class of solutions, namely equivariant ones. Precisely, in the case of wave maps it suffices to consider maps of the form

$$(r, \theta) \to (\phi(r), \theta), \qquad \phi(0) = 0, \quad \phi(\infty) = \pi$$

In the case of Schrödinger maps this class is not preserved by the flow. Instead one needs to consider a slightly larger class of maps of the form

$$(r, \theta) \to (\phi(r), \theta + \alpha(r)), \qquad \phi(0) = 0, \quad \phi(\infty) = \pi$$

The (still noncompact) restricted group of symmetries associated to equivariant maps contains only the scaling for wave maps, respectively the scaling and rotations for Schrödinger maps. The corresponding solitons are denoted by

$$Q_\lambda(r, \theta) = (2 \tan^{-1} \lambda r, \theta), \qquad Q_{\lambda,\alpha}(r, \theta) = (2 \tan^{-1} \lambda r, \theta + \phi)$$

Given initial data in the above class with energy less than $E(Q_1) + \epsilon$, one can identify parameters $\lambda(t)$ for wave maps, respectively $(\lambda(t), \alpha(t))$ for Schrödinger maps which describe how the solution moves along the soliton family. These are uniquely determined modulo an $O(\epsilon)$ error. Then we have

Open Problem 1. *Describe all possible dynamics of $\lambda(t)$ for wave maps, respectively $(\lambda(t), \alpha(t))$ for Schrödinger maps which correspond to maps with energy less than $E(Q_1) + \epsilon$.*

The scenario where $\lambda \to 0$ in finite time is prohibited. However, a finite focusing blow-up $\lambda \to \infty$ is possible, see [5],[7], [8], [6].

The aim of the talk was to introduce the above problem, and then discuss some progress made toward characterizing the behavior of large classes of solutions in [3] for Schrödinger maps, respectively in [4] for wave maps. Understanding what happens for all finite energy equivariant data as above seems still out of reach for now.

A key difficulty in the problem is caused by the presence of a resonant zero mode arising in the study of the linearized equations near the soliton.

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Smoothing properties for dispersive partial differential equations and systems of equations.

Nikolaos Tzirakis

(joint work with M. Burak Erdogan)

1. Main results

In this note we study local smoothing properties for dispersive partial differential equations and systems of equations. Examples include the periodic KdV, the periodic mKdV and the periodic KdV with a smooth, mean-zero, space-time potential

$$\begin{cases} u_t + u_{xxx} + (u^2 + Vu)_x = 0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ \quad u(x,0) = g(x) \in H^s(\mathbb{T}). \end{cases} \tag{1}$$

Another example is the periodic Zakharov system which consists of a complex field u (Schrödinger part) and a real field n (wave part) satisfying the equation:

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{s_1-1}(\mathbb{T}), \end{cases} \tag{2}$$

where $\alpha > 0$ and T is the time of existence of the solutions. Here α is the dispersion coefficient. In the literature it is standard to include the speed of an ion acoustic wave in a plasma as a coefficient β^{-2} in front of n_{tt} where $\beta > 0$. One can easily scale away this parameter and reduce the system to (2). For both equations, we prove that the difference of the nonlinear evolution with the linear evolution is smoother than the initial data. This smoothing property is not apparent if one views the nonlinear evolution as a perturbation of the linear flow and apply standard Picard iteration techniques to absorb the nonlinear derivative term. On the other hand the first Picard iteration for the KdV

$$e^{-t\partial_x^3}\int_0^t e^{t'\partial_x^3}\big[e^{-t'\partial_x^3}g\partial_x\big(e^{-t'\partial_x^3}g\big)\big]dt'$$

implies a possible full derivative smoothing for the nonlinearity, since on the Fourier side (ignoring zero modes):

$$\sum_{k_1+k_2=k}\int_0^t e^{-3ik_1k_2kt'}k_2\widehat{g}(k_1)\widehat{g}(k_2)dt' = \sum_{k_1+k_2=k}\frac{\widehat{g}(k_1)\widehat{g}(k_2)}{-3ikk_1}(e^{-3ik_1k_2kt}-1).$$

Therefore, if $g\in L^2$, then the correction term is in H^1. Our result, [7], validates this heuristic. It follows from a combination of the method of normal forms (through differentiation by parts) inspired by the result in [1] and the restricted norm method of Bourgain, [2]. As it is well-known, KdV is a completely integrable system with infinitely many conserved quantities. However, our method does not rely on the integrability structure of KdV, and thus it can be applied to other dispersive models.

The motivation for studying the Zakharov system in [9], comes from the fact when $\beta\to\infty$, the system reduces to the focusing cubic NLS. But there is no derivative smoothing for the NLS with power-type nonlinearity, [5, 6]. In [5], the author obtained a (local-in-time) smoothing estimate in the $\mathcal{F}\ell^p\to\mathcal{F}\ell^q$ setting for 1d cubic NLS. Here $\mathcal{F}\ell^p$ is the space of functions whose Fourier series is in ℓ^p. We, on the other hand, prove that for the Zakharov system there is derivative smoothing even though the resonances of the model are non trivial. Notice that if we integrate (1) and the wave part of (2) we obtain that the averages of the solutions are constant. Since the evolution does not change if we apply a trivial transformation and remove the constant average, we can safely assume mean zero solutions. This assumption is crucial. It removes the zero Fourier mode of the series solution and enables one to use the oscillatory character of the solution in the normal form transformations. On the $\mathbb{R}^n$ setting, the conservation of the average does not lead to the same result and it remains a challenging problem to extend our methods there. We state as an example the smoothing estimate for (1) (through Miura's transform they extend to the mKdV) but similar statements can be found in [9] for the Zakharov system where we distinguish the case of $\frac{1}{\alpha}\in\mathbb{N}$ and $\frac{1}{\alpha}\notin\mathbb{N}$.

Theorem 1. *Fix $s\geq 0$ and $s_1<s+1$. Consider (1) where $V\in C^\infty(\mathbb{T}\times\mathbb{R})$ is a mean zero real-valued potential with bounded derivatives and initial data $u(x,0)=g(x)\in H^s$. Assume that we have a growth bound $\|u(t)\|_{H^s}\leq C(\|g\|_{H^s})T(t)$ for some nondecreasing function T on $[0,\infty)$. Then $u(t)-e^{tL}g\in C_t^0H_x^{s_1}$ and*

$$\|u(t)-e^{tL}g\|_{H^{s_1}}\leq C(s,s_1,\|g\|_{H^s})(1+|t|)T(t)^9.$$

Here $L=-\partial_x^3+\big(\frac{1}{2\pi}\int_{-\pi}^{\pi}g\big)\partial_x$.

2. Applications

We highlight below some applications of the smoothing estimates for the KdV and Zakharov evolutions.

2.1. **Growth of higher order Sobolev norms.** Theorem 1 implies, that the smoothing estimates can be iterated (if a priori bounds can be established independently) to provide estimates on the growth of higher order Sobolev norms. These a priori bounds usually come from either the conservation laws of the equations or (like in our theorem) from "almost conservation laws", [4]. This method is not new and has been developed for many years now, [3], [13]. Our low regularity smoothing estimates supplement these results by providing simpler proofs of the known results or proving new bounds for low regularity solutions as in Theorem 1. For the Zakharov system the following Corollary can be proved which appears to be new.

Corollary 2. *For any $\alpha > 0$, and for any Sobolev exponents (s_0, s_1) that give rise to well defined local well-posed solutions with $s_0 \geq 1$, $s_1 \geq 0$, the global solution of* (2) *with $H^{s_0} \times H^{s_1} \times H^{s_1-1}$ data satisfies the growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n(t)\|_{H^{s_1}} + \|n_t(t)\|_{H^{s_1-1}} \leq C_1(1+|t|)^{C_2},$$

where C_1 depends on s_0, s_1, and $\|u_0\|_{H^{s_0}} + \|n_0\|_{H^{s_1}} + \|n_1\|_{H^{s_1-1}}$, and C_2 depends on s_0, s_1.

2.2. **Continuity of the KdV flow map.** For L^2 initial data g, Theorem 1 implies that $u - e^{tL}g \in C^0_t H^{1-}_x$, and hence is a continuous function of x and t. Using this remark and the following theorem of Oskolkov, [12], we obtain Corollary 4 below. We also note that using our theorem it is likely that other known properties of the Airy evolution could be extended to the KdV evolution.

Theorem 3. [12] *Let L be as in the previous theorems and assume that g is of bounded variation, then $e^{tL}g$ is a continuous function of x if $t/2\pi$ is an irrational number. For rational values of $t/2\pi$, it is a bounded function with at most countably many discontinuities. Moreover, if g is also continuous then $e^{tL}g \in C^0_t C^0_x$.*

Corollary 4. *Let u be the real valued solution of* (1) *with initial data $g \in BV \subset L^2$. Then, u is a continuous function of x if $t/2\pi$ is an irrational number. For rational values of $t/2\pi$, it is a bounded function with at most countably many discontinuities. Moreover, if g is also continuous then $u \in C^0_t C^0_x$.*

2.3. **Almost everywhere convergence of the KdV flow map.** It is well known, [11], that Strichartz estimates for linear dispersive PDE, can lead to an easy proof of the property of almost everywhere convergence of the linear evolution to the initial data $u_0 \in H^s$ for some $s \leq \frac{1}{2}$. For the periodic KdV a new result in [10] proves that for $s > 3/14$,

$$\|e^{-t\partial_x^3} g\|_{L^{14}_t L^{14}_x(\mathbb{T}\times\mathbb{T})} \leq C\|g\|_{H^s}.$$

This estimate implies, after some trivial calculations, that if $g \in H^s$, $s > 3/7$, then $e^{-t\partial_x^3} g$ converges to g almost everywhere as $t \to 0$. Theorem 1 establishes the same conclusion for the periodic KdV with or without a smooth space-time potential.

2.4. **Existence of global smooth attractors for weakly damped and forced equations and systems.** In many real situations one cannot neglect energy dissipation and external excitation when studying certain dispersive equations. For example for the periodic KdV problem one can study

$$\begin{cases} u_t + u_{xxx} + \gamma u + uu_x = f, \quad t \in \mathbb{R},\ x \in \mathbb{T} \\ u(x,0) = u_0(x) \in \dot{L}^2(\mathbb{T}) := \{g \in L^2(\mathbb{T}) : \int_{\mathbb{T}} g(x)dx = 0\}, \end{cases} \tag{3}$$

where $\gamma > 0$ and $f \in \dot{L}^2$. On a finite time interval the solution properties of the two models (1) (with $V = 0$) and (3) are identical. But the long time dynamics of the forced and weakly damped KdV is described by a finite dimensional attractor. Thus we want to study the influence of dissipation on the actual numbers of degrees of freedom of the infinite dimensional dynamical system. We note that inverse scattering theory describes the long time behavior of KdV as a truly infinite dimensional dynamical system. The problem of global attractors for nonlinear PDEs is concerned with the description of the nonlinear dynamics for a given problem as $t \to \infty$. In particular assuming that one has a well-posed problem for all times we can define the semigroup operator $U(t) : u_0 \in H \to u(t) \in H$ where H is the phase space. Dissipative systems are characterized by the existence of a bounded absorbing set into which all solutions enter eventually. The candidate for the attractor set is the omega limit set of an absorbing set, B, defined by $\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} U(t)B}$ where the closure is taken on H.

Definition 5. *We say that a compact subset $\mathcal{A}$ of H is a global attractor for the semigroup $\{U(t)\}_{t\geq 0}$ if $\mathcal{A}$ is invariant under the flow and if for every $u_0 \in H$, $d(U(t)u_0, \mathcal{A}) \to 0$ as $t \to \infty$.*

Definition 6. *We say a bounded subset $\mathcal{B}_0$ of H is absorbing if for any bounded $\mathcal{B} \subset H$ there exists $T = T(\mathcal{B})$ such that for all $t \geq T$, $U(t)\mathcal{B} \subset \mathcal{B}_0$.*

It is not hard to see that the existence of a global attractor $\mathcal{A}$ for a semigroup $U(t)$ implies the existence of an absorbing set. For the converse we need to prove the asymptotic compactness of the semigroup, which through Rellich's Theorem boils down to our smoothing estimates. In particular we proved that the solution of (3) decomposes into two parts; a linear one which decays to zero as time goes to infinity and a nonlinear one which always belongs to a smoother space. As a corollary we prove that all solutions are attracted by a ball in H^s, $s \in (0,1)$, whose radius depends only on s, the L^2 norm of the forcing term and the damping parameter. We record our theorem from [8]:

Theorem 7. *Consider the forced and weakly damped KdV equation* (3) *on $\mathbb{T} \times \mathbb{R}$ with $u(x,0) = u_0(x) \in \dot{L}^2$. Then the equation possesses a global attractor in $\dot{L}^2$. Moreover, for any $s \in (0,1)$, the global attractor is a compact subset of H^s, and it is bounded in H^s by a constant depending only on s, γ, and $\|f\|$.*

We prove a similar theorem, [9], for the energy solutions of the weakly damped and forced periodic Zakharov system:

$$\begin{cases} iu_t + u_{xx} + i\gamma u = nu + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} + \delta n_t = (|u|^2)_{xx} + g, \\ u(x,0) = u_0(x) \in H^1(\mathbb{T}), \\ n(x,0) = n_0(x) \in L^2(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{-1}(\mathbb{T}), \end{cases} \tag{4}$$

where $f \in H^1(\mathbb{T})$, $g \in L^2(\mathbb{T})$ are time-independent, g is mean-zero, and the damping coefficients δ, γ are positive.

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