Mathematisches Forschungsinstitut Oberwolfach

Report No. 39/2012

DOI: 10.4171/OWR/2012/39

Mathematical Aspects of Hydrodynamics

Organised by
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12th August – 18th August 2012

Abstract. The workshop dealt with the partial differential equations that describe fluid motion, namely the Euler equations and the Navier-Stokes equations. Many of the lectures concerned questions of existence and regularity of solutions, possibly weak solutions, of these equations and somewhat simpler variants.

Mathematics Subject Classification (2000): 76, 35.

Introduction by the Organisers

The workshop "Mathematical Aspects of Hydrodynamics" was held in August 12-18, 2012. For personal reasons the original organisers, Peter Constantin (Princeton) and Gregory Seregin (Oxford), were regrettably unable to attend the workshop. Two additional organisers, Susan Friedlander (Los Angeles) and Edriss S. Titi (Irvine and Rehovot) were added and they were present throughout the workshop. The scientific program consisted of 26 main talks (about 40 minutes each with additional 20 minutes were allocated for discussion) and 3 shorter contributions (posters plus 10 minute talks). There was plenty of time for discussions during and after the lectures and in private groups.

The emphasis of the meeting was on various mathematical facets of incompressible fluid dynamics, however one session was devoted to issues connected with compressible flows. A lecture on an important, very new result concerning the existence of dissipative, Holder continuous, incompressible solutions for the Euler flows opened the workshop. The main topics discussed at the meeting included: turbulence and weak solutions of the Euler equation, regularity questions for the Navier-Stokes equations, breaking solutions to the inviscid water wave equation, active scalar equations, issues of stability and instability, the limit of vanishing viscosity, aspects of magnetohydrodynamics, new numerical schemes for the Navier-Stokes equations, and stochastic systems related to fluid models.

There were 42 participants from 13 different countries, namely Germany, US, UK, Russia, France, Spain, Czech Republic, Lithuania, Canada, China, Japan and Korea. One organiser and one postdoc were women. Approximately 10 participants were young researchers who came to Oberwolfach for the first time.

The afternoon of the hike was very warm but a good number participated in the hike and everyone enjoyed the barbecue on the patio after the hike. A delightful musical trio provided entertainment.

The organisers thank the Institute staff for their great hospitality and support before and during the conference which was very well run. Financial support for young participants from the Leibniz Association and the National Science Foundation is gratefully acknowledged.

Workshop: Mathematical Aspects of Hydrodynamics

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Abstracts

Dissipative Hölder continuous Euler flows

László Székelyhidi Jr.
(joint work with Camillo De Lellis)

We consider the incompressible Euler equations on the torus $\mathbb{T}^3$, given by

$$\begin{aligned} \partial_t v + \operatorname{div}\, v \otimes v + \nabla p = 0, \\ \operatorname{div}\, v = 0. \end{aligned} \tag{1}$$

It is well known that classical solutions (say, $v \in C^1$) conserve the energy $E(t) = \frac{1}{2}\int |v|^2\, dx$. On the other hand in recent years there has been a lot of focus on understanding weak solutions, which do not necessarily conserve the energy. The motivation for this comes from the Kolmogorov-Onsager theory of homogeneous isotropic turbulence, laid down some 70 years ago.

In the formulation of Onsager the conjecture is that weak solutions of the Euler equations preserve the energy if the solution is Hölder-continuous with exponent $\alpha > 1/3$, whereas if $\alpha < 1/3$, weak solutions need not preserve the energy. Remarkably, the critical exponent $1/3$ matches exactly with exponents obtained by Kolmogorov for the decay of the energy spectrum in the inertial regime, even though Kolmogorov's calculations were for ensemble averages of solutions to the Navier-Stokes equations, based on the assumption that there is an energy cascade. In turn, the energy cascade links to an idea of Richardson concerning the structure of turbulent flows, which, roughly speaking, amounts to a cascade of vortex structures appearing in a self-similar fashion. Thus, the conjecture of Onsager, on the face of it a conjecture about optimal regularity, is linked to a (less precise) conjecture about a possible self-similar structure in weak solutions of the Euler equations. It is this problem that is addressed in the talk. Our main theorem is:

Theorem 1 ([3, 4])**.** *Let $\bar{e} : [0,1] \to \mathbb{R}$ a positive smooth function and $\theta < 1/10$. Then there is a weak solution (v,p) of the Euler equations* (1) *on $\mathbb{T}^3 \times [0,1]$ such that*

$$\bar{e}(t) = \int |v|^2(x,t)\, dx \qquad \forall t \in [0,1]$$

and there exists a constant C such that

$$\begin{aligned} |v(x,t) - v(x',t')| &\leq C(|x-x'|^\theta + |t-t'|^\theta) \\ |p(x,t) - p(x',t')| &\leq C(|x-x'|^{2\theta} + |t-t'|^{2\theta}) \end{aligned}$$

In the formulation of Theorem 1 we deliberately avoided mentioning the initial data. The reason for this stems from the fact, that, as has been argued in [1, 2], the Euler equations abide by Gromov's h-principle. A typical feature of the h-principle is that one can distinguish between a local and a global aspect. In some sense (that can be made precise, see [2]) the statement of Theorem 1 deals with the local aspect. The global aspect would involve matching initial/boundary conditions. For bounded solutions this has been done in [2], for continous solutions this is work in progress.

The basic idea of the proof is to use the so-called convex integration scheme as pioneered by J. Nash in the context of rough (C^1) isometric immersions. Though our proof shares several similarities with Nash's scheme, there are many points where the method departs dramatically from Nash's, due to some issues which are typical of the Euler equations and are not present for the isometric embeddings.

1) Perhaps the most important new aspect of our scheme is a "transport term" which arises, roughly speaking, as the linearization of (1): this term is typical of an evolution equation, whereas, instead, the equations for isometric embeddings are "static". At a first glance this transport term makes it impossible to use a scheme like the one of Nash. To overcome this obstruction we need to introduce a phase-function that acts as a kind of discrete Galilean transformation of the (stationary) Beltrami flows, and to introduce an "intermediate" scale along each iteration step on which this transformation acts.

2) The convex integration scheme of Nash and Gromov heavily relies on one-dimensional oscillations - the simple reason being that these can be "integrated", hence the name convex integration. In contrast, the main building blocks of our iteration scheme are Beltrami flows, which are truly three-dimensional oscillations. The issue of going beyond one-dimensional oscillations has been raised by Gromov as well as Kirchheim-Müller-Šverák, but as far as we know, there have been no such examples in the literature so far. In fact, it seems that with one-dimensional oscillations alone one cannot overcome the obstruction in 1).

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Finite time blow-up in supercritical drift-diffusion equations

Vlad Vicol
(joint work with Luis Silvestre, Andrej Zlatos)

We consider the (linear) drift-diffusion equation $\partial_t\theta+u\cdot\nabla\theta+(-\Delta)^s\theta = 0$. Here the divergence free drift u belongs to a supercritical space, and $0 < s \leq 1$. We prove that starting with smooth initial data solutions may become discontinuous in finite time. For $s < 1$ this may even be achieved with autonomous drift. On the other hand, for $s = 1$ and autonomous drift, in two space dimensions we obtain a modulus of continuity for the solution depending only on the L^1 norm of the drift, which is a supercritical quantity.

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Incompressible stirring and diffusion-less mixing

Charles R. Doering

Incompressible advection without diffusion preserves standard L^p norms of a scalar concentrations' deviations from the mean, so those norms cannot gauge the mixing effectiveness of stirring. On the other hand negative norms like H^{-1} can be depleted and serve as a *mix-norm*. The evolution of this mix-norm is studied via analysis and direct numerical simulations, and questions regarding optimal mixing under various flow constraints are considered.

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Recent results on supercritical SQG equation

Alexander Kiselev
(joint work with Michael Dabkowski, Luis Silvestre, Vlad Vicol)

I reviewed recent advances in understanding of the properties of solutions of the SQG equation in the supercritical regime. The supercritical SQG equation is given by

$$\partial_t\theta + (u\cdot\nabla)\theta + L\theta = 0 \tag{1}$$

where $u = \nabla^\perp(-\Delta)^{-1/2}\theta$ and L is a dissipative operator; for example $L = (-\Delta)^\alpha$, where $\alpha < 1/2$ corresponds to the supercritical case. Let us consider (1) on a two-dimensional torus.

Global regularity of (1) in the subcritical case is standard, and in the critical case has been established recently in [9, 1, 10, 2] There are three main kinds of results that are currently available in the supercritical regime.

1. The conditional regularity results. These are criteria which say that if a weak solution possesses a certain degree of regularity, then it is actually classical. One example of such results is due to Constantin and Wu [3], who show that $\theta(\cdot,t)\in C^\beta$, $\beta > 1-2\alpha$, on $[0,T]$, implies $\theta(\cdot,t)\in C^\infty(0,T]$. This result has been sharpened by Dong and Pavlovic to $\beta = 1-2\alpha$ [7].

2. Finite time regularization results. Finite time regularization is well known for many dissipative supercritical equations, for example for 3D Navier-Stokes

system. The proof of this result in the 3D Navier-Stokes case is straightforward: it follows from the energy inequality and global regularity result for small H^1 norm. For the SQG equation, the problem is harder. The norm that one can control from the analog of the energy inequality is not sufficient to establish small data global regularity result. First finite time regularization result is due to Silvestre [11], who proved it for slightly supercritical case $\alpha = 1/2 - \epsilon$. Dabkowski [4] extended finite time regularization result to the whole supercritical range $0 < \alpha < 1/2$. An alternative proof has been given by Kiselev [8]. All proofs are based on "regularity cascade", where regularity estimates are first established at large scales and propagate gradually to small ones, covering all scales in finite time.

3. Global regularity in the slightly supercritical case. Results of this sort so far have been obtained only for supercriticality weaker than any power [5, 6, 13]. Consider a generalized diffusion operator given by

$$Lf(x) = \int (f(x) - f(x+y)) \frac{m(y)}{|y|^d}\, dy,$$

where m is spherically symmetric, non-increasing, positive, and satisfies some additional technical assumptions. Note that $m(y) = |y|^{-2\alpha}$ corresponds to $L = (-\Delta)^\alpha$. The main result proved by Dabkowski, Kiselev, Silvestre and Vicol [6] states that if

$$\lim_{\epsilon \to 0} \int_\epsilon^1 m(y)\, dy = \infty, \tag{2}$$

then the solutions of (1) with sufficiently nice initial data are globally regular. This allows, in particular, for logarithmically supercritical $m(y) \sim \frac{1}{|y| \log |y|^{-1}}$ near zero. We note that we can also consider diffusive operator L given by Fourier multiplier $P(k)$. In this case the condition (2) translates into

$$\lim_{\epsilon \to 0} \int_\epsilon^1 P(|y|^{-1})\, dy = \infty,$$

and in particular allows $P(k) \sim |k|/\log|k|$ for large $|k|$. We note that global regularity in the slightly supercritical case (roughly by $(\log|k|)^{1/2}$) for the 3D Navier-Stokes system was proved by Terry Tao [12]. The argument is however very different from [6]: for the critical 3D Navier-Stokes, global regularity is proved by an L^2 argument, while for the critical SQG one needs L^∞-type argument. The method of the proof of [6] is based on the modulus of continuity method of [9], with necessary adjustments to accommodate loss of scaling.

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Large Data Analysis of Kolmogorov's Two Equation Model of Turbulence

Josef Málek

Kolmogorov seems to be the first who recognized (in 1941) that a two equation model of turbulence might be appropriate to turbulent flow prediction. We present the results (joint work with M. Bulíček) concerning long-time and large-data existence of weak solution to three-dimensional flows described by this Kolmogorov's two equation model of turbulence. Similar results (joint work with M. Bulíček and R. Lewandowski) associated with one equation model of turbulence (for turbulent kinetic energy, that is, for the kinetic energy of the velocity fluctuations) are presented as well. Finally, we state the results of the type "L^2-maximal regularity" for the evolutionary Stokes type problems (joint work with M. Bulíček and P. Kaplický) and formulate a conjecture concerning the full regularity for generalized turbulent kinetic energy model.

Navier-Stokes system with random force

Sergei Kuksin

In my lecture I discuss various asymptotic aspects of solutions for the Navier-Stokes system with a random force. Namely:

(1) Its asymptotical behaviour as $t \to \infty$.
(2) Dependence of solutions on external parameters, which turns out to be continuous uniformly in time.
(3) The inviscid limit "viscosity goes to zero", corresponding to the 2d turbulence.

(4) Behaviour of solution for the randomly forced 3d Navier-Stokes system in a thin spherical layer $S^2 \times (0, \epsilon)$. It turns out that when $\epsilon \to 0$, this behaviour may be described by the 2d Navier-Stokes system in the sphere S^2, uniformly in time and uniformly continuously in the parameters of the random force.

In some sense, (4) means that despite mathematically rigorous prediction of weather is impossible, still mathematically rigorous prediction of climate (statistics of the weather) is possible.

The talk is based on the recent book [1]

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Finite time singularities for the free boundary incompressible Euler equations

Diego Córdoba

(joint work with Angel Castro, Charles Fefferman, Francisco Gancedo, Javier Gomez-Serrano and Maria Lopez-Fernandez)

Numerical evidence showed (see [2]) that there exist solutions of the 2D water wave equation that start from a graph, turn over and collapse in a splash singularity (self intersecting curve in one point) in finite time. First (see [1]) we proved that there exist large initial data parameterized as a graph for which in finite time the interface reaches a regime in which it is no longer a graph. In [3] we exhibit smooth initial data for the 2D water wave equation for which we prove that smoothness of the interface breaks down in finite time (in a splash or splat singularity). In [4] we show that the surface tension does not prevent a finite time splash or splat singularity.

In further work would like to prove that an initially smooth water wave may start as a graph, then turn over, and finally produce a splash. To do so, our plan is to use interval arithmetic (together with a stability result) to produce a rigorous computer-assisted proof that, close to the approximate solution arising from our numerics, there exists an exact solution of water waves that ends in a splash.

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Phenomenology of intermittency and regularity problems in Navier-Stokes turbulence

Koji Ohkitani

1. Intermittency and the CKN theory re-examined (with Mark Dowker [1])

We study space-time integrals which appear in the Caffarelli-Kohn-Nirenberg (CKN) theory for the Navier-Stokes equations analytically and numerically. The key quantity is written in standard notations $\delta(r) = 1/(\nu r)\int_{Q_r} |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}\, dt$, which can be regarded as a local Reynolds number over a parabolic cylinder Q_r.

First, by re-examining the CKN integral we identify a cross-over scale $r_* \propto L\left(\frac{\|\nabla \boldsymbol{u}\|_{L^2}^2}{\|\nabla \boldsymbol{u}\|_{L^\infty}^2}\right)^{1/3}$, at which the CKN Reynolds number $\delta(r)$ changes its scaling behavior. This reproduces a result on the minimum scale $r_{\min}$ in turbulence: $r_{\min}^2\|\nabla \boldsymbol{u}\|_\infty \propto \nu$, consistent with a result of Henshaw *et al.* (1989). For the energy spectrum $E(k) \propto k^{-q}$ $(1 < q < 3)$, we show that $r_* \propto \nu^a$ with $a = \frac{4}{3(3-q)} - 1$. Parametric representations are then obtained as $\|\nabla \boldsymbol{u}\|_\infty \propto \nu^{-(1+3a)/2}$ and $r_{\min} \propto \nu^{3(a+1)/4}$. By the assumptions of the regularity and finite energy dissipation rate in the inviscid limit, we derive $\lim_{p\to\infty} \frac{\zeta_p}{p} = 1 - \zeta_2$ for any phenomenological models on intermittency, where ζ_p is the exponent of p-th order (longitudinal) velocity structure function. It follows that $\zeta_p \le (1-\zeta_2)(p-3)+1$ for any $p \ge 2$ *without* invoking fractal energy cascade.

Second, we determine the scaling behavior of $\delta(r)$ in direct numerical simulations of the Navier-Stokes equations. In isotropic turbulence around $R_\lambda = 100$ starting from random initial conditions, we have found that $\delta(r) \propto r^4$ *throughout the inertial range.* This can be explained by the smallness of $a \approx 0.26$. If the β-model is perfectly correct, the intermittency parameter a is related to the dissipation correlation exponent μ as $\mu = \frac{4a}{1+a} \approx 0.8$ which is larger than the observed $\mu \approx 0.20$. Corresponding integrals are studied using the Burgers vortex and the Burgers equations. The scale r_* offers a practical method of estimating intermittency.

This paper also sorts out a number of existing mathematical bounds and phenomenological models on the basis of the CKN Reynolds number.

2. Phenomenology of enstrophy bounds

We consider incompressible fluid dynamical equations with modified dissipativity $\nu_\alpha(-\triangle)^\alpha$ $(\alpha > 0)$ in d-spatial dimensions (with the exception of compressible fluid at $d = 1$). Heuristic arguments are given regarding the bounds for the H^α-norm on dimensional grounds. The arguments are formal as we use dimensional analysis.

It is well-known that for $\alpha \ge \frac{d+2}{4}$, global existence is guaranteed. On the other hand, Kiselev *et al.* have proved global regularity for the 1D Burgers hypoviscous equation with $\alpha \geq 1/2$ and blowup with $\alpha < 1/2$ [2]. It should be noted that their result is better than what the enstrophy bounds can cover, because $\frac{d+2}{4}\Big|_{d=1} = \frac{3}{4} > \frac{1}{2}$.

Generally speaking, for a possible blow-up we expect on dimensional grounds that the 'enstrophy' behaves as

$$Q_\alpha(t) \equiv \|\boldsymbol{u}\|^2_{H^\alpha(\mathbb{R}^d)} \geq c \frac{\nu_\alpha^{\frac{d+2-2\alpha}{2\alpha}}}{(t_* - t)^{\frac{6\alpha-d-2}{2\alpha}}}, \tag{1}$$

form which another branch $\alpha = \frac{d+2}{6}$ is identified. In fact, this is the borderline at which we lose the differential inequality for $Q_\alpha(t)$. It is well-known that this takes place at $d = 4$ for the Navier-Stokes equations, see e.g.[3]. Indeed, we expect by dimensional analysis a bound of the form

$$\frac{dQ_\alpha}{dt} \leq c\nu_\alpha^3 \underbrace{\left(\frac{Q_\alpha}{\nu_\alpha^2}\right)^{\frac{8\alpha-d-2}{6\alpha-d-2}}}_{=[L^{-2\alpha}]}. \tag{2}$$

Note that on the lower-branch a physical dimension of Q_α coincides with that of ν_α^2, hence it is impossible to form a length scale by their combination.

In the limit of small $\epsilon \equiv \frac{6\alpha-d-2}{2\alpha}$, it follows from (1) that

$$\|\boldsymbol{u}\|^2_{H^\alpha(\mathbb{R}^d)} \geq c\nu_\alpha^2 \left(1 + \epsilon \log \frac{1}{\nu_\alpha(t_* - t)}\right) \tag{3}$$

for a possible blowup, similar to the L^3-problem, see [4]. Passing to the limit $\alpha \to \frac{d+2}{6}$, we conjecture that

$$\|\boldsymbol{u}\|^2_{H^{\frac{d+2}{6}}(\mathbb{R}^d)} \to \infty \text{ as } t \to t_*$$

for a possible blowup. (We know that this never happens in $d = 1$.) Or, in other words, if

$$\|\boldsymbol{u}\|^2_{H^{\frac{d+2}{6}}(\mathbb{R}^d)} < \infty \text{ on } [0, T],$$

the flow is expected to remain regular on that time interval.

The lower-branch includes the Kiselev *et al.*'s regularity result on the 1D Burgers hypoviscous equation at $(d, \alpha) = (1, 1/2)$. It also includes the 4D Navier-Stokes equations at $(d, \alpha) = (4, 1)$, see Figure 1. Hence, a striking similarity between the 1D hypoviscous Burgers equation and the 4D Navier-Stokes equations is noted. It is of interest to study what is happening at the lower-branch, in particular, at the 4D Navier-Stokes equations. More details, including an alternative understanding of the lower-branch and a full bibliography, are to be reported elsewhere.

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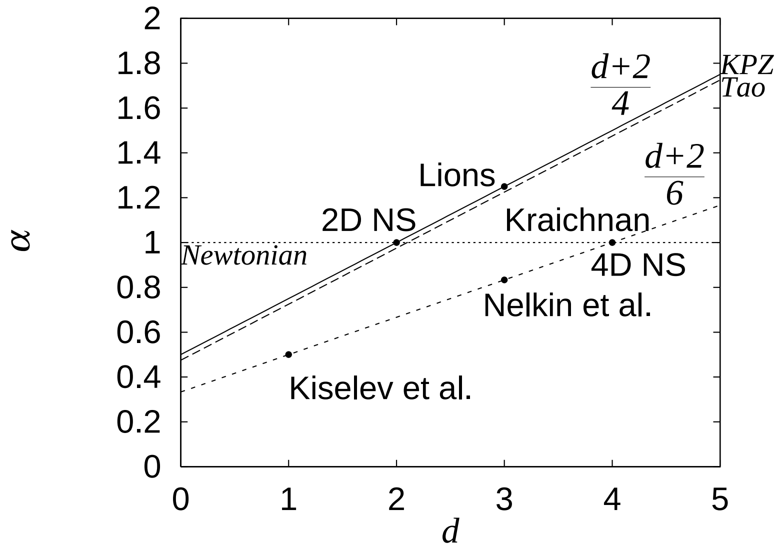

FIGURE 1. The α-d diagram: global existence for $\alpha \geq \frac{d+2}{4}$ and local existence for $\alpha > \frac{d+2}{6}$ according to the methods of the enstrophy bounds. The case $d = 2$ is exceptional, where we have global regularity for all $\alpha > 0$.

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Metastability in the two-dimensional Navier-Stokes equation

C. Eugene Wayne

(joint work with Margaret Beck)

Quasi-stationary, or metastable, states play an important role in weakly viscous systems like Burgers equation or the two-dimensional Navier-Stokes equation. In two-dimensional turbulent flows, they represent structures which emerge on time scales much shorter than the viscous time scale and which, once they have formed, dominate the flow for very long periods. We propose a dynamical systems explanation of these states in which the metastable states form an attractive, low-dimensional, manifold in the phase space of the system. After a short transient in which the solution approaches some point on this manifold, subsequent evolution corresponds to a slow motion close to this manifold until one reaches the long-time asymptotic state of the system.

For the one-dimensional Burgers equation, we use the Cole-Hopf transformation to give a global picture of this scenario. Almost every initial condition evolves to a state close to the metastable manifold, which in this case corresponds to the diffusive N-waves which were already identified as the primary metastable states by Kim and Tzavaras. We prove that the metastable manifold is locally attractive, so that once the solution is close to the manifold, the further evolution of the system consists of a flow along the manifold until the solution approaches the asymptotic state which is an explicit self-similar solution of the equation.

For the two-dimensional Navier-Stokes equation there are more possibilities for the metastable states. We focus on a family of states known as "bar" states, which are a simple shear flow, and which in numerical experiments of Yin, Montgomery and Clercx, and Bouchet and Morita, seem to capture the metastable behavior of solutions of the two-dimensional Navier-Stokes equation in many instances. To show that these bar states are locally attractive, we linearize about them and study the resulting linear evolution operator. Numerically computing the eigenvalues of this linearization, we find that (with the exception of a few zero eigenvalues) they all have strictly negative real parts and that if ν is the kinematic viscosity of the system, the real parts of the eigenvalues scale with $\sqrt{\nu}$ as $\nu \to 0$. This indicates that not only are these states stable, but they also introduce a new time scale into the problem, $t_{\text{meta}} \sim \nu^{-1/2}$, which is much shorter than the viscous time scale $t_{\text{visc}} \sim \nu^{-1}$. We are currently attempting to prove that the phenomena suggested by these numerics actually hold. Using Villani's theory of hypercoercivity (as adapted to the fluid mechanical setting by Gallagher, Gallay and Nier) we have shown that for a natural approximation of the evolution equation obtained by linearizing about the bar states, one does converge as $\exp(-\sqrt{\nu}t)$ - i.e. within this approximation, the bar states are locally attractive on the meta-stable time scale, rather than the much longer viscous time scale.

This is joint work with Margaret Beck (Heriot Watt University and Boston University).

Weak and strong solutions to the full Navier-Stokes-Fourier system

Eduard Feireisl

We discuss the concepts of weak and strong solutions for the full Navier-Stokes-Fourier system describing the time evolution of a compressible, viscous and heat conducting fluid. We show that the weak solutions satisfy a kind of relative entropy inequality and, consequently, coincide with the strong solutions as long as the latter exist. Finally, we introduce a regularity criterion in the class of weak solutions.

Stability of Vortex Sheets and Related Problems

Gui-Qiang G. Chen

Vortex sheets, along with shocks, rarefaction waves, and entropy waves, are fundamental waves in fluid mechanics. Understanding the behavior of vortex sheets is an important step towards our full understanding of fluid motions and solutions of fluid mechanical partial differential equations.

In this talk we present one of our recent research projects in the analysis of vortex sheets and related problems for the Euler equations in fluid mechanics. As is well known, incompressible vortex sheets are unstable in many physical situations. One of the main motivations for this project is to search for regularization mechanisms to stabilize the vortex sheets, besides the traditional viscosity regularization. In this talk we focus on two mechanisms: compressibility and magneticity, to examine whether and, if so, when they can serve as such regularization mechanisms.

We start with the nonlinear stability of compressible vortex sheets in two-dimensional steady supersonic Euler flows under a BV boundary perturbation of the boundary and/or the boundary data, since steady Euler flows are asymptotic states and may be global attractors of the corresponding unsteady Euler flows. It has been shown in [1, 2, 3, 4, 7] that such vortex sheets are always stable, no matter how small the the Mach number is.

Then we present that, for the two-dimensional time-dependent compressible Euler flows, the stability of vortex sheets does depend on the compressibility, that is, the Mach number (cf. [8, 9]); while the compressibility seems to play no significant role to stabilize for the vortex sheets in three-dimensions for inviscid fluids.

With these, we turn our discussion on whether the magneticity can stablize the vortex sheets in three-dimensions, which are unstable in the regime of pure gas dynamics, that is, whether they become stable under the magnetic effect in three-dimensional MHD. One of the main features is that the stability problem is equivalent to a free boundary problem whose free boundary is a characteristic surface, which is more delicate than noncharacteristic free boundary problems. Another feature is that the linearized problem for current-vortex sheets in MHD does not meet the uniform Kreiss-Lopatinskii condition. These features cause additional analytical difficulties and especially prevent a direct use of the standard Picard iteration to the nonlinear problem. We present a nonlinear approach in Chen-Wang [5, 6] to deal with these difficulties in three-dimensional MHD: The linearized problem is carefully formulated for the current-vortex sheets to show rigorously that the magnetic effect makes the problem weakly stable and to establish energy estimates in terms of the nonhomogeneous terms and variable coefficients without loss of the order in the linear level; and then these results are exploited to develop a suitable iteration scheme of Nash-Moser-Hörmander type in order to deal with the loss of the order of derivatives in the nonlinear level so that its convergence can be established, which leads to the existence and stability of current-vortex sheets, locally in time, in the three-dimensional MHD. Also see Trakhinin [10] for another independent proof.

Further remarks, trends, and open problems in this direction are also addressed.

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Finite-Time Blow-Up of Classical Solutions to The Full Compressible Navier-Stokes System with Vacuum

Zhouping Xin
(joint work with Wei Yan)

The full compressible Navier-Stokes system reads

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}\mathbb{J}, \qquad (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_t(\rho E) + \operatorname{div}(\rho u E + up) = \operatorname{div}(u\mathbb{T}) + \kappa\Delta\theta, \end{cases} \tag{1}$$

where Ω is a smooth domain in $\mathbb{R}^N$ or the periodic domain $\mathbb{T}^N$, ρ, u, θ, E, and p denote the density, velocity, absolute temperature, total energy, and pressure respectively,

$$\mathbb{J} = \mu(\nabla u + (\nabla u)^t) + \lambda(\operatorname{div} u)I,$$

$$E = \frac{1}{2}|u|^2 + e,$$

with e being the internal energy, and μ and λ being viscosity coefficients satisfying

$$\mu > 0, \qquad \lambda + \frac{2}{N}\mu \geq 0,$$

$k \geq 0$ is the coefficient of the heat conduction. For ideal polytropic fluids, the equations of state are

$$p = R\rho\theta, \quad p = (\gamma - 1)\rho e, \quad p(\rho, s) = Ae^{\frac{s}{c}}\rho^\gamma$$

with $\gamma > 1$, R, A, c being given positive constants.

As a fundamental system in continuum mechanics, the global in time well-posedness of solutions to the Cauchy problem or initial-boundary value problems for (1) has been studied extensively in the literature [4, 5, 3], yet the problem remains quite open for global smooth solutions for general initial and boundary data. One of the difficulties is due to the strong degeneracy of (1) in the presence of vacuum state whose possible apperance cannot be excluded in multi-space dimensions even in the case that the data contains no vacuum, in contrast to the one-dimensional case [2]. In the presence of vacuum, the system (1) may behave singularly. Indeed, as first observed by Xin in [8], in the case that the initial density contains vacuum at far fields, then any $C^1([0,t]; H^m(\mathbb{R}^N))$ solution to the Cauchy problem for (1) has to blow-up in finite time for $m > [\frac{N}{2}] + 2$, see also [1, 6]. In the analysis for finite time blow-up in [1, 6, 8], that the initial density approaches to vacuum at infinity and $u(\cdot, t) \in L^2(\mathbb{R}^N)$ plays the key role. It would be interesting to investigate whether indeed these two conditions are necessary for finite time blow-up of smooth solutions to (1).

Recently, in [7], we identify a class of initial data so that any classical solutions to (1) with such initial data will blow-up in finite time. More precisely, an initial data is said to have an isolated mass group if there is an open connected vacuum region in $\mathbb{R}^N$ which encloses a closed region where the initial density is not identically zero. Also, by a classical solution to (1), we mean that all the quantities are finite and all the derivatives appearing in (1) are continuous up to the boundary. Then the main results in [7] can be summarized as

Theorem. *Assume that $k = 0$, and the initial data has an isolated mass group. Then any classical solution to (1) with such initial data has to blow-up in finite time.*

There are few remarks in order.

Remark 1. Same results hold for $k > 0$.

Remark 2. The theorem applies to both Cauchy problems and periodic problem. In the case that Ω has a boundary, some natural constraints have to be proposed, see [7] for details.

Remark 3. The results generalize the previous results in [8] without the key assumptions that density has vacuum as far field and $u(\cdot, t) \in L^2(\mathbb{R}^N)$.

Remark 4. Our results implies that one cannot expect similar results on the existence of smooth solution with finite entropy and possible large oscillations as in the case of the isentropic compressible Navier-Stokes equations [3].

Remark 5. It would be interesting to investigate the time evolution of a vacuum core.

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The vanishing viscosity limit of the Navier-Stokes equations

Claude Bardos
(joint work with László Székelyhidi, Edriss S. Titi and Emile Wiedemann)

This talk is a report on ongoing work involving several colleagues and in particular László Székelyhidi, Edriss S. Titi and Emile Wiedemann.

The basic question is the convergence of solutions of the Navier-Stokes equations to solutions of the Euler equation when the viscosity goes to zero.

The recent papers of [3], which show the existence of infinite set of "admissible wild" solutions for the Euler equation contribute to the understanding of the problem. (admissible wild solutions correspond to initial data singular enough to support the non uniquenes of the corresponding Cauchy Problem). Keeping that in mind one observed that, in the absence of boundary , any smooth solution of the Euler equation is a viscosity limit of solutions of Navier-Stokes equation. On the other hand one may suppose that the viscosity limit would be a selection criteria among admissible wild solutions. Using a construction of [5] I have given an example (corresponding to [1]) where this is the case.

In the presence of boundary and in particular with the classical fluid mechanic "no slip boundary condition" the situation is much more subtle even in the presence of a smooth solution of the Euler equation.

In fact a short and clever paper of Kato [2] shows that pathological behavior (non convergence to the smooth solution) is equivalent to anomalous dissipation of energy. I elaborated on this issue considering the other alternative : non convergence to a weak solution (and then existence of a non trivial Reynolds Stress tensor) or concentration of oscillation near the boundary. In the course of the talk I stressed the similarity between the standard Reynold Stress tensor of the statistical theory of turbulence and the present one which results from weak convergence and coarse graining and which with the introduction of Wiener measure [4] generates a spectra which can be is with the standard spectra of statistical theory of turbulence.

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The 2D Boussinesq equations with partial dissipation

Jiahong Wu

The Boussinesq equations concerned here model geophysical flows such as atmospheric fronts and ocean circulations. Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and the Navier-Stokes equations such as the vortex stretching mechanism. The global regularity problem on the 2D Boussinesq equations with partial dissipation has attracted considerable attention in the last few years. In this talk we will summarize recent results on various cases of partial dissipation, present the work of Cao and Wu on the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion and explain the work of Chae and Wu on the logarithmically supercritical Boussinesq equations.

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Transport and diffusion of 3D divergence-free vector fields

Yann Brenier

1. Introduction

The usual diffusion equation for 3D divergence-free vector fields (DFVF, in short) reads $\partial_t B + \nabla \times \nabla \times B = 0$. Unfortunately, this equation is not "topology-preserving" (in the sense of Arnold and Moffatt, see [1]), since it cannot be written as

$$\partial_t B + \nabla \times (B \times v) = 0, \tag{1}$$

for some 3D vector field $v = v(t, x)$. This is in sharp contrast with the standard linear heat equation for positive density fields $\partial_t \rho = \Delta \rho$, which can be easily put in "transport" form

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad v = -\nabla(\log \rho). \tag{2}$$

It is therefore desirable to introduce "topology-preserving" diffusion equations for 3D DFVF. We consider the case $v = \frac{(\nabla \times B) \times B}{R}$, where R is some scalar positive function depending only on B and $\nabla \times B$, in which case the (degenerate) diffusion equation can be written

$$\partial_t B + \nabla \times \frac{(|B|^2 I - B \otimes B) \nabla \times B}{R} = 0, \tag{3}$$

where I is the 3×3 identity matrix.. We see that the diffusion process stops for all fields $B = B(x)$ such that $(\nabla \times B) \times B = 0$. Such fields are special stationary solutions of the 3D Euler equations and seem to play an important role in Turbulence Theory [6]. There are several ways of deriving (3), as done, for instance, by Nishiyama [7] with $R = 1$, or, recently, by the author [4]. with $R = |B|^2$. Here below, we sketch a different derivation following ideas of optimal transport theory [8]. More precisely, we generalize the derivation of the heat equation (2) by the so-called JKO scheme due to Jordan, Kinderlehrer and Otto [5]. Let us emphasize that the analysis of equations of type (3) seems to us widely open and very challenging.

2. Transport of 3D divergence-free vector fields

In an elementary way, every loop $s \in \mathbf{R}/\mathbf{Z} \to X(s) \in \mathbf{R}^3$ generates, in the sense of distributions, a DFVF (divergence-free vector field) $x \in \mathbf{R}^3 \to B(x) \in \mathbf{R}^3$

$$B(x) = \int_{s \in \mathbf{R}/\mathbf{Z}} X'(s) \delta(x - X(s)) ds.$$

Conversely, under suitable assumptions, a DFVF $x \in \mathbf{R}^3 \to B(x) \in \mathbf{R}^3$ can be seen (as shown, for instance, by S.Smirnov), as a superposition of loops, parameterized by some parameter a, so that

$$B(x) = \int_{s,a} \partial_s X(s, a) \delta(x - X(s, a)) ds da.$$

Whenever a time-dependent DFVF

$$B(t, x) = \int_{s,a} \partial_s X(t, s, a) \delta(x - X(t, s, a)) ds da$$

is transported by some velocity field $v(t, x)$, so that $\partial_t X(t, s, a) = v(t, X(t, s, a))$, we (formally) recover the transport equation (1). Next, we would like to define a concept of "transportation cost" for DFVF. For a single loop, a natural transportation cost is the space-time area spanned in the Minkowski space $\mathbf{R} \times \mathbf{R}^3$, namely $-\int\int \sqrt{|\partial_s X|^2 - |\partial_s X \times \partial_t X|^2}\, ds dt$. In terms of fields (B, v), we find

$$-\int\int \sqrt{|B|^2 - |B \times v|^2}\, dx dt \tag{4}$$

which defines for us the transport cost of B by v. Let us now substitute for v the new field $E = B \times v$ and rephrase (1,4) respectively by

$$\partial_t B + \nabla \times E = 0, \tag{5}$$

$$\int\int c_0(E,B)dxdt, \quad c_0(E,B) = -\sqrt{|B|^2 - |E|^2}, \quad E \cdot B = 0, \tag{6}$$

without loosing information. Remarkably enough, in these notations, "optimal transportation" of 3D DFVF turns out to be nothing but the (formal) limit $\lambda \to 0$ of the Born-Infeld model of electromagnetism in vacuum [2, 3] with "cost"

$$\int\int c_\lambda(E,B)dxdt, \quad c_\lambda(E,B) = -\sqrt{\lambda^2 + |B|^2 - |E|^2 - \lambda^{-2}(E\cdot B)^2} \tag{7}$$

(while the limit $\lambda \to \infty$ corresponds to the usual Maxwell model).

3. A JKO scheme for 3D DFVF

Let us now introduce our generalization of the JKO scheme [5] for DFVF. We are given a functional $B \to E(B)$ on the space of DFVD, the simplest example being

$$B \to E(B) = \frac{1}{2}\int_{\mathbf{R}^3} |B(x)|^2 dx \tag{8}$$

(which is always infinite for a single loop!). Given a time step $h > 0$, for each $n \in \mathbf{N}$, knowing $B((n-1)h,\cdot)$, we look for a critical point $(B,E)(t,x)$, on the time interval $](n-1)h, nh[$, of the following functional

$$E(B(nh,\cdot)) + \int_{(n-1)h}^{nh} dt \int_{\mathbf{R}^3} c_\lambda(E,B)\, dx, \tag{9}$$

subject to constraint (5). (Notice that we keep $\lambda > 0$ just for the sake of calculation, before setting it to zero.) After introducing a Lagrange multiplier $A(t,x)$ for constraint (5), we now look for a saddle-point (B,E,A) of

$$E(B(nh,\cdot)) + \int_{(n-1)h}^{nh}\int_{\mathbf{R}^3}\{c_\lambda(E,B) + \partial_t A\cdot B - (\nabla\times A)\cdot E\} - \int_{\mathbf{R}^3} B(nh,\cdot)\cdot A(nh,\cdot).$$

Optimizing first in E and A, we formally get, as necessary conditions

$$A(nh,\cdot) = E'(B(nh,\cdot)), \quad E = \partial_D c_\lambda^*(D,B), \quad D = \nabla \times A$$

where

$$c_\lambda^*(D,B) = \sup_{E\in\mathbf{R}^3} E\cdot D - c_\lambda(E,B) = \sqrt{(\lambda^2 + |B|^2)(1+|D|^2) - (B\cdot D)^2}.$$

Letting first h go to zero, we (formally) get a self-consistent evolution for B:

$$\partial_t B + \nabla \times E = 0, \quad E = \partial_D c_\lambda^*(D,B), \quad D = \nabla \times (E'(B)).$$

In the case of functional (8), we more explicitly find

$$E = \frac{(\lambda^2 + |B|^2)D - (B\cdot D)B}{\sqrt{(\lambda^2+|B|^2)(1+|D|^2) - (B\cdot D)^2}}, \quad D = \nabla \times B,$$

which, as $\lambda \to 0$, leads to equation (3) with $R = \sqrt{|B|^2 + |(\nabla \times B) \times B|^2}$.

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On free boundary problem of magnetohydrodynamics in multi-connected domains

Vsevolod A. Solonnikov

The communication is concerned with the solvability theory of evolution free boundary problems of magnetohydrodynamics for viscous incompressible capillary electrically conducting fluids. We consider the following problem: the fluid occupies a variable connected domain $\Omega_1(t)$ with a free boundary $\Gamma(t)$ surrounded by a vacuum region $\Omega_2(t)$. There is also a fixed domain Ω_3 with the boundary S_3 where electric current $\boldsymbol{j}(x,t)$ is given. The domain $\Omega = \bar{\Omega}_1(t) \cup \bar{\Omega}_3 \cup \Omega_2(t)$ is bounded by perfectly conduction surface S. It is assumed that the surfaces S, $\Gamma(t)$, S_3 have no common points. The case $S = \emptyset$ is not excluded.

The governing equations are the Navier-Stokes equation in $\Omega_1(t)$ and the Maxwell equation (in quasi-stationary approximation) in Ω:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot (T(\boldsymbol{v}, p) + T_M(\boldsymbol{H})) = \boldsymbol{f}(x,t), \\ \nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \Omega_1(t), \quad t > 0, \end{cases}$$

$$\begin{cases} \mu \boldsymbol{H}_t = -rot\boldsymbol{E}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad x \in \Omega_1(t) \cup \Omega_2(t) \cup \Omega_3, \\ rot\boldsymbol{H} = \alpha(\boldsymbol{E} + \mu(\boldsymbol{v} \times \boldsymbol{H}), \quad x \in \Omega_1(t), \\ rot\boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{E} = 0, \quad x \in \Omega_2(t), \\ rot\boldsymbol{H} = \alpha \boldsymbol{E} + \boldsymbol{j}(x,t), \quad x \in \Omega_3. \end{cases}$$

where

- $(\boldsymbol{v}, p)$ is the velocity and pressure of the fluid,
- $\boldsymbol{H}(x,t)$, $\boldsymbol{E}(x,t)$: magnetic and electric fields,
- μ is a piecewise constant positive function equal to μ_i in Ω_i,

- α is a piecewise constant function positive in $\Omega_1(t)$, Ω_3 and $\alpha = 0$ in $\Omega_2(t)$,
- $T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v})$: the viscous stress tensor,
- $S(\boldsymbol{v}) = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T$: the doubled rate-of-strain tensor,
- $T_M(\boldsymbol{H}) = \mu(\boldsymbol{H} \otimes \boldsymbol{H} - \frac{1}{2}|\boldsymbol{H}|^2 I)$: the magnetic stress tensor.

In addition, we have the boundary and jump conditions on S, S_3, $\Gamma(t)$:

$$\begin{cases} \boldsymbol{H} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{E}_\tau = 0, \quad x \in S, \\ [\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad [\boldsymbol{E}_\tau] = 0, \quad x \in S_3, \\ \big(T(\boldsymbol{v}, p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma \boldsymbol{n} H, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \\ [\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \\ \mathfrak{n}_t [\mu] \boldsymbol{H}_\tau + [\mathfrak{n}_x \times \boldsymbol{E}] = 0, \quad x \in \Gamma_t, \end{cases}$$

where

- $\boldsymbol{F}_\tau$ is the tangential component of the vector field $\boldsymbol{F}$ on S_3 or $\Gamma(t)$,
- $[\boldsymbol{F}]$: jump of $\boldsymbol{F}(x)$ on Γ_t or S_3,
- V_n: the velocity of evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$,
- $\mathfrak{n} = (\mathfrak{n}_x, \mathfrak{n}_t)$: normal to the surface $\mathfrak{G}_T = \{x \in \Gamma_t, \ t \in (0, T) \subset \mathbb{R}^4$,
- H: the doubled mean curvature of $\Gamma(t)$.

Finally, we have initial conditions

$$\boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x), \quad x \in \Omega_1(0), \quad \boldsymbol{H}(x, 0) = \boldsymbol{H}_0(x),$$
$$x \in \Omega_1(0) \cup \Omega_2(0) \cup \Omega_3,$$

and the orthogonality conditions for $\boldsymbol{E}$:

$$\int_{S_k} \boldsymbol{E} \cdot \boldsymbol{n} dS = 0.$$

where S_k are all the connected components of the boundary of $\Omega_2(t)$ (this normalization condition ensures the uniqueness of $\boldsymbol{E}$ in $\Omega_2(t)$).

It is proved that the above problem has a unique local solution in anisotropic Sobolev spaces, provided that the data possess some regularity properties and satisfy natural compatibility conditions, namely,

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0(x) = 0, \quad x \in \Omega_1(0), \\ &\nabla \cdot \boldsymbol{H}_0(x) = 0, \quad x \in \Omega_1(0) \cup \Omega_2(0) \cup \Omega_3, \\ &rot \boldsymbol{H}_0(x) = 0, \quad x \in \Omega_2(0), \\ &[(S(\boldsymbol{v}_0)\boldsymbol{n})_\tau] = 0, \quad [\boldsymbol{H}_{0\tau}] = 0, \\ &[\mu \boldsymbol{H}_0 \cdot \boldsymbol{N}] = 0, \quad x \in \Gamma(0), \\ &[\boldsymbol{H}_{0\tau}] = 0, \quad [\mu \boldsymbol{H}_0 \cdot \boldsymbol{n}] = 0, \quad x \in S_3, \\ &\boldsymbol{H}_0(x) \cdot \boldsymbol{n} = 0, \quad x \in S. \end{aligned}$$

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Existence, Stability and Coercivity of an extended Navier-Stokes system

Gautam Iyer

(joint work with Mihaela Ignatova, James P. Kelliher, Robert L. Pego and Arghir D. Zarnescu)

The numerical study of the incompressible Navier-Stokes equations in the presence of spatial boundaries is faced with many challenging issues. To address some of these issues, pressure-Poisson schemes replace the incompressibility constraint with an explicit Poisson equation for the pressure. The formal time-continuous limit of the time-discrete scheme proposed in [1] is equivalent to the system

$$\begin{cases} \partial_t u + \mathcal{P}(u \cdot \nabla u - f) = \nu(\mathcal{P}\Delta u + \nabla\nabla \cdot u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u\big|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

If the compatibility condition $\nabla \cdot u_0 = 0$ is imposed on the initial data, then the system (1) is exactly the incompressible Navier-Stokes equations.

This talk discusses existence, regularity, and well-posedness of (1) and stability of the associated time discrete scheme. The difficulty in proving existence of weak solutions to (1) with initial data that is not incompressible stems from the fact that the linear terms in (1) are not coercive with respect to the standard L^2 inner product.

We present two different ideas that can be used to work around this problem. The first idea [4] is to consider the H_0^1-orthogonal projection of u onto the subspace of divergence free H_0^1 fields, and treat the system as a perturbation of the Navier-Stokes equations. While this method can effectively be used to prove analogues of the classical results (global existence of weak solutions, uniqueness in $2D$, higher regularity), it does not appear to help in the stability analysis of the associated time discrete scheme.

The second idea [3] is to use an adjusted H_0^1-equivalent inner-product under which the linear terms in (1) are coercive. This crucially relies on a commutator estimate of [2]. This approach helps in the analysis of the time discrete scheme, but unfortunately requires the domain to be C^3 and only proves existence of strong solutions.

A direction for future study is to prove stability of the associated time discrete scheme in domains with corners, and for long time in 2D.

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Relative entropy applied to shocks for the compressible Euler equation

Alexis F. Vasseur
(joint work with Kyudong Choi, Nicholas Leger)

In this talk, we consider compressible inviscid fluid flows. As example, we can consider either the full compressible Euler system:

$$\begin{aligned}
&\partial_t \rho + \text{div } (\rho u) = 0,\\
&\partial_t(\rho u) + \text{div } (\rho u \otimes u) + \nabla(\rho\theta) = 0,\\
&\partial_t(\rho(\frac{|u|^2}{2} + \frac{3}{2}\theta)) + \text{div } ((\frac{\rho|u|^2}{2} + \frac{5}{2}\rho\theta)u) = 0,
\end{aligned}$$

or the isentropic case:

$$\begin{aligned}
&\partial_t \rho + \text{div } (\rho u) = 0,\\
&\partial_t(\rho u) + \text{div } (\rho u \otimes u) + \nabla\rho^\gamma = 0.
\end{aligned}$$

We are interesting in the stability of $1D$ shocks, that is discontinuous, piecwise constant solutions. This is closely related to the study of asymptotic limits to shock (for instance, from Navier-stokes to Euler). The method is based on relative entropy. The systems in play have entropies which are strictly convex with respect to the conserved quantities. For the full Euler system we have

$$U = (\rho, \rho u, \rho|u|^2/2 + \rho\theta), \qquad \eta(U) = \rho \ln(\rho/\theta^{3/2}).$$

For the isentropic case,

$$U = (\rho, \rho u), \qquad \eta(U) = \rho u^2/2 + \rho^\gamma/(\gamma - 1).$$

We define the relative entropy between two states $U_1, U_2 \in \mathcal{V}$

$$\eta(U_1|U_2) = \eta(U_1) - \eta(U_2) - \eta'(U_2)(U_1 - U_2).$$

From the strict convexity of η we have that

$$\eta(U_1|U_2) \approx |U_1 - U_2|^2.$$

The concept was introduced by Dafermos and DiPerna in 1979. They showed that if U_2 is a Lipschitz solution and U_1 is a weak solution, then

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(U_1|U_2)\, dx \leq C(U_2) \int_{\mathbb{R}} \eta(U_1|U_2)\, dx.$$

Especially, if at $t = 0$ $\int_{\mathbb{R}} \eta(U_1|U_2)\, dx \approx \varepsilon^2$, then at t: $\approx e^{Ct}\varepsilon^2$. When U_2 is a discontinuous solution, this is not true anymore. However, we can show that it is still true up to a drift:

Theorem 1. (Leger, V.) Consider (U_L, U_R, σ) a shock. Then there exist constants $C > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, and

$$\int_0^\infty |U_0(x) - S(x)|^2 \, dx \leq \varepsilon,$$

there exists a Lipschitzian map $x(t)$ such that for any $0 < t < T$:

$$\int_0^\infty |U(t,x) - S(x - x(t))|^2 \, dx \leq C\varepsilon(1+t),$$
$$|x(t) - \sigma t| \leq C\sqrt{\varepsilon t(1+t)}.$$

For $x < 0$, $S(x) = U_L$, for $x > 0$, $S(x) = U_R$.

We also present a first application to the result to some asymptotic limits in the scalar case.

Euler equations and turbulence: analytical approach to intermittency

Alexey Cheskidov
(joint work with Roman Shvydkoy)

Intermittency, if viewed as a measure of non-uniformity of the energy cascade, bears its signature on many statistical laws of a fully developed turbulence. A typical example is the power law for the energy density function,

$$E(\kappa) \sim \frac{\varepsilon^{2/3}}{\kappa^{5/3}} \left(\frac{\kappa_0}{\kappa}\right)^{1-\frac{d}{3}}, \tag{1}$$

where ε is the energy dissipation rate per unit mass, κ_0 the integral scale, and d is the dimension of a set that carries turbulent energy dissipation. One observes similar corrections for longitudinal structure functions or skewness and flatness factors of velocity gradients.

We introduce in precise mathematical terms some of the empirical concepts used to describe intermittency in a fully developed turbulence. We give definitions of the active turbulent region, volume, eddies, energy dissipation set, and derive rigorously some power laws of turbulence. In particular, we justify the formula for the Hausdorff dimension of the energy dissipation set and obtain upper/lower bounds on the energy spectrum.

Now we will outline our basic definitions and results. Let $u \in L^2([0,T] \times \mathbb{T}^3_L)$ be a divergence free time dependent vector field. We will use the Littlewood-Paley decomposition of u given by $u = \sum_{q \geq -1} u_q$. We interpret u_q as the collective velocity field of all eddies $\mathfrak{e}_q$ of dyadic size $\ell_q = L/2^q$. By bridging relations between the Euleran and Lagrangian description of the energy flux through scales ℓ_q we define the active volume as

$$V_q = L^3 \frac{\langle |u_q|^2 \rangle^3}{\langle |u_q|^3 \rangle^2}.$$

Here and throughout the bracket $\langle\cdot\rangle$ denotes the average over the space-time domain $\Omega_T = [0,T]\times\mathbb{T}_L^3$. The dimension d is then introduced as the following exponential type of the sequence $\{V_q\}$:

$$d = 3 - \liminf_{q\to\infty} \frac{\log_2(L^3/V_q)}{q}.$$

Note the dimension d is a number "attached" to the velocity field u without any assumptions on u. By the Hölder inequality we have one obvious bound $V_q \le L^3$, and in the stationary case Bernstein's inequality implies $V_q \ge c\ell_q^3$, where c is adimensional. The latter implies $d \ge 0$, which is a natural bound. However, in the time-dependent case, V_q can scale down faster due to possible temporal rarefaction of active regions giving negative values of the dimension d.

To define active regions that fill volume V_q we find that for an eddie to qualify as "active" it has to have speed at least proportional to

$$s_q \sim \frac{\langle |u_q|^3\rangle}{\langle |u_q|^2\rangle}.$$

The speeds are interpreted as magnitudes of coefficients in the atomic decomposition of u_q, and the supports of atoms are viewed as eddies. The collection of those eddies will constitute what we call an "active" region A_q. The number of active eddies is represented by V_q/ℓ_q^3. We prove that the Hausdorff dimension of

$$A = \limsup_{q\to\infty} A_q$$

is not to exceed d as predicted by the β-model of turbulence, and A_q supports most of the "active" part of the field u_q in terms of L^3-averages. We recover the energy law (1) in the form of an upper and lower bounds. The scaling of the second order structure function is as predicted by the β-model

$$S_2(\ell) \lesssim \varepsilon^{\frac{2}{3}}\ell^{\frac{2}{3}}\left[C_\delta\left(\frac{\ell}{L}\right)^{1-\frac{d}{3}-\delta} + \left(\frac{\ell}{L}\right)^{\frac{4}{3}}\right].$$

for all $\ell < L$, and any $\delta > 0$. Here and throughout, $\lesssim$ denotes inequality up to a universal constant.

We also show that A plays a role of an accumulant of the energy cascade for weak solutions to the Euler equations. We define the energy flux through scales ℓ_q by $\Pi_q = \langle\pi_q\rangle$ where density π_q is the trilinear term that contains all relevant interactions involved in the energy transfer. We prove that the complement of A takes passive part in the cascade process in the sense that there is a nested sequence of sets $G_p \to A$, in fact $G_p = \cup_{k>p} A_k$, such that for all $p > 0$

$$\lim_{q\to\infty} \langle |\pi_q|_{\Omega_T\setminus G_p}|\rangle = 0.$$

In other words, the turbulent cascade tends to dump the energy on the carrier A. These results are established under the natural Onsager regularity condition $\varepsilon = \limsup_{q\to\infty}\langle |u_q|^3\rangle/\ell_q < \infty$ which is known to be suitable for the turbulent interpretation of the field.

For the Navier-Stokes we also find a connection between the level of intermittency and the regularity of Leray-Hopf solutions. First, we define the dissipation wavenumber

$$\Lambda(t) = \min\left\{\lambda_q : \frac{\|u_p(t)\|_\infty \ell_p}{\nu} < c_0,\ \forall\ p > q, q \geq 0\right\},$$

and prove that Leray-Hopf solutions are regular as long as

$$\int_0^T \|\omega_{\leq \Lambda(t)}(t)\|_{B^0_{\infty,\infty}}\, dt < \infty,$$

where $\omega = \nabla \times u$. This criterion naturally turns into the Beale-Kato-Majda criterion in the inviscid case, and we show that it is weaker than every Ladyzhenskaya-Prodi-Serrin condition in the viscous case. If d is defined as a saturation level of Bernstein's inequality at the upper end of the inertial range then a solution with dimension d above 3/2 is automatically regular. Moreover, the time average of Λ is less than Kolmogorov's dissipation number κ_{d} corrected due to intermittency as predicted by the β-model of turbulence.

$$\frac{1}{T}\int_0^T \Lambda(t)\, dt \lesssim \kappa_{\mathrm{d}} = \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{d+1}} \kappa_0^{\frac{d-3}{d+1}},$$

where $\varepsilon = \frac{\nu}{T}\int_0^T \|\nabla u\|_2^2\, dt$.

Long-time asymptotics for the Navier-Stokes equation in a two-dimensional exterior domain

Thierry Gallay

(joint work with Yasunori Maekawa)

We consider the incompressible Navier-Stokes equations in a two-dimensional exterior domain Ω, with no-slip boundary conditions. Our initial data are of the form $u_0 = \alpha\Theta_0 + v_0$, where Θ_0 is the Oseen vortex with unit circulation at infinity and v_0 is a solenoidal perturbation belonging to $L^2(\Omega)^2 \cap L^q(\Omega)^2$ for some $q \in (1, 2)$. If $\alpha \in \mathbb{R}$ is sufficiently small, we show that the solution behaves asymptotically in time like the self-similar Oseen vortex with circulation α. This is a global stability result, in the sense that the perturbation v_0 can be arbitrarily large, and our smallness assumption on the circulation α is independent of the domain Ω.

Formation of singularities and global-in-time results for the Muskat problem

Francisco Gancedo
(joint work with A. Castro, P. Constantin, D. Córdoba, C. Fefferman, M. López-Fernández and R.M. Strain)

The Muskat problem models the dynamics of the interface between two incompressible immiscible fluids with different constant densities ρ^2 and ρ^1 in porous media. The phenomena is described using the experimental Darcy's law that is given in two dimensions by the following momentum equation:

$$u = -\nabla p - (0, \rho). \tag{1}$$

Here u is velocity, p pressure, and ρ density, a step function represented by

$$\rho(x,t) = \begin{cases} \rho^1, & x \in \Omega^1(t), \\ \rho^2, & x \in \Omega^2(t) = R^2 \setminus \Omega^1(t), \end{cases}$$

for $\Omega^i(t)$ two connected regions. As the density ρ is transported by the flow

$$\rho_t + u \cdot \nabla \rho = 0,$$

the free boundary evolves with the two dimensional velocity $u = (u_1, u_2)$. Suppose first that the interface is the graph of a function, $x_2 = f(x_1)$. This characterization is preserved by the system and f satisfies

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_R \frac{(\alpha-\beta)(\partial_\alpha f(\alpha,t) - \partial_\beta f(\beta,t))}{(\alpha-\beta)^2 + (f(\alpha,t) - f(\beta,t))^2} d\alpha, \tag{2}$$

$$f(\alpha,0) = f_0(\alpha), \quad \alpha \in R,$$

where "PV" denotes a principal value integral (the integral does not converge absolutely at infinity). Then it is well-known that the initial-value problem is locally well-possed for positive time, if and only if the heavy fluid lies below the interface and the light fluid lies above it (i.e., $\rho^2 > \rho^1$).If instead the heavy fluid lies above the light fluid ($\rho^1 > \rho^2$), then the initial-value problem is unstable. The problem has been shown to be ill-posed in the sense of Hadamard's point of view. If the interface is not the graph of a function, where the heavy fluid lies below the light fluid we find an unstable regime.

In this talk we show several results. First we give an $L^2(R)$ maximum principle, in the form of a new "log" conservation law

$$\|f\|^2_{L^2}(t) + \frac{\rho^2-\rho^1}{2\pi} \int_0^t ds \int_R d\alpha \int_R d\beta \ln\left(1 + \left(\frac{f(\beta,s)-f(\alpha,s)}{\beta-\alpha}\right)^2\right) = \|f_0\|^2_{L^2},$$

which is satisfied by the equation for the interface. Our second result is global existence for unique strong solutions if the initial data is smaller than an explicitly computable constant, for instance $\|f_0\|_1 \leq 1/5$ for $\rho^2 > \rho^1$. Previous results of this sort used a small constant $\epsilon \ll 1$ which was not explicit. Next, we present a global existence result for Lipschitz continuous solutions with initial data that satisfy $\|f_0\|_{L^\infty} < \infty$ and $\|\partial_\alpha f_0\|_{L^\infty} < 1$. We take advantage of the fact that the bound $\|\partial_\alpha f_0\|_{L^\infty} < 1$ is propagated by solutions, which grants strong compactness properties in comparison to the log conservation law.

For a general curve $z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t))$, the Muskat equation takes the form

$$z_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_R \frac{(z_1(\alpha,t)-z_1(\beta,t))(\partial_\alpha z(\alpha,t)-\partial_\beta z(\beta,t))}{|z(\alpha,t)-z(\beta,t)|^2} d\beta. \tag{3}$$

Lastly, with this formulation we show that there exist analytic initial data in the stable regime for the Muskat problem such that the solution turns to the unstable regime and later breaks down, i.e., no longer belongs to C^4. We exhibit a solution of the Muskat equation for which the interface is initially a smooth stable graph, but later enters an unstable regime (the interface is not a graph anymore), and still later develops a singularity. In [2] we proved that a turnover can occur. Observe that (assuming $\rho^2 > \rho^1$) the solution is stable if and only if the interface is the graph of a function. In particular the significance of a turnover is that the stability breaks down. Moreover, the Muskat solution exists and remains smooth for some time after the turnover, despite the instability of interfaces. The reason why the Muskat solution continues to exist in a highly unstable regime is that, by the time the turnover occurs, the interface has become real-analytic. In fact, starting from a smooth, stable configuration, the Muskat solution becomes instantly real-analytic. For $t > 0$, the function $z(\alpha,t)$ defining the interface $\partial\Omega^1(t) = \partial\Omega^2(t) = \{z(\alpha,t) : \alpha \in R\}$ continues analytically to a strip $\{\alpha \in C : |\Im\alpha| < h(t)\}$, with $h(t) > ct$ for small t. As t approaches the turnover time, the function $h(t)$ may decrease, but it remains strictly positive up to the moment when the turnover occurs (i.e. up to the appearance of a vertical tangent to the interface). Therefore, variants of the Cauchy-Kowalewski theorem show that a real-analytic Muskat solution continues to exist for a short time after the turnover. The interface becomes (presumably) more and more unstable as the turnover progresses. It is natural to believe that at some finite time T, the Muskat solution will break down. However, the technique in [2] is not strong enough to prove that such a breakdown occurs. In this talk, we show that a breakdown occurs, by proving the following result.

There exists a solution $z(\alpha,t)$ of the Muskat equation (3), defined for $t \in [t_0,t_2]$, $\alpha \in R$, such that the following hold:

A. At time t_0, the interface is a graph.
B. At some time $t_1 \in (t_0,t_2)$, the interface is no longer a graph.
C. For each time $t \in [t_0,t_2)$, the interface is real analytic.
D. At time t_2, the interface is C^3 smooth but not C^4 smooth.

Thus, at time t_2, there is no longer a C^4 solution of the Muskat equation. Note that (D) includes the assertion that the curve $\alpha \to z(\alpha,t_2)$ cannot be made C^4-smooth by reparametrization. It is not enough merely to show that $z(\alpha,t_2) \notin C^4_{loc}$. In our proof of the Main Theorem, the failure of C^4 smoothness in (D) will occur at a single point.

In [2], we made a rigorous analysis of the full nonlinear problem and established analytic continuation of Muskat solutions to the time-varying strip $\{|\Im\alpha| < h(t)\}$.

This allowed us to construct real-analytic Muskat solutions $z^0(\alpha, t)$ defined for $t \in [-T, T]$ that "turn over" at time 0. Here, we prove our result by constructing Muskat solutions $z(\alpha, t)$ analytic on a carefully chosen time-varying domain $\Omega(t) = \{|\Im\alpha| < h(\Re\alpha, t)\}$.

We take as initial datum the interface $z(\alpha, \tau)$ at some small positive time τ, and solve the Muskat equation backwards in time until we reach time $-\tau^2$. We take our initial datum $z(\alpha, \tau)$ to be a small perturbation of $z^0(\alpha, \tau)$, where z^0 is the real-analytic "turnover" solution constructed in [2].

Thus, we obtain Muskat solutions z analytic on $\Omega(t)$ and close to z^0 in a suitable Sobolev norm.

For all times $t \in [-\tau^2, \tau)$, we take the domain $\Omega(t)$ to be a neighborhood of the real axis. However, at time $t = \tau$, we take $\Omega(t)$ to be a domain that pinches to a single point.

In particular an initial datum $z(\alpha, \tau)$ may continue analytically to $\Omega(\tau)$, yet fail to belong to C^4 as a function of $\alpha \in R$. We take $z(\alpha, \tau)$ to belong to C^3 but not C^4. Solving the Muskat equation (backwards in time) on $\Omega(t)$ $(t \in [-\tau^2, \tau])$ for such an initial datum, we obtain a Muskat solution as in the statement of our result. In particular, assertions (A) and (B) of the result hold because z is a small perturbation of z^0, which is known to be a graph at time $t_0 = -\tau^2$, but not at time $t_1 = \tau^2$. (Recall that z^0 turns over at time zero.) Assertion (C) holds because $z(\alpha, t)$ continues analytically to $\Omega(t)$, for t less than $t_2 = \tau$. Finally, assertion (D) holds because we picked our initial datum $z(\cdot, \tau)$ to be in C^3 but not in C^4.

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Uniqueness criterion of weak solutions to the Navier-Stokes equations in general unbounded domains

Hideo Kozono

(joint work with Takahiro Okaba, Hidenori Takahashi)

In any domain $\Omega \subset \mathbb{R}^n$, possibly unbounded with non-compact boundary $\partial_t\Omega$, Masuda [1] had proved that if u is a weak solution of the Navier-Stokes equations in $L^\infty(0, T; L^n(\Omega))$ and if $u(\cdot, t)$ is continuous from the right in the $L^n(\Omega)$-norm for every $t \in [0, T)$, then u is the only weak solution on $\Omega \times [0, T)$. Later on, Kozono-Sohr [2] succeeded to remove such a restriction on u as right continuity in $L^n(\Omega)$ and showed that the class $L^\infty(0, T; L^n(\Omega))$ guarantees uniqueness of weak solutions provided Ω is the whole space $\mathbb{R}$, the half-space $\mathbb{R}^n_+$, and interior-exterior domains with the compact boundary. In this talk, we consider general unbounded domains

Ω with uniformly C^2-boundary $\partial\Omega$ and prove the uniqueness of weak solutions in the class $L^\infty(0,T;L^n(\Omega))$. The new function space $\tilde{L}^r(\Omega) = L^r(\Omega) + L^2(\Omega)$ for $1 < r \le 2$, $\tilde{L}^r(\Omega) = L^r(\Omega) \cap L^2(\Omega)$ for $2 \le r < \infty$ plays an important role for our proof.

Our result reads as follows:

Theorem. *Let Ω be a uniformly C^2 domain in $\mathbb{R}^n$. Suppose that u and v are two weak solutions of the the Navier-Stokes equations in $\Omega \times (0,T)$ in the Leray-Hopf class with same initial data $a \in L^2_\sigma(\Omega)$. Assume that*

$$u \in L^\infty(0,T;L^n(\Omega))$$

and that

$$\|v(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2(\Omega)} d\tau \le \|a\|^2_{L^2(\Omega)}, \quad 0 < t < T.$$

Then it holds that $u \equiv v$ in $\Omega \times [0,T)$.

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On one J. Leray's Problem in the Theory of Navier–Stokes Equations

Konstantin Pileckas

We consider the nonhomogeneous boundary value problem for the Navier–Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

in a domain $\Omega \subset \mathbb{R}^n$ with a multiply connected boundary $\partial\Omega$, consisting of N disjoint components Γ_j. The continuity equation in (1) implies the necessary solvability condition

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^N \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^N \mathcal{F}_j = 0, \tag{2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$. The compatibility condition (2) means that the net flux of the fluid over the boundary $\partial\Omega$ is zero.

First time problem (1) was studied in the famous J. Leray's paper published in 1933. J. Leray proved the existence of at least one weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1) under the stronger condition than (2) condition supposing all fluxes

F_j of the boundary value $\mathbf{a}$ to be zero separately across each component Γ_j of the boundary $\partial\Omega$:

$$\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a} \cdot \mathbf{n}\, dS = 0, \qquad j = 1, 2, \ldots, N. \tag{3}$$

After J. Leray's paper there were obtained many partial results concerning the solvability of problem (1). For example, the existence of the solution was proved assuming instead of (3) that the fluxes $\mathcal{F}_j$ are "sufficiently small. The most significant result is due to Ch. Amick (1984) who proved the solvability of problem (1) for arbitrary fluxes $\mathcal{F}_j$, satisfying only the necessary condition (2), in a symmetric two–dimensional domain assuming that boundary value $\mathbf{a}$ also has certain symmetry properties.

In spite of efforts of many mathematicians, in the general case, Leray's problem (1), (2) remains open until now. However, during the last several years some essentially new results were obtained. In the talk an overview of new results obtained by the author during the last several years will be presented.

First, problem (1) was studied in a two–dimensional bounded multiply connected domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, $\overline{\Omega}_2 \subset \Omega_1$. The solvability of problem (1) was proved without any restriction on the value of the flux $\mathcal{F} = \int\limits_{\Gamma_2} \mathbf{a}\cdot\mathbf{n} dS$ provided that $\mathcal{F} > 0$. Note that this is the first result on Leray's problem which does not require smallness of the net flux or symmetry conditions on the domain and boundary value. The proof of the this result is based on the Bernoulli law for a weak solution to the Euler equations, the one-side maximum principle for the total head pressure corresponding to this solution, fine properties of functions from the Sobolev space W_1^2 and on Morse-Sard and Luzin N-properties of Sobolev functions from W_1^2.

Next, using the same ideas, the nonhomogeneous boundary value problem (1), (2) was studied in a three–dimensional axially symmetric bounded domain with multiply connected Lipschitz boundary. Assuming that the boundary value is axially symmetric, it was proved, in particular, the existence of the solution for arbitrary large fluxes $\mathcal{F}_j$ of the boundary value, provided that all these components of the boundary intersect the axis of symmetry.

Problem (1) was also studied in two- and three–dimensional exterior domains. For the two-dimensional case the solvability of it was proved assuming the symmetry of the domain and the boundary value with respect to some axis (the same restrictions as in the paper of Ch. Amick). In the three-dimensional case the solvability for arbitrary fluxes is proved for the axially symmetric case.

All mentioned above results were obtained in series of papers by M. Korobkov, K. Pileckas and R. Russo.

The stationary Navier–Stokes system with nonhomogeneous boundary conditions is studied in a class of domains Ω having "paraboloidal" outlets to infinity. The boundary $\partial\Omega$ is multiply connected and consists of M infinite connected components S_m, which form the outer boundary, and I compact connected components Γ_i forming the inner boundary Γ. The boundary value $\mathbf{a}$ is assumed to have a compact support and it is supposed that the fluxes of $\mathbf{a}$ over the components Γ_i of the inner boundary are sufficiently small. We do not pose any restrictions on fluxes of **a** over the infinite components S_m. The existence of at least one weak solution to the Navier–Stokes problem is proved. The solution may have finite or infinite Dirichlet integral depending on geometrical properties of outlets to infinity. These results were obtained jointly with PhD student K. Kaulakyte.

Non-equilibrium steady states, a (progress) report

Jean-Pierre Eckmann

I will explain, starting from a linear chain, driven by stochastic heat baths at the ends, what other geometries can be studied. If time permits, I will also explain the recent work by L.E. Thomas on the wave equation version of this problem.

Hyperbolic approximation for the Navier–Stokes equations : convergence under minimal regularity requirements.

Pierre Gilles Lemarié-Rieusset
(joint work with Imene Hachicha, Valeria Banica)

In 2004, Brenier, Natalini and Puel [1] studied an hyperbolic approximation of the Navier–Stokes equations, where the parabolic operator $\partial_t - \Delta$ is approximated by a damped hyperbolic operator $\epsilon\partial_t^2 - \Delta/\partial_t$.

In the 2d case, they proved, through energy methods, existence of global solutions to the non-linear wave equation (with initial values in $H^2 \times H^1$) and convergence in L^2 norm when the initial data is in H^1.

Paicu and Raugel extended their work in 2007 [3]. Let $\sigma = d/2-1$ be the critical Sobolev exponent for the Navier–Stokes equations and let, for a real number p, p^+ be any number such that $p^+ > p$. . When the initial value belongs to H^σ (with small $\dot{H}^{1/2}$ norm if $d = 3$), they proved global existence in 2d or in 3d (with initial values in $H^{1^+} \times H^{0^+}$) by using Strichartz estimates; they studied convergence when the initial data is in H^1.

In my talk, I present recent results of Hachicha [2] which adapt the results of Brenier et al. [1] to the case of minimal regularity and to the dimensions $d = 2$ and $d = 3$. Hachicha proves global existence in 2d or in 3d (with initial values in $H^{1+\sigma^+} \times H^{\sigma^+}$) when the initial value belongs to H^σ (with small $\dot{H}^{1/2}$ norm if $d = 3$). If the initial value belongs to H^{σ^+}, she proves strong convergence in the H^σ norm (uniformly on bounded intervals of times). No Strichartz estimates are needed, but only a good lot of energy balances and interpolation inequalities.

References

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On the blow-up problem for the Euler equations and the Liouville type results in the fluid equations

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In this talk we discuss some observations connected with the blow-up problem in the 3D Euler equations. We first consider the scenarios of the self-similar blow-up and its generalizations. For the associated self-similar Euler equations we prove a Liouville type theorem by a simpler argument than the previous one, which shows that fast decaying vorticity at spatial infinity implies the triviality of solution. For an extreme case of the self-similar Euler equations, which corresponds to the Euler equations with damping, we show that any velocity decaying solution at spatial infinity(independent of the decay rate) is trivial. For the axisymmetric Euler equations we observe that the complex Riccati structure exists excluding the pressure term. In this case show that some uniformity condition for the pressure is not consistent with the global regularity. In the second part of talk we present Liouville type theorems for the steady Navier-Stokes equations for both of the incompressible and the compressible cases.

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