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Topologie

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Abstract. The Oberwolfach conference "Topologie" is one of the few occasions where researchers from many different areas in algebraic and geometric topology are able to meet and exchange ideas. Accordingly, the program covered a wide range of new developments in such fields as classification of manifolds, isomorphism conjectures, geometric topology, and homotopy theory. More specifically, we discussed progress on problems such as the Farrell-Jones conjecture, higher dimensional analogues of Harer's homological stability of automorphism groups of manifolds and new algebraic concepts for equivariant spectra, to mention just a few subjects. One of the highlights was a series of four talks on new methods and results about the Farrell-Jones conjecture by Arthur Bartels and Wolfgang Lück.

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Introduction by the Organisers

This conference was the second topology conference in Oberwolfach organized by Thomas Schick, Peter Teichner, Nathalie Wahl and Michael Weiss. About 50 mathematicians participated, working in many different areas of algebraic and geometric topology.

The 20 regular talks of the conference covered a wide range of topics such as stable homotopy theory, geometric group theory, algebraic K- and L-theory, and homotopy theory. One of the goals of the conference is to foster interaction between such different areas and the passage of methods from one to the other. Four of these talks were devoted to new developments concerning the Farrell-Jones isomorphism conjectures in algebraic K- and L-theory, allowing an in-depth discussion of the new ideas necessary for the breakthrough on a decades-old problem at the interface of topology and algebra.

In addition to the regular talks, to give the many young and very young participants the opportunity to present themselves and their work to a broader audience, a "gong show" was organized where five participants gave short overviews on their research efforts and results. Here, **Ryan Grady** from Boston reported on perturbative field theoretic constructions of topological invariants. **Holger Kammeyer** from Göttingen explained his calculation of L^2-invariants of non-cocompact lattices in higher rank Lie groups, in particular the vanishing of L^2-torsion in the even dimensional case and new estimates on Novikov-Shubin invariants. **Rosona Eldred** from Hamburg reported on the calculus of functors and its connections to nilpotence in topology. She obtains new calculations of the terms in the Goodwillie tower of certain functors, provides homotopy theoretically "correct" notions of nilpotence of spaces and uses the techniques to approach classification of such spaces. **Dmitri Pavlov** from Münster and **Daniel Berwick-Evans** from Berkeley discussed their joint work on 2-dimensional Yang-Mills theory and string topology. In particular, they obtain new local and functorial non-topological (volume-form dependent) field theories which extend Yang-Mills theory over a compact Lie group G. This is expected to be the quantization of a (yet to be constructed) local field theory given by the sigma model of the stack $*//G$ with connection.

We now report on some of the highlights of the regular talks, whose abstracts form the main part of this report.

Alexander Berglund and Oscar Randal-Williams gave two very different proofs of (differently formulated) homological stability theorems for automorphism groups of manifolds, stating that after sufficiently many connected sums with $S^n \times S^n$ the homology of the automorphism group of a highly connected manifold stabilizes, with or without dimension restrictions. (The limit was calculated by Galatius and Randal-Williams earlier). Berglund, in joint work with Madsen, does this by an explicit calculation of the rational homotopy type of the mapping space and they explicitly calculate its homology in terms of Lie algebras of certain symplectic derivations. Randal-Williams' approach is joint with Galatius and focuses on the adaptation to higher dimensions of the more classical approaches to homological stability. In particular, they identify explicit models for the classifying space of the diffeomorphism group in question (as a suitable space of embeddings of the model manifold into Euclidean space) and then use the geometry of these spaces to derive homological consequences.

In a talk on geometric group theory, Martin Bridson constructed new examples of finitely presented groups with "exotic" behavior as subgroups of quite easy groups, namely products of surface groups. The exotic behavior refers e.g. to the statement that conjugacy problem is unsolvable, or that the isomorphism problem for finitely presented subgroups is unsolvable. The approach consists of showing that all right angled Artin groups (RAAG) can be embedded as subgroups of surface groups, and then to show, using in particular methods from hyperbolic group theory, that the class of RAAGs is rich enough to display the desired properties.

Benson Farb described a new method (which he jointly develops with Tom Church and Jordan Ellenberg) to compute homological invariants (like Betti numbers) for sequences of spaces X_n with action of the symmetric group Σ_n, e.g. the space of configurations of n ordered points on any manifold. The key idea is an extension or variant of the concept of homological stability which involves representation theoretic patterns. These patterns can be studied using simple underlying structure shared by these and many other examples in algebra and topology. Stefan Schwede described a new branch of equivariant stable homotopy theory based in particular on orthogonal spectra, and presented fundamental calculations in the theory.

Wolfgang Steimle described a new construction of families of manifolds with positive scalar curvature, jointly carried out with Bernhard Hanke, Thomas Schick and Mark Walsh, which is based on new bundles of manifolds over the sphere where the $\hat{A}$-genus is not multiplicative. Also used is a breakthrough family version of the Gromov-Lawson surgery method due to Mark Walsh.

Other talks addressed for example the relation between periodicity in topological surgery and the (originally analytically defined) ρ-invariant, the application of a very functorial approach in generalized differential cohomology in order to construct a multiplicative Beilinson regulator in algebraic K-theory, the use of factorization homology to construct a new homology theory for links in 3-manifolds, or the use of operads to obtain and explain iterated loop space structures on spaces of long knots.

The famous Oberwolfach atmosphere helped to make this meeting exceptionally successful. Our thanks go to the institute for creating this atmosphere and making the conference possible.

Workshop: Topologie

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Abstracts

Motivating the Farrell-Jones Conjecture I & II

Wolfgang Lück

Let G be a discrete group and let R be an associative ring with unit. We begin with motivating the following version of the Farrell-Jones Conjecture

Farrell-Jones Conjecture. Let G be torsionfree and let R be regular. Then the assembly maps for algebraic K- and L-theory

$$\begin{aligned} H_n\big(BG;\mathbb{K}_R\big) &\to K_n(RG); \\ H_n\big(BG;\mathbb{L}_R^{\langle -\infty\rangle}\big) &\to L_n^{\langle -\infty\rangle}(RG), \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$.

There is a more complicate version of the Farrell-Jones Conjectures which makes sense for all groups and rings and allows twistings of the group ring, orientation characters, and the passage to finite wreath products. After we have explained the necessary input from equivariant homology theories, spectra over a groupoids and classifying spaces for families, we give a status report, i.e., we present the following result which summarizes joint work with Bartels, Echterhoff, Farrell, Reich, Rüping and Weinberger.

Theorem. Let $\mathcal{FJ}$ be the class of groups for which the Farrell-Jones Conjecture is true in its general form. Then:

(1) Hyperbolic groups belong to $\mathcal{FJ}$;
(2) CAT(0); groups belong to $\mathcal{FJ}$;
(3) Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;
(4) $Gl_n(R)$ belongs to $\mathcal{FJ}$ if the underlying additive group of the ring R is a finitely generated free abelian group;
(5) Arithmetic groups over number fields;
(6) Fundamental groups of (not necessarily compact) 3-manifolds possibly with boundary) belong to $\mathcal{FJ}$;
(7) If G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 * G_1$ and $G_0 \times G_1$;
(8) If G belongs to $\mathcal{FJ}$, then any subgroup of G belongs to $\mathcal{FJ}$;
(9) If $H \subseteq G$ has finite index and H belongs to $\mathcal{FJ}$ and then G belongs to $\mathcal{FJ}$;
(10) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). If each G_i belongs to $\mathcal{FJ}$, then also the direct limit of $\{G_i \mid i \in I\}$.
(11) Let $1 \to H \to G \xrightarrow{p} Q \to 1$ be an extension of groups. If Q and for all virtually cyclic subgroups $V \subseteq Q$ the preimage $p^{-1}(V)$ belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

For information about its proof we refer to the two lectures by Arthur Bartels.

Since certain prominent constructions of groups yield colimits of hyperbolic groups, the class $\mathcal{FJ}$ contains many interesting groups, e.g. limit groups, Tarski monsters, groups with expanders and so on. Some of these groups were regarded as possible counterexamples to the conjectures above but are now ruled out by the theorem above.

We explain and state the following conjectures and discuss their relevance. They all are consequences of the Farrell-Jones Conjecture above, where one has sometimes to make the assumption that the relevant dimensions are greater or equal to five.

Kaplanski Conjecture. If G is torsionfree and R is an integral domain, then 0 and 1 are the only idempotents in RG.

Conjecture. Suppose that G is torsionfree. Then $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Novikov Conjecture. Higher signatures are homotopy invariants.

Borel Conjecture. An aspherical closed manifold is topologically rigid.

There are also prominent constructions of closed aspherical manifolds with exotic properties, e.g., whose universal covering is not homeomorphic to Euclidean space, whose fundamental group is not residually finite or which admit no triangulation. All these constructions yield fundamental groups which are CAT(0) and hence yield topologically rigid manifolds.

Conjecture. If G is a finitely presented Poincaré duality group of dimension then it is the fundamental group of an aspherical homology ANR-manifold.

Conjecture If G is a hyperbolic group with S^n as boundary, then there is a closed aspherical manifold M whose fundamental group is G.

The Farrell-Jones Conjecture is open for instance for solvable groups, $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups or automorphism groups of finitely generated free groups.

There are also interesting versions for pseudo-isotopy, Waldhausen's A-theory, topological Hochschild homology, topological cyclic homology, rapid decay algebras which have to be investigated.

References

[1] **Bartels, Echterhoff, A. and Lück, W.**: *"Inheritance of Isomorphism Conjectures under colimits"*, Proceedings of the conference "K-theory and non-commutative geometry" in Valladolid, August/September 2006, EMS Series of Congress reports, editors: Cortinaz, Cuntz, Karoubi, Nest, Weibel, European Mathematical Society (2008) 41–70.

[2] **Bartels, A. and Farrell, T. and Lück, W.**: *"The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups*, arXiv:1101.0469 [math.GT], to appear in JAMS, (2011).

[3] **Bartels, A. and Lück, W.**: *"On crossed product rings with twisted involutions, their module categories and L-theory"*, Procedings of the summer school "Cohomology of groups and algebraic K-theory", Hangzhou, China, July 1 until July 12 in 2007, Advanced Lectures in Mathematics 12, International Press, 1-55 (2009).

[4] **Bartels, A. and Lück, W.**: *"The Borel Conjecture for hyperbolic and CAT(0)-groups"*, Annals of Mathematics 175, 631–689 (2012).
[5] **Bartels, A. and Lück, W.**: *"Geodesic flow for CAT(0)-groups"*, Geometry and Topology 16, 1345-1391 (2012).
[6] **Bartels, A. and Lück, W.**: *"The Farrell-Hsiang method revisited"*, arXiv:1101.0466 [math.GT], to appear in Math. Annalen, (2011).
[7] **Bartels, A., Lück, W. and Reich, H.**: *"Equivariant covers for hyperbolic groups"*, Geometry and Topology 12 (2008) 1799–1882.
[8] **Bartels, A., Lück, W. and Reich, H.**: *"The K-theoretic Farrell-Jones Conjecture for hyperbolic groups"*, Inventiones mathematicae 172 (2008), 29–70.
[9] **Bartels, A., Lück, W. and Reich, H.**: *"On the Farrell-Jones Conjecture and its applications"*, Journal of Topology 1 (2008), 57–86.
[10] **Bartels, A., Lück, W., and Weinberger, S.**: *"On hyperbolic groups with spheres as boundary"*, Journal of Differential Geometry 86 (1), 1–16, (2010) (2010).
[11] **Kreck, M. and Lück, W.**: *"Topological rigidity of non-aspherical manifolds"*, Pure and Applied Mathematics Quarterly 5 (3), special issue in honor of Friedrich Hirzebruch part 2, 873–914 (2009).
[12] **Lück, W.**: *"On the Farrell-Jones Conjecture and related conjectures"*, extended notes of six talks presented at the summer school "Cohomology of groups and algebraic K-theory", Hangzhou, China, July 1 until July 12 in 2007, arXiv:math.KT/0710.2269v1 (2007).
[13] **Lück, W. and Reich, H.**: *"The Baum-Connes and the Farrell-Jones Conjectures"*, Handbook of K-theory Volume 2, editors: É.M. Friedlander, D.R. Grayson, Springer (2005) 703 – 842.

On the proof of the Farrell-Jones Conjecture

Arthur Bartels
(joint work with Wolfgang Lück, Holger Reich)

These two talks gave an introduction to proofs of the Farrell-Jones Conjecture. An important tool for such proofs is controlled algebra and in particular the language of geometric modules as developed by Connell-Hollingsworth, Quinn, Pedersen and others. This theory can be used to describe the image of assembly maps. an instance of this is the following result.

Theorem. For every natural number N there is $\epsilon > 0$ such that the following holds.

Let E be a simplicial complex of dimension at most N. Let G act simplicially on E. If f is an automorphism of geometric $R[G]$-modules over E such that f and f^{-1} are ϵ-controlled, then the K-theory class of $[f] \in K_1(R[G])$ belongs to the image of the assembly map

$$H_1^G(E_F G; \mathbb{K}) \to K_1(R[G]).$$

I used this result to outline the proof of the following result from [1].

Theorem. Let G be a group that is finitely generated by S. Let F be a family of subgroups of G.

Assume that there is a natural number N such that for any $\epsilon > 0$ there exists the following:

(1) A compact, contractible, metrizable space X with an action of G by homeomorphisms. Moreover, for any $\delta > 0$ the space X is assumed to be δ-homotopy equivalent to a finite simplicial complex of dimension at most N.
(2) A map $f\colon X \to E$ where E is a simplicial complex of dimension at most N and G acts simplicially on such that the isotropy groups of the action belong to F. Moreover, the map f is assumed to be almost G-equivariant in the following sense: for $x \in X$, $g \in S \cup S^{-1}$ we have $d^1(f(gx), gf(x)) \leq \epsilon$.

Then the assembly map

$$H_*^G(E_F G; \mathbb{K}) \to K_*(R[G])$$

is bijective.

Using this result I outlined the proof of the Farrell-Jones Conjecture for hyerbolic groups. In particular I emphasized the role flow spaces for the Farrell-Jones conjecture. I also discussed the necessary changes in the case of $CAT(0)$-groups.

References

[1] A. Bartels, W. Lueck, H. Reich *The K-theoretic Farrell-Jones conjecture for hyperbolic group*, Invent. Math. **172** (2008), 29–70.
[2] A. Bartels, W. Lueck, *The Borel Conjecture for hyperbolic and CAT(0)-groups*, Ann. of Math. (2) **175** (2012), 631-689.

Homological stability for automorphisms of manifolds

Alexander Berglund
(joint work with Ib Madsen)

For an oriented manifold M, let $\mathrm{aut}(M) \supset \widetilde{\mathrm{Diff}}(M) \supset \mathrm{Diff}(M)$ denote the topological monoids of, respectively, homotopy automorphisms, block diffeomorphisms and diffeomorphisms, that are orientation preserving and restrict to the identity on the boundary ∂M.

Let M_g^{2d} denote the g-fold connected sum of $S^d \times S^d$ with itself,

$$M_g^{2d} = S^d \times S^d \# \cdots \# S^d \times S^d,$$

and let $M_{g,r}^{2d}$ be the manifold obtained from M_g^{2d} by removing the interiors of r disjoint embedded $2d$-disks. The manifold $M_{g+1,1}^{2d}$ can be obtained by gluing $M_{g,1}^{2d}$ and $M_{1,2}^{2d}$ along a boundary component, so every automorphism of $M_{g,1}^{2d}$ that restricts to the identity on the boundary can be extended to $M_{g+1,1}^{2d}$ by the identity on $M_{2,1}^{2d}$. This gives rise to a map on classifying spaces

$$\sigma_{d,g}^{\mathrm{aut}}\colon B\,\mathrm{aut}(M_{g,1}^{2d}) \to B\,\mathrm{aut}(M_{g+1,1}^{2d}).$$

Similarly, there are maps $\sigma_{d,g}^{\widetilde{\mathrm{Diff}}}$ and $\sigma_{d,g}^{\mathrm{Diff}}$.

For $2d = 2$ and $g > 1$, the manifold $M^2_{g,1}$ is a connected orientable genus g surface with one boundary circle, and the topological group $\mathrm{Diff}(M^2_{g,1})$ is homotopy equivalent to the discrete mapping class group $\pi_0\mathrm{Diff}(M^2_{g,1})$. A classical homological stability result for mapping class groups, originally due to Harer [4] and later improved by others (see [8] for a recent account), implies that $\sigma^{\mathrm{Diff}}_{1,g}$ induces an isomorphism in homology in degrees $< \frac{2g-2}{3}$. The same is then true for $\sigma^{\mathrm{aut}}_{1,g}$ because, as it turns out, the inclusion $\mathrm{Diff}(M^2_{g,1}) \to \mathrm{aut}(M^2_{g,1})$ is a homotopy equivalence for $g > 1$.

In joint work with Madsen [1, 2], we prove a counterpart of Harer's stability theorem in higher dimensions. For $2d > 2$, the classifying spaces $B\,\mathrm{aut}(M^{2d}_{g,1})$, $B\widetilde{\mathrm{Diff}}(M^{2d}_{g,1})$ and $B\mathrm{Diff}(M^{2d}_{g,1})$ have different homotopy types, and we need to treat each case separately.

Theorem 1. *For $d > 2$ the maps $\sigma^{\mathrm{aut}}_{d,g}$ and $\sigma^{\widetilde{\mathrm{Diff}}}_{d,g}$ induce isomorphisms in rational homology in degrees $< \frac{g-4}{2}$, and the map $\sigma^{\mathrm{Diff}}_{d,g}$ induces an isomorphism in rational homology in degrees $< \min(\frac{g-4}{2}, 2d-4)$*[1]*.*

The proof uses rational homotopy theory, the surgery exact sequence, and Morlet's lemma of disjunction. Along the way, we obtain the following general result about the rational homotopy type of classifying spaces of homotopy automorphisms of highly connected manifolds.

Theorem 2. *Let M be a $(d-1)$-connected $2d$-dimensional closed manifold and let N be the result of removing the interior of an embedded $2d$-disk from M. Suppose that $d > 1$ and $n := \operatorname{rank} \mathrm{H}_d(M) > 2$.*

(1) *The universal cover of $B\,\mathrm{aut}(N)$ is rationally homotopy equivalent to the classifying space of the graded Lie algebra*

$$\mathrm{Der}^+_\omega\,\mathbb{L}(\alpha_1,\ldots,\alpha_n)$$

of positive degree derivations on the free graded Lie algebra $\mathbb{L}(\alpha_1,\ldots,\alpha_n)$ that annihilate the element

$$\omega = \sum_{i,j} \omega_{ij}[\alpha_i,\alpha_j].$$

Here (ω_{ij}) is an $n \times n$-matrix representing the intersection form of M. The generators α_i have degree $d-1$.

(2) *The universal cover of $B\,\mathrm{aut}(M)$ is rationally homotopy equivalent to the classifying space of the graded Lie algebra*

$$\mathrm{OutDer}^+\,\mathbb{L}(\alpha_1,\ldots,\alpha_n)/(\omega)$$

of positive degree outer derivations on the quotient $\mathbb{L}(\alpha_1,\ldots,\alpha_n)/(\omega)$.

(3) *The fundamental groups of $B\,\mathrm{aut}(M)$ and $B\,\mathrm{aut}(N)$ are both commensurable to the group of linear automorphisms of $\mathrm{H}_d(M)$ that preserve the intersection form.*

[1] The range for $\sigma^{\mathrm{Diff}}_{d,g}$ has been improved to $< \frac{g-4}{2}$ by Galatius and Randal-Williams [3].

The 'classifying space' of a graded Lie algebra is a simply connected rational space whose homotopy Lie algebra — the direct sum of all homotopy groups together with the Whitehead product — is isomorphic to the given graded Lie algebra. The existence of such a classifying space was established by Quillen [7].

It is interesting to note that for $M = M_g^{2d}$ and d odd, the first Lie algebra appearing in the above theorem also appears in the work of Kontsevich on the cohomology of outer automorphisms of free groups [5, 6].

References

[1] A. Berglund, I. Madsen, *Homological stability of diffeomorphism groups*, arXiv:1203.4161v2 [math.AT]

[2] A. Berglund, I. Madsen *Rational homotopy theory of automorphisms of highly connected manifolds*, in preparation.

[3] S. Galatius, O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds*, arXiv:1203.6830 [math.AT]

[4] J.L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), no. 2, 215–249.

[5] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gelfand Mathematical Seminars, 1990–1992, p.173–187, Birkhäuser, 1993.

[6] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), 97–121, Progr. Math., 120, Birkhäuser, 1994.

[7] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969) 205–295.

[8] N. Wahl, *Homological stability for mapping class groups of surfaces*, arXiv:1006.4476v2 [math.GT]

Representation stability and FI-modules: a progress report

Benson Farb

(joint work with Tom Church and Jordan Ellenberg)

This talk is a summary of the theory of representation stability and FI-modules, initiated in [CF] and further applied and developed in [C, CEF, CEFN].

Let FI be the category whose objects are finite sets and whose morphisms are injections. An *FI-module* over a commutative ring k is a functor V from FI to the category of k-modules. We denote the k-module $V(\{1,\ldots,n\})$ by V_n. Since $\mathrm{EndFI}(n) = S_n$, any FI-module V determines a sequence of S_n-representations V_n with linear maps between them respecting the group actions. One theme of our work is the conceptual power of encoding this large amount of (potentially complicated) data into a single object V.

Many of the familiar notions from the theory of modules, such as submodule and quotient module, carry over to FI-modules. In particular, there is a natural notion of *finite generation* for FI-modules. An FI-module V is *finitely generated* if there is a finite set S of elements in $\coprod_i V_i$ so that no proper sub-FI-module of V contains S

Theorem 1: *If V is a finitely generated FI-module over a field of characteristic 0, there is an integer-valued polynomial $P \in \mathbb{Q}[T]$ and some $N \geq 0$ so that*

$$\dim(V_n) = P(n) \text{ for all } n \geq N.$$

Finitely generated FI-modules arise in a variety of contexts. Here are some important examples.

Theorem 2: *Each of the following sequences V_n of S_n-representations is a finitely generated FI-module (any parameter not equal to n should be considered fixed and nonnegative).*

1. $V_n = H^i(\mathrm{Conf}_n(M);\mathbb{Q})$, *where* $\mathrm{Conf}_n(M) =$ *is configuration space of n distinct ordered points on a connected, oriented manifold M.*
2. $V_n = R_J^{(r)}(n)$, *where* $J = (j_1, \dots j_r)$, $R^{(r)}(n) = \bigoplus_J R_J^{(r)}(n)$= *$r$-diagonal coinvariant algebra on r sets of n variables.*
3. $V_n = H^i(\mathcal{M}_{g,n};\mathbb{Q})$, *where $\mathcal{M}_{g,n}$ = moduli space of n-pointed genus $g \geq 2$ curves.*
4. $V_n = \mathcal{R}^i(\mathcal{M}_{g,n})$, *the i^{th} graded piece of tautological ring of $\mathcal{M}_{g,n}$*
5. $O(X_r(n))_i$, *the space of degree i polynomials on $X_r(n)$, the rank variety of $n \times n$ matrices of rank $\leq r$.*

Except for a few special (e.g. $M = \mathbb{R}^d$) and low-complexity cases, the dimensions of the vector spaces in Theorem 2 are not known, or even conjectured. Exact computations seem to be extremely difficult. By contrast, Theorem 1 and Theorem 2 together imply the following, which gives an answer, albeit a non-explicit one, in all of these cases.

Theorem 3: *Let $\{V_n\}$ be any of the sequences of vector spaces listed in Theorem 2. Then there exists an integer N and an integer-valued polynomial $P \in \mathbb{Q}[T]$ such that*

$$\dim(V_n) = P(n) \text{ for all } n \geq N.$$

References

[C] T. Church, Homological stability for configuration spaces of manifolds, *Invent. Math.*, 188 (2012) 2, 465–504.

[CEF] T. Church, J. S. Ellenberg and B. Farb, FI-modules: a new approach to stability for S_n-representations, arXiv:1204.4533, April 2012.

[CEFN] T. Church, J. S. Ellenberg, B. Farb and R. Nagpal, FI-modules over Noetherian rings, in preparation.

[CF] T. Church and B. Farb, Representation theory and homological stability, eprint, arXiv:1008.1368v2, revised October 2011.

A higher chromatic analogue of the J-homomorphism

Craig Westerland

The stable homotopy groups of a space X are defined as the colimit

$$\pi_j^S(X) = \lim_{m\to\infty} \pi_{j+m}(\Sigma^m X) = \lim_{m\to\infty} \pi_j(\Omega^m \Sigma^m X) = \pi_j(QX).$$

where $QX = \varinjlim \Omega^m \Sigma^m X$.

The J-homomorphism $J : \pi_j(O) \to \pi_j^S(S^0)$ may be regarded as a first approximation to the stable homotopy groups of S^0; here O denotes the infinite orthogonal group $\varinjlim O(m)$. It is induced in homotopy by the limit over m of maps

$$J_m : O(m) \to \Omega^m S^m,$$

where for a matrix $M \in O(m)$ regarded as a linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$, $J_m(M) = M \cup \{\infty\} : S^n \to S^n$. The homotopy groups of the domain is computable via Bott periodicity, and are

j mod 8	0	1	2	3	4	5	6	7
$\pi_j(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The work of Adams [Ada66] shows that for an odd prime p, in dimensions 3 mod 4, the p-torsion of the image in $\pi_j^S(S^0)$ of the cyclic group $\pi_j(O)$ is isomorphic to $\mathbb{Z}/p^{k+1}$, when we can write $j+1 = 2(p-1)p^k m$, with m coprime to p. For other j which are 3 mod 4, the p-torsion in the image of J is zero. As we are working away from $p = 2$, this computation may be done using U in place of O.

The aim of this talk is to explore an analogous result in the $K(n)$-local stable homotopy category. When $n = 1$, $K(1)$ is identified with (a split summand of) mod p K-theory. The fact that $\pi_*(U) = \pi_{*+1}(K)$ for $* > 0$ suggests that Adams' computation of the p-torsion in the image of J is related to $K(1)$-local homotopy theory. This is in fact the case; the localisation map

$$\pi_*^S(S^0) \to \pi_*(L_{K(1)} S^0)$$

carries $\operatorname{im} J$ isomorphically onto the codomain in positive degrees.

One substantial difference between the stable homotopy category and its $K(n)$-local variant is the existence of exotic invertible elements. In the stable homotopy category, the only spectra which admit inverses with respect to the smash product are spheres; thus the *Picard group* of equivalence classes of such spectra is isomorphic to $\mathbb{Z}$. In contrast, the Picard group of the $K(n)$-local category, Pic_n, includes p-complete factors as well as torsion (see, e.g., [HMS94, GHMR12]).

Our main result is a computation of part of the *Picard graded* homotopy of the $K(n)$-local sphere $S := L_{K(n)} S^0$ analogous to the image of J computation. Throughout, p will denote an odd prime; when localising with respect to $K(n)$, the prime p is implicitly used.

Theorem 1. *Let $\ell \in \mathbb{Z}$, and write $\ell = p^k m$, where m is coprime to p. Then the group $[S\langle\det\rangle^{\otimes \ell(p-1)}, L_{K(n)}S^1]$ contains a subgroup isomorphic to $\mathbb{Z}/p^{k+1}$. Furthermore, if $n^2 < 2p-3$, there is an exact sequence*

$$0 \to \mathbb{Z}/p^{k+1} \to [S\langle\det\rangle^{\otimes \ell(p-1)}, L_{K(n)}S^1] \to N_{k+1} \to 0$$

where $N_{k+1} \leq \pi_{-1}(S)$ is the subgroup of p^{k+1}-torsion elements.

Here, $S\langle\det\rangle \in \mathrm{Pic}_n$ was introduced by Goerss et al. in [GHMR12]; it is defined below. When $n = 1$ and $p > 2$, $S\langle\det\rangle$ may be identified as $L_{K(1)}S^2$, and so this result partially recovers the classical image of J computation. More generally, $S\langle\det\rangle$ may be identified as a shift of the Brown-Comenetz dual of the n^{th} monochromatic layer of the sphere spectrum if $\max\{2n+2, n^2\} < 2(p-1)$ (see [HG94]).

Morava's E-theories are Landweber exact cohomology theories E_n associated to the universal deformation of the Honda formal group Γ_n (defined over $\mathbb{F}_{p^n}$) to $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$. When $n = 1$, E_1 is precisely p-adic K-theory. The Goerss-Hopkins-Miller theorem [GH04, GH05] equips the spectrum E_n with a continuous action of the *Morava stabiliser group*

$$\Gamma_n := \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$$

which lifts the defining action in homotopy. The work of Devinatz-Hopkins [DH04] and Davis [Dav06, Dav09] then allows one to define continuous[1] homotopy fixed point spectra with respect to this action in a consistent way. The associated homotopy fixed point spectrum is the $K(n)$-local sphere: $E_n^{h\Gamma_n} \simeq L_{K(n)}S^0$.

The automorphism group $\mathrm{Aut}(\Gamma_n)$ is known to be the group of units of an order of a rank n^2 division algebra over $\mathbb{Q}_p$; the determinant of the action by left multiplication defines a homomorphism $\det : \Gamma_n \to \mathbb{Z}_p^\times$. We will write $S\Gamma_n$ for the kernel of this map. We may define the homotopy fixed point spectrum $E_n^{hS\Gamma_n}$ for the restricted action of this subgroup. This spectrum retains a residual action of $\mathbb{Z}_p^\times = \Gamma_n/S\Gamma_n$; for an element $k \in \mathbb{Z}_p^\times$, we will write the associated map as $\psi^k : E_n^{hS\Gamma_n} \to E_n^{hS\Gamma_n}$. The reader is encouraged to think of these automorphisms as analogues of Adams operations. Noting that $\mathbb{Z}_p^\times = \mu_{p-1} \times (1+p\mathbb{Z}_p)^\times$ is topologically cyclic with generator $g = \zeta_{p-1}(1+p)$, we define F_γ as the homotopy fibre of

$$\psi^g - \gamma : E_n^{hS\Gamma_n} \to E_n^{hS\Gamma_n}$$

for any $\gamma \in \mathbb{Z}_p^\times$. These spectra are always invertible, and in fact the construction $\gamma \mapsto F_\gamma$ defines a homomorphism $\mathbb{Z}_p^\times \to \mathrm{Pic}_n$. When $\gamma = 1$, the associated homotopy fibre defines the homotopy fixed point spectrum for the action of $\mathbb{Z}_p^\times$. and so

$$F_1 = (E_n^{hS\Gamma_n})^{h\mathbb{Z}_p^\times} \simeq E_n^{h\Gamma_n} \simeq L_{K(n)}S^0$$

In contrast, we define $S\langle\det\rangle := F_g$.

[1] All homotopy fixed point spectra considered in this article will be of the continuous sort.

References

[Ada66] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71. MR 0198470 (33 #6628)

[Dav06] Daniel G. Davis, *Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action*, J. Pure Appl. Algebra **206** (2006), no. 3, 322–354. MR 2235364 (2007b:55008)

[Dav09] ———, *Iterated homotopy fixed points for the Lubin-Tate spectrum*, Topology Appl. **156** (2009), no. 17, 2881–2898, With an appendix by Daniel G. Davis and Ben Wieland. MR 2556043 (2010j:55009)

[DH04] Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47. MR 2030586 (2004i:55012)

[GHMR12] Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk, *Picard groups for the prime 3 and chromatic level 2*, available online for download at the url `www.math.northwestern.edu/∼pgoerss/papers/picat3.pdf`, 2012.

[GH04] Paul G. Goerss and Michael J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. MR 2125040 (2006b:55010)

[GH05] Paul G. Goerss and Michael J. Hopkins, *Moduli problems for structured ring spectra*, `www.math.northwestern.edu/∼pgoerss/spectra/obstruct.pdf`, 2005.

[HMS94] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky, *Constructions of elements in Picard groups*, Topology and representation theory (Evanston, IL, 1992), Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 89–126. MR 1263713 (95a:55020)

[HG94] M. J. Hopkins and B. H. Gross, *The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), no. 1, 76–86. MR 1217353 (94k:55009)

Cubes, RAAGs, and Subgroups of Mapping Class Groups

Martin Bridson

1. Outline

How diverse and complicated can the finitely presented subgroups of a nice group be? ("Complicated" might, *in extremis*, be made precise by establishing the algorithmic undecidability of basic decision problems.) We'll address this question in the case where the ambient group is the mapping class group $\mathrm{Mod}(S)$ of a compact surface S of genus $g > 1$. The discussion will be used as a vehicle to introduce non-positively curved cube complexes and to explain why they have recently had such a great impact on low dimensional topology and group theory.

2. Subgroups of mapping class groups

Theorem A. *If the genus of S is sufficiently large, then the isomorphism problem for the finitely presented subgroups of $\mathrm{Mod}(S)$ is unsolvable.*

Theorem B. *If the genus of S is sufficiently large, then there is a finitely presented subgroup of $\mathrm{Mod}(S)$ with unsolvable conjugacy problem.*

Theorem C. *If the genus of S is sufficiently large, then there are finitely presented subgroups of $\mathrm{Mod}(S)$ for which the membership problem is unsolvable.*

Theorem D. *If the genus of S is sufficiently large, then there are finitely presented subgroups of* $\mathrm{Mod}(S)$ *whose Dehn functions are exponential.*

Theorem E. *If the genus of S is sufficiently large, then there exist finitely presented subgroups of* $\mathrm{Mod}(S)$ *with infinitely many conjugacy classes of torsion elements.*

For an account of the history of the problems settled by these theorems, see [16]. The first three of these theorems will be deduced from the corresponding results concerning subgroups of right-angled Artin groups. Any group which is virtually special in the sense of Haglund and Wise embeds in the mapping class group of infinitely many closed surfaces [8]. Theorem E was first proved in [7].

3. Subgroup of direct products of free groups

If $H_1, H_2 < \mathrm{Mod}(S)$ are supported on disjoint subsurfaces of S, then they commute. One can embed g disjoint once-punctured tori in a surface of genus g, and the mapping class group of a punctured-torus is $\mathrm{SL}(2,\mathbb{Z})$, which contains non-abelian free groups. Thus, if S has genus g, then $\mathrm{Mod}(S)$ contains the direct product D of g non-abelian free groups. In D, what (finitely generated of finitely presented) subgroups do we get?

Example 3.1. Let $Q = \langle A \mid R \rangle$ be a finitely presented group. Let F be the free group on A and let $p : F \to Q$ be the surjection implicit in the notation. The kernel of p is finitely generated if and only if Q is finite, but the **fibre product**

$$P = \{(u,v) \mid p(u) = p(v)\} < F \times F$$

is always (finitely!) generated by $\{(a,a),\ (r,1); a \in A,\ r \in R\}$.

This observation (which has a long history) provides complicated finitely generated subgroups of $F \times F$: if Q has an unsolvable word problem, then P has an unsolvable conjugacy problem; since $P = F \times F$ if and only if $Q = 1$, there is no algorithm that can determine which finite subsets of $F \times F$ generate; and there is no algorithm that, given a finite subset $S \subset F \times F$, can calculate the first homology of $\langle S \rangle$ (see [11]). But this does not provide us with complicated *finitely presented* subgroups of mapping class groups, because P is finitely presentable if and only if Q is finite. Indeed Bridson and Miller [12] (also [10]) show that f.p. subgroups of direct products of free groups are not so wild, so we must look elsewhere for the subgroups of Theorems A to C.

Theorem 3.2. [12] *The conjugacy and membership problems are algorithmically solvable for every finitely presented subgroup of a direct product of free (or surface) groups.*

Example 3.3. The Stallings-Bieri Groups. Let $h : F \times \cdots \times F \to \mathbb{Z}$ be a homomorphism that restricts to an epimorphism on each of the n factors. The kernel SB_n has a classifying space with a finite $(n-1)$-skeleton, but $H_n(\mathrm{SB}_n, \mathbb{Z})$ is not finitely generated.

Thus direct products of free groups supply us with some interesting finitely presented subgroups of mapping class groups, but not much beyond these examples:

Theorem 3.4. [12] *If $H < F_1 \times \cdots \times F_n = D$ is a subdirect product of finitely generated free groups that intersects each factor, then there is a subgroup of finite index $D_0 < D$ such that H contains the $(n-1)$st term of the lower central series of D_0.*

4. Fibre Products, Rips, and Decision Problems

We take up the theme of Example 3.1. Following Rips, we express Q as the quotient of a hyperbolic group rather than a free group, with a gain in the nature of the kernel.

Theorem 4.1. [20] *There is an algorithm that, given a finite group presentation $\mathcal{Q}$, will construct a short exact sequence*

$$1 \to N \to H \to Q \to 1$$

with H hyperbolic group, Q the group with presentation $\mathcal{Q}$, and N finitely generated.

In Rips's original argument, H satisfies a small cancellation condition (prescribed and as strict as you like). The construction is very flexible and subsequent variations by different authors have imposed extra conditions on H.
Template: In the last fifteen years, many results have been proved to the effect that finitely presented subgroups of direct products of hyperbolic (and related) groups can be remarkably wild (in contrast to the free group case). Most of these results rely on the template described I described in [9]: one employs a version of the Rips construction to construct $p : H \to Q$, then one forms the *fibre product* $P = \{(x, y) \mid p(x) = p(y)\} < H \times H$. In general, P will be finitely generated but not finitely presented. However, if Q is of *type F_3* (i.e. has a classifying space with finite 3-skeleton), then the 1-2-3 Theorem of [4] implies that P is finitely presentable. And if Q is a complicated group, its complications transfer to P in a manner one hopes to understand, cf. Example 3.1.

For example, it is proved in [4] that if $\mathcal{Q}$ is aspherical with an unsolvable word problem, then the conjugacy problem and membership problem for $P < H \times H$ are unsolvable. Less directly, one can deduce the following.

Theorem 4.2. [11] *Let $1 \to N \to H \to L \to 1$ be an exact sequence of groups. Suppose*

1. *H is torsion-free and hyperbolic,*
2. *N is infinite and finitely generated, and*
3. *L is a non-abelian free group.*

If F is a non-abelian free group, then the isomorphism problem for finitely presented subgroups of $H \times H \times F$ is unsolvable.

Desire: Find a version of the Rips construction so that direct products of the output groups H can be embedded in mapping class groups. The constructions given above will then prove Thoerems A to C.

5. RAAGs, cubes and CAT(0)

We take up the theme of Example 3.3.

The original proofs of Stallings and Bieri are essentially algebraic. Bestvina and Brady [6] discovered a geometric proof that motivated their theory of Morse theory for cubical complexes. Regard F as the fundamental group of a compact metric simplicial graph Y. Then $D = F \times \cdots \times F$ is the fundamental group of $X = Y \times \cdots \times Y$, which has a natural cubical structure. This **cube complex** is non-positively curved in the sense of Alexandrov, i.e. locally CAT(0). The vertex set of the universal cover is D and the homomorphism $h : D \to \mathbb{Z}$ extends linearly across cells to give a *Morse function* $\tilde{X} \to \mathbb{R}$. Bestvina and Brady determine the finiteness properties of the kernel of h by examining the way in which the sublevel sets of this Morse function change as one passes through critical points (vertices). This analysis extends to a larger class of cubical complexes.

A *right angled Artin group* (RAAG) is a group given by a presentation of the form

$$A_\Gamma = \langle v_1, \dots, v_n \mid [v_i, v_j] = 1 \, \forall (i,j) \in E \rangle.$$

Thus A_Γ is encoded by a graph Γ with vertex set $\{v_1, \dots, v_n\}$ and edge set $E \subset V \times V$. The prototype $F_2 \times \cdots \times F_2$ is the RAAG associated to the graph Γ that is the 1-skeleton of the join $\mathbb{S}0 * \cdots * \mathbb{S}0$. There is a natural classifying space for A_Γ obtained by gluing tori along coordinate faces according to the commuting relations in the presentation. The obvious cubical structure has non-positive curvature. We take from this story that *right angled Artin groups (RAAGs) are a natural generalisation of direct products of free groups, but their subgroup structure is much richer.*

5.1. RAAGS everywhere.

Whenever one has n automorphisms α_i of an object X, some of which commute, say $[\alpha_i, \alpha_j] = 1$ if $(i,j) \in E$, then one has an action of the RAAG associated to the n-vertex graph with edge-set E. This action will be faithful if the α_i that do not commute are unrelated. One such setting is that of surface automorphisms: if two simple closed curves on a surface are disjoint, then the Dehn twists in those curves commute, but if one has a set curves, no pair of which can be homotoped off each other, then high powers of the twists in those curves freely generate a free group.

Proposition 5.1. *Every RAAG embeds in the mapping class group of every surface of sufficiently high genus.*

Might this be the answer to our Desire?

5.2. Some Properties of RAAGs.

Subgroup of RAAG are free abelian or else map onto a non-abelian free group [2]; RAAGS are linear over $\mathbb{Z}$ and hence residually finite [14]; they are residually torsion-free nilpotent [15] and RFRS [1]; their quasi-convex subgroups are virtual retracts (so closed in the profinite topology) [17].

6. Questions and Answers

The preceding discussion begs certain questions:

- **(i)** which groups can be *cubulated*, i.e. act properly and cocompactly by isometries on a CAT(0) cube complex?
- **(ii)** can all (Gromov) hyperbolic groups be made to act in this way?
- **(iii)** can an arbitrary cube complex be completed in some way to the EA_Γ of some RAAG A_Γ, more precisely
- **(iii)'** does every compact non-positively curved cube complex X admit a local isometry to BA_Γ, or (weaker)
- **(iii)"** does $\pi_1 X$ embed in some A_Γ (which would imply that it is linear over $\mathbb{Z}$, embeds in mapping class groups etc. etc.), if not, then
- **(iv)** are there reasonable criteria that guarantee a positive answer?
- **(v)** which groups satisfy these criteria?

6.1. **(i) Cubulation.** The basic idea is due to Micah Sageev [21]: find subgroups with codimension one (i.e. $H < G$ so that the complement of a neighbourhood of H in the Cayley graph of G has at least two deep components). Given a collection $H_1, \dots, H_n < G$ of codimension-1 subgroups, Sageev builds a CAT(0) cube complex on which G acts, with hyperplane stabilisers conjugate to one of the given subgroups.

Proposition 6.1. [21] *If G is hyperbolic and $H_1, \dots, H_n$ are quasi-convex codim-1 subgroups, then the action of G on the associated cube complex is cocompact.*

It is harder to ensure that the action is proper. Crudely, one needs *enough* codimension-1 subgroups. One criterion for this was proved by Bergeron and Wise [5].

Theorem 6.2. [5] *Suppose G is hyperbolic. If each pair of distinct points in the Gromov boundary ∂G can be separated by the limit set of some quasi-convex subgroup, then there exists a finite collection of quasi-convex subgroups $H_1, \dots, H_n < G$ such that the action on the associated cube complex is proper and cocompact.*

A large family of examples come from an earlier theorem of Wise [22].

Theorem 6.3. [22] *$C'(1/6)$ small cancellation groups act properly and cocompactly on* CAT(0) *cube complexes.*

Hsu and Wise prove a general combination result that includes the following:

Theorem 6.4. *Let $G = A *_C B$ (of HNN) with G hyperbolic and $C \cong \mathbb{Z}$ malnormal. If A and B can be cubulated, so can G.*

6.2. **(ii) (T) is an obstruction.** A group with property (T) has a fixed point whenever it acts by isometries on a finite dimensional CAT(0) cube complex, and there are hyperbolic groups with property (T). Cocompact lattices in $\mathrm{Sp}(n,1)$ give concrete examples, and a probabilistic argument shows that they abound among random hyperbolic groups with a certain density of relations.

6.3. **(iii)" Lack of residual finiteness is an obstruction.** There are compact non-positively curved 2-complexes whose fundamental groups are not residually finite, indeed they can even be simple [13].

6.4. **(iv) Special Cube Complexes.** A NPC cube complex is **special** if it admits a locally isometric embedding into the cubical classifying space BA_Γ of some RAAG A_Γ, as described in the previous section.

We saw in answering (i) that hyperplanes are an important feature of NPC cube complexes. Hyperplanes in BA_Γ are well-behaved because the space is built out of tori in such an easy way. Haglund and Wise [18] focus on this and prove that it is sufficient to rule out four configurations of hyperplanes that would obstruct the existence of an embedding.

Theorem 6.5. [18] *A non-positively curved cube complex is* special *if and only if its hyperplanes are 2-sided, do not self-cross, do not self-osculate, and do not inter-osculate.*

Given a cube complex with well-behaved hyperplanes, there is an explicit procedure for building the RAAG.

6.5. **(v) Constructions.** Many groups are now proved to be **virtually special**. The crowning achievement, following much work of Wise and others, is Agol's theorem.

Theorem 6.6. [1] *If G is hyperbolic and the fundamental group of a compact NPC cube complex, then G has a subgroup of finite index that embeds in a RAAG.*

Corollary 6.7. *If a non-elementary hyperbolic group H is the fundamental group of a compact non-positively curved cube complex, then H is linear, large, and its quasi-convex subgroups are separable.*

Corollary 6.8. *All such groups embed in mapping class groups.*

Corollary 6.9. *The output from (most versions of) the Rips construction are virtually special and hence embed (algorithmically) in mapping class groups.*

Thus the Desire at the end of Section 4 is fulfilled, and the techniques described in that section can be used to prove Theorems A to C.

Remark 6.10. The proof of Theorems A-E in [8] predate Agol's work; they use the Haglund-Wise [18] version of the Rips construction to obtain embeddings into RAAGs.

7. Agol's Results on 3-Manifolds

The above theorem of Agol is the main result in his paper solving the virtually Haken conjecture (VHC). Why does it appear in a paper about 3-manifolds?

Post-geometrisation, standard arguments in 3-manifold topology reduce the VHC to the case of closed hyperbolic manifolds. The first major step in the proof, then, is the Surface Subgroup Theorem of Kahn and Markovic [19]. Their proof produces an abundance of surface subgroups:

Theorem 7.1. [19] *Let* Γ *be a cocompact Kleinian group and* C *a great circle in* $\partial\Gamma = \mathbb{S}^2$*. There exists a sequence of quasi-Fuchsian subgroups* $S_n \hookrightarrow \Gamma$ *whose limit sets converge to* C *in the Hausdorff topology.*

Following our earlier discussion of codimension-1 subgroups, particularly the Bergeron-Wise criterion, one sees that this enough to cubulate the given group. Now we see the point of Agol's theorem: he promotes *cubulated* to *virtually special*, from this he deduces that the groups are large, RFRS, LERF, etc. He can them deploy his earlier work on RFRS and virtual fibering (which in turn relies on work of many other people).

Theorem 7.2. [1] *Every closed aspherical 3-manifold has a finite sheeted cover that is Haken.*

Theorem 7.3. [1] *Every closed hyperbolic 3-manifold* M *has a finite sheeted cover that fibres over the circle. Moreover,* $\pi_1 M$ *is large and LERF.*

For a detailed overview of how these developments fit together, see the survey [3].

References

[1] I. Agol, *The virtual Haken conjecture*, (with appendix by I. Agol, D. Groves and J. Manning). ArXiv:1204.2810

[2] Y. Antolín and A. Minasyan, *Tits alternatives for graph products*, preprint (2011).

[3] Matthias Aschenbrenner, Stefan Friedl, Henry Wilton, 3-Manifold groups,arXiv:1205.0202.

[4] G. Baumslag, M. R. Bridson, C. F. Miller III and H. Short, *Fibre Products, non-positive curvature, and decision problems*, Comment. Math. Helv. **75** (2000), 457-477.

[5] N. Bergeron and D. Wise, *A boundary criterion for cubulation*, ArXiv 0908.3609.

[6] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), 445–470.

[7] N. Brady, M. Clay, Matt and P. Dani, *Morse theory and conjugacy classes of finite subgroups*, Geom. Dedicata **135** (2008), 15–22.

[8] M.R. Bridson, *On the subgroups of right angled Artin groups and mapping class groups*, arXiv:1205.5416.

[9] M.R. Bridson, *Non-positive curvature and complexity for finitely presented groups*, International Congress of Mathematicians. Vol. II, 961-987, Eur. Math. Soc., Zurich 2006.

[10] M. R. Bridson, J. Howie, C. F. Miller III and H. Short, *On the finite presentation of subdirect products and the nature of residually free groups.* Amer. J. Math., to appear. ArXiv:0809.3704.

[11] M. R. Bridson and C. F. Miller III, *Recognition of subgroups of direct products of hyperbolic groups*, Proc. Amer. Math. Soc., **132** (2003), 59–65.

[12] M. R. Bridson and C. F. Miller III, *Structure and finiteness properties of subdirect products of groups.* Proc. London Math. Soc. (3) **98** (2009), 631–651.

[13] M. Burger and S. Mozes, *Lattices in product of trees*, IHES Publ. Math. No. 92 (2000), 151–194.

[14] M. W. Davis and T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Appl. Algebra, 153 (2000), 229–235.

[15] G. Duchamp and D. Krob, *The lower central series of the free partially commutative group*, Semigroup Forum 45 (1992), no. 3, 385–394.

[16] B. Farb. Some problems on mapping class groups and moduli space. In *Problems on Mapping Class Groups and Related Topics* (B. Farb, editor), Proceedings of symposia in pure mathematics, vol 74. American Math. Soc., Providence RI, 2006, pp. 11–56.

[17] F. Haglund, *Finite index subgroups of graph products*, Geom. Dedicata 135 (2008), 167–209.
[18] F. Haglund and D.T. Wise, Special cube complexes. Geom. Funct. Anal. **17** (2008), 1551–1620.
[19] J. Kahn and V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math. 175 (2012), 1127–1190.
[20] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math Soc. **14** (1982), 45–47.
[21] M. Sageev, *Codimension-1 subgroups and splittings of groups*, J. Algebra, 189 (1997), no. 2, 377–389.
[22] D. T. Wise, *Cubulating small cancellation groups*, GAFA, 14(1):150–214, 2004.
[23] D.T. Wise, *The structure of groups with a quasi-convex hierarchy*, 181 pages, preprint (2012).

Homological stability for moduli spaces of manifolds

Oscar Randal-Williams
(joint work with Søren Galatius)

Fix a dimension $2n$ and consider the smooth closed $2n$-dimensional manifold $W_g := \#^g S^n \times S^n$. Choosing once and for all an embedding $D^{2n} \hookrightarrow W_g$, we can form the manifold with boundary

$$W_{g,1} := W_g \setminus \mathrm{int}(D^{2n}).$$

Let $\mathrm{Diff}_\partial(W_{g,1})$ denote the topological group of diffeomorphisms of $W_{g,1}$ which are the identity on a neighbourhood of the boundary. A choice of embedding $W_{g,1} \hookrightarrow W_{g+1,1}$ gives a continuous homomorphism $\mathrm{Diff}_\partial(W_{g,1}) \to \mathrm{Diff}_\partial(W_{g+1,1})$, and so a map $\mathcal{S}$ on classifying spaces.

In my talk I presented the proof of the following theorem, from [2].

Theorem A. *Suppose that $2n > 4$. Then the induced map*

$$\mathcal{S}_* : H_*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z}) \longrightarrow H_*(B\mathrm{Diff}_\partial(W_{g+1,1}); \mathbb{Z})$$

on integral homology is an isomorphism in degrees $* \leq \frac{g-4}{2}$.

This theorem is also true for $2n < 4$ (though with different stability ranges). If $2n = 0$, it is Nakaoka's stability theorem [5] for the homology of symmetric groups. If $2n = 2$, it is Harer's stability theorem [4] for the homology of mapping class groups of oriented surfaces.

Remark 1. Independently, Berglund and Madsen [1] have obtained a result similar to Theorem A, for rational cohomology in the range $* \leq \min(n-3, (g-6)/2)$. (for details see the contribution of A. Berglund to this volume.)

Our motivation for proving Theorem A is that in previous work [3] we have identified the ring

$$\varprojlim_{g \to \infty} H^*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z})$$

with the cohomology of an explicit infinite loop space, which allows for concrete calculations to be made. (This is too involved to explain here, but see [6] for a precis.) Along with Theorem A, this allows us to obtain interesting cohomological information about $H^*(B\mathrm{Diff}_\partial(W_{g,1}))$ in degrees $* \leq \frac{g-4}{2}$.

References

[1] Alexander Berglund and Ib Madsen, *Homological stability of diffeomorphism groups*, arXiv:1203.4161, 2012.

[2] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds*, arXiv:1203.6830, 2012.

[3] ———, *Stable moduli spaces of high dimensional manifolds*, arXiv:1201.3527, 2012.

[4] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces.* Ann. of Math. (2), 121(2):215–249, 1985.

[5] Minoru Nakaoka, *Decomposition theorem for homology groups of symmetric groups.* Ann. of Math. (2), 71:16–42, 1960.

[6] Oscar Randal-Williams, *Monoids of moduli spaces of manifolds, II*, Oberwolfach Reports. Vol. 7, no. 3, 2484–2486, 2010.

The additivity of the rho-invariant and periodicity in topological surgery

Tibor Macko

(joint work with Diarmuid Crowley)

The elements of the (simple) topological structure set $\mathcal{S}(M)$ of a closed manifold M are (simple) homotopy equivalences $h\colon N \to M$ of closed manifolds modulo the (s-cobordism) h-cobordism relation in the source. It is the principal object of study in surgery theory. Our result apply to both s and h versions.

A priori $\mathcal{S}(M)$ is just a pointed set. However, it also carries the structure of an abelian group, natural in some sense, which, however, still remains mysterious from the geometric point of view. This structure is obtained via the identification of the geometric surgery exact sequence (top row) and the algebraic surgery exact sequence (bottom row) [9, §18]:

$$\begin{array}{ccccccccc} \mathcal{N}_\partial(M \times D^1) & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathcal{S}(M) & \longrightarrow & \mathcal{N}(M) & \longrightarrow & L_n(\mathbb{Z}[\pi_1(M)]) \\ \downarrow\cong & & \downarrow\cong & & s\downarrow\cong & & t\downarrow\cong & & \downarrow\cong \\ H_{n+1}(M, \mathbf{L}_\bullet\langle 1\rangle) & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathbb{S}_{n+1}(M) & \longrightarrow & H_n(M, \mathbf{L}_\bullet\langle 1\rangle) & \longrightarrow & L_n(\mathbb{Z}[\pi_1(M)]) \end{array}$$

The exactness of the top row is to be understood in the sense of pointed sets. The bottom row is an exact sequence of abelian groups, in fact, all of these groups are L-groups of the appropriate algebraic bordism categories. This means that their elements are represented by quadratic chain complexes over some simplicial complex, say K, homotopy equivalent to M. The abelian group structure is given by the direct sum of quadratic chain complexes.

The vertical map $s\colon \mathcal{S}(M) \to \mathbb{S}_{n+1}(M)$ is obtained by choosing a homotopy equivalence $r\colon M \to K$ and making r and the composition $r \circ h$ transverse to the dual cells of K. This gives a compatible collection of degree one normal maps, one for each simplex of K, and such a collection gives a quadratic chain complex over K. Similar constructions are performed to obtain the other two vertical maps. It is proved in [9, §18] that they are bijections, but the group structure does not seem to have an illuminating geometric description in the top row.

Let M be a closed oriented $(2d-1)$-dimensional topological manifold and let $\lambda(M)\colon M \to BG$ be a map, G finite. The ρ-invariant

$$\rho(M,\lambda(M)) \in \mathbb{Q}R_{\widehat{G}}^{(-1)^d},$$

lies in a certain sub-quotient of the rationalised complex representation ring of G. It is defined as follows. Suppose that Z is a compact oriented $2d$-dimensional manifold with a map $\lambda(Z)\colon Z \to BG$, we call it an r-coboundary for $(M,\lambda(M))$ if $\partial(Z,\lambda(Z)) = \sqcup_r(M,\lambda(M))$ for some $r \geq 1$. From bordism theory we know that r-coboundaries always exist for some r. The G-signature of the induced G-covering $\widetilde{Z}$ is an element in the complex representation ring $R(G)$. It follows from the Atiyah-Singer G-index theorem [2], [11, §14B] that the expression

$$\rho(M,\lambda(M)) := (1/r)\cdot \text{G-sign}(\widetilde{Z})$$

becomes independent of the choice of Z and $r \geq 1$ after passing to the appropriate subquotient of the rationalisation of $R(G)$.

The ρ-invariant is a powerful invariant of odd-dimensional manifolds with torsion elements in their fundamental group, see [1], or [11, §14].

The reduced ρ-invariant defined by

$$\widetilde{\rho}\colon \mathcal{S}(M) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d},\ [h : N \to M] \ \longmapsto\ \rho(N,\lambda\circ h) - \rho(M,\lambda).$$

This is is a-priori just a set function. Wolfgang Lück asked whether it is in fact a homomorphism from the structure set to the underlying abelian group of the $\mathbb{Q}$-vector space and a positive answer to this question is our main theorem.

Theorem 1. *Let M be a closed oriented topological manifold of dimension $2d-1 \geq 5$ with a reference map $\lambda(M)\colon M \to BG$ where G is a finite group. Then the map*

$$\widetilde{\rho}\colon \mathcal{S}(M) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

is a homomorphism of abelian groups.

Clearly, it can be useful in computations of $\mathcal{S}(M)$ and this is the case in a forthcoming paper of Davis and Lück [4] about torus bundles over lens spaces.

The first step in the proof of Theorem 1 is to define, following [7], the rel boundary reduced ρ-invariant from the relative version of the structure set:

$$\widetilde{\rho}_\partial : \mathcal{S}_\partial(M \times D^l) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d},$$

The point is that the group structure on the relative version $\mathcal{S}_\partial(M \times D^l)$ is well understood from the geometric point of view (given by "stacking") which enables us to prove:

Proposition 2. *Let M be a closed oriented topological manifold of dimension n with a reference map $\lambda(M)\colon M \to BG$ for a finite group G, and let $n+l = 2d-1 \geq 5$. Then the map*

$$\widetilde{\rho}_\partial\colon \mathcal{S}_\partial(M \times D^l) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

is a homomorphism of abelian groups.

In the next step we recall that there is a near periodicity map $P^j : \mathcal{S}(M) \to \mathcal{S}_\partial(M \times D^{4j})$ defined in [10] and [9]. These definitions are again not illuminating from the geometric point of view. So a geometric passage from $\mathcal{S}(M)$ to $\mathcal{S}_\partial(M \times D^{4j})$ remained unclear until [3] where Cappell and Weinberger sketched maps $CW^j : \mathcal{S}(M) \to \mathcal{S}_\partial(M \times D^{4j})$ for $j = 1, 2$ or 4 (see also [5]). Much of our work goes into giving a detailed proof that the map CW^2 indeed realises the near periodicity map P^2 in the generality we need. We prove:

Proposition 3. *Let M be a closed topological manifold of dimension $n \geq 5$. The Cappell-Weinberger map gives an exact sequence of homomorphisms of abelian groups:*

$$0 \longrightarrow \mathcal{S}(M) \xrightarrow{CW^2} \mathcal{S}_\partial(M \times D^8) \longrightarrow H_0(M; \mathbb{Z}).$$

Finally we need one more proposition:

Proposition 4. *Let M be a closed topological manifold of dimension $(2d-1) \geq 5$ with a reference map $\lambda \colon M \to BG$ for a finite group G. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\quad CW^2 \quad} & \mathcal{S}_\partial(M \times D^8) \\ & \searrow^{\widetilde{\rho}} \qquad \swarrow^{\widetilde{\rho}_\partial} & \\ & \mathbb{Q}R_{\widehat{G}}^{(-1)^d} & \end{array}$$

Theorem 1 follows since $\widetilde{\rho}$ is expressed as a composition of two homomorphisms.

References

[1] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.

[2] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[3] Sylvain Cappell and Shmuel Weinberger. A geometric interpretation of Siebenmann's periodicity phenomenon. In *Geometry and topology (Athens, Ga., 1985)*, volume 105 of *Lecture Notes in Pure and Appl. Math.*, pages 47–52. Dekker, New York, 1987.

[4] Jim Davis and Wolfgang Lück. *in preparation*, 2010.

[5] Steven Hutt. On Siebenmann periodicity. *Pacific J. Math.*, 183(2):291–303, 1998.

[6] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[7] Ib Madsen and Melvin Rothenberg. On the classification of G-spheres. II. PL automorphism groups. *Math. Scand.*, 64(2):161–218, 1989.

[8] Andrew Ranicki. The algebraic theory of surgery. II. Applications to topology. *Proc. London Math. Soc. (3)*, 40(2):193–283, 1980.

[9] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[10] Laurence C. Siebenmann. Periodicity in topological surgery. *In* [6], pages 277–283, 1977.

[11] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1999.

Equivariant properties of symmetric products

Stefan Schwede

We let

$$Sp^\infty(X) \;=\; \bigcup_{n\geq 1} X^n/\Sigma_n$$

denote the infinite symmetric product, also known as the reduced free abelian monoid, of a based space X. It comes with a filtration by the finite symmetric products $Sp^n(X) = X^n/\Sigma_n$. We denote by

$$Sp^n = \{Sp^n(S^m)\}_{m\geq 0} \quad \text{respectively} \quad Sp^n/Sp^{n-1} = \{Sp^n(S^m)/Sp^{n-1}(S^m)\}$$

the orthogonal spectra whose terms are the stages respectively subquotients of the symmetric power filtration applied to spheres. The spectrum $Sp^1 = \mathbb{S}$ is the sphere spectrum. A celebrated theorem of Dold and Thom asserts that for $Sp^\infty(S^m)$ is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, m)$ for $m \geq 1$; so Sp^∞ is an Eilenberg-Mac Lane spectrum for the group $\mathbb{Z}$ of integers.

The symmetric product spectra have been much studied. The subquotient Sp^n/Sp^{n-1} is stably contractible unless n is a prime power. If p is a prime and $k \geq 1$, then Sp^{p^k}/Sp^{p^k-1} is p-torsion, and its mod-p cohomology has been completely worked out by Nakaoka [9]. The spectra Sp^{2^k}/Sp^{2^k-1} feature in the work of Mitchell and Priddy on stable splitting of classifying spaces $B(\mathbb{Z}/2)^k$ via Steinberg idempotents [8], and in Kuhn's solution of the Whitehead conjecture [4]. Lesh showed that Sp^n/Sp^{n-1} is stably equivalent to the suspensions spectrum of the unreduced suspension of the classifying space $B\mathcal{F}_n$ of the family $\mathcal{F}_n$ of non-transitive subgroups of the symmetric group Σ_n [5]. Arone and Dwyer relate these spectra to the partition complex, the homology of dual Lie representation and the Tits building [1].

This project is about the global equivariant features of the symmetric power filtration. Here 'global' refers to simultaneous and compatible actions of compact Lie groups. Various ways to formalize this idea have been explored in [6, Ch. II], [3, Sec. 5], [2]; we use a different approach via orthogonal spectra.

We recall that an *orthogonal spectrum* consists of:

- a sequence of based spaces X_n for $n \geq 0$,
- based, continuous left actions of the orthogonal groups $O(n)$ on X_n,
- based structure maps $\sigma_n : X_n \wedge S^1 \to X_{n+1}$.

This data is subject to the following condition: for all $n, m \geq 0$, the map

$$X_n \wedge S^m \;\to\; X_{n+m} \tag{1}$$

obtained by iterating the structure maps is $O(n) \times O(m)$-equivariant. Here $O(m)$ acts on S^m as the one-point compactification of $\mathbb{R}^m$, and $O(n) \times O(m)$ acts on the target by restriction along to block sum embedding into $O(n+m)$. A *morphism* $f : X \to Y$ of orthogonal spectra consists of $O(n)$-equivariant continuous based maps $f_n : X_n \to Y_n$, for $n \geq 0$, strictly compatible with the structure maps.

An orthogonal spectrum X can be made 'coordinate free' as follows. The *value* of X on a finite dimensional euclidean vector space V of dimension n is

$$X(V) \ = \ \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n \ ,$$

where $\mathbf{L}(\mathbb{R}^n, V)$ is the space of linear isometries from $\mathbb{R}^n$ to V. The orthogonal group $O(n)$ acts freely transitively on $\mathbf{L}(\mathbb{R}^n, V)$ by precomposition, so every choice of linear isometry from $\mathbb{R}^n$ to V gives rise to a homeomorphism from X_n to $X(V)$. The space $X(\mathbb{R}^n)$ is canonically homeomorphic to X_n, and the iterated structure maps (1) extend to *generalized structure maps*

$$X(V) \wedge S^W \ \to \ X(V \oplus W) \tag{2}$$

that are suitably associative and unital, where S^W is the one-point compactification, based at infinity, of W.

Now we let a compact Lie group G act on V by linear isometries. Then $X(V)$ becomes a G-space through the action on V. The underlying space of $X(V)$ depends, up to homeomorphism, only on the dimension of the representation V, but the G-action on V influences the G-action on $X(V)$. For any two orthogonal G-representations, the generalized structure map (2) is G-equivariant. So an orthogonal spectrum X yields an orthogonal G-spectrum, in the sense of [7], for every compact Lie group G. One should beware, though, that only very special orthogonal G-spectra are part of a 'global family', i.e., arise in this way from an orthogonal spectrum. An example that is not global is the equivariant suspension spectrum of a based G-space with non-trivial action.

As we just explained, an orthogonal spectrum has underlying equivariant spectra, and these have equivariant homotopy groups. For this project we only care about finite groups, and we recall the definition of the equivariant homotopy groups of an orthogonal spectrum X in this special case. The k-th *G-equivariant homotopy group*, for an integer k, is defined as

$$\pi_k^G X \ = \ \operatorname{colim}_n \, [S^{k+n\rho_G}, \ X(n\rho_G)]^G \ ,$$

where ρ_G is the regular representation of G and $[-,-]^G$ means G-equivariant homotopy classes of based G-maps. The colimit is taken along stabilization by the regular representation, using the generalized structure maps.

Definition 3. A morphism $f : X \to Y$ of orthogonal spectra is a *global equivalence* if the induced map

$$\pi_k^G f \ : \ \pi_k^G X \ \to \ \pi_k^G Y$$

is an isomorphism for all integers k and all finite groups G.

We define the *global stable homotopy category* by localizing the category of orthogonal spectra at the class of global equivalences. We emphasize that we are not inventing new objects – orthogonal spectra have been around for more than 10 years now – but we are looking at a substantially finer notion of equivalence than the usual stable equivalence. So the global stable homotopy category has way more homotopy types.

The global equivalences are part of a closed model structure, so the methods of homotopical algebra can be used to study the global homotopy category. This works more generally relative to a class $\mathcal{G}$ of compact Lie groups, where we define $\mathcal{G}$-global equivalences by requiring that $\pi_k^G f$ is an isomorphism for all integers and all groups in $\mathcal{G}$. If $\mathcal{G}$ satisfies certain mild closure properties, then there are two useful cofibration/fibration pairs that complement the $\mathcal{G}$-global equivalences to stable model structures. These model structures are useful for showing that the forgetful functor

$$(\mathcal{G}\text{-global stable homotopy category}) \to (\text{stable homotopy category})$$

has both a left and a right adjoint, and both are fully faithful. Besides finite groups, other interesting global families are the classes of all compact Lie groups, or all abelian compact Lie groups. The class of trivial groups is also admissible here, but then we just recover the 'traditional' stable category. When we look at the family of all compact Lie groups, the global sphere spectrum is in the image of the left adjoint. Global Borel cohomology theories are the image of the right adjoint. The 'natural' global versions of topological K-theory, algebraic K-theory, bordism, or Eilenberg-Mac Lane spectra of global functors are not in the image of either of the two adjoints.

The groups $\pi_k^G X$ have a lot of extra structure as the group G varies. Every group homomorphism $\alpha : K \to G$ gives rise to a *restriction map*

$$\alpha^* \ : \ \pi_k^G X \ \to \ \pi_k^K X \ .$$

We emphasize that, unlike in equivariant stable homotopy theory for one fixed group, α may have a non-trivial kernel. Finite index subgroups give *transfer maps*

$$\mathrm{tr}_H^G \ : \ \pi_k^H X \ \to \ \pi_k^G X \ .$$

The restriction and transfer maps satisfy various relations, among them transitivity and a double coset formula. In summary, for every orthogonal spectrum X the assignment $(\underline{\pi}_0 X)(G) = \pi_0^G X$ extends to a *global functor* $\underline{\pi}_0 X$, an additive functor on a certain 'Burnside category' with objects the finite groups and whose morphisms are Grothendieck groups of finite bisets that are free from one side (compare [12, p. 271] for details, where the term 'inflation functor' is used). The abelian category of global functors has been much studied in representation theory, and it is the natural home for homotopy groups in our context.

Now we calculate the 0-th equivariant homotopy groups $\pi_0^G(Sp^n)$ of the spectra in the symmetric power filtration. In the extreme cases $n = 1$ and $n = \infty$ the answer is well known. The value of the sphere spectrum $\mathbb{S} = Sp^1$ at a G-representation V is equivariantly homeomorphic to the representation sphere S^V. The groups $\pi_k^G(Sp^1)$ are thus the equivariant stable stems. Segal [10] identified the 0-th G-equivariant stable stem as the Burnside ring $A(G)$. In fact, $\underline{\pi}_0(Sp^1)$ is isomorphic, as a global functor, to the Burnside ring global functor A (which is representable, as a global functor, by the trivial group).

In [11], Segal argues that that for every G-representation V with $V^G \neq 0$ the G-space $Sp^\infty(S^V)$ is an equivariant Eilenberg-Mac Lane space of type $(\underline{\mathbb{Z}}, V)$, where $\underline{\mathbb{Z}}$ is the constant G-Mackey functor. So the orthogonal spectrum Sp^∞ is an Eilenberg-Mac Lane spectrum for the constant global functor with values $\mathbb{Z}$.

The symmetric power filtration of Sp^∞ thus yields a sequence of global functors

$$A = \underline{\pi}_0(Sp^1) \to \underline{\pi}_0(Sp^2) \to \ldots \to \underline{\pi}_0(Sp^n) \to \ldots \to \underline{\pi}_0(Sp^\infty) = \underline{\mathbb{Z}}$$

between the Burnside ring global functor and the constant global functor $\underline{\mathbb{Z}}$. The most elegant way to describe the intermediate terms is as the quotient of the Burnside ring global functor by one simple, explicit relation. We define an element t_n in the Burnside ring of the n-th symmetric group by

$$t_n \ = \ [\{1,\ldots,n\}] - n\cdot 1 \ = \ [\Sigma_n/\Sigma_{n-1}] - n\cdot[*] \ \in \ A(\Sigma_n),$$

the formal difference of the classes of the tautological Σ_n-set $\{1,\ldots,n\}$ and a trivial Σ_n-set with n elements. Since t_n has zero augmentation, the global subfunctor $\langle t_n\rangle$ generated by t_n lies in the augmentation ideal global functor I. The restriction of t_n to the Burnside ring of Σ_{n-1} equals t_{n-1}, so we obtain a nested sequence of global functors

$$0 = \langle t_1\rangle \ \subset \ \langle t_2\rangle \ \subset \ \ldots \ \subset \ \langle t_n\rangle \ \subset \ \ldots \ \subset I \ \subset \ A\ .$$

Each of these inclusions is proper and the global functors $\langle t_n\rangle$ exhaust the augmentation ideal functor.

Theorem 4. *The inclusion $Sp^1 \to Sp^n$ induces an isomorphism of global functors*

$$A/\langle t_n\rangle \ \cong \ \underline{\pi}_0(Sp^n)\ .$$

It is now a purely algebraic exercise to describe $\pi_0^G(Sp^n)$ as an explicit quotient of the Burnside ring $A(G)$: one has to enumerate all relations in $A(G)$ obtained by applying restrictions and transfers to the class t_n. The author thinks that the explicit answer for $\pi_0^G(Sp^n)$ is far less appealing than the global description above.

Theorem 4 is a fairly direct consequence of a global identification of the equivariant homotopy types of the subquotient spectra Sp^n/Sp^{n-1}:

Theorem 5. *The orthogonal spectrum of Sp^n/Sp^{n-1} is a global suspension spectrum of the unreduced suspension of a global classifying space $B_{\mathrm{gl}}\mathcal{F}_n$ for the family $\mathcal{F}_n$ of non-transitive subgroups of the symmetric group Σ_n.*

We will not define what a 'global classifying space' of a family is in general; instead we will explain the content of Theorem 5 at a specific finite group G. We let $\mathcal{F}_n(G)$ be the family of those subgroups K of $G\times\Sigma_n$ such that $K\cap(\{1\}\times\Sigma_n)$ is a non-transitive subgroup of Σ_n. We denote by

$$B\mathcal{F}_n(G) \ = \ (E\mathcal{F}_n(G))/\Sigma_n$$

the quotient of a universal $(G\times\Sigma_n)$-space $E\mathcal{F}_n(G)$ for the family $\mathcal{F}_n(G)$ by the action of Σ_n. This quotient is a G-CW-complex whose underlying non-equivariant space has the homotopy type of $B\mathcal{F}_n$, the classifying space of the family $\mathcal{F}_n$. However, the G-action on $B\mathcal{F}_n(G)$ is usually non-trivial, so this collection of classifying spaces is not 'constant in the global direction'. Theorem 5 says in particular that for every finite group G the underlying orthogonal G-spectrum of Sp^n/Sp^{n-1} is

G-equivariantly equivalent to the suspension spectrum of the unreduced suspension of the classifying space $B\mathcal{F}_n(G)$. When G is the trivial group, this reduces to Lesh's theorem [5].

For $n = 2$ the family $\mathcal{F}_2$ consist only of the trivial subgroup of Σ_2. In this case the global classifying space $B\mathcal{F}_2$ specializes to $B_{\mathrm{gl}}\Sigma_2$, the global classifying space of the group Σ_2. For a finite group G the orthogonal G-spectrum underlying this global homotopy type is the suspension spectrum of the projective space in a complete G-universe, a space that classifies principal Σ_2-bundles over G-spaces.

Theorem 4 can be deduced from Theorem 5 with the help of the long exact homotopy group sequence of the inclusion $Sp^{n-1} \subset Sp^n$. Because $B_{\mathrm{gl}}\mathcal{F}_n$ is a global space, hence globally connective, the sequence ends in a surjection $\underline{\pi}_0(Sp^{n-1}) \to \underline{\pi}_0(Sp^n)$. Moreover, the path component functor of global classifying spaces admit a group theoretic description; in the case at hand this implies that the global functor $\underline{\pi}_1(Sp^n/Sp^{n-1})$ is generated by a single element in the value at the symmetric group Σ_n. The final observation is that the connecting homomorphism sends the generator of the global functor $\underline{\pi}_1(Sp^n/Sp^{n-1})$ to the class t_n in $\pi_0^{\Sigma_n}(Sp^{n-1})$.

In contrast to the classical situation, the equivariant subquotients Sp^n/Sp^{n-1} are *not* rationally trivial (which can be seen already at the level of the 0-th equivariant homotopy groups). Even worse (or more interestingly?), the groups $\mathbb{Q} \otimes \pi_*^G(Sp^n/Sp^{n-1})$ are generally not concentrated in dimension 0, even though the initial and final terms in the symmetric power filtration are.

References

[1] G. Arone, W. Dwyer, *Partition complexes, Tits buildings, and symmetric products.* Proc. London Math. Soc. 82 (2001), 229–256.

[2] A. M. Bohmann, *Global orthogonal spectra.* `arXiv:1208.4997`

[3] J. P. C. Greenlees, J. P. May, *Localization and completion theorems for MU-module spectra.* Ann. of Math. (2) 146 (1997), 509–544.

[4] N. J. Kuhn, *A Kahn-Priddy sequence and a conjecture of G. W. Whitehead*. Math. Proc. Cambridge Philos. Soc. 92 (1982), 467–483. Corrigenda: Math. Proc. Cambridge Philos. Soc. 95 (1984), 189–190.

[5] K. Lesh, *A filtration of spectra arising from families of subgroups of symmetric groups.* Trans. Amer. Math. Soc. 352 (2000), no. 7, 3211–3237

[6] L. G. Lewis, Jr., J. P. May, M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, 1213, Springer-Verlag, 1986.

[7] M. A. Mandell, J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. 159 (2002), no. 755, x+108 pp.

[8] S. A. Mitchell, S. B. Priddy, *Stable splittings derived from the Steinberg module.* Topology 22 (1983), no. 3, 285–298.

[9] M. Nakaoka, *Cohomology mod p of symmetric products of spheres.* J. Inst. Polytech. Osaka City Univ. Ser. A 9 1958 1–18.

[10] G. Segal, *Equivariant stable homotopy theory.* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 59–63, 1971

[11] G. Segal, *Some results in equivariant homotopy theory.* Preprint, 1978.

[12] P. Webb, *Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces.* J. Pure Appl. Algebra 88 (1993), no. 1-3, 265–304.

The Structure of Tensor Categories via Local Topological Field Theories and Higher Categories

Christopher Schommer-Pries
(joint work with Christopher Douglas, Noah Snyder)

Fusion tensor categories arise in several areas of mathematics: as the representation categories of finite quantum groups, as the categories of positive energy representations of loop groups, and as the *basic invariants* of subfactor Von Neumann algebras. They have a complex and mysterious structure which is actively being explored today. In this work, joint with Chris Douglas and Noah Snyder, we show how much of this structure can be tied to 3-dimensional topology and to the structure of 3-dimensional fully-extended (a.k.a. local) topological field theories.

In more detail, *tensor categories* are monoidal abelian categories equipped with a compatible enrichment in finite dimensional vector spaces over a fixed field k which are *rigid*. That is every object x admits both a left and a right dual. This later means that there exists an object x^* and maps

$$\varepsilon : x^* \otimes x \to 1 \qquad \eta : 1 \to x \otimes x^*$$

satisfying the 'zig-zag' equations

$$(id_x \otimes \varepsilon) \circ (\eta \otimes id_x) = id_x,$$
$$(\varepsilon \otimes id_{x^*}) \circ (id_{x^*} \otimes \eta) = id_{x^*}.$$

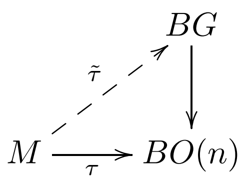
Such a category is *fusion* if in addition it is semisimple (every short exact sequence splits) and, up to isomorphism, there are only a finite number of simple objects.

One celebrated result in this area is the theorem of Etingof, Nikshych, and Ostrik [2] generalizing Radford's S^4-formula for Hopf algebras. This states that for fusion categories the quadruple dual endo-functor

$$x \mapsto x^{****}$$

is naturally and canonically isomorphic to the identity functor. The usual proof of this statement passes through the theory of weak Hopf algebras, leaving it unclear as to whether this is a general result or a particular coincidence. Our work shows that not only is this an instance of a very general result, but that this is essentially due to the topological fact that $\pi_1 SO(3) \cong \mathbb{Z}/2\mathbb{Z}$. The bridge between these algebraic structures and topology is achieved by using higher category theory (specifically symmetric monoidal $(\infty, 3)$-categories) and is inspired by Lurie's formulation [3] of the Baez-Dolan cobordism hypothesis.

For each (A_∞-)homomorphism $G \to O(n)$ we may speak of G-structures on manifolds of dimension $\leq n$. Such structures consist of lifts $\tilde{\tau}$ of the classifying map of the tangent bundle:



If the dimension of the manifold is strictly less than n we must first stabilize the tangent bundle with an appropriate trivial bundle. This gives rise to a symmetric monoidal $(\infty, 3)$-category, Bord_n^G, whose objects are compact 0-dimensional manifolds equipped with a G-structure, whose 1-morphisms consist of 1-dimensional bordisms equipped with a G-structure, whose 2-morphisms consist of 2-dimensional bordisms between the 1-dimensional bordisms equipped with a G-structure, etc. until dimension n. At this stage the topology of the diffeomorphism group of G-manifolds is incorporated into the symmetric monoidal $(\infty, 3)$-category Bord_n^G. The specifics of how this is accomplished depend on the particular model of $(\infty, 3)$-category one choses to work with, but this choice is inconsequential as all the most common models of (∞, n)-categories are known to be equivalent [1].

If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category, then we may define extended topological field theories with values in $\mathcal{C}$ (and with structure group G) as symmetric monoidal functors from Bord_n^G to $\mathcal{C}$. The *cobordism hypothesis*, as formulated by Lurie, consists of a pair of theorems which describe a universal property of the category Bord_n^G.

Theorem 1 ([3]). *For any symmetric monoidal (∞, n)-category $\mathcal{C}$, we have an equivalence:*

$$\mathrm{Fun}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq k(C^{\mathrm{fd}})$$

The left-hand side denotes the (∞, n)-category of tangentially framed topological field theories with target $\mathcal{C}$, while the right-hand side denotes the groupoid of fully-dualizable objects (the maximal $(\infty, 0)$-subcategory of the fully-dualizable subcategory).

Here the fully-dualizable subcategory of $\mathcal{C}$ is that in which every k-morphism $(1 \leq k \leq n-1)$ has both adjoints, and where every object has a dual. As $\mathrm{Bord}_n^{\mathrm{fr}}$ has an $O(n)$-action (by change of framing), the above theorem implies that the groupoid of fully dualizable objects $k(C^{\mathrm{fd}})$ admits an (A_∞-) $O(n)$-action. Thus for any homomorphism $G \to O(n)$, we also obtain a G-action on $k(C^{\mathrm{fd}})$. The second half of the cobordism hypothesis states:

Theorem 2 ([3]). *We further have an equivalence:*

$$\mathrm{Fun}(\mathrm{Bord}_n^{\mathrm{G}}, \mathcal{C}) \simeq [k(C^{\mathrm{fd}})]^{hG}$$

The left-hand side now denotes $\mathcal{C}$-valued tqfts for bordisms with G-structure, while the right hand space denotes the G-homotopy fixed points of $k(C^{\mathrm{fd}})$.

Lurie's proof of the cobordism hypothesis is inductive, so while it predicts the existence of an $O(n)$-action on the groupoid of fully-dualizable objects in any symmetric monoidal (∞, n)-category, it gives little insight into describing this action. In this talk I will describe joint work with Christopher Douglas and Noah Snyder in which we explicitly describe this action for symmetric monoidal 3-categories directly using the dualiability datum of that category. In particular we do not use the cobordism hypothesis, although it clearly inspires this work. Furthermore we show that fusion categories are precisely the fully-dualizable objects in a symmetric monoidal 3-category consisting of tensor categories, bimodule categories, functors, and natural transformations. Applying our results in this case allows us to deduce that a variety of algebraic results in the theory fusion categories are naturally explained by the existence of this $O(3)$-action. In particular we provide a new conceptual proof of the results of Etingof-Nikshych-Ostrik on the quadruple dual functor.

References

[1] C. Barwick and C. Schommer-Pries, *On the Unicity of the Homotopy Theory of Higher Categories*, preprint, ArXiv:1112.0040.
[2] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Ann. of Math. **(2)** 162 (2005), 581–642.
[3] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008 Int. Press, Somerville, MA (2009), 129–280.

Factorization knot homology

John Francis

Factorization homology, or topological chiral homology, forms an invariant of manifolds of a fixed dimension which enjoys an axiomatic characterization analogous to that of ordinary singular homology. This talk described joint work with David Ayala and Hiro Lee Tanaka on a type of factorization homology theory suitable to give invariants of n-manifolds with properly embedded k-dimensional submanifolds. These theories have an algebraic characterization, in terms of n-disk algebras acting on k-disk algebras. A particularly interesting case is that of links, 3-manifolds with 1-dimensional submanifolds, where these theories appear to want to model the structure of observables in a perturbative quantum field theory with 1-dimensional defects.

Homotopical applications of diagram spaces

Christian Schlichtkrull

In this talk we give a general introduction to the theory of diagram spaces and explain by examples how this notion can be used in various homotopical contexts.

Diagram spaces. Let $\mathcal{K}$ be a small category and let $\mathcal{S}$ be the category of "spaces" which can be interpreted either as simplicial sets or (compactly generated weak Hausdorff) topological spaces. By a *$\mathcal{K}$-space* we understand a functor $X\colon \mathcal{K} \to \mathcal{S}$ and we write $\mathcal{S}^{\mathcal{K}}$ for the category of $\mathcal{K}$-spaces. A monoidal structure on the index category $\mathcal{K}$ gives rise to a monoidal structure on $\mathcal{S}^{\mathcal{K}}$ and if $\mathcal{K}$ is braided or symmetric then so is $\mathcal{S}^{\mathcal{K}}$. We use the term *$\mathcal{K}$-space monoid* for a monoid in $\mathcal{S}^{\mathcal{K}}$. A $\mathcal{K}$-space X has an "underlying" space given by the homotopy colimit $X_{h\mathcal{K}}$ and we say that a map of $\mathcal{K}$-spaces $X \to Y$ is a *$\mathcal{K}$-equivalence* if the induced map $X_{h\mathcal{K}} \to Y_{h\mathcal{K}}$ is a weak homotopy equivalence.

Injections and E_∞ spaces. Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ and morphisms the injective maps between such sets. The $\mathcal{I}$-equivalences are the weak equivalences in a model structure on the category of $\mathcal{I}$-spaces $\mathcal{S}^{\mathcal{I}}$ with the property that the usual colimit/constant functor adjunction defines a Quillen equivalence $\mathcal{S}^{\mathcal{I}} \rightleftarrows \mathcal{S}$. In joint work with S. Sagave we prove that this equivalence can be refined to give a convenient model of the category of E_∞ spaces as strictly commutative monoids in the category $\mathcal{S}^{\mathcal{I}}$.

Theorem ([3]). There is a chain of Quillen equivalences relating the category $\mathcal{CS}^{\mathcal{I}}$ of commutative $\mathcal{I}$-space monoids to the category of E_∞ spaces.

This theorem implies that the homotopy category of $\mathcal{CS}^{\mathcal{I}}$ is equivalent to the homotopy category of E_∞ spaces. Under this equivalence a commutative $\mathcal{I}$-space monoid is mapped to its homotopy colimit which has a canonical action of the Barratt-Eccles operad. There are many situations where it is more convenient to work with strictly commutative monoids than with E_∞ spaces. For instance, the category of modules for a commutative $\mathcal{I}$-space monoid inherits a symmetric monoidal structure which is difficult to model in an E_∞ setting. Furthermore, many familiar E_∞ spaces have simple and explicit models as commutative $\mathcal{I}$-space monoids.

Braided injections and double loop spaces. Replacing the symmetric groups as automorphism groups for the category $\mathcal{I}$ with the braid groups we get the category $\mathcal{B}$ of *braided injections*. The corresponding category of $\mathcal{B}$-spaces has been analyzed in detail by M. Solberg. There again is a Quillen equivalence $\mathcal{S}^{\mathcal{B}} \rightleftarrows \mathcal{S}$, but now $\mathcal{S}^{\mathcal{B}}$ has the structure of a braided monoidal category which makes it useful for modeling E_2 spaces and in particular double loop spaces.

Theorem ([4]). The commutative monoids in $\mathcal{S}^{\mathcal{B}}$ model all E_2 spaces and the two-fold iterated bar construction in $\mathcal{S}^{\mathcal{B}}$ gives a two-fold classifying space functor.

The point of the theorem is that for a commutative monoid A in a braided monoidal category, the bar construction can be iterated: $A \mapsto B(A) \mapsto BB(A)$. Thus, by modeling E_2 spaces as commutative monoids in a braided monoidal category we get an independent proof of the fact that grouplike E_2 spaces are double loop spaces. In particular we get an explicit construction of the two-fold classifying space associated to the classifying space of a braided monoidal category.

Graded units and log structures. Motivated by applications to the theory of topological logarithmic structures introduced by J. Rognes [1], we have in joint work with S. Sagave defined a notion of *graded units* for symmetric ring spectra. The setting for this is again a type of diagram spaces where now the relevant index category $\mathcal{J}$ is the Quillen localization construction $\Sigma^{-1}\Sigma$ on the category Σ of finite sets and bijections.

Theorem ([3]). There is a chain of Quillen equivalences relating the category $\mathcal{CS}^{\mathcal{J}}$ of commutative $\mathcal{J}$-space monoids to the category of E_∞ spaces over $B\mathcal{J}$.

By work of Barratt, Priddy, and Quillen, it is known that $B\mathcal{J}$ is equivalent to $Q(S^0)$, so the above theorem allows us to interpret $\mathcal{CS}^{\mathcal{J}}$ as a model for the category of E_∞ spaces over $Q(S^0)$. This fits well with the general point of view that in a spectral context the sphere spectrum $\mathbb{S}$ takes the role played by the ring of integers $\mathbb{Z}$ in the traditional algebraic context. Indeed, in algebra a graded monoid is logically the same as a monoid A together with a monoid homomorphism $A \to \mathbb{Z}$ to the underlying additive group $(\mathbb{Z}, +, 0)$. In topology it is customary to think of $Q(S^0)$ as the "additive group" of $\mathbb{S}$ and hence we can think of commutative $\mathcal{J}$-space monoids as representing graded commutative spaces.

The relation to the category $\mathcal{C}Sp^{\Sigma}$ of commutative symmetric ring spectra is via the Quillen adjunction $\mathbb{S}^{\mathcal{J}}[-]\colon \mathcal{S}^{\mathcal{J}} \rightleftarrows Sp^{\Sigma}\colon \Omega^{\mathcal{J}}$ introduced in [3]. Given a symmetric ring spectrum R we define its graded units $\mathrm{Gl}_1^{\mathcal{J}}(R)$ to be the sub $\mathcal{J}$-space monoid of "homotopy units" in $\Omega^{\mathcal{J}}(R)$. With these notions in place we can transfer the algebraic notion of a *log ring* to the topological setting and in joint work with Rognes and Sagave [2] we introduce a corresponding notion of *logarithmic topological Hochschild homology.* The main advantage of the latter theory compared to ordinary topological Hochschild homology is that it gives rise to localization sequences analogous to those found in algebraic K-theory.

References

[1] J. Rognes, *Topological logarithmic structures*, in New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr. 16, 401–544, 2009

[2] J. Rognes, S. Sagave and C. Schlichtkrull, *Logarithmic topological Hochschild homology*, in preparation.

[3] S. Sagave and C. Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math., 231(3-4):2116–2193, 2012.

[4] C. Schlichtkrull and M. Solberg, *Braided injections and double loop spaces*, in preparation.

Spaces of long embeddings and iterated loop spaces

Kathryn Hess

(joint work with William G. Dwyer)

1. The main theorem

Let m and n be positive integers such that $m \leq n$, and let $e : \mathbb{R}^m \to \mathbb{R}^n$ be a linear embedding. The *space of long embeddings* of $\mathbb{R}^m$ into $\mathbb{R}^n$, denoted $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$, is the homotopy fiber, with respect to the baspoint e, of the inclusion

$$\mathrm{Emb}_c\big(\mathbb{R}^m, \mathbb{R}^n\big) \hookrightarrow \mathrm{Imm}_c\big(\mathbb{R}^m, \mathbb{R}^n\big),$$

where $\mathrm{Emb}_c\big(\mathbb{R}^m, \mathbb{R}^n\big)$ and $\mathrm{Imm}_c\big(\mathbb{R}^m, \mathbb{R}^n\big)$ are, respectively, the space of embeddings and the space of immersions of $\mathbb{R}^m$ into $\mathbb{R}^n$ that agree with e outside of a compact set.

Let $\mathscr{B}_m$ denote the operad of little m-balls, which detects m-fold loop spaces. The elements of the arity k component of $\mathscr{B}_m$ are standard embeddings of the disjoint union of k copies of the unit m-ball into itself, while the operad multiplication is given by embedding balls within balls, respecting ordering [1].

It is clear that $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ admits the structure of a $\mathscr{B}_m$-algebra. We have shown that it is in fact an $\mathscr{B}_{m+1}$-algebra, providing an explicit $(m+1)$-fold delooping as follows.

Theorem 1.1. For all $n > m+2$,

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1} \mathrm{Map}^{\mathrm{h}}_{\mathbf{Op}}(\mathscr{B}_m, \mathscr{B}_n)_{\varphi_{m,n}}.$$

Here, $\mathrm{Map}^{\mathrm{h}}_{\mathbf{Op}}$ denotes the derived mapping space of operad maps, while $\varphi_{m,n} : \mathscr{B}_m \to \mathscr{B}_n$ is the morphism of operads induced by the usual inclusion $\mathbb{R}^m \to \mathbb{R}^n$.

2. From geometry to algebra

Arone and Turchin provided a first translation of the geometric problem of understanding embeddings into an algebraic, homotopy-theoretic problem related to operads [1].

Let $\mathbf{V}$ be a symmetric monoidal category. A $\mathbf{V}$-operad is a monoid in the category of symmetric sequences of objects in $\mathbf{V}$, endowed with the nonsymmetric monoidal structure given by the composition product. If $\mathscr{P}$ is a $\mathbf{V}$-operad, then a $\mathscr{P}$-bimodule is a symmetric sequence equipped with compatible left and right actions of $\mathscr{P}$, with respect to the composition product.

Arone and Turchin constructed a model for the space of long embeddings in terms of a variant of the notion of $\mathscr{P}$-bimodule, which we call *linear $\mathscr{P}$-bimodules.* These are symmetric sequences $\mathscr{X}$ endowed with a right $\mathscr{P}$-action with respect to the composition product and an appropriately compatible, linear left $\mathscr{P}$-action: to every element of arity $k \geq 1$ in $\mathscr{P}$, every element of arity n in $\mathscr{X}$ and every $1 \leq i \leq k$, one associates an element of arity $n+k-1$ in $\mathscr{X}$. If $\mathscr{X}$ is a (linear) $\mathscr{P}$-bimodule with a distinguished "unit" element in $\mathscr{X}$, then we say that $\mathscr{X}$ is *pointed.* It is easy to see that an operad morphism $\varphi : \mathscr{P} \to \mathscr{Q}$ endows $\mathscr{Q}$ with the structure of a pointed $\mathscr{P}$-bimodule, while any pointed $\mathscr{P}$-bimodule can naturally be seen as a pointed, linear $\mathscr{P}$-bimodule. In particular, $\mathscr{B}_n$ is naturally a pointed, linear $\mathscr{B}_m$-bimodule, via the operad morphism $\varphi_{m,n} : \mathscr{B}_m \to \mathscr{B}_n$.

Theorem 2.1. [1] For all $n > m+2$,

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathrm{Map}^{\mathrm{h}}_{\mathbf{LBimod}_{\mathscr{B}_m}}(\mathscr{B}_m, \mathscr{B}_n).$$

Remark 2.2. When $\mathbf{V}$ is the category of simplicial sets, the categories $\mathbf{Bimod}_{\mathscr{P}}$, $\mathbf{LBimod}_{\mathscr{P}}$ and their pointed versions admit natural simplicial model category structures, compatible with the forgetful functor $U_{\mathscr{P}} : \mathbf{Bimod}^*_{\mathscr{P}} \to \mathbf{LBimod}^*_{\mathscr{P}}$. It is not hard to see that $U_{\mathscr{P}}$ admits a left adjoint $L_{\mathscr{P}} : \mathbf{LBimod}^*_{\mathscr{P}} \to \mathbf{Bimod}^*_{\mathscr{P}}$ for any operad $\mathscr{P}$.

3. Proof of the main theorem

It follows from Theorem 2.1 that the proof of Theorem 1.1 reduces to establishing the two results below.

Theorem 3.1. [2] For every morphism of operads $\varphi : \mathscr{P} \to \mathscr{Q}$,

$$\mathrm{Map}^{\mathrm{h}}_{\mathbf{Bimod}_{\mathscr{P}}}(\mathscr{P}, \mathscr{Q}) \simeq \Omega\, \mathrm{Map}^{\mathrm{h}}_{\mathbf{Op}}(\mathscr{P}, \mathscr{Q}).$$

Theorem 3.2. For every $m \geq 1$ and every pointed $\mathscr{B}_m$-bimodule $\mathscr{X}$,

$$\mathrm{Map}^{\mathrm{h}}_{\mathbf{LBimod}_{\mathscr{B}_m}}(\mathscr{B}_m, \mathscr{X}) \simeq \Omega^m\, \mathrm{Map}^{\mathrm{h}}_{\mathbf{Bimod}_{\mathscr{B}_m}}(\mathscr{B}_m, \mathscr{X}).$$

The case $m = 1$ of Theorem 3.2 was proved in [2]. We prove the case for arbitrary m by induction, the key to which is the following remarkable and difficult result.

Theorem 3.3. [3], [4] For all $m \geq 2$,

$$\mathscr{B}_m \simeq \mathscr{B}_1 \otimes^h \mathscr{B}_1 \otimes^h \cdots \otimes^h \mathscr{B}_1.$$

Here, $\otimes^h$ denotes a derived version of the Boardman-Vogt tensor product of operads $\otimes$, which endows the category of simplicial or topological operads with a symmetric monoidal structure.

To exploit Theorem 3.3 and prove Theorem 3.2 for all m, we first lift the Boardman-Vogt tensor product to bimodules, defining for every pair of operads $\mathscr{P}$ and $\mathscr{Q}$ a functor

$$-\widetilde{\otimes}- : \mathbf{Bimod}_{\mathscr{P}} \times \mathbf{Bimod}_{\mathscr{Q}} \to \mathbf{Bimod}_{\mathscr{P}\otimes\mathscr{Q}}$$

with many nice properties. In particular, there is a homotopy-pushout diagram

$$\begin{array}{ccc} L_{\mathscr{P}}(\mathscr{P})\widetilde{\otimes}\mathscr{Q} \cup \mathscr{P}\widetilde{\otimes}L_{\mathscr{Q}}(\mathscr{Q}) & \longrightarrow & \mathscr{P} \otimes \mathscr{Q} \\ \downarrow & & \downarrow \\ L_{\mathscr{P}}(\mathscr{P})\widetilde{\otimes}L_{\mathscr{Q}}(\mathscr{Q}) & \longrightarrow & L_{\mathscr{P}\otimes\mathscr{Q}}(\mathscr{P} \otimes \mathscr{Q}) \end{array} \tag{1}$$

in $\mathbf{Bimod}^*_{\mathscr{P}\otimes\mathscr{Q}}$. Note that $\mathscr{P}$ and $\mathscr{Q}$ each play several different roles in this diagram: as operads and as pointed (linear) bimodules over themselves.

Starting from the fact, proved in [2], that $L_{\mathscr{B}_1}(\mathscr{B}_1) \simeq S^1 \cdot \mathscr{B}_1$ (where $\cdot$ denotes tensorization), diagram (1) enables to give an inductive proof of Theorem 1.1, thanks to Theorem 3.3.

References

[1] G. Arone and V. Turchin, On the rational homology of high dimensional analogues of spaces of long knots, arXiv:1105.1576.

[2] W. G. Dwyer and K. Hess, *Long knots and maps between operads*, Geometry and Topology **16** (2012) 919-955.

[3] Z. Fiedorowicz and R. Vogt, *Interchanging A_∞ and E_n structures*, arXiv:1102.1311.

[4] J. Lurie, *Derived algebraic geometry VI: $\mathbb{E}[k]$-algebras*, arXiv:0911.0018.

[5] D. P. Sinha, *Operads and knot spaces*, J. Amer. Math. Soc. **19** (2006), no. 2, 461–486 (electronic).

A multiplicative Beilinson Regulator via differential forms

Ulrich Bunke
(joint work with Georg Tamme)

1. Bundles, Forms and regulators

The algebraic K-theory ring $K_*(X)$ of a regular separated scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ can be expressed as the homotopy groups of a connective commutative ring spectrum $K(X)$. One can use the evaluation of the cohomology theory represented by $K(X)$ on manifolds M in order to represent classes in $K_*(X)$. More precisely, a locally free, locally finitely generated $\mathrm{pr}_X^*\mathcal{O}_X$-module $\mathcal{V}$ on $M \times X$ (called bundle) determines a class $[\mathcal{V}] \in K(X)^0(M)$. An analysis of this class in the Atiyah-Hirzebruch spectral sequence (AHSS) with second page

$$E_2^{p,q} \cong H^p(M, K_{-q}(X))$$

and evaluation against classes in $H_*(M;\mathbb{Z})$ produces classes in $K_*(X)$. Note that the AHSS degenerates rationally at the second page. For example, a unit $\lambda \in \mathcal{O}_X^*(X)$ gives rise to a sheaf $\mathcal{V}(\lambda)$ on $S^1 \times X$ with holonomy λ such that $[\mathcal{V}(\lambda)] = 1 \oplus [\lambda]$ under the canonical isomorphism $K(X)^0(S^1) \cong K_0(X) \oplus K_1(X)$.

In order to detect those classes one can use Beilinson's regulator [Beĭ84]

$$r : K_*(X) \to H^*_{\mathcal{H}}(X)$$

mapping algebraic K-theory to absolute Hodge cohomology. To $M \times X$ we associate the commutative differential graded algebra $IDR(M \times X)$ which is defined as the subcomplex of the doubly graded complex

$$\prod_{p\geq 0} A([0,1] \times M \times X(\mathbb{C}))[2p]$$

of families of forms $(\omega(p))_{p\geq 0}$ which satisfy

1. $\omega(p)$ has logarithmic growth and belongs to the $2p$th step of the décalage of the weight filtration in the X-direction.
2. $(2\pi i)^{-p}\omega(p)_{|\{0\}}$ is real.
3. $\omega(p)_{|\{1\}}$ belongs to the pth step of the Hodge filtration.
4. ω is invariant under the natural $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action.

These conditions are modelled after [Bur94], [BW98], see also [BT], such that we have an isomorphism

$$H^*(IDR(M \times X)) \cong \bigoplus_{*=p+q} H^p(M; H^{-q}_{\mathcal{H}}(X)) \ . \qquad (1)$$

The complexification V of a bundle $\mathcal{V}$ is a complex vector bundle with a holomorphic structure $\bar\partial$ in the X and a flat connection ∇^I in the M-direction. A geometry on $\mathcal{V}$ is a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant connection $\tilde\nabla$ on the pull-back of V to $[0,1] \times M \times X(\mathbb{C})$ which extends to some compactification in the X-direction such that $\tilde\nabla_{|\{0\}}$ is unitary and $\tilde\nabla_{|\{1\}}$ extends the partial connection $\bar\partial + \nabla^I$.

The pair $(\mathcal{V}, \tilde{\nabla})$ will be called a geometric bundle. The family $(\mathbf{ch}_{2p}(\tilde{\nabla}))_{p\geq 0}$ of Chern forms is a zero cycle in $IDR^0(M\times X)$. If $[\mathcal{V}]_{\mathbb{Q}} \in \bigoplus_{p\geq 0} H^p(M, K_{-p}(X)\otimes\mathbb{Q})$ represents $[\mathcal{V}]\otimes\mathbb{Q}$ in the second AHSS-page for $K(X)^0(M)\otimes\mathbb{Q}$, then for a class $u\in H_p(M;\mathbb{Q})$ we have the equality in $H^{-p}_{\mathcal{H}}(X)$

$$r(\langle u, [\mathcal{V}]_{\mathbb{Q}}\rangle) = \langle u, \mathbf{ch}_{2p}(\tilde{\nabla})\rangle$$

where we use the decomposition (1).

2. Differential algebraic K-theory

Let H be the lax symmetric monoidal Eilenberg-MacLane correspondence from the ∞-catgeory of chain complexes to the ∞-category of spectra. We define the differential algebraic K-theory as a presheaf with values in the ∞-category of commutative ring spectra on the product $\mathcal{S}$ of the sites of manifolds (with open covering topology) and schemes X as above (with Zariski-topology) which evaluates at $M\times X$ as

$$\begin{array}{ccc} \hat{K}(M\times X) & \xrightarrow{R} & H(\sigma^{\geq 0} IDR(M\times X)) \\ \downarrow{\scriptstyle I} & & \downarrow \\ K(X)^M & \xrightarrow{r} & H(IDR(M\times X)) \end{array} .$$

The right vertical map is induced by the embedding of the stupid truncation $\sigma^{\geq 0}$ of a complex into itself. The construction of the lower horizontal arrow as a morphism between sheaves of ring spectra is the main result to be announced in this report. Its construction will be sketched in Section 4.

We define the differential algebraic K-theory by

$$\hat{K}^0(M\times X) := \pi_0(\hat{K}(M\times X)) .$$

It fits into a natural exact sequence

$$K(X)^{-1}(M) \xrightarrow{r} IDR(M\times X)^{-1} \xrightarrow{a} \hat{K}^0(M\times X) \xrightarrow{I} K(X)^0(M) \to 0 . \quad (2)$$

The construction of differential algebraic K-theory is designed such that there is an additive and multiplicative cycle map which sends a geometric bundle $(\mathcal{V}, \tilde{\nabla})$ to a class

$$[\mathcal{V}, \tilde{\nabla}] \in \hat{K}^0(M\times X)$$

with

$$R([\mathcal{V}, \tilde{\nabla}]) = (\mathbf{ch}_{2p}(\tilde{\nabla}))_{p\geq 0} , \quad I([\mathcal{V}, \tilde{\nabla}]) = [\mathcal{V}].$$

The homotopy fibre of R evaluated at $*\times X$ is a spectrum which represents arithmetic K-theory as introduced in [GS90] and further developed in [Tak05]. The constructions presented so far allow to construct classes in arithmetic K-theory in terms of virtual geometric bundles with vanishing curvature.

3. Why a multiplicative regulator

In this section we sketch a construction which motivates a multiplicative spectrum level Beilinson regulator r. Let R be a number ring and $X := \mathrm{Spec}(R)$. We let $R^\circ := R^* \cap (1 - R^*)$. To $\lambda \in R^\circ$ we associate a bundle $\mathcal{V}(\lambda)$ on $S^1 \times X$ and set $c(\lambda) := [\mathcal{V}(\lambda)] - 1 \in K(X)^0(S^1)$. The Steinberg relation implies that $\mathrm{pr}_1^* c(\lambda) \cup \mathrm{pr}_2^* c(1-\lambda) = 0$ in $K(X)^0(S^1 \times S^1)$. We now observe that we can equip $\mathcal{V}(\lambda)$ with a canonical geometry $\tilde{\nabla}(\lambda)$. We set $\hat{c}(\lambda) := [\mathcal{V}(\lambda), \tilde{\nabla}(\lambda)] \in \hat{K}^0(S^1 \times X)$. It turns out that

$$\mathrm{pr}_1^* \hat{c}(\lambda) \cup \mathrm{pr}_2^* \hat{c}(1-\lambda) + a(L(\lambda)) = 0$$

for a universal correction term $L(\lambda) \in IDR^{-1}(S^1 \times S^1 \times X)$. In order to construct it we replace R by $R^{univ} := \mathbb{Z}[\lambda^{univ}, (\lambda^{univ})^{-1}, (\lambda^{univ} - 1)^{-1}]$. There exists a unique (up to an additive constant) element $L^{univ} \in IDR^{-1}(S^1 \times S^1 \times \mathrm{Spec}(R^{univ}))$ such that

$$dL^{univ} = -R(\mathrm{pr}_1^* \hat{c}(\lambda^{univ}) \cup \mathrm{pr}_2^* \hat{c}(1 - \lambda^{univ})) \ .$$

The form L is essentially the Bloch-Wigner dilogarithm [Zag07]. We interpret λ as a map $X \to \mathrm{Spec}(R^{univ})$ and define $L(\lambda) := (\mathrm{id}_{S^1 \times S^1} \times \lambda)^* L^{univ}$.

The multiplicativity of the cycle map implies a factorization

$$\begin{array}{ccc} \mathbb{Z}[R^\circ] & \xrightarrow{\lambda \mapsto \mathrm{pr}_1^* \hat{c}(\lambda) \cup \mathrm{pr}_2^* \hat{c}(1-\lambda)} & \hat{K}^0(S^1 \times S^1 \times X) \\ {\scriptstyle \lambda \mapsto \lambda \wedge (1-\lambda)} \downarrow \phi & \nearrow & \\ R^* \wedge R^* & & \end{array} \ .$$

If $\sum_k n_k \lambda_k \in \mathbb{Z}[R^\circ]$ satisfies $\phi(\sum_k n_k \lambda_k) = 0$, i.e. it is a cycle in the Bloch complex [Bl], then $a(\sum_k n_k L(\lambda_k)) = 0$. This implies by (2) that there exists an element $x \in K(X)^{-1}(S^1 \times S^1)$ such that $r(x) = \sum_k n_k L(\lambda_k)$. In this way we can reproduce the construction [Bl] of elements in $K_3(R)$ from cycles in Bloch's complex whose regulator is given as a linear combination of values of the Bloch-Wigner dilogarithm.

4. Construction of the multiplicative regulator

We let Vect be the bimonoidal stack on $\mathcal{S}$ of bundles $\mathcal{V}$ as above. We apply the composition $\mathrm{K} := \mathrm{sp} \circ \Omega B \circ \mathrm{N}$ of the functors nerve N, ring completion, and identification of commutative ring spaces with connective ring spectra, and the sheafification $\mathcal{L}$ to Vect in order to define the sheaf of connective ring spectra $\mathbf{K} := \mathcal{L}(\mathrm{K}(\mathrm{Vect}))$. We have a natural equivalence $\mathbf{K}(M \times X) \cong K(X)^M$.

We refine Vect to the bimonoidal stack Vect^∇ of geometric bundles $(\mathcal{V}, \tilde{\nabla})$. Then we can define a morphism of presheaves of commutative ring spectra

$$\phi \colon \mathrm{K}(\mathrm{Vect}^\nabla) \to \mathrm{K}(\pi_0(\mathrm{Vect}^\nabla)) \stackrel{(\mathbf{ch}_{2p})_{p \geq 0}}{\longrightarrow} \mathrm{K}(Z^0(IDR)) \cong H(Z^0(IDR)) \to H(IDR)$$

where π_0 is the functor which sends a bimonoidal category to its semiring of isomorphism classes again considered as a bimonoidal category. We let $\bar{s}$ be the functor which maps a (pre)sheaf F on manifolds to the (pre)sheaf which evaluates at M as $\bar{s}F(M) := \mathrm{colim}_{\Delta} F(\Delta^{\bullet} \times M)$. There is a natural morphism $F \to \bar{s}F$ which is an equivalence if F was homotopy invariant. Using the flexibility of geometries one shows that the forgetful map

$$\bar{s}\mathrm{K}(\mathrm{Vect}^{\nabla}) \to \bar{s}\mathrm{K}(\mathrm{Vect})$$

is an equivalence. The construction of the multiplicative regulator now proceeds by the following diagram

$$\begin{array}{ccccccc}
\mathrm{K}(\mathrm{Vect}) & \longrightarrow & \bar{s}\mathrm{K}(\mathrm{Vect}) & \xleftarrow[\sim]{} & \bar{s}\mathrm{K}(\mathrm{Vect}^{\nabla}) & \xrightarrow{\phi} & \bar{s}H(IDR) \\
\downarrow & & & & & & \updownarrow \sim \\
\mathbf{K} & & & \xrightarrow{\quad r \quad} & & & H(IDR)
\end{array} .$$

The factorization of the upper-right-down composition over the left-down sheafification arrow exists since $H(IDR)$ is a sheaf.

References

[Beĭ84] A. A. Beĭlinson, *Higher regulators and values of L-functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR 760999 (86h:11103)

[Bur94] José Ignacio Burgos, *A C^∞ logarithmic Dolbeault complex*, Compositio Math. **92** (1994), no. 1, 61–86. MR 1275721 (95g:32056)

[Bl] Spencer Bloch, *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves.* CRM Monograph Series Vol 11, American Mathematical Society, Providence, RI, 2000.

[BW98] Jose Ignacio Burgos and Steve Wang, *Higher Bott-Chern forms and Beilinson's regulator*, Invent. Math. **132** (1998), no. 2, 261–305. MR 1621424 (99j:14008)

[BT] Ulrich Bunke and Georg Tame, *Regulators and cycle maps in higher-dimensional differential algebraic K-theory* ArXiv, 1209.6451.

[GS90] Henri Gillet and Christophe Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metric. I*, Ann. of Math. (2) **131** (1990), no. 1, 163–203. MR 1038362 (91m:14032a)

[Tak05] Yuichiro Takeda, *Higher arithmetic K-theory*, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 599–681. MR 2153537 (2006i:14022)

[Zag07] Don Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

On the space of positive scalar curvature metrics

Wolfgang Steimle

(joint work with Bernhard Hanke, Thomas Schick)

For a closed spin manifold M, let $\mathrm{Riem}^+(M)$ denote the space of Riemannian metrics of positive scalar curvature on M. It is topologized as an open subspace of the space $\mathrm{Riem}(M)$ of all Riemannian metrics, with its C^∞-topology.

While $\mathrm{Riem}(M)$, as a convex space, is homotopy equivalent to a point, not much is known on the homotopy type of the space $\mathrm{Riem}^+(M)$. A classical construction of Gromov–Lawson shows that for a spin manifold M of dimension $4n-1 \geq 7$, we have either $\mathrm{Riem}^+(M) = \emptyset$ or $|\pi_0 \,\mathrm{Riem}^+(M)| = \infty$ [7, Theorem IV.7.7]. On the other hand Hitchin [5, Theorem 4.7] constructed examples of spin manifolds M such that $\pi_1 \operatorname{Riem}^+(M)$ contains an element of order two.

Only recently some more information could be obtained on the higher homotopy type of $\operatorname{Riem}^+(M)$. Crowley–Schick showed [3, Corollary 1.5] that if M is any spin manifold with admits a metric of positive scalar curvature, there are non-trivial elements of order two in $\pi_* \operatorname{Riem}^+(M)$ in infinitely many degrees. In a somewhat different direction, Chernysh [2] and Walsh [9] showed that for spin manifolds, the homotopy type of $\operatorname{Riem}^+(M)$ is a spin cobordism invariant.

The following is the main result of the talk:

Theorem 1. *Let $k \geq 1$ be a natural number. Then there is a natural number $N(k)$ with the following property: For each $n \geq N(k)$ and each spin manifold M admitting a metric g of positive scalar curvature and of dimension $4n-k-1$, the homotopy group*

$$\pi_k(\operatorname{Riem}^+(M), g)$$

contains elements of infinite order.

To our knowledge, this is the first construction of infinite-order elements in higher homotopy groups of $\operatorname{Riem}^+(M)$. Moreover, in constrast to the results of Hitchin and Crowley–Schick, our construction is not based on the action of the diffeomorphism group $\operatorname{Diff}(M)$ on $\operatorname{Riem}^+(M)$. To express this fact, we make the following definition.

Definition A class $c \in \pi_k(\operatorname{Riem}^+(M), g)$ is called *geometrically insignificant* if c is represented by

$$\begin{array}{ccc} S^k & \to & \operatorname{Riem}^+(M) \\ t & \mapsto & f(t)^* g \end{array}$$

for some continuous pointed map $f \colon S^k \to \operatorname{Diff}(M)$. Otherwise, c is called *geometrically significant.*

Theorem 2. *Under the assumptions of Theorem 1, suppose moreover that all (but the 0th) rational Pontryagin classes of M vanish. Then the group $\pi_k(\operatorname{Riem}^+(M), g)$ contains elements of infinite order which are geometrically significant.*

Notice that the elements constructed by Hitchin and Crowley–Schick are, by their very construction, geometrically insignificant. The condition on the Pontryagin classes appearing in Theorem 2 is satisfied, for instance, if M is a sphere (or more generally a rational homology sphere), or if M is stably parallelizable.

Major ingredients of the proof of Theorems 1 and 2 are Igusa's fiberwise Morse theory [6] and a parametrized version of the Gromov–Lawson surgery method, as recently developed by Walsh [8]. Starting point of the construction is a sufficiently interesting bundle of cobordisms between two trivial sphere bundles. It is obtained using the following result of independent interest, whose proof relies on classical techniques from differential topology: Surgery theory, Casson's theory of prefibrations [1] and Hatcher's theory of concordance spaces [4].

Theorem 3. *Given $k, l \geq 1$, there is $N = N(k,l) \in \mathbb{N}_{\geq 0}$ with the following property: For all $n \geq N$, there is a $4n$-dimensional smooth closed spin manifold P which is the total space of a smooth fiber bundle*

$$F \hookrightarrow P \to S^k,$$

such that:

(1) *The $\hat{A}$-genus of P is non-zero,*
(2) *the fiber F is l-connected, and*
(3) *the bundle $P \to S^k$ has a smooth section $s\colon S^k \to P$ with trivial normal bundle.*

References

[1] Andrew J. Casson, *Fibrations over spheres*, Topology **6** (1967), 489-499.
[2] Vladislav Chernysh, *On the homotopy type of the space $\mathcal{R}^+(M)$*, online preprint (2004), arXiv:math/0405235.
[3] Diarmuid Crowley, Thomas Schick, *The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature*, preprint (2012), arXiv:1204.6474.
[4] Allen Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, Proceedings of Symposia in Pure Mathematics, Vol 32 (1978).
[5] Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.
[6] Kiyoshi Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), 1-355.
[7] H. Blaine Lawson, M.-L. Michelsohn, *Spin geometry*, Princeton University Press 1989.
[8] Mark Walsh, *Metrics of positive scalar curvature and generalized Morse functions, part II*, to appear in Trans. Am. Math. Soc.
[9] Mark Walsh, *Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics*, preprint (2011), arXiv:1109.6878.

A spectral sequence for the homology of a finite algebraic delooping

Birgit Richter
(joint work with Stephanie Ziegenhagen)

In the category of chain complexes E_n-algebras are the analogs of n-fold loop spaces. Examples of such algebras are given by ordinary commutative algebras, E_∞-algebras such as cochain complexes of topological spaces, Hochschild cochain complexes carry an E_2-algebra structure and chain complexes of n-fold loop spaces are E_n-algebras. We want to understand deloopings of such algebras.

There is a homology theory for E_n-algebras, E_n-homology. Benoit Fresse constructed an n-fold bar construction for such algebras and showed that E_n-homology calculates the homology of this n-fold bar construction [2]. In this sense, E_n-homology calculates the homology of an n-fold delooping.

We construct a spectral sequence whose E^2-term consists of derived functors of indecomposables with respect to a certain Gerstenhaber structure of the homology of the algebra in question and it converges to the E_n-homology of the algebra. We work relative to a ground field k of either characteristic two or zero. Over the rationals we consider E_n-algebras for arbitrary $n \geq 2$ whereas in characteristic two we restrict to the case $n = 2$. We work with augmented algebras $\varepsilon\colon A_* \to k$ and $\bar{A}_*$ denotes the augmentation ideal of A_*.

A crucial technical lemma identifies the homology of free E_2-algebras (E_n-algebras) in these situations as the free 1-restricted Gerstenhaber algebra on the homology for $\mathbb{F}_2$ and the free $(n-1)$-Gerstenhaber algebra on the homology over the rationals. Considering the standard resolution by free objects yields a resolution spectral sequence.

Theorem

- For any augmented E_n-algebra A_* over $\mathbb{Q}$ there is a spectral sequence
$$E^2_{p,q} = \mathbb{L}_p Q_{(n-1)G}(H_*(\bar{A}_*))_q \Rightarrow H^{E_n}_{p+q}(\bar{A}_*).$$
- If A_* is an augmented E_2-algebra over $\mathbb{F}_2$, then there is a spectral sequence
$$E^2_{p,q} = \mathbb{L}_p Q_{1rG}(H_*(\bar{A}_*))_q \Rightarrow H^{E_2}_{p+q}(\bar{A}_*).$$

Here Q denotes the functor of indecomposables, $\mathbb{L}_*$ is the corresponding derived functor, $1rG$ is the category of 1-restricted Gerstenhaber algebras and $(n-1)G$ is the category of $(n-1)$-Gerstenhaber algebras.

For an augmented commutative $\mathbb{Q}$-algebra, the spectral sequence collapses at the E^2-term and we obtain:
$$\bigoplus_{p+q=\ell} \mathbb{L}_p Q_{(n-1)G}(H_*(\bar{A}))_q \cong H^{E_n}_\ell(\bar{A}).$$

On the other hand one can identify E_n-homology with Hochschild homology of order n
$$H^{E_n}_\ell(\bar{A}) \cong HH^{[n]}_{\ell+n}(A;\mathbb{Q})$$
in the sense of Pirashvili. Over the rationals $HH^{[n]}_*$ possesses a Hodge decomposition for every $n \geq 1$ [3]. Comparing both decomposition helps to describe Hodge summands in terms of $(n-1)$-Gerstenhaber homology groups.

As we can express Gerstenhaber indecomposables as a composite of the indecomposables with respect to the multiplicative structure, Q_a, followed by the indecomposables with respect to the Lie structure, Q_{nL} or Q_{1rL}, we get Grothendieck-type spectral sequences in the non-additive context by the work of Blanc and Stover [1].

Theorem

- If the ground field is $\mathbb{F}_2$, then for any augmented 1-restricted Gerstenhaber algebra C there is a spectral sequence
$$E^2_{s,t} = (\mathbb{L}_s(\bar{Q}_{1rL})_t)(AQ_*(C|\mathbb{F}_2,\mathbb{F}_2)) \Rightarrow \mathbb{L}_{s+t}(Q_{1rG})\bar{C}).$$
- Over the rationals we have
$$(\mathbb{L}_s(\bar{Q}_{nL})_t)(AQ_*(C|\mathbb{Q},\mathbb{Q})) \Rightarrow \mathbb{L}_{s+t}(Q_{nG})\bar{C}).$$

Here, the $\bar{Q}$ denotes the extension of Q to the category of Π-Lie algebras.

A class of interesting examples of E_2-algebras is given by Hochschild cochains of associative algebras. For any vector space V, the tensor algebra TV is the free associative algebra generated by V. Taking the composition with the Hochschild cochains, $C^*(-,-)$, we assign to any vector space V the E_2-algebra $C^*(TV, TV)$. One can ask, how free this E_2-algebra is. For a free E_2-algebra on a vector space V, E_2-homology gives V back. Is the homology of the 2-fold delooping, *i.e.*, $H^{E_2}_*(C^*(TV, TV))$, close to V? We give a negative answer for a one-dimensional vector space over the rationals. For a vector space of arbitrary dimension, we can identify the input for the E^2-term of the Blanc-Stover spectral sequence.

The chain complex of an n-fold loop space carries an E_n-algebra structure. If the loop space is of the form $\Omega^n\Sigma^n X$ for $n \geq 2$ and connected X, then E_n-homology of the rational chain algebra hands back the reduced homology of $\Sigma^n X$. We identify the E^2-page of the resolution spectral sequence for rational chains on $\Omega^n X$ for any n-connected space X as

$$\mathbb{L}_p(Q_{(n-1)G})(H_*(\Omega^n X;\mathbb{Q}))_q \cong \mathrm{Tor}^{H_*(\Omega X;\mathbb{Q})}_{p+1,q+n-1}(\mathbb{Q},\mathbb{Q}).$$

References

[1] D. Blanc, C. Stover, *A generalized Grothendieck spectral sequence*, Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990), London Math. Soc. Lecture Note Ser. **175**, Cambridge Univ. Press (1992), 145–161.

[2] B. Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, Alg. Geom. Topol. **11** (2011), 747–838.

[3] T. Pirashvili, *Hodge decomposition for higher order Hochschild homology*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 151–179.

[4] B. Richter, S. Ziegenhagen, *A spectral sequence for the homology of a finite algebraic delooping*, in preparation.

Variety isomorphism in group cohomology and control of *p*–fusion

Jesper Grodal

(joint work with D. Benson, E. Henke)

This talk was a report on the joint work [1]. In this we show that if an inclusion of finite groups $H \leq G$ of index prime to p induces a homeomorphism of mod p cohomology varieties, or equivalently an F–isomorphism in mod p cohomology, then H controls p–fusion in G, if p is odd. This generalizes classical results of Quillen who proved this when H is a Sylow p-subgroup, and furthermore implies a hitherto difficult result of Mislin about cohomology isomorphisms. For $p = 2$ we give analogous results, at the cost of replacing mod p cohomology with higher chromatic cohomology theories.

The results are consequences of a general algebraic theorem we prove, that says that isomorphisms between p–fusion systems over the same finite p–group are detected on elementary abelian p–groups if p odd and abelian 2–groups of exponent at most 4 if $p = 2$.

References

[1] D. Benson, J. Grodal, and E. Henke, *Variety isomorphism in group cohomology and control of p–fusion*, arXiv:1210.1564 [math.AT].

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