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Mini-Workshop: Topology of Real Singularities and Motivic Aspects

Organised by
Georges Comte, Université de Savoie-Chambéry
Mihai Tibar, Université Lille I

30 September – 6 October 2012

Abstract. This workgroup focusses on some recent issues in real singularities, concerning the topology of the Milnor fibre of a singular map and several motivic aspects of singularities of sets definable in some structures over the reals or even over some valued field, with the ambition to develop the interplay between the two domains.

Mathematics Subject Classification (2000): 14B05, 14P25, 14P10, 32S55, 14E18.

Introduction by the Organisers

The workshop *Topology of Real Singularities and Motivic Aspects*, organised by Georges Comte (Université de Savoie) and Mihai Tibar (Université Lille I) was held 30 September – 6 October 2012. This workshop was well attended with 17 participants with broad geographic representation. 17 talks were given. The general topic of these talks was the topology of singular fibres of real mappings (or sets), in some tame category of sets such as algebraic sets, semi-algebraic sets, or sets definable in some structure over a given first order language.

• The existence of a stable fibration in the neighbourhood of an (isolated) singularity is fundamental for the understanding of the local structure of the space-function pair. In his well-known Princeton lecture notes, John Milnor studied the fibration of a complex analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. He explained the first steps of extending the study to a real analytic map germ $g : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with isolated singularity.

Many authors pursued ever since to implement and highly enrich Milnor's technique in other fields, such as non-isolated singularities of holomorphic functions, complete intersections with isolated singularities, singularities at infinity of polynomial mappings and meromorphic functions. Remarkably, the topological study of singularities and of their deformations gave new perspectives over a lot of other fields such as exotic structures, Lie algebras, mixed Hodge theory, equisingularity, Frobenius manifolds etc.

In the real setting, a recent progress took place by M. Oka's series of papers on "mixed singularities" i.e. real analytic map germs $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^2, 0)$. Oka develops Milnor's theory and finds real counterparts of some of his holomorphic results. For instance the existence of an open book fibration with binding the link $K = f^{-1}(0)$ is not insured like in the holomorphic setting, thus one researches the "best" conditions under which this happens, in the local setting and also in the global affine one. "Open book" appear under several names in different fields, for instance "Lefschetz fibrations" in algebraic and symplectic geometry, "fibered links" and "spinorial structures" in topology.

In the study of the numerous questions and conjectures which appeared, some interesting classes of singularities prove helpful: those with non-degenerate Newton principal part, and the radial and polar quasi-homogeneous singularities. We hope for instance that the Newton boundary techniques, locally and at infinity, would produce exotic examples like the clasical example by A'Campo in connexion to the Lefschetz number of the monodromy.

Along this path we aim to the study of the monodromy in the real setting, notably via the resolution by blow-ups and by toric methods.

• The second main stream of the workshop concerns motivic aspects of singularities.

To emphasize the perfect thematic continuity between the two domains into consideration during our workshop, one may start from A'Campo's formula for the Lefschetz number of the iterates of the monodromy of singularity in terms of the combinatorial data of a resolution of this singularity (including the Euler-Poincaré characteristic of the irreducible components of the exceptional divisor of the resolution). In the complex setting, this may be viewed as part of the theory of equisingularity, initiated by Zariski. In this context Zariski himself, then A'Campo, Briançon, Henry, Hironaka, Merle, Sabbah, Speder, Teissier, Lê-Dũng-Trang among others, established a number of results, relating different aspects of the geometry of the singularity (algebraic, topological, differential).

J. Denef and F. Loeser gave more recently a new point of view on this formula: in the spirit of Igusa's work in the p-adic framework, they proved that some formal series, the so-called motivic zeta function, built on some complex algebraic singularity, with coefficients in some formal Grothendieck ring of varieties, is a rational function and that the realization of this rational function via the Euler-Poincaré characteristic gives the Milnor number of the fibration. This rational function is therefore called the motivic Milnor fibre of the fibration. It would be crucial to completely understand the part of the topology in the proof of the rationality of the motivic Milnor fibre. Namely what kind of topological data is structurally encoded in the motivic Milnor fibre ? This issue contains for instance the so-called monodromy conjecture that aims to relate the poles of the motivic Milnor fibre and the eigenvalues of the monodromy function.

In this spirit, we would like to understand what topological invariants remain in the real motivic framework. Assuming that we would be able to establish a real substitute of the complex motivic Milnor fibre and prove a real version of the Denef-Loeser formula, is it still true that the Euler-Poincaré characteristic of semi-algebraic nearby fibres is contained in the topological and combinatorial data of a resolution? Indeed, in the real case, the combinatory of a resolution appears in a less trivial way after the possible realization of the real substitute of the complex motivic Milnor fibre and, consequently, a real formula probably would give more explicit views than in the complex case on the interplay between the topology of the nearby fibres and the combinatorial data of a resolution. To establish the real counterpart of the complex motivic Milnor fibre one has to find in particular the pertinent notion of Grothendieck ring of semialgebraic formulas.

The question underlying the question of a real motivic version of the complex Milnor fibre is the general problem of finding additive invariants on real algebraic or semi-algebraic sets, such as the Euler-Poincaré characteristic, the virtual Betti numbers etc. If this question gave rise to many successful attempts recently, much remains to do in this area.

Mini-Workshop: Topology of Real Singularities and Motivic Aspects

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Examples of mixed polynomials $f(x, y)$. Many arcs. Tchebyshev polynomials $f : \mathbb{C}^2 \to \mathbb{C}$

Norbert A'Campo

This talk has three parts. In the first part, we consider the mixed polynomial in holomorphic and anti-holomorphic variables $f(x, y) = (x^3 - y^2)^2 - x^4\bar{x}y$. The differentiable mapping $f : \mathbb{C}^2 \to \mathbb{C}$ has an isolated singularity at $0 \in \mathbb{C}^2$. This singularity shares many local properties with complex plane curve singularities:

– its local link is an iterated torus knot. Here we have the knot $(3, 2), (11, 2)$ which violates the Puiseux inequalities by 2.

– its local link is a divide link.

– the geometric monodromy is of finite type. The eigenvalues of the homological monodromy are roots of unity.

– the map f is locally near 0 symplectic in the following sense. The restriction of the standard symplectic form of $\mathbb{C}^2$ is at each smooth point of a fiber of f a symplectic form on the fiber.

Question. Do the above properties locally near an isolated singularity for all mixed polynomial mappings $f : \mathbb{C}^2 \to \mathbb{C}$ hold, if the mapping is locally symplectic near the singularity?

The polynomial $f(x, y) = \bar{x}y(x + y)$ is not symplectic near 0, the geometric monodromy is not of finite type, eigenvalues are not roots of unity, and the oriented link is not a divide link.

Second part. Let the (holomorphic) polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ have at $0 \in \mathbb{C}^{n+1}$ and above $0 \in \mathbb{C}$ an isolated singularity. We assume that 0 is the only critical value in $[0, 1] \subset \mathbb{C}$. We lift the constant vector field $X = -1$ on $\mathbb{C}$ above $]0, 1]$ to a vector field Y on $\mathbb{C}^{n+1}$ such that the conditions $(Df)_p(Y_p) = X_{f(p)}$ and $\omega(Y_p, Z_p) = 0$, $p \in f^{-1}([0, 1])$, $Z_p \in T_p(f^{-1}(f(p)))$, hold. Here, ω is the usual symplectic form on $\mathbb{C}^{n+1}$. For $p \in f^{-1}(1)$ let a_p be the closure of the flow line for Y_p over the time interval $[0, 1[$ starting at p. Now we have for every point in the regular fiber of f an parametrized arc that terminates at the singular fiber. Remark that the vector field Y explodes in norm at the singularity.

Question. Compare the system a_p, $p \in f^{-1}(1)$ of arcs with the motivic fiber?

Given an embedded resolution of the singularity at 0, the system a_p, $p \in f^{-1}$ does not lift to a continuous system of arcs in the total space: the end points $a_p(1)$ do not depend continuously on p.

Third part. Let Γ be an isotopy class of finite (vertex)-bicollared planar trees. Then using the Riemann mapping theorem for open sets in the complex line and Rouche's theorem one can constuct a polynomial $f_\Gamma : \mathbb{C} \to \mathbb{C}$ with critical values in the set $\{0, 1\}$ such that the planar tree $f_\Gamma^{-1}([0, 1])$ with vertices $f_\Gamma^{-1}(\{0, 1\})$ collared by 0 and 1 represents the class Γ. The polynomial f_Γ is unique up to affine coordinate change in the source. By a non unique affine coordinate change in the source we normalize further f_Γ to be monic with vanishing subleading coefficient. The coefficients of normalized f_Γ belong to a number field K_Γ. The classical Tchebyshev polynomial T_n are up to affine coordinate changes in source and target caracterized by degree$(T_n) = n$, the mapping $T_n : \mathbb{C} \to \mathbb{C}$ has 1 or 2 critical values and only Morse critical points. The corresponding planar tree is the linear tree A_{n-1}. The polynomials f_Γ are introduced by George Shabat [S] and called generalized Tchebyshev polynomials or Shabat polynomials. The Galois group of $\mathrm{Gal}(K_\Gamma/\mathbb{Q})$ acts on the coefficients of f_Γ. For $\sigma \in \mathrm{Gal}(K_\Gamma/\mathbb{Q})$ let f_Γ^σ be the resulting polynomial. Again $\Gamma^\sigma := (f_\Gamma^\sigma)^{-1}(\{0,1\})$ is a bicollared planar tree Γ^σ with corresponding polynomial f^σ. These action can be put together to a faithfull action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set of isotopy classes of planar trees.

We consider the mixed mapping $F_\Gamma := f_\Gamma \bar{f}_\Gamma : \mathbb{C} \to \mathbb{C}$. We identify the source $\mathbb{C}$ of F_Γ in the usual way with $\mathbb{R}^2$. All values of F_Γ are real, so we consider F_Γ as a polynomial mapping $F_\Gamma : \mathbb{R}^2 \to \mathbb{R}$. Let $T_\Gamma : \mathbb{C}^2 \to \mathbb{C}$ be the complexification of F_Γ.

Interestingly, the mapping $T_\Gamma : \mathbb{C}^2 \to \mathbb{C}$ has only two critical values, and can be called a complex Tchebyshev polynomial with two variables.

Problem. Compute the monodromies of $T_\Gamma : \mathbb{C}^2 \to \mathbb{C}$ arround $0, 1, \infty$. For $\sigma \in \mathrm{Gal}(K_\Gamma/\mathbb{Q})$ compare these monodromies T_Γ and T_{Γ^σ}.

Our aim is to find two mapping classes $a, b \in M_{g,r}$ that are not conjugated in the mapping class group, but which are in the same Galois orbit of the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the profinite completion of $M_{g,r}$. Geometric monodromies of $T_\Gamma, T_{\Gamma^\sigma}$ at ∞ are our candidates.

References

[1] Nikolai Adrianov, Georges B. Shabat, *Galois orbits of Plane Trees*, Geometric Galois Actions, ed. by Leila Schneps and Pierre Lochak, London Math. Soc. Lecture Notes Series 243, p. 13–24.
[2] Georges B. Shabat, *On the classification of plane trees by their Galois orbits*, The Grothendieck Theory od Dessins d'Enfants, ed. by L. Schneps, London Math. Soc. Lecture Notes Series 200, p. 169–177.

Gromov-Yomdin parametrizations in the non-archimedian context

Raf Cluckers
(joint work with Georges Comte, François Loeser)

After discussing mild parameterizations, and the impossibility for obtaining mild parameterizations in families of semi-algebraic (or more generally globally subanalytic) sets, we focus on parameterizations in the style of Gromov [3] and Yomdin [4]. Mild parameterizations are smooth functions with control on the size of all the partial derivatives; Gromov - Yomdin parameterizations control the size only up to a bounded (but possibly high) number of iterated partial derivatives. All these parameterizations have been used by J. Pila, A. Wilkie [5], [6], and others, to obtain striking bounds on the number of rational points of bounded height on various geometric objects, generalizing results of [1] to arbitrary dimensions. Mild parameterizations yield sharper conclusions (see [5]), but are, up to now, only applicable to low dimensional sets, although several conjectures (e.g. a conjecture by A. Wilkie) predict that these sharper upper bounds hold in many situations, regardless of dimension. In my talk, I focused on the recent work of G. Comte, F. Loeser and myself on transposing the line of results from Gromov-Yomdin parameterizations to its applications to counting in [6] to a non-archimedean context. We obtain parameterizations for complete Henselian valued fields of characteristic zero, in the style of Gromov-Yomdin, but with extra control for approximating by Taylor polynomials. Good approximation by Taylor polynomials comes for free in the real setting, but not so in the non-archimedean setting because of total disconnectedness, see e.g. [2]. In the case of a subanalytic set of $\mathbb{Q}_p$, upper bounds for the number of rational points of bounded height now follow in analogy to the real case. In the case of $\mathbb{C}((t))$, we consider elements of $\mathbb{C}[t]$ to be the analogue of integer points in $\mathbb{C}((t))$, and their degree in t as the analogue of the (logarithmic) height. Points of bounded height are no longer finite in cardinality, but they are finite in dimension over $\mathbb{C}$ (e.g. seen as vector space over $\mathbb{C}$). We then show that, on the transcendental part X^{trans} of a subanalytic set X in $\mathbb{C}((t))^n$, the dimension of the set of polynomials in t lying on X^{trans} and of degree at most ℓ in t, grows slower to infinity than $c_\epsilon + \epsilon\ell$ for any $\epsilon > 0$.

References

[1] E. Bombieri, J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J. **59** (1989), 337–357.

[2] R. Cluckers, G. Comte, F. Loeser, *Lipschitz continuity properties for p-adic semi-algebraic and subanalytic functions*, Geom. Funct. Anal. **20** (2010), 68–87.

[3] M. Gromov, *Entropy, homology and semialgebraic geometry.* Séminaire Bourbaki, Vol. 1985/86. Astérisque No. 145-146 **5** (1987), 225–240.

[4] G. Comte, Yomdin, *Tame Geometry with applications in smooth Analysis*, L.N.M. 1834, Springer (2004).

[5] J. Pila, *Counting rational points on a certain exponential-algebraic set*, Ann. Inst. Fourier **60** (2010), 489–514.

[6] J. Pila, A. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), 591–616.

On the topology of real Milnor-Lê fibrations

Nicolas Dutertre

(joint work with Raimundo Araújo dos Santos)

Let $F : (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be an analytic map that satisfies Milnor's conditions (a) and (b) introduced by D. Massey [1]. Let us remind first what these two conditions are. Let $V = F^{-1}(0)$. Let Σ_F be the set of critical points of F, i.e. the set of points where the gradients $\nabla f_1, \ldots, \nabla f_k$ are linearly dependent. Let ρ be the euclidian distance, we denote by $\Sigma_{F,\rho}$ be the set of critical points of (F, ρ), i. e. the set of points where the gradients $\nabla \rho, \nabla f_1, \ldots, \nabla f_k$ are linearly dependent.

Definition 0.1. *We say that F satisfies Milnor's condition (a) if $\Sigma_F \subset V$ near 0.*

We say that F satisfies Milnor's condition (b) if 0 is isolated in $V \cap \overline{\Sigma_{F,\rho} \cap V}$.

For $l \in \{1, \ldots, k\}$ and for $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, we denote by f_I the map $(f_{i_1}, \ldots, f_{i_l})$. Let us observe that

$$\Sigma_{f_I} \subset \Sigma_F \subset F^{-1}(0) \subset f_I^{-1}(0),$$

and so, that the map f_I satisfies Milnor's condition (a). We also see that $\Sigma_{f_I,\rho} \subset \Sigma_{F,\rho}$ and we have:

Lemma 0.2. *For $l \in \{1, \ldots, k\}$ and for $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, the map $f_I : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$ satisfies Milnor's condition (b).*

Let M_F be the Milnor fibre of the mapping F and, for $\#I \geq 2$, let M_{f_I} be the Milnor fibre of f_I.

Theorem 0.3. *We have $\chi(M_F) = \chi(M_\phi)$.*

If $I = \{i\}$, we denote by $M^+_{f_i}$ the positive Milnor fibre of f_i and by $M^-_{f_i}$ its negative Milnor fibre.

Corollary 0.4. *For every $j \in \{1, \ldots, k\}$, we have $\chi(M^+_{f_j}) = \chi(M^-_{f_j}) = \chi(M_F)$.*

We denote by $\mathcal{L}_I$ the link of the zero-set of f_I.

Theorem 0.5. *Let $l \in \{1, \ldots, k\}$ and let $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ be an l-tuple. If n is even, then we have:*

$$\chi(\mathcal{L}_I) = 2\chi(M_F) \text{ if } l \text{ is odd and } \chi(\mathcal{L}_I) = 0 \text{ if } l \text{ is even.}$$

If n is odd, then we have:

$$\chi(\mathcal{L}_I) = 2 - 2\chi(M_F) \text{ if } l \text{ is odd and } \chi(\mathcal{L}_I) = 2 \text{ if } l \text{ is even.}$$

□

References

[1] D. Massey, *Real analytic Milnor fibrations and a strong Lojasiewicz inequality*, Real and complex singularities, London Math. Soc. Lecture Note Ser., **380** Cambridge Univ. Press, Cambridge (2010), 268–292.

Motivic Real Milnor Fibres

Goulwen Fichou
(joint work with Georges Comte)

The Grothendieck ring of semi-algebraic sets is reduced to the ring of integers. Let $BSA_{\mathbb{R}}$ be the set of basic semialgebraic formulas, that is systems of real polynomial equations, inequalities $>$ and inequations $\neq$. We build a ring Grothendieck ring $K_0(BSA_{\mathbb{R}})$ that takes into account the formulas which define the basic semi-algebraic sets. We then produce a realization of this ring into

$$Z_f^{\epsilon}(T) \in (K_0(Var_{\mathbb{R}}) \otimes \mathbf{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]],$$

by considering natural coverings associated to the basic semi-algebraic sets. We then study motivic zeta functions

$$Z_f^{\epsilon}(T) \in (K_0(Var_{\mathbb{R}}) \otimes \mathbf{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]]$$

associated to a real polynomial germ f. The rationality of $Z_f^{\epsilon}(T)$ gives candidates for motivic versions of the real semialgebraic Milnor fibres.

To give an idea, consider the set of points in $\mathbb{R}^n$ satisfying the formula $R > 0$. It may be seen from two different points of view for our purpose

- Half of the constructible set $\{y^2 = R(x)\} \setminus \{R(x) = 0\}$
- The complement in $\mathbb{R}^n$ of half of $\{y^2 = -R(x)\}$
- The algebraic average of these two symetric points of view is

$$\frac{1}{2} \cdot \Big(\big(\frac{1}{2}[Y^2 = R] - [R = 0]\big) + \big(\mathbb{L}^n - \frac{1}{2}[Y^2 = -R]\big) \Big)$$

$$= \frac{1}{4}\big([Y^2 = R] - [Y^2 = -R]\big) + \frac{1}{2}[R \neq 0] \in K_0(Var_{\mathbb{R}}) \otimes \mathbf{Z}[\frac{1}{2}]$$

Now, let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a polynomial function, $\mathcal{L}$ the space of formal arcs $\gamma(t) = (\gamma_1(t), \cdots, \gamma_d(t))$, with $\gamma_j(0) = 0$ for $j \in \{1, \cdots, d\}$, $\mathcal{L}_n$ the space of $\mathcal{L}/(t^{n+1})$ truncated arcs.

For ϵ one of the 5 symbols $\{naive, -1, 1, >, <\}$ and via the realization of $K_0(BSA_{\mathbb{R}})$ in $K_0(Var_{\mathbb{R}}) \otimes \mathbf{Z}[\frac{1}{2}]$ one defines a zeta function $Z^{\epsilon}(T) \in (K_0(Var_{\mathbb{R}}) \otimes \mathbf{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]]$ by

$$Z_f^{\epsilon}(T) := \sum_{n \geq 1} [X_{n,f}^{\epsilon}] \mathbb{L}^{-nd} T^n, \text{ where}$$

- $X_{n,f}^{naive} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a \neq 0\}$,
- $X_{n,f}^{-1} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a = -1\}$,
- $X_{n,f}^{1} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a = 1\}$,
- $X_{n,f}^{>} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a > 0\}$,
- $X_{n,f}^{<} = \{\gamma \in \mathcal{L}_n;\ f(\gamma(t)) = at^n + \cdots, a < 0\}$.

By rationality of these zeta functions, we know that the limit

$$S_f^{\epsilon} := -\lim_{T \to \infty} Z_f^{\epsilon}(T)$$

exists, and we prove moreover that, evaluated via the Euler characteristics with compact supports, it coincides with the Euler characteristics with compact supports of the topological Milnor fibre.

References

[1] J. Denef, F. Loeser, *Computing certain invariants of topological spaces of dimension three*, Topology **41** (2002), no. 5, 1031–1040.

Motivic invariants of real polynomial functions and Newton polyhedron

Toshizumi Fukui
(joint work with Goulwen Fichou)

Motivation. We consider the classification problem for real polynomial function germs. A natural equivalence relation is defined by

$$f \underset{C^r}{\sim} g \iff \exists\ C^r\text{-diffeomorphism } h \text{ such that } f \circ h = g$$

for $r = 1, 2, \dots, \infty, \omega$. We call this equivalence relation C^r equivalence.

H. Whitney observed that C^1-equivalence relation causes a moduli, that is, setting $f_t(x, y) = xy(y - x)(y - tx)$, $0 < t < 1$, we are able to show that f_t and $f_{t'}$ are C^1-equivalent only if $t = t'$.

Topological analogue of the equivalence relation is defined by

$$f \underset{C^0}{\sim} g \iff \exists \text{ homeomorphism } h \text{ such that } f \circ h = g.$$

Since two functions $(x, y) \mapsto x^2 - y^3$ and $(x, y) \mapsto x$ are C^0-equivalent, it is hopeless to expect a decent classification theory for C^0-equivalence.

Suggested by the fact that Whitney's family becomes C^ω-trivial after blow up at the origin, T.-C. Kuo defined the notion of blow-analytic equivalence. He say that f and g are blow-analytic equivalent when there is a homeomorphism h with $f \circ h = g$ and h becomes an analytic isomorphism after finite composition of blow-ups with nonsingular centers. He call such homeomorphisms h by blow-analytic homeomorphisms. We feel better to call them by bi-blow-analytic homeomorphisms.

There is a bi-blow-analytic homeomorphism $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ which sends the cusp $C = \{(x, y) \mid x^2 - y^3 = 0\}$ to the line $L = \{(x, y) \mid x = 0\}$. This is obtained by constructing resolutions of C and L which have the same topological shape. By construction we observe that the jacobian of this bi-blow-analytic homeomorphism is not bounded away from infinity and zero.

We say that f and g are blow-Nash equivalence, if h can be chosen a semialgebraic homeomorphism and the jacobian determinant of h, which is defined on except a thin set, is bounded away from infinity and zero.

Motivic zeta functions, which are defined using virtual Poincare polynomials, seems to contain a lot of information. Actually Fichou showed that the type of quadratic parts of polynomials are recovered by zeta functions. He also showed

that analytic classification and blow-Nash classification are the same for simple singularities. We now motivate to seek a handable formulas for motivic zeta functions to decide whether given two polynomials are blow-Nash equivalent or not.
Results. Consider real polynomial germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. We denote by $\mathcal{A}_k(f)$ the set of analytic arcs $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ so that $f(\alpha(t)) = ct^k + \cdots$, $c \neq 0$. Take a Nash modification $\sigma : M \to \mathbb{R}^n$ so that the exceptional locus $E = \sum_{i\in I} E_i$ (irreducible decomposition) and $f \circ \sigma$ are normal crossing. Set

$$\det(d\sigma) = \sum_i (\nu_i - 1)E_i \quad \text{and} \quad (f \circ \sigma)_0 = X + \sum m_i E_i$$

where X is the strict transform. We set $X = \emptyset$ when f does not change the sign near 0. Set $E_J^\circ = \cap_{i\in J} E_i \setminus \cup_{j\notin J} E_j$ for $J \subset I$. For $a = (a_i)_{i\in I}$, set $I(a) = \{i \in I : a_i > 0\}$ and $|I(a)| = \#I(a)$.

Fichou showed the zeta function $\sum_{k\geq 1} \beta(\mathcal{A}_k(f)))t^k$ is a blow-Nash invariant where β denote the virtual Poincaré polynomial. Virtual Poincaré polynomials $\beta(\mathcal{A}_k(f))$ are considered as a representation of motivic invariant $[\mathcal{A}_k(f)]$.

We show the following formula:

$$\begin{aligned}[\mathcal{A}_k(f)] = &\sum_{a:m(a)=k} [E^\circ_{I(a)} \cap X](\mathbb{L}-1)^{|I(a)|}\mathbb{L}^{-s(a)} \\ &+ \sum_{a:m(a)<k} [E^\circ_{I(a)} \setminus X](\mathbb{L}-1)^{|I(a)|+1}\mathbb{L}^{-k+m(a)-s(a)}\end{aligned}$$

where $a = (a_j)_{j\in I}$, $m(a) = \sum_{i\in I} m_i$, $s(a) = \sum_{i\in I} \nu_i$. Here $\mathbb{L}$ denote the motif of $\mathbb{R}$. As a consequence, we obtain a strict linear degree bound for $\beta(\mathcal{A}_k(f))$ and recover the real log canonical threshold from the zeta function. We also show that the weights of convenient weighted homogeneous polynomials in three variables are recovered by the zeta function. Similar results for the zeta functions with signs are also discussed.

Inverse mapping theorem for bi-Lipschitz, blow-analytic, semi-algebraic homeomorphisms

Toshizumi Fukui
(joint work with Krzysztof Kurdyka, Adam Parusiński)

We say a map $f : \mathbb{R}^n, 0 \to \mathbb{R}^p$ is blow-analytic ([2]) if there is a composition $\sigma : M \to \mathbb{R}^n$ of locally finitely many blow-ups so that $f \circ \sigma$ is analytic.

We say a map $f : \mathbb{R}^n, 0 \to \mathbb{R}^p$ is arc-analytic ([3]) if $f \circ \alpha$ is analytic for any analytic map $\alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0$.

A blow-analytic map is clearly arc-analytic. It is known that ([1]) a semi-algebraic, arc analytic map is blow-analytic.

If a bi-Lipschitz subanalytic homeomorphism is arc-analytic, then the inverse is arc-analytic ([4]).

Let us consider a semi-algebraic homeomorphism $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$. We show the following conditions are equivalent.

- h is arc-analytic and h^{-1} is Lipschitz.
- h^{-1} is arc-analytic and h is Lipschitz.

The key step is to show that $\det(dh)$ is bounded away from infinity and zero. To show this, we need (at this moment at least) to compare virtual Poincaré polynomials (or motivic measures) of partitions of arc space $\mathcal{L}(\mathbb{R}^n, 0)$ with respect to certain Nash modification which sends everything normal crossing. In the talk, we describe the detailed proof of the following easier version: A (bi-)blow-analytic homeomotphism is bi-Lipschitz if it is Lipschitz and semi-algebraic.

References

[1] E. Bierstone and P. Milman, *Arc-analytic functions, Invent. math.*, **101** (1990), 411–424.
[2] T.-C. Kuo, *On classification of real singularities*, Invent. math., **82** (1985), 257–262.
[3] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann., **282**(1988), 445–462.
[4] T. Fukui, K. Kurdyka and L. Paunescu, *An inverse mapping theorem for arc-analytic homeomorphism*, in Geometric Singularity Theory (eds. Heisuke Hironaka, Stanislaw Janeczko, Stanislaw Lojasiewicz) Banach Center Publications **65** (2004), 49–56.

On the Euler characteristic of the real Milnor fibre(s)

Helmut A. Hamm

Let $f : (X, 0) \to (\mathbb{R}^k, 0)$ be a real analytic mapping between real analytic space germs with an isolated singularity, $\dim X = n$. Let L and K be the links of $(X, 0)$ resp. $(f^{-1}(0), 0)$. We look at cohomology with integral coefficients.

Similar to the complex case, $f : X \cap B_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\}) \to B_\alpha \setminus \{0\}$ is a locally trivial fibration (Milnor fibration), for $0 < \alpha \ll \epsilon \ll 1$.
The base space is connected if $k \geq 2$ but not if $k = 1$, so we treat these cases separately.

First we suppose $k \geq 2$. Then we can speak of the typical Milnor fibre F, and we have long exact sequences:
$\ldots \to H^m(L \setminus K) \to H^m(F) \to H^{m+2-k}(F) \to H^{m+1}(L \setminus K) \to \ldots$ (Wang sequence)
$\ldots \to H^m(K) \to H^m(F, \partial F) \to H^m(F) \to H^{m+1}(K) \to \ldots$
$\ldots \to H^m(L) \to H^m(F) \to H^{m-k+2}(F, \partial F) \to H^{m+1}(L) \to \ldots$
The last long exact sequence is the one for the pair (L, F): we can embed F in L.

Since one cannot expect good connectivity properties in the real case let us look at the Euler characteristic, we can conclude:
a) $\chi(L) = 0$ if n is even, $\chi(L) = 2\chi(F)$ if n is odd,
b) $\chi(K) = 0$ if $n - k$ is even, $\chi(K) = 2\chi(F)$ if $n - k$ is odd.
So $\chi(F)$ can be expressed by the Euler characteristic of a link except if k and n are both even.

In the case $k = 1$ we have two typical Milnor fibres: F_+, F_- . For the Euler characteristic we get the same result as before with $\chi(F_+) + \chi(F_-)$ instead of $2\chi(F)$; moreover $\chi(F_+) = \chi(F_-)$ if n is even.

It is difficult to conclude for individual cohomology groups but (with $k = 1$): Suppose that $n = 2m+1, m \geq 1$ and that F_+ and F_- have the homotopy type of a bouquet of m-spheres. Then $H^0(L) = \mathbb{Z}$, $H^l(L) = 0$ for $l \neq 0, m, 2m$, and $H^m(L)$ is free abelian. Furthermore $H^{2m}(L) \simeq \mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and L is non-orientable, otherwise $H^{2m}(L) \simeq \mathbb{Z}$.
Example: $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with isolated singularity, $X := \mathbb{C}^m \cap \{Im\, g = 0\}$, $f := Re\, g$.

Finally, using resolution of singularities we can calculate the Euler characteristic of the Milnor fibre(s). If $k \geq 2$ put $Y := X \cap \{f_1 = \ldots = f_{k-1} = 0\}$. Then the Milnor fibres of $f_k : (Y, 0) \to (\mathbb{R}, 0)$ coincide with the one of $f : (X, 0) \to (\mathbb{R}^k, 0)$, so we can reduce to the case $k = 1$ with $F_+ \simeq F_-$. So it is sufficient to look at the case $k = 1$.

In fact it is easier to note that the Milnor fibres of $f_k : (X, 0) \to (\mathbb{R}, 0)$ have the same Euler characteristic as the one of $f : (X, 0) \to (\mathbb{R}^k, 0)$, as remarked in the case $X = \mathbb{R}^n$ in [1], cf. conjecture by J.Milnor [3].

Now in the case $k = 1$ one can verify that the result of [2] about the computation of the Euler characteristic via resolution of singularities for $X = \mathbb{R}^n$ can be extended to the general case in a straightforward manner.

References

[1] R. Araújo dos Santos, D. Dreibelbis, N. Dutertre: Topology of the real Milnor fiber for isolated singularities. In: Contemp. Math. **569**, 2012, pp. 67-75.
[2] G. Comte, G. Fichou: Grothendieck ring of semialgebraic formulas and motivic real Milnor fibres. arXiv: 111.3181v3.
[3] J. Milnor: Singular points of complex hypersurfaces. Princeton Univ. Press: Princeton 1968.

Characteristic numbers of singular complex algebraic varieties

Laurentiu G. Maxim

The purpose of this lecture is to give a quick introduction to the theory of genera and characteristic numbers of *singular* complex algebraic varieties, e.g., see [2] and the references therein.

The first part deals with reviewing the basic definitions and classical examples of genera and characteristic numbers of compact manifolds (e.g., as explained in Hirzebruch's seminal book [1]). Such invariants of manifolds are intimately connected to the theory of cohomology characteristic classes of vector bundles.

The second part of the lecture is concerned with the singular setting. The aim is to provide possible answers to the following question of Goresky and MacPherson: *"Which characteristic numbers can be defined for compact complex algebraic varieties with singularities?"*

The discussion of invariants of singular complex algebraic can be divided into two distinct parts. First, we discuss invariants of log-terminal varieties. These are mildly singular varieties appearing in the minimal model program. Invariants of such varieties are generally defined in terms of resolutions of singularities. However, as resolutions are not unique, one also needs to show that the definition of invariants is independent of the choice of resolution. This is achieved by using either the weak factorization theorem or the theory of motivic integration with its change of variables formula.

Characteristic numbers of a singular complex algebraic variety (with any kind of singularities) can also be defined by making use of the intrinsic information encoded by the variety itself, e.g., by using the Deligne's (or Saito's) mixed Hodge structures on the (intersection) cohomology groups of the variety. Finally, certain characteristic numbers of compact varieties (e.g., Euler characteristic, Goresky-MacPherson signature, Hodge polynomial, etc.) can be interpreted by means of functorial characteristic class theories of singular varieties.

The end of the lecture is devoted to various computational aspects. For instance, we explain how to compute characteristic numbers of global orbifolds, and of spaces built out of a given variety (e.g., symmetric products, configuration spaces, or Hilbert schemes of points on an algebraic manifold).

References

[1] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, 1966.

[2] L. Maxim, *Characteristic numbers, genera, and resolutions*, in "Proceedings of the Seventh Congress of Romanian Mathematicians" (to appear).

Geometry of mixed functions of strongly polar weighted homogeneous face type

Mutsuo Oka

We consider mixed functions $f(\mathbf{z},\bar{\mathbf{z}})$. If $f(\mathbf{z},\bar{\mathbf{z}})$ has strongly non-degenerate Newton boundary, it has a Milnor fibration ([6]). Recall that $f(\mathbf{z},\bar{\mathbf{z}})$ is called a *polar weighted homogeneous polynomial* if there exist a weight vector $P = (p_1,\dots,p_n)$ and a non-zero integers d_p such that

$$f(\rho\circ\mathbf{z},\bar\rho\circ\bar{\mathbf{z}}) = \rho^{d_p} f(\mathbf{z},\bar{\mathbf{z}}),\quad \rho\circ\mathbf{z} = (\rho^{p_1}z_1,\dots,\rho^{p_n}z_n),\ \rho\in\mathbb{C},\ |\rho|=1.$$

Similarly $f(\mathbf{z},\bar{\mathbf{z}})$ is called a *radially weighted homogeneous polynomial* if there exist a weight vector $Q=(q_1,\dots,q_n)$ and a positive integer d_r such that

$$f(t\circ\mathbf{z},t\circ\bar{\mathbf{z}}) = t^{d_r} f(\mathbf{z},\bar{\mathbf{z}}),\quad t\circ\mathbf{z} = (t^{q_1}z_1,\dots,t^{q_n}z_n),\ t\in\mathbb{R}^+$$

A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called to be strongly polar weighted homogeneous if f is polar and radially weighted homogeneous and $p_j = q_j$ for $j=1,\dots,n$. A mixed analytic function is called of strongly polar weighted homogenous face type if the face function f_Δ is a strongly polar weighted homogenous polynomials for any face Δ (of any dimension) of the Newton boundary $\Gamma(f)$.

A typical example of such mixed functions is given as the pull-back $g(\mathbf{w}, \bar{\mathbf{w}}) = \varphi_{a,b}^* f(\mathbf{w}, \bar{\mathbf{w}})$ of a convenient non-degenerate holomorphic function $f(\mathbf{z})$ by a homogeneous mixed covering $\varphi_{a,b} : \mathbb{C}^n \to \mathbb{C}^n$ which is defined by $\varphi_{a,b}(\mathbf{w}, \bar{\mathbf{w}}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$. We call $g = \varphi_{a,b}^* f$ *the mixed homogeneous covering lift of* f. We use the notations: $\rho(\mathbf{z}) = |z_1|^2 + \cdots + |z_n|^2$,

$$B_\varepsilon = \rho^{-1}([0, \varepsilon^2]),\ S_\varepsilon = \rho^{-1}(\varepsilon^2),\ D_\delta^* := \{\eta \in \mathbb{C} \,|\, 0 < |\eta| \le \delta\}.$$

0.1. **A Varchenko type formula for the Zeta function.** Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate convenient mixed function of strongly polar weighted homogeneous face type. We consider the Milnor fibration $f : B_\varepsilon^{2n} \cap f^{-1}(D_\delta^*) \to D_\delta^*$ and its zeta function $\zeta(t)$. For $I \subset \{1, \dots, n\}$, let f^I be the restriction of f on the coordinate subspace $\mathbb{C}^I$ where

$$\mathbb{C}^I = \{\mathbf{z} \,|\, z_j = 0,\ j \notin I\},\ \mathbb{C}^{*I} = \{\mathbf{z} \,|\, z_j = 0 \iff j \notin I\}.$$

Let $\mathcal{S}_I$ be the set of primitive weight vectors $P = {}^t(p_i)_{i \in I}$ of the variables $\{z_i \,|\, i \in I\}$ such that $p_i > 0$ for all $i \in I$ and $\dim \Delta(P, f^I) = |I| - 1$. $P \in \mathcal{S}_I$ can be considered to be a weight vector of $\mathbf{z}$ putting $p_j = 0$, $j \notin I$. Then we can show the exact same formula of Varchenko type ([7, 3] for our mixed polynomial.

Theorem 1. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ a convenient non-degenerate mixed polynomial of strongly polar positive weighted homogeneous face type. Let $V = f^{-1}(V)$ be a germ of hypersurface at the origin and let $\widetilde{V}$ be the strict stransform of V to X. Then*
(1) $\tilde{V}$ is topologically smooth on $\pi^{-1}(\mathbf{0})$ and real analytic smooth variety outside of $\pi^{-1}(\mathbf{0})$.
(2) The zeta function of the Milnor fibration of $f(\mathbf{z}, \bar{\mathbf{z}})$ is given by the formula

$$\zeta(t) = \prod_I \zeta_I(t),\ \zeta_I(t) = \prod_{P \in \mathcal{S}_I} (1 - t^{\mathrm{pdeg}(P, f_P^I)})^{-\chi(P)/\mathrm{pdeg}(P, f_P^I)}$$

where $\mathrm{pdeg}_P f$ *is the polar degree of f_P^I and $\chi(P)$ is given by*

$$\chi(P) = \chi(F^*(P)),\ F_P^* := (f_P^I)^{-1}(1) \subset \mathbb{C}^{*I}.$$

0.2. **Contact structure.** Consider the contact structure on S_ε defined by the 1 form $\alpha = -d\rho \circ J$ where J is the complex structure. Let $\omega = d\alpha$. We consider the problem if the restriction of α gives an contact structure on the link of $g^{-1}(0)$. The following is a mix-link version of the result for the holomorphic link in [2, 1]

Theorem 2. *(1) Assume that $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function and let $g = \varphi_{a,b}^* f$ be the mixed homogeneous covering lift. Assume that $a > b > 0$ and consider the link $K_r := g^{-1}(0) \cap S_r$. Then there exists a positive number r_0 so that $K_r = g^{-1}(0) \cap S_r$ is a contact submanifold for any r, $0 < r \le r_0$.*

(2) (Symmplectic structure) Let $\alpha_c = \exp(-c|g|^2)\alpha$ and let $\omega_c = d\alpha_c$. Then there exists a sufficiently small r_0 and a sufficiently large $c > 0$ so that ω_c gives a

symmplectic structure on the fibers of the Milnor fibration

$$g/|g| : S_r \setminus K_\varepsilon \to S^1$$

where $0 < r \le r_0$ *and* $K_\varepsilon = g^{-1}(0) \cap S_r$.

References

[1] C. Caubel, A. Némethi, and P. Popescu-Pampu. Milnor open books and Milnor fillable contact 3-manifolds. *Topology*, 45(3):673–689, 2006.

[2] E. Giroux. Contact structures and symplectic fibrations over the circle, Notes of the summer school "*Holomorphic curves and contact topology*", Berder, June 2003

[3] M. Oka. *Non-degenerate complete intersection singularity.* Hermann, Paris, 1997. emMilnor J. Milnor. *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.

[4] M. Oka. Mixed functions of strongly polar weighted homogeneous face type, arxiv 1202.2166v1.

[5] M. Oka. Topology of polar weighted homogeneous hypersurfaces. *Kodai Math. J.*, 31(2):163–182, 2008.

[6] M. Oka. Non-degenerate mixed functions. *Kodai Math. J.*, 33(1):1–62, 2010.

[7] A. N. Varchenko. Zeta-function of monodromy and Newton's diagram. *Invent. Math.*, 37:253–262, 1976.

The weight filtration for real algebraic varieties

Adam Parusiński

(joint work with Clint McCrory)

Using the work of Guillén and Navarro Aznar we associate to each real algebraic variety a filtered chain complex, the weight complex, which is well-defined up to filtered quasi-isomorphism, and which induces on Borel-Moore (i.e. with closed supports) homology with $\mathbf{Z_2}$ coefficients an analog of the weight filtration for complex algebraic varieties.

$$0 = \mathcal{W}_{-k-1} H_k^{cl}(X) \subset \mathcal{W}_{-k} H_k^{cl}(X) \subset \cdots \subset \mathcal{W}_0 H_k^{cl}(X) = H_k^{cl}(X; \mathbf{Z_2}).$$

We work with homology rather than cohomology to take advantage of the topology of semialgebraic chains. We show that the weight complex can be also represented by a geometrically defined filtration on the complex of semialgebraic chains.

$$0 = \mathcal{G}_{-k-1} C_k^{cl}(X) \subset \mathcal{G}_{-k} C_k^{cl}(X) \subset \cdots \subset \mathcal{G}_0 C_k^{cl}(X) = C_k^{cl}(X; \mathbf{Z})$$

as follows. Given a semialgebraic chain of $C_k^{cl}(X)$ represented by a closed semialgebraic subset c of X. If X is not compact we embed it first in an algebraic compactification $X \subset \overline{X}$, and then consider the Zariski closure of c in $\overline{X}$. Denote this closure par Y. Suppose for simplicity that $Y = \overline{c}^Z$ is smooth and that c is locally a union of closed quadrants (if not we use the resolution of singularities). The group $G = \mathbf{Z_2^k}$ acts on these quadrants by reflections in coordinate hyperplanes. Then, $c \in \mathcal{G}_{-p} C_k^{cl}(X)$ if, locally at every point, c is the sum of orbits of subgroups of G of order 2^p. The elements of $G_{-k} C_k^{cl}(X)$ (pure weight cycles) are exactly those whose closure in $\overline{X}$ are arc-symmetric sets in the sense of Kurdyka.

The geometric filtration gives rise to a spectral sequence, the weight spectral sequence, whose initial term yields additive invariants for real algebraic varieties, the virtual Betti numbers. Unlike in the complex case with rational coefficients, this spectral sequence does not collapse at E^1 and hence the virtual Betti numbers cannot be computed from the weight filtration on homology.

The geometric filtration is functorial for proper regular morphisms, and for Nash maps (i. e. analytic with semi-algebraic graphs) defined on compact varieties. Thus as a corollary we obtain the invariance of virtual Betti numbers by regular homeomorphisms.

References

[1] F. Guillén, V. Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, IHES Publ. Math. **95** (2002), 1–83.

[2] C. McCrory, A. Parusiński, *Virtual Betti numbers of real algebraic varieties*, Comptes Rendus Acad. Sci. Paris, Ser. I, **336** (2003), 763–768. (See http://arxiv.org/pdf/math.AG/0210374 for an extended version.)

[3] C. McCrory, A. Parusiński,*The weight filtration for real algebraic varieties*, Topology of Stratified Spaces, MSRI Pub. **58**, Cambridge University Press, New York, 2011, 121–160

The weight filtration for real algebraic varieties

Adam Parusiński

(joint work with Clint McCrory)

To each real algebraic variety we associate a filtered chain complex, *the weight complex*, which is well-defined up to filtered quasi-isomorphism, and which induces on homology with $\mathbf{Z_2}$ coefficients an analog of the weight filtration for complex algebraic varieties. This complements our previous definition of the weight filtration of Borel-Moore homology. If X has dimension n then this filtration satisfies

$$0 = \mathcal{W}_{-n-1}H_k(X) \subset \mathcal{W}_{-n}H_k(X) \subset \cdots \subset \mathcal{W}_0 H_k(X) = H_k(X, \mathbf{Z_2}),$$

If X is compact then this filtration coincides with the weight filtration on $H_k^{cl}(X; \mathbf{Z_2})$ and thus $W_{-k-1}H_k(X) = 0$. If X is smooth then $W_{-k}H_k(X) = H_k(X)$.

We define the weight filtration of a smooth, possibly noncompact, variety X, in terms of a good compactification $\overline{X}$ with divisor D at infinity. We suppose $\overline{X}$ smooth with a divisor with simple normal crossings D and $X = \overline{X} \setminus D$. Let $D = \bigcup_{i \in I} D_i$, where each component D_i is the zero set of a regular section of a line bundle L_i. Let $\pi_{L_i} : S(L_i) \to \overline{X}$ denote the sphere bundle of L_i. It is an algebraic double covering. The pull-back of L_i on $S(L_i)$ is trivial and $\pi_{L_i}^{-1}(D_i)$ is the zero set of the regular function $\varphi_i : S(L_i) \to \mathbf{R}$. Let $\tilde{\pi} : \tilde{X} \to \overline{X}$ be the fiber product of the π_{L_i}, and let $\tilde{\varphi}_i : \tilde{X} \to \mathbf{R}$ denote the pullback of φ_i. The *corner compactification* of X associated to the good compactification $(\overline{X}, D)$ is the semialgebraic set $X' \subset \tilde{X}$ defined by

$$X' = \text{Closure}\{\tilde{x} \in \tilde{X} \mid \tilde{\varphi}_i(\tilde{x}) > 0,\ i \in I\}.$$

Intuitively it means that X' is obtained from $\overline{X}$ by cutting along the divisor D, or in local picture, by separating the closures of the connected components of the complement of D in $\overline{X}$.

The constructed above $\tilde{\pi} : \tilde{X} \to \overline{X}$ is a principal bundle with group a discrete torus $\{1, -1\}^{|I|}$. We define the *corner filtration* of the semialgebraic chain group of X', by taking the chains in $C_*(X')$ invariant by subgroups of $\{1, -1\}^{|I|}$ of order 2^p as generators of $-p$th term of this filtration. Finally, the filtered *weight complex* is obtained from the corner filtration by an algebraic construction, the Deligne shift.

Then we extend this fitration to arbitrary algebraic varieties using Guillén and Navarro Aznar's extension theorem.

References

[1] F. Guillén, V. Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, IHES Publ. Math. **95** (2002), 1–83.

[2] C. McCrory, A. Parusiński, *The weight filtration for real algebraic varieties*, Topology of Stratified Spaces, MSRI Pub. **58**, Cambridge University Press, New York, 2011, 121–160

[3] C. McCrory, A. Parusiński, *The weight filtration for real algebraic varieties II, classical homology*, to appear in Revista Acad. Serie A. Matematicas, volume dedicated to H. Hironaka on the occasion of his 80th birthday.

Splitting of Nash manifolds with involutions, and additive invariants for real algebraic varieties with involutions

Fabien Priziac

We prove that a connected compact affine Nash manifold with an algebraic involution can be split along an arc-symmetric subset, into two parts that would be the images of one another by the involution. We then apply this result to obtain an analog of the Smith exact sequence for involutions, taking in account the geometric filtration of C. McCrory and A. Parusiński. This filtration is functorial on the category of real algebraic varieties with proper regular morphisms, and realize the weight filtration for real algebraic varieties. Precisely, we show that when we cut an invariant chain, modulo its restriction to the fixed point set, into two chains, image of one another, we can control their degree in the filtration, so that it is not far from the degree of the original chain. We give also an interpretation of this "Smith geometric exact sequence" in the case of a free action on a compact variety, for which we extract a correspondance between the invariant chains and the ch ains of the arc-symmetric quotient, respecting the extended filtration on the category of arc-symmetric sets. Finally, we use the Smith geometric exact sequence to compute and try to understand some spectral sequences, constructed from, and which reflect, the very behaviour of actions on the geometry of varieties given by the geometric filtration. The Euler characteristic of the last term of these spectral sequences gives some additive invariants on the category of real algebraic varieties with algebraic involutions. In some cases, these "equivariant" additive invariants coincide with G. Fichou's equivariant virtual Betti numbers. In particular, they are equal in negative degree for all real algebraic varieties with involutions, and equal in any degree for compact varieties equipped with free actions.

References

[1] C. McCrory, A. Parusiński, *The weight filtration for real algebraic varieties*, Topology of stratified spaces, 121-160, Math. Sci. Res. Inst. Publ. **58**, Cambridge Univ. Press, Cambridge, 2011.

[2] G. Fichou, *Equivariant virtual Betti numbers*, Ann. de l'Inst. Fourier, **58**, no. 1 (2008), 1-27.

Singularities at infinity and motivic integration

Michel Raibaut

Let f be a regular function defined on a smooth complex algebraic variety U. Following ideas of Thom, there is a minimal finite set B_f, called *bifurcation set*, such that

$$f : U \setminus f^{-1}(B_f) \to \mathbb{C} \setminus B_f$$

is a topological locally trivial fibration. The bifurcation set is the union of the discriminant of f with the set of *atypical values* which are smooth and induced by the *"singularities at infinity"* of f. Good references for this topic are [2] and [6].

As a corollary, for each value a, there is $\eta > 0$ such that $f_{|D(a,\eta)^*}$ is a topological locally trivial fibration called *Milnor fibration of f at a*, its fiber is called *Milnor fiber of f for the value a* denoted by F_a and the action of $\pi_1(D(a,\eta)^*)$ on F_a is called *monodromy of f for the value a*. This monodromy action induces on each cohomology group $H_c^k(F_a, \mathbb{Q})$ a quasi-unipotent operator T_a.

Following constructions of Denef-Loeser [1] and more recently of Guibert-Loeser-Merle [3] we will explain below the construction of the motivic analog of (F_a, T_a) called *motivic Milnor fiber of f for the value a*, and denoted by $S_{f,a}$. Also, we will define motivic analogs of the atypical values and the bifurcation set. All the constructions and results are in [5].

The motivic Milnor fiber $S_{f,a}$ will be a *motive*, namely in our context, an element of a Grothendieck ring of varieties endowed with an action of the group of roots of unity $\hat{\mu}$. This ring is denoted by $\mathcal{M}^{\hat{\mu}}$. Roughly speaking, we consider first the ring $K_0(Var_{\mathbb{C}})$ generated by isomorphism classes $[X, \sigma]$ of such varieties and we quotient it by additivity and multiplicativity relations : for any varieties (X, σ_X), (Y, σ_Y), where σ_X and σ_Y are actions of $\hat{\mu}$:

(1) $[X, \sigma_X] = [F, \sigma_X] + [X \setminus F, \sigma_X]$, for F a closed subset of X stable under σ_X.

(2) $[X \times Y, \sigma_X \times \sigma_Y] = [X, \sigma_X].[Y, \sigma_Y]$, where $\sigma_X \times \sigma_Y$ is the diagonal action.

Then, the ring $\mathcal{M}^{\hat{\mu}}$ is the localization $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ where $\mathbb{L}$ is the class of the affine line $\mathbb{A}^1_{\mathbb{C}}$ endowed with the trivial action.

Let $(X, i, \hat{f})$ be a compactification of f, meaning $i : U \to X$ is an open dominant immersion and $\hat{f} : X \to \mathbb{P}^1$ is a proper map on X not necessary smooth, such that the following diagram is commutative

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ \mathbb{A}^1_{\mathbb{C}} & \xrightarrow{j} & \mathbb{P}^1_{\mathbb{C}} \end{array},$$

where $j(a) = [1 : a]$, for all value a. We denote by F the closed subset at infinity $X \setminus U$.

For $n \geq 1$ and $\gamma \geq 1$, we consider the arc space

$$X^{\gamma}_{n,a}(\hat{f}) = \{\varphi \in X(\mathbb{C}[[t]]) \mid \hat{f}(\varphi(t)) = a + 1.t^n + \ldots,\ \mathrm{ord}_t\, \varphi^*\mathcal{I}_F \leq n\gamma\}.$$

The condition $\mathrm{ord}_t\, \varphi^*\mathcal{I}_F \leq n\gamma$ is a contact order condition of the arc φ on the closed subset F. This scheme is endowed with an action of the group of n-roots of unity μ_n by $\lambda.\varphi(t) := \varphi(\lambda t)$.

We consider the *motivic zeta function*

$$Z^{\gamma}_{\hat{f}-a}(T) = \sum_{n\geq 1} \mathrm{mes}(X^{\gamma}_{n,a}(\hat{f}))T^n \in \mathcal{M}^{\mu}[[T]].$$

Let π_m be the troncation of arcs modulo t^{m+1}. The sequence $([\pi_m(X^{\gamma}_{n,a}(\hat{f}))]\mathbb{L}^{-m\dim U})$ is stationary by the contact condition, and its limit is the *motivic measure* $\mathrm{mes}(X^{\gamma}_{n,a}(\hat{f}))$. By Denef-Loeser results, this formal series is rational and admits a limit when T goes to infinity. This limit does not depend on $\gamma >> 1$.

Theorem 1. *We called* motivic Milnor fiber of f for the value a *the motive*

$$S_{f,a} = -\lim_{T\to\infty} Z^{\gamma}_{\hat{f}-a}(T) \in \mathcal{M}^{\hat{\mu}}.$$

This motive does not depend on the compactification $(X, i, \hat{f})$.

This motive was also differently introduced by Takeuchi and Matsui in [4]. It is the motivic analog of the Milnor fiber of f for the value a because it realizes on the class of the limit mixed Hodge structure of f for the value a and on the class of the nearby cycle sheaf of f for the value a, in appropriate Grothendieck rings ([1]§4.2.1, [3]§3.16, [4]§4.3, [5]§2.4).

In order to study the behavior at infinity of f, we consider for all $\gamma \geq 1$, and all $n \geq 1$

$$X^{\gamma,\infty}_{n,a}(\hat{f}) = \{\varphi \in X^{\gamma}_{n,a}(\hat{f}) \mid \varphi(0) \in F\}.$$

The associated motivic zeta function $Z^{\gamma,\infty}_{\hat{f}-a}(T)$ is also rational and has a limit

$$S^{\infty}_{f,a} := -\lim_{T\to\infty} Z^{\gamma,\infty}_{\hat{f}-a}(T) \in \mathcal{M}^{\hat{\mu}}$$

called *motivic vanishing cycles of f at infinity for the value a* which does not depend on $\gamma >> 1$ and on the compactification $(X, i, \hat{f})$. By definition a value a is *motivically atypical* if and only if $S^{\infty}_{f,a} \neq 0$.

Theorem 2. *The set of motivic atypical values is finite.*

The *motivic bifurcation set* of f is the union of the discriminant of f and its motivic atypical values. A regular function is called *motivically tame* if it has no motivic atypical value. For instance, a convenient and non-degenerate polynomial map for its Newton polyhedron at infinity is motivically tame.

References

[1] Jan Denef and François Loeser. Motivic Igusa zeta functions. *J. Algebraic Geom.*, 7(3):505–537, 1998.

[2] Alexandru Dimca. Monodromy and Hodge theory of regular functions. In *New developments in singularity theory (Cambridge, 2000)*, volume 21 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 257–278. Kluwer Acad. Publ., Dordrecht, 2001.

[3] Gil Guibert, François Loeser, and Michel Merle. Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink. *Duke Math. J.*, 132(3):409–457, 2006.

[4] Yutaka Matsui and Kiyoshi Takeuchi. Monodromy at infinity of polynomial map and mixed hodge modules. *arXiv:0912.5144v5.*

[5] Michel Raibaut. Singularités à l'infini et intégration motivique. *Bull. Soc. Math. France*, 140(1):51–100, 2012.

[6] Mihai Tibăr. *Polynomials and vanishing cycles*, volume 170 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2007.

On the relation between Chern- and Siefel-Whitney classes of singular spaces

Jörg Schürmann

First we explain the similarities (as well as some important differences) in the *Lagrangian* approach to characteristic classes of singular spaces. Here one is working in an embedded context with $X \subset M$ a closed subset in an ambient complex or real manifold M, which is analytic or algebraic of dimension m. In addition we fix a smooth Whitney stratification $\{S\}$ of $X = \uplus\, S$, with all closures $\bar{S}$ of the connected strata S complex analytic or algebraic resp. sub-analytic or semi-algebraic. Let $(T^*_S M)^{\circ} := T^*_S M \setminus \cup_{S' \neq S} \overline{T^*_{S'} M}$ be the set of *non-degenerate covectors* to the stratum S, with $\Lambda_X := \uplus_S\, T^*_S M \subset T^*M|_X$ the conic subset of all covectors to the strata, which is closed by the Whitney a-condition.

As a suitable homology theory $H_*(-)$ one can use either Chow groups $CH_*(-)$ or even degree Borel-Moore homology $H^{BM}_{2*}(-;\mathbb{Z})$ in the complex or $H^{BM}_{*}(-;\mathbb{Z}_2)$ in the real context. The corresponding group of *conic Lagrangian cycles* is by definition

$$L(X, M) := H_m(\Lambda_X) \subset \prod_S H_m\left((T^*_S M)^{\circ}\right) \ . \tag{1}$$

These are $\mathbb{C}^*$- resp. $\mathbb{R}^+$-conic in the complex resp. real context. For characteristic classes of these conic cycles, we need to consider in the real context the subgroup

$L^{sd}(X,M) \subset L(X,M)$ of all 'self-dual' $\mathbb{R}^*$-conic cycles which are invariant under push-down a_* for the antipodal map a of $T^*M|X$. Let $\mathcal{O}(-1) \to P(T^*M|_X \oplus 1) \overset{\pi}{\to} X$ be the tautological line bundle. Then the *Segre class* of $[C] \in L^{sd}(X,M)$ is defined by (with $[\hat{C}]$ the projective completion):

$$s_*([C]) := \sum_{j\geq 0} \pi_* \left(cl^1(\mathcal{O}(1))^j \cap [\hat{C}] \right) \in H_*(X)\,. \tag{2}$$

Here $cl^1(-)$ resp. $cl^*(-)$ is the first resp. total Chern or Stiefel-Whitney class. The corresponding characteristic class of a conic Lagrangian cycle is defined by

$$cl_* := cl^*(T^*M|X) \cap s_* :\; L^{sd}(X,M) \to H_*(X)\,. \tag{3}$$

Using 'stratified Morse theory for constructible functions' as a black box (see e.g. [2, Sect. 5.0.3] and [3]), on can associate to such a function $\alpha : X \to \mathbb{Z}$ (resp. $\mathbb{Z}_2$ in the real context), with α constant on all strata S, and a non-degenerate covector $\omega = df_x \in (T^*_S M)^\circ$ a *normal Morse index*

$$i(\omega,\alpha) := \alpha(x) - \chi(M^{(-)}_{f|N\cap X,x};\alpha)\,, \tag{4}$$

where $N \subset M$ is a transversal slice to S with $N \cap S = \{x\}$ and $f : (N \cap X, x) \to (\mathbb{K}, 0)$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ is a corresponding analytic function germ with *local (left) Milnor fiber* $M^{(-)}_{f|N\cap X,x}$. Then $i(\omega,\alpha)$ doesn't depend on the choice of f and is constant on any connected component $U \subset (T^*_S M)^\circ$. Then one can define the *characteristic cycle* map on the group $F(X,\{S\})$ of such constructible functions by (summing over all such connected components U for all strata S):

$$CC : F(X,\{S\}) \to L(X,M);\; \alpha \mapsto CC(\alpha) := \sum_{U,\omega\in U} (-1)^{dim(S)} \cdot i(\omega,\alpha)\cdot [U]\,. \tag{5}$$

That the right hand side defines also a *cycle* in the real context is not obvious and a deep theorem due to Kashiwara and also Fu. Since $i(\omega,\alpha) = \alpha(x)$ for $\omega = df_x \in (T^*_S M)^\circ$ for S an open stratum, one easily gets by induction that $CC : F(X,\{S\}) \to L(X,M)$ is an isomorphism. Finally, it is a beautiful observation of Fu and McCrory [1], that the 'self-duality' for $CC(\alpha)$ corresponds to Sullivan's classical *local Euler condition* for α. Then the composed group homomorphism

$$cl_* := cl_* \circ CC :\; F_{Eu}(X,\{S\}) \to H_*(X) \tag{6}$$

is nothing but the *(dual) MacPherson Chern class* c_* resp. *Sullivan Stiefel-Whitney class* transformation w_*, which has nice *functorial* properties for push-down under proper morphisms as well as specialization in one parameter families.

Let us finally assume that M, X and all closures $\bar{S}$ of strata are algebraic varieties *defined over* $\mathbb{R}$, so that the set of real points $X(\mathbb{R}) \subset M(\mathbb{R})$ gets an induced Whitney stratification with strata $S(\mathbb{R})$. Then one has by [2, eq. (5.86) on p.367]:

$$cl_R(CC(1_X)) = CC(1_{X(R)})\,, \text{with} \quad cl_R : CH_*(-/\mathbb{R}) \to H^{BM}_*(-(\mathbb{R}),\mathbb{Z}_2) \tag{7}$$

the *real fundamental class* map of Borel-Haefliger viewed as a group homomorphism on the Chow group of algebraic cycles defined over $\mathbb{R}$. This implies the following relation between the Chern class of X and the Stiefel-Whitney class of $X(\mathbb{R})$:

$$cl_R(c_*(1_X)) = w_*(1_{X(R)})\,. \tag{8}$$

As an application one gets for a toric variety X (viewed in the standard way as an algebraic variety over $\mathbb{R}$) the following nice formula for the Stiefel-Whitney class of $X(\mathbb{R})$ from the corresponding result for the Chern class $c_*(1_X) \in CH_*(X/\mathbb{R})$ of X due to Aluffi:

$$w_*(1_{X(R)}) = \sum_{O(R)} \left[\overline{O(\mathbb{R})}\right] \in H_*^{BM}(X(\mathbb{R}), \mathbb{Z}_2) \tag{9}$$

where the sum is over all real orbits $O(\mathbb{R}) \simeq (\mathbb{R}^*)^k$ (for some k).

References

[1] J. Fu, C. McCrory, *Stiefel-Whitney classes and the conormal cycle of a singular variety*, Trans. of the Amer. Math. Soc. 349 (1997), 809–835.

[2] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), 63. Birkhauser Verlag, Basel, 2003.

[3] J. Schürmann, M. Tibar, *Index formula for MacPherson cycles of affine algebraic varieties*, Tohoku Math. J. 62 (2010), 29–44.

Monodromies at infinity of tame and non-tame polynomials

Kiyoshi Takeuchi

Let $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map. Then there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1}$$

of f is a locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this property. Let $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ $(R >> 0)$ be a sufficiently large circle in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. Then by restricting the above locally trivial fibration to C_R we obtain a geometric monodromy automorphism $\Phi_f^\infty\colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R)$ and the linear maps

$$\Phi_j^\infty\colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \quad (j = 0, 1, \ldots) \tag{2}$$

associated to it. We call Φ_j^∞'s the (cohomological) monodromies at infinity of f. Many mathematicians studied Φ_j^∞'s from various points of view. In particular in [1] Broughton proved that if f is tame at infinity we have the concentration

$$H^j(f^{-1}(R);\mathbb{C}) = 0 \qquad (j \neq 0, n-1) \tag{3}$$

for the generic fiber $f^{-1}(R)$ $(R >> 0)$ of f. Therefore in the tame case, Φ_{n-1}^∞ is the only non-trivial monodromy at infinity to study and Libgober-Sperber [5] obtained a beautiful formula which expresses the semisimple part (i.e. the eigenvalues) of Φ_{n-1}^∞ in terms of the Newton polyhedron at infinity of f (see [6] for its generalizations). Moreover in [7] (see also [3]) we proved the formula for its nilpotent part (i.e. its Jordan normal form) by introducing the motivic Milnor fiber at infinity of f as in Denef-Loeser [2] and Guibert-Loeser-Merle [4] etc. Note that the same notion was introduced also by Raibaut [9] independently. However if we do not assume the tameness at infinity of f, we can not expect to have the above cohomological concentration in general. This is a very serious obstruction in applying our methods in [7] and [3] to non-tame polynomials.

Recently in [11] we partially overcame this problem by showing that the desired cohomological concentration holds for the generalized eigenspaces of Φ_j^∞ for "good" eigenvalues. Namely if we avoid some "bad" eigenvalues associated to f, we can generalize the results in [7] to non-tame polynomials and completely determine the Jordan normal forms of Φ_{n-1}^∞. For a polynomial $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ we denote by $NP(f)$ the Newton polytope of f. The convex hull $\Gamma_\infty(f) \subset \mathbb{R}_+^n$ of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ is called the Newton polyhedron at infinity of f. Assume that $\dim \Gamma_\infty(f) = n$ and f is non-degenerate at infinity (for the definition see [5] and [11] etc.). If we assume moreover that f is convenient, then by a result of [1] it is tame at infinity. So, in order to treat the non-tame case, we assume here that f is "not" convenient.

Definition 0.6. *We say that a face $\Gamma \prec \Gamma_\infty(f)$ is atypical if $0 \in \Gamma$ and the cone which corresponds to it in the dual fan of $\Gamma_\infty(f)$ is not contained in the first quadrant $\mathbb{R}_+^n$.*

Note that this definition of atypical faces is closely related to that of bad faces in [8] and [12].

Definition 0.7. *We say that a complex number $\lambda \in \mathbb{C}$ is an atypical eigenvalue of f if there exists a face $\gamma \prec \Gamma_\infty(f)$ contained in an atypical one $\Gamma \prec \Gamma_\infty(f)$ such that $0 \notin \gamma$ and $\lambda^{d_\gamma} = 1$. Here $d_\gamma \in \mathbb{Z}_{>0}$ is the lattice distance of γ from the origin $0 \in \mathbb{R}^n$ (for the definition see [7] and [11] etc.). We denote by $A_f \subset \mathbb{C}$ the set of the atypical eigenvalues of f.*

Then we have the following main theorem in [11]. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $H^j(f^{-1}(R); \mathbb{C})_\lambda \subset H^j(f^{-1}(R); \mathbb{C})$ be the generalized eigenspace for the eigenvalue λ of Φ_j^∞ $(R >> 0)$.

Theorem 0.8. *In the situation as above, for any $\lambda \notin A_f$ we have the concentration*

$$H^j(f^{-1}(R); \mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{4}$$

for the generic fiber $f^{-1}(R) \subset \mathbb{C}^n$ $(R >> 0)$ of f.

This theorem was proved by refining the proof of Sabbah's theorem [10, Theorem 13.1]. With Theorem 0.8 at hand, by using the results in [7, Section 2] we can easily obtain the analogues of [7, Theorems 5.9, 5.14 and 5.16] for the non-tame polynomial f and completely determine the λ-part of the Jordan normal form of Φ_{n-1}^∞ for any $\lambda \notin A_f$. For the details see [11].

References

[1] Broughton, S. A. "Milnor numbers and the topology of polynomial hypersurfaces", *Invent. Math.*, 92 (1988): 217-241.

[2] Denef, J. and Loeser, F. "Geometry on arc spaces of algebraic varieties", *Progr. Math.*, 201 (2001): 327-348.

[3] Esterov, A. and Takeuchi, K. "Motivic Milnor fibers over complete intersection varieties and their virtual Betti numbers", *Int. Math. Res. Not.*, Vol. 2012, No. 15 (2012): 3567-3613.

[4] Guibert, G., Loeser, F. and Merle, M. "Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink", *Duke Math. J.*, 132 (2006): 409-457.

[5] Libgober, A. and Sperber, S. "On the zeta function of monodromy of a polynomial map", *Compositio Math.*, 95 (1995): 287-307.

[6] Matsui, Y. and Takeuchi, K. "Monodromy zeta functions at infinity, Newton polyhedra and constructible sheaves", *Mathematische Zeitschrift*, 268 (2011): 409-439.

[7] Matsui, Y. and Takeuchi, K. "Monodromy at infinity of polynomial maps and Newton polyhedra, with Appendix by C. Sabbah", to appear in *Int. Math. Res. Not.*

[8] Némethi, A. and Zaharia, A. "On the bifurcation set of a polynomial function and Newton boundary", *Publ. Res. Inst. Math. Sci.*, 26 (1990): 681-689.

[9] Raibaut, M. "Fibre de Milnor motivique à l'infini", *C. R. Acad. Sci. Paris Sér. I Math.*, 348 (2010): 419-422.

[10] Sabbah, C. "Hypergeometric periods for a tame polynomial", *Port. Math.*, 63 (2006): 173-226.

[11] Takeuchi, K. and Tibăr, M. "Monodromies at infinity of non-tame polynomials", arXiv:1208.4584 v1, preprint.

[12] Zaharia, A. "On the bifurcation set of a polynomial function and Newton boundary II", *Kodai Math. J.*, 19 (1996): 218-233.

Regularity of real mappings and non-isolated singularities

Mihai Tibăr

The title refers to regularity conditions which one may impose in order to obtain the local triviality of the fibration defined by a mapping, either in the local setting or in the global one. Let us first consider a real analytic mapping germ $\psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, and let $V := \psi^{-1}(0)$. The isolated singularity case $\operatorname{Sing} \psi = \{0\}$ has been settled by Milnor; his result [5, Theorem 11.2] tells that there exists a mapping $S^{m-1}_\varepsilon \setminus K_\varepsilon \to S^{p-1}_\delta$ which is a locally trivial fibration and its diffeomorphism type is independent of the small enough chosen constants $0 < \delta \ll \varepsilon$, where $K_\varepsilon := V \cap S^{m-1}_\varepsilon$. It turns out that the sphere S^{m-1}_ε is moreover endowed with a *higher open book structure* with binding K_ε, as defined in [2, Definition 2.1]. In case the singularities are non-isolated but included in V, we introduce the following definition [3]: *We say that the pair (K, θ) is a higher open book structure with singular binding on an analytic manifold M of dimension $m - 1 \geq p \geq 2$ if $K \subset M$ is a singular real subvariety of codimension p and $\theta : M \setminus K \to S^{p-1}_1$ is a locally trivial smooth fibration such that K admits a neighbourhood N for which the restriction $\theta_{|N \setminus K}$ is the composition $N \setminus K \stackrel{h}{\to} B^p \setminus \{0\} \stackrel{s/\|s\|}{\to} S^{p-1}_1$ where h is a locally trivial fibration.* For some open set U containing 0, let $M(\psi) := \{x \in U \mid \rho \not\pitchfork_x \psi\}$ and $M(\frac{\psi}{\|\psi\|}) := \text{closure}\{x \in U \setminus V \mid \rho \not\pitchfork_x \frac{\psi}{\|\psi\|}\}$.

We then have the following result, essentially extracted from Milnor's proofs [5], see [3]: *Let* $\psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ *be an analytic mapping germ,* $m > p \geq 2$*, such that* codim V = p. *If* $(*) : \overline{M(\psi) \setminus V} \cap V = \{0\}$ *then there exists an open book structure on* S_ε^{m-1} *with singular binding* K_ε*.* If moreover $M(\frac{\psi}{\|\psi\|}) = \emptyset$, then it is shown in [3] that this open book structure is induced by the mapping $\frac{\psi}{\|\psi\|}$.

The regularity condition $(*)$ insures the transversality of the fibres of ψ to the small enough spheres S_ε^{m-1} in the neighbourhood of the link $K := V \cap S_\varepsilon^{m-1}$. It allows non-isolated singularities, more precisely it implies $\operatorname{Sing} \psi \subset M(\psi) = A \cup B$, where $A \subset V$ and $B \cap V \subset \{0\}$, and where both A and B may be of positive dimension. The above result thus represents a simultaneous extension of [2, Theorem 2.2], where the singular locus was of type B, and of [1, Proposition 5.3], where the singular locus was of type A.

It is also clear (and this observation goes back at least to the '70s) that the Thom regularity along V implies the condition $(*)$. The later can be directly verified in many situations, such as the following (we refer to Oka's talk on mixed functions for the employed notions):

Theorem. [3] *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a non-constant mixed polynomial which is polar weighted-homogeneous,* $n \geq 2$*, such that* $\operatorname{codim}_{\mathbb{R}} \mathrm{V} = 2$*. Then* $\overline{M(\psi) \setminus V} \cap V = \{0\}$ *and* $(K_\varepsilon, \frac{f}{\|f\|})$ *is an open book structure with singular binding on* S_ε^{2n-1}*.*

Nevertheless, Thom regularity is a strictly stronger condition: $h(x, y) = xy\bar{x}$ has property $(*)$ but it is not Thom regular along V. A natural class of mixed functions which have Thom regularity along V is $f\bar{g}$, for holomorphic f and g in 2 variables and such that $f\bar{g}$ has 0 as isolated singular value. In more than 2 variables, the same property has been recently claimed in [6] but its proof appears to be false. I've explained the issue and I've asked for a concrete counter-example. At the end of my talk, Adam Parusiński showed a simple such example which is a 3-variables deformation of the above function h, and discussed it during the "problem session".

In the last part of my talk, I've briefly presented recent results obtained jointly in [4] about regularity conditions at infinity for mappings and about their relations: Gaffney-Malgrange condition, Rabier and Kurdyka-Orro-Simon condition, the t-regularity and the ρ-regularity.

References

[1] R.N. Araújo dos Santos, M. Tibăr, *Real mapping germs and higher open books*, December 7, 2007. preprint arXiv:0801.3328.

[2] R.N. Araújo dos Santos, M. Tibăr, *Real mapping germs and higher open book structures*, Geometria Dedicata **147** (2010), 177–185.

[3] R.N. Araújo dos Santos, Y. Chen, M. Tibăr, *Singular open book structures from real mappings*, arXiv:1011.1757, to appear in Cent. Eur. J. Math.

[4] L.R.G. Dias, M.A.S. Ruas, M. Tibăr, *Regularity at infinity of real mappings and a Morse-Sard theorem*, J. Topology **5** (2012), no. 2, 323–340.

[5] J. Milnor, *Singular points of complex hypersurfaces,* Ann. of Math. Studies 61, Princeton University Press, 1968.

[6] A. Pichon, J. Seade, *Milnor fibrations and the Thom property for maps $f\overline{g}$*, J. Singul. **3** (2011), 144–150.

Integration in real closed fields

Yimu Yin

Towards the end of the introduction of [4] three hopes for the future of the theory of motivic integration are mentioned. We realize one of them in this talk: integration in real closed fields. Of course all the real closed fields that we shall consider are non-archimedean. A typical example is $\mathbb{R}(\!(\mathbb{Q})\!)$. We could work with any polynomial-bounded o-minimal expansion of the theory of real closed fields. However, for concreteness and simplicity, we shall work in $\mathbb{R}(\!(\mathbb{Q})\!)$ and concentrate on one particular polynomial-bounded theory of it.

We begin with the theory $\mathrm{RCF}_{\mathrm{an}}$ as described in [3]; that is, $\mathbb{R}(\!(\mathbb{Q})\!)$ is considered as a model of the theory of real closed fields with restricted analytic functions. We write VF for the only sort. It admits quantifier elimination in the language $\mathcal{L}_{\mathrm{an}}$. It also carries a natural valued field structure, where the maximal ideal $\mathcal{M}$ consists of all the Puiseux series whose leading degrees are greater than 0. So we may add an additional sort $\mathrm{RV} = \mathrm{VF}^{\times} / 1 + \mathcal{M}$ and the natural map $\mathrm{rv} : \mathrm{VF}^{\times} \longrightarrow \mathrm{RV}$ to capture this structure, as described in [5]. The valuation ring $\mathcal{O}$ is T-convex in the sense of [2]; that is, $\mathcal{O}$ is closed under all $\emptyset$-$\mathcal{L}_{\mathrm{an}}$-definable continuous functions $\mathrm{VF} \longrightarrow \mathrm{VF}$. The construction of the integration map relies on many technicalities worked out in [1],[2].

Let VF_* and $\mathrm{RV}[*]$ be two suitable categories of definable sets that are respectively associated with the VF-sort and the RV-sort. We want to construct a canonical homomorphism $\mathbf{K}\,\mathrm{VF}_* \longrightarrow \mathbf{K}\,\mathrm{RV}[*]/\,\mathrm{I}_{\mathrm{sp}}$ between the Grothendieck rings, where I_{sp} is a suitable ideal. There are three steps:

- *Step 1.* There is a natural lifting map $\mathbb{L}$ from the set of objects of $\mathrm{RV}[*]$ into the set of objects of VF_*. We show that $\mathbb{L}$ hits every isomorphism class of VF_*.
- *Step 2.* We show that $\mathbb{L}$ induces a ring homomorphism from $\mathbf{K}\,\mathrm{RV}[*]$ into $\mathbf{K}\,\mathrm{VF}_*$, which is also denoted by $\mathbb{L}$.
- *Step 3.* In order to obtain a precise description of the kernel of $\mathbb{L}$, we introduce two operations: special bijection in the VF-sort and blowup in the RV-sort. In a sense these two operations mirror each other. Using this correlation we show that, for any objects $\mathbf{U}_1$, $\mathbf{U}_2$ in $\mathrm{RV}[*]$, there are isomorphic blowups $\mathbf{U}_1^\sharp$, $\mathbf{U}_2^\sharp$ of $\mathbf{U}_1$, $\mathbf{U}_2$ if and only if $\mathbb{L}(\mathbf{U}_1)$, $\mathbb{L}(\mathbf{U}_2)$ are isomorphic.

Through certain standard algebraic manipulations, the inverse of $\mathbb{L}$ gives rise to various ring homomorphisms and module homomorphisms. These are understood as generalized Euler characteristic or, if volume forms are present, integration.

We hope that this construction will give rise to various invariants of real Milnor fibers, including the Euler characteristic.

References

[1] Lou van den Dries, *T-convexity and tame extensions II*, Journal of Symbolic Logic **62** (1997), no. 1, 14–34.

[2] Lou van den Dries and Adam H. Lewenberg, *T-convexity and tame extensions*, Journal of Symbolic Logic **60** (1995), no. 1, 74–102.

[3] Lou van den Dries, Angus Macintyre, and David Marker, *The elementary theory of restricted analytic fields with exponentiation*, Annals of Mathematics **140** (1994), no. 1, 183–205.

[4] Ehud Hrushovski and David Kazhdan, *Integration in valued fields*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser, Boston, MA, 2006, math.AG/0510133, pp. 261–405.

[5] Yimu Yin, *Special transformations in algebraically closed valued fields*, Annals of Pure and Applied Logic **161** (2010), no. 12, 1541–1564, arXiv:1006.2467.

Reporter: Georges Comte

Participants

Prof. Dr. Norbert A'Campo
Dorneckweg 14
4108 Witterswil
SWITZERLAND

Prof. Dr. Raf Cluckers
Laboratoire Paul Painleve, UMR 8524
Université des Sciences et
Technologie de Lille 1
Cite Scientifique, Bat. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Georges Comte
Laboratoire de Mathématiques
Université de Savoie
73376 Le Bourget du Lac Cedex
FRANCE

Prof. Dr. Nicolas Dutertre
LATP, UMR-CNRS 6632
Centre de Mathématiques et Inform.
Université de Provence, Aix-Mars. I
39, rue Joliot Curie
13453 Marseille Cedex 13
FRANCE

Prof. Dr. Goulwen Fichou
U. F. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Prof. Dr. Toshizumi Fukui
Department of Mathematics
Faculty of Education
Saitama University
255 Shimo-Ookubo
Urawa, Saitama 338-8570
JAPAN

Prof. Dr. Helmut Arend Hamm
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Laurentiu-G. Maxim
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Prof. Dr. Mutsuo Oka
Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601
JAPAN

Prof. Dr. Adam Parusinski
Laboratoire J.-A. Dieudonne
Université de Nice
Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

Fabien Priziac
U. F. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Dr. Michel Raibaut
38 Av. Caravadossi
06000 Nice
FRANCE

Dr. Jörg Schürmann
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Dirk Siersma
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
3508 TA Utrecht
NETHERLANDS

Prof. Dr. Kiyoshi Takeuchi
Institute of Mathematics
University of Tsukuba
Tsukuba-shi
Ibaraki 305-8571
JAPAN

Prof. Dr. Mihai Tibar
Mathématiques
UMR 8524 CNRS
Université de Lille 1
59655 Villeneuve d'Ascq
FRANCE

Dr. Yimu Yin
Institut de Mathématiques de Jussieu
UMR 7586 du CNRS
Universié Pierre et Marie Curie
4 place Jussieu
75252 Paris Cedex 05
FRANCE