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Model Theory: Groups, Geometry, and Combinatorics

Organised by
Ehud Hrushovski, Jerusalem
Anand Pillay, Leeds
Katrin Tent, Münster
Frank Olaf Wagner, Lyon

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Abstract. Overall this was a high quality meeting, with carefully chosen talks fitting in with the announced themes of the workshop.

Mathematics Subject Classification (2010): 03C,20A.

Introduction by the Organisers

1. Approximate subgroups

A major theme was the use of model theoretic and "nonstandard" methods in (generalized) additive combinatorics, especially the study of "approximate subgroups" and related problems, as well as the relationship with established methods such as asymptotic cones. Freiman's classification of approximate subgroups of $\mathbb{Z}$ is a cornerstone of additive combinatorics. Similar questions concerning approximate subrings of fields (the sum-product phenomenon) have been very recently resolved by Bourgain, Tao and others, with applications by Bourgain, Gamburd, Wigderson and others; the contribution of the model theorist Chris Miller was essential. Tao defined and studied approximate subgroups of arbitrary groups. In recent work by Hrushovski, model theoretic techniques provided a breakthrough that later led to decisive results concerning the case of linear groups, and is likely to lead to such results in general. The novelty of the methods was not only in using "nonstandard models" or ultraproducts, from model theory, but also applying techniques from the study of definable groups in "tame" first order theories (stable, simple,..) even though the cases at hand are far from tame.

The connection with simplicity theory is deep, and one can hope to develop more of the methods of this part of model theory in a pseudo-finite setting.

On the other hand, a very important aspect has been the connection with Lie groups. While the question of connected quotient groups arose in the simple setting, it is still not known if they actually occur there; a question that deserves another look. But connected quotient groups, often Lie groups, have been essential in recent developments in theories without the independence property. It would be very valuable to put together the expertise from these different pursuits, and attempt to amalgamate them. (This connects with topic 4 below.) One significant theoretical issue is the generalization from absolutely to relatively bounded index quotients, which would have great interest for applications.

Earlier use of nonstandard methods in broadly similar frameworks was seen with the Van den Dries-Wilkie treatment of Gromov's theorem on finitely generated groups of polynomial growth, in which a model-theoretic approach to "asymptotic cones" of metric spaces (viewing a metric space from afar) played an important role. What is also common in these examples (Hrushovski, Gromov) is the appearance and construction of a Lie group as a key feature of the proof. So far, the deeper methods of Lie theory have been used as a black box, although on the other hand the shortest and most general treatments of substantial parts Montgomery-Zippin-Gleason-Yamabe work have been given by model theorists (Hirschfeld, Goldbring.) It would be important again to combine more organically what has been understood by different works.

Other broadly related developments in recent years include the classification of asymptotic cones of semisimple Lie groups (Kramer, Tent, and others).

So the meeting in part focussed on an exposition of the above methods, exploring the common features, and then hearing about the most recent advances.

We had several excellent talks in the workshop on this topic. There was first a tutorial by Pierre Simon and Emmanuel Halupczok on Hrushovski's work on approximate subgroups. Subsequently a talk by Breuillard on his work with Green and Tao giving a definitive description if approximate subgroups (but influenced by Hrushovski). Also a talk by Hrushovski, generalizing the results to approximate equivalence relations, as well as discussing other work on related themes of the workshop, such as how close is G/G^{00} to a compact Lie group, for suitable definable groups G in NIP theories.

The organizers included in the meeting several other related topics:

2. Stability, free groups, and hyperbolic groups

Free groups (as well as torsion-free hyperbolic groups) were recently proved by Sela to be structures whose first order theory is stable. This gives very important new examples of stable groups. Secondly it suggests that the "true" algebraic geometry of the free group should be the study of the category of definable sets in the free group. Understanding the definable sets will have a great impact on other questions about the free group as work of Bestvina and Feighn shows.

It is also natural to apply methods from 1. (especially asymptotic cones) to the free group and the interaction with stability looks to be fascinating. By studying the dependence relation in free groups one can also hope to give another proof of the stability of free groups.

In the workshop Ould-Houcine first gave a short tutorial on methods in geometric group theory (such as the JSJ decomposition). There were later talks on current research by Ould-Houcine, Tent, Sklinos, and Sela. Sela described some of his recent work on free products solving many long-standing conjectures.

3. Classification of first order theories

Stable first order theories have been mentioned in 1 and 2. Much of the theory needed in recent applications of model theory (as in 1) has come from the extension of methods from the study of stable first order theories to unstable but "tame" first order theories, such as simple theories, theories without the independence property, etc. In fact this theme and the connection to valued fields and rigid geometry was the basis of the January 2010 Model Theory meeting at Oberwolfach. In any case, this is a very active area on the "purer" side of model theory (e.g. Banff workshops in 2009, 2012) and will feed in to the other aspects of the meeting.

There were a variety of research talks around this theme: Krupinski, Newelski, Simon, Starchenko. In particular Pierre Simon's talk on invariant types in NIP theories, presented some very strong results. Berarducci also gave a research talk connected with this theme, around groups defined in certain 2-sorted structures, with some striking results.

4. Connected components, non G-compact theories

Recent work (Conversano, Pillay) has produced new examples of "connected components" of groups (in suitable model-theoretic senses), arising from ultraproducts of universal covers of suitable semisimple Lie groups. This ties up with theme 3 above (giving for example new non G-compact theories) and also the methods in theme 1.

Conversano presented some of her results. Gismatullin gave a short talk presenting new noncommutative examples of G^{00}/G^{000}. And Kaplan gave a beautiful talk on his recent work with Miller and Simon on Lascar strong types and descriptive set theory.

5. Other

Other talks in the workshop were on important contemporary themes such as valued fields, difference fields, groups of finite Morley rank, and the search for "bad groups". (Chatzidakis, Halupczok, Hils, Loeser, Poizat, Zilber).

Workshop: Model Theory: Groups, Geometry, and Combinatorics

Table of Contents

Pierre Simon
Approximate subgroups I 11

Martin Hils (joint with Artem Chernikov)
Valued difference fields and NTP_2 12

Abderezak Ould Houcine
Introduction to actions on trees and JSJ-decompositions 14

Immanuel Halupczok
Approximate subgroups II 15

Itay Kaplan (joint with Benjamin Miller and Pierre Simon)
The Borel cardinality of Lascar strong types 16

Zoé Chatzidakis
Algebraic dynamics and difference fields 18

Gilbert Levitt (joint with Vincent Guirardel)
Extendable automorphisms 21

Sergei Starchenko
VC-density and (p,q)-theorems* 22

Pierre Simon
Invariant types in NIP structures 24

Bruno Poizat
Groupes lineaires de rang de Morley fini 25

Ehud Hrushovski
Approximate equivalence relations 27

Emmanuel Breuillard (joint with Ben Green and Terence Tao)
The structure of approximate groups 31

François Loeser
Motivic counting of points of bounded height 34

Rizos Sklinos (joint with Chloé Perin, Larsen Louder)
Hyperbolic towers in the free group 34

Immanuel Halupczok
Isometries in valued fields 35

Itaï Ben Yaacov (joint with Ehud Hrushovski)
Globally valued fields 37

Annalisa Conversano (joint with Marcello Mamino, Anand Pillay)
Groups definable in o-minimal structures 38

Zlil Sela
The Elementary Theory of Free Products of Groups 42

Ludomir Newelski
Topological dynamics of stable groups 44

Alessandro Berarducci (joint with Marcello Mamino)
Groups definable in two orthogonal sorts 47

Boris Zilber (joint with K.Tent)
A non-desarguesian projective plane of an analytic origin 49

Abderezak Ould Houcine (joint with K. Tent)
Ampleness in free groups 51

Jakub Gismatullin
On model-theoretic connected components 54

Krzysztof Krupiński (joint with T. Gogacz, P. Tanović, F. Wagner)
Minimality and quasi-minimality in the context of groups and fields 57

Chris Miller (joint with Philipp Hieronymi)
Minkowski dimension and definability 60

Abstracts

Approximate subgroups I

Pierre Simon

This talk was the first of a series of three presenting the work of Hrushovski and of Breuillard-Green-Tao on approximate subgroups. My presentation focussed on the model-theoretic framework and in particular the use of definable measures to study pseudofinite structures. Everything I mentioned is due to Hrushovski and comes from [1] and [2].

Given a sequence of finite structures $(M_i : i < \omega)$, we consider the ultraproduct $M = \prod_{\mathcal{U}} M_i$, where $\mathcal{U}$ is some non-principal ultrafilter on ω. For our purposes it is convenient to work with a very large language L, say one containing any *internal* subset of M as a predicate. For some measure-theoretic arguments, it is useful to have a countable language, in which case we can take a sufficiently rich sublanguage of L.

Now, on each structure M_i, we have a measure μ_i which is the normalized counting measure. Those give rise to a limit measure μ which is a $[0,1]$-valued finitely additive measure of definable sets of M. Having taken the language rich enough, that measure is *definable*. This means that for every $r \in (0,1)$, the condition $\mu(\phi(x;b)) > r$ is an open condition in $tp(b/\emptyset)$.

We say that an invariant relation $R(x,y)$ is *stable* if there does not exist an indiscernible sequence $(a_i, b_i : i < \omega)$ such that $R(a_i, b_j)$ holds if and only if $i \leq j$. An easy, but fundamental, observation is that for any two formulas $\phi(x;y)$ and $\psi(x;z)$, the relation $R(a,b) \equiv \mu(\phi(x;a) \cap \psi(x;b)) = 0$ is stable. In fact, this remains true if we replace 0 by any other value, but it is more difficult to prove. One can then adapt stability theory to work in this context (and also in the slightly more general context which we will need for applications to near-subgroups). More precisely, one shows that if $R(x,y)$ is a stable invariant relation, if a, b, b' are points in M such that b and b' have the same Lascar-strong type, if neither $tp(b/a)$ nor $tp(b'/a)$ forks over $\emptyset$, then $R(a,b)$ holds if and only if $R(a,b')$ holds. If furthermore R is type-definable, then one can replace Lascar-strong types by compact (also called *Kim-Pillay*) strong types.

Given a group G equipped with a finitely additive G-invariant measure μ (taking values in $\mathbb{R}^+ \cup \{\infty\}$), a subset X of G is said to be a *near-subgroup* if $X^{-1} = X$ and we have $0 < \mu(X), \mu(X^{\cdot 3}) < \infty$, where $X^{\cdot 3}$ denotes the set of products of 3 elements of X.

The main theorem is the following: Let $X \subset G$ be a near-subgroup (with respect to some definable measure μ), then there is a μ-wide type-definable subgroup $S \subseteq X^{\cdot 4}$ of bounded index.

To prove this, we must first slightly adapt the context described above to allow for measures taking unbounded—and even infinite—values. One then works in the $\bigvee$-definable set $\tilde{G}$ defined as the group generated by X and considers its reduct to $\tilde{G}$-translation-invariant relations. In particular, the group of automorphisms acts transitively on elements. The required subgroup S is then simply taken to be the set of elements which have same compact strong type as the identity.

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Valued difference fields and $\mathbf{NTP_2}$

Martin Hils

(joint work with Artem Chernikov)

Every non-principal ultrapoduct of structures of the form $(\mathbb{F}_p^a, Frob_p)$ is an algebraically closed field with a generic automorphism. This is a deep result of Hrushovski [8] which required a twisted version of the Lang-Weil estimates. One may consider the non-standard Frobenius acting on an algebraically closed *valued* field of residue characteristic 0, i.e. the limit theory of the Frobenius automorphism acting on an algebraically closed valued field of characteristic p (where p tends to infinity). Hrushovski gave a natural axiomatisation of this limit theory in the language of valued difference fields (denoted by VFA in the sequel). Durhan (formerly Azgın) [1] obtained an alternative axiomatisation, as well as an Ax-Kochen-Ershov principle for a certain class of valued difference fields.

Recall that a valued difference field is a valued field (K, k, Γ) together with an automorphism σ such that $\sigma(\mathcal{O}) = \mathcal{O}$, where $\mathcal{O} = \{a \in K \mid \mathrm{val}(a) \geq 0\}$. The automorphism σ induces an automorphism $\overline{\sigma}$ of the residue field k and an automorphism σ_Γ of the value group Γ. We treat valued difference fields in the 3-sorted language of Pas, augmented by symbols for σ, $\overline{\sigma}$ and σ_Γ, and where we require that angular component map ac satisfies $\mathrm{ac} \circ \sigma = \overline{\sigma} \circ \mathrm{ac}$), together with

The theory VFA is interesting from an algebraic point of view. The induced automorphism $\bar{\sigma}$ on the residue field is generic, by the aforementioned result of Hrushovski. The induced automorphism σ_Γ on the value group Γ is ω*-increasing* (i.e. $\sigma_\Gamma(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \geq 1$; valued difference fields satisfying this property will be called *contractive*). Thus, Γ gets the structure of a divisible torsion free ordered $\mathbb{Z}[\sigma]$-module (i.e. an ordered vector space over $\mathbb{Q}(\sigma)$, where $\sigma >> 1$ is an indeterminate). It is sufficient to add a σ*-Hensel property* to obtain an axiomatisation of VFA. The Ax-Kochen-Ershov principle of Durhan then holds for contractive σ-henselian valued difference fields of residue characteristic 0.

Note that the theory VFA is neither simple (due to the total order in the value group) nor NIP (since the residue difference field has the independence property). It appears that Shelah has defined another property called NTP_2 (*not the tree property of the second kind*). This class generalises both simple and NIP theories, and contains new examples (any ultraproduct of p-adics is NTP_2 [5]). Recently it has attracted attention, largely motivated by Pillay's question on equality of forking and dividing over models in NIP, and a good theory of forking for NTP_2 theories has been developed [7, 5, 4].

Definition. *Let T be a complete theory. We say that $\varphi(x,y)$ has* TP_2 *if there $k \in \omega$ and $(a_{ij})_{i,j\in\omega}$ (in some monster model of T) such that:*

(1) $\{\varphi(x,a_{ij})\}_{j\in\omega}$ *is k-inconsistent for every $i \in \omega$.*

(2) $\{\varphi(x,a_{if(i)})\}_{i\in\omega}$ *is consistent for every $f : \omega \to \omega$.*

The theory T is called NTP_2 *if no formula has* TP_2*.*

Here is the main result of our work.

Theorem 1. *Let $\mathcal{K} = (K,\Gamma,k,\sigma)$ be a contractive σ-henselian valued difference field of residue characteristic* 0*, with residue field k and value group Γ. Assume that* $\mathrm{Th}(k,\bar\sigma)$ *and* $\mathrm{Th}(\Gamma,\sigma_\Gamma)$ *are* NTP_2*. Then* $\mathrm{Th}(\mathcal{K})$ *is* NTP_2*.*

Corollary. *Every completion of* VFA *is* NTP_2*.*

The corollary follows from the theorem. Indeed, $\overline{\sigma}$ is generic, so $\mathrm{Th}(k,\bar\sigma)$ is simple and in particular NTP_2. The value group together with the induced automorphism σ_Γ is an ordered vector space over $\mathbb{Q}(\sigma)$, so *o*-minimal. In particular, (Γ,σ_Γ) is NIP and thus NTP_2.

In the isometric case, where one requires $\sigma_\Gamma = \mathrm{id}$, an Ax-Kochen-Ershov principle is known to hold as well [9, 3, 2]. In this setting, we prove the following statement which is analogous to Theorem 1.

Theorem 2. *Let $\mathcal{K} = (K,k,\Gamma,\sigma)$ be a σ-henselian valued difference field of residue characteristic 0, with σ an isometry and where we assume that there are enough constants. Then* $\mathrm{Th}(\mathcal{K})$ *is* NTP_2 *if and only if* $\mathrm{Th}(k,\overline{\sigma})$ *is* NTP_2*.*

For p a prime number, let $W(\mathbb{F}_p^{alg})$ be the quotient field of the ring of Witt vectors with coefficients from $\mathbb{F}_p^{alg}$, with its natural valuation. On the valued field $W(\mathbb{F}_p^{alg})$, there is a natural isometry, namely the *Witt-Frobenius* automorphism which we denote by $\widetilde{Frob_p}$, sending $x = \sum_n a_n p^n \in W(\mathbb{F}_p^{alg})$ to $\sum_n a_n^p p^n$. Letting $\mathrm{ac}(x) := a_{\mathrm{val}(x)}$, we get an ac-valued difference field $\mathcal{W}_p = (W(\mathbb{F}_p^{alg}), \mathbb{Z}, \mathbb{F}_p^{alg}, \widetilde{Frob_p})$. Let $\mathcal{U}$ be a non-principal ultrafilter on the set of prime numbers. Then $\prod_{\mathcal{U}} \mathcal{W}_p$ satisfies the hypotheses of Theorem 2 (see [3]). Thus, these valued difference fields are NTP_2 by our result.

Theorem 2 is an analogue of Delon's theorem for preservation of NIP in henselian valued fields of residue characterstic 0. Preservation for NTP_2 in this setting is due to Chernikov [5].

The proofs of Theorem 1 and Theorem 2 are quite technical. They combine a new result on extending indiscernible arrays by parameters coming from stably embedded NTP_2 sorts with the back-and-forth system used to eliminate field quantifiers. In this way, one reduces to a situation where one deals with immediate extensions, and these are controlled by NIP formulas.

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Introduction to actions on trees and JSJ-decompositions

Abderezak Ould Houcine

In this talk we introduce actions on simplicial trees and JSJ-decompositions. We explain the structure of groups acting on simplicial trees as given by Bass-Serre theory. Bass-Serre theory shows that a group acts on a simplicial tree if and only if it is an iteration of amalgamated free products and of HNN-extensions. By the notion of graph of groups, this theory gives a complete description of groups acting on simplicial trees.

In geometric group theory, the following general problem is studied : "Given a finitely generated group G and a class of subgroups $\mathcal{C}$, what is the relationship between the various graphs of groups decompositions of G, where the associated edges groups are in $\mathcal{C}$? "

Schematically, a JSJ-decomposition (or JSJ-splitting), relative to $\mathcal{C}$, corresponds to the most possible canonical decomposition of G from which all other decompositions (relative to $\mathcal{C}$) can be obtained. We explain JSJ-decompositions.

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Approximate subgroups II

Immanuel Halupczok

Freiman's inverse problem consists in characterizing those finite subsets X of a group G such that $X \cdot X$ is not much bigger than X. More precisely, assuming that $|X \cdot X| \leq K \cdot |X|$ for some fixed $K \in \mathbb{N}$, one would like to obtain a description of X whose "complexity" only depends on K. For simplicity, one usually also assumes $X = X^{-1}$ and $1 \in X$. The oldest result of this type, dating from the 60ies, is due to Freiman himself and describes such sets X in $G = \mathbb{Z}$. Later, there have been several generalizations until finally, in 2009, Breuillard-Green-Tao obtained such a result in arbitrary groups G. It turns out that if G is non-commutative, then to obtain good descriptions of X, it is better to impose a somewhat stronger condition than just $|X \cdot X| \leq K \cdot |X|$: either one requires that $X \cdot X$ can be covered by a fixed number K of (say, left-) translates of X (such an X is called a *K-approximate subgroup*), or one requires that $|X \cdot X \cdot X| \leq K \cdot |X|$ for some fixed K (such an X is said to have "small tripling"). The result of Breuillard-Green-Tao is valid in both settings.

There are two basic types of approximate subgroups. One of them are actual subgroups of G. For the other one, suppose first that we have elements $a_1, \dots, a_n \in G$ such that the generated group $G' := \langle a_1, \dots, a_d \rangle$ is abelian. Moreover, choose $n_1, \dots, n_d \in \mathbb{N}$ and consider $P := \{a_1^{r_1} \cdots a_d^{r_d} \mid -n_i \leq r_i \leq n_i\}$. This set P is a K-approximate subgroup where K only depends on d (one can take $K = 2^d$). If the generated group G' is nilpotent, then by imposing some additional conditions on the a_i and r_i, one can also ensure that P is an approximate subgroup; such P are called *nilprogressions*.

The two above types of approximate subgroups can be combined: if we have $H \lhd G' \subseteq G$ with G'/H nilpotent and P' is a nilprogression in G'/H, then the preimage $P \subseteq G$ of P' is also an approximate subgroup. The result of Breuillard-Green-Tao says that any approximate subgroup X "is close to" a P of this form, where the d appearing in the definition of P' only depends on K (but neither on X nor on G). One possible definition of what it means for X and P to be "close to each other" is to require that $X \cdot X$ and P are *C-commensurable* for some C depending only on K: each of the two sets $X \cdot X$ and P can be covered by C-many translates of the other one.

In the talk, I sketched a model theoretic variant of the proof of Breuillard-Green-Tao; this variant is due to Hrushovski and can be found in lecture notes on his web page. The first step is to reformulate the main result using ultraproducts as follows. One fixes $K \in \mathbb{N}$ and assumes that G is an ultraproduct of groups G_i and $X \subseteq G$ is an ultraproduct of K-approximate subgroups $X_i \subseteq G_i$. (In fact, Hrushovski uses the small tripling condition instead of approximate subgroups.) The claim is that then, there exist definable $H \lhd G' \subseteq G$, $P \subseteq G'$ satisfying conditions similar to the ones in the classical version of the result for some $d, C \in \mathbb{N}$. There is no condition anymore that d, C only depend on K; this has been replaced by the fact that d and C are not allowed to be non-standard numbers.

The main steps of the proof of this ultraproduct version of the result are the following:

- The main theorem of the talk *Approximate Subgroups I* can be applied to X and yields a $\wedge$-definable group S such that in particular, the quotient $L := \langle X\rangle/S$ has bounded cardinality. From this, one deduces that L is a locally compact group.
- By a theorem of Gleason-Yamabe, L has a subquotient L' which is a Lie-group. We can write it as $L' = \tilde{G}/S'$ for some $\vee$-definable $\tilde{G} \subseteq \langle X\rangle$ and some $\wedge$-definable $S' \supseteq S$.
- Using that $\tilde{G}$ is generated by a definable set, one finds a smallest definable group G' containing $\tilde{G}$ (in an appropriate language). With considerably more work, one also finds a largest definable group H contained in S'. The heart of the proof is then to show that G'/H is nilpotent. When doing this, one more or less automatically obtains the nilprogression P'; the reason is that the construction of $\tilde{G}$ and S' ensures that *any* definable subset of $\tilde{G}$ containing S' is commensurable to $X \cdot X$.
- We suppose without loss $H = 1$ (by passing to the quotient). To prove that G' is nilpotent, we define a "distance to 1" in G'. One the one hand, the commutator of two elements close to 1 is even closer to 1 (this is true in the Lie group L', and a central part of the work is to show that it is also true in G'). On the other hand, using that X is pseudo-finite, we can find an element $a \in G' \setminus \{1\}$ that has mininimal distance to 1. This implies that a is central in G'. By repeating this argument (and using an induction over $\dim L$), one obtains that G' is nilpotent.

The Borel cardinality of Lascar strong types

Itay Kaplan

(joint work with Benjamin Miller and Pierre Simon)

To any complete first-order theory T we can associate natural equivalence relations, or strong types. A *strong type* (over $\emptyset$) is a class of an automorphism-invariant equivalence relation on $\mathfrak{C}^\alpha$ which is bounded (i.e., the quotient has small cardinality) and refines equality of types. The phrase "strong type" by itself often refers to a *Shelah strong type,* which is simply a type over the algebraic closure of $\emptyset$ (in T^{eq}). In other words, two tuples have the same Shelah strong type if they are equivalent with respect to every definable equivalence relation with finitely many classes. Refining this is the notion of *KP strong type* ($\equiv^\alpha_{KP}$), in which two tuples are equivalent if they are equivalent with respect to every bounded type-definable equivalence relation. Finally, the *Lascar strong type* ($\equiv^\alpha_L$) is the finest notion of strong type. The classes of $\equiv^\alpha_L$ coincide with the connected components of the *Lascar graph* on $\mathfrak{C}^\alpha$, in which two tuples are neighbors if they lie along an infinite indiscernible sequence. The *Lascar distance* d_L is the associated graph distance.

In [New03], Newelski established the following fundamental facts:

Fact 1. [New03] *Suppose that T is a complete first-order theory and α is an ordinal.*

(1) *A Lascar strong type is type definable iff it has finite diameter.*
(2) *If Y is an $\equiv_L^\alpha$-invariant closed set, contained in some $p \in S_\alpha(\emptyset)$, on which every $\equiv_L^\alpha$-class has infinite diameter, then Y contains at least $2^{\aleph_0}$-many $\equiv_L^\alpha$-classes.*
(3) *Lascar strong types of unbounded diameter are not G_δ sets (see below).*
(4) *If T is small (i.e., T is countable and the number of finitary types over $\emptyset$ is countable), then $\equiv_L^n = \equiv_{KP}^n$ for all $n < \omega$ (i.e., the two notions of type agree on finite sequences).*

As opposed to Shelah and KP strong types, the space of Lascar strong types does not come equipped with a Hausdorff topology. It is therefore unclear to what category this quotient belongs. In [KPS12], the authors suggest viewing it through the framework of descriptive set theory (this idea was already mentioned in [CLPZ01]).

Given two Polish spaces X and X' and two Borel equivalence relations E and E' respectively on X and X', we say that E is *Borel reducible* to E' if there is a Borel map f from X to X' such that $x \mathrel{E} y \iff f(x) \mathrel{E'} f(y)$ for all $x, y \in X$. Two relations are *Borel bi-reducible* if each is Borel reducible to the other. The quasi-order of Borel reducibility is a well-studied object in descriptive set theory, and enjoys a number of remarkable properties. One of them is given by the Harrington-Kechris-Louveau dichotomy, which asserts that a Borel equivalence relation is either smooth (Borel reducible to equality on 2^ω) or at least as complicated as $\mathbb{E}_0$ (eventual equality on 2^ω).

Suppose that T is a complete countable first-order theory, α a countable ordinal. Then for a countable model M, $S_\alpha(M)$ is a Polish space. If $p, q \in S_\alpha(M)$, we write $p \equiv_L^{M,\alpha} q$ (respectively $p \equiv_{KP}^{M,\alpha} q$) when there are realizations $a \models p$ and $b \models q$ such that $a \equiv_L^\alpha b$ (respectively $a \equiv_{KP}^\alpha b$). It is not hard to see that $\equiv_L^{M,\alpha}$ is an F_σ equivalence relation, and that $\equiv_{KP}^{M,\alpha}$ is a closed equivalence relation.

It is shown in [KPS12] that up to Borel bi-reducibility, $\equiv_L^{M,\alpha}$ does not depend on the choice of M, even when restricted to a KP-class. This observation extends to a larger class of sets. Suppose that $Y \subseteq \mathfrak{C}^\alpha$ is G_δ — a countable intersection of $\bigvee$-definable sets — and that Y is closed under $\equiv_L^\alpha$. Then for a countable model M, let $Y_M = \{p \in S_\alpha(M) \mid \exists a \in Y\, (a \models p)\}$. One can show that Y_M is a G_δ subset of $S_\alpha(M)$, and that up to Borel bi-reducibility $\equiv_L^{M,\alpha} \restriction Y_M$ is independent of the choice of M (so we may write $\equiv_L^\alpha \restriction Y$).

Our main result is the following solution to the main conjecture of [KPS12].

Theorem 2. *Suppose T is a complete countable first-order theory, α a countable ordinal, and suppose Y is a G_δ subset of $\mathfrak{C}^\alpha$ which is closed under $\equiv_L^\alpha$. If for some $a \in Y$, there is no bound on d_L restricted to the class $[a]_{\equiv_L^\alpha}$ then $\equiv_L^\alpha \restriction Y$ is non-smooth: $\mathbb{E}_0 \leq_B \equiv_L^\alpha \restriction Y$.*

Note that by Fact 1 (1) the assumption that there is no bound on d_L in the class $[a]_{\equiv_L^\alpha}$ is equivalent to saying that $[a]_{\equiv_L^\alpha}$ is not type-definable. However, the proof of the theorem does not use this factKPLN.

Corollary 3. *Suppose T and α are as above. Suppose $K \subseteq \mathfrak{C}^\alpha$ is a KP strong type. If $\equiv_L^\alpha \restriction K$ is not trivial, then it is non-smooth.*

For uncountable theories, the ambient type spaces are no longer Polish, so we get:

Theorem 4. *Suppose T is a complete first-order theory, α a small ordinal, $Y \subseteq \mathfrak{C}^\alpha$ is either G_δ or closed (i.e., $\bigwedge$-definable) and Y is closed under $\equiv_L^\alpha$. If for some $a \in Y$, there is no bound on d_L restricted to the class $[a]_{\equiv_L^\alpha}$ then $|Y/{\equiv_L^\alpha}| \geq 2^{\aleph_0}$.*

To prove it we use the notion of a (strong) Choquet space, and in factKPLN allow Y to be any "strong Choquet" space (this includes closed and G_δ sets). From this theorem we can recover Fact 1.

Problem 5. What are the possible Borel cardinalities of $\equiv_L^\alpha$? Note that by definition, this relation is K_σ (a countable union of compacts), so by [Kan08, Theorem 6.6.1] it is Borel reducible to ℓ^∞ (this is an equivalence relation on $\mathbb{R}^\omega$, where a and b are ℓ^∞-equivalent if there is some $k \in \mathbb{N}$ such that $|a(i) - b(i)| < k$ for all $i < \omega$).

See [KMS13] for the full paper.

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Algebraic dynamics and difference fields

Zoé Chatzidakis

(Most of this is joint work with E. Hrushovski, [2, 3], and was partially supported by ANR-06-BLAN-0183)

Algebraic dynamics. An *algebraic dynamics* is a pair (V, ϕ) where V is a (quasi-projective, irreducible) variety, and $\phi : V \to V$ is a rational dominant map (i.e., locally, it is given by tuples of rational functions; it is defined outside some Zariski close set, and $\phi(V)$ is Zariski dense in V).

Assume that (V, ϕ) is defined over the field k, and consider the function field $k(V)$ of V. Thinking of the elements of $k(V)$ as of functions on V which are defined almost everywhere, we obtain a map $\phi^* : k(V) \to k(V)$, $f \mapsto f\phi$. As ϕ is dominant, this map is an injection of $k(V)$ into itself, and $[k(V) : \phi^* k(V)]$ is finite, called the *degree of* ϕ.

We work in some large algebraically closed field Ω, over which everything is defined. Algebraic dynamics form a category, in which morphisms from (V, ϕ) to (W, ψ) are given by rational maps $h : V \to W$ such that $\psi f = g\phi$.

Difference fields. A *difference field* is a field K endowed with an endomorphism σ (which is necessarily injective, but not necessarily onto). Examples are (K, σ) where σ is any automorphism of K, for instance the Frobenius map $x \mapsto x^p$ if the characteristic is $p > 0$. And ... $(k(V), \phi^*)$ when (V, ϕ) is an algebraic dynamics, and ϕ^* is defined as above. Note that ϕ^* is the identity on K.

Conversely, if (L, σ) is a difference field, which is finitely generated (as a field) over some subfield k on which σ is the identity, then $(L, \sigma) \simeq (k(V), \phi^*)$ for some algebraic dynamics (V, ϕ) as above. Indeed, let a be a finite tuple such that $L = k(a)$, and let V be the algebraic (affine) variety defined over k of which a is a generic (so that $k(a)$ is naturally isomorphic to $k(V)$). Since L is closed under σ, we have $\sigma(a) \in k(a)$, and for some tuple f of rational functions over k, we have $\sigma(a) = f(a)$. As $k(a) \simeq k(\sigma(a))$, $\sigma(a)$ is also a generic point of V, so that the tuple f induces naturally a rational dominant map $\phi : V \to V$.

The original question. Let (V, ϕ) as above be defined over the field $K = k(t)$, t transcendental over k. For each natural number n, let us denote by $\phi^{(n)}$ the n-time iterate of the function ϕ, and let us denote by K_n the set of elements of $k(t)$ which can be represented by quotients of polynomials of degree $\leq n$. Assume that for some $N > 0$, for every $m > 0$, the set

$$V(K_N) \cap \phi^{-1} V(K_N) \cap \cdots \cap \phi^{-(m)} V(K_N)$$

is Zariski dense in V. What can one say about (V, ϕ)?

Answer M. Baker ([1]): If $\deg \phi > 1$, $V = \mathbb{P}^1_K$, then $(V, \phi) \simeq (V_0, \phi_0)$, where (V, ϕ_0) is defined over k.

Crucial reduction. Under the above hypotheses, there is a positive integer ℓ, an algebraic dynamics (W, ψ) defined over k, and a dominant map $g : (W, \psi) \to (V, \phi^{(\ell)})$.

Explanation: K_N is bijectively isomorphic (via a map π) to a definable subset of k^{2N+2}, and the map ϕ induces a map Φ which sends $V(K_N)$ to $V(K_{N'})$, where $N' - N$ is the sup of the degrees of the rational maps defining ϕ. For each m, we set

$$S_m = \pi^{-1}(V(K_N) \cap \phi^{-1} V(K_N) \cap \cdots \cap \phi^{-(m)} V(K_N));$$

we know that each S_m is infinite, and they form a decreasing sequence. If S is the intersection of the Zariski closures of the S_m's, then S is infinite, and Φ restricts to a dominant map $S \to S$. This map permutes the irreducible components of S, and if W is a component of maximal dimension of S and whose image under π is Zariski dense in V, we take ℓ such that $\Phi^{(\ell)}$ stabilises W. The constructible map π then induces a rational dominant map $g : W \to V$, and $g\Phi^{(\ell)} = \phi^{(\ell)}g$ as desired.

New question. Let $K = k(t)$, (V, ϕ) and (W, ψ) algebraic dynamics, $g : (W, \psi) \to (V, \phi)$ a dominant map, and we suppose that everything is defined over K, and in addition (W, ψ) is defined over k.

What can we say about (V, ϕ)?

Answer: (V, ϕ) has a quotient (V_0, ϕ_0) defined over k, with deg $\phi_0 =$ deg ϕ.

Remark: this does not say much when deg $\phi = 1$, since one can then take the dominant map to be a constant map and $\dim(V_0) = 0$. However, when deg $\phi > 1$, the variety V_0 must be of positive dimension.

Translation in terms of difference fields.
Let a be a generic of W over K (in some large algebraically closed field), and let $b = g(a)$. Then b is a generic of V. Define the endomorphism σ of $K(a)$ by letting σ be the identity on K, and $\sigma(a) = \psi(a)$; then $\sigma(b) = \phi(b)$. We want to find some $c \in K(b)$ such that $\sigma(c) \in k(c)$ and $c \perp_k K$ ($k(c)$ is free from K over k). This will be given by (an adaptation of the proof of) the following model-theoretic result:

Theorem (T a complete supersimple theory, having the CBP, and with $T = T^{eq}$; finite rank). Let $B_0 \subset B$ be such that $tp(B_0/B)$ is almost-$\mathcal{S}$-internal for some set $\mathcal{S}$ of rank 1 types. Let a_1 be a tuple, $a_1 \perp_{B_0} B$, and let a_2 be a tuple such that $a_2 \in \operatorname{acl}(Ba_1)$. Then there is $e \in \operatorname{dcl}(Ba_2)$ such that $e \perp_{B_0} B$ and $tp(a_2/B_0e)$ is almost-$\mathcal{S}$-internal.

Idea of the proof. If it were true that dcl coincided with "the difference field generated by", this would give us almost directly the result: take $B_0 = k$, $B = k(t)$, $\mathcal{S}$ the generic type of $\operatorname{Fix}(\sigma)$, and $a_1 = a$ the generic of W, $a_2 = b = g(a_1)$ the generic of V. The tuple e above, if chosen well, will satisfy $\sigma(e) \in k(e)$, and will give us the desired (V_0, ϕ_0). The assertion on the degree of ϕ_0 comes from the fact that if $L \subset M$ with $tp(M/L)$ almost-$\mathcal{S}$-internal, then the limit degree of the extension is 1 (i.e., $M = L\sigma(M)$).

But this is false: dcl is in general much bigger than "the difference field" generated by. Some tricks need to be used, which also give the result for (V, ϕ) when the dominating map sends (W, ψ) to $(V, \phi^{(\ell)})$ with $\ell > 1$.

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Extendable automorphisms

Gilbert Levitt
(joint work with Vincent Guirardel)

Rather than looking for subgroups invariant (up to conjugacy) under a given automorphism of a group G, one can fix a subgroup $H \subset G$ and consider automorphisms leaving H invariant.

Define $\mathrm{Out}(H \upharpoonleft G) \subset \mathrm{Out}(H)$ as the group of outer automorphisms of H which extend to G. A general question (that we heard from D. Calegari at an Oberwolfach meeting) is to understand $\mathrm{Out}(H \upharpoonleft G)$: what does it contain? is it finitely generated? finitely presented?...

If H is a vertex group G_v in a graph of groups decomposition of G, it is easy to see that $\mathrm{Out}(H \upharpoonleft G)$ contains $\mathrm{Out}(H; \mathrm{Inc}_v)$, the group of automorphisms of H which act trivially (i.e. as an inner automorphism of H) on each incident edge group.

When H is a malnormal subgroup of a free group F_n, this construction accounts for almost all of $\mathrm{Out}(H \upharpoonleft G)$:

Theorem 1. Let H be a finitely generated malnormal subgroup of F_n. If $\mathrm{Out}(H \upharpoonleft F_n)$ is infinite, then H is a vertex group in a graph of groups decomposition of G with finitely generated edge groups. Moreover, $\mathrm{Out}(H; \mathrm{Inc}_v)$ has finite index in $\mathrm{Out}(H \upharpoonleft F_n)$.

It follows that $\mathrm{Out}(H \upharpoonleft F_n)$ is finitely presented (it has a finite index subgroup with a finite classifying space).

The theorem also holds when H is a malnormal quasiconvex subgroup of a hyperbolic group G (this ensures that G is hyperbolic relative to H).

When H is not malnormal, one can still prove:

Theorem 2. Let $H \subset F_n$ be finitely generated, not cyclic. If every automorphism of H extends to an automorphism of F_n, then H is a free factor.

I sketched the proof of Theorem 2 when $H = \langle a, b \rangle$ has rank 2. In this case H is a free factor provided that the automorphism α sending a to ab and b to bab extends to an automorphism $\bar{\alpha}$ of F_n. The key property of α is that powers of the commutator $\gamma = aba^{-1}b^{-1}$ are the only periodic conjugacy classes.

The main tool in the proof is the canonical cyclic JSJ decomposition of F_n relative to γ. Assuming that H is not contained in a proper free factor, this is an $\bar{\alpha}$-invariant graph of groups Γ, with infinite cyclic edge groups and γ contained in a vertex group. Moreover, vertex groups are cyclic, surface groups (quadratically hanging), or rigid.

The choice of α ensures that H is contained in a vertex group. One then shows that this vertex group is of surface type, and that F_n is an amalgam $H *_{\langle \gamma \rangle} H'$. It follows that H is a free factor.

VC*-density and (p,q)-theorems

Sergei Starchenko

In this talk we survey VC-density and its relation with (p,q)-theorems.

We fix a complete first order theory T. Let $\varphi(x;y)$ be a partitioned formula. For $\mathcal{M} \models T$ and $A \subseteq M^{\ell(y)}$ let $S_\varphi(A)$ be the set of all complete φ-types over A: maximal consistent subsets of $\{\varphi(x,a) \colon a \in A\} \cup \{\neg\varphi(x,a) \colon a \in A\}$.

Let $\pi^*_\varphi \colon \mathbb{N} \to \mathbb{N}$ be the function defined as

$$\pi^*_\varphi(n) = \max\{|S_\varphi(A)| \colon A \subseteq M^{\ell(y)}, |A| = n\}.$$

By Sauer-Shelah dichotomy we have that either $\pi^*_\varphi(n) = 2^n$ for all n or $\pi^*_\varphi(n) \leq Cn^d$ for some d.

A complete theory T has NIP if for any formula $\varphi(x;y)$ there is $d \in \mathbb{N}$ such that $|S_\varphi(A)| < 2^d$ for any $A \subseteq M^{\ell(y)}$.

Assume T has NIP. For a formula $\varphi(x;y)$ and $r \in \mathbb{R}$ we write $vc^*(\varphi) \leq r$ if there is $C \in \mathbb{R}$ with $|S_\varphi(A)| \leq C|A|^r$ for all finite $A \subseteq M^{\ell(y)}$. We define the dual VC-density of $\varphi(x;y)$ as

$$vc^*(\varphi) = \inf\{r \in \mathbb{R} \colon vc^*(\varphi) \leq r\}.$$

In general it is not easy to compute vc^*-densities.

Theorem 0.1.

(1) *Let T be a weakly o-minimal theory. Then $vc^*(\varphi(x;y)) \leq \ell(x)$.*

(2) *Let T be a theory of finite U-rank without f.c.p. Then $vc^*(\varphi(x;y)) \leq \ell(x)U(T)$.*
In particular, if T is $\aleph_1$-categorical then $vc^*(\varphi(x;y)) \leq \ell(x)RM(T)$.

Many open questions.

Conjecture. If $T = Th(\mathbb{Q}_p)$ or $T = ACVF_{(p,q)}$ then $vc^*(\varphi(x;y)) \leq \ell(x)$.

The best known bound for $\mathbb{Q}_p$ is $vc^*(\varphi(x;y)) \leq 2\ell(x) - 1$, and the best known bound for $ACF_{(0,0)}$ is $vc^*(\varphi(x;y)) \leq 2\ell(x)$.

Open question. Assume T has NIP. Assume there is k such that for all $\varphi(x;y)$ with $\ell(x) = 1$ we have $vc^*(\varphi) \leq k$.
Is it true that for all n there is $k(n)$ such that $vc^*(\varphi(x;y)) \leq k(\ell(x))$?

Conjecture Let T be a VC-minimal theory. Then $vc^*(\varphi(x;y)) \leq \ell(x)$.

Theorem 0.2 (V.Guingona)**.** *If T is VC-minimal and $\varphi(x;y)$ a formula with $\ell(x) \leq 2$ then $vc^*(\varphi) \leq 2$.*

1. VC*-density and forking

Observation Assume T has NIP. Let $\varphi(x;y)$ be a formula and $k \in \mathbb{N}$ with $k > vc^*(\varphi)$. Let $\vec{a} = (a_i)_{i\in\omega}$ be an indiscernible sequence. If the family $\{\varphi(x;a_i) \colon i \in \omega\}$ is k-consistent then it is consistent.

From Forking=Dividing (Chernikov, Kaplan) we obtain:

Claim 1.1. *Assume T has NIP. Let $\mathbb{U}$ be a large saturated model of T and $\mathcal{M} \prec \mathbb{U}$ a smal substructure. Let $\varphi(x;y)$ be a formula and $k \in \mathbb{N}$ with $k > vc^*(\varphi)$.*

For $a \in \mathbb{U}$ the following are equivalent.

(1) *$\varphi(x;a)$ does not fork over M.*
(2) *The family $\{\varphi(x;a') \colon a' \equiv_M a\}$ is consistent.*
(3) *The family $\{\varphi(x;a') \colon a' \equiv_M a\}$ is k-consistent.*

1.1. On (p,q)-Theorem.

Theorem 1.2 ((k,k)-theorem, Matoušek)**.** *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a VC-class and $k \in \mathbb{N}$ with $k > vc^*(\mathcal{F})$. Then there is $t \in \mathbb{N}$ such that for any finite k-consistent $\mathcal{F}_0 \subseteq \mathcal{F}$ there are $p_1, \ldots, p_t \in X$ such that every $F \in \mathcal{F}_0$ contains at least one p_i.*

Corollary 1.3 (A form of weak f.c.p.)**.** *Assume T has NIP. Let $\varphi(x;y)$ be a formula and $k \in \mathbb{N}$ with $k > vc^*(\varphi)$. Then there is $t \in \mathbb{N}$ such that for any set A if the family $\{\varphi(x,a) \colon a \in A\}$ is k-consistent then the family*

$$\left\{ \bigvee_{i=1}^{t} \varphi(x_i;a) \colon a \in A \right\}$$

is finitely consistent.

Observation. Let X be a compact topological space. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a VC-class of closed subsets of X, and $k \in \mathbb{N}$ with $k > vc^*(\mathcal{F})$. Then there is $t \in \mathbb{N}$ such that for any k-consistent $\mathcal{F}_0 \subseteq \mathcal{F}$ there are $p_1, \ldots, p_t \in X$ such that every $F \in \mathcal{F}_0$ contains at least one p_i.

Using the above observation we obtained the following theorem.

Theorem 1.4. *Assume T has NIP. Let $\mathbb{U}$ be a large saturated model of T and $\mathcal{M} \prec \mathbb{U}$ a small substructure. Let $\varphi(x;y)$ and $\theta(y)$ be formulas over $\mathcal{M}$ and $k \in \mathbb{N}$ with $k > vc^*(\varphi)$. Assume the family $\{\varphi(x;a) \colon a \in \theta(M)\}$ is k-consistent. Then there are $\mathcal{M}$-invariant types $p_1, \ldots, p_t \in S_x(\mathbb{U})$ such that every $\varphi(x;a)$, for $a \in \theta(\mathbb{U})$, is contained in one of p_i.*

Open question. Do we have a definable (k,k)-theorem?

Assume T has NIP. Let $\varphi(x,y)$ be a formula and $d \in \mathbb{N}$ with $d > vc^*(\varphi)$. Let $\theta(y)$ be a formula such that the family $\{\varphi(x;a) \colon \models \theta(a)\}$ is d-consistent. Can we find formulas $\theta_i(y), i = 1, \ldots, t$ such that $\theta(y) \to \bigvee \theta_i(y)$, and each family $\{\varphi(x;a) \colon \models \theta_i(a)\}$ is consistent?

By logical compactness, the above question is equivalent to the following.

Assume $\varphi(x;a)$ does not fork over $\mathcal{M}$. Can we find $\theta(y) \in tp(a/M)$ such that the family $\{\varphi(y;b) \colon b \in \theta(M)\}$ is consistent.

Theorem 1.5 (Simon)**.** *Assume T is dp-minimal. Then definable (k,k)-theorem holds for $\varphi(x,y)$ with $\ell(x) \leq 2$.*

Invariant types in NIP structures

Pierre Simon

In this talk, I presented some constructions and results concerning invariant types in NIP theories. The underlying hope of this work is that one can analyze invariant types in terms of definable and finitely satisfiable ones.

A central notion here is that of *commuting* types. Two invariant types p and q commute if realizing first p then the invariant extension of q is the same as realizing first q and then the invariant extension of p. It is well known that a finitely satisfiable type and a definable one always commute. The converse is also true: if an invariant type commutes to all finitely satisfiable types, then it is definable. Shelah has shown in [2] that a general type over a saturated NIP structure can be decomposed into a finitely satisfiable one and a *directed* quotient. Putting those two facts together, one can hope to decompose an invariant type in an NIP theory into a finitely satisfiable component, and a definable quotient.

This strategy works at least in the case of dp-minimal types, where actually the decomposition collapses to one extreme or the other. We prove that an invariant dp-minimal type is either finitely satisfiable or definable. This theorem is stated in [1], but I presented a different proof during the talk which involves defining two functions that map invariant types to finitely satisfiable ones.

From now on, we assume NIP. Let $M \prec N$, N is $|M|^+$-saturated. Let p be an M-invariant type. One can define two M-finitely satisfiable types $R_M(p)$ and $F_M(p)$ as follows:

$R_N(p)$ is taken to be the eventual type (over $\mathbb{C}$) of Morley sequences of p lying inside N.

To define $F_N(p)$, one first works in a pair (N', N) where N' is $|N|^+$-saturated. Then consider a saturated elementary extension (N'_1, N_1). The type $F_N(p)$ is defined as the eventual type over N' of Morley sequences of p lying inside N_1.

The intuition is that p has some N-finitely satisfiable part and that this part is maintained in $F_N(p)$ and somehow reversed in $R_N(p)$. It is always the case that p and $R_N(p)$ commute. The key to proving the aforementioned theorem is to determine what it means for $F_N(p)$ to commute with either p or $R_N(p)$. (One should think of commuting types as being somehow 'far away', and in particular as having no finitely-satisfiable part in common).

The following properties hold:

$F_N(p)$ commutes with p if and only if p is definable, if and only if $F_N(p) = R_N(p)$.

$F_N(p)$ commutes with $R_N(p)$ if and only if p is finitely satisfiable, if and only if $F_N(p) = p$.

To prove the theorem, we now only need to observe that if q, r are invariant commuting types and p is dp-minimal, then p must commute with either q or r.

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Groupes lineaires de rang de Morley fini

Bruno Poizat

Suivant la problèmatique de la Conjecture de Cherlin-Zilber, beaucoup d'ènergie a été dépensée ces dernières années en vue de montrer l'existence, dans un quelconque groupe de rang de Morley fini, de sous-groupes définissables connexes nilpotents généreux (comme sont les tores maximaux d'un groupe algébrique simple).

Nous montrons que c'est vrai pour les groupes de rang de Morley fini linéaires, c'est-à-dire ceux qui se plongent, pour un certain entier n, dans $GL_n(K)$, où K est un corps qu'on peut supposer gratuitement algébriquement clos. Nous commençons par montrer que, si le groupe est en outre simple (ou, plus généralement, s'il n'a pas de sous-groupe normal unipotent non-trivial) les centralisateurs de ses points génériques sont des groupes abéliens divisibles et généreux, qui sont tous conjugués.

Nous rappelons les définitions suivantes :

- une partie A du groupe G est dite générique si G est recouvert par un nombre fini de translatés de A; autrement dit $G = a_1 \cdot A \cdot b_1 \cup \cdots \cup a_n \cdot A \cdot b_n$; si A est définissable et si G est de rang de Morley fini, cela signifie que A a même rang que G;
- une partie A du groupe G est dite généreuse si la réunion de ses conjuguées est générique.

Notre méthode d'investigation repose sur deux types d'ingrédients :

- des propriétés bien connues des groupes algébriques
- la théorie de Wagner et de Nevelski de la généricité pour les sous-groupes non-définissables d'un groupe stable, qui permet de transférer à un groupe linéaire certaines des propriétés de sa clôture de Zariski.

Nous rendons compte ici de cette théorie dans le cadre qui nous est utile. La clôture de Zariski d'un sous-groupe H d'un groupe algébrique est le plus petit sous-groupe de ce dernier contenant H qui soit définissable (les géomètres disent constructible) au sens de la théorie du corps algébriquement clos K.

Lemme de Wagner. *Soit G un groupe algébrique, et soit H un sous-groupe de G dont il soit la clôture de Zariski; alors tout type générique de $S_1(G)$ est finiment satisfaisable dans H.*

Corollaire 1. *Sous les hypothèses du lemme précédent, si A est une partie constructible générique de G, on peut trouver un uplet $a_0, \ldots a_n$ d'éléments de H tels que $G = a_0 \cdot A \cup \cdots \cup a_n \cdot A$.*

Corollaire 2. *Sous ces mêmes hypothèses, et si A est une partie constructible de G, A est générique dans G si et seulement si $B = A \cup H$ est générique dans H.*

Corollaire 3. *On reprend les hypothèses du Lemme de Wagner en supposant en outre que H est un groupe de rang de Morley fini, qui soit connexe relative-ment à sa propre théorie, et que la paire (G, H) est suffisamment saturée. Alors on peut trouver g dans $H \cup G^o$ qui soit générique (sur $\emptyset$, ou sur un ensemble convenu de paramètres dans H) à la fois au sens de G et au sens de H.*

On considère alors un groupe infini simple H de rang de Morley fini, qui soit linéaire: pour un certain entier n, H est un sous-groupe de $GL_n(K)$, où K est un corps algébriquement clos. Nous notons G la clôture de Zariski de H dans $GL_n(K)$. En minimisant sa dimension, on se ramène au cas où G est un groupe algébrique simple. Quitte à la remplacer par une extension élémentaire, nous pouvons supposer que la paire (G, H) est autant saturée qu'on veut, si bien que nous trouvons un point g de H qui est générique à la fois au sens de G et au sens de H. Il est bien connu que le centralisateur T de g dans G est un tore connexe, soit encore un groupe commutatif divisible, qui est généreux dans G. Nous montrons alors, grâce à des calculs de rang à la Jaligot, que le centralisateur de g dans H, qui est l'intersection de H et de T, est généreux dans H, et qu'il est connexe au sens de la théorie de H. Cette démonstration fonctionne en fait quant G est seulement réductif; on en conclut que tout groupe linéaire H de rang de Morley fini possède un sous-groupe définissable connexe résoluble et généreux, puis, par transitivité, un tel sous-groupe nilpotent, car cela est vrai pour tout groupe résoluble de rang de Morley fini d'après les travaux de Frécon et Jaligot.

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Approximate equivalence relations

Ehud Hrushovski

1. Preliminaries; probability logic

Given a graph (Ω, R), we define a *metric* $d_R(x,y) = \min n.\ (\exists x = x_1, \dots, x_n = y, R(x_i, x_{i+1}))$.

Let μ be the normalized counting measure $\mu(X) = |X|/d_\Gamma$. $\mu(\Omega)$ will generally be infinite, but we assume balls have finite measure. We similarly have compatible measures μ_m on Ω^m; in the ultraproduct, they will usually measure a bigger algebra than the measure algebra.

We will use *probability logic*, see [8]. The most convenient form is within continuous logic, cf. e.g. [1]: relations are viewed as real-valued, Boolean connectives $\{0,1\}^n \to \{0,1\}$ are replaced by continuous functions $[0,1]^n \to [0,1]$. In probability logic, quantifiers are further replaced by integral operators: $\phi(x,y) \mapsto I_y\phi(x,y)$, intended to denote the μ_y-measure of $\{y : \phi(x,y)\}$. A formula $I_{y_1} \cdots I_{y_n}\phi$, with ϕ quantifier-free, is said to be in normal form. A quantifier-elimination result of Hoover (1982) asserts (in the present setting) that any formula can be approximated by one in normal form. [1] Hoover's theorem can be proved as follows: the normal form sentences of a theory T induce a probability distribution on the space of L-structures on $\mathbb{N}$; a random L-structure on $\mathbb{N}$ is then a model of all of T; moreover, probability agrees with frequency ($I_y\phi(x,y) = \lim_n \frac{1}{n}|\{i \leq n : \phi(x,a_i)\}$).

In the case of *complete separable metric spaces* A, with the distance viewed as a real-valued relation, the completion of any random L-structure will be isomorphic to the support of the measure on A; this is Vershik's proof of Gromov's reconstruction theorem for measured-metric spaces, see [5], [10].

In the case of *finite graphs*, with normalized counting measures, the (normal form) theory of a graph Ω can be described as follows. Let $Gr(m)$ be the set of graphs on $m+1$ vertices. Given $a \in \Omega$, and $\gamma \in Gr(m)$, let $C(\gamma, a)$ be the set of graph embeddings $\gamma \to \Omega$ with $0 \mapsto a$. Define the *local statistics function* $LS_m : \Omega \to [0,1]^{Gr(m)}$: $LS_m(a)(\gamma) = \mu_m(C(\gamma,a)$. Compare [2] in the case of sentences, [3] for formulas.

Definition 1.1. *(Ω, R) is m, ϵ-*homogeneous *if the range of LS_m is concentrated in an ϵ-ball (for sup metric on $\mathbb{R}^N$.)*

2. The stabilizer theorem for equivalence relations

We will be interested in graphs of large finite degree, and larger, or infinite diameter; our main results will be *modulo* bounded degree graphs.

Say two metrics d, d' are *commensurable at scale* α if an α-ball of d' is contained in finitely many α- balls of d, and vice versa.

[1] As we only assume balls have finite measure, we require a slight modification where only formulas of radius 1 can be measured by a quantifier; $I_y(\phi(x,y)\&R_1(x_1,y))$ would be a typical case.

A metric space is *k-doubling at scale α* if $d, (1/2)d$ are commensurable at scale α.

Let R be a graph, with a measure assigning finite measure to balls. R is a *k-approximate equivalence relation* if

- d_R-balls of radius 1 have measure of a a fixed order of magnitude d_Γ, $(1/k)d_\Gamma \leq \mu(B_1(a)) \leq kd_\Gamma$.
- d_R is k-doubling at scale 1: every 2-ball is a union of k 1-balls.

An ϵ- *slice* is a set Z whose intersection with every 1-ball has measure $< \epsilon$.

Theorem 2.1 (Stabilizer theorem). *Let R be a k- approximate equivalence relation. Then there exists a graph S on the same set of vertices, such that $S^{\circ 8} \subset R^{\circ 4}$, and for all $a \in \Omega$ outside an ϵ-slice U, $|S(a)| \geq O_k(1)|R(a)|$.*

Moreover S is 0-definable, uniformly in (Ω, R), in an appropriate logic; in particular $Aut(\Omega, R)$ leaves U, S invariant.

This generalizes the stabilizer theorem of [7] and of Sanders-Croot-Sisask, cf. [9]. Namely, given a subset X of a group G, define $R_X(x, y) \iff xy^{-1} \in X$. Then X is an *approximate subgroup* of G iff R_X is an approximate equiv. relation. *canonical* statements about R, such as the theorem above, translate back to statements about X.

sketch of proof:

- xS_ny iff $\mu\{z : |\mu(R(x)\triangle R(z)) - \mu(R(y)\triangle R(z))| \geq 2^{-n}\} \leq 2^{-n}\}$
- At limit, $\cap_n S_n$: for almost all z, $\mu(R(x)\triangle R(z)) = \mu(R(y)\triangle R(z))$. It is cobounded.
- $S_{n+1} \circ S_{n+1} \subset S_n$. (Away from measure 0).
- $S_n \subset R^{\circ 4}$, for large n.
- S_n is definable in terms of R using probability logic . This definability will be essential, showing that (approximate) symmetries of the graph, are (approximate) symmetries of the associated refining metric.
- The proof uses stability: $\mu(R(x)\triangle R(z))$ is a stable real-valued formula.

A locally compact limit.

Proposition 2.2. *Take an ultraproduct (Ω, R) of k-approximate equivalence relations. Define a finer metric d: $d(x, y) = 2^{-m}$ if $S_m(x, y)$ but not $S_{m+1}(x, y)$. Factor out the equivalence relation: $d(x, y)$ infinitesimal. Then (Ω, R) is a disjoint union of components; each is a* locally compact metric space Y*, with a* locally finite measure.

Proposition 2.3. *Let (Ω, μ, R) be an approximate equivalence relation, with respect to a measure μ. Then up to measure* 0*, the completion with respect to d is determined by the local statistics of Ω. Moreover, if $a, b \in \Omega$ and $LS(a) = LS(b)$ then there exists an isometry with $a \mapsto b$.*

Proof. (Vershik style). Suppose (Ω', μ', R') has the same local statistics. Let (a_n) be a random sequence in Ω, and (b_n) a random sequence in Ω', with $R(a_i, a_j) \iff R'(b_i, b_j)$. Then the map $a_n \to b_n$ is an isomorphism preserving not only R, but also any probability-logic definable relation; in particular S_n. Hence it is an isometry; extend it to the completion. The same holds in the pointed case; in particular for (Ω, a) vs. (Ω, b). $\square$

3. Sharpening the focus

We managed to raise the resolution of a metric given at scale 1; but at finer scales, we lost sight of the doubling property. We next aim to show that assuming approximate symmetry, we can maintain doubling at arbitrarily fine scales.

A metric $d : \Omega^2 \to \mathbb{N}$ *admits a fine structure of dimension e, scale s, distortion c* if there exists a pseudo-metric [2] $d' : \Omega^2 \to 2^{-s}\mathbb{N}$, such that

- The 2^e-doubling condition holds at every scale $2^{-s}, \ldots, 1$.
- d, d' are c-commensurable at scale 1, up to a $1/c$-slice.

Fix a degree of approximateness K, also a fast growing function Ψ.

Theorem 3.1. *For some $c, e \in \mathbb{N}$, for any K- approximate equivalence relation (X, R), the fibers of $LS : X \to \mathbb{R}^N$ admit a fine structure of dimension $\leq c$, , distortion $\leq e$, and scale $\Psi(c + e)$.*

While there are no groups in hypothesis or conclusion, the proof uses group theory (Peter-Weyl, Gleason, Yamabe, ...).

Corollary 3.2. *For some $c \in \mathbb{N}$, for any $(c, 1/c)$-homogeneous K- approximate equivalence relation (X, R) admits a fine structure of dimension $\leq c$,distortion $\leq c$, and scale $\Psi(c)$. In fact, any sequence of increasingly homogeneous K- approximate equivalence relation has a subsequence approaching (after distortion) a Riemannian homogeneous space.*

Compare [4], Theorem 2, in the case of the circle.

Proof

- Ultraproduct. Obtain two equivalence relations: $\widetilde{E}$ = finite distance. Γ = infinitesimal distance.
- Let Ω be a class of $\widetilde{E}$; then Ω/Γ is locally compact.
- $G := Aut(\Omega/\Gamma)$ acts transitively on Ω, by isometries of the fine metric. Keisler,Gromov-Vershik,
- A locally compact structure on G (compact-open topology.) The stabilizer of a point is compact.
- By Gleason-Yamabe, an open subgroup H, a small normal compact subgroup N, with H/N a Lie group.
- From Ω to an H-orbit: locally bounded distortion. (R induces a graph of bounded degree on Ω/H.)
- Factor out N. Obtain a coarser equivalence relation than the original distance-zero, but still contained in $d_R \leq 4$.

[2] we allow $d(x, y) = 0$ without $x = y$; in other words we factor out a (precise) equivalence relation, contained entirely in $R^{\circ 4}$. SImilarly we allow $d(x, y) = \infty$, thus fibering over a bounded valence graph.

- Now the Lie group H/N acts transitively on Ω/Γ, compact point stabilizer. Find an invariant Riemannian metric. This metric is doubling up to distance 1, and the “distance -1” relation is commensurable with d_R.
- Return information to finite factors, up to scale $\Psi(c)$.

It is to be hoped that stronger Riemannian statements, for instance of Bishop-Gromov type, can also be brought down.

4. Lovász-Szegedy and the NIP route to finite dimensionality

Recall our locally compact limit Y, constructed at the end of §1. When $|\Omega| = O(d_\Gamma)$, this is precisely the space that Lovász-Szegedi call the *graphon.* (They use the L_1 metric: $d(x,y) = |\mu(R(x)\triangle R(z)) - \mu(R(y)\triangle R(z)).)$ (See [2], [6]).

Whereas we used a doubling condition, approximate symmetry and Gleason-Yamabe to obtain finite dimensionality, [6] obtain a similar result under an assumption of NIP. Recall: $NIP(R) \leq k$ if there is no set Y of size k such that every subset fo Y has the form $R(b) \cap Y$, for some b. A metric space has *packing dimension* $\leq r$, if for large N, one cannot find N^r disjoint $1/N$-balls. Lovász-Szegedi show: if R has NIP, the associated locally compact space has *finite packing dimension* r.

Definition 4.1. *A* nearly-Lie group *is a connected locally compact group G, with a profinite subgroup H, and G/H Lie. Note that nearly Lie groups are topologically finitely generated.*

Restoring the approximate symmetry assumption, we obtain a (NIP) version of Corollary 3.2 without distortion.

Theorem 4.2. *A NIP, k-approximate, increasingly symmetric sequence of graphs approaches a homogeneous space for a nearly Lie group, fibered over a graph of bounded valency.*

We also obtain an answer to a question raised earlier by Pillay.

Theorem 4.3. *Let G be a definable group in a NIP theory. Assume μ is a left-invariant generically stable measure on G. Let $\phi(x,y)$ be a formula, and let G^{00}_ϕ be the μ-stabilizer of ϕ: $g \in G^{00}_\phi \iff \mu(\phi(x,a)\Delta\phi(gx,a)) = 0$ for all a. Then G^0/G^{00}_ϕ is a nearly Lie group.*

It is natural to ask: given a saturated model M of a NIP theory and a formula $\phi(x,y)$, does there exist a homomorphism $h : Aut(M) \to G$ into a nearly-Lie group, whose kernel fixes all ϕ-types over M that do not fork over $\emptyset$?

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The structure of approximate groups

Emmanuel Breuillard
(joint work with Ben Green and Terence Tao)

In the talk, I discussed the notion of *approximate subgroup*. The concept was introduced a few years ago by T. Tao in [6] as a tool to tackle the so-called non-commutative Freiman problem. This problem can be described as asking for a rough classification of finite subsets A of an ambient group G with the property that the size of the product set $AA = \{a_1a_2|a_1, a_2 \in A\}$ is not much bigger that the size of A itself in that $|AA| \leq K|A|$, where $K \geq 1$ is a fixed parameter. Such sets are said to be of *doubling at most* K. Rough means that we will consider two such sets A and B to be essentially equivalent for the purpose of this classification if each set can cover the other set by few (left and right) translates of it, where few means a number less than a constant $f(K)$ depending on K only. The celebrated Freiman Theorem from the 1960's (see e.g. [4]) answered this problem in the case when $G = \mathbb{Z}$. For some historical background and a presentation of the non-commutative Freiman problem, we refer the reader to T.Tao's blog. Let $K \geq 1$ be a parameter.

Definition 1 (Approximate subgroup)**.** *A* K-approximate subgroup *of an ambient group G is a finite subset A of G such that A is symmetric (i.e. stable under inverse $A = A^{-1}$), contains the identity and verifies $AA \subset XA$ for some symmetric subset $X \subset G$ with $|X| \leq K$.*

Using various tools from additive number theory and the combinatorics of sumsets, T. Tao [6] reduced the non-commutative Freiman problem to the classification of approximate subgroups of a given ambient group.

Basic examples of approximate groups include finite groups, arithmetic progressions, or more generally nilpotent progressions.

Definition 2 (progression)**.** *Given an ambient group G, elements $e_1, \dots, e_r$ in G and integers $L_1, \dots, L_r > 0$, a* progression $P = P(e_1, \dots, e_r; L_1, \dots, L_r)$ *with generators $e_1, \dots, e_r$ and side lengths $L_1, \dots, L_r$ is defined as the subset of all elements g in G that can be written as a word in the letters $e_i^{\pm 1}$ using at most L_i letters e_i or e_i^{-1} for each $i = 1, \dots, r$. The number r is called the rank of the nilprogression.*

Definition 3 (nilprogression)**.** *A nilprogression of rank r and step s is a progression $P = P(e_1, \dots, e_r; L_1, \dots, L_r)$ in a given ambient group G, such that the subgroup generated by the e_i's is nilpotent of nilpotency class at most s.*

The above definition is from [2]. A related concept was introduced in [1]. The two notions are roughly equivalent in the sense given below.

Definition 4 (coset nilprogression)**.** *A coset nilprogression of rank r and step s is the inverse image of a nilprogression of rank r and step s under a group homomorphism with finite kernel.*

It is a rather simple matter to prove that coset nilprogressions of rank r and step s are K-approximate subgroups with a parameter K which depends only on r and s, and not on the side lengths L_i's, the generators e_i's, nor on the ambient group G.

In 2009, E. Hrushovksi wrote the paper [5] in which he addressed the general problem in an arbitrary group by tools that had not been considered before in this context as they pertain to model theory and stability theory in mathematical logic.

Making key use of the Gleason-Montgomerry-Zippin structure theory of locally compact groups, he was able to give the first general structure theorem valid for all approximate groups showing that they always exhibit some nilpotent behavior close to what B. Green and T. Sanders had been calling a *Bourgain system*. He also gave complete answers to the non-commutative Freiman problem in several non-trivial cases: for groups with bounded exponent (there every approximate group is roughly equivalent to a finite subgroup), for subgroups of GL_n over a field (there every approximate group is roughly equivalent to a solvable approximate group -by- a finite semisimple group), for finitely generated groups that are exhausted by an increasing union of K-approximate subgroups (they must be nilpotent-by-finite and this improves Gromov's well-known polynomial growth theorem). Hrushovski deduced all these properties from what Green, Tao and I now call the "Hrushovski Lie model theorem". This associates to any infinite sequence of K-approximate groups a certain canonically defined connected Lie group. For this result, we refer to Hrushovski's lecture notes available on his web-page, as well as the extended abstracts of Pierre Simon and Immanuel Halupczok corresponding to the talks they gave on Hrushovski's work on approximate groups at this workshop.

Recently together with B. Green and T. Tao, we managed to extend Hrushovski's methods in order to get a more complete description of an arbitrary approximate subgroup in an arbitrary group. We say that two subsets $A, A' \subset G$ are K'-roughly equivalent if

$$|A \cap A'| \geq \frac{1}{K'} \max\{|A|, |A'|\}$$

A simple argument (Ruzsa covering lemma) shows that if A, A' are K'-roughly equivalent, then A' can be covered by at most K' left translates of AA^{-1} and also by K' right translates of $A^{-1}A$.

We prove:

Theorem 1 ([2])**.** *Let A be a K-approximate subgroup in an ambient group G. Then A is $C(K)$-roughly equivalent to a coset nilprogression P of rank and nilpotency class $O(\log K)$. The constant $C(K)$ depends only K and not on A nor G.*

In fact we may choose P so that $P \subset A^D$, where $D = D(K)$. Moreover, we may even require P to belong to A^{12} at the expense of getting only a polynomial bound $O(K^C)$ on the rank and step of P. See [2].

In the talk, we described how one can deduce this theorem from Hrushovski's Lie model theorem by analysing the proof by Gleason and Yamabe of the structure theorem for locally compact groups (proved in the 1950's) and mimicking some of these techniques to make them work in the approximate group setting. These old arguments of Gleason and Yamabe are best understood using model theory, as was first spelled out by J. Hirshfeld. We refer the reader lecture notes by Goldbring an Van den Dries for the non-standard treatment of the theorems of Gleason and Yamabe and to the original paper [2] for a detailed description of our method.

Theorem 1 has several consequences. For example it gives another proof as well as a strengthening of Gromov's theorem on groups with polynomial growth. It also allows one to prove a conjecture of Gromov regarding a so-called "generalised Margulis lemma" about discrete subgroups of isometries in general metric spaces. Finally it has the following consequence, which is worth noting:

Corollary 1. *Any K-approximate subgroup A of a local group has a subset of size at least $|A|/C(K)$ which can be embedded in a global group.*

This is the approximate group analogue of a recent result of I. Goldbring [3], which answered a long standing basic problem regarding local groups by showing that the same holds for compact neighborhoods of the identity in arbitrary locally compact local groups.

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Motivic counting of points of bounded height

François Loeser

The aim of the talk was to present some recent results that can be seen as "motivic" versions of various results about counting points of bounded height in diophantine geometry. In a classical paper [1], Bombieri and Pila used the determinant method to give bounds for points of bounded height and transcendental and algebraic curves. This was later extended to higher dimensions by Pila and Wilkie [6] and Pila [5], respectively. We have presented work in progress with R. Cluckers and G. Comte on non-archimedean versions of such results. Our approach relies on a non-archimedean analogue of the Yomdin-Gromov Lemma [3]. In the second part of the talk, we have presented joint work with A. Chamber-Loir on a motivic analogue of his recent result with Y. Tschinkel on counting integral points of bounded height on partial equivariant compactifications of vector groups [2]. Our approach in based on the motivic Poisson summation of Hrushovski and Kazhdan [4].

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Hyperbolic towers in the free group

Rizos Sklinos
(joint work with Chloé Perin, Larsen Louder)

We use hyperbolic towers to answer some model theoretic questions around the generic type in the theory of free groups. We note that the notion of a hyperbolic tower was first introduced by Sela in his characterization of finitely generated groups that have the same elementary theory as a non abelian free group.

The understanding of the generic types of a stable group is of great importance in model theory. Towards this end one is able to see using a result of Poizat (connectedness of F_ω) that the free group has essentialy a unique generic type, which from now on we call p_0.

It was shown by Pillay and Sklinos (independently) that p_0 has infinite weight. Note that this is strictly stronger than the non-superstability of the free group. Moreover, Sklinos showed that p_0 is non-isolated. In other words, in the light of a result of Pillay, i.e. p_0 defines the set of primitives elements, our result is equivalent to saying that the set of primitives is not $\emptyset$-definable. As a matter of fact in this talk we show that no complete type over the empty set is isolated.

We show that all the finitely generated models of this theory realize the generic type p_0, but that there is a finitely generated model which omits $p_0^{(2)}$. We exhibit a finitely generated model in which there are two maximal independent sets of realizations of the generic type which have different cardinalities, i.e. the free product of $\mathbb{Z}$ with the fundamental group of the connected sum of two tori. We show that the fundamental group of the connected sum of two tori is homogeneous. Thus, we also exhibit a free product of homogeneous groups which is not homogeneous.

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Isometries in valued fields

Immanuel Halupczok

To any definable set $X \subseteq \mathbb{Z}_p^n$ in the p-adic integers, one can associate a *Poincaré series*, which is defined as follows. Let N_λ be the number of points of the image of X in $(\mathbb{Z}/p^\lambda\mathbb{Z})^n$. Then the Poincaré series is the formal sum

$$P_X(t) = \sum_{\lambda=0}^{\infty} N_\lambda t^\lambda \in \mathbb{Q}[[t]].$$

In the 80ies, Denef proved that this series is a rational function, i.e., $P_X(t) \in \mathbb{Q}(t)$. It is easy to verify that $P_X(t)$ only depends on the isometry type of X (with respect to the ultrametric maximum norm on $\mathbb{Z}_p^n$), so Denef's result can be regarded as a result about isometry classes of definable sets. However, the precise meaning of this result for the isometry classes is not very intuitive. The goal of these notes is to present a geometric description of definable sets up to isometry under the assumption that p is sufficiently big; that description will be precise enough to imply the result by Denef.

Our main result lives in Henselian valued fields K of characteristic $(0,0)$. (Recall that a valued field is called "Henselian" if it satisfies Hensel's Lemma, i.e., if zeros of polynomials in the residue field can be lifted to K.) To obtain a corresponding result in $\mathbb{Q}_p$ for big p, we will use a standard compactness argument. The result says that for every definable set $X \subseteq K^n$ there exists a *t-stratification*. Roughly, a t-stratification for X is a definable partition $K^n = S_0 \dot\cup \cdots \dot\cup S_n$, where S_i is of dimension i, which satisfies the following condition. Suppose that $d \le n$ and that $B \subseteq K^n$ is a ball with $B \cap (S_0 \cup \cdots \cup S_{d-1}) = \emptyset$. (We are still using the maximum metric, so a ball is the same as a cube.) Then there exists a definable isometry $f\colon B \to B$ such that $f(X \cap B) = B_0 \times X'$ where B_0 is the projection of B to K^d and X' is a definable subset of K^{n-d}. In other words, for each $d \le n$, away from a subset of dimension $< d$, X is "isometrically trivial in d dimensions".

Note that this is rather different from what happens in the archimedean world. For example, no open subset of the parabola $\{(x, x^2) \mid x \in \mathbb{R}\}$ is isometric to a straight line; however, writing $\mathcal{O}_K$ for the valuation ring of K, it is an easy computation to verify that the projection $\{(x, x^2) \mid x \in \mathcal{O}_K\} \to O_K$ to the first coordinate is an isometry.

The precise definition of a t-stratification is a bit stronger than what is written above. The following additional properties will be needed to deduce the rationality of Poincaré series.

(a) The same isometry $f\colon B \to B$ that sends $X \cap B$ to $B_0 \times X'$ also sends $S_i \cap B$ (for $i \ge d$) to $B_0 \times S_i'$ for some $S_i' \subseteq K^{n-d}$. One can deduce that the partition $S_d' \dot\cup \cdots \dot\cup S_n'$ of the projection of B to K^{n-d} is a t-stratification for X'.

(b) After possibly permuting coordinates, X' and S_i' can be chosen to be fibers of the projection $\pi\colon B \to B_0$. More precisely, we can fix any $a \in B_0$ and choose $X' := \{x \in K^{n-d} \mid (a, x) \in X\}$ and $S_i' := \{x \in K^{n-d} \mid (a, x) \in S_i\}$.

Now let me sketch how this implies the rationality of the Poincaré series. The main result holds uniformly for all K, so for any formula ϕ, we can apply compactness to obtain that there exists a t-stratification for $X := \phi(\mathbb{Q}_p^n)$ as soon as p is big enough.

Let us denote by $\mathcal{B}$ the set of all balls $B \subseteq \mathbb{Q}_p^n$ satisfying $B \cap S_0 = \emptyset$ and that are maximal with this property: for any $B' \supsetneq B$, we have $B' \cap S_0 \ne \emptyset$. Then $\mathbb{Q}_p^n$ is the disjoint union of all $B \in \mathcal{B}$ and S_0. Since S_0 is 0-dimensional and hence finite, this allows us to compute $P_X(t)$ from the series $P_{X \cap B}(t)$, where B runs over $\mathcal{B}$. Since $B \cap S_0 = \emptyset$, the definition of t-stratification yields an isometry $X \cap B \to B_0 \times X_B'$ for some $X_B' \subseteq K^{n-1}$ (and where B_0 is the projection of B to K) and since Poincaré series are not affected by isometries, we have $P_{X \cap B}(t) = P_{B_0 \times X_B'}(t)$. Now $P_{B_0 \times X_B'}(t)$ can easily be computed from $P_{X_B'}(t)$, and using induction, we can assume that $P_{X_B'}(t)$ is a rational function.

The last missing ingredient to obtain rationality of $P_X(t)$ is to understand how $P_{X_B'}(t)$ depends on B (when B runs through $\mathcal{B}$). Each set X_B' is definable individually but unfortunately, these sets are not definable uniformly in B. However, by property (b) above, we have a definable family $X_{B,a}'$ parametrized by

$\{(B,a) \mid B \in \mathcal{B}, a \in \pi(B)\}$. (Different $B \in \mathcal{B}$ might need different choices of the coordinate projection $\pi\colon K^n \to K$, but since there are only finitely many possibilities, this is not a problem.) Moreover, by (a) each set $X'_{B,a}$ comes with a t-stratification $S'_{1,B,a}, \dots, S'_{n,B,a}$, and this is uniform in (B,a). This allows us to deduce that the dependence of $P_{X'_{B,a}}(t)$ on (B,a) is also uniform in a suitable sense, which (together with the fact that $P_{X'_{B,a}}(t)$ does not depend on a) finally allows us to deduce rationality of $P_X(t)$.

One final remark: Our main result in Henselian valued fields resembles a result from real and complex geometry about the existence of *Whitney stratifications*, which describe singularities of subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$. In fact, our result actually implies the existence of Whitney stratifications. The idea is that an elementary extension K of $\mathbb{R}$ or $\mathbb{C}$ can be turned into a Henselian valued field in a natural way; if one applies our result in K, the obtained t-stratification induces a Whitney stratification in $\mathbb{R}$ resp. $\mathbb{C}$.

Globally valued fields

Itaï Ben Yaacov

(joint work with Ehud Hrushovski)

In order to fix terminology, by a *valuation* on a field k we mean a map $v\colon k \to (\Gamma, +, \leq) \cup \{\infty\}$, most often with $\Gamma \subseteq \mathbf{R}$, satisfying

- $v(ab) = v(a) + v(b)$,
- $v(a+b) \geq v(a) \wedge v(b) - v(2)^-$,
- $v(a) = \infty$ if and only if $a = 0$,

where $\wedge$ denotes the minimum, and $t^- = -(t \wedge 0)$ is the negative part of t. If $v(t)^- \neq 0$ (i.e., if $v(2) < 0$) then the valuation is said to be *Archimedean*, and there exists an embedding $k \subseteq \mathbf{C}$ under which $|x| = 2^{v(x)/v(2)}$. Otherwise, the corrective term $-v(2)^-$ vanishes, and v is an "ordinary" ultra-metric valuation.

The *sum formula* is a relation linking the valuations on a number or function field, asserting that for every $x \neq 0$ the sum of all valuations of x, appropriately normalised, must vanish:

$$\sum_p v_p(x) = 0.$$

This property (often stated in multiplicative notation as the *product formula* $\prod_p |x|_p = 1$) is extremely useful, e.g. for the definition of heights etc. In contrast, the Approximation Theorem asserts that a finite family of nonequivalent valuations on a given field can satisfy no relation between them.

In order to cast the "sum formula" into a model-theoretic formalism we need the following facts and observations:

- In the definition of a valuation, one may replace the hypothesis that Γ is totally ordered with Γ being lattice-ordered.

- We recall that an L^1 *lattice* is a real Banach lattice (namely a Banach space equipped with a compatible lattice ordering), such that in addition $\big\||x| + |y|\big\| = \|x\| + \|y\|$. For every measure space X, the space $L^1(X)$ of real summable functions is an L^1 lattice, and every L^1 lattice is of this form (see Meyer-Nieberg [Mey91]).
- The class of L^1 lattices is elementary in continuous logic. It is moreover stable (see [BBH11]).
- In an L^1 lattice one can recover integration as $\int f = \|f \vee 0\| - \|f \wedge 0\|$.

We therefore define a *globally valued field (GVF)* as a triplet (K, E, v), where:

- K is a field.
- E is an L^1 lattice.
- $v \colon K \to E \cup \{\infty\}$ is a lattice-valued valuation, such that for every $x \in K^\times$ we have $\int v(x) = 0$, i.e., $\|v(x) \vee 0\|$, $\|v(x) \wedge 0\|$ (and, as an aside, the *height* of x is defined as this common value: $h(x) = \|v(x) \vee 0\|$).

The class of GVFs is elementary in continuous logic, and inductive: the (completed) union of an increasing chain of GVFs is one as well. Example include $\mathbf{Q}$, $k(t)$, as well as any algebraic extension thereof (in the case of infinite algebraic extensions, the set of valuations is no longer discrete).

At this stage the natural questions arise:

- Does the theory GVF admit a model companion? We know that GVF does not have the amalgamation property, so a model companion, if it exists, cannot have quantifier elimination.
- Is the model companion stable? (Notice that the "value group", namely the L^1 lattice, is known to be stable.)
- Are $\widetilde{\mathbf{Q}}$ and $\widetilde{k(t)}$ models of the model companion, i.e., are they e.c.?

For the time being, the only definitive positive answer we can give is that $\widetilde{k(t)}$ is e.c. Partial results toward answering the other two involve capacity theory and a theory of lattice-valued vector spaces over globally valued fields.

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Groups definable in o-minimal structures

Annalisa Conversano

(joint work with Marcello Mamino, Anand Pillay)

Several classes of connected real Lie groups can be defined in an o-minimal structure, such as compact groups, algebraic groups, Nash groups and linear semisimple groups. Semisimple groups which are not linear have infinite discrete centre, so they cannot be defined in an o-minimal structure. However they can be defined in the two-sorted structure $((\mathbb{Z},+),(\mathbb{R},+,\cdot))$ (where there are no additional basic relations between the sorts) [11], which has NIP as well.

In [9] Fornasiero notices how, in all the cases mentioned above, the groups can be defined in a d-minimal structure, and asks if this is the case for every connected Lie group. The negative answer is provided by the following:

Theorem 1. [5] A first-order structure $\mathcal{M}$ interprets all connected real Lie groups if and only if $\mathcal{M}$ interprets the real field expanded with a predicate for the integers.

In the proof, a connected 3-dimensional nilpotent Lie group L is presented, which is bi-interpretable with the real field expanded with a predicate for the integers (where the projective sets can be defined). Therefore L *interprets every connected Lie group.*

The classical Levi decomposition of connected Lie groups finds an analogue in the o-minimal context, where the appropriate notion of a Levi subgroup is given by the following:

Definition 2 ([8])**.**

(i) A group S is **ind-definable semisimple** if S is ind-definable definably connected and $S/Z(S)$ is definable semisimple (a definable group is semisimple if has no infinite abelian normal subgroups).

(ii) A maximal ind-definable semisimple subgroup of a definable group G is called a **ind-definable Levi subgroup** of G.

Theorem 3 (Levi decomposition [8]). Let G be a definably connected group definable in an o-minimal expansion of a field. Then G has a ind-definable Levi subgroup S, unique up to conjugation, and

$$G = R \cdot S,$$

where R is the solvable radical of G. Moreover $R \cap S \subseteq Z(S)$, and $Z(S)$ is finitely generated.

The study of the homotopy type of a connected real Lie group is reduced to the compact case, because Lie groups have maximal compact subgroups (all conjugate) which are homotopy equivalent to the whole group [12]. Groups definable in o-minimal structures in general do not have a maximal definably compact, definable subgroup. However, a reduction to the compact case still exists, considering the quotient $G/\mathcal{N}(G)$ by the maximal normal definable torsion-free subgroup $\mathcal{N}(G)$. (The existence of $\mathcal{N}(G)$ is proved in [6])

Theorem 4. [3] Let G be a definably connected group definable in an o-minimal structure, and $\mathcal{N}(G)$ its maximal normal definable torsion-free subgroup.

Then the quotient $G/\mathcal{N}(G)$ has a maximal definably compact definable subgroup K, which is definably connected and unique up to conjugation. Moreover,

$$G/\mathcal{N}(G) = K \cdot H,$$

where H is a (maximal) definable torsion-free subgroup, and $K \cap H = \{e\}$.

By Theorem 4 and previous work on torsion-free definable groups [14], it follows that G *and* K *have the same o-minimal homotopy type.*

On the other hand, even though G might not contain a definable subgroup isomorphic to K, denoting by $\pi\colon G \to G/\mathcal{N}(G)$ the canonical projection, it is possible to show that the exact sequence $1 \to \mathcal{N}(G) \to \pi^{-1}(K) \to K \to 1$ always splits abstractly [4]. Therefore:

Theorem 5. [4] Every definably connected group G, definable in an o-minimal structure, can be decomposed as

$$G = K_1 \cdot H_1 \quad \text{with} \quad K_1 \cap H_1 = \{e\},$$

where $H_1 = \pi^{-1}(H)$ is a maximal definable torsion-free subgroup of G, and K_1 is abstractly isomorphic to the maximal definably compact subgroup K of $G/\mathcal{N}(G)$.

It is known that given a definable group G in a saturated o-minimal expansion of a field, there is a canonical homomorphism from G to a compact real Lie group G/G^{00} , where G^{00} is the smallest type-definable subgroup of G of bounded index [2]. It follows by Theorem 4 and [1] (where K and K/K^{00} are proved to be homotopy equivalent) that G and G/G^{00} have the same homotopy type if and only if G/G^{00} is Lie-isomorphic to K/K^{00}. So it is natural to ask when this is the case and, more generally, whether there is a relation between the compact Lie groups G/G^{00} and K/K^{00}.

It turns out that G/G^{00} is always a quotient of K/K^{00} [7]. Moreover the problem of determining when these two Lie groups are isomorphic is related to other model-theoretic notions (see Theorem 8 below) that are now briefly recalled.

Let T be an arbitrary theory, and M a model of T. If X is a definable set in M, then a **Keisler measure** μ on X (over M) is a finitely additive probability measure on the family of subsets of X which are definable (with parameters) in M.

When $X = G$ is a definable group, then $G(M)$ acts (on both the left and right) on the set of Keisler measures μ on G over M: if Y is an M-definable subset of G, then $(g \cdot \mu)(Y) = \mu(g^{-1} \cdot Y)$. In particular it makes sense for a Keisler measure μ on G over M to be left (or right) $G(M)$-invariant.

Note that if N is another model of T (assuming G is definable without parameters), then there is a $G(N)$-invariant Keisler measure on G over N if and only if there is a $G(M)$-invariant Keisler measure on G over M [10]. So the following definition does not depend on the model:

Definition 6. [10] A definable group G is **definably amenable** if has a left G-invariant Keisler measure.

Definition 7. [13] Let $\bar{M}$ be a saturated structure. A definable group G **has a bounded orbit** if there is some $p \in S_G(\bar{M})$ whose stabilizer $Stab(p) = \{g \in G(\bar{M}) : gp = p\}$ has bounded index in $G(\bar{M})$.

In [6], the two notions above are shown to be equivalent for groups definable in a saturated o-minimal expansions of a field, giving a positive answer to a conjecture of Newelski and Petrykowski [13] in this special case. Moreover:

Theorem 8. [7] Let G be a definably connected group definable in a saturated o-minimal expansion of a real closed field. Then the following are equivalent:

- G is definably amenable;
- G has a bounded orbit;
- G/G^{00} is Lie isomorphic to K/K^{00};
- $G/\mathcal{N}(G)$ is definably compact;
- G^{00} is torsion-free.

When the equivalent conditions in Theorem 8 are satisfied, then the connected components G^{00} and G^{000} coincide [7], where G^{000} denotes the smallest invariant type-definable subgroup of bounded index in G. The first known example of a definable group G where $G^{00} \neq G^{000}$ appears in [6]. The general analysis of the connected components and related quotients is continued in [7].

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The Elementary Theory of Free Products of Groups

Zlil Sela

We study the first order theory of free products of arbitrary groups. In a joint work with E. Jaligot, we started this study, by analyzing the set of solutions to systems of equations over an arbitrary free product. For that purpose, we used limit groups over free products, and with each system of equations, or alternatively, with each finitely presented group, we have associated (non-canonically) a Makanin-Razborov diagram over free products. This Makanin-Razborov diagram encodes the set of solutions to the finite system of equations over an arbitrary free product, or alternatively describes all the quotients of a given finitely presented group that are free products.

We start this work by studying systems of equations with parameters over arbitrary free products. We generalize the notion of a graded limit group from free groups to free products, and then define rigid and (weakly) solid limit groups over free products, generalizing the corresponding notions over free and hyperbolic groups. Unfortunately, the boundedness results that were proved for the number of rigid and strictly solid families of homomorphisms over free and hyperbolic groups, can no longer be valid over free products. However, we manage to prove a combinatorial boundedness for rigid and (weakly) strictly solid families, that plays an essential role in studying the first order theory of free products, successfully replacing the strong boundedness results that hold in free and hyperbolic groups.

In further prove a general form of Merzlyakov theorem (over free groups) on the existence of formal solutions for sentences and formulas over varieties that are defined over free products. In particular, we show how to associate (non-canonically) a formal Makanin-Razborov diagram with a given AE sentence or formula over free products, generalizing the results over free groups.

By applying the techniques that were used in proving quantifier elimination in the theory of a free group, we finally associate (non-canonically) finitely many graded resolutions with a given coefficient-free formula over free products. This finite collection of resolutions is non-canonical, but it is universal, and it is good for all non-trivial free products apart from the infinite dihedral group, D_∞. In principle, the finite collection of resolutions enables one to reduce a sentence or a formula from an ambient free product to its factors. Indeed, we show that any given coefficient-free sentence over free products is equivalent to a finite disjunction of conjunctions of (coefficient-free) sentences over the factors of the free product. Furthermore, any given coefficient-free formula over free products is equivalent to a coefficient-free formula in an extended language, that involves finitely many quantifiers over the factors of the free product, and only 3 quantifiers over the ambient free product. Note that since the resolutions that we associated with a coefficient-free predicate are universal, the reduction of sentences and formulas from the ambient free product to its factors is uniform, i.e., it is good for all free products, and it does not depend on any particular given one.

The uniform reduction of sentences and formulas, and the resolutions that are associated with a given (coefficient-free) formula, enable us to prove some basic results on the first order theory of free products. S. Feferman and R. Vaught studied the first order properties of certain products of structures. Their methods, that look at the cartesian product of given structures, do not cover free products of groups (as they indeed indicated in their paper). This and his work with A. Tarski, led R. L. Vaught to ask the following question that we answer affirmatively:

Theorem 1. Let A_1, B_1, A_2, B_2 be groups. Suppose that A_1 is elementarily equivalent to A_2, and B_1 is elementarily equivalent to B_2. Then $A_1 * B_1$ is elementarily equivalent to $A_2 * B_2$.

The existence of graded resolutions that are associated with a given sentence over free products enables one to prove the following theorem, that generalizes Tarski's problem for free groups.

Theorem 2. Let A, B be non-trivial groups, and suppose that either A or B is not Z_2. Let F be a (possibly cyclic) free group. Then $A * B$ is elementarily equivalent to $A * B * F$.

The resolutions that are associated with coefficient-free formulas and sentences over free products, that enable a uniform reduction from the ambient free product to its factors, allow us to prove other uniform properties of sentences over free products.

Theorem 3. Let Φ be a coefficient free sentence over groups. There exists an integer, $k(\Phi)$, so that for every group, H, Φ is a truth sentence over $H_1 * \ldots * H_{k(\Phi)}$, $H_i \simeq H$, if and only if Φ is a truth sentence over $H_1 * \ldots * H_n$, $H_i \simeq H$, for every $n \geq k(\Phi)$.

Note that the integer $k(\Phi)$ depends on the coefficient free sentence, Φ, but it does not depend on the group, H. It is easy to see that $k(\Phi)$ can not be chosen to be a universal constant, e.g., we can take Φ_m to be a sentence that specifies if the number of conjugacy classes of involutions in the group is at least m. For such a sentence, Φ_m, $k(\Phi_m) = m$.

Theorem 3 can be further strengthened for sequences of groups. Let Φ be a coefficient free sentence over groups. Given any sequence of groups, $G_1, G_2, \ldots$, we set $M_1 = G_1$, $M_2 = G_1 * G_2$, $M_3 = G_1 * G_2 * G_3$, and so on. The sentence Φ may be truth or false on any of the groups (free products) M_i, $i = 1, \ldots$. Here one can (clearly) not guarantee that the sentence Φ is constantly truth or constantly false staring at a bounded index (of the M_i's). However, one can prove the following.

Theorem 4. There exists an integer $c(\Phi)$, so that for every sequence of groups, $G_1, G_2, \ldots$, the sentence Φ over the sequence of groups, $M_1 = G_1, M_2 = G_1 * G_2, \ldots$ may change signs (from truth to false or vice versa) at most $c(\Phi)$ times.

We proceed by using the resolutions that are associated with a coefficient-free formula over free products, and combine them with a modification of the strategy that was applied to prove the stability of free groups, to prove that free products of stable groups is stable.

Theorem 5. Let A and B be stable groups. Then $A * B$ is stable.

In fact we prove a slightly stronger result, and show that a free product of a countable collection of groups that are uniformly stable, is stable.

Theorem 6. Let $G_1, G_2, \ldots$ be a sequence of groups. Suppose that every sentence Φ is uniformly stable over the sequence $\{G_i\}$. Then the countable free product, $G_1 * G_2 * \ldots$, is stable.

Finally, it is worth noting that our results for free products of groups, or slight strengthenings of them that are still valid over groups, can be shown to be false for free products of semigroups, using techniques of Quine and Durnev. e.g., a free product of finite semigroups is in general unstable (although it is stable if the finite semigroups happen to be groups). Hence, it seems that model theoretic techniques that handle products of general structures, like the ones that were used by Feferman and Vaught, can not suffice to analyze the elementary theory of free products of groups.

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Topological dynamics of stable groups

Ludomir Newelski

Assume T is a complete stable theory in language L, $M \models T$ and $G \subseteq M$ is a 0-definable group in M. As usual we work within a monster model $\mathfrak{C}$ of T. $S_G(M)$ denotes the set $\{tp(a/M) : a \in G^{\mathfrak{C}}\}$.

For the types $p, q \in S_G(M)$ we define their free product $p * q$ as the type $tp(a \cdot b/M)$, where $a \models p$ and $b \models q$ are independent over M. $(S_G(M), *)$ is a semigroup. This semigroup was investigated already in [1]. Here we look at it from the point of view of topological dynamics.

Topological dynamics

$S_G(M)$ is a point-transitive G-flow, meaning that $S_G(M)$ is a compact topological space, upon which G acts by homeomorphisms (here the action is induced by left translation), with a dense G-orbit. In our situation we have that Gen, the set of generic types $p \in S_G(M)$, is the unique minimal sub-flow of $S_G(M)$. It is also the unique minimal left ideal in $S_G(M)$ and a maximal subgroup of $S_G(M)$.

Assume X is an arbitrary point-transitive G-flow. We associate with it its Ellis semigroup $E(X)$ which is the topological closure of the set $\{\pi_g : g \in G\}$ in the space of functions $X \to X$ with the topology of pointwise convergence. $\pi_g, g \in G$

are the homeomorphisms of X given by the action of G on X. $E(X)$ is also a point-transitive G-flow in its own right.

In our situation we have that $S_G(M)$ is isomorphic to its Ellis semigroup $E(S_G(M))$ both as a semi-group (with respect to $*$) and a G-flow. We locate the semigroup $S_G(M)$ within the definable realm of M. The main tools for this are definability of types in stable theories [2] and the following functional interpretation of the Ellis semigroup.

Functional interpretation

Assume $\mathcal{A} \subseteq \mathcal{P}(G)$ is a G-algebra of sets (meaning that it is a Boolean algebra of subsets of G with an action of G preserving the Boolean operations; the action again is the left translation). Then naturally $S(\mathcal{A})$, the Stone space of $\mathcal{A}$, is a point-transitive G-flow.

For every $p \in S(\mathcal{A})$ we define a function $d_p : \mathcal{A} \to \mathcal{P}(G)$ by:

$$d_p(U) = \{g \in G : g^{-1}U \in p\}$$

We say that $\mathcal{A}$ is d-closed if it is closed under d_p for every $p \in S(\mathcal{A})$. For example, $\mathcal{A} = Def_G(M) = \{U \subseteq G : U \text{ is definable (with parameters) in } M\}$ is a d-closed G-algebra of subsets of G, because every type in T is definable. Also, $S(\mathcal{A}) = S_G(M)$. Assume $\mathcal{A}$ is d-closed and let $End(\mathcal{A})$ denote the semigroup of all G-endomorphisms of $\mathcal{A}$. We have the following facts.

- For $p \in S(\mathcal{A})$, $d_p \in End(\mathcal{A})$.
- Let $d : S(\mathcal{A}) \to End(\mathcal{A})$ be the function mapping p to d_p. Then d is a bijection.
- d induces an operation $*$ on $S(\mathcal{A})$ so that $d : (S(\mathcal{A}), *) \to (End(\mathcal{A}), \circ)$ is an isomorphism of semigroups. Also the semigroup $(S(\mathcal{A}), *)$ is isomorphic to its Ellis semigroup $E(S(\mathcal{A}))$ via the function $p \mapsto l_p$, where l_p is the left translation by p.
- If $\mathcal{A} = Def_G(M)$, then the operation $*$ induced by d is just the free multiplication of types.

To locate the semigroup $S_G(M)$ in the definable realm of M we follow the classical path to define the generic types of G. For $\Delta \subseteq L$ let $Def_{G,\Delta}(M)$ be the algebra of relatively Δ-definable subsets of G. We say that Δ is invariant if the family of subsets of G relatively defined by M-instances of formulas from Δ is invariant under both left- and right-translation. Let Inv be the family of all finite invariant sets $\Delta \subseteq L$. It is co-final in $[L]^{<\omega}$.

Let $\Delta \in Inv$. Then we have the following facts:

- $Def_{G,\Delta}(M)$ is a d-closed G-algebra of sets.
- $S_{G,\Delta}(M) = S(Def_{G,\Delta}(M))$ is a point-transitive G-flow and we have a semigroup operation $*$ on it so that it is isomorphic to $(End(Def_{G,\Delta}(M)), \circ)$ and to its Ellis semigroup.
- $Def_{G,\Delta}(M) = \bigcup_{\Delta \in Inv} Def_{G,\Delta}(M)$ and $S_G(M)$ is an inverse limit of $S_{G,\Delta}(M), \Delta \in Inv$, both as a G-flow and a semigroup. The connecting functions are restrictions.

Using the definability lemma and the isomorphisms d we get that the semigroup $(S_{G,\delta}(M),*)$ is type-definable in M^{eq}.

Maximal subgroups of $S_{G,\Delta}(M)$

Maximal subgroups of Ellis semigroups are important in topological dynamics. Here we are able to describe explicitly the maximal subgroups of $S_{G,\Delta}(M)$, where $\Delta \in Inv$.

Let $H < G$ be a Δ-definable Δ-connected subgroup. This means that $Mlt_\Delta(H) = 1$ and implies that there is a unique type $p \in S_{G,\Delta}(M)$ that is generic in H. Let $N = N_G(H)$ and let $S_p = \{n \cdot p : n \in N\} \subseteq S_{G,\Delta}(M)$. Then S_p is a maximal subgroup of $S_{G,\Delta}(M)$. Moreover it is definable in M^{eq} (and also definable in $S_{G,\Delta}(M)$ in the pure semigroup structure) and is definably isomorphic to the group N/H. We have the following:

- All maximal subgroups of $S_{G,\Delta}(M)$ are of the above form, they are pairwise disjoint and every subgroup of $S_{G,\Delta}(M)$ is contained in a unique maximal one.
- if S is a maximal subgroup of $S_G(M)$, then S is an inverse limit of some maximal subgroups $S_\Delta \subseteq S_{G,\Delta}(M), \Delta \in Inv$ (the connecting functions are restrictions).

Let $p \in S_G(M)$. We may consider the sequence of $*$ powers $p^{*n} = p * \cdots * p$ (n times) of p. We prove that in this respect p is "profinitely many steps away" from a translate of a generic type of a connected subgroup of G. To make this explicit, for $\Delta \in Inv$ let $n_\Delta = RM_\Delta(G)$ and $p_\Delta = p|_\Delta \in S_{G,\Delta}(M)$. Then for every $\Delta \in Inv$ there is a maximal subgroup S_Δ of $S_{G,\Delta}(M)$ such that $p^{*n} \in S_\Delta$ for every $n \geq n_\Delta$. So $S_\Delta = N_\Delta/H_\Delta$ as explained above.

Let S be the maximal subgroup of $S_G(M)$ being the inverse limit of the groups S_Δ and let $H = \bigcap_\Delta H_\Delta$. Then H is a connected type-definable subgroup of G and the types p^{*n} eventually converge to translates of the generic type of H. Here an essential role is played by the functional interpretation of types as endomorphisms (via the functions d). Let $p \in S_G(M)$. Then d_p is an endomorphism of $Def_G(M)$. We associate with it its kernel and image. The sizes of the kernels and images of functions d_p are strictly correlated with their local ranks and then with forking. These new objects provide a new way to measure types in $S_G(M)$.

This talk is based on my paper [4].

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Groups definable in two orthogonal sorts

Alessandro Berarducci
(joint work with Marcello Mamino)

This work can be thought as a contribution to the model-theory of covers of groups in the spirit of [7, 2, 1]. We classify the groups G which are interpretable in the disjoint union of two structures (seen as a two-sorted structure). We show that if one of the two structure is superstable of finite SU-rank and the SU-rank is definable, then G is an extension of a group interpretable in the (possibly) unstable sort by a group interpretable in the stable sort.

We assume some familiarity with the basic notions of model theory. A good recent reference is [6]. Given two structures Z and R, let (Z, R) be the two-sorted structure with a sort for Z and another sort for R in a disjoint language (no connections between the two sorts). Note that Z and R are then *fully orthogonal* in the following sense: any definable subset of $Z^m \times R^n$ is a finite union of sets of the form $A \times B$ with A a definable subset of Z^m and B a definable subset of R^n.

Our aim is to study the groups G which are interpretable in (Z, R), or equivalently definable in $(Z, R)^{eq}$. Obvious examples are the direct products of groups $H \times K$ with H definable in Z and K definable in R. More generally one can have a quotient of $H \times K$ by a finite subgroup. There are however more interesting examples. Indeed by [2] the universal cover $f : G \to H$ of a real Lie group H definable in an o-minimal expansion R of the real field is interpretable in $((\mathbb{Z}, +), R)$. This shows that a group G interpretable in $(Z, R)^{eq}$ does not need to arise from a direct product.

The next natural question is whether G is always an extension of a group definable in one sort by a group definable in the other sort. As already mentioned we will show this is indeed true under a suitable stability assumption on Z, but let us first show that in full generality the question has a negative answer. To this aim we take $Z = R = (\mathbb{R}, +, <)$. So we have two structures Z and R which are at the same time "equal" and orthogonal. There is of course no contradiction: indeed strictly speaking in (Z, R) we only have an isomorphic copy of Z and an isomorphic copy of R with the isomorphism not definable in (Z, R). We are going to describe a group G definable in $(Z, R)^{eq}$ with no infinite definable subgroup internal to one of the two sorts. So in particular G cannot be a definable extension of a group internal to one sort by a group internal to the other sort. The construction is based on [3, Example 5.2]. Take $G = (Z \times R)/\Lambda$ with $\mathbb{Z}^2 \cong \Lambda < Z \times R$ and Λ in sufficiently generic position. Note that Λ is not definable. However we can define G in $(Z, R)^{eq}$ taking a definable set $X \subseteq Z \times R$ such that $X + \Lambda = G$ and $X \cap \Lambda$ is finite (a big enough square $X = [0, a] \times [0, a]$ will do) and identifying G with X/Γ (where X/Λ is the quotient of X by the equivalent relation "to be in the same coset"). Since we only need a finite portion of Λ to define X/Λ we obtain a definition in $(Z, R)^{eq}$. This is exactly the example in [3] except that in that paper the authors work with only one sort (which amounts to have the identity map from Z to R at disposal). They prove that in the one-sort setting G has no definable proper infinite subgroups. This holds *a fortiori* in the two-sorted setting since we have fewer definable sets. Thus clearly G has no infinite subgroups internal to one of the two sorts.

In the example just given it is important to have the order relation $<$ in the language, so the structures are unstable (in the model-theoretic sense). We will show that under a suitable stability assumption on the Z-sort any group interpretable in (Z, R) is an extension of a group interpretable in R by a group interpretable in Z. Our main result is:

Theorem. Let Z be a superstable structure of finite SU-rank and assume that the SU-rank is definable. Let R be an arbitrary structure. Given a group $(G, \cdot)$ definable in $(Z, R)^{eq}$, there is a Z-internal definable normal subgroup $\Gamma \lhd G$ such that G/Γ is R-internal.

Note that in any superstable structure Z of SU-rank 1 (for instance $(\mathbb{C}, +, \cdot)$ or $(\mathbb{Z}, +)$, or $(\mathbb{R}, +)$), the SU-rank is definable (see [5, Corollary 5.11]), and therefore Z satisfies the assumption of the theorem.

The subgroup Γ that we are going to describe is not canonical, namely it depends on how G sits in the ambient space $(Z, R)^{eq}$. For instance if G is the universal cover of the circle group $\mathbb{R}/\mathbb{Z}$, then G can be naturally interpreted in $((\mathbb{Z}, +), \mathbb{R})$, but it has neither a minimal nor maximal Z-internal normal subgroup. Indeed in this example Γ is Z-internal if and only if 2Γ is such, and G/Γ is R-internal if and only if $G/2\Gamma$ is such, so there is no reason to prefer Γ over 2Γ.

We give below the definition of Γ, omitting the proof that Γ has the desired properties. Let us first deal with the case when Γ is definable in (Z, R) rather than $(Z, R)^{eq}$. So we have $G \subseteq Z^m \times R^n$ for some $m, n \in \mathbb{N}$. Let $\pi_R : Z^m \times R^n \to R^n$ be the natural projection. We define:

$$\Gamma \quad = \quad \big\{ g \in G : (\mathrm{Most}\, y)(\mathrm{Most}\, x)\big(\pi_R(xg^y) = \pi_R(g^y x) = \pi_R(x)\big)\big\}$$

where $g^y = ygy^{-1}$ and $(\mathrm{Most}\, y)\phi(y)$ means that the projection on Z^m of the set of $y \in G$ such that $\phi(y)$ fails has lower SU-rank than the projection of the whole of G.

The case when G is definable in $(Z, R)^{eq}$ is similar, but we need to define π_R in a suitable way. To do this we first show that there is a finite-to-one definable function f from any given sort of $(Z, R)^{eq}$ to a sort of $Z^{eq} \times R^{eq}$. To define π_R in the imaginary case we first apply f and then we use the projection from $Z^{eq} \times R^{eq}$ to Z^{eq}. The same definition of Γ will then work.

We can also show that if R is an o-minimal structure, then every group G interpretable in $((\mathbb{Z}, +), R)$ admits a unique "t-topology" in analogy with the o-minimal case [4]. In particular, if R is based on the reals, then G has a natural Lie group structure. As a corollary of the main theorem we then obtain:

Corollary. Let R be an o-minimal structure. Then any group definable in $((\mathbb{Z}, +), R)$ is a cover of a group definable in R.

Here by "cover" we mean a definable morphism which is continuous and open in the t-topology and has a discrete kernel. Note that it is not generally true that an extension of a group definable in $\mathbb{R} = (\mathbb{R}, +, \cdot)$ by a group definable in $(\mathbb{Z}, +)$ is a cover of a group definable in $\mathbb{R}$ (see [1, Theorem 3.12]). Among all the extensions, only the covers will be definable in $((\mathbb{Z}, +), \mathbb{R})^{eq}$.

Acknowledgements. In my talk in Oberwolfach I only spoke about the SU-rank 1 case. The finite rank case can be obtained with the same construction adding the definability assumption on the rank (which in the SU-rank 1 case comes for free). We thank Anand Pillay for suggesting this possibility. Preliminary versions of the results were presented at the "Konstanz-Naples Model Theory Days" (Konstanz 6-8 Dec. 2012).

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A non-desarguesian projective plane of an analytic origin

Boris Zilber
(joint work with K.Tent)

Hrushovski's construction of "new" strongly minimal structures and more generally "new" stable structures proved very effective in providing a number of examples to classification problems in stability theory. For example, J.Baldwin used this method to construct a non-desarguesian projective plane of Morley rank 2 (see e.g. [3]). But there is still a classification problem of similar type which resists all attempt of solution, the Algebraicity (or Cherlin-Zilber) Conjecture. At present there is a growing belief that there must exists a simple group of finite Morley rank which is not isomorphic to a group of the form $G(\mathbb{F})$ for G an algebraic group and $\mathbb{F}$ an algebraically closed field (a *bad* group).

The speaker developed an alternative interpretation of the "new" stable structures obtained by Hrushovski's construction, see e.g. [5]. In this interpretation the universe M of the structure is represented by a complex manifold and relation by some subsets of M^n explained in terms of the analytic structure on M. In this interpretation Hrushovski's predimension inequality corresponds to a form of (generalised) Schanuel's conjecture. We argue that looking for stable structures of analytic origin is potentially a better way of producing new stable structures.

Below we briefly explain a construction of a new non-desarguesian projective plane that originates in a complex analytic structure. The new, in comparison with previous examples of e.g. "green fields" (see [6]) is that we have to use a non-trivial collapse procedure.

Consider structures $K_f = (K, +, \cdot, f)$, where $(K, +, \cdot)$ is a field and $f : K \to K$ a unary function.

Let L^{alg} be a relational language for structures of the form K_f, relations of which are those of L along with all the relations corresponding to Zariski closed 0-definable subsets of K^n. We always assume that K is a field of characteristic 0. Let $C(K_f)$ be the class of all finite L^{alg}-structures that can be embedded in K_f.

Note that in the language L^{alg} we can say for an n-tuple X and a variety W over $\mathbb{Q}$ that $X \in W$. So the expression $\mathrm{tr.deg}(X) = m$ means that m is the dimension of the smallest variety W over $\mathbb{Q}$ such that $X \in W$.

Below we use the terminology of [3].

Theorem 1. (A.Wilkie[4] and P.Koiran [2]) The structure $\mathbb{C}_f = (\mathbb{C}, +\cdot, f)$, where f is an entire **Liouville function,** is a model of the first order theory T_f of a rich structure for the class of finite L^{alg}-structures satisfying, for every finite subset X, the Hrushovski inequality

$$\delta(X) \geq 0, \text{ where } \delta(X) := \mathrm{tr.deg}(X \cup f(X)) - |X|.$$

We add to this the following.

Theorem 2. $\mathbb{C}_f$ is ω-saturated. Moreover, $\mathbb{C}_f$ is the unique model of T_f of cardinality continuum which satisfies the countable closure property.

Consider the class $C(\mathbb{C}_f)$. This is an amalgamation class with respect to strong embeddings $\leq$ determined by δ.

Let μ be a Hrushovski function satisfying $\mu(\alpha) = 1$, for any α, which is a code of a pair $(x, x_1, y_1, x_2, y_2/a_1, a_2, b)$ in a substructure $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\}$ that satisfies relations

$$\begin{array}{l} a_1 x = x_1, \ a_2 x = x_2, \\ f(x_1) = y_1, \ f(x_2) = y_2, \\ y_1 - y_2 = b \end{array}$$

Note, that the code of type α says that $f(a_1 x) - f(a_2 x) = b$, and $\mu(\alpha) = 1$ amounts to saying that the latter has at most one solution in x.

Consider the corresponding subclass $C_\mu(\mathbb{C}_f)$. We want to prove that this class has AP with respect to $\leq$.

Theorem. $C_\mu(\mathbb{C}_f)$ is an amalgamation class. There exists a countable rich structure K_f for class $C_\mu(\mathbb{C}_f)$.

(i) K_f is an algebraically closed field with a function f.

(ii) Given $a_1 \neq a_2$ and b in K_f, there is a unique solution to the equation

$$f(a_1 x) - f(a_2 x) = b.$$

In particular, f is a bijection on K.

(iii) K_f is embeddable in $\mathbb{C}_f$.

(iv) depending on μ, the theory of K_f is ω-stable of rank ω or strongly minimal.

The difficult part of the proof is to establish that $C_\mu(\mathbb{C}_f)$ is an amalgamation class. The rest follows by standard arguments.

Lemma. The ternary operation $T(a, b, x)$ on K_f determine a ternary ring. That is
the following hold:
(T1) $T(1, a, 0) = T(a, 1, 0) = a$ for all $a \in K_f$;
(T2) $T(a, 0, c) = T(0, a, c) = c$ for all $a, c \in K_f$;
(T3) If $a, b, c \in K_f$, the equation $T(a, b, y) = c$ has a unique solution y;
(T4) If $a, a', b, b' \in K_f$ and $a \neq a'$, the equations $T(x, a, b) = T(x, a', b')$ have a unique solution x in R;
(T5) If $a, a', b, b' \in K_f$ and $a \neq a'$, the equations
$T(a, x, y) = b,\ T(a', x, y) = b'$ have a unique solution x, y in K_f.

It is well-known (see [1]) that with every ternary ring there is an associated projective plane (which is definable in the ring). Every Desarguesian plane has a unique associated ternary ring, which is an associative division ring.

Corollary. The projective plane $\mathbb{P}_T(K_f)$ associated with the ternary ring (K_f, T) is not desarguesian. The Morley rank of $\mathbb{P}_T(K_f)$ is equal to 2-times the Morley rank of K_f.

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Ampleness in free groups

Abderezak Ould Houcine
(joint work with K. Tent)

Pillay [6] defined the notion of ampleness in a stable theory. It is a property that reflects the existence of geometric configurations behaving very much like projective space over a field and so any theory interpreting an infinite field is in fact ample (see Pillay [6]). We use here the slightly stronger definition given by Evans in [1]. As usual we write $A \mathop{\smile\!\!\!\!\!\!|}_C B$ to express that the sets A and B are independent over the set C; we denote by $\mathrm{acl}^{\mathrm{eq}}(\bar{a})$ the algebraic closure of $\bar{a}$ with respect to T^{eq} (see Section 2 and [8] for more background).

Definition 0.1. [1] *Suppose T is a complete stable theory and $n \geq 1$ is a natural number. Then T is n-ample if (in some model of T, possibly after naming some parameters) there exist tuples $a_0, \ldots, a_n$ such that:*
(i) $a_n \not\perp a_0$*;*
(ii) $a_0 \ldots a_{i-1} \perp_{a_i} a_{i+1} \ldots a_n$ *for* $1 \leq i < n$*;*
(iii) $\mathrm{acl}^{\mathrm{eq}}(a_0) \cap \mathrm{acl}^{\mathrm{eq}}(a_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$*;*
(iv) $\mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1} a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1})$ *for* $1 \leq i < n$*.*

We call T ample *if it is n-ample for all $n \geq 1$.* □

Pillay [5] proved that the elementary theory of nonabelian free groups is 2-ample and conjectured that it is not 3-ample. We here refute Pillay's conjecture by showing the following more general result.

Theorem 0.2. [3] *The elementary theory of any nonabelian torsion-free hyperbolic group is ample.*

Since nonabelian free groups of finite rank are torsion-free and hyperbolic it follows that their elementary theory is ample. In fact in our proof we go from free groups to torsion-free hyperbolic groups, we show ampleness in free groups and then we transfer this result to torsion-free hyperbolic groups.

By [2] the algebraic closure is closely related to graph of groups decompositions and JSJ-decompositions. We construct a graph of groups decomposition (over cyclic subgroups) of the free group in such a way that certain vertex groups then witness ampleness. Since we also need to study the imaginary algebraic closure, we use Sela's elimination of imaginaries to reduce the problem to the usual algebraic closure and the algebraic closure relative to the conjugacy classes. The necessary properties of the imaginary algebraic closure are established using again its close relation to JSJ- decompositions.

In what follows, we shall explain the strategy of the proof of Theorem 0.2 in the case of free groups. We use the following result of Sela about elimination of imaginaries.

Theorem 0.3. [7] *The elementary theory of nonabelian free groups has geometric elimination of imaginaries relative to conjugation, right (left) cosets of cyclic subgroups and double cosets of cyclic groups.* □

We reduce the problem to the study of the ordinary algebraic closure and the imaginary algebraic closure relative to conjugacy. We let

$$\mathrm{acl}^c(A) = \mathrm{acl}^{\mathrm{eq}}(A) \cap S_{E_0}$$

here S_{E_0} is the sort of the conjugacy relation and

$$\mathrm{acl}(A)^c = \{a^F | a \in A\}.$$

Proposition 0.4. [3] *Let F be a nonabelian free group. For finite tuples $\bar{a}, \bar{b}, \bar{c} \in F$ we have*

$$\mathrm{acl}^{\mathrm{eq}}(\bar{a}) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{b}) = \mathrm{acl}^{\mathrm{eq}}(\bar{c})$$

if and only if

$$\mathrm{acl}^c(\bar{a}) \cap \mathrm{acl}^c(\bar{b}) = \mathrm{acl}^c(\bar{c})$$

and

$$\mathrm{acl}(\bar{a}) \cap \mathrm{acl}(\bar{b}) = \mathrm{acl}(\bar{c}).$$

$\square$

To understand the ordinary algebraic closure we use results from [2] where in particular the following link with the JSJ-decomposition is given.

Theorem 0.5. [2] *If F is a free group of finite rank with nonabelian subgroup A, then $\mathrm{acl}(A)$ coincides with the vertex group containing A in the generalized malnormal (cyclic) JSJ-decomposition of F relative to A.* $\square$

It turns out that the algebraic closure relative to the conjugacy classes is also related to JSJ-decompositions. We prove the following.

Proposition 0.6. [3] *Let F be a free group of finite rank, A a nonabelian subgroup of F and $c \in F$. The following are equivalent:*

(1) $c^F \in \mathrm{acl}^c(A)$.

(2) *There exists finitely many automorphisms $f_1, \ldots, f_p \in Aut_A(F)$ such that for any $f \in Aut_A(F)$, $f(c)$ is conjugate in F to some $f_i(c)$.*

(3) *c is malnormaly universally elliptic relative to A.*

(4) *In any generalized cyclic JSJ-decomposition of F relative to A, either c is conjugate to some element of the elliptic abelian neighborhood of a rigid vertex group or it is conjugate to an element of a boundary subgroup of a surface type vertex group.* $\square$

To deal with the independence relation we use the following characterization of Perin and Sklinos.

Proposition 0.7. [4] Let F be a free group of finite rank, $\bar{a}, \bar{b}$ be finite tuples from F and C a free factor of F. Then

$$\bar{a} \mathop{\perp}_{C} \bar{b}$$

if and only if

$$F = A * C * B \text{ with } \bar{a} \in A * C \text{ and } \bar{b} \in C * B.$$

$\square$

Sequences witnessing ampleness in free groups are defined as follows. Let $H_i = \langle c_i, d_i, a_i, b_i \mid c_i d_i [a_i, b_i] = 1 \rangle$, that is H_i is the fundamental group of an orientable surface with 2 boundary components and genus-1, where c_i and d_i are the generators of boundary subgroups. Let $P_n = H_0 * H_1 * \cdots * H_{n-1} * H_n$, and $G_0 = P_0 = H_0$, $G_n = \langle P_n, t_i, 0 \leq i \leq n-1 \mid d_i^{t_i} = c_{i+1} \rangle$ for $n \geq 1$ Then G_{2n} is a free group and the sequence $(a_0, b_0, c_0), (a_2, b_2, c_2), \cdots, (a_{2n}, b_{2n}, c_{2n})$ is a witness of the n-ample property.

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On model-theoretic connected components

Jakub Gismatullin

Suppose $(G, \cdot, \ldots)$ is an infinite group with some first order structure. By G^* we denote sufficiently saturated elementary extension of G (i.e. a monster model). In model theory we consider several kinds of *model-theoretic connected components* of G. For a small set of parameters $A \subset G^*$ define (see [2] and references therein):

- ${G^*}^0_A$ (the connected component of G over B) is the intersection of all A-definable subgroups of G^* which have finite index in G^*,
- ${G^*}^{00}_A$ (the type-connected component of G over A) is the smallest subgroup of 'bounded index' in G^* (bounded relative to $|G^*|$), that is, type-definable over A,
- ${G^*}^{000}_A$ (the ∞-connected component of G) is the smallest subgroup of bounded index in G^*, that is, A-invariant (invariant under the automorphisms of G^* fixing A pointwise).

In the literature the component ${G^*}^{000}_A$ is sometimes denoted by ${G^*}^{\infty}_A$. We say, that ${G^*}^{000}$ exists, if for every small $A \subset G^*$, ${G^*}^{000}_A = {G^*}^{000}_\emptyset$ (likewise ${G^*}^{00}$ and ${G^*}^0$). The quotients $G^*/{G^*}^{\infty}_A$, $G^*/{G^*}^{00}_A$ and $G^*/{G^*}^0_A$ with the logic topology are compact topological groups which are invariants of the theory $\mathrm{Th}(G)$ of G. That is, they do not depend on the choice of saturated extension G^*. We have ${G^*}^{\infty}_A \subseteq {G^*}^{00}_A \subseteq {G^*}^0_A$. Moreover,

- $G^*/{G^*}^0_A$ is a profinite group,
- $G^*/{G^*}^{00}_A$ is a compact Hausdorff group,
- $G^*/{G^*}^{\infty}_A$ is a quasi-compact group, that is, compact but not necessary Hausdorff.

For example, if $\mathrm{Th}(G)$ is stable, then ${G^*}^0_A = {G^*}^{00}_A = {G^*}^{000}_A$ is just the connected component ${G^*}^0$ of G, which does not depend on parameter set A. For a non-stable group, model-theoretic connected components may be distinct. For example, for $G = S^1 = \mathrm{SO}_2(\mathbb{R})$ viewed as a group definable in $(\mathbb{R}, +, \cdot, 0, 1)$, $G^* = {G^*}^0_A$

but $G^* \neq G^{*00}_A = G^{*000}_A$ are infinitesimals and G^*/G^{*00}_A is just S^1. Suppose $G = \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is the topological universal cover of $\mathrm{SL}_2(\mathbb{R})$. A. Conversano and A. Pillay proved in [1] that G^{*00}/G^{*000} is $\widehat{\mathbb{Z}}/\mathbb{Z}$ (where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$). In [5] a general result on model-theoretic connected components of central extensions is proved. Our aim is to understand what is the structure of the quotient G^{*00}_A/G^{*000}_A. Can it be non-abelian? (in examples from [1, 5] all such quotients are abelian). What groups may appear as a quotient G^{*00}_A/G^{*000}_A? In [3] we constructed examples of type-absolutely connected groups (see below) with some additional structure, for which G^{*00}_A/G^{*000}_A is far from being abelian.

Compactifications and type-absolutely connected groups

Equip G with the discrete topology. Then for all $A \subseteq G$ the mappings $G \to G^*/G^{*00}_A$ and $G \to G^*/G^{*0}_A$ are group-compactifications of G, that is, the image of G is dense in G^*/G^{*00}_A and in G^*/G^{*0}_A. In [3] we introduced notions of *absolutely connected* and *type-absolutely connected* group. The latter notion has the following topological interpretation

Fact ([3, Theorem 5.10.]) *Let G be an infinite group. Then every homomorphism $G \to C$ to a compact group C is trivial (i.e. the Bohr compactification of G with discrete topology is trivial) if and only if G is type-absolutely connected, that is $G^{*00}_B = G^*$ for all sufficiently saturated extensions G^* of an arbitrary expansion of G, and all small $B \subseteq G$.*

In the literature groups with the trivial Bohr compactification are called *minimally almost periodic* (with respect to the discrete topology). Using results of S. Rothman and A. Shtern [7] and our Fact one can characterize type-absolutely connected Lie groups:

> Suppose $G = SR$ (where R is the solvable radical and S is a Levi subgroup) is a Levi decomposition of a connected Lie group G. Then G is minimally almost periodic (that is, type-absolutely connected) if and only if S has no nontrivial compact simple factors and $R = [G, R]$.

In particular, every semisimple connected Lie group without compact factors (e.g. $\widetilde{\mathrm{SL}_2(\mathbb{R})}$) is type-absolutely connected.

Another source of type-absolutely connected groups is the Peter-Weyl-van Kampen theorem: suppose C is a compact Hausdorff group, then continuous finite dimensional unitary representations separate points of C. Certain properties of unitary groups and above theorem give [4]:

1. ([4, 2.4]) *The following groups are type-absolutely connected*
   - (a) *a simple group of cardinality $> 2^{\aleph_0}$,*
   - (b) *a group G satisfying the following: for every $x \in G$, there are infinitely many distinct finite nonabelian simple groups H such that $x \in H \leq G$.*

(2) ([4, 2.5]) *Suppose G is an infinite group with finite exponent $m \in \mathbb{N}$, i.e. $g^m = e$, for every $g \in G$. Then ${G^*}^{00}_A = {G^*}^0_A$ for an arbitrary monster model $G^* \succ G$ of an arbitrary expansion of G and small $A \subset G^*$.*

Non-abelian ${G^*}^{00}_A/{G^*}^{000}_A$

Using type-absolutely connected groups [3, Section 3], generalized quasimorphisms [3, Section 4] and a method of recovering of a compact group C from its dense subgroup D (as a quotient of a monster model D^* with some structure by some type-definable bounded index subgroup, [3, Proposition 4.7]) we proved in [3] the following:

Fact ([3, Theorem 4.9]) *Suppose C is a compact Hausdorff group and $D < C$ is a dense subgroup generated by $D = \langle d_i \rangle_{i \in I}$. Let $S = \bigcup_{k=-3}^{3} \bigcup_{i \in I} {d_i^k}^D$, where $a^b = b^{-1}ab$. Then there exists a type-absolutely connected group G with some first order structure, and an epimorphism*

$$ {G^*}^{00}_\emptyset / {G^*}^{000}_\emptyset \to C / \langle \overline{S} \rangle , $$

where $\overline{S}$ is the closure of S in C.

Let $C = \widehat{\mathbb{F}_2}$ be the profinite completion of a 2-generated free group $\mathbb{F}_2 = \langle a, b \rangle$ and let D be the smallest normal subgroup of $\widehat{\mathbb{F}_2}$ generated by a, b. Applying our result we obtain a group G and an epimorphism ${G^*}^{00}_\emptyset/{G^*}^{000}_\emptyset \to C/D$. One can ask: is C/D non-abelian? Nikolov and Segal recently proved [6] that C/D is abelian, so one cannot deduce in this case that ${G^*}^{00}_\emptyset/{G^*}^{000}_\emptyset$ is non-abelian. Nevertheless, this example might give a way how to study groups from [6] using model-theoretic connected components.

An example of G with non-abelian ${G^*}^{00}_\emptyset/{G^*}^{000}_\emptyset$ can be derived from:

Fact ([3, Theorem 4.11]) *Suppose $\{G_n\}_{n \in \mathbb{N}}$ is any family of compact connected Hausdorff topological groups of weight $\leq 2^{\aleph_0}$ (e.g. compact connected Lie groups). Then there exists a type-absolutely connected group G such that $\prod_{n \in \mathbb{N}} G_n / \bigoplus_{n \in \mathbb{N}} G_n$ is a homomorphic image of ${G^*}^{00}_\emptyset/{G^*}^{000}_\emptyset$.*

In particular, if infinitely many of the G_n's are non abelian, then ${G^*}^{00}_\emptyset/{G^*}^{\infty}_\emptyset$ is also non abelian. Also, an ultraproduct $\prod_{n \in \mathbb{N}} G_n/\mathcal{U}$ is a homomorphic image of $\prod_{n \in \mathbb{N}} G_n / \bigoplus_{n \in \mathbb{N}} G_n$. Therefore, our result implies that there exists a type-absolutely connected G such that ${G^*}^{00}_\emptyset/{G^*}^{\infty}_\emptyset$ is, for example, non solvable.

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Minimality and quasi-minimality in the context of groups and fields

Krzysztof Krupiński
(joint work with T. Gogacz, P. Tanović, F. Wagner)

The part of this research concerning regular groups and fields comes from [1], and the part with more specific results concerning minimal groups and fields comes from [2].

Recall that a minimal structure is an infinite structure whose all definable subsets are finite or co-finite. A quasi-minimal structure is an uncountable structure in a countable language whose all definable subsets are countable or co-countable.

Basic examples of minimal structures are various orders (e.g. $(\omega, <)$) and strongly minimal structures (e.g. algebraically closed fields). Among the main examples of quasi-minimal structures there are various orders (e.g. $\omega_1 \times \mathbb{Q}$), strongly minimal structures expanded by some orders, and Zilber's pseudoexponential fields. A well-known conjecture of Boris Zilber predicts that the complex exponential field $(\mathbb{C}, +, \cdot, 0, 1, \exp)$ is quasi-minimal.

A common generalization of minimal and quasi-minimal groups [or fields] are regular groups [or fields], i.e., groups [fields] possessing a global regular type (which is necessarily a unique generic type). The notion of a regular type has been introduced in [3], and it generalizes the classical notion of regularity from the stable situation to the context of arbitrary theories.

Our goal is to understand the structure of regular groups and fields, or, more specifically, of minimal or quasi-minimal groups and fields.

A fundamental theorem of Reineke tells us that each minimal group is abelian. Surprisingly, an analogous statement for quasi-minimal groups seems hard to prove.

Conjecture 0.1. *Each quasi-minimal group is abelian.*

A more general question has been formulated in [3, Section 3].

Question 0.2. *Is every regular group abelian?*

We reduce the problem to the case of groups with only one non-trivial conjugacy class. Then we notice that a standard construction (involving HNN-extensions) of an uncountable group with a unique non-trivial conjugacy class does not lead to a quasi-minimal group, because the centralizers of all non-trivial elements of the resulting group are uncountable (and so also co-uncountable). Motivated by this obstacle, we construct a group of cardinality ω_1 with only one non-trivial conjugacy class, in which the centralizers of all non-trivial elements are countable.

We leave as an open question whether the group we constructed is quasi-minimal, or at least regular.

One of the oldest unsolved problems in algebraic model theory is Podewski's conjecture predicting that each minimal field is algebraically closed. Known to be true in positive characteristic [4], it remains wide open in the zero characteristic case.

One has an obvious analog of Podewski's conjecture for quasi-minimal fields.

Conjecture 0.3. *Each quasi-minimal field is algebraically closed.*

The above conjecture is open even in positive characteristic. A common generalization of Podewski's conjecture and Conjecture 0.3 is:

Conjecture 0.4. *Each regular field is algebraically closed.*

We say that a regular group is generically stable if its unique generic type is generically stable (see [3, 1] for the definition of a generically stable type). We prove the above conjecture in the generically stable situation.

Theorem 0.5. *Each generically stable regular field is algebraically closed. In particular, each generically stable minimal or quasi-minimal field is algebriacally closed.*

As a consequence, one gets that each quasi-minimal field of cardinality greater than ω_1 is algebraically closed, and a similar result for regular fields with NSOP. The case of minimal or quasi-minimal fields with NIP is still open.

Having Theorem 0.5, a natural question arises whether there exists a regular field which is not generically stable. We conjecture that there are no such minimal fields (see Conjecture 0.6 below), but in the quasi-minimal context such a filed exists [3, Example 5.1].

Assume now that K is a regular field which is not generically stable. Suppose that K is not algebrically closed, i.e., it has a finite extension L of degree n. Then L is naturally interpreted as K^n with coordinate-wise addition and some definable multiplication. Let p be the global generic type of K and $p^{(n)}$ its n-th power. We prove that the orbit of $p^{(n)}$ under the multiplicaive group of L is unbounded, which one can hope to be useful to get a final contradiction for some (e.g. NIP) fields.

From now on, we focus on minimal groups and fields. We say that a minimal structure is ordered if there is a definable order on singletons having an infinite chain. We obtain a dichotomy saying that a minimal group [field] is either generically stable or it is ordered. By virtue of Theorem 0.5, this reduces Podewski's conjecture to the case of ordered minimal fields (ordered in the above sense, and not in the sense of field theory!). So, each of the following two conjectures implies Podewski's conjecture.

Conjecture 0.6. *There is no minimal ordered field [of characteristic zero].*

Conjecture 0.7. *Each minimal ordered group is a torsion group.*

Analogous conjectures in the quasi-minimal context are false by [3, Example 5.1]. We introduce a notion of almost linear structure, and we prove these two conjectures in the almost linear situation (in quasi-minimal context, even in almost linear situation they are false).

Definition 0.8. *1) A definable order $<$ on M with an infinite chain is almost linear if the incomparability relation $\sim$ defined on M by*

$$x \sim y \iff \neg(x < y \vee y < x)$$

is an equivalence relation on M.
2) A minimal structure is almost linear if such an order exists.

We proved:

Theorem 0.9. *An almost linear minimal group G is either elementary abelian of exponent p or a finite sum of Prüfer p-groups for a fixed prime p. In particular, it is a torsion group.*

This implies immediately:

Theorem 0.10. *There is no almost linear minimal field.*

We have also found a complete classification of minimal almost linear groups as certain valued groups, which yields examples showing that all possibilities from the conclusion of Theorem 0.9 can be realized.

Theorem 0.10 implies that a possible counter-example to Conjecture 0.6 would have to be a field which is not almost linear. It is hard to believe that such structures exist. In particular, all known examples of minimal ordered structures are almost linear.

Question 0.11. *Does there exist a minimal ordered structure [group] which is not almost linear?*

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Minkowski dimension and definability

Chris Miller
(joint work with Philipp Hieronymi)

My research is currently focussed on the study of tameness in expansions $\mathfrak{R}$ of the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$.

It is well known that nondefinability of the set of nonnegative integers $\mathbb{N}$ is justifiably regarded as necessary for the definability theory of $\mathfrak{R}$ to be worthy of study. This suggests the question: What can be said about $\mathfrak{R}$ if it does not define $\mathbb{N}$? Of course, we should be most interested in cases where $\mathfrak{R}$ is obtained by expanding $\overline{\mathbb{R}}$ by mathematically "reasonable", "natural" or "interesting" sets. I focus here on a basic case: *Assume from now on that $\mathfrak{R}$ is an expansion of $\overline{\mathbb{R}}$ by constructible sets*, that is, $\mathfrak{R}$ is of the form $(\overline{\mathbb{R}}, (X_i)_{i\in I})$ where I is an index set and each $X_i \subseteq \mathbb{R}^{n_i}$ is a boolean combination of open sets. Note that if $X \subseteq \mathbb{R}^m$ is constructible and $f\colon X \to \mathbb{R}^n$ is continuous, then the graph of f is constructible.

Abstractly, there is a largest such $\mathfrak{R}$; up to interdefinability, it just $(\overline{\mathbb{R}}, \mathbb{N})$. By cell decomposition, every o-minimal expansion of $\overline{\mathbb{R}}$ is an expansion of $\overline{\mathbb{R}}$ by constructible sets; in the context of the tameness program, I regard all o-minimal structures as equally well behaved. Thus, the point is to understand what can be said about $\mathfrak{R}$ if it neither defines $\mathbb{N}$ nor is o-minimal (see [2, 3, 4] for some examples).

If $X \subseteq \mathbb{R}^n$ is constructible, then X either has (nonempty) interior or is nowhere dense. The question arises: When does this hold for all sets definable in $\mathfrak{R}$? (A number of nice properties then follow—see [3]—but it would take us too far afield to discuss this here.) It certainly fails if $\mathfrak{R}$ defines $\mathbb{N}$, as then $\mathbb{Q}$ is definable. The question arises: *If $\mathfrak{R}$ does not define $\mathbb{N}$, does every definable set either have interior or be nowhere dense?* This question turns out to be equivalent to one of interest in its own right concerning a coincidence of natural dimensions on definable sets, as I now explain.

Given $\emptyset \neq E \subseteq \mathbb{R}^n$, let $\dim E$ be the euclidean dimension of E, that is, the maximal $m \in \mathbb{N}$ such that some coordinate projection of E on $\mathbb{R}^m$ has interior. We also put $\operatorname{dimcl} E = \dim \operatorname{cl}(E)$, where $\operatorname{cl}(E)$ is the usual topological closure of E. If E is also bounded, then the upper Minkowski dimension, $\overline{\dim}_{\mathrm{M}} E$, of E is defined by

$$\overline{\dim}_{\mathrm{M}} E = n - \varliminf_{r\downarrow 0} \frac{\log \mu\{\, x \in \mathbb{R}^n : \operatorname{dis}(x, E) < r \,\}}{\log r}$$

where $\operatorname{dis}(x, E)$ denotes the distance of x to E and μ denotes Lebesgue measure in $\mathbb{R}^n$. (There are many different names and equivalent formulations of $\overline{\dim}_{\mathrm{M}}$ in the literature.) For convenience, we put $\overline{\dim}_{\mathrm{M}} \emptyset = \dim \emptyset = -\infty$, and extend the definition of $\overline{\dim}_{\mathrm{M}}$ to arbitrary $E \subseteq \mathbb{R}^n$ by setting

$$\overline{\dim}_{\mathrm{M}} E = \sup\{\, \overline{\dim}_{\mathrm{M}} E' : E' \text{ is a bounded subset of } E \,\}.$$

It is easy to see that $\dim E \leq \operatorname{dimcl} E \leq \overline{\dim}_{\mathrm{M}} \operatorname{cl}(E) = \overline{\dim}_{\mathrm{M}} E \leq n$. Most notions of dimension that are commonly encountered in geometric measure theory (e.g., capacitary, Hausdorff and packing dimensions) are bounded below by dim and above by $\overline{\dim}_{\mathrm{M}}$. Hence, if $\dim E = \overline{\dim}_{\mathrm{M}} E$, then it is fair to say that essentially all dimensions normally encountered in geometric measure theory coincide on E with both dim and dimcl. Observe that $\dim = \overline{\dim}_{\mathrm{M}}$ on all open sets, but it is easy to construct examples of closed sets where this fails (e.g., $\overline{\dim}_{\mathrm{M}} \operatorname{cl}\{1/n : n \in \mathbb{N}^{>0}\} = 1/2$).

It is an exercise to see that every definable set either has interior or is nowhere dense iff dim = dimcl on all definable sets. Thus, if $\dim = \overline{\dim}_{\mathrm{M}}$ on all definable sets, then every definable set either has interior or is nowhere dense. Recently, P. Hieronymi and I (building on earlier joint work with A. Fornasiero [1]) have established the converse by showing that if $\mathfrak{S}$ is any expansion of $\overline{\mathbb{R}}$ that does not define $\mathbb{N}$, then $\dim = \overline{\dim}_{\mathrm{M}}$ on all coordinate projections of closed sets definable in $\mathfrak{S}$. (If every set definable in $\mathfrak{S}$ either has interior or is nowhere dense, then $\mathfrak{S}$ does not define $\mathbb{N}$. All closed sets are trivially projections of themselves.)

Hieronymi and I are currently working to extend our result mentioned above. We hope to show that if $\mathfrak{R}$ does not define $\mathbb{N}$, then $\dim = \overline{\dim}_{\mathrm{M}}$ on all sets definable in $\mathfrak{R}$ (thus showing also that every definable set either has interior or is nowhere dense). At the workshop, I announced that we had managed to show that $\dim = \overline{\dim}_{\mathrm{M}}$ on all boolean combinations of projections of closed definable sets, but we have since discovered a gap in our proof. Thus, our most immediate goal is to attempt to repair this gap.

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Reporter: Javier de la Nuez González

Participants

Prof. Dr. Tuna Altinel
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Andreas Baudisch
Institut für Mathematik
Humboldt-Universität Berlin
Rudower Chaussee 25
10099 Berlin
GERMANY

Prof. Dr. Itai Ben Yaacov
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Alessandro Berarducci
Dip. di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Dr. Elisabeth Bouscaren
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Emmanuel Breuillard
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Enrique Casanovas
Departament de Logica, Historia i
Filosofia de la Ciencia
Universitat de Barcelona
Montalegre 6
08001 Barcelona
SPAIN

Prof. Dr. Zoe Chatzidakis
U.F.R. de Mathématiques
Case 7012
Université Paris 7
75205 Paris Cedex 13
FRANCE

Prof. Dr. Gregory L. Cherlin
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Dr. Artem Chernikov
Institute of Mathematics
The Hebrew University of Jerusalem
Givat-Ram
91904 Jerusalem
ISRAEL

Dr. Annalisa Conversano
Institute of Natural & Math. Sciences
Massey University
Private Bag 102-904
North Shore City Mail Centre
Auckland
NEW ZEALAND

Javier de la Nuez González
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. David M. Evans
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
UNITED KINGDOM

Dr. Antongiulio Fornasiero
Dipartimento di Matematica
Seconda Universita di Napoli
Via Vivaldi 43
81100 Caserta
ITALY

Dr. Jakub Gismatullin
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Isaac Goldbring
Dept. of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
851 S. Morgan St.
Chicago, IL 60607-7045
UNITED STATES

Prof. Dr. Vincent Guingona
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame IN 46556-4618
UNITED STATES

Dr. Immanuel Halupczok
Institut für Mathematische
Logik und Grundlagenforschung
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Deirdre Haskell
Dept. of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton, Ont. L8S 4K1
CANADA

Dr. Martin Hils
U.F.R. de Mathématiques
Case 7012
Université Paris 7
75205 Paris Cedex 13
FRANCE

Prof. Dr. Ehud Hrushovski
Department of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL

Franziska Jahnke
Mathematical Institute
Oxford University
24-29 St. Giles
Oxford OX1 3LB
UNITED KINGDOM

Prof. Dr. Eric Jaligot
Laboratoire de Mathématiques
Université de Grenoble I
Institut Fourier
B.P. 74
38402 Saint-Martin-d'Heres Cedex
FRANCE

Dr. Itay Kaplan
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Byunghan Kim
Department of Mathematics
Yonsei University
Seoul 120-749
KOREA, REPUBLIC OF

Prof. Dr. Piotr Kowalski
Department of Mathematics
University of Wroclaw
50 384 Wroclaw
POLAND

Prof. Dr. Krzysztof Krupinski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Salma Kuhlmann
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz
GERMANY

Prof. Dr. M. Chris Laskowski
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Prof. Dr. Gilbert Levitt
Dépt. de Mathématiques
Université de Caen
LMNO
BP 5186
14032 Caen Cedex
FRANCE

Prof. Dr. Francois Loeser
Institut de Mathématiques de Jussieu
Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. H. Dugald MacPherson
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Dr. Amador Martin-Pizarro
Institut Camille Jordan
C.N.R.S.
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Chris Miller
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Ludomir Newelski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Alf Onshuus
Depto. de Matematicas
Universidad de Los Andes
Carrera 1a. No. 18-A-10
Apartado Aero 4976-12340
Bogota
COLOMBIA

Prof. Dr. Abderezak Ould-Houcine
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Dr. Davide Penazzi
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Prof. Dr. Kobi Peterzil
Dept. of Mathematics & Computer Sciences
University of Haifa
Mount Carmel
Haifa 31905
ISRAEL

Prof. Dr. Anand Pillay
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Prof. Dr. Francoise Point
Institut de Mathématiques
175, rue du Chevaleret
75013 Paris
FRANCE

Prof. Dr. Bruno Poizat
Institut Girard Desargues
Université Lyon 1
Batiment Braconnier
21 Avenue Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Dr. Zlil Sela
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Dr. Pierre Simon
Institute of Mathematics
The Hebrew University of Jerusalem
Givat-Ram
91904 Jerusalem
ISRAEL

Dr. Rizos Sklinos
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Givat Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Sergei S. Starchenko
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-4618
UNITED STATES

Prof. Dr. Charles Steinhorn
Vassar College
Box 453
Poughkeepsie, NY 12604
UNITED STATES

Prof. Dr. Katrin Tent
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Alex Usvyatsov
CMAF & University of Lisbon
Centro de Matematica e Aplicacoes
Fundamentais
Av. Prof. Gama Pinto, 2
Lisboa 1649-003
PORTUGAL

Prof. Dr. Frank Olaf Wagner
Institut Camille Jordan
Université Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Boris I. Zilber
Mathematical Institute
Oxford University
24-29 St. Giles
Oxford OX1 3LB
UNITED KINGDOM