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Progress in Surface Theory

Organised by
Uwe Abresch, Bochum
Franz Pedit, Tübingen
Masaaki Umehara, Tokyo

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Abstract. Over the last 30 years global surface theory has become pivotal in the understanding of low dimensional global phenomena. At the same time surface geometry became a platform on which seemingly different areas of mathematics – such as geometric and topological analysis, integrable systems, algebraic geometry of curves, and mathematical physics – coalesce to produce far reaching ideas, conjectures, methods and results. The workshop hosted talks on the resolutions of famous conjectures in surface geometry, including the Willmore conjecture, and on exciting new progress in the understanding of moduli spaces of special surface classes.

Mathematics Subject Classification (2010): 53, 58, 14Hxx, 51Pxx.

Introduction by the Organisers

The workshop *Progress in Surface Theory* brought together 26 participants including PhD and PostDoc researchers. The 21 talks were evenly spread over the duration of the workshop and generally lasted 60 minutes, except for shorter 50 or 30 minute presentations by junior researchers. The schedule allowed ample time for discussions and ongoing and emerging collaborations between participants. The workshop also hosted a well attended and lively problem session. Incidentally, the workshop *Geometric Knot Theory* took place during the same week and the morning session on Wednesday was held jointly with talks by Andre Neves and Joel Hass.

A central theme of the meeting was the study of moduli spaces of special surface classes, including Willmore, minimal, and constant mean/Gauss curvature surfaces, by different methods such as geometric analysis, integrable systems and algebraic curve theory. Researchers in these fields benefit significantly from mutual interactions and the workshop provided a stimulating atmosphere for such exchanges.

A number of lectures addressed the recent resolutions of famous conjectures in global surface geometry. Andre Neves gave a series of extraordinary talks on applications of min-max theory, including the solution of the Willmore conjecture and a solution of a conjecture about the minimal Möbius energy link configuration. A further highlight was the resolution of a generalized version of the Lawson conjecture (first mentioned by Pinkall and Sterling), stating that the only embedded constant mean curvature tori in the 3-sphere are the rotational ones. The proof, which uses non-collapsing arguments developed for the mean curvature flow and ideas from Brendle's proof of the Lawson conjecture, was beautifully explained in a lecture by Ben Andrews. Using techniques from algebro-geometric integrable systems, Martin Schmidt characterized the moduli space of constant mean curvature cylinders of finite spectral genus in the 3-sphere in terms of their hyper-elliptic spectral curves. When applied to the special situation of constant mean curvature tori this description provides a conceptually different proof of the (generalized) Lawson conjecture. Finally, Wilhelm Klingenberg outlined elements for a proof of the Caratheodory conjecture (in joint work with Brendan Guilfoyle) by combining Lagrangian geometry and mean curvature flow.

A significant advance in the understanding of higher genus constant mean curvature surfaces in the 3-sphere was presented by Sebastian Heller, who outlined a program to understand their moduli via Abelianization of flat connections. A first glimpse of the progress made was a description of Lawson's genus 2 minimal surface (and its constant mean curvature deformations) in terms of an explicit family of Fuchsian connections over the Riemann sphere and the (as yet numerical) solution of its accessory parameter problem.

There were several additional lectures at the interface of geometric analysis and integrable systems. In one such lecture, Francis Burstall explained how to associate a Lagrangian density to a map into the space of lines (known already to Darboux) and characterized the "harmonic maps" in this setting. Special cases include the mean curvature sphere congruence of Willmore surfaces and a recently studied functional on Lagrangian surfaces (arising from a cubic differential) in the complex projective plane. Christoph Bohle used the Weierstrass representation of a conformal immersion given by solutions of the Dirac operator (associated to the induced spin bundle of the immersion) with mean curvature potential to construct constant mean curvature disks with prescribed (bi-normal) boundary values. On a related topic Ulrich Pinkall defined a gradient flow of the Willmore functional on the submanifold of mean curvature potentials giving rise to conformal immersions. Thus, rather than flowing the geometric object (the conformal immersion) directly the flow acts on an infinitesimal invariant of the surface. By design the flow preserves the conformal structure of the immersion and its fixed points are constrained Willmore surfaces. It is known that constrained Willmore tori can be obtained from spectral curves of finite genus. Lynn Heller characterized all constrained Willmore tori of spectral genus $g \leq 2$ in terms of constrained elastic curves on the 2-sphere. She also conjectured that the minimizers for the Willmore energy in conformal classes near the Clifford torus should be among those constrained Willmore Hopf tori. Drawing upon an analogue between the Riemann mapping theorem and the Plateau problem for minimal surfaces, Laura Desideri outlined an approach to solve the Plateau problem with analytic boundary via the universal Schlesinger system. The latter is a generalization of the classical Schlesinger system arising in the study of isomonodromic deformations of Fuchsian connections. Atsufumi Honda studied (extrinsically) flat fronts, that is flat surfaces with special singularities, in Euclidean space and the 3-sphere and also described a transformation theory of these surfaces. Finally, Mark Haskins reported on some recent progress in the construction of compact 7-manifolds with G_2-holonomy via twisted connected sums of asymptotical cylindrical Calabi-Yau 3-folds.

The 3-dimensional spaces of constant curvature are the classical target spaces for surface geometry. Recent years have seen continued interest in the study of surfaces in other 3-dimensional target spaces, especially those which are homogeneous. Besides intrinsic motivations for their study, one can apply rescaling arguments to construct special surfaces (e.g. minimal) in the classical target spaces. Josef Dorfmeister explained a loop group description for minimal surfaces in the 3-dimensional Heisenberg group. Joaquin Perez studied constant mean curvature surfaces in 3-dimensional Lie groups endowed with a left-invariant metric and related the isoperimetric problem to the Cheeger constant and the critical mean curvature of the ambient space. Considering periodic minimal surfaces as maps into tori, Toshihiro Shoda computed the index and nullity of families arising from the Abel maps of hyper-elliptic Riemann surfaces. Using the Lie quadric as a target space, Udo Hertrich-Jeromin discretized linear Weingarten surfaces (whose Gauss and mean curvature satisfy an affine relation) in any of the space forms (also with signatures) and indicated their integrable structure and transformation theory.

Three lectures addressed recent progress in the theory of isoparametric hypersurfaces in spheres. First, Hui Ma studied the Gauss maps of isoparametric hypersurfaces as special examples of minimal Lagrangian submanifolds in the complex quadric. For instance, the Gauss maps of homogeneous isoparametric hypersurfaces turn out to be Hamiltonian stable. In a related talk, Reiko Miyaoka interpreted the Karcher-Münzer-Ferus polynomials, which arise in the study of non-homogeneous isoparametric hypersurfaces, as moment maps for the Spin action. Finally, Anna Siffert used the relationship (given by the Gauss map) between hypersurfaces in spheres and Lagrangian submanifolds in the complex quadric to outline a structural approach to the classification problem of isoparametric hypersurfaces.

The talks gave a balanced view of current results and ongoing research in the field of differential geometry of surfaces. At the same time the lectures demonstrated that the tandem of geometric analytical and integrable systems techniques can significantly deepen our understanding of the properties of special surface classes. In this sense, the workshop provided an ideal backdrop for the exchange of ideas, understanding of techniques, stimulation of collaboration, and development of new approaches in the field. For instance, it would not be surprising if the interior ball curvature estimates were more widely applicable and could lead to more general rigidity results, including a different proof of the Willmore conjecture; or if the integrable systems techniques combined with gradient flow methods could be used to give a more detailed picture (and perhaps even a proof) of a yet to be formulated constrained Willmore conjecture; or if the moduli of holomorphic and flat bundles over complex curves could provide the correct setting to obtain a complete picture of constant mean curvature surfaces of higher genus and with ends.

Workshop: Progress in Surface Theory

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Abstracts

Non-collapsing and the Lawson and Pinkall-Sterling conjectures

Ben Andrews

In this talk I aim to present the recent proof by Simon Brendle [4] of the Lawson conjecture. In particular I describe some of the geometric background to his proof, including the 'non-collapsing' estimate which first appeared in the context of mean curvature flow, and show how it is used in Brendle's proof as well as in my proof with Haizhong Li [3] of the Pinkall-Sterling conjecture.

The Lawson conjecture [7] is as follows:

Every embedded minimal torus in the three-dimensional sphere is congruent to the Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$.

Later Pinkall and Sterling [9] made the following conjecture:

Every embedded constant mean curvature torus in the three-dimensional sphere is axially symmetric.

Part of the difficulty of these conjectures arises from the need to use both the global torus topology (which appears naturally in descriptions of the minimal surface using complex analysis and the Hopf differential, but is hard to detect using the local geometry of the surface) and the embeddedness of the surface (which is very difficult to relate to the Hopf differential). Without the torus topology there are many examples of embedded minimal surfaces (constructed first by Lawson [6]). With torus topology but without embeddedness the result also fails: Examples of immersed minimal tori can be constructed as surfaces of rotation [8] or more generally as solutions of an integrable system [9].

There are several ingredients that go into Brendle's proof of the Lawson conjecture: The torus topology comes into the proof in just one place: The fact that a minimal torus has no umbilic points, which can be deduced from properties of the Hopf differential. The embeddedness comes in through a geometric argument which I call 'non-collapsing', which first appeared in my work on mean curvature flow [1]. This method allows comparison of the curvature of the largest touching ball at each point to other geometric quantities involving the pointwise curvatures of the surface. In the case of mean curvature flow the result of [1] states that for a mean-convex hypersurface moving by mean curvature flow, the curvature of the largest touching ball at each point remains bounded by a multiple of the mean curvature, if this is true initially. This is proved by a maximum principle argument, applied to a function of two points x and y in the hypersurface: The curvature of the largest ball touching at x is given by

$$\overline{k}(x) = \sup_{y \neq x} \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2}.$$

The estimate follows by applying the maximum principle to the function

$$Z(x,y) = \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2 H(x)}$$

where H is the mean curvature. The computation is somewhat lengthy but otherwise straightforward. In [2] this computation was interpreted as saying that the function $\overline{k}$ is a viscosity subsolution of the linearised mean curvature flow, while H is a solution of the same equation.

In applying this method to minimal surfaces, Brendle had the wonderful idea of comparing the curvature of the largest touching ball to the maximum principal curvature. That is, his computation is equivalent to applying a maximum principle to the function $Z(x,y)$ above, where $H(x)$ is replaced by the maximum principal curvature $\kappa(x)$. Since there are no umbilic points, κ is a smooth positive function, and the Simons' identity can be written as an elliptic partial differential equation satisfied by κ, which plays the same role here as the linearised mean curvature flow did in the non-collapsing result for mean curvature flow. The computation is more delicate than in the mean curvature flow case, and one must use all the available terms, but the end result is that $\overline{k}$ is a strict viscosity subsolution of the same elliptic PDE at points where Z is greater than 1. The maximum principle implies that $\overline{k} \leq \kappa$ everywhere. But on the other hand any touching ball can have curvature no smaller than κ, so necessarily $\overline{k} = \kappa$ identically. This is a beautiful argument and a spectacular result!

To finish the proof, Brendle observes that every point x has a touching ball of curvature $\kappa(x)$, so along the corresponding principal direction this ball agrees with the surface up to second order. It follows that the derivative of κ along this direction must vanish (since otherwise the surface crosses the boundary of the ball in the direction where κ increases). Thus one component of the derivative of the second fundamental form vanishes. Now repeating the argument with balls touching on the other side of the surface shows that the other component also vanishes (a minimal surface only has two independent components of the derivative of second fundamental form). Therefore the second fundamental form is parallel, and the conclusion that the surface is Clifford follows easily.

Later Haizhong Li and I adapted the argument to constant mean curvature tori. We were partly motivated by a computation of Randol and Perdomo which produced examples of axially symmetric embedded tori of constant mean curvature H, for any value of H other than 0 and $1/\sqrt{3}$. We thought the argument might work only in the case $H = 1/\sqrt{3}$, perhaps. However the story worked out rather differently: We were able to prove that if $H > 0$, then $\overline{k} = \kappa$, where κ is the larger principal curvature (consideration of the Hopf differential again implies there are no umbilic points in this case). The argument is similar, but instead of comparing $\overline{k}$ to κ, we have to compare $\overline{k} - H$ to $\kappa - H$. This gives as before vanishing of one of the components of the derivative of second fundamental form. However if we touch with balls on the other side of the surface then the computation no longer works, since the mean curvature changes sign and becomes negative. However we were able to deduce from this weaker information that the surfaces are axially symmetric. We later found out that this was an explicit conjecture made by Pinkall and Sterling in their 1989 paper. By also understanding the ODE corresponding to axially symmetric CMC surfaces, we were able to completely classify the embedded CMC tori in the three-sphere.

Finally, I report on some more recent work: With Xuzhong Chen (ECNU) I have been looking at how the argument applies to 'Weingarten tori' where other curvature equations are satisfied. The nonlinearity of the problem introduces some additional complications, but some of these were previously handled for flows by nonlinear functions of curvature in my joint work with Langford and McCoy [2]. The technique works for a reasonably wide range of curvature functions: We can compare $\overline{k}-\kappa$ to the difference between the principal curvatures, $\kappa-\kappa_2$, and deduce that the ratio is in fact zero (note that this choice agrees with the ones that worked for minimal surfaces and for constant mean curvature surfaces). We need to impose some conditions, which include monotonicity to ensure that the equation is elliptic, as well as some further convexity conditions. Fortunately Bryant [5] has proved that for an amazingly wide class of such equations, Weingarten tori do not have umbilic points. Thus all embedded Weingarten tori satisfying an equation in this class are axially symmetric. A particular case of interest is where $\kappa_1 + a\kappa_2 = b$. Here the method works provided $0 < a \leq 1$ and $b \geq 0$, so we can deduce axial symmetry for embedded tori of this kind. In general we cannot deduce that these tori are Clifford, since the argument breaks down for balls touching on the other side of the surface. However if $b = 0$ we can use the fact that we already know axial symmetry to make the argument work on the other side of the surface and deduce that the surface is Clifford. Note that in this case the restriction $a < 1$ is superfluous, since we can reverse the normal direction to replace a by $1/a$.

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Spectral geometry of immersed discs

Christoph Bohle
(joint work with Ulrich Pinkall)

Spectral geometry in its usual sense refers to the relation between spectral properties of geometrically defined operators and the intrinsic geometry of the underlying spaces. In this short note (based on [3]) we deal with another kind of spectral geometry, the extrinsic spectral geometry of immersed spheres and discs.

Attached to every conformal immersion $f\colon M \to \mathbb{R}^3$ of a compact oriented surface M, possibly with $\partial M \neq \emptyset$, is a *Dirac operator* $\mathcal{D}_f$ acting on quaternion valued functions $\lambda\colon M \to \mathbb{H}$ by

$$\mathcal{D}_f\lambda = -\frac{df \wedge d\lambda}{|df|^2}, \tag{1}$$

where $\mathbb{R}^3$ is identified with the imaginary quaternions and $|df|^2$ denotes the volume form induced by f. This operator quite naturally appears in the description [4, 5] of conformal deformations: an immersion $\tilde{f}\colon M \to \mathbb{R}^3$ induces the same conformal structure as f and is regularly homotopic to f if and only if

$$d\tilde{f} = \bar{\lambda} df \lambda \tag{2}$$

with $\lambda\colon M \to \mathbb{H}_* \cong \mathbb{R}_+ Spin(3)$. Conversely, a function λ yields, via (2), a closed form $d\tilde{f}$ (and hence locally a conformal deformation $\tilde{f}$ of f) if and only if

$$\mathcal{D}_f\lambda = \rho\lambda \tag{3}$$

for a real valued function ρ. The function ρ describes, via $\tilde{H}|d\tilde{f}| = H|df| + \rho|df|$, the change of mean curvature half density. The operator $\mathcal{D}_f$ is elliptic and formally self–adjoint, and differs from an Atiyah–Singer–Dirac– or $\bar{\partial}$–operator by a potential proportional to the mean curvature half density $H|df|$.

Case $\partial M = \emptyset$: Because $\mathcal{D}_f$ is elliptic and formally self–adjoint, if $\partial M = \emptyset$ the spectrum of $\mathcal{D}_f$ is a sequence of real numbers and there is an L^2–orthonormal basis of eigenspinors. In particular, in the simply connected case $M = S^2$ the spectrum can be geometrically realized by the (possibly branched) immersions $\tilde{f}$ obtained via (2) from the eigenspinors of $\mathcal{D}_f$, see Figure 1.

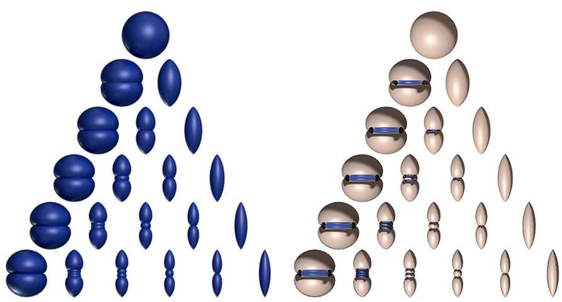

Figure 1: The periodic table of Dirac spheres: a geometric realization, via (2), of the spectrum of $\mathcal{D}_f$ for f the immersion of the round 2–sphere. The first row corresponds to the eigenvalues $\mu = 0$ and -2, the second row to the eigenvalues $\mu = 1$ and -3, the third row to $\mu = 2$ and -4 ... (Pictures by Keenan Crane)

Recently, interest in the above approach to conformal deformations has been generated by the computer graphics algorithm of [5]. Its idea is to describe a conformal deformation $\tilde{f}$ of f by the potential ρ, i.e., by the change of mean curvature half density. For generic ρ the operator $\mathcal{D}_f - \rho$ will have trivial kernel. However, $\mathcal{D}_f - \rho$ being self–adjoint and elliptic, spectral theory suggests taking an eigenspinor λ with $(\mathcal{D}_f - \rho)\lambda = \mu\lambda$ for μ an eigenvalue of $\mathcal{D}_f - \rho$ with smallest possible modulus as the "best approximation" to a section in the kernel of $\mathcal{D}_f - \rho$.

One can think of $\tilde{f}$ thus defined as a solution to the following "physical model": for a given conformal structure and "energy distribution" prescribed by the change of mean curvature half density, one tries to find a realization $\tilde{f}$ in space. For a generic energy distribution ρ this is only possible after a suitable small deformation by adding a constant $\rho \rightsquigarrow \rho + \mu$. The spectrum of $\mathcal{D}_f$ itself corresponds to the constant energy distributions μ that allow a realization in space. (That only discrete values of μ are realizable can be "understood" from Figure 1, if one assumes that the formation of new loops needs a certain increase in energy...)

Case $\partial M \neq \emptyset$: The aim of this note is to explain how an analogue of the story told so far about surfaces without boundary can be obtained for surfaces with boundary, if one imposes the right boundary conditions.

The boundary conditions we impose for $\mathcal{D}_f$ are *local*, i.e., the restriction $\lambda_{|\partial M}$ of λ to ∂M has to take values in a prescribed orientable subbundle E of the trivial $\mathbb{H}$–bundle over ∂M. The most important case is when E is two–dimensional and hence of the form

$$E = \{\lambda \in \mathbb{H} \mid V\lambda = \lambda\tilde{V}\} \tag{4}$$

for a pair of functions V, $\tilde{V} \colon \partial M \to S^2$ (unique up to a common factor ± 1).

The *canonical frame* of f along ∂M is (T, N, B) with N the Gauss–map, T the positive unit vector field tangent to $f_{|\partial M}$, and $B = T \times N$ the binormal field. It gives rise to three canonical types of local boundary conditions: if $V = T$, N or B, then $\tilde{f}$ given by (2) with boundary condition (4) has prescribed $\tilde{T} = \tilde{V}$, $\tilde{N} = \tilde{V}$, or $\tilde{B} = \tilde{V}$, respectively.

Theorem ([3]) *If $\partial M \neq \emptyset$, a local boundary condition E for $\mathcal{D}_f$ is*

- elliptic *iff E is 2–dimensional and $V_p \neq \pm N_p$ for all $p \in \partial M$,*
- self–adjoint *iff E is 2–dimensional and $V_p \perp T_p$ for all $p \in \partial M$.*

If one imposes a self–adjoint, elliptic boundary condition for $\mathcal{D}_f$, then as in the case $\partial M = \emptyset$ the spectrum is a sequence of real numbers and there is an L^2–orthonormal basis of eigenspinors.

The most natural choice of self–adjoint, elliptic boundary condition for $\mathcal{D}_f$ is arguably the *binormal boundary condition* $(V, \tilde{V}) = (B, B)$. (Thus, the "physical model" above carries over to surfaces with non–empty boundary if one prescribes, in addition to the conformal structure and the change of mean curvature half density, the binormal vector field of $\tilde{f}$.)

We define the *spectrum* for immersions of surfaces with boundary as the spectrum of $\mathcal{D}_f$ with this boundary condition. In the simply connected case, i.e., when $M = D$ is a disc, the spectrum can again be geometrically realized by the (possible branched) immersions $\tilde{f}$ obtained via (2) from the eigenspinors of $\mathcal{D}_f$. All immersions $\tilde{f}$ thus obtained have the same binormal vector fields, and their mean curvature half densities differ by constant multiples of $|df|$.

An additional feature only present if $\partial M \neq \emptyset$ is the canonical deformation of the spectrum obtained through the loop of self–adjoint, elliptic boundary conditions

$$(V, \tilde{V}_t) = (B, \cos(t)B - \sin(t)N), \qquad t \in \mathbb{R}/2\pi\mathbb{Z} = S^1. \tag{5}$$

Going around one full period in this loop does not change the spectrum, but continuously following the ordered sequence of eigenvalues during the deformation might result in a shift of the spectrum known as *spectral flow* [1]. For $M = D$ a disc, this shift can be computed by the following theorem (which more generally holds for arbitrary periodic families $(V, \tilde{V}_t)$ of self–adjoint, elliptic boundary conditions):

Theorem ([3]) *If $M = D$ is a disc, the spectral flow of $\mathcal{D}_f$ with a periodic family of boundary conditions $(V, \tilde{V}_t)$ equals the degree of $\tilde{V}$ seen as a map from $T^2 = S^1 \times \partial M$ to S^2.*

For all immersed discs that are rotational symmetric on a small neighborhood of their boundary and have vertical binormals, the family (5) of boundary conditions has a non–trivial spectral flow. In fact, its spectral flow is ± 1 for each *half*–rotation of the binormal vector field (cf. the proof of Theorem 4 in [3]; although the boundary condition is only 2π–periodic, the operators with boundary conditions $t = 0$ and $t = \pi$ are equivalent, because $\tilde{B}_{t=0} = i$ and $\tilde{B}_{t=\pi} = -i$ or vice versa.)

To finish the paper, we discuss an example for which the spectrum can be explicitly computed: if f parametrizes the round half–sphere, the spectrum of $\mathcal{D}_f$ with binormal boundary condition coincides with that of the full sphere (in fact, the proof given in [3] goes through for all immersed discs that are one "half" of a reflectional symmetric immersion of the sphere with rotational symmetry.)

Numerical experiments suggest that in this example one can, under the deformation (5) of binormal boundary conditions, continuously follow the flow of certain eigenvalues such that the family of immersed discs obtained via (2) from corresponding eigenspinors is rotational symmetric. This would amount to "wrapping up" the half sphere around its boundary. In the resulting family of immersed discs, the immersions with vertical binormals are "halfs" of the immersions shown on the left hand side of the pyramid in Fig. 1 and each half–rotation of the binormals corresponds to moving up or down one of the rows in Fig. 1. Because Dirac spheres are related to mKdV–solitons, see [2], from this perspective spectral flow appears as a continuous geometric realization of the process of "adding solitons".

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Projective geometry and harmonic maps

Francis Burstall

There are a number of variational problems arising in classical differential geometry that share common features: there is an associated Gauss map whose harmonic map energy coincides with the given functional and, moreover, solutions to the problem are characterised by harmonicity of this Gauss map. Examples include the Willmore functional in conformal geometry [1, 2] and the projective/Lie sphere area in projective/Lie sphere geometry [1, 3].

In this note, we sketch a uniform approach to these matters based on the observation that all the surfaces participating in these theories may be viewed as *line congruences*.

Line congruences. We begin with an n-dimensional complex projective space $\mathbb{P}^n = \mathbb{P}(V)$ and denote by $G_2(V)$ the Grassmannian of (projective) lines in $\mathbb{P}(V)$.

Definition. A *line congruence in* $\mathbb{P}(V)$ is a map $L : \Sigma \to G_2(V)$ of a (real) surface which admits *focal surfaces*, thus $X, Y : \Sigma \to \mathbb{P}(V)$ such that, for each $p \in \Sigma$, $L(p)$ is tangent to X and Y at p.

Thus, to be a line congruence is a first order condition on a map $L : \Sigma \to G_2(V)$. This condition is automatically satisfied for generic L when $n = 3$.

If L is a line congruence with focal surfaces X and Y then we have a decomposition $T\Sigma^{\mathbb{C}} = T^+\Sigma \oplus T^-\Sigma$ defined by the requirement that $\mathrm{d}_{U^-}X$ and $\mathrm{d}_{U^+}Y$ lie along L, for $U^\pm \in T^\pm$. We note:

(1) Σ acquires a (possibly indefinite) conformal structure with null directions $T^\pm$.

(2) $T^\pm$ are orthogonal with respect to the second fundamental forms of X and Y (in classical terms, we have *conjugate nets* on X and Y).

Let $\mathrm{d} = \partial^+ + \partial^-$ with $\partial^\pm : \Omega^0 \to \Gamma((T^\pm)^*) =: \Omega^\pm$ and let $\xi, \eta : \Sigma \to V$ be lifts of X, Y: $X = [\xi]$, $Y = [\eta]$. Our requirements read

$$\partial^-\xi = \alpha\xi + \beta\eta$$
$$\partial^+\eta = \gamma\xi + \delta\eta,$$

with $\alpha, \beta \in \Omega^-$ and $\gamma, \delta \in \Omega^+$.

Definition ([5, Part 2, Chapter II])**.** The *Laplace invariant* of a line congruence L is the (complex) 2-form $\ell(L) := \beta \wedge \gamma$.

We view ℓ as a Lagrangian density and so define a $\mathrm{PGL}(V)$-invariant functional on line congruences:

$$W(L) = \tfrac{1}{2}\int_\Sigma \ell(L).$$

As we will see, when reality conditions are imposed on L, this functional is already familiar in a variety of contexts.

Examples.

Conformal geometry. Let $V = \mathbb{H}^2$ or, equivalently, $V = \mathbb{C}^4$ along with a quaternionic structure j. The quaternionic projective line $\mathbb{HP}^1 \cong S^4$ may be identified with the j-stable elements of $G_2(V)$ so that maps $\Sigma \to \mathbb{HP}^1$ are the same as j-stable line congruences. For such a line congruence, our functional coincides (up to a topological term) with the Willmore functional.

Klein correspondence. Let $V = \mathbb{C}^4$ and recall that $G_2(V)$ can be identified with a quadric Q in $\mathbb{P}(\bigwedge^2 V)$ via $W \mapsto \bigwedge^2 W$. Lines in Q correspond to the pencil of lines through a fixed point of $\mathbb{P}(V)$ and lying in a fixed plane. Thus we identify the space Z of lines in Q with the space of contact elements (incident pairs of points and planes) in $\mathbb{P}(V)$. Now a map $L : \Sigma \to Z \subset G_2(\bigwedge^2 V)$ is the same as a map $(f, f^*) : \Sigma \to \mathbb{P}(V) \times \mathbb{P}(V^*)$ with $f \leq f^*$. One checks that L is a line congruence if and only if (f, f^*) is the contact lift of f: $\mathrm{d}\phi \in \Omega^1(f^*)$, for any $\phi \in \Gamma f$.

Now impose reality conditions on V: if V is real, then a line congruences in Q is the contact lift of $f : \Sigma \to \mathbb{RP}^3$ and $W(L)$ is the projective area of Blaschke and Thomsen [1] and the conformal structure on Σ is that for which the asymptotic directions of f are null.

Again, the presence of a Hermitian structure of signature $(2, 2)$ on V induces (via the Hermitian Hodge $*$-operator), a real structure on $\bigwedge^2 V$ so that Q is defined by a metric of signature $(4, 2)$. This is the context of Lie sphere geometry [4] and a real line congruence L in Q can be viewed as the contact lift of a surface $F : \Sigma \to S^3$ and $W(L)$ is the Lie sphere area of F [1] (for a modern treatment, see [3]).

Finally, a Hermitian structure of signature $(3, 1)$ induces a quaternionic structure j on $\bigwedge^2 V$ preserving Q. Now j-stable line congruences in Q amount to contact lifts of the form $(f, f^\perp)$ and these, in turn, amount to Legendre maps into the CR 5-sphere. Such a map projects to a Lagrangian surface $F : \Sigma \to \mathbb{CP}^2$ and contracting the second fundamental form of F with the Kähler form of $\mathbb{CP}^2$ gives a symmetric cubic form C. We now have

$$W(L) = \tfrac{1}{2} \int_\Sigma |C^{3,0}|^2.$$

This functional has been recently discussed by Wang [7].

Gauss map. When $n = 2k + 1$ is odd, we can construct a Gauss map for line congruences as follows. Set

$$S^+ = \langle (\mathrm{d}_{U^+})^j \xi \colon 0 \leq j \leq k \rangle$$
$$S^- = \langle (\mathrm{d}_{U^-})^j \eta \colon 0 \leq j \leq k \rangle$$

and assume that $S^{\pm}$ are pointwise complementary: $S^{+}(p) \oplus S^{-}(p) = V$, for all $p \in \Sigma$. Then $S = (S^{+}, S^{-})$ defines a map into the space

$$\mathcal{S} = \{(W^{+}, W^{-})\colon W^{\pm} \leq V, \dim W^{\pm} = k+1, V = W^{+} \oplus W^{-}\}$$

which is a complexified (para-)Hermitian symmetric space.

By an old argument of Lichnerowicz [6], the harmonic map energy $E(S)$ of S splits into holomorphic and antiholomorphic parts $E(S) = E^{+}(S) + E^{-}(S)$ whose difference is topological. We now have:

- S is conformal with respect to the conformal structure induced by L.
- $E^{+}(S) = W(L)$. Thus if S is harmonic, L is W-critical with respect to variations through line congruences.
- In fact, if L is W-critical with respect to such variations, S is harmonic.

Thus the well-developed theory of harmonic maps may be applied to the study of W-critical line congruences.

This Gauss map has familiar geometrical content in our examples:

(1) When $L : \Sigma \to \mathbb{HP}^1$, $\mathcal{S}_j$, the j-stable part of $\mathcal{S}$ in which S takes values, is the space of 2-spheres in S^4 and S is thereby identified with the central sphere congruence of Blaschke–Thomsen [1] or , equivalently, the conformal Gauss map of Bryant [2].

(2) When $L : \Sigma \to Q$ is a contact lift in projective, resp. Lie sphere geometry, S amounts to the congruence of Lie quadrics, resp. cyclides (see [3] for more details).

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”Universal” Schlesinger system and the Plateau problem

Laura Desideri

Following Garnier’s ideas, we obtained a more constructive resolution to the Plateau problem for polygonal boundary curves, based on deformations by the Schlesinger system of minimal disks spanned by polygons. We present it here, with its possible generalization to rectifiable boundary curves: the aim would be to exhibit an infinite-dimensional integrable system, which would describe deformations of minimal disks with rectifiable boundary curves. This is still a work in progress.

1. Motivation

There is a strong analogy between Riemann conformal mapping theorem and the Plateau problem, and their different approaches — and even more than an analogy since the former can be seen as a consequence of the latter in the case of planar boundary curves. Concerning Riemann theorem, there are mainly three resolutions:

(1) the variational method, by Riemann himself, working for any continuous boundary, very powerful, but not constructive,
(2) the Schwarz–Christoffel solution, which only works for polygonal boundary curves, but which is merely explicit (except for the determination of the lengths of the polygon),
(3) a more recent resolution due to Wiegmann and Zabrodin [4] for analytic curves, based on an integrable hierarchy (namely the dispersionless 2D Toda hierarchy).

The corresponding resolutions of the Plateau problem are, for (1) of course the Douglas–Radò solution. The resolution we obtained in [5] following Garnier’s point of view is exactly the generalization of (2) – it relies on deformations governed by the Schlesinger system whose dimension coincides with the number of vertices. It might thus be natural to wonder whether there exists an analog of (3) for the Plateau problem, that is to say an infinite-dimensional integrable system that would generalize the Schlesinger system, and that would describe minimal disks with rectifiable (or analytic) boundary curves. Such a resolution would inherit the constructive nature of the proofs for polygons, but would be much more general. The aim is thus not to carry out a polygonal approximation technique and to study convergence of minimal immersions as we already did with R. Jakob [6], but really to extend Garnier’s point of view to a larger class of boundary curves.

What makes also this question relevant is that there exists a natural candidate for the system we are looking for: the“universal” Schlesinger system given by D. Korotkin and H. Samtleben in [3], an infinite-dimensional Schlesinger system, whose geometrical meaning is not clear up to now.

2. Garnier’s approach to the Plateau problem

What follows is an overwiev of the paper [5], which is a new proof of the polygonal Plateau problem which has been initiated by Garnier [1]. The method relies on the fact that, thanks to the spinor Weierstrass representation, we can associate locally any minimal surface in Euclidean $\mathbb{R}^3$ with an ordinary 2×2 differential system (A)

$$Y' = A(x)Y.$$

When the minimal surface is a minimal disk with a polygonal boundary curve, it appears that many geometrical properties of the surface can be read on its system: the correspondence become explicit, which means that it is relevant to build and describe such minimal disks through their associated differential systems.

Let us denote by N the number of vertices of the polygonal curves, and by $D = (D_1, \dots, D_N)$ their oriented directions. As for the Schwarz–Christoffel solution, we explicitly prescribe D, but not the lengths. We thus introduce the spaces:

$$\mathcal{M}_D^N = \{\text{minimal disks with a polygonal boundary curve of direction } D\}$$
$$\mathcal{A}_D^N = \left\{\text{differential systems } (A) \text{ associated with an } M \in \mathcal{M}_D^N\right\},$$

which are in a 1-to-1 correspondence. The proof is then in three steps.

(1) Characterization of $\mathcal{A}_D^N$: we prove that (A) belongs to $\mathcal{A}_D^N$ if and only if:

(a) (A) is a Fuchsian system on the Riemann sphere $\overline{\mathbb{C}}$, which means it writes

$$(A) \qquad Y' = A(x)Y, \qquad A(x) = \sum_{i=1}^{N} \frac{A_i}{x - t_i},$$

(b) its monodromy is determined by the direction D,
(c) it satisfies a reality condition.

(2) Explicit description of $\mathcal{A}_D^N$: the Schlesinger system describes the variations of the residue matrices $A_i = A_i(t)$ under isomonodromic deformations of Fuchsian systems (A) of parameter the position $t = (t_1, \dots, t_N)$ of its singularities:

$$(1) \qquad \begin{aligned} \frac{\partial A_i}{\partial t_j} &= \frac{[A_i, A_j]}{t_i - t_j} \qquad (i \neq j) \\ \frac{\partial A_i}{\partial t_i} &= -\sum_{j \neq i} \frac{[A_i, A_j]}{t_i - t_j}. \end{aligned}$$

This provides us with an explicit parametrization of the minimal disks:

$$\mathcal{M}_D^N = \left(M_D(t) \mid t \in \mathbb{R}^N, t_1 < \cdots < t_N\right).$$

(3) Study of the lengths of $M_D(t)$: to end the proof, we need to show that during the deformation, all the possible values for the lengths of the boundary curves $P_D(t) = \partial M_D(t)$ are reached. This is the most technical part of the resolution. It relies on a careful study of the confluence of singularities in the Schlesinger system (see [2]).

We recover in this construction the Schwarz–Christoffel solution when the system (A) is diagonal and its monodromy is reducible.

3. Universal Schlesinger system

Let us briefly introduce the universal Schlesinger system obtained by Korotkin and Samtleben in [3]. Their aim is to investigate symmetries of the Schlesinger system (1): they find a symmetric uniform formulation of it, that they then use to generalize Okamoto's equation to the case of an arbitrary number of poles.

Starting with a Fuchsian system (A) non singular at infinity, they introduce new dependent variables

$$B_n = \sum_{i=1}^{N} t_i^n A_i = \operatorname{Res}\big(x^n A(x), x = \infty\big), \qquad n \geq 0$$

and new differential operators

$$L_m = \sum_{i=1}^{N} t_i^{m+1} \frac{\partial}{\partial t_i}, \qquad m \geq -1,$$

which satisfy the commutation relations of the Virasoro algebra. They then prove that, if the residue matrices $A_i(t)$ solve the Schlesinger system (1), then the L_m act on the B_n as follows

$$L_m B_n = \sum_{k=1}^{n-1} [B_k, B_{m+n-k}] + nB_{m+n} \tag{2}$$

for all $m \geq -1$, $n \geq 0$. This infinite set of equations is of course in this setting not independent. We recover the Schlesinger system (1) from the equations corresponding to $m, n \leq N$. What is remarkable in system (2), is that its form is independent for the number and position of the poles t_i, which only enter in the definition of the L_m. That is the reason why Korotkin and Samtleben name it the "universal" Schlesinger system. Considering the B_n not as dependent variables, but as independent ones, one gets an infinite-dimensional system, whose geometrical meaning in terms of isomonodromic deformations is not clear, but which should be the natural candidate to generalize the Schlesinger system to an infinite set of poles. This system would certainly describe the deformation of the differential systems (A) associated with minimal disks spanned by an analytic curve.

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Minimal surfaces in Nil_3 via loop groups

Josef Dorfmeister
(joint work with Junichi Inoguchi and Shimpei Kobayashi)

Introduction. Loop group methods are well known for surfaces of constant mean curvature and for surfaces of constant curvature in space forms. In recent years much research has been devoted to the study of minimal surfaces also in other three-dimensional manifolds, like the spaces $M(\kappa, \tau)$.

The basic set-up. We consider the simply connected three-dimensional Heisenberg group $Nil_3 = M(0, 1/2)$ together with the non-degenerate, left-invariant metric of $M(0, 1/2)$.

For the Lie algebra $\mathfrak{nil}_3$ of Nil_3 we choose the natural orthonormal basis e_1, e_2 and e_3 and have the bracket operations $[e_1, e_2] = e_3$ and $[e_2, e_3] = [e_3, e_1] = 0$

The starting point for this talk is the following

Proposition 1. *Let $f : M \to \mathrm{Nil}_3$ be a conformal immersion with the conformal factor e^u. Moreover, set $\Phi = f^{-1} f_z = \sum_{k=1}^{\ell} \phi_k e_k$. Then the following statements hold*:

$$f_z = f\Phi, \qquad f_{\bar{z}} = f\bar{\Phi}, \tag{1}$$

$$\sum_{k=1}^{\ell} \phi_k^2 = 0, \quad \text{and} \quad \Phi_{\bar{z}} - \bar{\Phi}_z + [\bar{\Phi}, \Phi] = 0. \tag{2}$$

Conversely, let $\mathbb{D}$ be a simply-connected domain and $\Phi = \sum_{k=1}^{\ell} \phi_k e_k$ a non-zero 1-form on $\mathbb{D}$ which takes values in the complexification $\mathfrak{nil}_3^{\mathbb{C}}$ of $\mathfrak{nil}_3$ satisfying the conditions (2)*. Then for any initial condition in* Nil_3 *given at some base point in $\mathbb{D}$ there exists a unique conformal immersion f into* Nil_3*.*

Note, for a conformal immersion $f : M \to \mathrm{Nil}_3$ one also has the structure equation

$$\Phi_{\bar{z}} + \bar{\Phi}_z + \{\Phi, \bar{\Phi}\} = e^u f^{-1} \boldsymbol{H}, \tag{3}$$

where $\{\cdot, \cdot\}$ denotes the bilinear symmetric map defined by $\{X, Y\} = \nabla_X Y + \nabla_Y X$ for $X, Y \in \mathfrak{g}$ and $\boldsymbol{H}$ denotes the mean curvature vector field.

From conformal immersions to non-linear Dirac equations. In view of (2), following an approach pioneered by Konopelchenko and Taimanov, we write $f^{-1} f_z = \Phi$ in the form of a Weierstrass representation by using (complex valued) spinors ψ_1 and ψ_2:

$$\phi_1 = (\overline{\psi_2})^2 - \psi_1^2, \quad \phi_2 = i((\overline{\psi_2})^2 + \psi_1^2), \quad \phi_3 = 2\psi_1 \overline{\psi_2}, \tag{4}$$

where $\overline{\psi_2}$ denotes the complex conjugate of ψ_2.

Then the conformal factor e^u of the induced metric $\langle df, df \rangle$, the pulled back unit normal vector field $f^{-1} N$ and the mean curvature H can be expressed in terms of these spinors.

We define the *support* function h of f by $h = 2(|\psi_1|^2 - |\psi_2|^2)$. For this talk we assume that h takes positive values everywhere. Then the Dirac potentials

$$\mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h, \tag{5}$$

stated in [4], are defined and one obtains:

Theorem 1. *The equations* (2) *and* (3) *are equivalent to the nonlinear Dirac equation,*

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} := \begin{pmatrix}\partial_z\psi_2 + \mathcal{U}\psi_1\\ -\partial_{\bar z}\psi_1 + \mathcal{V}\psi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{6}$$

From the Dirac equation to the Berdinskii system. For the case of Nil_3 Berdinskii has rephrased the non-linear Dirac equation in terms of a "Lax pair" kind of equation. In this system of equations also the "Abresch-Rosenberg" quadratic differential B enters. It was shown in [1] that B is holomorphic, if f is of constant mean curvature. The converse is almost true

Theorem 2. *Let $f : M \to \mathrm{Nil}_3$ be a conformal immersion and assume that the Abresch-Rosenberg differential is holomorphic. Then f is of constant mean curvature, or f is a Hopf cylinder.*

Theorem 3 ([3]). *Let $\mathbb{D}$ be a simply connected domain in $\mathbb{C}$, $f : \mathbb{D} \to \mathrm{Nil}_3$ a conformal immersion and w the complex valued function defined in* (5). *Then the vector $\tilde\psi = (\psi_1, \psi_2)$ satisfies the system of equations*

$$\tilde\psi_z = \tilde\psi\tilde U, \;\; \tilde\psi_{\bar z} = \tilde\psi\tilde V, \tag{7}$$

where

(8)
$$\tilde U = \begin{pmatrix}\frac12 w_z + \frac12 H_z e^{-w/2+u/2} & -e^{w/2}\\ Be^{-w/2} & 0\end{pmatrix}, \;\; \tilde V = \begin{pmatrix}0 & -\bar B e^{-w/2}\\ e^{w/2} & \frac12 w_{\bar z} + \frac12 H_{\bar z} e^{-w/2+u/2}\end{pmatrix}.$$

Conversely, every vector solution $\tilde\psi$ to (7)*, where all terms are expressed by ψ_1 and ψ_2, is a solution to the nonlinear Dirac equation* (6).

After gauging this system by $diag(e^{-w/4}, e^{-w/4})$ we obtain for constant mean curvature H a system of equations for which the coefficient matrices have the form

(9)
$$U(\lambda)(:= U^\lambda) = \begin{pmatrix}\frac14 w_z & -\lambda^{-1}e^{w/2}\\ \lambda^{-1}Be^{-w/2} & -\frac14 w_z\end{pmatrix}, \;\; V(\lambda)(:= V^\lambda) = \begin{pmatrix}-\frac14 w_{\bar z} & -\lambda\bar B e^{-w/2}\\ \lambda e^{w/2} & \frac14 w_{\bar z}\end{pmatrix}.$$

Note, here we have introduced in addition a "loop parameter" $\lambda \in S^1$. One can show that "constant mean curvature" can be characterized by the integrability of the corresponding system for all $\lambda \in S^1$. However, the corresponding "associated family of surfaces" f_λ can not have exclusively values in Nil_3, as follows from an idea of Berdinskii [2]; also see [7], Proposition 4.5.

Minimal surfaces in Nil_3**.** Minimal surfaces can be characterized among all constant mean curvature surfaces in the following manner.

Theorem 4. *Let* f *be a surface of constant mean curvature in* Nil_3. *Then the following statements are mutually equivalent*:

(1) f *is a minimal surface.*
(2) $e^{w/2} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h$ *is purely imaginary.*
(3) *The matrices* $U(\lambda)$ *and* $V(\lambda)$ *satisfy*

$$V(\lambda) = -\sigma_3 \overline{U(1/\bar{\lambda})}^t \sigma_3, \quad \textit{where } \sigma_3 = diag(1,-1), \tag{10}$$

(4) *The Maurer-Cartan form* $\alpha^\lambda = U(\lambda)dz + V(\lambda)d\bar{z}$ *takes values in the real Lie subalgebra* $\mathfrak{su}(1,1)$.
(5) *The stereographic projection* g *of* $f^{-1}N$ *from the south-pole is a harmonic map into the hyperbolic space* $\mathbb{H}^2$.

Since g is a harmonic map ("normal Gauss map") into the space form $\mathbb{H}^2$, it can be constructed by a known loop group procedure [5]. The actual immersion into Nil_3 can be obtained by differentiation for λ. For more details see [7] or [6].

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Recent Progress in G_2 Geometry

Mark Haskins

(joint work with Alessio Corti, Johannes Nordström, Tommaso Pacini)

In this talk we discuss recent progress in the construction of compact Riemannian 7-manifolds with holonomy group the compact exceptional Lie group G_2. Every such G_2-holonomy manifold is a (non-flat) Ricci-flat manifold and these G_2-manifolds provide one of the very few sources of odd-dimensional compact Ricci-flat manifolds. We concentrate on recent progress which makes it possible to determine for the first time the diffeomorphism type of the underlying smooth 7-manifolds. This relies on the diffeomorphism (and almost diffeomorphism) classification of 2-connected 7-manifolds. We show that many G_2-manifolds attained by the twisted connected sum construction (due originally to Donaldson and Kovalev) are 2-connected and in many cases determine their diffeomorphism type. The main ingredients in the proof are:

1. the construction of asymptotically cylindrical Calabi-Yau 3-folds from appropriate smooth projective 3-folds (weak Fano or semi-Fano 3-folds),
2. the deformation theory of semi-Fano 3-folds
3. the twisted connected sum of G_2-manifolds
4. understanding the topology (integral cohomology, characteristic classes) of the projective 3-folds.

We explain how our work suggests many further open questions and suggest some potential approaches to them.

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Constrained Willmore Hopf tori

Lynn Heller

Constrained Willmore tori are conformal immersions $f : T^2 \to S^3$ which are critical points of the Willmore energy $\mathcal{W} = \int(H^2 + 1)dA$ under conformal variations. Constant mean curvature (CMC) immersions in any 3−dimensional space form are examples of constrained Willmore tori. Immersions minimizing the Willmore energy $\mathcal{W}$ for a fixed conformal class can be viewed as the optimal realization of the underlying Riemann surface in 3−space. Existence and regularity of such a minimizer is shown in [6] under the provision that the infimum of the Willmore energy in the conformal class is below 8π. It is still open whether the above statement is true for general conformal classes. Recent results in [4] and [1] show that the only embedded CMC tori in 3−dimensional space forms are rotational symmetric. In particular, they have rectangular conformal type. Non embedded CMC tori have Willmore energy above 8π and can therefore not be the candidates for the minimizers in their respective conformal class (at least in the conformal classes near the one of the Clifford torus). In order to find candidates for minimizers for all conformal classes, it is necessary to find constrained Willmore tori which do not have constant mean curvature in a space form.

It is shown in [3] that constrained Willmore tori in S^3 form an integrable System. This means we can associate to every constrained Willmore immersion a compact Riemann surface Σ - the spectral curve - and we can recover the immersion from algebraic data on the spectral curve. The genus g of Σ is called the spectral genus of the immersion and the complexity of the immersion increases with g. On Σ there exist two natural involutions σ and ρ. If the torus is a CMC torus in a space form then σ is the hyperelliptic involution of Σ and $\rho \circ \sigma$ has fixpoints. For a generic immersion in the moduli space of constrained Willmore tori, i.e., for which a certain holomorphic line bundle on the spectral curve has degree $g+3$ (we refer to these as simple immersions in the following), we can show the reverse.

Theorem 1 ([8]). *Let $f : T^2 \to S^3$ be a simple constrained Willmore torus and let Σ the spectral curve of f. If $\Sigma/\sigma \cong \mathbb{C}P^1$ and if the involution $\rho \circ \sigma$ has fixed points, then f is a CMC torus in a space form.*

The condition that the involution $\rho \circ \sigma$ has fixed points is always valid if Σ has even genus. Further for $g = 1$ the case where $\rho \circ \sigma$ has no fixed points is also understood, see [9] and [2].

Corollary 1 ([7] and [8]). *Let f be a simple constrained Willmore torus and let g be its spectral genus. Then the following holds:*

- *$g = 0$ if and only if f is homogenous.*
- *$g = 1$ if and only if f is CMC in a space form and lies in the associated family of a cylinder of revolution as a constrained Willmore and isothermic surface.*
- *$g = 2$ if and only if f is either CMC or lies in the constrained Willmore associated family of a constrained Willmore Hopf cylinder.*

Constrained Willmore Hopf tori are given by the preimages of closed constrained elastic curves (critical points of the energy functional $\int \kappa^2 ds$ with prescribed length and enclosed area) under the Hopf fibration[1]. The definition of the constrained Willmore associated family can be found in [7] or [5]. Constrained Willmore Hopf tori are never isothermic and hence never CMC in a space form unless they are homogenous, see [7]. Further, the Euler-Lagrange equation for constrained Willmore tori reduces to the Euler-Lagrange equation for constrained elastic curves

$$\kappa'' + \tfrac{1}{2}\kappa^3 + (\mu + G)\kappa + \lambda = 0, \tag{1}$$

where κ is the geodesic curvature of the curve in the round S^2 of constant curvature G and μ and λ are real Lagrange multipliers. Willmore Hopf tori correspond to curves with $\mu = -\frac{G}{2} < 0$ and $\lambda = 0$. Solutions to $\lambda = 0$ are elastic curves. These were also discussed in [11]. We can proof the following:

Theorem 2 ([9]). *The minimizer of the Willmore energy in the class of Hopf tori exist for every conformal type of the torus. In particular, every conformal class of the torus can be realized as a constrained Willmore (Hopf) torus.*

By restricting to the smaller class of Hopf tori, we can show therein the existence of minimizers of the Willmore energy with prescribed conformal class without any energy bound as it is needed in [6]. Near the conformal class of the Clifford torus we conjecture the constrained Willmore Hopf minimizers to be the global minimizers of the Willmore energy in their respective conformal classes.

[1]Hopf tori and Willmore Hopf tori were first considered in [12].

Constrained Willmore Hopf tori can be parametrized explicitly using elliptic functions. By multiplying equation (1) with κ' and integrate equation (1) holds if and only if there exist a $\nu \in \mathbb{R}$ with

$$(\kappa')^2 = -\tfrac{1}{4}\kappa^4 - (\mu + G)\kappa^2 - 2\lambda\kappa - \nu.$$

Since we are only interested in periodic solutions, we can choose $\kappa'(0) = 0$ without loss of generality. The necessary and sufficient condition for the above equation to have periodic solution is that the polynomial in κ on the righthand side has real roots. If all roots are simple the ODE can be solved by

$$\kappa = \sqrt{-8\mathrm{Re}(\wp(x + x_0)) - 4\wp(\rho) + G},$$

where $\wp$ is the Weierstrass $\wp$ function defined on a torus $\mathbb{C}/\Gamma$ given by the real lattice invariants

$$g_2 = \frac{1}{12}(\mu + G)^2 + \frac{1}{4}\nu \quad \text{and} \quad g_3 = \frac{1}{216}(\mu + G)^3 + \frac{1}{24}\nu(\mu + G) + \frac{1}{16}\lambda^2$$

and $x_0, \rho \in i\mathbb{R}_*$. For the multiple roots case see [9]. For constrained elastic curves the torus $\mathbb{C}/\Gamma$ on which the $\wp-$function is defined can be viewed as the spectral curve of the curve in S^2. Let ω_1 denote the half lattice point of Γ on the real axis. The curve is closed if and only if there exist integers m and n such that

$$g(\rho) := \zeta(\omega_1)\rho + \zeta(\rho)\omega_1 = \frac{m}{n}\pi i.$$

The function g is purely imaginary valued and non constant. Thus we always obtain infinite many closed curves for given g_2, g_3 and x_0. We transformed the parameters (μ, G, λ, ν) into other parameters (g_2, g_3, ρ, x_0) such that varying x_0 for fixed g_2, g_3 and ρ preserves the closeness of the constrained elastic curve. Further, there is a x_0 unique up to sign in $i\mathbb{R}/\Gamma$ such that the corresponding curve is elastic. The integer m is the tangent turning number of the curve and n is the number of intrinsic periods which we call lobe number. In fact for every $n \geq 2$ there is a family of embedded closed closed curves with lobe number n starting at the equator (which corresponds to the Clifford torus) and oscillating around it. The isospectral deformation shifts these $n-$lobed curves such that they oscillate around circles instead of the equator. What we get are $2-$parameter families of closed and embedded but non isothermic constrained Willmore tori near the Clifford torus, see [9] and [10].

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Higher genus CMC surfaces via integrable systems

Sebastian Heller

The moduli spaces of CMC spheres and CMC tori in the round 3-sphere are quite well-understood by now. CMC spheres are totally umbilic due to the vanishing Hopf differential. Based on the ideas of Brendle's proof of the Lawson conjecture [3] Andrews and Li showed that all embedded CMC tori in S^3 are rotational and therefore classified [1]. Additionally, all CMC immersions from a torus into 3-dimensional space forms are given rather explicitly in terms of algebro-geometric data on their associated spectral curves [13, 10, 2]. This is in stark contrast to the case of higher genus CMC surfaces in S^3. There exist a few compact examples for every genus, like the Lawson minimal surfaces [11], but these examples have been constructed by implicit methods from geometric analysis. Moreover, there is no theory which describes the space of all CMC surfaces of higher genus, nor is there any starting point for classification of the embedded ones.

The study of CMC tori via integrable systems is based on the associated family

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a fixed hermitian rank 2 bundle [10]. The complex linear endomorphism valued 1-form Φ is nowhere vanishing and nilpotent and Φ^* is its adjoint with respect to the unitary metric. For minimal surfaces in S^3 the flatness of this family of connections is just a gauge theoretic reformulation of the harmonic map equation. For CMC surfaces, the family of flat connections is induced by the Lawson correspondence. The connections ∇^λ are unitary for $\lambda \in S^1 \subset \mathbb{C}^*$ and trivial at two Sym points $\lambda_1 \neq \lambda_2 \in S^1$. The immersion can be obtained as the gauge between ∇^{λ_1} and ∇^{λ_2}, and its mean curvature is given by $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$.

Compact oriented CMC surfaces can be coarsely distinguished by the complexity of their associated family of monodromy representations: The easiest case is when all connections ∇^λ are trivial. This happens only for spheres, see [10]. A more complicated case is given when all monodromy representations are abelian but generically not trivial. Then, the connections split generically into the direct sum of flat line bundles, which can be parametrized on a double covering of the spectral plane. For tori with their abelian first fundamental group, this is the starting point of Hitchin's spectral curve theory for harmonic tori [10]. It was shown in [12, 6, 5] independently, that the generic connection ∇^λ of a compact CMC surface of genus $g \geq 2$ is irreducible. This is clearly the most complicated case and a naive generalization of the spectral curve approach does not work for higher genus CMC surfaces.

Of fundamental importance for the construction of new examples are loop group factorization methods. They have first been applied to surface geometry by Dorfmeister, Pedit and Wu [4] in their description of simply connected CMC surfaces via holomorphic (Weierstrass) data. In our situation they yield:

Theorem 1 ([7]). *Let $\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of flat $SL(2,\mathbb{C})$-connections over a compact Riemann surface M of genus $g \geq 2$ such that*

- *the asymptotic at $\lambda = 0$ is given by $\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla} + \ldots$ where $\Psi \in \Gamma(M, KEnd_0(V))$ is nowhere vanishing and nilpotent;*
- *for all $\lambda \in S^1 \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^\lambda$ is unitary with respect to this metric;*
- *$\tilde{\nabla}^\lambda$ is trivial for $\lambda_1 \neq \lambda_2 \in S^1$.*

Then there exists a unique CMC surface $f\colon M \to S^3$ such that its associated family of flat connections ∇^λ and the family $\tilde{\nabla}^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g(\lambda) = \tilde{\nabla}^\lambda$ for all λ.

We now restrict to a class of CMC surfaces which we call Lawson symmetric. These CMC surfaces are symmetric with respect to those symmetries of the Lawson surface which are orientation preserving on both, the extrinsic space and the surface (Lawson symmetries). Instead of parametrizing parallel eigenlines (which do not exist) we consider the holomorphic eigenlines of symmetric Higgs fields, i.e., complex linear endomorphism valued 1-forms which are equivariant with respect to the Lawson symmetries. These eigenlines do only exist on a double covering - the Hitchin curve - and are elements in an affine Prym variety, see [9]. In our case, the Prym variety can be identified with the Jacobian of a torus T^2 which is the quotient of the Hitchin curve by the Lawson symmetries. It is shown in [7] that there is a two-to-one correspondence Ψ between flat line bundles on T^2 and Lawson symmetric flat $\mathrm{SL}(2,\mathbb{C})$-connections. Moreover, as a consequence of the Narasimhan-Seshadri theorem, there exists for every holomorphic line bundle in the Prym variety exactly one compatible flat line bundle connection such that the corresponding $\mathrm{SL}(2,\mathbb{C})$-connection is unitary. This gives rise to a real analytic section a^u of the affine bundle of flat line bundle connections modulo gauge equivalence over the Jacobian. We then have:

Theorem 2 ([7]). *Let $\lambda \mapsto \nabla^\lambda$ be the associated family of a Lawson symmetric CMC surface of genus $g \geq 2$. Then there exists a Riemann surface $p\colon \Sigma \to \mathbb{C}$ which double covers the spectral plane $\mathbb{C}$ together with a holomorphic map $\mathcal{L}\colon \Sigma \to$*

$Jac(T^2)$ and a meromorphic lift $\mathcal{D}$ with a first order pole over $\lambda = 0$ into the affine moduli space of flat line bundles on T^2 such that $\Psi \circ \mathcal{D}(\mu)$ is gauge equivalent to $\nabla^{p(\mu)}$ for all $\mu \in \Sigma$. The spectral curve branches over $\lambda = 0$ and the spectral data satisfy the reality condition $a^u(\mathcal{L}(\mu)) = \mathcal{D}(\mu)$ for all $\mu \in p^{-1}(S^1)$.

Conversely, spectral data $(\Sigma, \mathcal{L}, \mathcal{D})$ as above which satisfy the reality condition $a^u(\mathcal{L}(\mu)) = \mathcal{D}(\mu)$ for all $\mu \in p^{-1}(S^1)$ and the extrinsic closing condition that $\Psi \circ \mathcal{D}(\mu)$ is trivial for all $\mu \in p^{-1}(\{\lambda_{1,2}\})$ give rise to Lawson symmetric CMC surfaces.

Up to now, the reality condition is not understood explicitly. Nevertheless, we have been able to apply these spectral methods in computer experiments in order to investigate the moduli space of Lawson symmetric CMC surfaces, see [8]. As a result we have obtained the numerical existence of real 1-parameter families of Lawson symmetric CMC surfaces in S^3 passing through the Lawson surfaces $\xi_{g,1}$ analogous to the Whitham deformation of the Clifford torus via homogeneous CMC tori.

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Discrete linear Weingarten surfaces

Udo Hertrich-Jeromin
(joint work with Francis E Burstall, Wayne Rossman)

Recall that a surface $\sigma : M^2 \to \mathbb{R}^3$ is called a *linear Weingarten surface* if there is a non-trivial triple of coefficients $a, b, c \in \mathbb{R}$ so that

$$a\,K + 2b\,H + c = 0, \tag{W}$$

where $K = k_1 k_2$ and $H = \frac{k_1+k_2}{2}$ denote the Gauss and mean curvatures of the surface, respectively. Examples of linear Weingarten surfaces clearly include: surfaces of constant mean curvature H, where $(a, b, c) = (0, 1, -2H)$; surfaces of constant Gauss curvature K, where $(a, b, c) = (1, 0, -K)$; "tubular" surfaces with a constant principal curvature, where the discriminant $\Delta := ac - b^2 = 0$ vanishes so that the equation (W) factorizes.

The mission of the talk was twofold: firstly, to discuss a notion of "discrete linear Weingarten surfaces" and, secondly, to generalize to arbitrary ambient space forms (possibly with other signatures). Using a Lie geometric approach, both goals are achieved in a straightforward way — in the process it is seen that discrete linear Weingarten nets/surfaces are intimately related to a discrete version of Demoulin's Ω-surfaces, the Lie geometric analogue of isothermic surfaces in Möbius geometry, cf [2, 3].

A discretization of linear Weingarten surfaces in Euclidean geometry is straightforward from [1]: any *circular* net $\sigma : \mathbb{Z}^2 \supset \Gamma \to \mathbb{R}^3$ — that is, every face $(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ has a circumcircle — admits a "Gauss map" $\nu : \Gamma \to S^2$ with parallel edges, that is, the pair (σ, ν) satisfies a discrete version of Rodrigues' equation

$$0 = d\nu_{ij} + k_{ij} d\sigma_{ij}. \tag{R}$$

Then the "parallel nets" $\sigma^t := \sigma + t\nu$ of σ are (edge-) parallel circular nets (with Gauss map ν). Note that circularity is, in general, necessary for the existence of such a Gauss map, cf [1, Thm 18], and that the Gauss map is not unique. Thus we call a pair (σ, ν) of a circular net with Gauss map a *principal net*. The Gauss and mean curvatures of a principal net can then be defined (as functions on faces) via Steiner's formula

$$A^t = (1 - 2t\,H + t^2 K)\,A, \quad \text{where} \quad A_{ijkl} = \delta\sigma_{ik} \times \delta\sigma_{jl} \tag{S}$$

yields the (directed) area of a face in terms of the cross product of its diagonals. A *discrete linear Weingarten net* is a principal net satisfying (W), cf [1, Thm 17].

Principal nets $(\sigma, \nu) : \Gamma \to \mathbb{R}^3 \times S^2$ naturally lift to "Legendre maps" (or, "principal contact element nets", cf [1]) in Lie sphere geometry: fixing a "point-sphere complex" $p \in \mathbb{R}^{4,2}$ with $(p, p) = -1$ and an isotropic "space form vector" $q \in \mathbb{R}^{4,2}$ with $q \perp p$ we set

$$\begin{aligned} Q^3 &:= \{X \in \mathbb{R}^{4,2} \,|\, (X, X) = 0, (X, p) = 0, (X, q) = -1\}, \\ P^3 &:= \{X \in \mathbb{R}^{4,2} \,|\, (X, X) = 0, (X, p) = -1, (X, q) = 0\}; \end{aligned}$$

further fixing an "origin" $o \in Q^3$ we obtain an isometry

$$\mathbb{R}^3 \cong \{o, q, p\}^\perp \ni x \mapsto X := o + x + \tfrac{1}{2}(x, x)\, q \in Q^3,$$

and a unit tangent vector $y \in T_x\mathbb{R}^3$ can be identified with the hyperplane in Q^3 through $X \in Q^3$ by

$$S^2 \ni y \mapsto Y := y + (y, x)\, q + p \in P^3.$$

The line through $\langle X, Y\rangle \subset \mathbb{P}(\mathcal{L}^5)$ in the Lie quadric (the projectivized light cone of $\mathbb{R}^{4,2}$) then defines a contact element, and $\Lambda = \langle \Sigma, \mathrm{N}\rangle$,

$$\Sigma = o + \sigma + \tfrac{1}{2}(\sigma, \sigma)\, q \ \text{ and } \ \mathrm{N} = \nu + (\nu, \sigma)\, q + p,$$

is the *Legendre lift* of a principal net (σ, ν) in $\mathbb{R}^3$. The characteristic property of such Legendre lifts is to possess *edge curvature spheres* κ_{ij}:

$$0 = d\mathrm{N}_{ij} + k_{ij} d\Sigma_{ij} \quad \Leftrightarrow \quad \mathrm{N}_j + k_{ij}\Sigma_j = \mathrm{N}_i + k_{ij}\Sigma_i =: \kappa_{ij} \in \Lambda_i \cap \Lambda_j. \tag{R$'$}$$

Thus a map Λ into the space of contact elements, that is, lines in the Lie quadric, will be called a *Legendre map* if adjacent lines intersect.

Apart from projection issues, these Legendre maps are exactly the Legendre lifts of principal nets — in $\mathbb{R}^3$ or, more generally, space forms: dropping the assumption on q to be isotropic (and on the sign of (p, p) to include Lorentzian space forms), we shall call a pair $(\Sigma, \mathrm{N}) : \Gamma \to Q^3 \times P^3$ a *space form projection* of a Legendre map Λ if $\Lambda = \langle \Sigma, \mathrm{N}\rangle$. By (R$'$), we may again use Steiner's formula (S) to define the Gauss and mean curvatures of (Σ, N), when replacing the cross product by wedge product, $A(\Sigma, \Sigma)_{ijkl} = \delta\Sigma_{ik} \wedge \delta\Sigma_{jl}$, to encode the (directed) area. Note that the involved wedge products decompose,

$$A(\Sigma, \Sigma) = A(\sigma, \sigma) + (\dots) \wedge q \in \Lambda^2\mathbb{R}^3 \oplus (\mathbb{R}^3 \wedge \langle q\rangle), \ \text{ etc.},$$

showing that this notion of Gauss and mean curvature for space form projections of Legendre maps generalizes the earlier one for principal nets in $\mathbb{R}^3$. Hence we say that a space form projection (Σ, N) of a Legendre map Λ is a *linear Weingarten net* if there is a non-trivial triple of coefficients $a, b, c \in \mathbb{R}$ so that

$$a\, A(\mathrm{N}, \mathrm{N}) - 2b\, A(\mathrm{N}, \Sigma) + c\, A(\Sigma, \Sigma) = 0, \tag{W$'$}$$

where A is thought of as a bilinear form obtained from the quadratic area form by polarization.

Now observe: if (Σ, N) is a non-tubular linear Weingarten net, $\Delta = ac - b^2 \neq 0$, we may factorize the linear Weingarten condition to obtain edge-parallel nets $\Sigma^\pm$ taking values in linear sphere complexes $k^\pm \in \mathbb{R}^{4,2} \setminus \{0\}$, where $\langle \Sigma, \mathrm{N}\rangle = \langle \Sigma^+, \Sigma^-\rangle$,

$$A(\Sigma^+, \Sigma^-) = a\, A(\mathrm{N}, \mathrm{N}) - 2b\, A(\mathrm{N}, \Sigma) + c\, A(\Sigma, \Sigma) = 0 \ \text{ and } \ \Sigma^\pm \perp k^\pm. \tag{C}$$

For example: in the constant mean curvature case $(a, b, c) = (0, 1, -2H)$ we may take $(\Sigma^+, \Sigma^-) = (\mathrm{N} + H\,\Sigma, \Sigma)$ and $(k^+, k^-) = (q - H\,p, p)$ to obtain the original net together with its mean curvature sphere congruence; in the constant Gauss curvature case $(a, b, c) = (1, 0, -K)$ factorization yields (possibly complex conjugate) nets $\Sigma^\pm = \mathrm{N} \pm \sqrt{K}\,\Sigma$ and $k^\pm = q \mp \sqrt{K}\,p$.

As *constant* linear combinations of Σ and N the obtained nets $\Sigma^{\pm}$ are edge-parallel and, factorizing the linear Weingarten condition, their opposite diagonals are parallel, that is, $\Sigma^{\pm}$ are Königs dual nets in $\mathbb{R}^{4,2}$, cf [1, Thm 13]. Hence $\Sigma^{\pm}$ project to Königs nets in $\mathbb{R}P^5$ and, taking values in a quadric, to isothermic nets in the Lie quadric, cf [6]. Analogy with the smooth case, cf [3], then suggests a discretization of Demoulin's Ω-surfaces, cf [5]: a discrete Legendre map Λ will be called an *Ω-net* if $\Lambda = \langle \Sigma^+, \Sigma^- \rangle$ for two isothermic sphere congruences that admit Königs dual lifts $\Sigma^{\pm}$. Thus we obtain a characterization of linear Weingarten nets as space form projections of special Ω-nets:

The Legendre lift $\Lambda = \langle \Sigma, \mathrm{N} \rangle$ of a linear Weingarten net (Σ, N) is an Ω-net; its enveloped isothermic sphere congruences $\Sigma^{\pm}$ take values in linear sphere complexes, $\Sigma^{\pm} \perp k^{\pm}$.

Conversely: if $\Lambda = \langle \Sigma^+, \Sigma^- \rangle$ is an Ω-net with $\Sigma^{\pm} \perp k^{\pm}$, then any space form projection (Σ, N) with $\langle p, q \rangle = \langle k^+, k^- \rangle$ yields a linear Weingarten net.

This characterization of linear Weingarten surfaces/nets provides a novel approach to understanding the integrable nature of this class of nets/surfaces, in particular, the rich isothermic transformation theory descends to linear Weingarten surfaces/nets to provide new insight into their transformations. Moreover, a simple geometric approach to the Weierstrass type representations for certain linear Weingarten surfaces/nets is obtained. For more details the interested reader is referred to our recent paper [4], the smooth case is discussed in two notes [2, 3].

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Transformations and orientability of extrinsically flat surfaces

Atsufumi Honda

By the Hartman-Nirenberg theorem, any complete flat surface in the Euclidean 3-space $\mathbb{R}^3$ must be a cylinder over a complete plane curve. This fact implies that the global theory of flat surfaces in $\mathbb{R}^3$ is trivial. However, if we admit some singularities, there exist many nontrivial flat surfaces. Murata-Umehara [3] investigated flat surfaces with admissible singularities called "*flat fronts*" and proved the following.

Fact 1 ([3])**.** A complete flat front in the Euclidean 3-space whose singular point set is non-empty has no umbilics, is orientable and co-orientable. Moreover, if its ends are embedded, there exist at least four singular points other than cuspidal edges.

This estimate is sharp (see Figure 1).

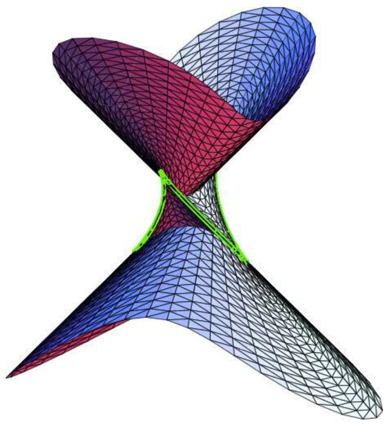

Figure 1. A complete flat front in $\mathbb{R}^3$ which has four singular points other than cuspidal edges.

Here, we recall some terminologies about wave fronts. Let M^2 be a smooth 2-manifold. A smooth map $f : M^2 \to \mathbb{R}^3$ is called a *wave front* (or a *front*) if, for any point $p \in M^2$, there exist a neighborhood $U_p \subset M^2$ of p, and a smooth map $\nu : U_p \to S^2$ such that $\langle df(T_q M^2), \nu(q)\rangle = 0$ holds for all $q \in U_p$ and

$$L := (f, \nu) : U_p \to \mathbb{R}^3 \times S^2$$

is an immersion, where we denote by $\langle\ ,\ \rangle$ the standard inner product of $\mathbb{R}^3$. Regarding $\mathbb{R}^3 \times S^2$ as the unit tangent bundle $T_1\mathbb{R}^3$ of $\mathbb{R}^3$, L is a Legendrian immersion with respect to the canonical contact structure of $T_1\mathbb{R}^3$. We call L the *Legendrian lift* of f. Equipping the Sasakian metric $\langle\ ,\ \rangle_{T_1R^3}$ with $T_1\mathbb{R}^3$, the pullback metric $g_L := L^*\langle\ ,\ \rangle_{T_1R^3}$ defines a Riemannian metric on M^2. We call g_L the *lift metric*. We remark that a parallel surface of some immersed surfaces is a front. Conversely, we can prove that any front is locally given in this way. Then, a point $p \in M^2$ is called

- *singular* point of f, if $\operatorname{rank}(df)_p < 2$,
- *umbilic* point of f, if p is umbilic for some parallel surface of f.

A front $f : M^2 \to \mathbb{R}^3$ is called

- *flat*, if $\operatorname{rank}(d\nu)_p < 2$ for all $p \in M^2$,
- *orientable*, if M^2 is orientable,
- *co-orientable*, if ν is globally defined on M^2,
- *weakly complete*, if its lift metric g_L is complete,

and *complete*, if there exist a symmetric covariant 2-tensor T^2 with compact support on M^2 such that $ds^2 + T^2$ gives a complete Riemannian metric on M^2, where

$ds^2 = \langle df, df \rangle$ is the first fundamental form of f. By definition, completeness implies weakly completeness.

On the other hand, let $f : M^2 \to \mathbb{R}^3$ be an immersed surface with one principal curvature a constant c, that is

$$(\lambda_1 - c)(\lambda_2 - c) = 0$$

holds on M^2, where we denote by λ_1, λ_2 the principal curvatures of f. If $c = 0$, it is a flat immersion. In the case of $c \neq 0$, Shiohama-Takagi proved the following.

Fact 2 ([5])**.** A complete surface with one principal curvature a nonzero constant in the Euclidean 3-space is either a totally umbilical or umbilic free. In the latter case, such a surface is a tube of some complete regular curve.

Therefore, such a surface is trivial. Thus, we consider a front with one principal curvature a constant c. A front $f : M^2 \to \mathbb{R}^3$ is called *with one principal curvature a constant* c, if $\mathrm{rank}(d\nu + c\, df) < 2$ holds on M^2. Then, we have the following.

Theorem 1 ([1])**.** *A weakly complete front with one principal curvature a nonzero constant in the Euclidean* 3*-space is either a totally umbilical or umbilic free. In the latter case, such a front is a tube of some complete regular curve. Moreover, such a front must be orientable.*

While an immersed surface with one principal curvature a nonzero constant is orientable, a front with one principal curvature a nonzero constant is co-orientable. We remark that there exist *non-orientable* examples (see Figure 2).

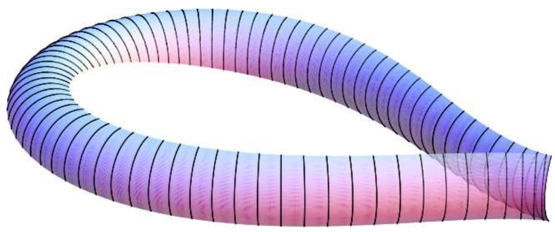

Figure 2. A front with one principal curvature a nonzero constant which is not orientable but co-orientable.

In the case of the 3-sphere S^3, we consider *extrinsically flat surfaces*, where an immersed surface $f : M^2 \to S^3$ is called *extrinsically flat* (or *e-flat*, for short) if its extrinsic curvature $K_{\mathrm{ext}} = \lambda_1\lambda_2$ vanishes identically on M^2. The Gauss equation $K = K_{\mathrm{ext}} + 1$ implies that e-flat surfaces are equivalent to surfaces of constant Gaussian curvature one. O'Neill-Stiel [4] proved that *any complete extrinsically flat surface must be totally geodesic.* On the other hand, there exist many nontrivial e-flat fronts (see Figure 3).

Applying the method used to prove Theorem 1, we have the following.

Theorem 2 ([2])**.** *A weakly complete extrinsically flat front in the* 3*-sphere is either a totally umbilical or umbilic free. Moreover, a co-orientable weakly complete extrinsically flat front must be orientable.*

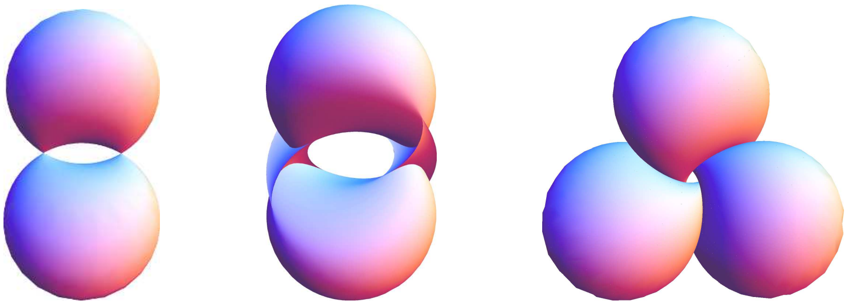

FIGURE 3. Extrinsically flat fronts in S^3.

We remark that an e-flat front without umbilics is developable, and is a tube of a regular curve of radius $\pi/2$. There exist compact *non-co-orientable* e-flat fronts, and compact e-flat fronts which has no singular points other than cuspidal edge. These examples imply that global properties of e-flat fronts in S^3 are different from those of flat fronts in $\mathbb{R}^3$.

Moreover, we construct two transformations among e-flat fronts in S^3 which we call *dual* and *caustic*, and prove some properties about them. In particular, we classify weakly complete self-dual e-flat fronts. Moreover, the commutativity of such two transformations are proved.

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Proof of the Global Caratheodory Conjecture and a resulting Local Umbilic Index Bound

WILHELM KLINGENBERG

(joint work with Brendan Guilfoyle)

1. Let $\mathcal{S} := \{S \hookrightarrow \mathbb{R}^3 |$ strictly convex closed $C^{3,\alpha}$ – smooth spheres$\}$, endowed with the Hölder space $C^{3,\alpha}$ -topology.

Theorem 1. [1] **"The Global Carathéodory Conjecture"** *For every $S \in \mathcal{S}$:* $\#\{\text{umbilic points s} \in \text{S}\} \geq 2$.

Theorem 2. [2] "**Local Umbilic Index Estimate**" *For every isolated umbilic $s \in S \in \mathcal{S}$:* $\text{Index}_{\text{s}}(\text{S}) \leq 3/2$.

Note that Theorem 2 implies Theorem 1 by the Poincaré-Hopf index theorem when applying to one of the (in general non-orientable) principal curvature foliations of S. Note that the singularities thereof are umbilics, where the foliation take on half-integer valued 'umbilic indices'. We believe that above index bound is sharp and are lead to the following Conjecture: *There exist $S_E \in \mathcal{S}$ with an 'exotic' umbilic $s_E \in S_E$ of maximal index*: $\mathrm{Index}_{\mathrm{s_E}}(\mathrm{S_E}) = 3/2$. By Hamburger's Theorem [3], who proved an upper umbilic index bound of 1 for real analytic surfaces S, and thereby indeed the real analytic Carathéodory Conjecture, the surface S_E can not be real-analytic. The above conjecture therefore marks a split between the smooth and real analytic categories in classical differential geometry.

2. Our **proof of Theorem 1** is indirect, proceeds by assuming that there exists an $S_\infty \in \mathcal{S}$ with only one umbilic point, and arrives at the resulting contradiction between Theorem 3 and Theorem 4 below. It takes place in the space $L(\mathbb{R}^3) :=$ {all oriented lines in $\mathbb{R}^3$} $\equiv$ {oriented geodesics in euclidean $\mathbb{E}^3$}, which is seen to be an open 4 - manifold diffeomorphic to TS^2. Here, $S^2 \subset \mathbb{R}^3$ stands for the round sphere. The manifold L is furthermore equipped with a complex structure $J(l) : T_lL \to T_lL$ given by $J(l) = dR_l$, where $R(l) : (L, l) \to (L.l)$ is induced by rotation of $\mathbb{R}^3$ by $\pi/2$ about the oriented line $l \subset \mathbb{R}^3$. It can be seen that $L \approx TS^2$ and $(L, J) \approx T^{10}S^2$. Next we consider vector fields on S^2, viewed as surfaces in L. $\mathcal{L} :=$ { all $C^{2,\alpha}$ – smooth global sections Σ of $TS^2 \to S^2$}, endowed with the Hölder space $C^{2,\alpha}$ -topology. For $S \in \mathcal{S}$ we consider $N(S) \in \mathcal{L}$ as defined by $N(S) :=$ { all $l \in L(\mathbb{R}^3)$ which are normal to S}. Then $N(S) \in \mathcal{L}$ and the salient feature of this transformation N of surfaces is :

$s \in S \hookrightarrow \mathbb{R}^3$ is an umbilic point $\iff$ $N(s) \in N(S) \hookrightarrow (L, J)$ is a complex point.

For $\Sigma \in \mathcal{L}$, we set $\Sigma^* := \Sigma -$ {complex points}, and a relative class $A \in H^2(L; \Sigma^*)$, define a fibre bundle $\mathcal{F}_A \to \mathcal{L}$. Its fibre over $\Sigma \in \mathcal{L}$ are those Sobolev maps f of the disc $D \subset \mathbb{C}$ in $H^{1,s}((D, \partial D); (L, \Sigma^*))$ that realize the class A : $[f(D), f(\partial D)] = A$. The proof of Theorem 1 follows by contradiction from the following two results about the subbundle $\pi : \mathcal{F}_A \supset \mathcal{H}_A \to \mathcal{L}$ consisting of holomorphic discs in $\mathcal{F}_A$, namely those $f \in \mathcal{F}_A$ satisfying the Cauchy-Riemann equations $df \circ i = J \circ df$.

Theorem 3. " Non-Existence near $N(S_\infty) \in \mathcal{L}$ " *Assume that there exists an $S_\infty \in \mathcal{S}$ with* **only one** *umbilic point. Then there exists a sequence of $\Sigma_k \in \mathcal{L} - N(S_\infty)$ which converges to $N(S_\infty) \in \mathcal{L}$ and such that for the class A representing the disc $N(S_\infty) \setminus$ {unique complex point} we have: $\pi^{-1}(N(\Sigma_k)) = \emptyset$.*

Theorem 4. " Existence Theorem " *Assume that $S \in \mathcal{S}$ has an umbilic-free hemisphere. Then $\pi^{-1}(N(S)) \neq \emptyset$. Furthermore, there exists a neighborhood $\mathcal{U} \subset \mathcal{L}$ of $N(S)$ such that for all $\Sigma \in \mathcal{U}$, we have $\pi^{-1}(\Sigma) \neq \emptyset$.*

3. The **proof of Theorem 3** follows by applying the Theorem of Sard-Smale to the Fredholm map π in a neighborhood of $N(S_\infty) \in \mathcal{L}$. It is achieved by applying the Theorem of Riemann-Roch to determine the analytic index of π in terms of the complex points of the boundary condition Σ_0 and relies on the following

Theorem 5. *" π is Fredholm near $N(S_\infty)$ " Assume that $S_\infty \in \mathcal{S}$ has only one umbilic. Then there exists an open set $N(S_\infty) \in \mathcal{U} \subset \mathcal{L}$ such that $\pi : \mathcal{H}_A \supset \pi^{-1}\mathcal{U} \to \mathcal{U}$ is a Fredholm map of Banach manifolds.*

We now define the subset $\mathcal{L}_0 \subset \mathcal{L}$ given by all those $\Sigma_0 \in \mathcal{L}$ which contain the point $\xi = \eta = 0$ in L. Furthermore we have the set of all tangent vector fields of Σ_0, which we denote by $\Gamma T\Sigma_0$, and the subset $\Gamma_0 T\Sigma_0 \subset \Gamma T\Sigma_0$ of all vector fields that vanish in the fibre over the north pole with coordinates $\xi = \eta = 0$. Note that the euclidean group $Aut(\mathbb{R}^3)$ of $\mathbb{R}^3$ acts on $\mathcal{L}$, and its linearization $dAut(\mathbb{R}^3)$ acts on $\Gamma T\Sigma_0$. The following follows from observing that $Aut(\mathbb{R}^3)$ acts transitively on L and $dAut(\mathbb{R}^3)$ acts transitively on $T_{\Sigma_0}L$, and $\mathcal{L}_0 \cong \mathcal{L}/Aut(\mathbb{R}^3$, and $\Gamma_0 T\Sigma_0 \cong \Gamma T\Sigma_0/dAut(\mathbb{R}^3)$. *Let $\Sigma_\infty \in \mathcal{L}$ be a surface with* **only one** *complex point, namely $(0,0) \in L$. Then there exist open neighborhoods $0 \in \mathcal{B} \subset \Gamma_0 T\Sigma_\infty$, and $\Sigma_\infty \in \mathcal{U}_1 \subset \mathcal{L}_0$ such that $exp : \mathcal{B} \to \mathcal{U}_1$ is a homeomorphism. Here, exp denotes the exponential map in (L, G).* The proof of this Lemma uses the following string of isomorphisms: $\Gamma_0 T\Sigma_\infty \cong \Gamma_0 JT\Sigma_\infty \cong \Gamma_0 N\Sigma_\infty \cong \Gamma N\Sigma_\infty/dAut(\mathbb{R}^3)) \cong \{$ variations of Σ_∞ in $\mathcal{L}/dAut(\mathbb{R}^3)\}$. This implies Theorem 3. Remark 0: The first isomorphism uses the fact that J is an isomorphism in each fibre. Remark 1: N stands for the normal bundle. The second isomorphism follows since $JT\Sigma_\infty$ is transversal to $T\Sigma_\infty$ except at np, where $JT_0\Sigma_\infty = T_0\Sigma_\infty$, therefore $JT_0\Sigma_\infty/T_0\Sigma_\infty = 0$. Remark 2: The second isomorphism is not true if L is replaced by S, i.e. if we work in Lagrangian category. Remark 3: The third isomorphism follows from $dAut(R3)$ being transitive on $T\Sigma_0$. Remark 4: The second and third isomorphisms do not hold if Σ_∞ has more then one complex point.
4. The **proof of Theorem 4** relies on a geometrization of the open manifold (L, J) as introduced by the authors in 2005 , which we will now describe. **(i)** Define an indefinite **distance function** d on L by $d(l_l, l_2) = D \cdot (\pi_1(l_1) \times \pi_1(l_2))$. Here, $D \in \mathbb{R}^3$ is the vector spanning the minimal distance from $l_1 \subset \mathbb{R}^3$ to $l_2 \subset \mathbb{R}^3$ and $\pi_1 : L \to S^2 \subset \mathbb{R}^3$ is the Gauss map. **(ii)** For l_1 fixed, the linearization at l_1 of $d(l_1, \cdot)$ defines a quadratic form $G_{l_1} : T_{l_1}L \otimes T_{l_1}L \to \mathbb{R}$, which can be seen to have signature $(2,2)$ and gives rise to a **neutral metric** on L. **(iii)** Let now $TS^2 \to T^*S^2$ be the diffeomorphism given by fibrewise pairing with the round metric on S^2. Define the symplectic form Ω by pulling back the canonical structure : $(L \approx TS^2, \Omega) \to (T^*S^2, \omega_{CAN})$. **(iv)** *It can be seen that $(L, G(\cdot, J)) \approx (TS^2, \Omega)$ and that (L, J, G) is a* **neutral Kähler surface** *and that $Isometry\ Group_0(\mathbb{E}^3) \approx Isometry\ Group_0(L, G)$.* **(v)** Let $\Sigma \hookrightarrow (L, \Omega, G)$ be any $C^{2,\alpha}$ - smooth section of $TS^2 \approx L$. Then Σ is Lagrangian $\iff \{l^\perp \subset \mathbb{R}^3 | l \in \Sigma\}$ is an integrable plane field $\iff \exists S \in \mathcal{S} : \Sigma = N(S) \implies \Sigma \hookrightarrow (L, G)$ is Lorentzian. $\{l^\perp \subset \mathbb{R}^3 | l \in \Sigma\}$ is a contact structure of $\mathbb{R}^3 \iff \Sigma \hookrightarrow (L, G)$ is spacelike. **(vi)** In the proof of Theorem 4 we use local holomorphic coordinates on on the complement of the fibre over the south-pole of S^2: $N \approx T^{10}S^2 \ni \eta\partial/\partial\xi \to (\xi, \eta) \in \mathbb{C}^2$. Here, $\xi : S^2 -$southpole $\to \mathbb{C}$ is the stereographic projection. These allow us to define a local biholomorphic map $\Phi_{C_0} : (L, J) \to (L, J)$ given by $\Phi_{C_0}(\xi, \eta) := (\xi, \eta - iC_0\xi)$ for fixed $C_0 \in \mathbb{R}$. This map has the effect of "twisting" a family of lines $N(S) \subset L$ which we will use as follows.

i) *Let* $S_+ \subset \mathbb{R}^3$ *be an open striclty convex and umbilic - free surface whose Gauss image is the upper hemishpere in* S^2*,*
ii) $N(S_+) \hookrightarrow L$ *be its surface of normals,*
iii) $\tilde{\Sigma} := \Phi_{C_0} N(S_+) \hookrightarrow L$ *be its twisting*
iv) $\tilde{\Sigma}_+ \subset \tilde{\Sigma} \subset (L, G)$ *be its spacelike subset.*

Then for every C_0*,* $\tilde{\Sigma}$ *is totally real in* (L, J) *and for* $C_0 \to \infty$ *, the Gauss image of* $\tilde{\Sigma}_+$ *covers the upper hemisphere of* S^2*.* **(vii)** In coordinates, J, Ω, G have the form $J\frac{\partial}{\partial \xi} = i\frac{\partial}{\partial \xi}, J\frac{\partial}{\partial \eta} = i\frac{\partial}{\partial \eta}$, $\Omega = 4(1+\xi\bar{\xi})^{-2}\mathbb{R}\mathrm{e}\left(d\eta \wedge d\bar{\xi} - \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}} d\xi \wedge d\bar{\xi}\right)$,
$\mathbb{G} = 4(1+\xi\bar{\xi})^{-2}\mathbb{I}\mathrm{m}\left(d\bar{\eta}d\xi + \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}} d\xi d\bar{\xi}\right)$.

5. Theorem 4 is proved using Mean Curvature Flow with mixed Dirichlet-Neumann boundary conditions of a spacelike surface with boundary in geometrized line space (L, G) as introduced in the previous section. Specifically, we study the following **Initial Boundary Value Problem**.

Consider for $s \in [0, s_0)$ *a family of spacelike real surfaces in* (L, G)*,* $f_s \in C^{2,\alpha}(\overline{D})$ *such that* $(df/ds)^\perp = H_f$*, with the following initial and boundary conditions:*
(i) $f_0(D) = \Sigma_0$
(ii) $f_s(\partial D) \subset \tilde{\Sigma}$
(iii) the hyperbolic angle B *between the spacelike planes* $Tf_s(\overline{D})$ *and* $T\tilde{\Sigma}$ *is constant along* $f_s(\partial D)$
(iv) $f_s(\partial D)$ *is asymptotically holomorphic:* $|\bar{\partial} f_s| = C/(1+s)$*.*
Here, H_s *is the mean curvature vector of* $f_s(D)$ *in* (L, G)*, and* $\tilde{\Sigma}$ *and* Σ_0 *are spacelike surfaces in* (L, G)*.*

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Hypersurfaces in spheres and Lagrangian submanifolds in complex hyperquadrics

Hui Ma

(joint work with Yoshihiro Ohnita)

In this talk, we focus on the interesting relation between hypersurfaces in spheres and Lagrangian submanifolds in complex hyperquadrics. Let $Q_n(\mathbb{C})$ be a complex hyperquadric of $\mathbf{C}P^{n+1}$ defined by the homogeneous quadric equation

$z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$. Let $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ (resp. $Gr_2(\mathbf{R}^{n+2})$) be the real Grassmann manifold of oriented 2-dimensional vector subspaces (resp. 2-dimensional vector subspaces) of $\mathbf{R}^{n+2}$. Denote by $[W]$ a 2-dimensional vector subspace W of $\mathbf{R}^{n+2}$ equipped with an orientation. Then we have the identification

$$Q_n(\mathbf{C}) \ni [\mathbf{a} + \sqrt{-1}\mathbf{b}] \longleftrightarrow [W] = \mathbf{a} \wedge \mathbf{b} \in \widetilde{Gr}_2(\mathbf{R}^{n+2}),$$

where $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of W compatible with the orientation of $[W]$. $Q_n(\mathbf{C})$ is diffeomorphic to $SO(n+2)/(SO(2) \times SO(n))$, which are compact irreducible Hermitian symmetric space of rank 2 if $n \geq 3$ and $S^2 \times S^2$ if $n = 2$. $Q_n(\mathbf{C})$ can also be regarded as the space of oriented geodesics of $S^{n+1}(1)$.

Let N^n be an oriented hypersurface immersed in the unit standard sphere $S^{n+1}(1) \subset \mathbf{R}^{n+2}$. Denote by $\mathbf{x}$ its position vector of a point p of N and $\mathbf{n}$ the unit normal vector field of N in $S^{n+1}(1)$. It is a fundamental fact in symplectic geometry that the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$$

is always a Lagrangian immersion into the complex hyperquadric $Q_n(\mathbf{C})$. Let $\kappa_i (i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$ and H denote the mean curvature vector field of the Gauss map $\mathcal{G}$. Palmer showed the following mean curvature form formula ([14]):

$$(1) \qquad \alpha_{\mathrm{H}} = -d\left(\sum_{i=1}^{n} \operatorname{arc}\cot \kappa_i\right) = d\left(\operatorname{Im}\left(\log \prod_{i=1}^{n}(1 + \sqrt{-1}\kappa_i)\right)\right).$$

Hence, if N^n is an oriented austere hypersurface in $S^{n+1}(1)$, introduced by Harvey-Lawson ([4]), then its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. In particular, the Gauss map of a minimal surface in $S^3(1)$ is a minimal Lagrangian immersion in $Q_2(\mathbf{C}) \cong S^2 \times S^2$ ([2]).

A hypersurface immersed in the standard sphere is called isoparametric if it has constant principal curvatures, which can be regarded as the generalization of geodesic spheres in the standard spheres. The theory of isoparametric hypersurfaces in spheres was started by Élie Cartan and well developed since then. As the most fundamental result, Münzner ([9], [10]) showed that the number g of distinct principal curvatures of an isoparametric hypersurface N^n in $S^{n+1}(1)$ must be $g = 1, 2, 3, 4, 6$ and their multiplicities satisfy $m_1 = m_3 = \cdots \leq m_2 = m_4 = \cdots$. It follows from (1) that the Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion ([14]). Concerning about the Gauss image $\mathcal{G}(N^n)$, we get

Theorem 1 ([7, 11]). (1) *The Gauss image $\mathcal{G}(N^n)$ is a compact smooth minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$, which is preserved by the deck transformation group $\mathbf{Z}_2$ of universal cover $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \to Gr_2(\mathbf{R}^{n+2})$.*

(2) *The Gauss map $\mathcal{G} : N^n \to \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ as a map onto its image is a covering map with the deck transformation group $\mathbf{Z}_g$.*

(3) *$\mathcal{G}(N^n)$ is orientable if and only if the integer $2n/g$ is even.*

(4) *$\mathcal{G}(N^n)$ is a monotone and cyclic embedded Lagrangian submanifold in $Q_n(\mathbf{C})$ with minimal Maslov number $2n/g$.*

Every homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of a linear isotropy representation of a compact rank 2 Riemannian symmetric pair (U, K), by Hsiang-Lawson ([5]) and Takagi-Takahashi ([15]). Only in the case of $g = 4$ are there known to exist non-homogeneous isoparametric hypersurfaces, which were discovered first by Ozeki-Takeuchi ([12], [13]) and extensively generalized by Ferus-Karcher-Münzner ([3]). We know that an isoparametric hypersurface in $S^{n+1}(1)$ is homogeneous if and only if its Gauss image is a homogeneous Lagrangian submanifold in $Q_n(\mathbf{C})$ ([7]). Based on the link with the theory of homogeneous isoparametric hypersurfaces in spheres, we classified all compact homogeneous Lagrangian submanifolds in $Q_n(\mathbf{C})$.

Theorem 2 ([7]). *For any compact homogeneous Lagragian submanifold L in $Q_n(\mathbf{C})$, there exists a unique compact homogeneous isoparametric hypersurface N^n in $S^{n+1}(1)$ corresponding to a compact rank 2 Riemannian symmetric pair (U, K) such that $L = \mathcal{G}(N^n)$ or L belongs to a Lagrangian deformation of $\mathcal{G}(N)$ consisting of compact homgeneous Lagrangian submanifolds. Actually, there exists such a non-trivial Lagrangian deformation of $\mathcal{G}(N^n)$ only when (U, K) is one of*

(1) $(S^1 \times SO(3), SO(2))$,
(2) $(SO(3) \times SO(3), SO(2) \times SO(2))$,
(3) $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 3)$,
(4) $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2, m \geq 3)$.

Hamiltonian volume variational problem is a proper constraint volume variational problem for Lagrangian submanifolds in Kähler manifolds, which was introduced by Y.G. Oh in 1990s. It is interesting to construct and classify compact Hamiltonian stable minimal or Hamiltonian minimal Lagrangian submanifolds in specific Kähler manifolds. Applying the spherical function theory of compact homogeneous spaces and fibrations on homogeneous isoparametric hypersurfaces, we completely determined the strict Hamiltonian stability of the Gauss images $\mathcal{G}(N^n)$ of all homogeneous isoparametric hypersurfaces N^n in $S^{n+1}(1)$.

Theorem 3 ([7, 8]). *The Gauss image $\mathcal{G}(N)$ of homogeneous isoparametric hypersurface N^n is Hamiltonian stable if and only if $m_2 - m_1 \geq 3$ or N^n is a principal orbit of the isotropy representation of the Riemannian symmetric pair of type EIII (in this case $(m_1, m_2) = (6, 9)$).*

Further questions:

(1) Study further relations between hypersurfaces in M and Lagrangian submanifolds in $\mathrm{Geod}^+(M)$. Note that $\mathrm{Geod}^+(\mathbf{C}P^n) = SU(n+1)/(T^2 \cdot SU(n-1))$, $\mathrm{Geod}^+(\mathbf{H}P^n) = Sp(n+1)/T^1 \cdot Sp(1) \cdot Sp(n-1)$, $\mathrm{Geod}^+(\mathbf{O}P^2) = F_4/T^1 \cdot Spin(7)$ ([1]).
(2) Study the Hamiltonian stability of the Gauss images of compact non-homogenous isoparametric hypersurfaces (OT-FKM type, embedded in spheres with $g = 4$).
(3) Study other properties of the Gauss images in complex hyperquadrics, e.g., curvature property, intersection theory, Lagrangian Floer theory.

(4) Study the relation between our Gauss image construction and Karigiannis-Min-Oo's results ([6]).

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Moment map expression of isoprametric hypersurfaces

Reiko Miyaoka

We express all the known Cartan-Münzner polynomials $F(x)$ of degree four in terms of the moment map of certain group actions.

A hypersurface M in S^n with g distinct constant principal curvatures is called isoprametric, which is expressed as a level set of $F(x)$ of degree g restricted to S^n. The multiplicities of the principal curvatures are given by a pair (m_1, m_2).

The classification has been done except for the case $g = 4$, $(m_1, m_2) = (7, 8)$ (see [12], and a corrected version [13], thanks to U. Abresch and A. Siffert).

g	1	2	3	4^*	6
M	S^{n-1} (hom.)	$S^{k-1} \times S^{n-k}$ (hom.)	Cartan h's $C_{\mathbb{F}}$ (hom.)	hom. or OT-FKM	$SO(4)$ or G_2-orbits (hom.)

It turns out that non-homogeneous cases occur only when $g = 4$, where Ozeki-Takeuchi first [17], and then Ferus-Karcher-Münzner [6] constructed infinitely many non-homogeneous and homogeneous examples, called of OT-FKM type.

Classification of $g = 4$ except for $(7, 8)$ ([2], [9], [3], [4])

<table>
<tr><td rowspan="3">OT-FKM type</td><td>non-homogeneous</td><td>$(m_1, m_2) = (3, 4k), (7, 8k)$, etc.</td></tr>
<tr><td rowspan="4">homogeneous:
isotropy orbits of
rank 2 symmetric space
G/K</td><td>G/K : non-Hermitian
$(4, 4k-1)$</td></tr>
<tr><td>*Hermitian
$(1, k), (2, 2k-1), (9, 6)$</td></tr>
<tr><td rowspan="2">non OT-FKM</td><td>*Hermitian $(4, 5)$</td></tr>
<tr><td>non-Hermitian $(2, 2)$</td></tr>
</table>

Recently, Fujii [7], and F-Tamaru [8] give an expression of Cartan-Münzner polynomials of degree four corresponding to the isotropy orbits of rank two Hermitian symmetric spaces (* in the table) by using the square norm of the moment map of the isotropy action. However, their method is not valid for the remaining cases. In [14], we give a new expression in all the known cases of $g = 4$.

The OT-FKM type hypersurfaces are given by a Clifford system $P_0, \dots, P_m \in O(2l)$ on $\mathbb{R}^{2l}$, by which we mean those satisfying $P_i P_j + P_j P_i = 2\delta_{ij}\mathrm{id}$. The pairs (m, l) exist in infinite series. The inner product $\langle P, Q\rangle = \frac{1}{2l}\mathrm{Tr}(P^t Q)$ makes $P_0, \dots, P_m$ an orthonormal basis of the linear space V spanned by themselves.

Fact 3 ([6])**.** For a given Clifford system $P_0, \dots, P_m$,

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2 \tag{1}$$

is a Cartan-Münzner polynomial of degree four. If $l - m - 1 > 0$, $F|_{S^{2l-1}}$ defines isoparametric hypersurfaces in S^{2l-1} with $g = 4$ and $m_1 = m$, $m_2 = l - m - 1$.

In this case, $P_i P_j$, $0 \le i < j \le m$, generate a Lie subalgebra $\mathfrak{o}(m+1)$ of $\mathfrak{o}(2l)$, which induces a spin action on $\mathbb{R}^{2l}$.

Fact 4 ([6])**.** $Spin(m+1)$ acts on $\mathbb{R}^{2l}$, and preserves $F(x)$.

When $K \subset O(n)$ acts on $\mathbb{R}^n$, we extend it naturally to an action on $T\mathbb{R}^n$. For $\zeta \in \mathfrak{o}(n)$, the fundamental vector field is given by $X_\zeta = \zeta x$.

Fact 5 ([14])**.** The K-action is a Hamiltonian action with the moment map $\mu : T\mathbb{R}^n \to \mathfrak{k}^*$ given by

$$\mu(x, Y)(\zeta) = -\langle \zeta x, Y\rangle, \quad \zeta \in \mathfrak{o}(n).$$

This is a generalization of the angular momentum of the classical mechanics. Thus our $Spin(m+1)$ action on $T\mathbb{R}^{2l}$ has the moment map $\mu : T\mathbb{R}^{2l} \to \mathfrak{o}^*(m+1)$ given by

$$\mu(x, Y) = -\sum_{0\le i<j\le m} \langle P_iP_jx, Y\rangle P_iP_j \in \mathfrak{o}(m+1) \cong \mathfrak{o}^*(m+1),$$

where P_iP_j generate an orthonormal frame of $\mathfrak{o}(m+1)$. It follows immediately that $\|\mu(x,Y)\|^2 = \sum_{0\le i<j\le m}\langle P_iP_jx, Y\rangle^2$. Comparing this with the second term of $F(x)$; $\sum\langle P_ix, x\rangle^2$, we find an appropriate vector field Y_x which satisfies:

Theorem 1 ([14])**.** *Define a vector field $Y : \mathbb{R}^{2l} \to T\mathbb{R}^{2l}$ (not necessarily continuous) by*

$$Y_x = \begin{cases} P_0x, & \text{if } \langle P_0x, x\rangle = 0 \\ \dfrac{\langle P_1x,x\rangle P_0x - \langle P_0x,x\rangle P_1x}{\sqrt{\langle P_1x,x\rangle^2 + \langle P_0x,x\rangle^2}}, & \text{if } \langle P_0x, x\rangle \ne 0. \end{cases}$$

Then the Cartan-Münzner polynomial is expressed as

$$F(x) = \|x\|^2 - 2\|\mu(x, Y_x)\|^2, \tag{2}$$

where μ is the moment map of the $Spin(m+1)$ action naturally extended to $T\mathbb{R}^{2l}$.

Remark 1. In fact, Y_x can be replaced by any Px orthogonal to x, $P \in \Sigma$, where Σ is the unit sphere of V, called the Clifford sphere. From this view point, we give a more natural statement. If we identify the oriented 2-plane Grassmannian on $\mathbb{R}^{2l}$ with the complex hyperquadratic $Q^{2l-2}(\mathbb{C})$, $Spin(m+1)$ action is naturally extended to $Q^{2l-2}(\mathbb{C})$. This is a Hamiltonian action with the moment map μ_Q given by $\mu_Q([x\wedge Px]) = \mu(x, Px)$ for $P \in \Sigma$ such that $\langle Px, x\rangle = 0$. Thus we obtain:

Theorem 2 ([15])**.** *Let $E \to S^{2l-1}$ be a singular sphere bundle with fiber $E_x = \{Px \mid P\in\Sigma, \langle Px,x\rangle = 0\}$. Define $\varphi : E \to Q^{2l-2}(\mathbb{C})$ by $\varphi(x\wedge Px) = [x\wedge Px]$. Then $F(x)$ is expressed as*

$$F(x) = \|x\|^2 - 2\|\mu_Q([x\wedge Px])\|^2, \tag{3}$$

where μ_Q is the moment map of the spin action naturally extended to $Q^{2l-2}(\mathbb{C})$.

Moreover, we obtain:

Theorem 3 ([15])**.** *Under the above situation, $\mu_Q^{-1}(0)$ is a coisotropic submanifold of $Q^{2l-2}(\mathbb{C})$, and $\mu_Q^{-1}(0)/Spin(m+1)$ is a symplectic reduction.*

The complex hyperquadratic $Q^{n-2}(\mathbb{C})$ is important as the target space of the Gauss map of a hypersurface M in S^{n-1} (see Hui Ma's report in this volume).
The non-OT-FKM type isoparametric hypersurfaces are isotropy orbits of $SO(5)\times SO(5)/SO(5)$, $(m_1,m_2) = (2,2)$, and of $SO(10)/U(5)$,$(m_1,m_2) = (4,5)$. In these cases ([14]), as well as in the cases $g = 1,2,3,6$ ([15]), we give an expression of $F(x)$ in terms of the moment map of the isotropy action.

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Min-max Theory and the Willmore conjecture

André Neves
(joint work with Fernando Marques)

The bending energy (or Willmore energy) of a closed surface Σ immersed in Euclidean three-space is the total integral of the square of the mean curvature:

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\Sigma.$$

It was already known to Blaschke and Thomsen in the 1920s that this energy is conformally invariant, i.e., $\mathcal{W}(F(\Sigma)) = \mathcal{W}(\Sigma)$ for any $F \in \text{Conf}(\mathbb{R}^3)$. The bending energy appears naturally in some physical contexts as the bending energy of elastic membranes.

In the 1960s, Willmore proved the following result:

Theorem. (Willmore) *Let Σ be a smooth closed surface in $\mathbb{R}^3$. Then $\mathcal{W}(\Sigma) \geq 4\pi$, and equality holds if and only if Σ is a round sphere.* The geometric content of

the result is that every compact surface bends more then a sphere and if it bends as much as the sphere, then the compact surface must be a sphere.

It is then natural to ask what is the optimal shape among all surfaces of some fixed topological type. Motivated by the analysis of circular tori of revolution, Willmore made a conjecture for the case of genus one:

Willmore Conjecture (1965). *The integral of the square of the mean curvature of a torus immersed in $\mathbb{R}^3$ is at least $2\pi^2$.*

The equality is achieved by the torus of revolution whose generating circle has radius 1 and center at distance $\sqrt{2}$ from the axis of revolution:

$$(u,v) \mapsto \big((\sqrt{2}+\cos u)\cos v, (\sqrt{2}+\cos u)\sin v, \sin u\big) \in \mathbb{R}^3.$$

In the talk I explained how to prove this conjecture using the min-max theory of minimal surfaces.

The key idea consists in associating to every compact surface Σ e a continuous 5-parameter family of surfaces (integral 2-currents with boundary zero, to be more precise) in S^3 such that the area of each surface in the family is bounded above by $\mathcal{W}(\Sigma)$. This family is parametrized by a map Φ defined on I^5, and is constructed so that

- $\Phi(x,0) = \Phi(x,1) = 0$ (trivial surface) for any $x \in I^4$,
- $\Phi(x,t)$ is an oriented round sphere in S^3 for any $x \in \partial I^4$, $t \in [0,1]$,
- $\{\Phi(x,t)\}_{t\in[0,1]}$ is a homotopically nontrivial sweepout of S^3 for any $x \in I^4$,
- $\sup\{\text{area}(\Phi(x,t)) : (x,t) \in I^5\} \leq \mathcal{W}(\Sigma)$.

This map Φ has the crucial property that, if the genus of Σ is positive, its restriction to $\partial I^4 \times \{1/2\}$ is a homotopically nontrivial map into the space of oriented great spheres, which is homeomorphic to S^3. Therefore the min-max theory developed in [1] shows that

$$2\pi^2 \leq \sup\{\text{area}(\Phi(x,t)) : (x,t) \in I^5\} \leq \mathcal{W}(\Sigma).$$

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Min-Max Theory and the Energy of Links

André Neves
(joint work with Ian Agol, Fernando Marques)

Let $\gamma_i : S^1 \to R^3$, $i = 1, 2$, be a 2-component link. The Möbius cross energy of the link (γ_1, γ_2) is defined to be

$$E(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{|\gamma_1'(t)||\gamma_2'(t)|}{|\gamma_1'(t) - \gamma_2'(t)|^2}\ ds\ dt$$

The Möbius energy has the remarkable property of being invariant under conformal transformations of $\mathbb{R}^3$ [1]. In the case of knots other energies were considered by O'Hara [5].

It is not difficult to check that $E(\gamma_1, \gamma_2) \geq 4\pi \mathrm{lk}(\gamma_1, \gamma_2)|$, where $\mathrm{lk}(\gamma_1, \gamma_2)$ denotes the linking number of (γ_1, γ_2). This is an immediate consequence of the Gauss formula:

$$\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\gamma_1'(s), \gamma_2'(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|3}\ ds\ dt.$$

By considering pairs of circles which are very far from each other, we see that the cross energy can be made arbitrarily small. If the linking number of (γ_1, γ_2) is nonzero, the estimate says that $E(\gamma_1, \gamma_2) \geq 4\pi$. It is natural to search for the optimal configuration in that case.

It was conjectured by Freedman, He and Wang [1], in 1994, that the Möbius energy should be minimized, among the class of all nontrivial links in $\mathbb{R}^3$, by the stereographic projection of the standard Hopf link. The standard Hopf link $(\hat{\gamma}_1, \hat{\gamma}_2)$ is described by

$$\hat{\gamma}_1(s) = (\cos(s), \sin(s), 0, 0) \in S^3 \quad \text{and} \quad \hat{\gamma}_2(t) = (0, 0, \cos(t), \sin(t)) \in S^3,$$

and it is simple to check that $E(\hat{\gamma}_1, \hat{\gamma}_2) = 2\pi^2$. Here we note that the definition of the energy and the conformal invariance property extend to any 2-component link in $\mathbb{R}^n$ [3]. A previous result of He proved that the minimizer must be isotopic to a Hopf link [2].

In the talk I explained how to prove this conjecture using the min-max theory of minimal surfaces. We now briefly sketch the proof. For any link (γ_1, γ_2) in $\mathbb{R}^3$, we associate a continuous 5-parameter family of surfaces (integral 2-currents with boundary zero, to be more precise) in S^3 such that the area of each surface in the family is bounded above by $E(\gamma_1, \gamma_2)$. This family is parametrized by a map Φ defined on I^5, and is constructed so that

- $\Phi(x, 0) = \Phi(x, 1) = 0$ (trivial surface) for any $x \in I^4$,
- $\Phi(x, t)$ is an oriented round sphere in S^3 for any $x \in \partial I^4, t \in [0, 1]$,
- $\{\Phi(x, t)\}_{t \in [0,1]}$ is a homotopically nontrivial sweepout of S^3 for any $x \in I^4$,
- $\sup\{\mathrm{area}(\Phi(x, t)) : (x, t) \in I^5\} \leq E(\gamma_1, \gamma_2)$.

This map Φ has the crucial property that its restriction to $\partial I^4 \times \{1/2\}$ is a homotopically nontrivial map into the space of oriented great spheres, which is homeomorphic to S^3. Therefore the min-max theory developed in [4] shows that $2\pi^2 \leq \sup\{\text{area}(\Phi(x,t)) : (x,t) \in I^5\} \leq E(\gamma_1, \gamma_2)$.

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Isoperimetric domains with large volume in simply connected homogeneous three-manifolds

Joaquín Pérez

(joint work with William H. Meeks III, Pablo Mira, Antonio Ros)

Minimal and constant mean curvature (CMC) surfaces have been deeply studied in different three-dimensional ambient spaces. The classical framework for this study is when the ambient space is a *space form,* whose six-dimensional isometry group allows to use a wide range of techniques when analyzing CMC surfaces as the Alexandrov reflection method, the holomorphicity of the Hopf differential or the Lawson correspondence. Since the discovery in 2004 by Abresch and Rosenberg of a holomorphic quadratic differential for CMC surfaces in homogeneous spaces with four dimensional isometry group, many results have been produced in this more general framework where the ambient space is a Riemannian submersion with constant bundle curvature $\tau \in \mathbb{R}$ over a surface of constant curvature $\kappa \in \mathbb{R}$ (these are the so-called $\mathbb{E}(\kappa, \tau)$ spaces). Space forms and $\mathbb{E}(\kappa, \tau)$ spaces are special cases of simply connected homogeneous three-manifolds, but the generic case of such a Riemannian ambient space is when the isometry group is merely three-dimensional. All simply connected homogeneous three manifolds are Lie groups endowed with left invariant metrics (with the exception of $\mathbb{S}^2(\kappa) \times \mathbb{R} = \mathbb{E}(\kappa, 0)$ for $\kappa > 0$, which has isometry group of dimension four). In this talk we will explore certain aspects of the theory of CMC surfaces in non-compact simply connected homogeneous three-manifolds, connecting the isoperimetric problem with two numbers that appear naturally: the *Cheeger constant* and the *critical mean curvature* of the ambient space.

In the sequel, X will denote a non-compact, simply connected homogeneous three-manifold. The *isoperimetric profile* of X is the function $I\colon (0,\infty) \to (0,\infty)$ given by

$$I(t) = \inf\{\text{Area}(\partial\mathcal{D}) : \overline{\mathcal{D}} \subset X \text{ is a smooth compact domain with Volume}(\mathcal{D}) = t\}.$$

Since X is homogeneous and has dimension less than 8, for every value of $t \in (0,\infty)$ there exists at least one smooth compact domain $\Omega \subset X$ of volume t and area $I(t)$ (called an *isoperimetric domain*), and the boundary of every such a domain has non-negative constant mean curvature with respect to the inward pointing unit normal vector.

Definition. Let $\mathcal{A}$ be the collection of all compact, immersed surfaces in X. Given a surface $\Sigma \in \mathcal{A}$, let $|H_\Sigma|\colon \Sigma \to [0,\infty)$ be the absolute mean curvature function of Σ. The *critical mean curvature* of X is the non-negative number

$$H(X) = \inf\{\max_{\Sigma} |H_\Sigma| \;:\; \Sigma \in \mathcal{A}\}.$$

Definition. The *Cheeger constant* of a Riemannian manifold Y with infinite volume is the non-negative number

$$\mathrm{Ch}(Y) = \inf\left\{\frac{\mathrm{Area}(\partial\mathcal{D})}{\mathrm{Vol}(\mathcal{D})} \;:\; \overline{\mathcal{D}} \subset Y \text{ is a smooth compact domain}\right\}.$$

By definition of the Cheeger constant, $\mathrm{Ch}(X) = \inf\{\frac{I(t)}{t} \mid t \in (0,\infty)\}$ for every non-compact, simply connected homogeneous three-manifold X, where I is the isoperimetric profile of X. The main result explained in this talk is the following one, whose proof can be found in [1].

Theorem 1. *Let X be a non-compact, simply connected homogeneous three-manifold.*

(1) *Suppose that X is not isometric to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ of a two-sphere of constant curvature $\kappa > 0$ with the real line. If $\Omega \subset X$ is an isoperimetric domain in X with volume t, then $\partial\Omega$ is connected and*

$$\mathrm{Ch}(X) < \min\left\{2H, \frac{I(t)}{t}\right\},$$

where $H > 0$ is the constant mean curvature of the boundary of Ω with respect to the inward pointing unit normal.

(2) $\mathrm{Ch}(X) = 2H(X) = \lim_{t\to\infty} \frac{I(t)}{t} = \lim_{t\to\infty} I'_-(t) = \lim_{t\to\infty} I'_+(t)$, *where $I'_-(t), I'_+(t)$ denote the left and right derivatives of the isoperimetric profile I of X.*

(3) *Given any sequence of isoperimetric domains $\Omega_n \subset X$ with volumes tending to infinity, as $n \to \infty$, the sequence of constant mean curvatures of their boundaries converges to $H(X)$ and the sequence of radii of these domains diverges to infinity*[1].

(4) *If $X \neq \mathbb{S}^2(\kappa) \times \mathbb{R}$, then there exist two 1-parameter subgroups $\Gamma, \widetilde{\Gamma}$ and a $(\mathbb{Z}\times\mathbb{Z})$-subgroup Δ of the isometry group* $\mathrm{Iso}(X)$ *of X, both acting freely on X, and a topologically product foliation $\mathcal{F}$ of X by properly embedded stable surfaces of constant mean curvature $H(X)$, with the following properties.*

[1]The *radius* of a compact Riemannian manifold with boundary is the maximum distance from points in the manifold to its boundary.

(a) $\mathcal{F} = \{\phi(\Sigma) \mid \phi \in \widetilde{\Gamma}\}$, *where* Σ *is any particular leaf of* $\mathcal{F}$. *In particular, all leaves of* $\mathcal{F}$ *are congruent.*
(b) *Each of the leaves of* $\mathcal{F}$ *is invariant under* Γ *and* Δ.
(c) *Each orbit of the left action of* $\widetilde{\Gamma}$ *on* X *intersects every leaf of* $\mathcal{F}$ *transversely at a single point.*
(d) *If* $\mathrm{Ch}(X) > 0$, *then given a sequence* $\{\Omega_n\}_n$ *of isoperimetric domains in* X *with volumes tending to infinity, there exist open sets* $S_n \subset \partial\Omega_n$ *with* $\frac{\mathrm{Area}(S_n)}{\mathrm{Area}(\partial\Omega_n)} \to 1$ *as* $n \to \infty$, *such that for any sequence of points* $q_n \in S_n$, *there exists a subsequence of the surfaces* $\{q_n^{-1}\partial\Omega_n\}_n$ *that converges smoothly (in the uniform topology on compact sets of* X*) to the leaf* Σ_a *of some congruent foliation* $a\mathcal{F}$ *to* $\mathcal{F}$ *passing through the identity element* e *of* X. *Furthermore, for this subsequence, the domains* $q_n^{-1}\Omega_n$ *converge to the closure of the mean convex component of* $X - \Sigma_a$.

It is worth explaining some details about item 4 of the last theorem. Classification results (see e.g. Milnor [3] or Meeks and Pérez [2]) insure that under the conditions of item 4, X is isometric to one of the following two Lie groups endowed with left invariant metrics:

A: The semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some 2×2 real matrix A. This means $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ with the group operation $(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A}\mathbf{p}_2, z_1 + z_2)$, where $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^2$ and $z_1, z_2 \in \mathbb{R}$, equipped with its canonical metric (i.e., the left invariant extension of the standard Euclidean inner product after identifying $\mathbb{R}^3$ with the tangent space to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ at the identity element $(0,0,0)$).

B: The universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the special linear group endowed with a left invariant metric (there is a three-parameter family of such metrics).

In case **A** above, one can define the objects in item 4 of the last theorem as follows. $\Sigma = \mathbb{R}^2 \rtimes_A \{0\}$, $\widetilde{\Gamma} = z$-axis, Γ is any straight line contained in Σ and Δ is the $(\mathbb{Z} \times \mathbb{Z})$-lattice generated by the translations by two linearly independent points $a_1 \in \Gamma$ and $a_2 \in \Sigma - \Gamma$.

In case **B**, if the left invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has isometry group of dimension four, then one can take Σ as a vertical horocylinder (i.e., the lifting to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of a horocycle α in $\mathbb{H}^2$ by the natural Riemannian submersion $\Pi\colon \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathbb{H}^2$), Γ is the unique 1-parameter parabolic subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ contained in Σ, $\widetilde{\Gamma}$ is the 1-parameter subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ obtained after lifting the hyperbolic translations along a geodesic in $\mathbb{H}^2$ one of whose ends is the point at $\partial_\infty\mathbb{H}^2$ of α, and Δ is the $(\mathbb{Z}\times\mathbb{Z})$-lattice generated by the left translations by a non-trivial element $a_1 \in \Gamma$ and by the generator a_2 of the center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Finally, if in case **B** the left invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has three-dimensional isometry group, then one can take $\Gamma, \widetilde{\Gamma}$ and Δ as in the last paragraph, and Σ can be defined as a certain entire doubly periodic graph over a vertical horocylinder (this notion of graph refers to the property that Σ intersects exactly once to every integral curve of the Killing vector field associated to the hyperbolic 1-parameter subgroup $\widetilde{\Gamma}$). In this case, the quotient surfaces $\phi(\Sigma)/\Delta$ with $\phi \in \widetilde{\Gamma}$ form a foliation by CMC tori of the locally homogeneous three-manifold $W = \widetilde{\mathrm{SL}}(2,\mathbb{R})/\Delta$ (diffeomorphic to the product of a torus with the real line, with one end of finite volume and another end of infinite volume) which give the complete solution to the isoperimetric problem in W, in the sense that given $V > 0$, the unique isoperimetric domain in W enclosing volume V is the end representative $\Omega(V)$ of the end of W with finite volume whose boundary is one of the CMC tori $\phi(\Sigma)/\Delta$, $\phi = \phi(V) \in \widetilde{\Gamma}$.

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Conformal Willmore flow

Ulrich Pinkall

(joint work with Keenan Crane, Peter Schroeder)

It is known [1] that every compact Riemann surface M admits a conformal immersion $f : M \to \mathbb{R}^3$. It then seems natural to ask for an optimal such realization in the sense of minimal Willmore functional

$$W(f) = \int_M H^2.$$

A minimizer f is known to exist [5] provided the conformal type is such that the infimum of the Willmore functional is below 8π.

The usual setup for treating such questions would be to use the space of all immersions f as the space over which the functional W is to be minimized. Here we propose a "change of variable": We look for an optimal conformal immersion f by following the gradient flow of

$$\int_M \mu^2$$

on the space $\mathcal{M}$ consisting of all half densities μ on M that can be realized as the mean curvature half-density $\mu = H|df|$ of some conformal immersion $f : M \to \mathbb{R}^3$. Here we denote by $|df|$ the half density $|df| = \sqrt{\sigma}$ where σ is the volume 2-form of the metric on M induced by f.

We claim that away from certain (rather well understood) singularities $\mathcal{M}$ is a submanifold of codimension $6g + 1$ (g being the genus of M) in the euclidean vector space formed by all half-densities on M. The gradient flow of the Willmore functional (which here is just the squared distance to the origin) basically is just an ordinary differential equation on $\mathcal{M}$: No partial derivatives on M are involved.

Based on earlier work on spin-transformations of surfaces [2, 3] we implemented [4] an extremely efficient numerical algorithm for this *conformal Willmore flow*. Figure 1 shows an example.

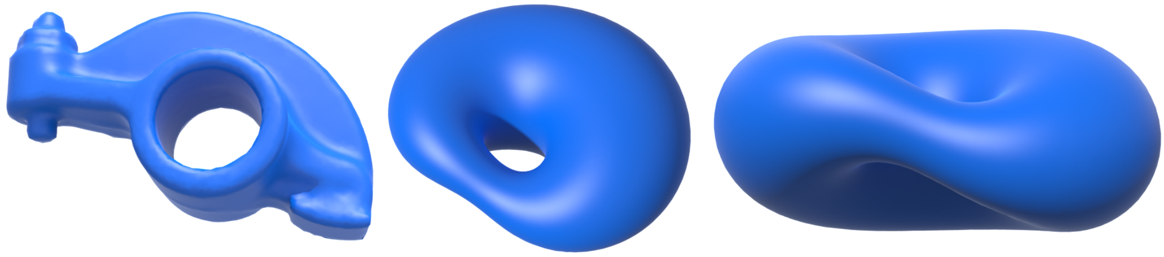

FIGURE 1. An initial torus and the limiting surface under the flow (center and right show different perspectives).

It is very instructive to consider also the analogous situation for closed plane curves. Here we look at the space $\mathcal{M}$ consisting of all L-periodic functions κ that are the curvature function of a closed plane γ of length L. Figure 2 shows an example of following the gradient flow of $\int \kappa^2$.

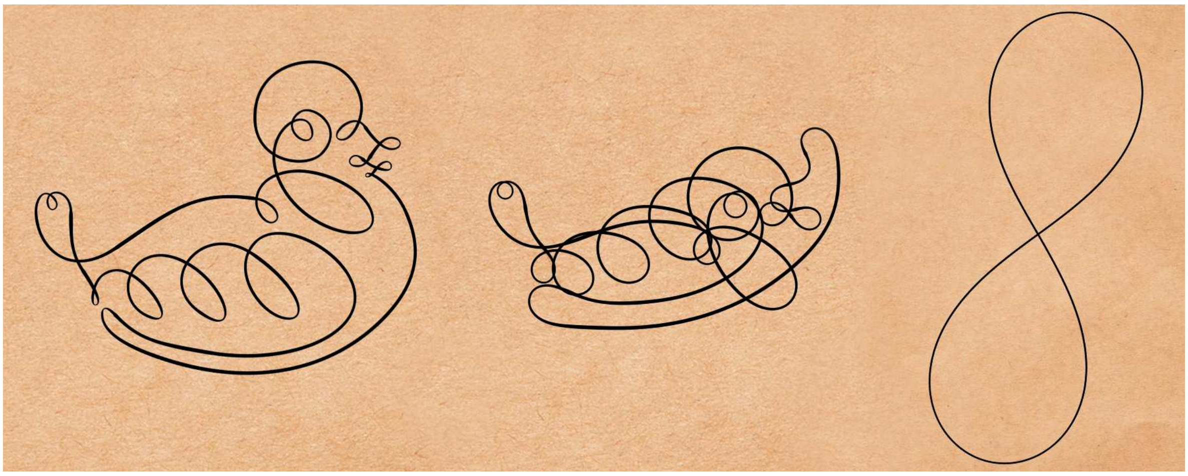

FIGURE 2. An initial curve and two stages of the flow.

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On the Moduli of CMC Annuli of finite type in $\mathbb{S}^3$

Martin U. Schmidt
(joint work with Laurent Hauswirth and Martin Kilian)

Pinkall and Sterling [5] constructed constant mean curvature (CMC) tori in $\mathbb{R}^3$ and independently Hitchin [4] harmonic maps from tori into $\mathbb{S}^3$ in terms of a real algebraic curve Σ and a holomorphic line bundle L on Σ via integrable system theory. This algebraic geometric correspondence between pairs (Σ, L) and geometric objects turned out to be a powerful tool for the construction of new examples. We want to enhance these methods in order to classify the corresponding geometric objects. One example are embedded CMC tori and annuli in homogenous three-dimensional geometries. Here we focus on Alexandrov embedded annuli in $\mathbb{S}^3$. In [3] we considered embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. For this purpose we have to understand the set of algebraic data (Σ, L) which correspond to these geometric objects. For fixed spectral curves Σ we call the set of line bundles L of algebraic data (Σ, L) isospectral set. Since this compact degree of freedom is well understood we focus on the remaining degrees of freedom. Consequently the main issue is the parametrisation of the moduli space. This is the space of all spectral curves Σ, such that (Σ, L) are algebraic data for some line bundle L.

Our main task is the investigation of the moduli space $\mathcal{M}$ of CMC annuli of finite type in the 3-sphere. We consider parabolic CMC annuli which have constant Hopf differential and bounded curvature. Such CMC annuli are said to be of finite type. The spectral curves of such finite type CMC annuli are real hyperelliptic curves. They are described by a polynomial a, whose roots are the branch points of the corresponding two-sheeted covering over $\mathbb{CP}^1$. The global geometry of an annulus is encoded in a meromorphic differential dh with specified properties. For given Σ with polynomial a the differential dh is described by another polynomial b. Furthermore two marked points λ_1 and λ_2 parametrise the mean curvature and the Hopf differential. We call the quadruples $(a, b, \lambda_1, \lambda_2)$ spectral data, and identify $\mathcal{M}$ with a subspace of such quadruples. We define vector fields on the space of quadruples $(a, b, \lambda_1, \lambda_2)$, which preserve the subspace $\mathcal{M}$. The corresponding flows simultaneously deform Σ, dh and the marked points λ_1 and λ_2. Along this flow the curvature can blow up. It turns out that the curvature stays uniformly bounded on the annulus as long as the roots of a stay away from the poles of dh. A typical limit of a curvature blow up is a chain of spheres touching each other at points which are limits of shrunken necks.

Another accident of the flow are coalescing roots of a. The limits are higher order roots of a and called singularities of Σ, since the corresponding algebraic variety is not any more a complex manifold. The dimension of the isospectral set of Σ is the genus and equal to half the number of roots of a. It turns out that in case of roots of a coalescing on $\mathbb{S}^1$ (in the real part) one dimension of the isospectral set shrinks to a point. In this case the limit of the isospectral sets coincides with the isospectral set of the desingularised curve Σ and there is no geometric accident. In case of roots of a coalescing at points away from $\mathbb{S}^1$ the limit of the isospectral sets is a union of the compact isospectral set of the desingularised curve and an extra higher-dimensional non-compact part. The lower-dimensional compact part is the closure of the higher-dimensional non-compact part and the union is still compact. The movement from the isospectral set of the desingularised curve to the extra non-compact part drastically changes the corresponding geometric annulus. In this case there is a geometric accident.

As an application of the deformation of spectral data we establish global properties of the moduli space. In the main Theorem 1 we construct for all spectral curves of CMC annuli a path in the moduli space, which starts at the given spectral curves and ends at spectral curves of spectral genus zero. Along this path the curvature stays bounded and no geometric accident happens. The spectral curves of genus zero and their annuli can effectively be classified by explicit calculation. In fact we determine all spectral data of spectral genus zero corresponding to mean convex Alexandrov embedded annuli. Moreover, we extend the corresponding family to a two-dimensional family of spectral data of spectral genus at most one. The corresponding CMC annuli are rotational mean convex Alexandrov embedded annuli. We call this family of spectral data rotational family. We show that the rotational family is connected with spectral curves outside of this family only by a movement from the lower-dimensional compact part of a non-singular spectral curve into the higher-dimensional extra non-compact part of a singular spectral curve as explained above. Here the desingularised curve of the latter spectral curve is the former spectral curve. As a consequence of Theorem 1 we characterise in Theorem 2 the rotational family as a subset of the moduli space of spectral curves of CMC annuli of finite type by three properties.

In the second part we apply our results on the moduli space and classify mean convex Alexandrov embedded annuli of finite type in $\mathbb{S}^3$ in Theorem 3. These annuli are exactly the rotational annuli, since the corresponding moduli space is shown to have the three properties of Theorem 2. We show that mean convex Alexandrov embedded surfaces with constant mean curvature have collars with depths uniformly bounded from below. For this purpose we use a 'maximum principle at infinity' which was communicated to us by Harold Rosenberg. As a consequence we establish in Theorem 4 the third property. This means that continuous deformations preserve the spectral data of mean convex Alexandrov embedded annuli, as long as the degree of a is preserved.

Recently Brendle proved with elementary methods Lawson's conjecture, which states that the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$ [2]. Andrews and Li [1] extended these arguments and confirmed the conjecture by Pinkall and Sterling, which states that all embedded CMC tori in $\mathbb{S}^3$ are surfaces of revolution. Both results follow from Theorem 3.

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Morse indices of minimal surfaces in flat tori

Toshihiro Shoda
(joint work with Norio Ejiri)

Our object is a properly n-periodic minimal immersion of an oriented surface into an n-dimensional Euclidean space $\mathbb{R}^n$. It can be considered as a minimal immersion of a compact oriented surface into an n-dimensional flat torus $\mathbb{R}^n/\Lambda$, and the conformal structure induced by the immersion makes the surface a Riemann surface. It is usually called *a conformal minimal immersion*. Now we study the latter for the case $n = 3$, 4 in terms of the Morse index and nullity. Recall that the Morse index of a compact oriented minimal surface in a flat torus is defined as the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues of the second variational operator of area. The nullity is the dimension of the 0-eigenspace.

First, we consider the case $n = 3$. In 1991, S. Montiel and A. Ros [6] established an impressive theory. By their results, we can show that both Morse index and nullity of Schwarz' CLP surface are 3. After that, in 1992, M. Ross obtained that each Morse index of Schwarz' P surface, D surface, and Schoen's Gyroid is 1. Moreover, Ross' argument implies that each nullity of the three surfaces is 3. On the other hand, N. Ejiri [3, 4] considered a moduli theory of compact oriented minimal surfaces in flat tori in terms of differential geometry and studied it from the point of view of the Morse index and nullity. This theory includes an algorithm to compute the Morse index under some assumptions. In the previous work, we carried out Ejiri's algorithm and computed the Morse index of examples as follows.

Main result 1

For $a \in (0, 1)$, let M be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z(z^3 - a^3)\left(z^3 - \frac{1}{a^3}\right)$ and f a conformal minimal immersion given by

$$f(p) = \mathrm{Re}\int_{p_0}^{p} i(1 - z^2,\, i(1 + z^2),\, 2z)^t\frac{dz}{w}.$$

Then there exist $0 < a_1 < a_2 < 1$ satisfying the following properties:
(i) *index* $= 2$ and *nullity* $= 3$ for $a \in (0, a_1)$,
(ii) *index* $= 1$ and *nullity* $= 3$ for $a \in (a_1, a_2)$,

(iii) *index* $= 3$ and *nullity* $= 3$ for $a \in (a_2, 1)$,
where $a_1 \sim 0.497010$, $a_2 \sim 0.714792$.

Note that we obtain the similar results for $a > 1$. This family of minimal surfaces is called *H family*.

Main result 2

For $a \in (0, 1]$, let M be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z(z^3 - a^3)\left(z^3 + \frac{1}{a^3}\right)$ and f a conformal minimal immersion given by

$$f(p) = \mathrm{Re}\int_{p_0}^{p} i(1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{dz}{w}.$$

Then there exist $0 < a_1 < 1$ satisfying the following properties:
(i) *index* $= 2$ and *nullity* $= 3$ for $a \in (0, a_1)$,
(ii) *index* $= 1$ and *nullity* $= 3$ for $a \in (a_1, 1]$,
where $a_1 \sim 0.494722$.

Note that we obtain the similar results for $a \geq 1$. This family of minimal surfaces is called *rPD family*.

Main result 3

For $a \in (2, \infty)$, let M be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z^8 + az^4 + 1$ and f a conformal minimal immersion given by

$$f(p) = \mathrm{Re}\int_{p_0}^{p} (1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{dz}{w}.$$

Then there exist $2 < a_1 < 14 < a_2 < \infty$ satisfying the following properties:
(i) *index* $= 2$ and *nullity* $= 3$ for $a \in (2, a_1)$,
(ii) *index* $= 1$ and *nullity* $= 3$ for $a \in (a_1, a_2)$,
(iii) *index* $= 2$ and *nullity* $= 3$ for $a \in (a_2, \infty)$,
where $a_1 \sim 7.40284$, $a_2 \sim 28.7783$.

This family of minimal surfaces is called *tP family*. Moreover, *tD family* is defined as a family of conjugate surfaces of minimal surfaces in tP family, and we find the same result for tD family.

Main result 4

For $a \in (-2, 2)$, let M be a hyperelliptic Riemann surface of genus 3 defined by $w^2 = z^8 + az^4 + 1$ and f a conformal minimal immersion given by

$$f(p) = \mathrm{Re}\int_{p_0}^{p} (1 - z^2,\, i(1 + z^2),\, 2z)^t \frac{dz}{w}.$$

Then, for an arbitrary $a \in (-2, 2)$, *index* $= 3$ and *nullity* $= 3$.
This family of minimal surfaces is called *tCLP family*.

Next we consider the case $n = 4$. We showed the existence of non-holomorphic hyperelliptic minimal surfaces of even genus in 4-dimensional flat tori, and moreover, we computed their Morse indices. Now we mention our motivation on it.

The existence of hyperelliptic minimal surfaces of odd genus in flat tori was given by C. Arezzo and G. P. Pirola [1]. Their technique is using deformations of a hyperelliptic minimal surface in a 3-dimensional flat torus to a hyperelliptic minimal surface in an n-dimensional flat torus ($n > 3$). Recall that there is a topological obstruction for a hyperelliptic minimal surface in a 3-dimensional flat torus. In fact, a hyperelliptic curve of even genus cannot be minimally immersed into any 3-dimensional flat torus [5, 7]. Thus Arezzo-Pirola treat only odd genus case. From this point of view, we consider the following problem: are there any hyperelliptic minimal surfaces of even genus in n-dimensional flat tori for $n > 3$? We focus on the simplest case, that is, the case $n = 4$. From E. Colombo and Pirola's argument [2], we can see the existence of holomorphic immersions of hyperelliptic curves of even genus into 4-dimensional flat tori. As a result, we improve our problem as follows: are there any non-holomorphic hyperelliptic minimal surfaces of even genus in n-dimensional flat tori for $n > 3$? We obtained a partial answer for this problem.

Main result 5

For $0 < a < 1$, let M be a hyperelliptic curve of genus 4 defined by

$$w^2 = z\left(z^4 + a^4\right)\left(z^4 + \frac{1}{a^4}\right).$$

Suppose that f is given by

$$\begin{aligned} f \ : M &\longrightarrow \mathbf{R}^4 \\ p &\longmapsto \mathrm{Re}\int_{p_0}^{p} \left(1 - z^3,\ i\,(1 + z^3),\ z^2 + z,\ i\,(z^2 - z)\right)^T \frac{dz}{w} \end{aligned}$$

By suitable deformations of M or f, there exist countably infinite non-holomorphic minimal surfaces in 4-dimensional flat tori.

Moreover, there exist $0 < a_1 < a_2 < a_3 < a_4 < 1$ satisfying the following properties:

(i) $index = 5$ and $nullity = 4$ for $a \in (0,\ a_1) \cup (a_4,\ 1)$,
(ii) $index = 3$ and $nullity = 4$ for $a \in (a_1,\ a_2) \cup (a_3,\ a_4)$,
(iii) $index = 2$ and $nullity = 4$ for $a \in (a_2,\ a_3)$.

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A structural approach towards the classification of isoparametric hypersurfaces in spheres

Anna Siffert

The problem of classifying isoparametric hypersurfaces in spheres is still not completely solved. Recent contributions to the case with six different principal curvatures seem to contain fundamental gaps. In my paper I develop a structural approach towards the classification of isoparametric hypersurfaces in spheres.

The basic idea for the structural approach is the following: instead of working with the family of parallel surfaces $F_t : M^n \to Sph^{n+1}$, with normal field $\nu_t \in \Gamma(\nu M^n)$, one considers the associated Lagrangian submanifold

$$\iota : M^n \to Q^n \subset \mathbb{CP}^{n+1}$$

of the complex quadric Q^n. The t-independent map ι is the composition of the map $\widehat{F}_t = F_t + i\,\nu_t$, which is a horizontal immersion into the Stiefel manifold $St_2(\mathbb{R}^{n+2}) \subset \mathbb{C}^{n+2}$, and the standard projection $\pi : St_2(\mathbb{R}^{n+2}) \to Q^n$. The key fact is that all the relevant geometric invariants of $F_t(M^n)$, i.e. the metric g_t, the Weingarten map A_t, and its covariant derivative ${}^t\nabla A_t$, translate into natural geometric invariants $(g, \alpha, B \otimes B^{-1})$ of the Lagrangian submanifold $\iota(M^n) \subset Q^n$ which are now independent of t. The symmetric tensor is defined via

$$\alpha(X, Y, Z) = g_t(({}^t\nabla_X A_t)Y, Z),$$

where A_t denotes the shape operator of F_t with respect to ν_t. The tensor α is one of the fundamental invariants used in the previous classification approaches though usually encoded in some much less invariant Maurer-Cartan forms Λ_t. The really interesting fact is that α coincides - up to a factor of two - with the second fundamental form of the Lagrangian submanifold. This yields a new, geometric interpretation for this important tensor. In order to define $B \otimes B^{-1}$ we introduce the operator

$$B_t = (A_t + i\mathbb{1})(A_t - i\mathbb{1})^{-1}.$$

Here, the crucial observation is that the expression $B_t \otimes B_t^{-1}$ is independent of t. The invariant $B \otimes B^{-1} := B_t \otimes B_t^{-1}$ encodes the Bochner tensor of Q^n and thus also has a natural geometric interpretation.

The classical Weyl identities depend on several indices. In terms of the invariants described above, these multiple identities unify into one tensor identity. This, in particular, makes it feasible to consider higher derivatives of the Weyl identity, which is not possible in the classical approaches. Moreover, the classical identities expressing the reflections through each of the focal manifolds, sometimes also referred to as global identities, translate into the fact that the pullback of α under certain symmetries coincides with the negative of α. So far, all these considerations are completely general; they work for any number g of distinct principal curvatures.

For the special case where the number g of distinct principal curvatures equals six, the goal consists in the proof of the homogeneity of the isoparametric hypersurfaces. I deduce several easy tensor identities which are equivalent to this homogeneity. In particular, I show that the integrability of $D_j \oplus D_{j+3}$ is a necessary and sufficient condition. Furthermore, I prove that this homogeneity is equivalent to the statement that certain sectional curvatures of the Lagrangian submanifold vanish. This is a new, geometric interpretation of the goal that provides an interesting link to some global metric properties of the quadric $Q^n \subset \mathbb{CP}^{n+1}$.

Furthermore, I develop a more efficient calculus for studying the linear isospectral families associated to the second fundamental form of the focal submanifolds. With this calculus I manage to classify the linear isospectral families in the case $g = 6$ and all multiplicities equal to one, completely. Thereby I reprove the classification theorem by Dorfmeister and Neher in a somewhat different way. It turns out thereof that the linear isospectral family encodes just some of the classical Weyl identities and some higher derivatives of them. Thus we expect that it will be crucial to evaluate the higher derivatives of the Weyl identities in a more efficient way.

Naturally, all the classical approaches to the classification problem can be translated into this more structural picture. Thereby the proofs become easier and allow more geometric insights.

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Reporter: Lynn Heller

Participants

Prof. Dr. Uwe Abresch
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Ben Andrews
CMA
School of Mathematical Sciences
Australian National University
Canberra ACT 0200
AUSTRALIA

Prof. Dr. Christoph Bohle
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Francis E. Burstall
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Dr. Laura Desideri
Laboratoire Paul Painleve, UMR 8524
Université des Sciences et
Technologie de Lille 1
Cite Scientifique, Bat. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Josef F. Dorfmeister
Zentrum Mathematik M8
TU München
Boltzmannstr. 3
85747 Garching
GERMANY

Prof. Dr. Karsten Grosse-Brauckmann
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

Dr. Mark Haskins
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Dr. Lynn Heller
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Sebastian Heller
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Udo Hertrich-Jeromin
Institut f. Diskrete Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
1040 Wien
AUSTRIA

Atsufumi Honda
Miyakonojo National College of
Technology
473-1 Yoshio-cho, Miyakonojo-shi
Miyazaki 885-8567
JAPAN

Prof. Dr. Wilhelm H. Klingenberg
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Hui Ma
Department of Mathematical Sciences
Tsinghua University
Beijing 100 084
CHINA

Prof. Dr. Reiko Miyaoka
Mathematical Institute
Faculty of Science
Tohoku University
6-3 Aoba
Sendai 980-8578
JAPAN

Prof. Dr. Andre Neves
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Yoshihiro Ohnita
Department of Mathematics
Osaka City University
Advance Mathematical Institute
(OCAMI)
Sugimoto, Sumiyoshi-ku
Osaka 558-8585
JAPAN

Prof. Dr. Franz Pedit
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Joaquin Perez
Dept. of Geometry and Topology
University of Granada
Avenida Fuentenueva s/n
18071 Granada
SPAIN

Prof. Dr. Ulrich Pinkall
Institut für Mathematik
Technische Universität Berlin
Sekr. MA 3-2
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Wayne Rossman
Department of Mathematics
Faculty of Science
Kobe University
1-1, Rokkodai, Nada-ku
Kobe 657-8501
JAPAN

Prof. Dr. Martin Ulrich Schmidt
Lehrstuhl für Mathematik III
Fak. für Mathematik und Informatik
Universität Mannheim
Seminargebäude A 5
68159 Mannheim
GERMANY

Prof. Dr. Toshihiro Shoda
Faculty of Culture and Education
Saga University
1 Honjo-cho, Saga-City
Saga 840-8502
JAPAN

Dr. Anna Siffert
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Masaaki Umehara
Dept. of Mathematical & Information Science
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8550
JAPAN

Prof. Dr. Kotaro Yamada
Department of Mathematics
Tokyo Institute of Technology
2-12-1 Oh-okayama,
Meguro-ku
Tokyo 152-8551
JAPAN