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Algebraic K-theory and Motivic Cohomology

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Abstract. Algebraic K-theory and motivic cohomology are strongly related tools providing a systematic way of producing invariants for algebraic or geometric structures. The definition and methods are taken from algebraic topology, but there have been particularly fruitful applications to problems of algebraic geometry, number theory or quadratic forms. 19 one-hour talks presented a wide range of latest results on the theory and its applications.

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Introduction by the Organisers

Algebraic K-theory and motivic cohomology are both tools providing a systematic way of producing invariants for algebraic or geometric structures. Their definition and many methods are taken from algebraic topology, but they have found particularly fruitful applications for problems of algebraic geometry, number theory, quadratic forms, or group theory. Motivic cohomology and algebraic K-theory are closely related by a spectral sequence, but have different special features.

The workshop program presented a varied series of lectures on the latest developments in the field. The 51 participants came mostly from Europe and North America, but there were also participants from Japan, Korea, and South America. The participants ranged from leading experts in the field to younger researchers and also some graduate students. 19 one-hour talks presented a wide range of latest results on the theory and its applications, reflecting a good mix of nationalities and age groups as well.

We now want to describe in more detail the topics which were touched.

Computations in K-theory and $\mathbb{A}^1$-homotopy theory. Cortiñas applied the Borel regulator to obtain results on the assembly map for groups, and Vishik described unstable operations on algebraic cobordism and, more generally, so-called theories of rational type. Asok gave a description of the unstable $\mathbb{A}^1$-homotopy groups $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ and a conjectural description of $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ for $n \geq 4$.

Categorical constructions for K-theory and motives. Tamme introduced a new theory, differential algebraic K-theory, which combines spectra of manifolds with algebraic K-theories and is related to analytic index theorems as conjectured by Lott as well as constructions of explicit elements in algebraic K-theory. Hesselholt introduced a new theory, called real algebraic K-theory, which is a theory for exact categories with duality, which also works if the prime 2 is not invertible in the coefficients. Ivorra extended some work of Kahn and Yamazaki on reprocity functors, which are certain Nisnevich sheaves with transfers, which are not necessarily $\mathbb{A}^1$-invariant, and defined an analogue of the Somekawa K-group for them. Bondarko discussed how to extend the definition of Kahn and Sujatha of a birational category of motives over a field to rather general base schemes.

Categories of mixed motives, and motivic cohomology. Kelly outlined how the results of Gabber on alterations (refining the previous results by de Jong) allow to prove all results on Voevodsky motives over a field k without using resolution, as long as one inverts the exponential characteristic p in the coefficients. Spitzweck defined a theory of categories of mixed motives for schemes of finite type over Dedekind rings (of possibly mixed characteristic), satisfying the six functor formalism.

$\mathbb{A}^1$-homotopy theory. Wendt discussed the failure of homotopy invariance and even weak homotopy invariance for the homology of algebraic groups. For any scheme X of finite type over a subfield k of $\mathbb{C}$, Drew constructed Hodge realizations on the Morel-Voevodsky stable homotopy category $SH(X)$, with values in M. Saito's category of mixed Hodge modules, which is compatible with Grothendieck's six functors in this setting. Weibel settled a conjecture of Voevodsky concerning the slices of $KGL^{\wedge n}$ in the stable homotopy category over rather general bases.

Chow groups and algebraic cycles. Pirutka proved that the (suitably defined) integral Tate conjecture holds for cubic fourfolds over a finite field of characteristic at least 5. For simple CM abelian varieties A over $\overline{\mathbb{Q}}$, Sugiyama discussed the relationship between the validity of the Hodge conjecture for A and the Tate conjecture on the reduction A_0 of A at a non-archimedean place w of $\overline{\mathbb{Q}}$. Zhong studied the torsion of Chow groups of complete flag varieties for linear groups over a field k and in particular a bound on the exponent. For a prime p, Totaro studied the mod-p-Chow ring $CH^*(BG)/p$ of an affine group scheme over $\mathbb{C}$ as a model case for mod p Chow groups in general.

Arithmetic. Lichtenbaum proposed a new conjecture on the special values of the zeta functions of schemes of finite type over $\mathbb{Z}$. Schmidt defined a reciprocity map from Weil étale Suslin homology of an arbitrary variety over a finite field to the abelianized tame fundamental group and showed that it is an isomorphism after completion provided resolution of singularities holds over the field.

Foundations. Voevodsky, cofounder of the theory of motivic cohomology and $\mathbb{A}^1$-homotopy theory, has more recently proposed a new logical foundation for mathematics and a formal language supposed to enable computer-based proof checks. Grayson gave an introduction into this theory, which is intuitively linked to the language of homotopy theory and was the topic of a special year at the Institute for Advanced Study at Princeton culminating in a 600 page book on this theory.

Workshop: Algebraic K-theory and Motivic Cohomology

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Abstracts

On the failure of weak homotopy invariance

Matthias Wendt
(joint work with Kevin Hutchinson)

Homotopy invariance of algebraic K-theory can be stated as group homology isomorphisms $H_\bullet(GL_\infty(R),\mathbb{Z}) \cong H_\bullet(GL_\infty(R[T]),\mathbb{Z})$ for R an arbitrary regular ring. Examples of Krstić-McCool for SL_2 and Wendt for SL_3 have shown that such isomorphisms typically do not exist for algebraic groups unless R is a field. These examples motivate the following weaker version of homotopy invariance: let k be a field and $k[\Delta^\bullet]$ be the standard simplicial k-algebra with n-simplices $k[\Delta^n] = k[X_0,\dots,X_n]/(\sum X_i - 1)$. For an algebraic group G over k, consider the simplicial group $G(k[\Delta^\bullet])$ and define the *group homology of G made $\mathbb{A}^1$-invariant* as $H_\bullet(BG(k[\Delta^\bullet]),\mathbb{Z})$. There is a natural change-of-topology morphism $G(k) \to G(k[\Delta^\bullet])$. We say that *group homology of G has weak homotopy invariance for the field k* if the change-of-topology morphism induces an isomorphism

$$H_\bullet(G(k),\mathbb{Z}) \to H_\bullet(BG(k[\Delta^\bullet]),\mathbb{Z}).$$

If true, weak homotopy invariance would allow to use $\mathbb{A}^1$-homotopy theory to prove theorems or do computations in group homology. However, we have the following negative results for the third homology of SL_2, cf. [HW13]:

(1) For k a number field and ℓ an odd prime, the kernel of the change-of-topology morphism

$$H_3(SL_2(k),\mathbb{Z}/\ell) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}/\ell)$$

is not finitely generated.

(2) For k a field complete with respect to a discrete valuation, with residue field $\overline{k}$ (assume for simplicity algebraically closed), the change of topology morphism

$$H_3(SL_2(k),\mathbb{Z}[1/2]) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}[1/2])$$

factors through $K_3^{\mathrm{ind}}(k)\otimes\mathbb{Z}[1/2]$, and its kernel surjects onto the pre-Bloch group $\mathcal{P}(\overline{k})\otimes\mathbb{Z}[1/2]$.

The proof of these results (and others of a similar nature) combines two ingredients: on the one hand, the group homology of SL_2 made $\mathbb{A}^1$-invariant can be studied via $\mathbb{A}^1$-homotopy theory. On the other hand, a series of recent papers by Kevin Hutchinson [Hut11a, Hut11b] provides a lot of knowledge about $H_3(SL_2(k),\mathbb{Z}[1/2])$, its relation with refined pre-Bloch groups as well as the existence of residue maps for pre-Bloch groups.

The basis of assertion (1) is a size comparison: fibre sequences in $\mathbb{A}^1$-homotopy theory can be used to establish stabilization results for $H_\bullet(BSp_{2n}(k[\Delta^\bullet]),\mathbb{Z})$ which imply that $H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}[1/2]) \cong H_3(Sp_\infty(k),\mathbb{Z}[1/2])$ is a finitely generated $\mathbb{Z}[1/2]$-module for k a number field. On the other hand, the residue maps on refined Bloch groups defined by Hutchinson imply that for k a number field there is a morphism

$$H_3(SL_2(k),\mathbb{Z}[1/2]) \to \bigoplus_{\mathfrak{p}\subseteq\mathcal{O}_k \text{ prime}} \mathcal{P}(\mathcal{O}_k/\mathfrak{p})\otimes\mathbb{Z}[1/2]$$

with finite cokernel, hence $H_3(SL_2(k),\mathbb{Z}[1/2])$ is not finitely generated. The assertion (1) follows using the fact that $\#\mathcal{P}(\mathbb{F}_q)=q+1$ and Chebotarev density.

Assertion (2) follows from a comparison of $\mathbb{Z}[k^\times/(k^\times)^2]$-module structures on both sides. The natural $k^\times$-action by conjugation descends to the square class ring on both sides. Moreover, on the $\mathbb{A}^1$-homotopy side, it descends even further to a module structure under the Grothendieck-Witt ring $GW(k)$. Therefore, the change-of-topology map factors as

$$H_3(SL_2(k),\mathbb{Z}) \to H_3(SL_2(k),\mathbb{Z})\otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$$

In the situation of (2) and taking $\mathbb{Z}[1/2]$-coefficients, the group in the middle can be identified with $H_3(SL_2(k),\mathbb{Z}[1/2])\otimes_{\mathbb{Z}[k^\times/(k^\times)^2]}\mathbb{Z}\cong K_3^{\mathrm{ind}}(k)\otimes\mathbb{Z}[1/2]$ and the claim follows again from residue map computations of Hutchinson.

In particular, our results imply that even the weak homotopy invariance with finite coefficients can fail for fields which are not algebraically closed. However, it should be emphasized that the methods do not apply to k algebraically closed.

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Differential algebraic *K*-theory

Georg Tamme

(joint work with Ulrich Bunke)

1. Motivation

Let R be the ring of integers in a number field and denote by Σ the set of embeddings $R\hookrightarrow\mathbb{C}$. If V is a locally constant sheaf of finitely generated projective R-modules on a manifold M, called *bundle* for short, we get, for each $\sigma\in\Sigma$, a flat $\mathbb{C}$-vector bundle $V_\sigma\to M$. By a *geometry* g^V on V we mean a collection of hermitian metrics on the V_σ which is compatible with the action of complex conjugation. The choice of a geometry g^V on V allows one to construct a characteristic form $\omega(V,g^V)$ given by the Kamber-Tondeur forms of the flat bundles V_σ with hermitian metrics. These are closed odd differential forms which represent the Borel regulator class of V.

Lott [5] defined a secondary K-group $\overline{KR}^0(M)$ in terms of generators and relations: A generator is a triple $\hat{V} := (V, g^V, \eta)$ where V is a bundle as above, g^V is a geometry on V, and η is an even differential form (modulo exact forms), such that $d\eta = \omega(V, g^V)$. There is a relation $\hat{V}_0 - \hat{V}_1 + \hat{V}_2 = 0$ if there is an exact sequence of the underlying bundles $\underline{V} : 0 \to V_0 \to V_1 \to V_2 \to 0$ such that $\eta_0 - \eta_1 + \eta_2 \equiv \mathcal{T}(\underline{V}, g^{\underline{V}})$ modulo exact forms, where $g^{\underline{V}} = (g^{V_0}, g^{V_1}, g^{V_2})$ and $\mathcal{T}(\underline{V}, g^{\underline{V}})$ is a version of the Bismut-Lott higher analytic torsion form associated to the sequence $\underline{V}$ with geometries $g^{V_0}, g^{V_1}, g^{V_2}$ [1]. Lott proves that the functor $\overline{KR}^0$ is homotopy invariant. However, it is not part of a generalized cohomology theory.

Given a proper submersion of manifolds $\pi\colon W \to B$ with metric and connection, Lott defines an analytic push-forward $\pi_!\colon \overline{KR}^0(W) \to \overline{KR}^0(B)$. He shows that it is independent of the additional geometric choices, and hence of topological nature.

Denote by $KR\mathbb{R}/\mathbb{Z}^*$ the cohomology theory defined by the $\mathbb{R}/\mathbb{Z}$-version of the algebraic K-theory spectrum KR of R. Lott conjectures the following secondary index theorem:

Conjecture (Lott)**.** *There is a natural transformation $\overline{KR}^0 \to KR\mathbb{R}/\mathbb{Z}^{-1}$ under which the analytic push-forward $\pi_!$ corresponds to the Becker-Gottlieb-Dold transfer on the right hand side.*

The main result of this report is the proof of the first part of this conjecture concerning the existence of such a natural transformation.

2. Differential algebraic K-theory for number rings

A bundle V on the manifold M defines a class $[V] \in KR^0(M)$ in the cohomology theory defined by the spectrum KR. The main feature of differential algebraic K-theory $\widehat{KR}^0(M)$ is that a class $\hat{x} \in \widehat{KR}^0(M)$ combines the information about an underlying K-theory class, denoted $I(\hat{x}) \in KR^0(M)$, and an odd differential form, denoted $R(\hat{x})$, representing the Borel regulator class of $I(\hat{x})$, with secondary data. The precise construction is due to Bunke-Gepner [2] (see also Section 3 below).

There is a cycle map which associates to a bundle V with geometry g^V a class $\mathrm{cycl}(V, g^V) \in \widehat{KR}^0(M)$ such that $I(\mathrm{cycl}(V, g^V)) = [V]$ and $R(\mathrm{cycl}(V, g^V)) = \omega(V, g^V)$. By construction, there is also a map a from even differential forms (modulo exact forms) to $\widehat{KR}^0$ such that $R \circ a$ is the exterior differential d of forms and $\mathrm{im}(a) = \ker(I)$. Moreover, the so-called flat part $\widehat{KR}^0_{flat}(M) = \{\hat{x} \in \widehat{KR}^0(M) \mid R(\hat{x}) = 0\}$ turns out to be naturally isomorphic to $KR\mathbb{R}/\mathbb{Z}^{-1}(M)$.

Theorem. *If $\underline{V} : 0 \to V_0 \to V_1 \to V_2 \to 0$ is an exact sequence of locally constant sheaves of finitely generated projective R-modules on the manifold M, and $g^{\underline{V}} = (g^{V_0}, g^{V_1}, g^{V_2})$ is a collection of geometries on the V_i's, then the relation*

$$\mathrm{cycl}(V_0, g^{V_0}) - \mathrm{cycl}(V_1, g^{V_1}) + \mathrm{cycl}(V_2, g^{V_2}) = a(\mathcal{T}(\underline{V}, g^{\underline{V}}))$$

holds true in $\widehat{KR}^0(M)$.

The theorem implies that sending a generator (V, g^V, η) for Lott's secondary K-group to the class $\mathrm{cycl}(V, g^V) - a(\eta) \in \widehat{KR}^0(M)$ gives a well defined map $\overline{KR}^0(M) \to \widehat{KR}^0_{flat}(M) \cong KR\mathbb{R}/\mathbb{Z}^{-1}(M)$ and hence settles the first part of Lott's conjecture.

The proof of the theorem uses in an essential way the extension of differential algebraic K-theory from number rings to higher dimensional schemes developed in [3].

3. The case of higher dimensional schemes

In this generalized setting, differential algebraic K-theory is a functor of two variables, a smooth manifold M and a regular separated scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$, denoted by $M \times X \mapsto \widehat{K}^0(M \times X)$. The idea is that the Borel regulator in the number ring case should now be replaced by Beilinson's regulator. To this end, we introduce a complex $\mathrm{DR}(M \times X)$ built from smooth differential forms on the manifold $M \times X(\mathbb{C})$ which computes the cohomology of M with coefficients in the absolute Hodge cohomology of X.

In this setting, a *bundle* is locally free sheaf of finitely generated $\mathrm{pr}_X^{-1}\mathcal{O}_X$-modules on the topological space $M \times X$. Such a bundle V gives rise to a class $[V] \in KX^0(M)$, where KX is the algebraic K-theory spectrum of X. To V we can associate a $\mathbb{C}$-vector bundle $V_{\mathbb{C}} \to M \times X(\mathbb{C})$ which is naturally equipped with a flat partial connection ∇^I in the M-direction and holomorphic structure $\bar{\partial}$ in the X-direction. A *geometry* g^V on V is given by a pair of a hermitian metric and a connection that extends the partial connection $\nabla^I + \bar{\partial}$, such that the metric and the connection locally on M extend to some compactification of X. Using a geometry g^V on V we can construct a characteristic form $\omega(V, g^V) \in \mathrm{DR}(M \times X)$ which represents the Beilinson regulator of $[V]$.

The construction of the complex DR and the characteristic form $\omega(V, g^V)$ is inspired by work of Burgos-Wang [4].

Denote by H the Eilenberg-MacLane functor from chain complexes to spectra. In order to define $\widehat{K}^0(M \times X)$ we construct sheaf of spectra $\mathbf{K}$ on the category of pairs (M, X), where M is a smooth manifold and X a scheme as above, such that $\mathbf{K}(M \times X)$ is a model of the function spectrum KX^M, and a map of sheaves of spectra $r \colon \mathbf{K} \to H(\mathrm{DR})$ which induces Beilinson's regulator on homotopy groups. We then define the presheaf of spectra $\widehat{\mathbf{K}}$ as the homotopy pull-back

$$\begin{array}{ccc} \widehat{\mathbf{K}} & \xrightarrow{R} & H(\sigma^{\geq 0}\,\mathrm{DR}) \\ {\scriptstyle I}\downarrow & & \downarrow \\ \mathbf{K} & \xrightarrow{r} & H(\mathrm{DR}), \end{array}$$

where $\sigma^{\geq 0}$ denotes the stupid truncation in degree 0, and define

$$\widehat{K}^0(M \times X) := \pi_0(\widehat{\mathbf{K}}(M \times X)).$$

There exist maps a, R, and I as in the number ring case and their properties follow formally from the definition via a homotopy pull-back.

There is a cycle map that associates a differential algebraic K-theory class $\mathrm{cycl}(V, g^V) \in \widehat{KR}^0(M \times X)$ to a bundle V on $M \times X$ with geometry g^V. It has similar properties as before.

4. The proof of the Theorem

For the proof of the theorem, we let $X := \mathrm{Spec}(R)$. The extension of bundles of R-modules $\underline{V}$ on M with geometries $g^{\underline{V}}$ as in the Theorem corresponds to an extension of bundles with geometry on $M \times X$ as in Section 3.

We naturally construct a sheaf W with geometry g^W on $M \times X \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}}$ which restricts to $(V_0 \oplus V_2, g^{V_0} \oplus g^{V_2})$ at 0 and to (V_1, g^{V_1}) at ∞. Then

$$\sum_i (-1)^i \,\mathrm{cycl}(V_i, g^{V_i}) = \mathrm{cycl}(W, g^W)\big|_0 - \mathrm{cycl}(W, g^W)\big|_\infty .$$

There is a natural homotopy operator

$$\int \colon \mathrm{DR}(M \times X \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}})^* \to \mathrm{DR}(M \times X)^{*-1}$$

and one shows that

$$\mathrm{cycl}(W, g^W)\big|_0 - \mathrm{cycl}(W, g^W)\big|_\infty = a\left(\int R\left(\mathrm{cycl}(W, g^W)\right)\right) = a\left(\int \omega(W, g^W)\right).$$

To conclude, we observe that the naturality of the construction together with the axiomatic characterization of the Bismut-Lott torsion form imply that $\int \omega(W, g^W) \equiv \mathcal{T}(\underline{V}, g^{\underline{V}})$ modulo exact forms.

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An application of Borel regulators to the K-theory of group rings

Guillermo Cortiñas
(joint work with Gisela Tartaglia)

The talk was about our joint article arXiv:1305.1771. Let G be a group, $\mathcal{F}in$ the family of its finite subgroups, and $\mathcal{E}(G, \mathcal{F}in)$ the classifying space. Let $\mathcal{L}^1$ be the algebra of trace-class operators in an infinite dimensional, separable Hilbert space over the complex numbers. Consider the rational assembly map in homotopy algebraic K-theory

$$H_p^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \to KH_p(\mathcal{L}^1[G]) \otimes \mathbb{Q}.$$

The rational KH-isomorphism conjecture predicts that the map above is an isomorphism; it follows from a theorem of Yu (see arXiv:1106.3796, arXiv:1202.4999) that it is always injective. We prove the following.

Theorem 1. *Assume that the map above is surjective. Let* $n \equiv p + 1 \mod 2$. *Then:*

i) *The rational assembly map for the trivial family*

$$H_n^G(\mathcal{E}(G, \{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} \to K_n(\mathbb{Z}[G]) \otimes \mathbb{Q}$$

is injective.

ii) *For every number field* F, *the rational assembly map*

$$H_n^G(\mathcal{E}(G, \mathcal{F}in), K(F)) \otimes \mathbb{Q} \to K_n(F[G]) \otimes \mathbb{Q}$$

is injective.

We remark that the K-theory Novikov conjecture asserts that part i) of the theorem above holds for all G, and that part ii) is equivalent to the rational injectivity part of the K-theory Farrell-Jones conjecture for number fields.

The idea of the proof of the theorem above is to use an algebraic, equivariant version of Karoubi's multiplicative Chern character, which we introduce in the article. Our character is defined for all $\mathbb{C}$-algebras; in the case of finite dimensional Banach algebras, it agrees with Karoubi's. In particular, by work of Karoubi, the Borel regulator can be recovered from the multiplicative character applied to $\mathbb{C}$. This allows us to relate the assembly map with $\mathcal{L}^1$ coefficients to those with coefficients in number fields and in $\mathbb{Z}$.

Real algebraic *K*-theory

LARS HESSELHOLT
(joint work with Ib Madsen)

We define a 2-functor that to an exact category with duality $(\mathscr{C}, D, \eta)$ assigns a real symmetric spectrum $KR(\mathscr{C}, D, \eta)$ and prove real versions of both the group completion theorem and the additivity theorem. We stress that we do *not* require that 2 be invertible in the sense that $\mathscr{C}$ be enriched in $\mathbb{Z}[1/2]$-modules.

We recall that a (strong) duality structure on a category $\mathscr{C}$ is a pair (D, η) of a functor $D\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ and a natural isomorphism $\eta\colon \mathrm{id}_{\mathscr{C}} \to D \circ D^{\mathrm{op}}$ such that the quadruple $(D^{\mathrm{op}}, D, \eta, \eta^{\mathrm{op}})$ is an adjoint equivalence of categories from $\mathscr{C}$ to $\mathscr{C}^{\mathrm{op}}$. For instance, if A is a ring, and if (L, α) is a pair of a right $A \otimes A$-module $L = L_{12}$ and an $A \otimes A$-linear map $\alpha\colon L_{12} \to L_{21}$ such that (1) the right A-modules L_1 and L_2 are finitely generated and projective; (2) $\alpha \circ \alpha = \mathrm{id}_L$; and (3) the unique map of A-A-bimodules ${}_1A_2 \to {}_1\mathrm{Hom}_A(L_2, L_1)_2$ that to 1 assigns α is an isomorphism, then there is a duality structure on the category $\mathscr{P}(A)$ of (small) finitely generated projective right A-modules with $D(P) = \mathrm{Hom}_A(P, L_1)_2$ and $\eta_P(x)(f) = \alpha(f(x))$, and every duality structure on $\mathscr{P}(A)$ is, up to equivalence, of this form.

A real space is a left G-space and a real map is a G-equivariant map for the group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For example, the one-point compactifications $S^{2,1}$, $S^{1,0}$, and $S^{1,1}$ of $\mathbb{C}$, $\mathbb{R}$, and $i\,\mathbb{R}$ are pointed real spaces. A real symmetric spectrum is a symmetric spectrum $E = \{E_r, \sigma_r\colon E_r \wedge S^{2,1} \to E_{r+1}\}$ in the category of pointed real spaces with respect to the sphere $S^{2,1}$. The category of real symmetric spectra has a model structure defined by Mandell, the associated homotopy category of which is the G-stable homotopy category. The real algebraic K-groups of $(\mathscr{C}, D, \eta)$ are defined to be the ($RO(G)$-equivariant) homotopy groups

$$KR_{p,q}(\mathscr{C}, D, \eta) = [S^{p,q}, KR(\mathscr{C}, D, \eta)]_R$$

given by the abelian groups of morphisms in the homotopy category of real symmetric spectra from $S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q}$ to $KR(\mathscr{C}, D, \eta)$. In particular, the groups $KR_{p,0}(\mathscr{C}, D, \eta)$ are the hermitian K-groups of $(\mathscr{C}, D, \eta)$.

The construction of $KR(\mathscr{C}, D, \eta)$ is based on the real Waldhausen construction

$$(\mathscr{C}, D, \eta) \longmapsto (S^{2,1}\mathscr{C}[-], D[-], \eta[-])$$

that to a pointed exact category with duality associates a simplicial pointed exact category with duality. (By the latter we mean a simplicial pointed exact category $S^{2,1}\mathscr{C}[-]$ together with a duality structure $(D[n], \eta[n])$ on $S^{2,1}\mathscr{C}[n]$ such that

$$\begin{array}{ccc} S^{2,1}\mathscr{C}[n]^{\mathrm{op}} & \xrightarrow{D[n]} & S^{2,1}\mathscr{C}[n] \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \bar{\theta}^*} \\ S^{2,1}\mathscr{C}[m]^{\mathrm{op}} & \xrightarrow{D[m]} & S^{2,1}\mathscr{C}[m] \end{array}$$

commutes for every $\theta\colon [m] \to [n]$. Here $\bar{\theta}\colon [m] \to [n]$ is given by $\bar{\theta}(i) = n - \theta(m-i)$. Note that this is not a simplicial object in the category of pointed exact categories with duality.) By definition, the category

$$S^{2,1}\mathscr{C}[n] \subset \mathsf{Cat}(\mathsf{Cat}([2],[n]),\mathscr{C})$$

is the full subcategory of functors $A\colon \mathsf{Cat}([2],[n]) \to \mathscr{C}$ that satisfy that

(1) for all $\mu\colon [1] \to [n]$,

$$A(s_0\mu) = A(s_1\mu) = 0,$$

a fixed null-object; and

(2) for all $\sigma\colon [3] \to [n]$, the sequence

$$A(d_3\sigma) \xrightarrow{f} A(d_2\sigma) \xrightarrow{g} A(d_1\sigma) \xrightarrow{h} A(d_0\sigma)$$

is 4-term exact in the sense that f is an admissible monomorphism, h an admissible epimorphism, and g induces an isomorphism of a cokernel of f onto a kernel of h;

and $D[n]$ and $\eta[n]$ are defined by $D[n](A)(\theta) = D(A(\bar{\theta})^{\mathrm{op}})$ and $\eta[n]_\theta = \eta$. We note that the construction is similar to the usual Waldhausen construction, which we will write $(S^{1,1}\mathscr{C}[-], D[-], \eta[-])$, substituting $\mathsf{Cat}([2],[n])$ for $\mathsf{Cat}([1],[n])$. We also remark that $S^{2,1}\mathscr{C}[0]$ and $S^{2,1}\mathscr{C}[1]$ consist of a single object and a single morphism, while $S^{2,1}\mathscr{C}[2]$ is isomorphic to $\mathscr{C}$ via the functor that takes A to $A(\mathrm{id}_{[2]})$.

Definition. The real algebraic K-theory spectrum of a pointed exact category with duality $(\mathscr{C}, D, \eta)$ is the real symmetric spectrum with rth space

$$KR(\mathscr{C}, D, \eta)_r = |N(iS^{2r,r}\mathscr{C}[-], D[-], \eta[-])[-]|_R$$

given by the realization of the nerve of the subcategory of isomorphisms in the r-simplicial pointed exact category with duality obtained by applying the real Waldhausen construction r times to $(\mathscr{C}, D, \eta)$ and with the rth structure map

$$KR(\mathscr{C}, D, \eta)_r \wedge S^{2,1} \xrightarrow{\sigma_r} KR(\mathscr{C}, D, \eta)_{r+1}$$

induced by the inclusion of the 2-skeleton in the last $S^{2,1}$-direction.

The involution on $KR(\mathscr{C}, D, \eta)_r$ is obtained from the duality structure in a way that was explained in the lecture. In particular, the subspace of G-fixed points in the 0th space $KR(\mathscr{C}, D, \eta)_0$ is naturally weakly equivalent to the realization of the nerve of the category $\mathrm{Sym}(i\mathscr{C}, D, \eta)$ of symmetric spaces in $(i\mathscr{C}, D, \eta)$. It has objects pairs (c, b) of an object c and an isomorphism $b\colon c \to D(c^{\mathrm{op}})$ in $\mathscr{C}$ such that b is symmetric in the sense that it is equal to its adjoint $D(b^{\mathrm{op}}) \circ \eta_c$ and has morphisms $g\colon (c, b) \to (c', b')$ the isomorphisms $g\colon c \to c'$ such that

$$\begin{array}{ccc} c & \xrightarrow{b} & D(c^{\mathrm{op}}) \\ \downarrow{\scriptstyle g} & & \uparrow{\scriptstyle D(g^{\mathrm{op}})} \\ c' & \xrightarrow{b'} & D(c'^{\mathrm{op}}) \end{array}$$

commutes.

Theorem (Real group completion theorem). *Let $(\mathscr{C}, D, \eta)$ be a pointed exact category with duality and suppose that the exact category $\mathscr{C}$ is split-exact. Then for every $H \subset G$ and positive integer r, the canonical ring homomorphism*

$$H_*(KR(\mathscr{C}, D, \eta)_0^H)[\pi_0(KR(\mathscr{C}, D, \eta)_0^H)^{-1}] \to H_*((\Omega^{2r,r} KR(\mathscr{C}, D, \eta)_r)^H)$$

is an isomorphism.

The following statement was identified by Schlichting as the appropriate version of the additivity theorem for real algebraic K-theory.

Theorem (Real additivity theorem). *Let $(\mathscr{C}, D, \eta)$ be a pointed exact category with duality. Then the duality-preserving functor*

$$(S^{1,1}\mathscr{C}[3], D[3], \eta[3]) \longrightarrow (\mathscr{C} \times \mathscr{C} \times \mathscr{C}, \gamma_{13} \circ (D \times D \times D), \eta \times \eta \times \eta)$$

that to $A\colon \mathsf{Cat}([1],[3]) \to \mathscr{C}$ associates $(A(01), A(12), A(23))$ induces a level weak equivalence of real algebraic K-theory spectra.

The proof proceeds by exhibiting (four) explicit (real) simplicial homotopies, and is modelled on McCarthy's proof of the additivity theorem for Waldhausen algebraic K-theory. As an important corollary, we show that the real algebraic K-theory spectrum is positively fibrant in the following sense.

Corollary. *Let $(\mathscr{C}, D, \eta)$ be a pointed exact category with duality. Then for every positive integer r, the adjoint structure map*

$$KR(\mathscr{C}, D, \eta)_r \xrightarrow{\tilde{\sigma}_r} \Omega^{2,1} KR(\mathscr{C}, D, \eta)_{r+1}$$

is a real weak equivalence.

ldh descent for Voevodsky motives

Shane Kelly

The assumption that resolution of singularities is true litters Voevodsky's work on motives. While it has been proven to hold over a characteristic zero field, in positive characteristic resolution of singularities remains one of the most important open problems in algebraic geometry.

In 1996 de Jong published a theorem which can be used to replace some resolution of singularities arguments if we are willing to work with rational coefficients. More recently, Gabber has a theorem in the same spirit which provides an alternative if we forego knowledge of torsion equal to the characteristic.

A weak version of this theorem of Gabber is the following.

Theorem 1 (Gabber [1, Theorem 3]). *Let X be a separated scheme of finite type over a perfect field k and l a prime distinct from the characteristic of k. There exists a smooth quasi-projective k scheme X', and a k-morphism $X' \to X$ that is proper, surjective, such that every generic point of X' maps to a generic point of X, and such that the field extensions at generic points are finite of degree prime to l.*

In this talk we outline how this theorem of Gabber can be used to remove the assumption of resolution of singularities, if we work with $\mathbb{Z}[1/p]$-coefficients (where p is the exponential characteristic of the base field).

To apply resolution of singularities, Voevodsky and Suslin introduced the cdh topology. We describe in this talk a slightly finer topology which allows one to apply the above theorem of Gabber.

Definition 1. *The* ldh topology *on the category of separated schemes of finite type over a noetherian base scheme is the coarsest topology such that cdh covering families are covering families, and so are singletons $\{f : Y \to X\}$ containing a finite flat surjective morphism of constant degree prime to l.*

The difficult applications of the resolution of singularities hypothesis in [2] are all via the following theorem. Recall that $\underline{C}_\bullet(F)(-)$ is the complex of presheaves $F(\Delta^\bullet \times -)$, where $\Delta^\bullet$ is the canonical cosimplicial scheme which has $\Delta^n = Spec(k[t_0, \dots, t_n]/(\sum_{i=0}^n t_i = 1))$.

Theorem 2 ([2, Chapter 5, Theorem 4.1.2])**.** *Suppose k is a perfect field that satisfies resolution of singularities, and let F be a presheaf with transfers on the category of separated schemes of finite type over k. If the cdh associated sheaf F_{cdh} is zero, then the complex $F(\Delta^\bullet \times -)$ on the full subcategory of smooth schemes is exact as a complex of Nisnevich sheaves.*

The most important theorem presented in this talk is the following.

Theorem 3 ([3, Theorem 5.3.1])**.** *Suppose k is a perfect field of exponential characteristic p, and let F be a presheaf of $\mathbb{Z}_{(l)}$-modules with transfers on the category of separated schemes of finite type over k (with $l \neq p$). If the ldh associated sheaf F_{ldh} is zero, then the complex $F(\Delta^\bullet \times -)$ on the full subcategory of smooth schemes is exact as a complex of Nisnevich sheaves.*

A consequence is the following.

Corollary 1. *All the results in* [2] *remain true without the hypothesis of resolution of singularities, if $\mathbb{Z}[1/p]$-coefficients are used.*

Via a formal adjunction argument, Theorem 3 follows once one has shown that in the Morel-Voevodsky stable homotopy category $SH(k)$, any module over the object $H\mathbb{Z}_{(l)}$ that represents motivic cohomology with $\mathbb{Z}_{(l)}$-coefficients satisfies descent for the ldh topology.

The first main technical theorem used to show this descent statement is the following.

Theorem 4 ([3, Theorem 3.8.1])**.** *Suppose that F is a Nisnevich sheaf of $\mathbb{Z}_{(l)}$-modules on the category of separated schemes of finite type over a perfect field k of exponential characteristic $p \neq l$. We suppose further that*

(1) *F is unramified in the sense of Morel, i.e., for every open immersion of smooth schemes $U \to X$ the morphism $F(X) \to F(U)$ is*
   (a) *injective if U contains all the points of codimension zero of X, and*

(b) *an isomorphism if U contains all the points of codimension ≤ 1 of X,*

(2) *F has a "structure of traces", i.e., attached to every finite flat surjective morphism $f : Y \to X$ there is a morphism $Tr_f : F(Y) \to F(X)$, and these morphisms satisfy appropriate additivity, functoriality, base-change, and degree-formula axioms (see* [3, Definition 3.3.1] *for a precise definition),*

(3) *$F(X) \to F(X_{red})$ is an isomorphism for every scheme X.*

Then for every $n \geq 0$ the canonical morphism $H^n_{cdh}(-, F_{cdh}) \to H^n_{ldh}(-, F_{ldh})$ is an isomorphism of presheaves, and this presheaf has a canonical structure of presheaf with transfers.

This theorem is applied to show that the cdh and ldh descent spectral sequences associated to an object of $SH(k)$ are isomorphic, if the object has a structure of traces (for a suitable notion of a structure of traces, see [3, Definition 4.3.1]). The second technical theorem is the following.

Theorem 5 ([3, Corollary 5.2.4])**.** *Suppose that k is a perfect field of exponential characteristic $p \neq l$. For any object $E \in SH(k)$ the object $H\mathbb{Z}_{(l)} \wedge E$ has a canonical structure of traces.*

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Unstable operations in small theories

Alexander Vishik

In Algebraic Geometry there are two types of theories: "large" ones represented by some spectra in the $\mathbb{A}^1$-homotopy category, and indexed by two numbers $A^{j,i}$, and "small" ones typically given by *pure parts* $A^{2*,*}$ of "large" theories. The latter theories are called *oriented cohomology theories* and can be defined axiomatically (see [6, Definition 2.1]) by the standard axioms of D.Quillen - see [2, Definition 1.1.2] plus the *localisation* axiom where one requires *right exactness* only. The good thing about them is that the set of such theories is stable under change of coefficients. Examples of "small" theories are CH^*, K_0, and the Algebraic Cobordism of M.Levine-F.Morel Ω^* which is the *universal* such theory.

The construction of cohomological operations in such theories, and especially, in Algebraic Cobordism was an important open problem for some time. The case of *stable operations* can be dealt with due to *universality result* of M.Levine-Morel ([2, Theorem 1.2.6]) and *reorientation procedure* of I.Panin-A.Smirnov ([4, 3]) , which produces Landweber-Novikov operations. But the *unstable* case was completely open, as aside from couple of isolated examples nothing was known.

Our new approach permits to describe/construct all additive (unstable) operations $A^n \to B^m$, as long as the source theory is "good".

We say that a theory A^* is *constant* if it satisfies:

$$(CONST) \qquad A = A^*(\mathrm{Spec}(k)) \stackrel{\cong}{\to} A^*(\mathrm{Spec}(L)), \quad \text{for any f.g. field ext. } L/k.$$

This implies that one has canonical splitting $A^*(X) = A \oplus \overline{A}^*(X)$, where the second summand consists of classes having positive codimension of support.

We say that a theory A^* is *of rational type*, if it is *constant* and the natural sequence:

$$\bigoplus_{W \to X \times \mathbb{P}^1} A_{*+1}(W) \stackrel{i_0^* - i_1^*}{\to} \lim_{V \to X} A_*(V) \to \overline{A}_*(X) \to 0$$

is exact, where the first sum is taken over all projective maps with smooth W of dimension $\leq \dim(X)$, and such that the preimages $W_0 \hookrightarrow W$ and $W_1 \hookrightarrow W$ of 0 and 1 are divisors with strict normal crossing, and the second limit is taken over the category of projective maps with smooth V of dimension $< \dim(X)$. This way, a theory is defined inductively on the dimension of X.

It appears that such theories are exactly ones obtained from Algebraic Cobordism of M.Levine-F.Morel by change of coefficients:

Proposition ([6, Proposition 4.8])
A^* is of rational type $\Leftrightarrow$ $A^* = \Omega^* \otimes_{\mathbb{L}} A$.

In particular, Ω^*, CH^*, K_0 are of rational type.

Our main result is the following:

Main Theorem. ([6, Theorem 5.1])
Let A^* be a theory of rational type, and B^* - any theory. There is 1-to-1 correspondence between additive (unstable) operations $A^n \to B^m$ and transformations: $A^n((\mathbb{P}^\infty)^{\times r}) \to B^m((\mathbb{P}^\infty)^{\times r})$, $r \in \mathbb{Z}_{\geq 0}$ commuting with the pull-backs for:

- (*i*) *the action of $\mathfrak{S}_r$;*
- (*ii*) *the partial diagonals;*
- (*iii*) *the partial Segre embeddings;*
- (*iv*) $(\mathrm{Spec}(k) \hookrightarrow \mathbb{P}^\infty) \times (\mathbb{P}^\infty)^{\times s}$, $\forall s$.

In Topology, an analogous result was obtained by T.Kashiwabara in [1]. Our methods though are quite different, as we are working not with spectra, but with theories themselves (using induction on the dimension of X).

This result permits to reduce the study of additive operations from a theory of rational type to "Algebra". As applications we get the following results:

Theorem 1 ([6, Theorem 6.1])
Additive (unstable) operations $\Omega^n \to \Omega^m$ are exactly those $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$-linear combinations of the Landweber-Novikov operations which take integral values on all

$(\mathbb{P}^\infty)^{\times r}$.

We also show that *stable* ones among them are exactly $\mathbb{L}$-linear combinations, but this result does not require the above technique.

Theorem 2 ([6, Theorem 6.8])
Let A^* be a theory of rational type, and B^* - any theory. Then multiplicative operations $A^* \to B^*$ are in 1-to-1 correspondence with morphisms of formal group laws $(A, F_A) \to (B, F_B)$.

As an application, we significantly extend the result of Panin-Smirnov-Levine-Morel:

Theorem 3 ([6, Theorem 6.9])
Let B^* be any theory, and $b_0 \in B$ be not a zero divisor. Let $\gamma = b_0x + b_1x^2 + b_2x^3 + \ldots \in B[[x]]$. Then there exists a multiplicative operation $\Omega^* \stackrel{G}{\to} B^*$ with $\gamma_G = \gamma$ if and only if the shifted FGL $F_B^\gamma \in B[b_0^{-1}][[x,y]]$ has coefficients in B (no denominators). In such a case, the operation is unique.

In the original result, b_0 was invertible (which restricted applications to stable operations only). As an important example of situation with non-invertible b_0, we can now construct T.tom Dieck - style Steenrod operations in Algebraic Cobordism - see [6, Theorem 6.17]:

$$\Omega^* \stackrel{Sq}{\to} \Omega^*[[t]]/\left(\frac{p \cdot_\Omega t}{t}\right).$$

Another important application of Theorem 2 is the construction of Integral (!) Adams Operations for all theories of rational type:

Theorem 4 ([6, Theorem 6.15])
For any $A^* = \Omega^* \otimes_{\mathbb{L}} A$, and any $k \in \mathbb{Z}$, there exists unique unstable (for $k \neq 1$) multiplicative A-linear operation $\Psi_k : A^* \to A^*$ with $\gamma_{\Psi_k} = k \cdot_A x$ ("formal" product). In the case of K_0 these are classical Adams operations.

In Topology, such operations were constructed by W.Wilson in [7, Theorem 11.53].

Finally, we can produce *Symmetric Operations* for all primes p - see [6, Theorem 6.18], which was the main motivation behind the current work. These operations have applications to questions of rationality of cycles - see [5].

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A new conjecture on special values of scheme zeta-functions

Stephen Lichtenbaum

Let X be a regular scheme, projective, connected and flat over Spec $\mathbb{Z}$, of Krull dimension d. A point x of $|X|$ is closed if and only if the residue field $\kappa(x)$ is finite, in which case we define the norm $N(x)$ of x to be the cardinality of $\kappa(x)$. Let $s \in \mathbb{C}$. The scheme zeta-function of X $\zeta(X,s)$ is defined to be $\prod_x (1 - N(x)^{-s})^{-1}$. This product is convergent for $Res > d$, and defines an analytic function of s in this range. A well-known conjecture asserts that $\zeta(X,s)$ can be analytically continued to a meromorphic function of s in the entire complex plane, and we wil be tacitly assuming the truth of this conjecture when we talk about special values of $\zeta(X,s)$ at integers $\leq d$.

We are interested in the behavior of $\zeta(X,s)$ at integral points n, so we would like to give formulas for the order $a(X,n)$ of the zero of $\zeta(X,s)$ at $s = n$, and the leading term $\zeta^*(X,n)$ defined to be $lim_{s \to n}(\zeta(X,s)(s-n)^{-a(X,n)}$.

There is a beautiful conjecture of Soulé which asserts that for any connected X of finite type over Spec $\mathbb{Z}$ of Krull dimension d, $a(X,n) = \sum_{i=0}(-1)^i dim$ $(K'_i(X)^{(d-n)} \otimes \mathbb{Q})$. Here $K_i(X)^{(r)}$ denotes the subgroup of $K_i(X)$ consisting of elements of Adams weight r, and is well-defined up to torsion. Since this talk will be focused on motivic cohomology, we will replace $K'_i(X)^{(r)}$ by its conjectured equivalent motivic cohomology group $H_M^{2r-i}(X, \mathbb{Z}(r))$. Since after tensoring with $\mathbb{Q}$ the Zariski and étale motivic cohomology groups are the same, at this point we may use either one, although for the leading term, étale cohomology is necessary. For us, motivic cohomology will mean the hypercohomology of the sheafification of Bloch's higher Chow groups complex [1]

We will concentrate for the rest of this talk on formula for the leading term $\zeta^*(X,n)$ which we hope are valid up to sign and powers of 2. Previously, conjectured special-value formulas have been given by Bloch and Kato ([2]) for suffiiently positive integers and Fontaine and Perrin-Riou ([3]) for all integers. fontaine asserts in [3] that these two conjectures agree when the Bloch-Kato one is defined, but does not give a complete proof.

In any case, both of these conjectures involve the Hasse-Weil L-functions $L_i(X,s)$ of X, rather than the scheme zeta-function. The $L_i(X,s)$ only depend on the generic fiber $X_{\mathbb{Q}}$ of X, and are related to $\zeta(X,s)$ by $\zeta(X,s) = \prod_i (L'_i(X,s)^{(-1)^i}$, where $L_i'(X,s) = L_i(X,s)$ if X is smooth over Spec $\mathbb{Z}$. In general, the two L-functions differ only by products of polynomials in p^{-s}, where p is the characteristic of a prime where X has bad reduction. Whether our conjecture is compatible with the previous conjectures is in general a non-trivial question. We do not give any conjectures involving $L_i'(X,s)$, as it is not clear how to distribute the torsion terms in the formula for $\zeta^*(X,s)$

Our conjectured formula involves the orders of finite groups and also determinants (regulators). We begin with the finite groups:

Let $|N|$ denote the order of the finite group N, and let $H^i(X,\mathbb{Z}(r))$ denote the étale motivic cohomology of X. (We put $\mathbb{Z}(r) = 0$ if r is negative.)

Let $A(X,r) = \prod_{i=0}^{2r+1}(|H^i(X,\mathbb{Z}(r))_{tor}|)^{(-1)^{i+1}}$.

Remark: the natural conjecture here is that if $i \leq 2r+1$, $H^i(X,\mathbb{Z}(r))$ is finitely generated, so its torsion subgroup is finite.

Let $B(X,r) = \prod_{i=0}^{2d+3-2r}(|H^i(X,\mathbb{Z}(d+1-r)_{tor})|^{(-1)^{i+1}})$.

Let $X_{\mathbb{C}}$ denote the fibered product $X \times_{\mathbb{Z}} \mathbb{C}$, with its complex analytic structure.

Let $C(X,r) = \prod_{i+0}^{2d}|(H_B^i)^+(X_{\mathbb{C}},\mathbb{Z}(r))_{tor}|^{(-1)^i}$, where H_B denotes the usual singular cohomology of $X_{\mathbb{C}}$. If r is even, H_B^+ denotes the subgroup of H_B left fixed by the automorphism σ of $X_{\mathbb{C}}$ induced by complex conjugation on $\mathbb{C}$. If r is odd, H_B^+ denotes the subgroup of elements x in H_B such that $\sigma(x) = -x$.

$$\begin{aligned} \text{Let } d(X,i,j,k) &= |H^j(\mathbb{Z}, R^k\pi_*\underline{\lambda}^i(\Omega_X))_{tor}|, \\ \chi(X,i) &= \prod_{k=0}^{d}\prod_{j=0}^{d} d(X,i,j,k)^{(-1)^{(j+k)}}. \end{aligned}$$

Let $D(X,r) = \prod_{i=0}^{r}\chi(X,i)^{(-1)^i}$.

Let $E_1(X,r) = A(X,r)B(X,r)C(X,r)D(x,r)$. Then $E_1(X,r)$ represents the torsion contribution to $\zeta^*(X,r)$.

We now consider the regulator terms:

We first recall that if (*) =

$$0 \to V_0 \to V_1 \cdots \to V_n \to 0$$

is an exact sequence of complex vector spaces, and we give ourselves bases B_i for V_i, we can define the determinant of (*) with respect to the B_i, generalizing the usual definition of the determinant for $n = 1$. If each of the V_i is given to us as $(M_i)_{\mathbb{C}} = M_i \otimes_{\mathbb{Z}} \mathbb{C}$, where M_i is a finitely generated abelian group, and we take bases of the V_i coming from generators for the M_i modulo torsion, then, up to sign, the determinant of (*) is independent of the choice of generators.

Now let M be the motive $H^i(X,r)$, and consider the six-term, conjecturally exact sequence of complex vector spaces which Fontaine associates with M:

$$0 \to H_f^0(M)_{\mathbb{C}} \to Ker(\alpha_M) \to H_c^1(M)_{\mathbb{C}} \to H_f^1(M)_{\mathbb{C}} \to Coker(\alpha_M) \to H_c^2(M)_{\mathbb{C}} \to 0$$

wnere α_M is the map from $H_B(M)^+$ to $H_{DR}(M)/F_0$ induced by the period map. We give our interepretations of integral bases for the groups in this exact sequence, somewhat modified from Fontaine:

$H^i_B(M)(r)$ is the vector space $H^I_B(M)_{\mathbb{C}}$, with a basis given by a basis for $H_B(M)$ multiplied by $(2\pi i)^r/\Gamma^*(r)$. Here $\Gamma^*(r)$ is the leading term of $\Gamma(s)$ at $s = r$, and is equal to $(r-1)!$ if $r \geq 1$ and $((-r)!)^{(-1)}$ if $r \leq 0$.

$H^i(X_{\mathbb{C}}, \Omega^r)$ has a basis given by $H^i(X, \lambda^I\Omega_{X/\mathbb{Z}})$, where λ^i denotes derived exterior power.

$H^1_f(M) = H^{i+1}_M(X, \mathbb{Z}(r))$, if $i \leq 2r-2$ and equals the subgroup $(H^{2r}_{et}(X, \mathbb{Z}(r))_0$ of $H^{2r}_{et}(X, \mathbb{Z}(r))$ consisting of étale cycles homologically equivalent to zero, if $i = 2r-1$. If $i \geq 2r-2$, let $H^1_f(M) = Hom(KerH^{2d-i}_{et}(X, \mathbb{Z}(d-r), \mathbb{Z})$

$M^*(1) = H^{2d-i-2}(X, d-r)$, and $H^i_c(M) = H^{(2-i)}_f(M^*(1))$. Note that this is the definition of $M^*(1)$ given in Flach ([4]), while Fontaine ([3]) gives $M^*(1) = H^i(X, i+1-r)$. Flach's definition seems better.

If $i = 2r$, $H^0_f(M) = H^{2r}_{et}(X, \mathbb{Z}(r)/\sim$, where $\sim$ denotes homological equivalence. Otherwise $H^0_f(M) = 0$.

For r fixed let $Z_i(X, r)$ be the determinant of the six-term sequence corresponding to the motive $H^i(X, r)$, with the indicated integral structures., and let $Z(X, r)$ be the alternating product $\prod_i Z_i(X, r)^{(-1)^i}$. Then the final conjecture is

$$\zeta^*(X, r) = E(X, r)Z(X, r)$$

up to sign and powers of 2.

Warning: this conjecture should be taken with many grains of salt. I have not carefully checked it for misprints and obvious errors, nor calculated it in enough different kinds of examples to be thoroughly convinced. It should be possible, making standard assumptions, to check its compatibility with the standard conjectural functional equation of Serre, but I have not done that. I welcome any comments and corrections.

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The Tate conjecture for integral classes on cubic fourfolds

Alena Pirutka
(joint work with F.Charles)

Let $\mathbb{F}$ be a finite field and let X be a smooth and projective variety over $\mathbb{F}$. Denote $\bar{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$ and $G = Gal(\bar{\mathbb{F}}/\mathbb{F})$. The Tate conjecture [10] predicts that the cycle class map

$$CH^i(\bar{X}) \otimes \mathbb{Q}_\ell \to \bigcup_U H^{2i}_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell(i))^U,$$

where the union is over all open subgroups U of G, $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$ and $\ell \neq char(\mathbb{F})$, is surjective. By a restriction-corestriction argument, this statement is also equivalent to the surjectivity of the map

$$CH^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell(i))^G.$$

In the integral version one is interested in the cokernel of the following map

$$CH^i(\bar{X}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^{2i}_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell(i))^U, \tag{1}$$

and, as a stronger version, in the cokernel of the map

$$CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell(i))^G \tag{2}$$

or of the map

$$CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}_{\acute{e}t}(X, \mathbb{Z}_\ell(i)). \tag{3}$$

The map (1) is not surjective in general: the counterexamples of Atiyah-Hirzebruch [1], revisited by Totaro [11], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs a torsion class α in $H^4(\bar{X}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety X constructed as a quotient of a smooth complete intersection in $\mathbb{P}^n$ by a free action of a finite group. To establish that α is not algeraic, one uses Steenrod operations.

In the case of curve classes, i.e. for $i = \dim(X) - 1$, Schoen established in [9] that the map (1) is surjective if the Tate conjecture holds for divisors on surfaces.

The cokernel of the map (3) for codimension 2 cycles has been also expressed in terms of the third unramified cohomology group in recent works [5, 4]. More precisely, if

$$M = Coker[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))]$$

then the torsion subgroup M_{tors} of M is isomorphic to the quotient of the group $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup. Using this description, Parimala and Suresh [8] establish that the map (3) is surjective for codimension 2 cycles on threefolds X fibred in conics over a geometrically ruled surface S over $\mathbb{F}$. For quadric fibrations over a surface S over $\mathbb{F}$, the map (3) is not surjective in general: one can construct examples of non-algebraic non-torsion classes for $i = 2$ in the case when the general fibre is a quadric of dimension 3 and $S = \mathbb{P}^2_{\mathbb{F}}$

(see [7, 6]). However, the case when the general fibre in a quadric of dimension 2 remains open.

In a joint work with F.Charles [2], we establish that the first version of the integral Tate conjecture holds for codimension 2 cycles on cubic fourfolds X over $\mathbb{F}$, if the $char\mathbb{F}$ is at least 5 : for such X, the map (1) is surjective for $i = 2$. The goal of this talk is to explain our approach, which is also inspired by the work of Claire Voisin [12] where she establishes the integral Hodge conjecture for cubic fourfolds over $\mathbb{C}$.

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Hodge realizations of triangulated mixed motives

Brad Drew

Fix a subfield $\mathbf{k} \subseteq \mathbf{C}$ and a finite type $\mathbf{k}$-scheme X. In [2], J. Ayoub constructs a Betti realization functor from $\mathrm{SH}(X)$, the Morel-Voevodsky stable homotopy category of X, to the derived category of analytic sheaves of abelian groups on $X(\mathbf{C})$. This realization functor is compatible with Grothendieck's six functors f^*, f_*, $f_!$, $f^!$, $\otimes$ and $\underline{\mathrm{hom}}$. We describe an analogue of M. Saito's derived category of mixed Hodge modules on X that receives a "Hodge realization" functor from $\mathrm{SH}(X)$ compatible with Grothendieck's six functors and Ayoub's Betti realization. We also construct a de Rham realization functor whose target is the derived category of holonomic $\mathcal{D}_X$-modules.

Notation.
- For simplicity, fix a quasi-projective $\mathbf{k}$-scheme X and let $\mathrm{Sm}\downarrow X$ denote the category of smooth, quasi-projective X-schemes.
- Let MHS denote the category of polarizable $\mathbf{Q}$-mixed Hodge structures and $\mathcal{M} := \mathrm{Cplx}(\mathrm{Ind}(\mathrm{MHS}))$ that of unbounded cochain complexes of ind-objects of MHS.
- Let $\mathbf{y}_{\mathcal{M},X} : \mathrm{Sm}\downarrow X \to \mathrm{Psh}(\mathrm{Sm}\downarrow X, \mathcal{M})$ denote the $\mathcal{M}$-enriched Yoneda embedding, which takes Y to the presheaf given by $W \mapsto \bigoplus_{\mathrm{mor}_X(W,Y)} \mathbf{1}_{\mathcal{M}}$, where $\mathbf{1}_{\mathcal{M}}$ is the unit of the symmetric monoidal category $\mathcal{M}$.
- Let $\sigma_1 : X \to \mathbf{G}_{\mathrm{m},X}$ be the unit section, $\mathbf{1}_X(1) := \mathrm{coker}(\mathbf{y}_{\mathcal{M},X}(\sigma_1))[-1]$ the Tate object and $\mathrm{Sp}^{\mathfrak{S}}(X, \mathcal{M})$ the category of symmetric $\mathbf{1}_X(1)$-spectra in $\mathrm{PSh}(\mathrm{Sm}\downarrow X, \mathcal{M})$.

Theorem 1. *There exists a combinatorial, stable, left proper, symmetric monoidal model structure satisfying the monoid axiom on* $\mathrm{Sp}^{\mathfrak{S}}(X, \mathcal{M})$ *whose homotopy category* $\mathrm{SH}_{\mathcal{M}}(X)$ *is the* $\mathcal{M}$*-enriched stable homotopy category of* [1, 4.5.24].

Specifically, $\mathrm{SH}_{\mathcal{M}}(X)$ is the category obtained by localizing $\mathrm{PSh}(\mathrm{Sm}\downarrow X, \mathcal{M})$ with respect to quasi-isomorphisms, forcing Nisnevich descent and $\mathbf{A}^1$-homotopy invariance, and inverting the Tate object with respect to the tensor product.

Developing ideas of [1] and [3], one obtains a six functor formalism for $\mathrm{SH}_{\mathcal{M}}(-)$. In order to study this functoriality, it is useful to observe that the "trivial $\pi_1(\mathrm{MHS})$-representation" functor $\alpha^* : \mathrm{Mod}(\mathbf{Q}) \to \mathrm{Ind}(\mathrm{MHS})$ and any fiber functor $\omega : \mathrm{MHS} \to \mathrm{Mod}(\mathbf{Q})$ induce Quillen adjunctions

$$\mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q}) \underset{\alpha_*}{\overset{\alpha^*}{\rightleftarrows}} \mathrm{Sp}^{\mathfrak{S}}(X, \mathcal{M}) \underset{\omega_*}{\overset{\omega^*}{\rightleftarrows}} \mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q}),$$

where $\mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q})$ is obtained by replacing $\mathcal{M}$ by $\mathrm{Cplx}(\mathrm{Mod}(\mathbf{Q}))$ in the construction of $\mathrm{Sp}^{\mathfrak{S}}(X, \mathcal{M})$. In particular, $\mathrm{SH}_{\mathbf{Q}}(X) := \mathrm{ho}(\mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q}))$ is the $\mathbf{Q}$-localization of $\mathrm{SH}(X)$. Furthermore, the derived functor $\mathbf{L}\omega^*$ is conservative and $\mathbf{L}\alpha^*$ and $\mathbf{L}\omega^*$ are compatible with the six functors.

Theorem 2. *Assume* $\mathbf{k} = \mathbf{C}$.

(1) *There exists a commutative algebra* $E \in \mathrm{CAlg}(\mathrm{PSh}(\mathrm{Sm}\downarrow\mathbf{C}, \mathcal{M}))$ *such that the* r*th cohomology object* $\mathfrak{h}^r E(X)$ *is isomorphic to the Betti cohomology* $\mathrm{H}^r_{\mathrm{Betti}}(X, \mathbf{Q})$ *equipped with the* $\mathbf{Q}$*-mixed Hodge structure of Deligne* [4] *for all* $r \in \mathbf{Z}$*,* $X \in \mathrm{Sm}\downarrow\mathbf{C}$.

(2) *There exists* $\mathcal{E} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathfrak{S}}(\mathbf{C}, \mathcal{M}))$ *such that*

$$\mathbf{R}\mathrm{hom}^{\mathcal{M}}(\mathbf{y}_{\mathcal{M},\mathbf{C}}(X), \mathcal{E}(r)) \cong E(X) \otimes \mathbf{1}(r)$$

for all $r \in \mathbf{Z}$*,* $X \in \mathrm{Sm}\downarrow\mathbf{C}$*, where* $\mathbf{R}\mathrm{hom}^{\mathcal{M}}$ *denotes the* $\mathcal{M}$*-enriched derived mapping space and* $\mathbf{1}(r)$ *the* r*th Tate twist in* MHS.

The proof of the first assertion relies on the theory of $(\infty, 1)$-categories as developed in [6, 7] and in particular on the rectification results contained therein.

If we define $\mathcal{E}_X := \mathbf{L}\pi^*\mathcal{E} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathfrak{S}}(X, \mathcal{M}))$ for all $\pi\colon X \to \mathrm{Spec}(\mathbf{C})$, then the category $\mathrm{D}(\mathcal{E}_X) := \mathrm{ho}(\mathrm{Mod}(\mathcal{E}_X))$ is a variant of M. Saito's derived category of mixed Hodge modules. Indeed, $\mathrm{D}(\mathcal{E}_{\mathrm{Spec}(\mathbf{C})}) \cong \mathrm{D}(\mathrm{Ind}(\mathrm{MHS}))$ and $\hom_{\mathrm{D}(\mathcal{E}_X)}(\mathcal{E}_X, \mathcal{E}_X(r)[s])$ is the absolute Hodge cohomology of X.

The six functor formalism on $\mathrm{SH}_{\mathcal{M}}(-)$ induces another relating the categories $\mathrm{D}(\mathcal{E}_X)$ and one technical advantage of the latter category vis-à-vis M. Saito's category is the possibility of defining these six functors using Quillen adjunctions rather than via constructions at the level of triangulated categories. Moreover, the desired Hodge realization functor is quite natural in this setting: it suffices to consider the canonical functor $(-)\otimes^{\mathbf{L}}\mathcal{E}_X : \mathrm{SH}_{\mathbf{Q}}(X) \to \mathrm{D}(\mathcal{E}_X)$, which is compatible with Grothendieck's six functors.

One can recover the data "of geometric origin" in $\mathrm{D}(\mathcal{E}_X)$ from $\mathrm{SH}(X)$ as follows. Let $\mathcal{E}_{\mathrm{abs}} := \mathbf{R}\alpha_*\mathcal{E}$ and $\mathrm{D}(\mathcal{E}_{\mathrm{abs},X}) := \mathrm{ho}(\mathrm{Mod}(\mathcal{E}_{\mathrm{abs},X}))$. The canonical functor $(-) \otimes^{\mathbf{L}} \mathcal{E}_{\mathrm{abs},X} \colon \mathrm{SH}_{\mathbf{Q}}(X) \to \mathrm{D}(\mathcal{E}_{\mathrm{abs},X})$ factors through the category of Beĭlinson motives $\mathrm{DM}_{\mathcal{B}}(X)$ defined in [3, 14.2.1]. The arguments of [5] apply *mutatis mutandis* to $\mathrm{D}(\mathcal{E}_{\mathrm{abs},X})$ to give a weight structure compatible with that of *loc. cit.* on $\mathrm{DM}_{\mathcal{B}}(X)$.

If $\mathcal{T}$ denotes a triangulated category with small coproducts, let $\mathcal{T}_{\mathrm{c}}$ denote the full subcategory of $\aleph_0$-compact objects. Also, we let $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ denote the derived category of complexes of $\mathcal{D}_X$-modules with bounded holonomic cohomology.

Theorem 3. *For any nonsingular, quasi-projective* $\mathbf{C}$*-scheme* X*, there exists a canonical symmetric monoidal, triangulated functor*

$$\varrho^*_{\mathrm{dR},X} \colon \mathrm{SH}_{\mathrm{c}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$$

that commutes with f^* *for* $f : X \to Y$ *any morphism of two such schemes,* f_* *for* f *projective and* $f_!$ *for* f *smooth. This functor* $\varrho^*_{\mathrm{dR},X}$ *induces a fully faithful symmetric monoidal, triangulated functor*

$$\chi^*_{\mathrm{dR},X} \colon \mathrm{D}_{\mathrm{c}}(\mathcal{E}_{\mathrm{dR},X}) \hookrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$$

compatible with f^*, f_* *for* f *projective and* $f_!$ *for* f *smooth, where the commutative ring spectrum* $\mathcal{E}_{\mathrm{dR},X} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q}))$ *represents algebraic de Rham cohomology.*

The proof of this theorem relies heavily on the theory of $(\infty,1)$-categories and the universal property of the $(\infty,1)$-categorical stable homotopy category [8, Corollary 1.2]. Also, note that $\varrho^*_{\mathrm{dR},X}$ and $\chi^*_{\mathrm{dR},X}$ are symmetric monoidal with respect to a monoidal structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ Verdier dual to the usual one.

If $\mathcal{T}(X)$ is a family of symmetric monoidal, triangulated categories indexed by nonsingular quasi-projective $\mathbf{C}$-schemes endowed with a six functor formalism, then, for each such X, $\mathcal{T}_{\mathrm{gm}}(X)$ denotes the full subcategory of objects of geometric origin, i.e. the thick triangulated subcategory of $\mathcal{T}(X)$ generated by the objects $f_*\mathbf{1}_Y$ for all $f : Y \to X$ projective, where $\mathbf{1}_Y$ is the monoidal unit object of $\mathcal{T}(Y)$.

Let $\mathcal{E}_{\mathrm{Betti},X} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathfrak{S}}(X, \mathbf{Q}))$ represent Betti cohomology with rational coefficients. The period isomorphism induces an equivalence of commutative ring spectra $\mathcal{E}_{\mathrm{Betti},X} \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathcal{E}_{\mathrm{dR},X}$ and the equivalence $\mathrm{D}_{\mathrm{gm}}(\mathcal{E}_{\mathrm{Betti},X} \otimes_{\mathbf{Q}} \mathbf{C}) \cong \mathrm{D}_{\mathrm{gm}}(X(\mathbf{C}), \mathbf{C})$ of [3, 17.2.22] therefore implies the following.

Corollary (Riemann-Hilbert). *Let X be a nonsingular quasi-projective $\mathbf{C}$-scheme. There exists an equivalence of symmetric monoidal, triangulated categories*

$$\mathrm{D}_{\mathrm{gm}}(X(\mathbf{C}), \mathbf{C}) \cong \mathrm{D}_{\mathrm{gm}}(\mathcal{D}_X)$$

compatible with f^*, f_* for f projective and $f_!$ for f smooth.

In fact, the theory of modules over $\mathcal{E}_{\mathrm{dR},X}$ makes sense for singular $\mathbf{C}$-schemes and this Riemann-Hilbert correspondence extends immediately to the singular case if we define $\mathrm{D}_{\mathrm{c}}(\mathcal{E}_{\mathrm{dR},X})$ to be the derived category of $\mathcal{D}_X$-modules of geometric origin on the possibly singular $\mathbf{C}$-scheme X.

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A motivic Eilenberg-MacLane spectrum in mixed characteristic

Markus Spitzweck

In our talk we presented a family of motivic spectra over base schemes which can be viewed as a good candidate for the motivic Eilenberg-MacLane spectra.

We first give some overview over the story so far.

- If k is a field and X a smooth scheme over k, then
$$\mathrm{Hom}_{\mathrm{DM(k)}}([X], \mathbb{Z}(i)[j]) \cong \mathrm{Hom}_{\mathrm{SH(k)}}(\Sigma^\infty_T X_+, \Sigma^{j,i}\mathsf{M}\mathbb{Z})$$
$$\cong \mathrm{CH}^i(X, 2i-j),$$
where the last isomorphism is a result of Voevodsky [1].
- In general Voevodsky defined a motivic Eilenberg-MacLane spectrum over any base scheme, but its properties are so far not very well known (there are results of Cisinski-Deglise for its rationalization).
- If k has characteristic 0 there is an equivalence
$$\mathrm{Ho}(\mathsf{M}\mathbb{Z}\text{−Mod}) \simeq \mathrm{DM}(k),$$
see [2].

- For general base schemes X Cisinski-Deglise defined a category of so-called Beilinson-motives $\mathrm{DM}_B(X)$ satisfying the six functor formalism (by work of Ayoub [3]), see [5].

We outlined the contruction of a motivic spectrum $\mathsf{M}\mathbb{Z}_S$ over the spectrum S of a given Dedekind domain of mixed characteristic which enjoys the following properties:

(1) If X is a smooth scheme over S, then

$$\mathrm{Hom}(\Sigma_T^\infty X_+, \Sigma^{j,i}\mathsf{M}\mathbb{Z}_S) = H^j_{\mathcal{M}}(X, \mathbb{Z}(i)),$$

where the latter group denotes Levine's motivic cohomology defined using Bloch's cycle complexes.

(2) $\mathsf{M}\mathbb{Z}_S$ has an E_∞-structure (which then can be strictified to a strictly commutative symmetric motivic ring spectrum by a theorem of Hornbostel).

(3) If $p \in S$ is a point, $f\colon \mathrm{Spec}(\kappa(p))$ the canonical morphism, then

$$f^*\mathsf{M}\mathbb{Z}_S \cong \mathsf{M}\mathbb{Z}_{\kappa(p)},$$

where the latter spectrum denotes the usual motivic Eilenberg-MacLane spectrum over $\kappa(p)$.

(4) If f is as above and p is closed, then

$$f^!\mathsf{M}\mathbb{Z}_S \cong \mathsf{M}\mathbb{Z}_{\kappa(p)}(-1)[-2].$$

As application we can define motivic triangulated categories with integral coefficients over any base scheme which have many expected properties. To do so let X be a Noetherian seperated finite dimensional scheme and $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. Let

$$\mathsf{M}\mathbb{Z}_X := f^*\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$$

and

$$\mathrm{DM}(X) := \mathrm{Ho}(\mathsf{M}\mathbb{Z}_X\text{−Mod}).$$

Then again by the work of Ayoub the assignment

$$X \mapsto \mathrm{DM}(X)$$

satisfies the six functor formalism (this is true for any cartesian family of E_∞-spectra).

Let X be a smooth scheme over a Dedekind ring of mixed characteristic and Y be smooth over X. Then we have

$$\mathrm{Hom}_{\mathrm{DM}(X)}([Y], \mathbb{Z}(i)[j]) \cong H^j_{\mathcal{M}}(Y, \mathbb{Z}(i)).$$

Let X be an arbitrary base scheme, $i\colon Z \hookrightarrow X$ a closed inclusion and $j\colon U \subset X$ the open complement. Then we have for $F \in \mathrm{DM}(X)$ an exact triangle

$$j_!j^*F \to F \to i_*i^*F \to j_!j^*F[1].$$

In the talk we gave a sketch of the construction of $\mathsf{M}\mathbb{Z}_S$:

One uses the Bloch-Kato conjecture (now a theorem due to Voevodsky and coworkers) and work of Geisser [4] to define $\mathsf{M}\mathbb{Z}/p^n$ outside characteristic p by truncated étale sheaves and corrects it at characteristic p using logarithmic de Rham-Witt sheaves.

Taking the limit one gets the p-completions $\mathsf{M}\mathbb{Z}^{\wedge p}$.

These then are glued with the Beilinson-spectrum along an arithmetic square.

This is done such that property (3) is satisfied.

By construction property (1) holds with finite coefficients. To prove it with integral coefficients one introduces a second motivic spectrum by a strictification procedure out of the motivic cycle complexes and compares it via étale cycle class maps to $\mathsf{M}\mathbb{Z}_S$.

The proof of property (3) makes use of results of Bloch-Kato [6].

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Lefschetz classes of simple abelian varieties

Rin Sugiyama

We consider a certain property for abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ and for abelian varieties over $\mathbb{F}$. Here $\mathbb{Q}^{\mathrm{alg}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\mathbb{F}$ denotes an algebraic closure of a finite field $\mathbb{F}_p$ with p-elements. In this paper, an abelian variety over $\mathbb{Q}^{\mathrm{alg}}$ means an abelian variety over $\mathbb{C}$ which is defined over a number field. An abelian variety A over $\mathbb{C}$ is said to be *of CM-type* if the reduced degree of the $\mathbb{Q}$-algebra $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is equal to $2 \dim A$. By a result of Serre–Tate [4, Theorem 6], for an abelian variety A of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ and for any prime w of $\mathbb{Q}^{\mathrm{alg}}$ dividing p, one can consider the reduction $A_0/\mathbb{F}$ of A at w. We then discuss a relationship between a certain property for an abelian variety A of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ and for its reduction A_0 at a prime of $\mathbb{Q}^{\mathrm{alg}}$.

For a relationship between the Hodge conjecture and the Tate conjecture for abelian varieties, we know a result of Milne [3, Theorem 1.2]: Let A be an abelian variety of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ and let A_0 be its reduction at a prime of $\mathbb{Q}^{\mathrm{alg}}$. Under an assumption, if the Hodge conjecture holds for all powers of A, then the Tate conjecture holds for all powers of A_0.

Instead of the conjectures, we consider property $(*)$ for A_0 (resp. A) which is in fact a sufficient condition for the Tate conjecture (resp. the Hodge conjecture). Our main result (Theorem 1) gives a kind of answer of the following questions.

Questions: (1) Does property $(*)$ for A imply property $(*)$ for the reduction of A at any prime of $\mathbb{Q}^{\mathrm{alg}}$?

(2) Are there any class of abelian varieties A of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ such that property $(*)$ for its reduction A_0 implies property $(*)$ for A?

To state property $(*)$, we define some cohomology classes:

Definition. (1) Let A_1 be an abelian variety of dimension g over a finite subfield $\mathbb{F}_q$ of $\mathbb{F}$. Let A_0 be the abelian variety $A_1 \otimes_{\mathbb{F}_q} \mathbb{F}$. Let ℓ be a prime number different from p. For each integer i with $0 \leq i \leq g$, we define the space of *the ℓ-adic Tate classes* of degree i on A_0 as follows:

$$\mathcal{T}_\ell^i(A_0) := \varinjlim_{L/\mathbb{F}_q\ :\ \text{finite}} H^{2i}(A_0, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\mathbb{F}/L)}.$$

Let A be an abelian variety of dimension g over $\mathbb{C}$. For each integer i with $0 \leq i \leq g$, we define the space of *the Hodge classes* of degree i on A as follows:

$$H^i_{\mathrm{hodge}}(A) := H^{2i}_{\mathrm{Betti}}(A, \mathbb{Q}) \cap H^i(A, \Omega^i)$$

(2) A Tate (resp. Hodge) class is said to be *algebraic* if it belongs to the image of the cycle class map.

Tate conjecture. All Tate classes are algebraic on A_0.

Hodge conjecture. All Hodge classes are algebraic on A.

Definition. The elements of the $\mathbb{Q}_\ell$-subalgebra (resp. the $\mathbb{Q}$-subalgebra) of

$$\bigoplus_{i=0}^{g} \mathcal{T}_\ell^i(A_0) \qquad \Big(\text{resp. } \bigoplus_{i=0}^{g} H^i_{\mathrm{hodge}}(A)\Big)$$

generated by all Tate (resp. Hodge) classes of degree one are called **the Lefschetz classes** on A_0 (resp. A).

We consider the following property about Lefschetz classes:

Property $(*)$: All Tate classes are Lefschetz on all powers of A_0.
(All Hodge classes are Lefschetz on all powers of A)

By a result of Tate, if property $(*)$ holds for A_0, then the Tate conjecture holds for all powers of A_0. Similarly, by the Lefschetz–Hodge theorem, if property $(*)$ holds for A, then the Hodge conjecture holds for all powers of A. For example, property $(*)$ holds for products of elliptic curves, which is proved by Spiess in case over $\mathbb{F}$, by Tate, Murasaki, Imai and Murty in case over $\mathbb{C}$. However, there are examples of abelian varieties for which property $(*)$ does not hold, but the Tate (or Hodge) conjecture holds ([1], [6], [3, Example 1.8]).

Theorem 1. *Let A be a simple abelian variety of CM-type over $\mathbb{Q}^{\mathrm{alg}}$.*

(1) *Assume that the CM-field $\mathrm{End}^0(A)$ is an abelian extension of $\mathbb{Q}$. If property $(*)$ holds for A, then for any prime w of $\mathbb{Q}^{\mathrm{alg}}$, property $(*)$ holds for a simple factor of the reduction of A at w.*

(2) *Let w be a prime of $\mathbb{Q}^{\mathrm{alg}}$. Let A_0 be the reduction of A at w. Assume that the restriction of w to the reflex field of A is unramified over $\mathbb{Q}$ and its absolute degree is one.*

(a) *If the Hodge conjecture holds for all powers of A, then the Tate conjecture holds for all powers of A_0.*

(b) *Property $(*)$ holds for A if and only if property $(*)$ holds for A_0.*

Let $\mathcal{C}$ be a class of simple abelian varieties A of CM-type over $\mathbb{Q}^{\mathrm{alg}}$ whose CM-field $\mathrm{End}^0(A)$ is an abelian extension of $\mathbb{Q}$. Then by Theorem 1, this class $\mathcal{C}$ gives an answer of the above questions in the following sense: for any A in $\mathcal{C}$, property $(*)$ holds for A **if and only if for any prime** w of $\mathbb{Q}^{\mathrm{alg}}$, property $(*)$ holds for simple factors of the reduction of A at w.

The keys of proof of main result:

(1) To give necessary and sufficient conditions for property $(*)$ for A and for A_0 by "arithmetic words" (see lemmas below);

(2) To compare the conditions for A and for A_0 using a result of Shimura–Taniyama on the prime ideal decomposition of Frobenius endomorphism [5].

Lemma 1 (cf. [2])**.** *Let A be a simple abelian variety with many endomorphisms over $\mathbb{Q}^{\mathrm{alg}}$. Assume that the CM-field $E := \mathrm{End}^0(A)$ is an abelian extension over $\mathbb{Q}$. Let G be the Galois group $\mathrm{Gal}(E/\mathbb{Q})$. Then property $(*)$ holds for A if and only if*

$$\sum_{\sigma \in G} \varphi(\sigma)\chi(\sigma) \neq 0$$

for any character χ of G such that $\chi(\iota) = -1$. Here $\varphi : G \to \{0, 1\}$ be the map defined by the representation of E on the tangent space of A at zero.

Lemma 2 ([7])**.** *Let A_0 be a simple abelian variety over $\mathbb{F}$. Assume that the center C_0 of $\mathrm{End}^0(\mathrm{A}_0)$ is an abelian extension of $\mathbb{Q}$. Let G_0 be the Galois group $\mathrm{Gal}(C_0/\mathbb{Q})$. Let $\mathfrak{p}$ be a prime of C_0 dividing p. Then property $(*)$ holds for A_0 if and only if*

$$\sum_{\sigma \in G_0} \mathrm{ord}_{\mathfrak{p}}(\sigma\pi)\chi(\sigma) \neq 0$$

for any character χ of G_0 such that $\chi(\iota) = -1$. Here $\pi \in C_0$ is the Frobenius endomorpihsm defined over a finite subfield of $\mathbb{F}$.

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Toward a meta-stable range in $\mathbb{A}^1$-homotopy theory of punctured affine spaces

Aravind Asok
(joint work with Jean Fasel)

Suppose k is a perfect field having characteristic unequal to 2. Write $\mathcal{S}_k$ for the category of schemes that are separated, smooth and of finite type over k. Write $\mathcal{H}_\cdot(k)$ for the Morel-Voevodsky unstable pointed $\mathbb{A}^1$-homotopy category [MoVo99]. A (pointed) k-space $\mathcal{X}$ (resp. $(\mathcal{X}, x)$) is a (pointed) simplicial Nisnevich sheaf on $\mathcal{S}_k$. Given two pointed k-spaces $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$, we write $[(\mathcal{X}, x), (\mathcal{Y}, y)]_{\mathbb{A}^1}$ for $\hom_{\mathcal{H}_\cdot(k)}(\mathcal{X}, \mathcal{Y})$. If $(\mathcal{X}, x)$ is a pointed k-space, write $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ for the Nisnevich sheaf associated with the presheaf $U \mapsto [S_s^i \wedge U_+, (\mathcal{X}, x)]_{\mathbb{A}^1}$.

Point $\mathbb{A}^n \setminus 0$ by $(1, 0, \ldots, 0)$, and suppress this base-point from notation. Results of Morel yield a description of $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$, $n \geq 2$, as the sheaf $\mathbf{K}_n^{MW}$ of "unramified Milnor-Witt K-theory." In previous work, the authors provided a description of $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ [AsFa12a, AsFa12b]. The goal of the talk was to provide a conjectural description of $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ for $n \geq 4$. The proposed description is in two parts.

Suslin matrices and the degree map. Schlichting and Tripathi constructed an orthogonal Grassmannian OGr and showed that $\mathbb{Z} \times OGr$ represents Hermitian K-theory in the unstable $\mathbb{A}^1$-homotopy category [ScTr12]. They also establish a geometric form of Bott periodicity in Hermitian K-theory that identifies various loop spaces of $\mathbb{Z} \times OGr$ in terms of other natural spaces; we summarize this result as follows.

Proposition 1. *There are weak equivalences of the form*

$$\Omega_s^1 \Omega_{\mathbb{P}^1}^i(\mathbb{Z} \times OGr) \xrightarrow{\sim} \begin{cases} O & \text{if } i \equiv 0 \mod 4 \\ GL/Sp & \text{if } i \equiv 1 \mod 4 \\ Sp & \text{if } i \equiv 2 \mod 4, \text{ and} \\ GL/O & \text{if } i \equiv 3 \mod 4; \end{cases}$$

Here $O := \operatorname{colim}_n O(q_{2n})$, *where* q_{2n} *is the standard hyperbolic form,* $Sp := \operatorname{colim}_n Sp_{2n}$, $GL/Sp := \operatorname{colim}_n GL_{2n}/Sp_{2n}$, *and* $GL/O := \operatorname{colim}_n GL_{2n}/O(q_{2n})$.

The class of $\langle 1\rangle \in GW(k)$ yields a distinguished element in $GW(k) = [\operatorname{Spec} k_+, \mathbb{Z}\times OGr]_{\mathbb{A}^1}$. An adjunction argument can be used to show that this element corresponds to a distinguished class in $[\mathbb{A}^n \setminus 0, P_n]_{\mathbb{A}^1}$, where P_n is either O, GL/O, Sp or GL/Sp depending on whether n is congruent to 0, 1, 2 or 3 modulo 4.

Let Q_{2n-1} be the smooth affine quadric defined as a hypersurface in $\mathbb{A}^{2n}$ given by the equation $\sum_i x_i x_{n+i} = 1$. There is an $\mathbb{A}^1$-weak equivalence $Q_{2n-1} \to \mathbb{A}^n \setminus 0$ defined by projecting onto the first n variables. Each variety P_n is an ind-algebraic variety, and Suslin inductively defined certain matrices S_n that correspond to morphisms $s_n : Q_{2n-1} \to P_n$ [Su77].

Proposition 2. *The distinguished homotopy classes $[\mathbb{A}^n \setminus 0, P_n]_{\mathbb{A}^1}$ described in the previous paragraph is represented by the morphism $s_n : Q_{2n-1} \to P_n$ given by the matrix S_n.*

The $\mathbb{A}^1$-homotopy sheaves of $O, GL/O, Sp$ and GL/Sp can be identified in terms of the Nisnevich sheafification of Schlichting's higher Grothendieck-Witt groups. Indeed, $\pi_i^{\mathbb{A}^1}(O) \cong \mathbf{GW}^0_{i+1}$, $\pi_i^{\mathbb{A}^1}(GL/O) \cong \mathbf{GW}^1_{i+1}$, $\pi_i^{\mathbb{A}^1}(Sp) \cong \mathbf{GW}^2_0$ and $\pi_i^{\mathbb{A}^1}(GL/Sp) \cong \mathbf{GW}^3_{i+1}$. In general, the sheaves $\mathbf{GW}^j_i$ are viewed as 4 periodic in j. Therefore, the morphism s_n yields, upon applying the functor $\pi_n^{\mathbb{A}^1}(\cdot)$, a morphism

$$s_{n*} : \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}^n_{n+1}.$$

This morphism is not surjective for $n \geq 4$, but it does coincide with a corresponding morphism constructed in the computations of $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$.

Recall the contraction of a sheaf $\mathcal{F}$ is defined by the formula $\mathcal{F}_{-1}(U) := \ker((id\times e)^* : \mathcal{F}(\mathbf{G}_m \times U) \to \mathcal{F}(U))$, where $e : \operatorname{Spec} k \to \mathbf{G}_m$ is the unit section. One defines $\mathcal{F}_{-i}$ inductively as $(\mathcal{F}_{-(i-1)})_{-1}$.

Theorem 3. *The morphism s_{n*} becomes surjective after $(n-3)$-fold contraction and split surjective after n-fold contraction.*

Motivic Hopf maps and the kernel of the degree map. In [AsFa12b], we introduced the geometric Hopf map $\nu : \mathbb{A}^4 \setminus 0 \to \mathbb{P}^{1\wedge 2}$ and showed that it was $\mathbb{P}^1$-stably essential (i.e., is not null $\mathbb{A}^1$-homotopic after repeated $\mathbb{P}^1$-suspension). For any integer $n \geq 2$, set

$$\nu_n := \Sigma^{n-2}_{\mathbb{P}^1}\nu : \mathbb{A}^{n+2} \setminus 0 \longrightarrow \mathbb{P}^{1\wedge n}.$$

Applying $\pi_n^{\mathbb{A}^1}(\cdot)$ to the above morphism yields a map

$$(\nu_n)_* : \mathbf{K}^{MW}_{n+2} \longrightarrow \pi_{n+1}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge n}).$$

For $n \geq 4$, Morel's Freudenthal suspension theorem yields isomorphisms

$$\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \pi_{n+1}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge n}),$$

so in this range, we can view $(\nu_n)_*$ as giving a map $\mathbf{K}^{MW}_{n+2} \to \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$.

For $n = 3$, Morel's Freudenthal suspension theorem only yields an epimorphism. We can refine this result to provide an analog of the beginning of the EHP sequence in $\mathbb{A}^1$-homotopy theory. A particular case of the general result we can establish can be stated as follows.

Theorem 4. *There is an exact sequence of the form*

$$\pi_5^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3}) \xrightarrow{H} \pi_5^{\mathbb{A}^1}(\Sigma_s^1(\mathbb{A}^3 \setminus 0)^{\wedge 2}) \xrightarrow{P} \pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \xrightarrow{E} \pi_4^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3}) \longrightarrow 0.$$

The morphism H in the above exact sequence *conjecturally* admits a description as a variant of the Hopf invariant in Chow-Witt theory. Assuming this, the results we have proven on $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ show that ν_{3*} factors through an explicit quotient of $\mathbf{K}_5^{MW}$. In turn, this (conjectural) computation suggests the following conjecture.

Conjecture 5. *For any integer $n \geq 3$, the morphism ν_{n*} factors through a morphism $\mathbf{K}_{n+2}^{M}/24 \to \pi_{n+1}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge n})$.*

The structure of $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$. We now study the relationship between the two morphisms constructed above. Using an obstruction theory argument, one can demonstrate the following result.

Proposition 6. *For any integer $n \geq 4$, the composite map*

$$\mathbf{K}_{n+2}^{MW} \longrightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}_{n+1}^{n}$$

is zero.

Combining everything discussed so far, one is led to the following conjecture.

Conjecture 7. *For any integer $n \geq 4$, there is an exact sequence of sheaves of the form*

$$\mathbf{K}_{n+2}^{M}/24 \longrightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}_{n+1}^{n}.$$

The sequence becomes short exact after n-fold contraction.

Remark 8. *The conjecture above stabilizes to an unpublished conjecture of F. Morel on the stable motivic π_1 of the motivic sphere spectrum. Using the motivic Adams(-Novikov) spectral sequence, K. Ormsby and P.-A. Østvær have checked that after taking sections over fields having 2-cohomological dimension ≤ 2, the 2-primary part of the stable conjecture is true. Nevertheless, the stable conjecture does not imply the conjecture above (even for large n) because of a lack of a Freudenthal suspension theorem for $\mathbb{P}^1$-suspension. On the other hand, the conjecture above for every n sufficiently large implies the stable conjecture.*

Remark 9. *By the results of* [AsFa12b]*, the above conjecture immediately implies "Murthy's conjecture:" if X is a smooth affine $(d+1)$-fold over an algebraically closed field k, and $\mathcal{E}$ is a rank d vector bundle on X, then $\mathcal{E}$ splits off a free rank 1 summand if and only if $0 = c_d(\mathcal{E}) \in CH^d(X)$. However, the conjecture is much stronger: it gives the complete secondary obstruction to splitting a free rank 1 summand of a vector bundle on a smooth affine scheme.*

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On the torsion of Chow group of twisted complete spin flags

Changlong Zhong
(joint work with Sanghoon Baek, Kirill Zainoulline)

Theorem 1. *Let k be an arbitrary field, and let G be a split simple simply connected linear algebraic group over k. Suppose that G is of Dynkin type B_n or D_n. Let $X = G/B$ be the variety of complete flags of G, and let $\xi \in H^1(k, G)$. Then for $1 \leq d \leq 2n-3$, the torsion of $CH^d({}_\xi X)$ is annihilated by*

$$M_d := (d-1)! \prod_{i=2}^{d} 2^{i+1}[\frac{i}{2}]!(i-1)!.$$

Here ${}_\xi X$ is the twisted variety of X twisted by the 1-cocycle ξ. This theorem, together with all theorems below are due to Baek–Neher–Zainoulline [1] for $2 \leq d \leq 4$, and are due to the author jointly with Baek and Zainoulline [2] for $5 \leq d \leq 2n-3$.
Remark.

(1) Clearly the integer M_d does not depend on the rank of the group G.
(2) Note that $CH^1({}_\xi X) \cong Pic({}_\xi X)$, so there is no torsion at codimension 1. On the other hand, if G is of type A_n or C_n, then there is no torsion part in $CH^d({}_\xi X)$ for any $d \geq 1$.
(4) There are some computations by Karpenko on the order of the torsion of CH^d for $2 \leq d \leq 4$. On the other hand, Karpenko and Merkurjev constructed some example of quadrics whose CH^4 has infinite torsion part.

We sketch the proof as follows.
Step 1. Let Λ be the group of characters of G, and let W be the Weyl group of G. Let $I_a \subset S^*(\Lambda)$ and $I_m \subset \mathbb{Z}[\Lambda]$ be the augmentation ideals, respectively. There exists an isomorphism $\phi_d : \mathbb{Z}[\Lambda]/I_m^{d+1} \to S^*(\Lambda)/I_a^{d+1}$. Let I_m^W (resp. I_a^W) be the ideal of $\mathbb{Z}[\Lambda]$ (resp. $S^*(\Lambda)$) generated by W-invariant elements in I_m (resp. I_a).

Definition 2 ([1]). *The smallest integer τ_d such that*

$$\tau_d \cdot (I_a^W/(I_a^W \cap I_a^{d+1})) \subset \phi_d(I_m^W/(I_m^W \cap I_m^{d+1}))$$

is called the d-th exponent of the W-action.

The existence of τ_d is proved in [1]. Mover, $\tau_1 = 1$ for all G, and $\tau_d = 1$ for all $d \geq 1$ if G is of type A or C. Moreover

Theorem 3. *If G is of type B_n or D_n, and if $2 \leq d \leq 2n-3$, then $\tau_d | 2$.*

Step 2. There exists a commutative diagram

(1)
$$\begin{array}{ccc} I_m^d/I_m^{d+1} & \xrightarrow{(-1)^{d-1}(d-1)!\phi_d} & S^d(\Lambda) \\ {\scriptstyle c_m}\downarrow & & \downarrow{\scriptstyle c_a} \\ \gamma^{d/d+1}(X) & \xrightarrow{c_d} & CH^d(X). \end{array}$$

Here c_a and c_m are the characteristic maps of CH and K_0, respectively, and c_d is the d-th Chern class. $\gamma^{d/d+1}(X)$ is the d-th associated quotient of the γ-filtration. It is known that c_m is surjective with $\ker c_m = I_m^W$, and $I_a^W \subset \ker c_a$.

Theorem 4. *Let G be of type B_n or D_n, and suppose $2 \leq d \leq 2n-3$. Then*

1. *The index of the embedding $(I_a^W)^{(d)} \subset (\ker c_a)^{(d)}$ has an upper bound $(d-1)!\eta_d\tau_d$.*
2. *The torsion part of $\gamma^{d/d+1}({}_\xi X)$ is killed by $(d-1)!\tau_d\eta_d$.*

The proof of part (1) uses the basic polynomial invariants in $S^*(\Lambda)$, and that of part (2) follows from diagram chasing of diagram (1) and the isomorphism $\gamma^{d/d+1}(X) \cong \gamma^{d/d+1}({}_\xi X)$ in [3].
Step 3. Let $\tau^*({}_\xi X)$ be the topological filtration. Via the relationship $\gamma^d({}_\xi X) \subset \tau^d({}_\xi X)$, one obtain an upper bound of the annihilator of the torsion of $\tau^{d/d+1}({}_\xi X)$. Finally, using the Riemann–Roch Theorem without denominators, we obtain the upper bound in Theorem 1.

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Non-commutative localizations and weight structures; applications to birational motives

Mikhail Bondarko
(joint work with Vladimir Sosnilo)

In my talk (based on the recent preprint [BoS13]) I explained that weight structures in (localizations of) triangulated categories are closely related with *non-commutative localizations* of arbitrary additive categories. Localizing an arbitrary triangulated C by a set S of morphisms in the heart Hw of a weight structure w for it one obtains a triangulated category endowed with a weight structure w'. Note here: though the definition and several properties of weight structures are quite similar to those for t-structures, the obvious analogue of this localization statement for t-structures is certainly wrong.

The heart Hw' of the weight structure w' obtained is a certain idempotent completion of the ('non-commutative') localization $Hw[S^{-1}]_{add}$ of Hw by S. Here $Hw[S^{-1}]_{add}$ is the natural categorical version of the Cohn's localization for a ring (see [Coh85]) i.e. the functor $Hw \to Hw[S^{-1}]_{add}$ is universal among all the additive functors that make the elements of S invertible.

In particular, taking $C = K^b(A)$ for an additive A we obtain a very efficient tool for computing $A[S^{-1}]_{add}$. Using it (together with the yoga of *weight decompositions*) we generalize the calculations of [Ger82] and of [Mal82] from the case of categories of finitely generated projective modules over a ring to the one of arbitrary additive categories. Note here: $A[S^{-1}]_{add}$ coincides with the 'abstract' localization $A[S^{-1}]$ (as constructed in [GaC67]) if S contains all A-identities and is closed with respect to direct sums.

The motivating example for our work was the triangulated category of birational motives. We define the latter generalizing the definition given in [KaS02] to the case of a (more or less) arbitrary base scheme U; so, birational equivalences are inverted in (a version of) effective geometric Voevodsky's motives over U. We obtain: there exists a weight structure w_{bir} on the category $DM^o(U)$ obtained. Hw_{bir} is given by retracts of birational motives of (smooth) U-schemes; our results yield certain new formulas for morphism groups in $Hw_{bir} \subset DM^o(U)$. The existence of w_{bir} previously was only known for U being (the spectrum of) a perfect field; even in this case we obtain a new 'elementary' proof of this fact. As shown in previous papers (starting from [Bon10]), the existence of a weight structure yields functorial weight filtrations and weight spectral sequences for any cohomology theory that factorizes through birational motives, and a conservative exact weight complex functor whose target is $K^b(Hw_{bir})$. We also calculate the Grothendieck group of $DM^o(U)$. Lastly, we note that our (new, general) results on localizations of triangulated categories endowed with *adjacent* weight and t-structures yield: restricting the canonical t-structure for the derived category of presheaves with transfers (over an arbitrary U) to the category of *birational motivic complexes* yields the homotopy t-structure for it. We also obtain a description of the heart of the latter t-structure; again, this generalizes the corresponding results of [KaS02] to the case of an arbitrary U.

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Chow rings of finite groups and modules over the Steenrod algebra

Burt Totaro

Morel-Voevodsky and I constructed an algebro-geometric model of the classifying space of a group [4, 6]. For any affine group scheme G of finite type over a field k, we get a ring CH^*BG, the Chow ring of algebraic cycles on the classifying space of G. Each group CH^iBG coincides with CH^i of a certain smooth k-variety (the quotient by G of an open subset of a vector space). The definition is natural, in that the ring CH^*BG is exactly the ring of characteristic classes with values in the Chow ring for principal G-bundles (in the fppf topology) over smooth k-schemes. We focus on the case where G is a finite group, viewed as an algebraic group over k.

If the field k is the complex numbers, then the topological realization of BG is the classifying space of G as a topological group, and so we have a ring homomorphism

$$CH^*BG \to H^*(BG, \mathbf{Z}).$$

This is always an isomorphism after tensoring with the rationals, but it need not be an isomorphism integrally. The fact that not all the cohomology of a finite group is algebraic can be viewed as the source of Atiyah-Hirzebruch's counterexamples to the integral Hodge conjecture for certain quotient varieties [1]. Fix a prime number p, and write $CH^*_G = CH^*(BG)/p$ and $H^*_G = H^*(BG, \mathbf{F}_p)$. The problem of computing CH^*_G is a model case for the mod p Chow groups of smooth varieties more generally.

Some fundamental questions about the Chow ring of a finite group are open. It is not known whether the mod p Chow groups CH^i_G are finite for all i. We do know that the Chow ring CH^*_G is generated by elements of bounded degree, and so finiteness of each group would imply that CH^*_G is a finitely generated $\mathbf{F}_p$-algebra. Also, CH^*_G is generated by transferred Euler classes in many examples, but not much is known about when that happens. (An n-dimensional representation V of a group H has an Euler class $\chi(V) = c_n(V)$ in CH^n_H, and we consider the transfer maps $\mathrm{tr}^G_H : CH^*_H \to CH^*_G$ for all subgroups H of G.) Guillot showed that CH^3_G does not consist of transferred Euler classes for the extraspecial group of order 2^7 (a subgroup of $Spin(7)$), but no such example is known at odd primes [2].

The talk studied Chow rings using Henn-Lannes-Schwartz's ideas about the cohomology of finite groups [3]. Given an unstable module N over the Steenrod algebra, the *topological nilpotence degree* $d_0(N)$ is the supremum of the natural numbers d such that N contains the dth suspension of a nonzero unstable module. The mod p cohomology of any space is an unstable module over the Steenrod algebra, and so in particular H^*_G is an unstable module. Henn-Lannes-Schwartz's basic result is that for any compact Lie group G, $d_0(H^*_G)$ is equal to the smallest number d such that the map

$$H^*_G \to \prod_{V\subset G} H^*_V \otimes_{\mathbf{F}_p} H^{\leq d}_{C_G(V)}$$

is injective. The product runs over the elementary abelian p-subgroups V of G, and the map comes from the group homomorphism $V \times C_G(V) \to G$. Since the cohomology of elementary abelian groups is known, we can say that H^*_G is detected using the cohomology of G and certain subgroups of G in degrees at most $d_0(H^*_G)$.

Henn-Lannes-Schwartz showed that for every prime number p and every finite group G with a faithful complex representation of dimension n, $d_0(H^*_G)$ is at most n^2. We show that in fact $d_0(H^*_G)$ is less than $2n$, as well as more precise bounds [7]. That should be a useful tool for computing group cohomology. Next, for a finite group G, we define the *topological nilpotence degree* of the mod p Chow ring, $d_0(CH^*_G)$, to be the smallest number d such that CH^*_G is detected as above in degrees at most d. We conjecture that this number coincides with the largest number d such that CH^*_G, as a module over the Steenrod algebra, contains the dth suspension of a nonzero unstable module.

Although that conjecture is open, having an upper bound for $d_0(CH^*_G)$ is immediately useful for computing the Chow ring. We prove that for G with a faithful representation of dimension n over k, $d_0(CH^*_G)$ is less than n, as well as more precise bounds. We also prove the analogue for Chow rings of Symonds's theorem on degrees of generators for group cohomology [5]. As a result, we compute the Chow ring completely for several classes of p-groups: the 14 groups of order 16, the 5 groups of order p^3, 12 of the 15 groups of order p^4 for an odd prime p, and several infinite families of p-groups. Also, for all 51 groups of order 32 and all 15 groups of order p^4 with p odd, the Chow ring consists of transferred Euler classes [7].

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K-groups of reciprocity functors

Florian Ivorra
(joint work with Kay Rülling)

In this talk we give a survey of the main results of [3].

In [3] a notion of reciprocity functors is introduced. Early in the 90's, Kahn suggested to use the local symbols of Rosenlicht and Serre [8, 10] for smooth commutative algebraic groups in order to develop a theory which contains algebraic groups and homotopy invariant Nisnevich sheaves with transfers, see e.g. [5]. Our approach is inspired by his idea, and in [3] reciprocity functors are introduced as functors defined over finitely generated field extensions of F and regular curves over them.

Given reciprocity functors $\mathscr{M}_1, \dots, \mathscr{M}_n$, our main construction is a reciprocity functor $\mathrm{T}(\mathscr{M}_1, \dots, \mathscr{M}_n)$ that we call the K-group of $\mathscr{M}_1, \dots, \mathscr{M}_n$, although it is much more than a group, it is a reciprocity functor. This construction is related to the K-group associated by Somekawa [11] with a family of semi-Abelian varieties and its variants introduced in [7, 1, 4].

1. Around the theorem of Nesterenko-Suslin

In [4], Kahn and Yamazaki have constructed an isomorphism

$$(*) \qquad \mathrm{K}(F, \mathscr{F}_1, \dots, \mathscr{F}_n) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}(\mathbb{Z}, \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n)$$

where the $\mathscr{F}_i$'s are homotopy invariant Nisnevich sheaves with transfers, the left hand side is a Somekawa type K-group and the right hand side is the group of morphisms in Voevodsky's category of effective motivic complexes. This result implies in particular the generalizations of the theorem of Nesterenko-Suslin [6] due to Raskind-Spieß [7] and Akhtar [1].

Bloch and Esnault have also proved an additive variant of Nesterenko-Suslin's theorem (see [2]) where the absolute Kähler differentials replace the Milnor K-groups (this result has been generalized by Rülling in [9]).

2. Reciprocity functors

Let F be a perfect field. Let **Reg** be the category of regular F-schemes of dimension $\leqslant 1$, that are separated and of finite type over some finitely generated field extension k/F. A reciprocity functor $\mathscr{M}$ is a Nisnevich sheaf with transfers on **Reg** satisfying certain conditions. The most important of them is the *modulus* condition that may be stated as follows. For all regular projective and connected curves C/k over some finitely generated extension k/F, all non-empty open subsets $U \subseteq C$ and sections $a \in \mathscr{M}(U)$, there exists an effective divisor $\mathfrak{m}$ with support equal to $C \setminus U$ and such that

$$\sum_{P \in U} v_P(f) \mathrm{Tr}_{\kappa(P)/\kappa_C} s_P(a) = 0,$$

where $f \in K^\times$ is any non-zero element in the function field K of C, which is congruent to 1 modulo $\mathfrak{m}$ (i.e. $\operatorname{div}(f-1) \geqslant \mathfrak{m}$). Here v_P is the discrete valuation associated with the closed point $P \in C$, $k_C = H^0(C, \mathscr{O}_C)$, and $s_P : \mathscr{M}(U) \to \mathscr{M}(\kappa(P))$, $\operatorname{Tr}_{\kappa(P)/\kappa_C} : \mathscr{M}(\kappa(P)) \to \mathscr{M}(\kappa_C)$ are the morphisms given by the structure of presheaf with transfers.

As suggested by B. Kahn, examples of reciprocity functors are: (a) smooth commutative algebraic groups over F (this essentially follows from a theorem of M. Rosenlicht [8]); (b) homotopy invariant Nisnevich sheaves with transfers (more precisely, by going to the generic stalks, each homotopy invariant Nisnevich sheaf with transfers $\mathscr{F}$ defines a reciprocity functor $\hat{\mathscr{F}}$); (c) Rost's cycle modules; (d) absolute Kähler differentials.

One can use the same computations as in [10] to show that a reciprocity functor $\mathscr{M} \in \mathbf{RF}$ has local symbols that satisfy a reciprocity law for regular projective curves over finitely generated extensions k/F. More precisely, let C be such a curve (with function field K) and $P \in C$ be a closed point, the local symbol at P is a bilinear map

$$(-,-)_P : \mathscr{M}(K) \times K^\times \to \mathscr{M}(k),$$

which is continuous, when $\mathscr{M}(K)$ and $\mathscr{M}(k)$ are equipped with the discrete topology and $K^\times$ with the $\mathfrak{m}_P$-adic topology. These local symbols provide an increasing and exhaustive filtration $\operatorname{Fil}^\bullet_P \mathscr{M}(K)$, where $\operatorname{Fil}^0_P \mathscr{M}(K) = \mathscr{M}_{C,P}$ and $\operatorname{Fil}^n_P \mathscr{M}(K)$ is the subgroup consisting of the elements $a \in \mathscr{M}(K)$ such that $(a, 1+\mathfrak{m}_P^n)_P = 0$.

3. K-groups of reciprocity functors

Now let $\mathscr{M}_1, \ldots, \mathscr{M}_n$ and $\mathscr{N}$ be reciprocity functors. Then a n-linear map of reciprocity functors $\Phi : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{N}$ is a n-linear map of sheaves, which is compatible with pullback, satisfies a projection formula, and the following condition

$$\text{(L3)} \quad \Phi(\operatorname{Fil}^{r_1}_P \mathscr{M}_1(K) \times \cdots \times \operatorname{Fil}^{r_n}_P \mathscr{M}_n(K)) \subset \operatorname{Fil}^{\max\{r_1,\ldots,r_n\}}_P \mathscr{N}(K),$$

for all regular projective curves C with function field K, all closed points $P \in C$ and all positive integers $r_1, \ldots, r_n \geqslant 1$. We denote by $n - \operatorname{Lin}(\mathscr{M}_1, \ldots, \mathscr{M}_n; \mathscr{N})$ the group of n-linear maps as above. The main theorem of [3] is the following:

Theorem. *The functor* $\mathbf{RF} \to$ *(Abelian groups),* $\mathscr{N} \mapsto n - \operatorname{Lin}(\mathscr{M}_1, \ldots, \mathscr{M}_n; \mathscr{N})$ *is representable by a reciprocity functor*

$$\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n).$$

We call $\mathrm{T}(\mathscr{M}_1, \ldots, \mathscr{M}_n)$ the K-group of $\mathscr{M}_1, \ldots, \mathscr{M}_n$, although it is much more than a group, it is a reciprocity functor. We would like to call this a tensor product, unfortunately it is not clear whether associativity is satisfied (one reason is the condition (L3)). But other properties of a tensor product hold: we have commutativity, compatibility with direct sums, the constant reciprocity functor $\mathbb{Z}$ is a unit object.

4. Some computations

One of the main computation is the following theorem:

Theorem. *Let $\mathscr{F}_1, \ldots, \mathscr{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be homotopy invariant Nisnevich sheaves with transfers. There exists a canonical and functorial isomorphism of reciprocity functors*

$$\mathrm{T}(\hat{\mathscr{F}}_1, \ldots, \hat{\mathscr{F}}_n) \xrightarrow{\sim} (\mathscr{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathscr{F}_n)\widehat{}\,.$$

Let us emphasize that the definition of the K-group of reciprocity functors is different from the Somekawa type one. In particular the above theorem does not follow from the isomorphism $(*)$.

Theorem. *Assume F has characteristic zero. Then there is an isomorphism for all finitely generated extension k/F*

$$\theta : \Omega^n_{k/\mathbb{Z}} \xrightarrow{\simeq} \mathrm{T}(\mathbb{G}_a, \underbrace{\mathbb{G}_m, \ldots, \mathbb{G}_m}_{n\ copies})(k).$$

The above theorem provides a link with [2]. This result does not hold in positive characteristic. B. Kahn conjectured that one should have $\mathrm{T}(\mathbb{G}_a, \mathbb{G}_a) = 0$. This is indeed the case. More generally we prove the following vanishing result:

Theorem. *Assume* $\mathrm{char}(F) \neq 2$. *Let $\mathscr{M}_1, \ldots, \mathscr{M}_n$ be reciprocity functors. Then*

$$\mathrm{T}(\mathbb{G}_a, \mathbb{G}_a, \mathscr{M}_1, \ldots, \mathscr{M}_n) = 0.$$

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Suslin Homology and Class Field Theory

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(joint work with T. Geisser)

Let k be a field of characteristic $p \geq 0$. We denote the category of separated schemes of finite type over k by Sch/k. The *tame fundamental group* $\pi_1^t(X)$ of $X \in Sch(X)$ classifies (curve-)tame finite étale coverings of X (cf. [KSc]) and is a quotient of the usual étale fundamental group $\pi_1^{\mathrm{et}}(X)$ in a natural way. Dually, for every $m \in \mathbb{N}$, we have the subgroup $H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \subset H_{\mathrm{et}}^1(X, \mathbb{Z}/m\mathbb{Z})$ which classifies isomorphism classes of tame $\mathbb{Z}/m\mathbb{Z}$-torsors over X. The inclusion is equality if X is proper or if $p \nmid m$.

Let $k = \mathbb{F}$ be a finite field. Sending a closed point $x \in X$ to its Frobenius automorphism Frob_x defines a homomorphism

$$Z_0(X) \longrightarrow \pi_1^{\mathrm{ab}}(X)$$

from the group of zero cycles of X to its abelianized fundamental group. Let $C_n(X) = \mathrm{Cor}(\Delta^n, X)$ denote the group of finite correspondences from the n-dimensional standard simplex to X. The integral Suslin homology of X is defined by $H_n^S(X, \mathbb{Z}) = H_n(C_\bullet(X))$, see [SV]. By [Sc, Thm. 8.1] the composite $Z_0(X) \to \pi_1^{\mathrm{ab}}(X) \twoheadrightarrow \pi_1^{t,\mathrm{ab}}(X)$ factors through $H_0^S(X, \mathbb{Z})$ inducing

$$rec_X : H_0^S(X, \mathbb{Z}) \to \pi_1^{t,\mathrm{ab}}(X). \tag{1}$$

Theorem 1 (Schmidt/Spiess [SS],[Sc])**.** *If X is smooth, then rec_X fits into an exact sequence*

$$0 \longrightarrow H_0^S(X, \mathbb{Z}) \xrightarrow{rec} \pi_1^{t,\mathrm{ab}}(X) \longrightarrow \widehat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0.$$

The induced map on the degree zero subgroups $rec_X^0 : H_0^S(X, \mathbb{Z})^0 \to \pi_1^{t,\mathrm{ab}}(X)^0$ is an isomorphism of finite abelian groups.

Theorem 1 generalizes the unramified class field theory of Kato and Saito [KaS], [Sa] to the case of smooth, not necessarily proper schemes. Recently, Kerz and Saito [KeS] found a generalization which describes the full fundamental group $\pi_1^{\mathrm{ab}}(X)$ by using "Chow groups with modulus" instead of Suslin homology.

Note that the assumption on X being smooth is vital in Theorem 1. The cokernel of rec_X classifies completely split coverings and might be large if X is not geometrically unibranch. Furthermore, even for proper, normal schemes there are examples where rec_X is not injective [MAS].

Next we are going to construct a reciprocity map for varieties over algebraically closed fields. Let k be algebraically closed, U, $X \in Sch(k)$, U regular, and $\alpha \in \mathrm{Cor}(U, X)$ a finite correspondence. For any $m \in \mathbb{N}$, we construct a functor

$$\alpha^* : \mathrm{PHS}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{PHS}(U, \mathbb{Z}/m\mathbb{Z})$$

from the category of étale $\mathbb{Z}/m\mathbb{Z}$-torsors on X to those on U which gives back the usual pull-back map $\alpha^* : H_{\mathrm{et}}^1(X, \mathbb{Z}/m\mathbb{Z}) \to H_{\mathrm{et}}^1(U, \mathbb{Z}/m\mathbb{Z})$ on isomorphism classes and which sends tame torsors to tame torsors. For a tame $\mathbb{Z}/m\mathbb{Z}$-torsor $\mathcal{T}$ on X and a finite correspondence $\alpha : \Delta^1 \to X$ we obtain the tame, hence trivial torsor $\alpha^*(\mathcal{T})$ on $\Delta^1 \cong \mathbb{A}^1$. Parallel transport therefore induces an isomorphism

$$\Phi_{par} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T}))$$

of $\mathbb{Z}/m\mathbb{Z}$-torsors over Δ^0. If α represents a 1-cocycle in the mod m Suslin complex, we furthermore obtain a tautological identification

$$\Phi_{taut} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T})).$$

Hence there is a unique $\langle \alpha, \mathcal{T} \rangle \in \mathbb{Z}/m\mathbb{Z}$ such that

$$\Phi_{par} = (\text{translation by } \langle \alpha, \mathcal{T} \rangle) \circ \Phi_{taut}.$$

Theorem 2. *For any $X \in Sch/k$ the assignment $(\alpha, \mathcal{T}) \mapsto \langle \alpha, \mathcal{T} \rangle$ induces a pairing*

$$H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

of finite abelian groups. If $p \mid m$ assume that resolution of singularities holds over k. Then the pairing $\langle\ ,\ \rangle$ is perfect and we obtain a reciprocity isomorphism

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \pi_1^{t,\mathrm{ab}}(X)/m.$$

For $(m,p) = 1$ we have the comparison isomorphism of Suslin-Voevodsky [SV] $\alpha_X : H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} H^1_S(X, \mathbb{Z}/m\mathbb{Z})$. Therefore, for $(m,p) = 1$, the source and the target of rec_X are known to be isomorphic abelian groups from the very beginning. However, the isomorphism α_X of [SV] zig-zags through Ext-groups in various categories and is difficult to understand. The merit of Theorem 2 should be seen in constructing an explicit isomorphism which naturally extends to the case that m is divisible by p. However, we also have

Theorem 3. *For $(m,p) = 1$, rec_X coincides with the dual of the Suslin-Voevodsky isomorphism α_X.*

Returning to the case that $k = \mathbb{F}$ is finite, we recall the notion of Weil-Suslin homology introduced by Geisser [Ge]: Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, $X \in Sch/\mathbb{F}$ and $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$. The Frobenius automorphism $\mathrm{Frob} \in \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$ acts on $C_n(\bar{X}) = \mathrm{Cor}(\bar{\Delta}^n, \bar{X})$ for all n and the Weil-Suslin homology of X with values in an abelian group A is defined by

$$H_n^{WS}(X, A) = H_n\big(\operatorname{cone}(C_\bullet(\bar{X}) \otimes A \xrightarrow{1-\mathrm{Frob}} C_\bullet(\bar{X}) \otimes A)\big).$$

The obvious homomorphism $H_0^S(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z})$ is conjectured to be an isomorphism if X is smooth.

In a similar spirit as above, one constructs compatible pairings for all m

$$H_1^{WS}(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

These pairings and the natural maps $H_1^{WS}(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z}/m\mathbb{Z})$ induce a homomorphism

$$rec_X^{WS} : H_1^{WS}(X, \mathbb{Z}) \to \pi_1^{t,\mathrm{ab}}(X) \tag{2}$$

such that composition with $H^0_S(X,\mathbb{Z}) \to H_1^{WS}(X,\mathbb{Z})$ is the map rec_X defined in (1) above.

Theorem 4. *Assume that resolution of singularities holds over* $\mathbb{F}$. *Then, for any* $X \in Sch/\mathbb{F}$ *the map* rec_X^{WS} *induces an isomorphism*

$$H_1^{WS}(X,\mathbb{Z})^\wedge \to \pi_1^{t,\mathrm{ab}}(X)$$

on profinite completions.

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Slices of co-operations for *KGL*

Chuck Weibel

(joint work with Pablo Pelaez)

We settle a conjecture of Voevodsky [1] concerning the slices of $KGL \wedge KGL$ and $KGL^{\wedge n}$ in the stable homotopy category over a base S which is smooth over a perfect field, and more generally over any finite-dimensional noetherian base satisfying mild conditions.

Adams showed that the ring $\pi_*(KU \wedge KU)$ is not just an algebra over KU_* but a Hopf algebroid; it follows that the cosimplicial ring $n \mapsto \pi_*(KU^{\wedge n+1})$ is the cobar complex over this algebroid. This forms the background for the conjecture.

The slices s_qE of a motivic spectrum E were defined by Voevodsky in [1]. If E is a ring spectrum, the direct sum $s_*E = \oplus s_qE$ of the slices s_qE of E form a graded ring spectrum. For example, $s_*KGL \cong s_0(KGL)[u,u^{-1}]$ by periodicity.

Let the tensor product $E \otimes A$ of a spectrum E with an abelian group A have its usual meaning, and let $E \otimes \pi_{2*}KU^{\wedge n+1}$ denote the direct sum of the motivic spectra $(T^q \wedge E) \otimes \pi_{2q}KU^{\wedge n+1}$. Finally, let $H\mathbb{Z}$ denote the spectrum representing motivic cohomology. The following result verifies Voevodsky's conjecture.

Theorem: *Suppose that S is smooth over a perfect field. Then there is an isomorphism of motivic spectra in $SH(S)$. Then there is an isomorphism of cosimplicial motivic spectra:*

$$s_*(KGL^{\wedge *+1}) \cong H\mathbb{Z} \otimes \pi_{2*} KU^{\wedge *+1}.$$

Voevodsky's conjecture is intertwined with other conjectures of Voevodsky, that the maps $H\mathbb{Z} \leftarrow \mathbf{1} \rightarrow KGL$ induce isomorphisms on zero-slices. Here is a more general result; as usual, $[E, F]$ denotes homotpy classes of maps $E \rightarrow F$.

Theorem: *Suppose that S is a finite-dimensional noetherian scheme. Then:*
(a) There are isomorphisms for all $n \geq 0$:

$$s_0(KGL) \otimes \pi_{2*} KU^{\wedge n} \xrightarrow{\cong} s_*(KGL^{\wedge n}).$$

(b) Suppose in addition that $s_0(\mathbf{1}) \cong s_0(KGL)$ and that $[s_0(\mathbf{1}), s_0(\mathbf{1})]$ is torsionfree. Then the maps in (a) are the components of an isomorphism of cosimplicial motivic ring spectra:

$$s_*(KGL^{\wedge *+1}) \cong s_0(\mathbf{1}) \otimes \pi_{2*} KU^{\wedge *+1}.$$

Here are the main steps in the proof. The isomorphisms in (a) are proven using a toy version of the theorem, namely that

$$s_*(KGL \wedge \mathbb{P}^{\infty \wedge *}) \cong s_0(KGL) \otimes \pi_{2*}(KU \wedge \mathbb{CP}^{\infty \wedge *})$$

The ring F of numerical polynomials is a subring of $\mathbb{Q}[t]$ with basis the polynomials $\binom{t}{n}$, and $KU_*(\mathbb{CP}^\infty) \cong KU_* \otimes F$. The heart of the argument is that the projective bundle theorem that $KGL \wedge \mathbb{P}^\infty \cong KGL \otimes F$ identifies the product

$$KGL \wedge \mathbb{P}^\infty \wedge \mathbb{P}^\infty \rightarrow KGL \wedge \mathbb{P}^\infty$$

with the map $KGL \otimes F \otimes F \rightarrow KGL \otimes F$ given by the product $F \otimes F \rightarrow F$. By a result of Gepner-Snaith-Spitzweck-Østvær, this implies that $KGL \wedge KGL \cong KGL \otimes F[1/t]$.

Everything in (b) but compatibility with the coface ∂^0 and codegeneracy σ^0 follows easily from (a). We use the hypothesis that $s_0(\mathbf{1}) \cong s_0(KGL)$ to show compatibility with ∂^0, and the hypothesis that $[s_0(\mathbf{1}), s_0(\mathbf{1})]$ is torsionfree to show compatibility with σ^0.

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Homotopy type theory

Daniel R. Grayson

Homotopy type theory with the univalence axiom of Voevodsky provides both a new logical foundation for mathematics and a formal language usable with computers for checking the proofs mathematicians make daily. As a foundation, it replaces Zermelo-Fraenkel set theory with a framework where sets are defined in terms of a primitive notion called "type". As a formal language, it encodes the axioms of mathematics and the rules of logic simultaneously, and promises to make the extraction of algorithms and values from constructive proofs easy. With a semantic interpretation in homotopy theory, it offers an alternative world where the proofs of basic theorems of homotopy theory can be formalized with minimal verbosity and verified by computer.

The aim of the talk was to expose these recent ideas and developments, due to various participants in the just-concluded special year at the Institute for Advanced Study in Princeton. A 600 page book has been produced explaining homotopy type theory and how to make proofs using the new foundations, entitled "Homotopy Type Theory: Univalent Foundations of Mathematics" and available at http://homotopytypetheory.org/book/.

One of the evident annoyances of the use of set theory as a foundation for mathematics is the existence of stupid propositions, such as whether 2 is an element of 3, or whether the natural number 5 is equal to the integer 5. The truth of such propositions depends on the precise definitions used in setting up the mathematical objects to be discussed, but is ultimately irrelevant to mathematicians. Type theory avoids the first sort of stupid proposition above by discarding the relation that says that one set is an element of another: instead, elements and the "types" they belong to are different kinds of things, and each element is born knowing its type; that information is now part of the grammar of the theory, rather than part of the mathematical content. Type theory avoids the second sort of stupid proposition by using an equality relation that can be applied only to compare elements of the same type.

Homotopy type theory is a more fundamental version of type theory, which refrains from positing that two proofs of the same equality are equal. It reveals a world where types are like topological spaces, elements are like points, proofs of equality between two elements are like paths, and proofs of equality between two proofs of equality are like homotopies between paths. In this world, we regard a proof that $x = y$ as providing a way to "identify" x with y; the "identification" chosen matters in subsequent reasoning. A dictionary is constructed that links certain propositions to traditional notions of homotopy theory. In particular, one may describe the types that replace the propositions of set theory (they are like spaces that are empty or contractible), the types that replace the sets of set theory (they are like spaces every component of which is contractible), and the maps between types that are "equivalences" (they are like homotopy equivalences).

The Univalence Axiom of Voevodsky posits a way to convert equivalences to identifications. As a consequence, in its presence, there are no stupid propositions, because every definition, every proposition, and every proof can be transported from one type to any equivalent type. Voevodsky's theorem states that there is a model in simplicial sets that demonstrates the consistency of homotopy type theory with the axiom added (relative to the consistency of traditional mathematics). Voevodsky's conjecture is that constructive proofs using the axiom remain computable; establishing it is an important problem.

The higher inductive types of Lumsdaine and Shulman allow new types to be constructed synthetically, cell by cell, and allow new proofs of traditional theorems of homotopy theory, such as the Freudenthal Suspension Theorem and the Blakers-Massey Theorem, to be proven in new and beautiful ways. For details, see Chapter 8 of the book.

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