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Differentialgeometrie im Großen

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Abstract. The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Global Riemannian geometry with its connections to topology, geometric group theory and geometric analysis remained an important focus of the conference. Special emphasis was given to Kähler manifolds, geometric flows and singular spaces of non-positive curvature.

Mathematics Subject Classification (2010): 53Cxx, 51Fxx, 51Mxx, 57Sxx, 58Exx, 32Qxx, 22Exx.

Introduction by the Organisers

The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Traditionally, the conference series covers a wide scope of different aspects of global differential geometry and its connections with geometric analysis, topology and geometric group theory. The Riemannian aspect is emphasized, but the interactions with the developments in complex geometry, symplectic/contact geometry/topology and physics play an important role as well. Within this spectrum, each particular conference gives special attention to two or three topics of particular current relevance.

Similar to the last conference, the scientific program consisted of 22 one hour talks. This allowed to include many of the interesting talk proposals in the schedule while still leaving ample time for informal discussions.

Among the broad spectrum of topics presented at the workshop, a prominent theme were Kähler manifolds, in particular conditions for the existence of Kähler-Einstein metrics, gluing constructions for Einstein manifolds, and singular Kähler metrics on polyhedral manifolds and complex surfaces.

Another area of focus were geometric flows, with several talks concerned about the existence and regularity as well as curvature estimates for graphical mean curvature flows and mean convex hypersurfaces. Furthermore, the (finiteness of) singularities of Ricci flow in dimension three and in the Kähler-Ricci case were studied.

Moreover, singular spaces with curvature bounds played an important role. Special attention was given to different notions of non-positive curvature, involving the existence of a geodesic bicombing or coarse medians, a quasi-isometry invariant concept applicable to Cayley graphs. The implications of group actions exhibiting some hyperbolic dynamics were also considered.

Apart from these topics, other talks presented results from min-max theory (in particular the Willmore conjecture), Lorentzian geometry (globally hyperbolic manifolds and manifolds with noncompact isometry groups), complete affine manifolds, the action of discrete subgroups of higher rank Lie groups on the boundary of the associated symmetric space (with emphasis on domains of discontinuity and cocompactness), a geometric characterization of certain representations of surface groups into higher rank Lie groups of Hermitian type, and geometrically formal 4-manifolds.

There were 51 participants from 8 countries, more specifically, 21 participants from Germany, 11 from the United States of America, 8 from France, 4 from England, 4 from Switzerland and respectively 1 from Japan, Spain and Russia. There were 2 women among the participants. 33% of the participants (17) were young researchers (less than 10 years after diploma or B.A.), both on doctoral and postdoctoral level.

The organizers would like to thank the institute staff for their great hospitality and support before and during the conference. The financial support for young participants, in particular from the Leibniz Association and from the National Science Foundation, is gratefully acknowledged.

Workshop: Differentialgeometrie im Großen

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Abstracts

Kähler-Einstein metrics on Fano manifolds

Song Sun
(joint work with Xiuxiong Chen, Simon Donaldson)

Let X be a Fano manifold. A Kähler metric $\omega \in 2\pi c_1(X)$ is *Kähler-Einstein* if it satisfies the equation $Ric(\omega) = \omega$. In this lecture we discuss the following theorem.

Theorem 0.1 ([2, 3, 4, 5]). *X admits a Kähler-Einstein metric if X is K-stable.*

It is well-known that the Kähler-Einstein condition imposes restrictions on the Lie algebra $\mathfrak{g}$ of holomorphic vector fields:

- Matsushima: $\mathfrak{g}$ must be reductive.
- Futaki: The Futaki invariant $Fut : \mathfrak{g} \to \mathbb{C}$ must vanish.

In complex dimension 2 these are indeed equivalent to the existence of a Kähler-Einstein metric [13], but this is no longer true in higher dimension [15]. Yau [16] first proposed that there should be further algebro-geometric stability conditions that are equivalent to the existence of a Kähler-Einstein metric. The precise notion of K-stability was later introduced in [15] and [6]. The definition involves the notion of a *test configuration.* Choose an arbitrary projective embedding of X into some $\mathbb{P}^N$ by sections of K_X^{-r}, and a one parameter subgroup $\lambda(t)$ of $PGL(N+1;\mathbb{C})$. Then we obtain a limit scheme $X_0 := \lim_{t\to 0} \lambda(t).X$. Putting these together yields a $\mathbb{C}^*$ equivariant family $\mathcal{X}$ of polarized schemes over $\mathbb{C}$ with general fiber X and central fiber X_0. $\mathcal{X}$ is then called a test configuration for X. It has a *Futaki invariant* $Fut(\mathcal{X})$, defined in terms of the coefficients occurring in the asymptotic expansion of the dimension of $H^0(X, K_X^{-rk})$ and the total weight of the induced $\mathbb{C}^*$ action. X is called *K-stable* if $Fut(\mathcal{X}) > 0$ for all $\mathcal{X}$ such that the central fiber X_0 has KLT singularities and X_0 is not isomorphic to X (the fact that one can restrict the singularities follows from the proof of Theorem 0.1 and also see [9] for an algebro-geometric study). Formally a Kähler-Einstein metric is a critical point of a geodesically convex functional (defined by Mabuchi) on an infinite dimensional space and the notion of K-stability can be viewed as an algebraization of the condition that the derivative at infinity of this functional along any rational geodesic ray is positive. The converse to Theorem 0.1 is also known (see for example [1] and the references therein).

The strategy we use in the proof of Theorem 0.1 is the one proposed in [7], involving metrics with cone singularities along a divisor. This is a variant of the classical Aubin-Yau continuity method. Choose $\lambda > 1$ and fix a smooth divisor $D \in |-\lambda K_X|$. Consider the family of equations

$$Ric(\omega_\beta) = r_\beta \omega_\beta + 2\pi(1-\beta)[D], \tag{1}$$

where $r_\beta = 1-(1-\beta)\lambda$ and $\beta \in (0,1]$. The metrics ω_β are required to be smooth away from D and have cone angle $2\pi\beta$ along D. For $\beta = 1/N$ with N a big integer we have $r_\beta < 0$, and one can obtain a solution of (1) by the Aubin-Yau theorem and an orbifold trick. Then the idea is to show that we can deform these solutions as we deform β all the way up to 1. The openness is proved in [7] (with a technical point clarified in [11] concerning the kernel of the linearized operator) and the main difficulty is to then show the closedness. The following theorem is the essential ingredient in the proof of Theorem 0.1.

Theorem 0.2. *Given a sequence $\beta_i \to \beta_\infty \in (0,1]$ and a corresponding sequence of conical Kähler-Einstein metrics ω_i.*

- *By passing to a subsequence we obtain a Gromov-Hausdorff limit X_∞ together with a closed subset D_∞.*
- *There is an integer k depending only on β_∞, and projective embeddings $T_i : X \to \mathbb{P}^{N_k}$ defined by orthonormal bases of $H^0(X, K_X^{-k})$ (with respect to the Hermitian metric defined by ω_i), so that T_i converges to a map $T_\infty : X_\infty \to \mathbb{P}^{N_k}$.*
- *T_∞ is a homeomorphism onto a $\mathbb{Q}$-Fano variety W, and it maps D_∞ onto a Weil divisor Δ, so that $(W, (1-\beta_\infty)\Delta)$ is a KLT pair.*
- *W admits a weak conical Kähler-Einstein metric ω_∞ with cone angle $2\pi\beta_\infty$ along Δ, in the sense of pluripotential theory. In particular, it has a locally continuous potential function that is smooth in $W \setminus (\Delta \cup W^{sing})$.*

Given i and k, the Bergman density of state function on X is defined by

$$\rho_{k,\omega_i}(x) = \sup\{|s(x)| : s \in H^0(X, K_X^{-k}), ||s||_{L^2} = 1\},$$

where the norms are defined by the metric ω_i. The following technical theorem is the key to prove Theorem 0.2.

Theorem 0.3. *There are k_0 and $\epsilon_0 > 0$ depending only on β_∞ so that $\rho_{k,\omega_i}(x) \geq \epsilon_0$ for all i and $x \in X$.*

For a sequence of smooth Kähler-Einstein Fano manifolds, Theorem 0.3 and 0.2 are proved in [8] in general dimension (this was conjectured by Tian [14] and also proved by Tian [13] in dimension two). The proof of Theorem 0.3 involves a combination of convergence theory of Riemannian manifolds developed by Cheeger-Colding and the Hörmander L^2 method for constructing holomorphic sections. At the presence of cone singularities many more technical difficulties arise and these occupy the main content in the series of papers [3, 4, 5].

Next we use the K-stability assumption to prove that $(W, (1-\beta_\infty)\Delta)$ is isomorphic to $(X, (1-\beta_\infty)D)$ (This means when $\beta_\infty = 1$ we ignore the divisors). As in the smooth case, there is also a corresponding notion of K-stability for the triple (X, D, β), where we consider the same notion of a test configuration $\mathcal{X}$, but need to modify the definition of Futaki invariant to

$$Fut_\beta(\mathcal{X}) = Fut(\mathcal{X}) + (1-\beta)C(X_0, D_0). \tag{2}$$

A crucial fact is that the constant C depends only on X_0 and D_0, so that Fut_β depends linearly on β. By [12], [10] $(X, D, 0)$ is K-stable, so under the assumption that X is K-stable we know (X, D, β) is K-stable for all $\beta \in (0,1]$. Suppose

$(W,(1-\beta_\infty)\Delta)$ is not isomorphic to $(X,(1-\beta_\infty)D)$, then we want to construct a test configuration $\mathcal{X}$ with $Fut_{\beta_\infty}(X,D,\beta_\infty)=0$. Clearly the above limit $(W,(1-\beta_\infty)\Delta)$ serves as a natural candidate for the central fiber, but to construct $\mathcal{X}$ one needs to strengthen the sequential convergence into a $\mathbb{C}^*$ orbit closure. This can be regarded as a finite-dimensional question about group orbits but in this generality the assertion is not true. However, it is true if we know $Aut(W,(1-\beta_\infty)\Delta)$ is reductive. Now we see that we are back to prove the Matsushima and Futaki theorem for the weak singular Kähler-Einstein pair $(W,(1-\beta_\infty)\Delta)$. The presence of singularities of W causes serious difficulty if one would like to directly extend the standard proof to prove the Matsushima theorem. In [5] we instead make use of the recent advance in pluripotential theory due to Berndtsson and others.

Therefore we have proved X admits a weak conical Kähler-Einstein metric with cone angle $2\pi\beta_\infty$ along D. To continue the deformation we need to prove higher regularity. When $\beta_\infty<1$ we need to prove a priori C^α regularity on ω_i so that the limit metric ω_∞ has the correct geometry near D and fits into the assumption in the openness theorem in [7]. In [4] we achieved this by a rescaling argument involving Cheeger-Colding theory again. When $\beta_\infty=1$ we have proved ω_∞ is smooth away from Δ and we need to prove a removable singularity theorem [5], similar to a result of Trudinger, and finally we obtain a smooth Kähler-Einstein metric on X.

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Ricci flow on quasiprojective manifolds

John Lott
(joint work with Zhou Zhang)

We consider the Kähler-Ricci flow on a complete Kähler metric that lives on a quasiprojective manifold $X = \overline{X} - D$, where $\overline{X}$ is a compact complex manifold that admits a Kähler metric and D is a divisor in $\overline{X}$. In a series of papers, Tian and Yau gave sufficient conditions for X to admit a complete Kähler-Einstein metric [2, 3, 4]. We consider the flow of an initial metric satisfying the same spatial asymptotics as in the Tian-Yau papers.

In earlier work [1], we considered an initial metric which has finite volume, with spatial asymptotics like those considered in [2]. Going to infinity in a given direction, the metric looks like a family of products of hyperbolic surface cusps. We showed that at a later time, the metric has similar asymptotics, taking into account the Kähler-Ricci flow on the divisor. We computed the first singularity time (if there is one) and gave a sufficient condition for the singularity to be type-II.

We now consider initial metrics with spatial asymptotics of the type that were considered in [3, 4] to build complete Ricci-flat Kähler metrics. There are three cases :

(1) Families of cylinders [3].
(2) Bulging asymptotics [3].
(3) Conical asymptotics [4].

In case (1), we again show that at a later time, the ensuing metric has similar asymptotics to the initial metric, taking into account the Kähler-Ricci flow on the divisor.

In case (2), we show that at a later time, the ensuing metric has the exact same spatial asymptotics as the initial metric. Under a mild curvature assumption, we show that one sees the divisor flow when taking parabolic blowdowns based at points in the time-zero manifold that go to spatial infinity.

In case (3), we use pseudolocality to show that there is a blowdown limit flow, which evolves from a metric cone. In known cases, this blowdown limit flow is an expanding soliton (possibly with a singular point). We show that an expanding soliton always exists in the sense of a formal asymptotic expansion in the inverse of the radial variable, and that it is the formal blowdown limit flow.

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There are finitely many surgeries in Perelman's Ricci flow

Richard H. Bamler

In this talk we analyze the long-time behavior of Ricci flows with surgery on 3 dimensional manifolds without imposing any assumptions on the topology of the initial manifold. In particular, we will prove the following theorem which resolves a conjecture raised by Perelman:

Theorem 0.1. *There is a continuous function $\delta : [0,\infty) \to (0,\infty)$ such that the following holds: Let $\mathcal{M}$ be a Ricci flow with surgery with normalized initial conditions which is performed by at least $\delta(t)$-precise cutoff. Then $\mathcal{M}$ has only finitely many surgeries and there are constants $T, C < \infty$ such that $|\mathrm{Rm}_t| < Ct^{-1}$ on $\mathcal{M}(t)$ for all $t \geq T$.*

We mention two important direct consequences of Theorem 0.1 which can be expressed in a more elementary way:

Corollary 0.2. *Let $(M, (g_t)_{t\in[0,\infty)})$ be a non-singular, long-time existent Ricci flow on a compact 3-manifold M, i.e. $\partial_t g_t = -2\,\mathrm{Ric}_{g_t}$. Then there is a constant $C < \infty$ such that*

$$|\mathrm{Rm}_t| < \frac{C}{t+1} \qquad \text{for all} \qquad t \geq 0.$$

Moreover, we obtain the following result which ensures that the condition of the previous Corollary can be satisfied.

Corollary 0.3. *Let M be a compact, orientable 3-manifold. There exists a long-time existent Ricci flow $(g_t)_{t\in[0,\infty)}$ on M if and only if M is irreducible and aspherical.*

Both results are new and follow directly from Theorem 0.1. In fact, Corollary 0.2 is just the statement of the Theorem for non-singular Ricci flows. Note that its proof in the non-singular case cannot be simplified essentially. The existence of the non-singular Ricci flow $(g_t)_{t\in[0,\infty)}$ in Corollary 0.3 can be established as follows: Choose an arbitrary normalized metric g_0 on M. By Perelman's work there is a Ricci flow with surgery $\mathcal{M}$ which is performed by $\delta(t)$-cutoff and whose initial time slice is (M, g_0). Theorem 0.1 implies that $\mathcal{M}$ is non-singular on the time-interval $[T, \infty)$ for some large T. The topological assumption ensures that the topology of the underlying manifold does not change between any two time-slices. So shifting this Ricci flow restricted to $[T,\infty)$ back in time by $-T$ yields the desired flow. The reverse direction is well known.

The Ricci flow with surgery has been used by Perelman to solve the Poincaré and Geometrization Conjecture ([Per1], [Per2], [Per3]). Given any initial metric on a closed 3-manifold, Perelman managed to construct a solution to the Ricci flow with surgery on a maximal time-interval and showed that its surgery times do not accumulate. Hence every finite time-interval contains only a finite number of surgery times. Furthermore, he could prove that if the given manifold is a homotopy sphere (or more generally a connected sum of prime, non-aspherical manifolds), then this flow goes extinct in finite time. This implies that the initial manifold is a sphere if it is simply connected and hence establishes the Poincaré Conjecture. On the other hand, if the Ricci flow continues to exist for infinite time, Perelman could show that the manifold decomposes into a thick part which approaches a hyperbolic metric and a thin part which becomes arbitrarily collapsed on local scales. Based on this collapse, it is then possible to show that the thin part can be decomposed into more concrete pieces ([ShY], [MT], [KL]). This decomposition can then be reorganized to a geometric decomposition, establishing the Geometrization Conjecture.

Observe that although the Ricci flow with surgery was used to solve such hard and important problems, some of its basic properties remained unknown, because they surprisingly turned out to be irrelevant for these problems. Perelman conjectured that in the long-time existent case there are finitely many surgery times, i.e. that after some time the flow can be continued by a conventional smooth, non-singular Ricci flow defined up to time infinity. Furthermore, it is still unknown whether and in what way the Ricci flow exhibits the full geometric decomposition of the underlying manifold as $t \to \infty$.

In [Lot1], [Lot2] and [LS], Lott and Lott-Sesum could give a description of the long-time behavior of certain Ricci flows on manifolds which consist of a single component in their geometric decomposition. However, they needed to make additional curvature and diameter or symmetry assumptions. In [Bam1], the author proved that under a purely topological condition $\mathcal{T}_1$, which roughly states that the manifold only consists of hyperbolic components, there are only finitely many surgeries and the curvature is bounded by Ct^{-1} after some time. In [Bam2], this condition was generalized to a far more general topological condition $\mathcal{T}_2$, which requires the non-hyperbolic pieces in the geometric decomposition of the underlying manifold to contain sufficiently many incompressible surfaces. For example, manifolds of the form $\Sigma \times S^1$ for closed, orientable surfaces Σ—such as the 3-torus T^3—satisfy property $\mathcal{T}_2$, but the Heisenberg manifold does not. We refer to [Bam2, sec 1.2] for a precise definition and discussion of the conditions $\mathcal{T}_1$ and $\mathcal{T}_2$.

Theorem 0.1 announced in this talk does not require any extra topological assumption on the initial manifold anymore and hence confirms Perelman's conjecture in its full generality. It can essentially be applied to all Ricci flows with surgery constructed by Perelman. The only restriction is that the function $\delta(t)$ which bounds the preciseness under which the surgeries are performed might be smaller than the function that Perelman had to assume in the construction process. Note however that the choice of $\delta(t)$ in Perelman's work was highly non-explicit and its behavior essentially cannot be controlled whatsoever.

The curvature bound Ct^{-1} in Theorem 0.1 is optimal in the sense that its asymptotics are realized by most known examples of Ricci flows, e.g. the sectional curvatures on a Ricci flow starting from a hyperbolic manifold behave asymptotically like $-\frac{1}{4t}$ for $t \to \infty$.

In the course of the proof of Theorem 0.1 we also obtain a more detailed description of the geometry of the time-slices $\mathcal{M}(t)$ for large t. A more precise characterization of the long-time behavior would however still be desirable and it is very likely that the new curvature bound enables us to describe this behavior using more analytical tools. A central question in this direction, which still remains unanswered, is whether the diameter of the manifold is bounded by $C\sqrt{t}$ for large t whenever the underlying manifold consists of a single geometric piece.

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Weakly maximal representations and causal structures

Anna Wienhard

(joint work with G. Ben Simon, M. Burger, T. Hartnick, A. Iozzi)

Given a compact oriented surface Σ of negative Euler characteristic, possibly with boundary, a general theme is to study the space of representations $\hom(\pi_1(\Sigma), G)$ of the fundamental group of Σ into a semisimple Lie group G, and in particular to distinguish subsets of geometric significance, such as holonomy representations of geometric structures. Classical examples include the set of Fuchsian representations in $\hom(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ or the set of quasi-Fuchsian representations in $\hom(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{C}))$, where the target group is of real rank one. In recent years these studies have been extended to the case where G is of higher rank. They are summarized under the terminus *higher Teichmüller theory*. We refer the reader to [4] for a survey and references to the literature.

Here we discuss an extension of higher Teichmüller theory in the Hermitian context. For more details and proofs we refer the reader to [1]. Recall that a semisimple Lie group G is called *Hermitian* if the associated symmetric space $\mathcal{X}$ admits a G-invariant Kähler form $\omega_{\mathcal{X}}$. This Kähler form can be used to define a continuous function $T : \hom(\pi_1(\Sigma), G) \to \mathbb{R}$ on the representation variety; the invariant $T(\rho)$ is called the *Toledo number* of the representation ρ (see [6]). The Toledo number is subject to a *Milnor-Wood type inequality* of the form

$$|T(\rho)| \leq ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|, \tag{1}$$

where $\kappa_G^{\mathrm{b}} \in H^2_{cb}(G;\mathbb{R})$ denotes the bounded Kähler class of G, i.e. the class corresponding to $\omega_{\mathcal{X}}$ under the isomorphisms $H^2_{cb}(G;\mathbb{R}) \cong H^2_c(G;\mathbb{R}) \cong \Omega^2(\mathcal{X})^G$, and $||\cdot||$ denotes the seminorm in continuous bounded cohomology (see [6]). The class of representations ρ with maximal Toledo invariant $T(\rho) = ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|$, or *maximal representations* for short, has been the main object of study in higher Teichmüller theory with Hermitian target groups.

Our starting point here is the observation that the inequality (1) can be refined into the chain of inequalities

$$|T(\rho)| \leq ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)| \leq ||\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|.$$

In particular, a representation is maximal iff it satisfies both $||\rho^*\kappa_G^{\mathrm{b}}|| = ||\kappa_G^{\mathrm{b}}||$ and $T(\rho) = ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|$. Representations satisfying $||\rho^*\kappa_G^{\mathrm{b}}|| = ||\kappa_G^{\mathrm{b}}||$ are called *tight*; these have been investigated in much greater generality in [5]. Here we are interested in representations satisfying the complementary property (see [10]):

Definition 0.1. A representation $\rho : \pi_1(\Sigma) \to G$ is *weakly maximal* if it satisfies

$$T(\rho) = ||\rho^*\kappa_G^{\mathrm{b}}|| \cdot |\chi(\Sigma)|. \tag{2}$$

By definition a representation is maximal iff it is weakly maximal and tight.

Various general structure theorems for maximal representations have been established in [6]. An essential part of these structure theorems holds similarly for weakly maximal representations. For example:

Theorem 0.2. *Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal representation and $T(\rho) \neq 0$. Then ρ is faithful with discrete image.*

An important step in the proof of Theorem 0.2 is the realization that a representation ρ is weakly maximal iff there exists $\lambda \geq 0$ such that

$$\rho^*\kappa_G^{\mathrm{b}} = \lambda \cdot \kappa_\Sigma^b, \tag{3}$$

where $\kappa^b_\Sigma \in H^2_b(\Gamma)$ is the bounded fundamental class of the surface Σ as introduced in [6]. We prove in [1] that the constant λ has in fact to be *rational*. This provides severe restrictions on the kernel and range of ρ.

Causal structures

It turns out that techniques from [3] can be used to provide a geometric characterization of weakly maximal representations with nonzero Toledo invariant in terms of bi-invariant orders. To simplify the formulation we will only spell out the results in the case where the target group G is of *tube type*. We will also assume that G is *adjoint simple.*

We now fix an adjoint simple Hermitian Lie group G of tube type and denote by $\widehat{G} = \widetilde{G}/\pi_1(G)^{tor}$ the unique central $\mathbb{Z}$-extension of G. Then causal geometry gives rise to a bi-invariant partial order on $\widehat{G}$ (see [2] for a discussion of this and various related bi-invariant partial orders on Lie groups). A prototypical example arises from the action of $G = \mathrm{PU}(1,1)$ on the boundary of the Poincaré disk $\mathbb{D}$; this action lifts to an action of the universal covering $\widehat{G} = \widetilde{\mathrm{PU}(1,1)}$ on $\mathbb{R}$, hence induces a bi-invariant partial order on $\widehat{G}$ by setting

$$g \leq h :\Leftrightarrow \forall x \in \mathbb{R} :\ g.x \leq h.x.$$

In the general case one utilizes the fact that by the tube type assumption there exists a unique pair $\pm\mathcal{C}$ of G-invariant causal structures on the Shilov boundary $\check{S}$ of the bounded symmetric domain associated with G (see [9]). Here, by a causal structure $\mathcal{C}$ we mean a family of *closed* cones $\mathcal{C}_x \subset T_x\check{S}$ with non-empty interior, and invariance is understood in the sense that $g_*\mathcal{C}_x = \mathcal{C}_{gx}$. The causal structures $\pm\mathcal{C}$ lift to $\widehat{G}$-invariant causal structures on the universal covering $\check{R}$ of $\check{S}$, which in turn induce a pair of mutually inverse (closed) partial orders on $\check{R}$ via causal curves. Let us denote by $\preceq$ the partial order which is compatible with the orientation given by the Kähler class. We then obtain a bi-invariant partial order on $\widehat{G}$ by setting

$$g \leq_{\widehat{G}} h :\Leftrightarrow \forall x \in \check{R} :\ g.x \preceq h.x.$$

The *dominant set* $\widehat{G}^{++}$ (in the sense of [7, 3]) of this bi-invariant order is given by the formula

$$\widehat{G}^{++} := \{g \in \widehat{G} \,|\, \forall h \in G \exists n \in \mathbb{N} :\ g^n \geq_{\widehat{G}} h\},$$

We provide the following simple description in terms of the causal structure:

Theorem 0.3. *If $\widehat{G}$ is of tube type then*

$$\widehat{G}^{++} = \{g \in \widehat{G} \,|\, \forall x \in \check{R} :\ g.x \succ x\}.$$

We now provide an interpretation of weakly-maximal representations in terms of dominant sets. Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with n boundary components. We always assume that $\chi(\Sigma)_{g,n} < 0$ so that there exists a hyperbolization $\rho : \Gamma_{g,n} := \pi_1(\Sigma_{g,n}) \to \mathrm{PU}(1,1)$. If $n \geq 1$, then $\Gamma_{g,n}$ is a free group, hence ρ admits a lift $\widetilde{\rho} : \Gamma_{g,n} \to \widetilde{\mathrm{PU}(1,1)}$ whose restriction to the group of homologically trivial loops $\Lambda_{g,n} := [\Gamma_{g,n}, \Gamma_{g,n}]$ is unique. In particular, the translation number quasimorphism on $\widetilde{\mathrm{PU}(1,1)}$ pulls back to a quasimorphism $f_{\Sigma_{g,n}}$ on $\Lambda_{g,n}$. It turns out that this quasimorphism is independent of the choice of hyperbolization ρ; in fact it admits a topological description in terms of winding numbers [8]. In the case in which $n = 0$, one cannot perform this construction on $\Gamma_{g,0}$, but one has to pass to the central extension $\overline{\Gamma_{g,0}}$ that corresponds to the generator of $\mathrm{H}^2(\Gamma_{g,0}, \mathbb{Z})$ or, equivalently, can be realized as the fundamental group of the S^1-bundles over Σ_g of Euler number one. One then obtains in the same way as above a canonical quasimorphism $f_{\Sigma_{g,0}}$ on $\Lambda_{g,0} := [\overline{\Gamma}_{g,0}, \overline{\Gamma}_{g,0}]$. We emphasize that the quasimorphism $f_{\Sigma_{g,n}}$ depends on the topological surface $\Sigma_{g,n}$, not just the abstract group $\Gamma_{g,n}$.

Theorem 0.4. *Let G be an adjoint simple Hermitian Lie group of tube type and let $\widehat{G}$, $\widehat{G}^{++}$ as above. Let $\Sigma_{g,n}$ be a surface of negative Euler characteristic and $\Gamma_{g,n} := \pi_1(\Sigma_{g,n})$. Then a representation $\rho : \Gamma_{g,n} \to G$ is weakly maximal with $T(\rho) \neq 0$ iff for the unique lift $\widetilde{\rho} : \Lambda_{g,n} \to \widehat{G}$ there exists $N > 0$ such that*

$$f_{\Sigma_{g,n}}(\gamma) > N \Rightarrow \widetilde{\rho}(\gamma) \in \widehat{G}^{++} \quad (\gamma \in \Lambda_{g,n}). \tag{4}$$

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Dynamics at infinity of discrete subgroups of isometries of higher rank symmetric spaces

Joan Porti
(joint work with M. Kapovich and B. Leeb)

Let $X = G/K$ be a symmetric space of noncompact type. For a discrete subgroup $\Gamma \subset G$, we look for domains of discontinuity of Γ in G-orbits in the geometric boundary $\partial_\infty X$, if possible cocompact. When X has rank one, then G acts transitively on $\partial_\infty X$ and $\Omega = \partial_\infty X - \Lambda_\Gamma$ is a domain of discontinuity for Γ, where $\Lambda_\Gamma = \overline{\Gamma x} \cap \partial_\infty X$ denotes the limit set. Moreover, if Γ is convex cocompact, then the action of Γ on Ω is cocompact.

When G has higher rank, the situation is more involved. A theorem of Karlsson [2] for CAT(0) spaces asserts that the action of Γ is properly discontinuous on $\partial_\infty X - \overline{\mathcal{N}_{\pi/2}(\Lambda_\Gamma)}$. Our aim is to improve this result by using the Tits building structure of $\partial_\infty X$, considering the action on G-orbits and, in addition, discussing cocompactness. The idea will be to remove the analog of a tubular neighborhood of the limit set.

To simplify, we assume that Γ satisfies a regularity assumption, so that one can define "chamber convergence" and there is a well defined chamber limit set $\Lambda_{\mathrm{Ch}}(\Gamma) \subset \partial_F X$, where $\partial_F X$ denotes the Fürstenberg boundary of X. This can be relaxed so that one can define a limit set on other partial flag manifolds. For a chamber $\sigma \in \partial_F X$, one defines a relative position

$$\mathrm{pos}(\cdot, \sigma) : \partial_\infty X \to a^{mod},$$

where a^{mod} denotes the model apartment. When restricted to a G-orbit, $\mathrm{pos}(\cdot,\sigma)$ takes values on a W-orbit of the model apartment. Its level sets are the so-called Schubert cells, whose closure are Schubert cycles and define a partial ordering on W-orbits, by inclusion. More precisely, for $\xi_1, \xi_2 \in W\xi$, we say that $\xi_1 \leq \xi_2$ when $\{\mathrm{pos}(\cdot,\sigma) = \xi_1\} \subset \overline{\{\mathrm{pos}(\cdot,\sigma) = \xi_2\}}$.

A thickening is a subset $\mathrm{Th} \subset W\xi$ closed by this partial ordering: if $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathrm{Th}$, then $\xi_1 \in \mathrm{Th}$, so that $\{\mathrm{pos}(\cdot,\sigma) \in \mathrm{Th}\}$ is a closed subset of the G-orbit at $\partial_\infty X$. Let $w_0 \in W$ denote the longest element for the Bruhat order. We say that Th is *fat* if $W\xi = \mathrm{Th} \cup w_0\mathrm{Th}$, *slim* if $\mathrm{Th} \cap w_0 \mathrm{Th} = \emptyset$, and *balanced* if it is both slim and fat.

Theorem 1. *Let Γ be a regular discrete subgroup of G and $Th \subset W\bar{\xi}$ a thickening. Then $G\xi - \bigcup_{\sigma \in \Lambda_{Ch}(\Gamma)} \{\xi \mid \mathrm{pos}(\xi,\sigma) \in Th\}$ is:*

- *a domain of discontinuity for Γ if Th is fat;*
- *Γ-cocompact if Th is slim and Γ is RCA.*

We say that Γ is RCA if it is regular, chamber conical (limit chambers can be approached by orbit points in a tubular neighborhood of sectors) and antipodal (different chambers in $\Lambda_{\mathrm{Ch}}(\Gamma)$ are antipodal).

We can show that regular orbits always have balanced thickenings. Furthermore, this domain is non-empty for all irreducible types other than B_2 and G_2.

We also show that a group is RCA if and only if it is a hyperbolic group that is Anosov in the sense of Labourie [3] and Guichard-Wienhard [4].

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Complete affine manifolds: a survey

William M. Goldman

An *affinely flat manifold* (or just *affine manifold*) is a manifold with a distinguished coordinate atlas with locally affine coordinate changes. Equivalently M is a manifold equipped with an affine connection with vanishing curvature and torsion. A *complete affine manifold* M is a quotient E/Γ where $\Gamma \subset \mathsf{Aff}(E)$ is a discrete group of affine transformations acting properly on E. This is equivalent to geodesic completeness of the connection. In this case, the universal covering space of M is affinely diffeomorphic to E, and the group $\pi_1(M)$ of deck transformations identifies with the affine holonomy group Γ.

Flat Riemannian manifolds are special cases where Γ is a group of Euclidean isometries. The classical theorems of Bieberbach provide a very satisfactory picture of such structures: every compact flat Riemannian manifold is finitely covered by a flat torus E/Λ where $\Lambda \subset G$ is a lattice in the group G of translations of E. Furthermore every *complete* flat Riemannian manifold is a flat orthogonal vector bundle over its *soul,* a totally geodesic flat Riemannian manifold. (See, for example, Wolf [29].)

An immediate consequence is $\chi(M) = 0$ if M is compact (or even if Γ is just nontrivial). This follows immediately from the intrinsic Gauß-Bonnet theorem of Chern [11], who conjectured that the Euler characteristic of a closed affine manifold vanishes. (Chern-Gauß-Bonnet applies only to *orthogonal connections* and not to linear connections.) In this generality, Chern's conjecture remains unsolved.

Affine manifolds are considerably more complicated than Riemannian manifolds, where metric completeness is equivalent to geodesic completeness. In particular, simple examples such as a Hopf manifold $\mathbb{R}^n \setminus \{0\}/\langle\gamma\rangle$, where γ is a linear expansion of $\mathbb{R}^n$ illustrate that closed affine manifolds need not be complete. For this reason we restrict only to *geodesically complete manifolds.*

Kostant and Sullivan [20] proved Chern's conjecture when M is complete. In other directions, Milnor [24] found flat oriented $\mathbb{R}^2$-bundles over surfaces with nonzero Euler class. Using Milnor's examples, Smillie [26] constructed flat affine connections on some manifolds of nonzero Euler characteristic. (Although the curvature vanishes, it seems hard to control the torsion.)

Auslander's flawed proof [4] of Kostant-Sullivan still contains interesting ideas. Auslander claimed that every closed complete affine manifold is finitely covered by a *complete affine solvmanifold* G/Γ, where $G \subset \mathsf{Aff}(E)$ is (necessarily solvable) closed subgroup of affine automorphisms of E. This generalizes Bieberbach's structure theorem for flat Riemannian manifolds. Whether every closed complete affine manifold has this form is a fundamental question in its own right, and this question is now known as the *"Auslander Conjecture."* ([16]). It has now been established in all dimensions $n < 7$ by Abels-Margulis-Soifer [2, 3].

Milnor's paper [25] clarified the situation. Influenced by Tits [28] he asked whether any discrete subgroup of $\mathsf{Aff}(E)$ which acts properly on E must be virtually polycyclic. If so then complete affine manifolds admit a simple structure, and can be classified by techniques similar to the Bieberbach theorems. Tits's theorem implies that either Γ is virtually polycyclic or it must contain a free subgroup of rank two. Thus Milnor's question is equivalent to whether $\mathbb{Z} \star \mathbb{Z}$ admits a proper affine action.

Margulis [21, 22] showed that indeed nonabelian free groups can act properly and affinely on affine spaces of all dimensions > 2. In dimension 3, Fried-Goldman [16] showed that if $\Gamma \subset \mathsf{Aff}(E)$ is discrete and acts properly, then either Γ is polycyclic *or* the linear holonomy homomorphism L maps Γ faithfully onto a discrete subgroup of a subgroup conjugate to the special orthogonal group $\mathsf{SO}(2,1) \subset \mathsf{GL}(3,\mathbb{R})$. In particular $\Sigma := \mathsf{H}^2/L(\Gamma)$ is a complete hyperbolic surface homotopy-equivalent to $M^3 = E/\Gamma$.

Already this implies Auslander's Conjecture in dimension 3: Since M^3 is closed, Γ has cohomological dimension 3, contradicting Γ being the fundamental group of a surface Σ. Much deeper is the fact that Σ cannot be closed (Mess [23]). Therefore Γ must itself be a free group.

Since Margulis's examples admit complete flat Lorentzian metrics, quotients E/Γ where Γ is free of rank > 2, have been called *Margulis spacetimes.*

Which groups admit proper affine actions in higher dimension remains an intriguing and mysterious question. The Bieberbach theorems imply that any discrete group of Euclidean isometries is finitely presented. The class of properly acting discrete affine groups contains $\mathbb{Z} \star \mathbb{Z}$, and is closed under Cartesian products and taking subgroups. Thus properly discontinuous affine groups needn't be finitely generated, and even finitely generated properly discontinuous affine groups needn't admit finite presentations (Stallings [27]). *The only hyperbolic groups known to admit proper affine actions are free.*

In his 1990 doctoral thesis [14], Drumm gave a geometric proof of Margulis's result and sharpened it. Using polyhedral hypersurfaces in $\mathbb{R}^3$ called *crooked planes,* he built fundamental polyhedra for proper affine actions of discrete groups. Therefore his examples are homeomorphic to solid handlebodies. (This has been recently proved in general, for convex cocompact $L(\Gamma)$, by Choi-Goldman [12] and Danciger-Guéritaud-Kassel [13] independently.)

Using crooked planes, Drumm [15] showed that Mess's theorem is the *only* obstruction for the existence of a proper affine deformation: *Every* noncompact hyperbolic surface admits a proper affine deformation with a fundamental polyhedron bounded by crooked planes. Using much different dynamical methods, Goldman-Labourie-Margulis [18] identify the space of proper affine deformations of a convex cocompact Fuchsian group as an open convex cone in a vector space.

Our joint work [6, 7, 5, 8] with Charette and Drumm classifies Margulis spacetimes where $\Gamma \cong \mathbb{Z} \star \mathbb{Z}$ using crooked planes. Recently Danciger-Guéritaud-Kassel announced that *every* Margulis spacetime with convex cocompact $L(\Gamma)$ admits a crooked fundamental polyhedron.

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The geometry of the initial singularity of space-times of constant curvature

Thierry Barbot

A lorentz metric g on a manifold M is a pseudo-Riemannian metric of signature $(-,+,...,+)$. We denote by n the number of $+$, so that M has dimension $n+1$. An immersed curve $c : I \to M$ is *time-like* (respectively *causal*) if the g-norm of every $\dot{c}(t)$ is negative (respectively non positive).

At every point p, the set of time-like vectors in T_pM has two connected component; a *time orientation* on M is a selection of one of these connected components continuous in p. It is tantamount to the choice of a time-like everywhere continuous vector field ξ. A *space-time* is a time oriented lorentz manifold.

On a given space-time (M,g) one can define a partial order - the *causal order* $\preceq$ on M, where $p \preceq q$ means that there is a future causal curve starting from p and finishing at q.

The causality theory is nowadays well developed, with contributions by several authors; let us mention here Kronheimer, Penrose ([11]), Harris ([8]), Bernal, Sanchez ([5]). A crucial notion in this theory is the notion of *global hyperbolicity,* which emerged from General Relativity, ie. the study of Einstein Equations. It has been introduced by J. Leray ([9]), and developed by Choquet-Bruhat ([6]) and Geroch ([7]). A particularly interesting case is the case of *spatially compact* globally hyperbolic space-times (abbrev. GHC). A brief and accurate way to define GHC space-times is to characterize them as space-times admitting a time function whose level sets are all compact.

In the 90's, G. Mess classified GHC space-times of dimension $2+1$ of constant curvature ([10]) which are *maximal*, ie. cannot be extended to a bigger GHC spacetimes of constant curvature. His main discovery is a 1-1 correspondence between maximal GHC space-times of dimension $2+1$ and measured geodesic laminations on closed surfaces. This work has then been extended in any dimension ([12], [1]).

The main focus of the talk is to present a recent result by M. Belraouti ([2]): let $T : M \to (0, +\infty)$ be a time function on a globally hyperbolic flat space-time of dimension $2+1$. Assume that T is in expansion, ie. that every level set of T is convex. Then, when $t \to 0$, the level sets $T^{-1}(t)$ converge in the Gromov equivariant topology to the real tree dual to the measured geodesic lamination defining M. This result answers a question by R. Benedetti and R. Guadagnini ([4]).

In higher dimensions, there is a similar result ([3]): the level sets still converge to some CAT(0) space, but which in general may not be a real tree.

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Lorentz manifolds with noncompact isometry group

Charles Frances

It is a vague general principle, stated for instance in [3] and [2], that rigid geometric structures having a large group of automorphisms, for instance a noncompact group when the underlying manifold is compact, must be peculiar enough to be classified. Pseudo-Riemannian structures are examples of rigid structures on which the principle can be tested and illustrated. Because the isometry group of a compact Riemannian manifold is a compact Lie transformation group (this is a theorem of Myers and Steenrod proved in [4]), the first signature which is interesting for our purpose is the Lorentzian one. And indeed, some nice compact Lorentz manifolds may admit a noncompact group of isometries. For instance, if $\Sigma = \mathbb{H}^2/\Gamma$ is a complete hyperbolic surface, there is on the unit tangent bundle $T^1(\Sigma)$ a natural Lorentz metric of constant curvature -1 which is preserved both by the geodesic and the horocyclic flows. Also, the flow obtained by suspending a hyperbolic linear transformation on the 2-torus preserves a (flat) Lorentz metric on the associated solmanifold. This is a first hint of the richness that Lorentz dynamics can display.

In [5], A. Zeghib obtained a complete classification of 3-dimensional compact Lorentz manifolds (M^3, g) admitting a noncompact isometry group. Actually, Zeghib's assumption was that the *connected component of the identity* $\mathrm{Iso}^o(M, g)$ was noncompact. From the classification he gave, we will only retain the following relevant consequence:

Theorem ([5]) *Let (M, g) be a compact, 3-dimensional Lorentz manifold. If the group $Iso^o(M, g)$ is noncompact, then (M, g) is locally homogeneous.*

Zeghib's conclusion can be compared to a result proved by Gromov in [3], saying that whenever the automorphism group of a rigid geometric A-structure (for instance a Lorentz manifold) has a dense orbit, then the structure must be locally homogeneous on a dense open set.

One can wonder if Zeghib's result remains true under the weaker assumption that $\mathrm{Iso}(M, g)$ is noncompact. Indeed, one could imagine that the isometry group $\mathrm{Iso}(M, g)$ is noncompact, still having a compact identity component (for instance, $\mathrm{Iso}(M, g)$ may be infinite discrete). One interesting, and maybe rather unexpected, feature is that new phenomena can happen when the isometry group is noncompact with a compact $\mathrm{Iso}^o(M, g)$. For instance, we give examples of Lorentz metrics on the 3-torus having a noncompact isometry group, and which are locally homogeneous on no open subset of $\mathbb{T}^3$. Also, it is possible to build more complicated examples (still with a noncompact isometry group), where infinitely many open subsets of $\mathbb{T}^3$ are locally homogeneous, but pairwise locally nonisometric, and other subsets are not locally homogeneous. Those more complicated examples are smooth, but not analytic. In the analytic case, we show that the possibilities are more restricted, and prove the following:

Theorem *Let (M, g) be a compact, 3-dimensional Lorentz manifold. If the group $Iso(M, g)$ is noncompact, then we are in exactly one of the following situations.*

(1) *The manifold (M, g) is locally homogeneous.*
(2) *Around each point of M, the Lie algebra of local Killing vector fields is isomorphic to the 3-dimensional algebra $\mathfrak{sol}$. The $Kill^{loc}$-orbits are flat Lorentz tori. Up to finite index, $Iso(M, g)$ is a semi-direct product of $\mathbb{Z}$ and $\mathbb{T}^2$.*

(3) *Around each point of M, the Lie algebra of local Killing vector fields is isomorphic to the Heisenberg algebra $\mathfrak{heis}(3)$. The $Kill^{loc}$-orbits are degenerate tori. Up to finite index, $Iso(M,g)$ is a semi-direct product of $\mathbb{Z}$ and $\mathbb{T}^2$.*

One deduces from the theorem that 3-dimensional compact analytic Lorentz manifolds, the fundamental group of which is not solvable, must have a compact isometry group.

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Curvature Decay Estimate of Graphical Mean Curvature Flow in Higher Codimensions

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(joint work with Knut Smoczyk, Mu-Tao Wang)

This report gives a short account of results on curvature decay estimate of graphical mean curvature flow obtained in [2]. Let Σ_1 and Σ_2 be two compact Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product manifold. We consider a smooth map $f : \Sigma_1 \to \Sigma_2$ and denote the graph of f by Σ; Σ is a submanifold of M by the embedding $id \times f$. We study the deformation of f by the mean curvature flow. The idea is to deform Σ along its mean curvature vector in M with the hope that Σ will remain a graph. This is the negative gradient flow of the volume functional and a stationary point is a "minimal map" introduced by Schoen in [1].

To describe previous results, we recall the differential of f, df, at each point of Σ_1 is a linear map between the tangent spaces. The Riemannian structures enables us to define the adjoint of df. Let $\{\lambda_i\}$ denote the eigenvalues of $\sqrt{(df)^T df}$, or the singular values of df, where $(df)^T$ is the adjoint of df. Note that λ_i is always nonnegative. We say f is an *area decreasing map* if $\lambda_i\lambda_j < 1$ for any $i \neq j$ at each point. In particular, f is area decreasing if df has rank one everywhere.

In [3], we prove that the area decreasing condition is preserved along the mean curvature flow and the following global existence and convergence theorem.

Theorem A. *Let Σ_1 and Σ_2 be compact Riemannian manifolds of constant curvature k_1 and k_2 respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 \geq 0$ and $dim(\Sigma_1) \geq 2$. If*

f is a smooth area decreasing map from Σ_1 to Σ_2, the mean curvature flow of the graph of f remains the graph of an area decreasing map and exists for all time. Moreover, if $k_1 + k_2 > 0$ then it converges smoothly to the graph of a constant map.

In general, the global existence and convergence of a mean curvature flow relies on the boundedness of the second fundamental form. In the above theorem, the boundedness of the second fundamental form is obtained by an indirect blow-up argument (see [3, 4, 5]). While the idea of the proof of convergence is to use the positivity of $k_1 + k_2$ (or k_1) to show that the gradient of f is approaching zero, which in turn gives the boundedness of the second fundamental form when the flow exists for sufficiently long time. In the following theorem, we prove pointwise estimates without making any smallness assumption. As a result, the convergence result when $\Sigma_1 = \Sigma_2 = T^2$ follows. We have the following results.

Theorem B([2]) (Joint work with Mu-Tao Wang and Knut Smoczyk)
Suppose Σ is the graph of a area decreasing map $f : T^2 \to T^2$ as a submanifold of $M = T^2 \times T^2$ and Σ^t is the mean curvature flow with initial surface $\Sigma^0 = \Sigma$. Then Σ^t remains the graph of an area decreasing map f_t along the mean curvature flow. The flow exists smoothly for all time and Σ^t converges smoothly to a totally geodesic submanifold as $t \to \infty$. Moreover, we have the following mean curvature decay estimate

$$t|H|^2 \leq \frac{1}{\alpha}$$

where $\alpha = inf_{t=0} \frac{(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} > 0$. If Σ is a Lagrangian submanifold in $T^2 \times T^2$ then a stronger estimate can be obtained:

$$t|A|^2 \leq 8(\frac{1}{\alpha^2} + \frac{64^2}{\alpha^5}).$$

We will briefly sketch the proof of the mean curvature decay estimate. Let $u = \frac{(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}$. Then u satisfies the following differential inequality

$$(\frac{d}{dt} - \Delta)\ln u \geq 2|A|^2 + \frac{|\nabla \ln u|^2}{2}.$$

On the other hand, we can also derive

$$(\frac{d}{dt} - \Delta)\ln(t|H|^2 + 1) \leq 2|A|^2 + \frac{|\nabla \ln(t|H|^2+1)|^2}{2}.$$

Combining these two inequalities, we have

$$(\frac{d}{dt} - \Delta)\ln\Big(\frac{t|H|^2+1}{u}\Big) \leq \frac{\nabla \ln(\frac{t|H|^2+1}{u}) \cdot \nabla \ln\Big(u(t|H|^2+1)\Big)}{2}.$$

By the maximum principle, we have $\frac{t|H|^2+1}{u} \leq sup_{t=0}\frac{1}{u}$. Thus $\frac{t|H|^2}{u} \leq sup_{t=0}\frac{1}{u}$ and

$$t|H|^2 \leq \frac{u}{inf_{t=0}u} \leq \frac{1}{\alpha}$$

where $\alpha = inf_{t=0}\frac{(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} > 0$. Here we have also used the fact that $u = \frac{(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} > 0$ is preserved by MCF and $u \leq 1$.

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Smoothing singular extremal Kähler surfaces and minimal Lagrangians

Yann Rollin

(joint work with Olivier Biquard)

We consider smoothings of a complex surface with singularities of class T and no nontrivial holomorphic vector field. Under a hypothesis of non degeneracy of the smoothing at each singular point, we prove that if the singular surface admits an extremal metric, then the smoothings also admit extremal metrics in nearby Kähler classes.
In addition, we construct small Lagrangian stationary spheres which represent Lagrangian vanishing cycles for surfaces close to the singular one.

Min-max theory in Geometry

André Neves

(joint work with Fernando Marques)

Min-max theory was first used in Geometry by Birkhoff in the 20's to show that every sphere admits a closed embedded geodesic. Since then the technique was explored to show that every sphere admits three closed embedded geodesics (Lusternick and Shnirelmann), that every manifold of dimension no bigger than eight admits a smooth embedded minimal hypersurface (Pitts and Schoen-Simon for the regularity), and that every 3-sphere admits an embedded minimal sphere (Simon-Smith).

Recently, Fernando and I used this technique to prove the Willmore conjecture and, with Agol, we also used this technique to solve a conjecture regarding two component links with least Mobius energy.

In the end, I mentioned my new result with Fernando Marques, where we show that manifolds with dimension no bigger than eight having a metric of positive Ricci curvature, admit an infinite number of minimal embedded hypersurfaces.

Mean curvature flow without singularities

Oliver Schnürer
(joint work with Mariel Sáez)

We study the evolution of complete graphs under mean curvature flow. This is illustrated by three examples:

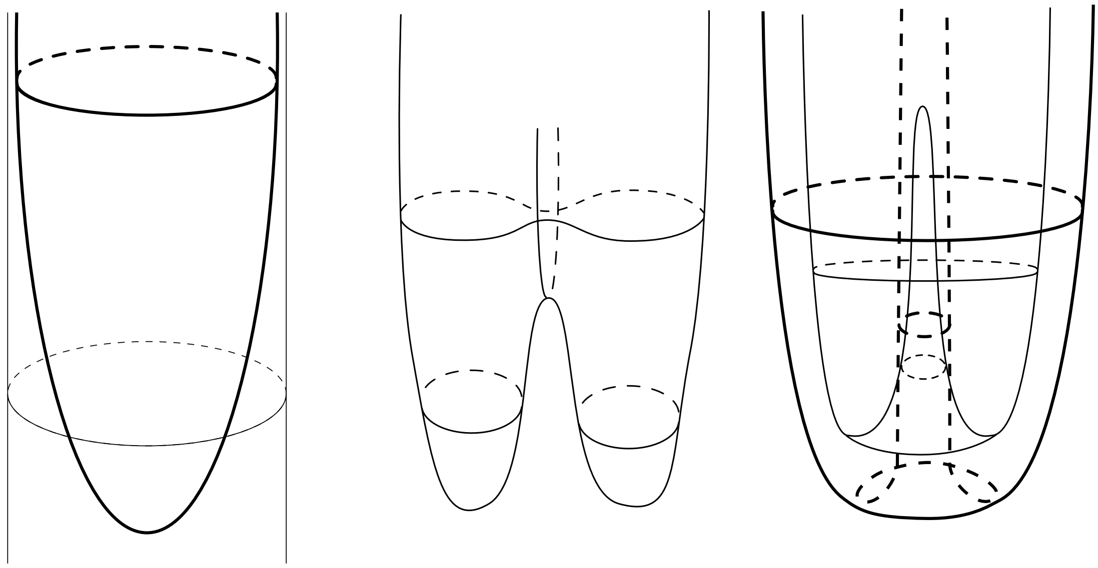

Figure 1. Examples of evolution

(1) Picture on the left: A graph (thick) inside a cylinder (thin) disappears to infinity at the time the cylinder collapses.
(2) Picture in the middle: The middle part of the 4-dimensional graph disappears to infinity and avoids the formation of a neck-pinch.
(3) Picture on the right: Before the cylinder inside the surface (thick) degenerates to a line, a "cap at infinity" is being added to the surface that moves downwards very quickly. The thin surface depicts the surface shortly after that.

If $u_0\colon \Omega_0 \to \mathbb{R}$ is locally Lipschitz, defined on a bounded domain $\Omega_0 \subset \mathbb{R}^{n+1}$, $u_0(x) \to \infty$ as $x \to \partial\Omega_0$, and u_0 is bounded below, then there exists a maximal smooth solution u to graphical mean curvature flow with initial value u_0.

The orthogonal projections $\mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ of the evolving graphs yield a level set solution to mean curvature flow.

This allows to define weak solutions as projections of classical graphical solutions.

Critical metrics on connected sums of Einstein four-manifolds

Jeff A. Viaclovsky
(joint work with Matthew J. Gursky)

A Riemannian manifold (M^4, g) in dimension four is critical for the Einstein-Hilbert functional

$$\mathcal{R}(g) = Vol(g)^{-1/2} \int_M R_g dV_g, \tag{1}$$

where R_g is the scalar curvature, if and only if it satisfies

$$Ric(g) = \lambda \cdot g, \tag{2}$$

where λ is a constant; such Riemannian manifolds are called *Einstein manifolds*. Non-collapsing limits of Einstein manifolds have been studied in great depth. In particular, with certain geometric conditions, the limit space is an orbifold, with asymptotically locally Euclidean (ALE) spaces bubbling off at the singular points. A natural question is whether it is possible to reverse this process: can one start with the limit space, and glue on a bubble in order to obtain an Einstein metric? A recent article of Olivier Biquard makes great strides in the Poincaré-Einstein setting [Biq11]. In this work it is shown that a $\mathbb{Z}/2\mathbb{Z}$-orbifold singularity p of a non-degenerate Poincaré-Einstein orbifold (M, g) has a Poinaré-Einstein resolution obtained by gluing on an Eguchi-Hanson metric if and only if the condition

$$\det(\mathbf{R}^+(p)) = 0 \tag{3}$$

is satisfied, where $\mathbf{R}^+(p) : \Lambda^2_+ \to \Lambda^2_+$ is the purely self-dual part of the curvature operator at p. The self-adjointness of this gluing problem is overcome by the freedom of changing the boundary data of the Poincaré-Einstein metric.

However, there is not much known about gluing compact manifolds together in the Einstein case. In this work, we will replace the Einstein equations with a generalization of the Einstein condition. Namely, we ask whether it is possible to glue together Einstein metrics and produce a critical point of a certain Riemannian functional generalizing the Einstein-Hilbert functional. It turns out that there is a family of such functionals; this gives an extra parameter which will allow us to overcome the self-adjointness of this problem. The particular functional will then depend on the global geometry of the gluing factors.

To describe the functionals, let M be a closed manifold of dimension 4. We will consider functionals on the space of Riemannian metrics $\mathcal{M}$ which are quadratic in the curvature. Such functionals have also been widely studied in physics under the name "fourth-order," "critical," or "quadratic" gravity. In previous work, the authors have studied rigidity and stability properties of Einstein metrics for quadratic curvature functionals [GV11]; those results play a crucial role in this work.

Using the standard decomposition of the curvature tensor Rm into the Weyl, Ricci and scalar curvature components (denoted by W, Ric, and R, respectively), a basis for the space of quadratic curvature functionals is

$$\mathcal{W} = \int |W|^2 \, dV, \quad \rho = \int |Ric|^2 \, dV, \quad \mathcal{S} = \int R^2 \, dV, \tag{4}$$

where we use the tensor norm. In dimension four, the Chern-Gauss-Bonnet formula

$$32\pi^2\chi(M) = \int |W|^2 \, dV - 2\int |Ric|^2 \, dV + \frac{2}{3}\int R^2 \, dV \tag{5}$$

implies that ρ can be written as a linear combination of the other two (plus a topological term). Consequently, we will be interested in the functional

$$\mathcal{B}_t[g] = \int |W|^2 \, dV + t\int R^2 \, dV \tag{6}$$

(with $t = \infty$ formally corresponding to $\int R^2 dV$).

The Euler-Lagrange equations of $\mathcal{B}_t$ are given by

$$B^t \equiv B + tC = 0, \tag{7}$$

where B is the *Bach tensor* defined by

$$B_{ij} \equiv -4\Big(\nabla^k\nabla^l W_{ikjl} + \frac{1}{2}R^{kl}W_{ikjl}\Big) = 0, \tag{8}$$

and C is the tensor defined by

$$C_{ij} = 2\nabla_i\nabla_j R - 2(\Delta R)g_{ij} - 2RR_{ij} + \frac{1}{2}R^2 g_{ij}. \tag{9}$$

It follows that any Einstein metric is critical for $\mathcal{B}_t$. We will refer to such a critical metric as a *B^t-flat metric.* Note that by taking a trace of (7), it follows that the scalar curvature of a B^t-flat metric on a compact manifold is necessarily constant. Therefore a B^t-flat metric satisfies the equation

$$B = 2tR \cdot E, \tag{10}$$

where E denotes the traceless Ricci tensor. That is, the Bach tensor is a constant multiple of the traceless Ricci tensor.

Our main theorem involving the existence of critical metrics is the following:

Theorem 1 (Gursky-Viaclovsky [GV13])**.** *A B^t-flat metric exists on the manifolds in the table for some t near the indicated value of t_0.*

Topology of connected sum	Value(s) of t_0
$\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$	$-1/3$
$S^2 \times S^2\#\overline{\mathbb{CP}}^2 = \mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$	$-1/3$, $-(9m_1)^{-1}$
$2\#S^2 \times S^2$	$-2(9m_1)^{-1}$

The constant m_1 is a geometric invariant called the *mass* of a certain asymptotically flat metric: the Green's function metric of the product metric $S^2 \times S^2$.

Employing various symmetries, it is possible to produce many more examples on other 4-manifolds. For more examples, and details of the proof, we refer the reader to [GV13].

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Group actions on quasi-trees

Koji Fujiwara

(joint work with Mladen Bestvina, Ken Bromberg)

Let X be a geodesic space with a base point x. Let g be an isometry of X. We say g is *hyperbolic* if there is $C > 0$ such that for all n, $d(x, g^n(x)) \geq C|n|$. Let O_x be the orbit of x by g, namely, $\langle g \rangle(x)$. Let $\pi : X \to O_x$ be the nearest points projection. Maybe the image of a point is more than one point. We say O_x is D-*strongly-contracting* if for any metric ball $B \subset X$ disjoint from O_x, $diam(\pi(B)) \leq D$. We say g is *strongly-contracting* if O_x is D-strongly-contracting for some D, [2]. One can show that it does not depend on the choice of x (maybe the constant D does).

Here are some examples. If X is δ-hyperbolic, then any hyperbolic isometry is strongly-contracting. If X is a proper CAT(0) space, then a hyperbolic isometry is rank-1 (in the sense of Ballmann) iff it is strongly-contracting. Moreover, if X is the Teichmuller space of the Teichmuller metric, then a pseudo-Anosov map is hyperbolic and strongly contracting (Minsky).

Now I discuss an application. A *quasi-tree* is a graph which is quasi-isometric to a simplicial tree. In [1] we studied group actions on quasi-trees and found interesting applications. For example we showed that mapping class groups have finite asymptotic dimension. In [1] we obtained a set of conditions (or Axioms) from which we can produce quasi-trees and group actions on them. It gives many natural examples for hyperbolic groups, mapping class groups and the outer automorphism groups of free groups.

The theorem I discuss in the talk from [2] is that if G acts on a geodesic space X with a hyperbolic and strongly contracting element g, and moreover if g is a "weakly proper" element then G acts on a quasi-tree with g a hyperbolic and weakly proper element. This gives a unified approach to the examples in the previous paragraph. Once we have such an action, the results in [1] and [2] will apply to G.

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Weak global notions of nonpositive curvature

Urs Lang

By a *geodesic bicombing* σ on a metric space (X, d) we mean a map

$$\sigma\colon X \times X \times [0,1] \to X$$

such that for every pair $(x, y) \in X \times X$, the map $\sigma_{xy} := \sigma(x, y, \cdot)\colon [0,1] \to X$ is a constant speed geodesic with $\sigma_{xy}(0) = x$ and $\sigma_{xy}(1) = y$. We call a geodesic bicombing σ on X

- *convex* if the function $s \mapsto d(\sigma_{xy}(s), \sigma_{x'y'}(s))$ is convex on $[0,1]$ for all $x, y, x', y' \in X$;
- *conical* if $d(\sigma_{xy}(s), \sigma_{x'y'}(s)) \leq (1-s)d(x, x') + sd(y, y')$ for all $x, y, x', y' \in X$ and $s \in [0,1]$;
- *consistent* if $\sigma_{x'y'}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$ whenever $x, y \in X$, $0 \leq s < t \leq 1$, $x' := \sigma_{xy}(s)$, $y' := \sigma_{xy}(t)$, and $\lambda \in [0,1]$;

Convex geodesic bicombings are conical, and every conical and consistent geodesic bicombing is convex. However, there exist geodesic bicombings that are conical but not convex. If X is a linearly convex subset of a normed space, then $\sigma_{xy}(s) := (1-s)x + sy$ defines a convex and consistent geodesic bicombing on X. If $\bar{X}$ is a metric space with a conical geodesic bicombing $\bar{\sigma}$, and if $\rho\colon \bar{X} \to X$ is a 1-Lipschitz retraction onto some subspace $X \subset \bar{X}$, then $\sigma := \rho \circ \bar{\sigma}|_{X \times X \times [0,1]}$ is a conical geodesic bicombing on X.

The existence of a convex or conical geodesic bicombing on a metric space X may be seen as weak global nonpositive curvature condition. This leads to the following hierarchy of properties for a geodesic metric space X:

(A) X is a CAT(0) space;
(B) X is globally convex in the sense of Busemann, that is, for every pair of constant speed geodesics $\sigma, \tau\colon [0,1] \to X$, the function $s \mapsto d(\sigma(s), \tau(s))$ is convex on $[0,1]$;
(C) X admits a convex and consistent geodesic bicombing;
(D) X admits a convex geodesic bicombing;
(E) X admits a conical geodesic bicombing.

Clearly the implications

$$(A) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D) \Rightarrow (E)$$

hold. We prove the following two results that allow to "go back" from (E) to (D) and from (D) to (C), under suitable additional assumptions.

Theorem 1. *Let X be a proper metric space with a conical geodesic bicombing. Then X also admits a convex geodesic bicombing.*

Theorem 2. *Let X be a metric space of finite combinatorial dimension in the sense of Dress* [1] *(see below), and suppose that X has a convex geodesic bicombing σ. Then σ is consistent and furthermore unique, that is, σ is the only convex geodesic bicombing on X.*

Our interest in these results comes from the fact that above condition (E) is satisfied for every metric space X that is *injective* as an object in the category of metric spaces and 1-Lipschitz maps. This means that for every isometric embedding $\rho\colon A \to B$ of metric spaces and every 1-Lipschitz map $f\colon A \to X$ there exists a 1-Lipschitz extension $\bar{f}\colon B \to X$, so that $\bar{f} \circ \rho = f$; equivalently, X is an absolute 1-Lipschitz retract. Basic examples of injective metric spaces are the real line, every L_∞ space, and all metric ($\mathbb{R}$-)trees; however, this list is by far not exhaustive. Indeed, Isbell [2] showed that every metric space X possesses an essentially unique *injective hull* $\mathrm{E}(X)$. If X is compact, then so is $\mathrm{E}(X)$, and if X is finite, then $\mathrm{E}(X)$ is a finite polyhedral complex with l_∞ metrics on the cells. Isbell's construction was rediscovered twenty years later by Dress [1], who gave it the name *tight span.* The combinatorial dimension of a metric space X is the supremum of the dimensions of the polyhedral complexes $E(S)$ for all finite subsets S of X.

In [3], we proved that if Γ is a Gromov hyperbolic group, endowed with the word metric with respect to some finite generating set of Γ, then the injective hull $\mathrm{E}(\Gamma)$ is a proper, finite-dimensional polyhedral complex with finitely many isometry types of (l_∞) cells, and Γ acts properly and cocompactly on $\mathrm{E}(\Gamma_S)$ by cellular isometries. Since $\mathrm{E}(\Gamma)$ satisfies condition (E) from the above list, Theorems 1 and 2 now show that $\mathrm{E}(\Gamma)$ possesses in fact a convex and consistent geodesic bicombing that is furthermore unique and hence equivariant with respect to the action of Γ. In other words, every word hyperbolic group acts properly and cocompactly on a proper, finite-dimensional polyhedral complex with a Γ-equivariant convex structure of type (C).

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Median algebras and coarse non-positive curvature

Brian H. Bowditch

There are several ways in which one can formulate a notion of non-positive curvature for a general geodesic metric space. The most frequently used is the notion of a CAT(0) space, described by Gromov, and based on work of Aleksandrov and Toponogov. This hypothesis states that any geodesic triangle in the space is at least as thin, in the appropriate sense, as the comparison triangle in the euclidean plane.

Similarly, one can formulate a notion of negative curvature in terms of the CAT(−1) property, where the euclidean plane is replaced by the hyperbolic plane. This latter condition has a natural coarse variation, namely that of a hyperbolic space, in the sense of Gromov. Unlike the CAT conditions, hyperbolicity is quasi-isometry invariant, and hence applicable to finitely generated groups via their Cayley graphs.

Ideally, one might search for a notion of non-positive curvature which is also quasi-isometry invariant. While there is no one preferred choice, various candidates involving combings etc. have been proposed. Here, we introduce the notion of a "coarse median space". It includes a number of naturally occuring examples, and is closed under various natural operations; though it is not clear exactly which spaces admit such a structure.

A coarse median space is a geodesic space equipped with a ternary operation satisfying the axioms of a median algebra up to bounded distance. This can be applied to a broad class of groups. Many results about such groups can be viewed in these terms. The idea was inspired by work of Behrstock and Minsky, and other people, on the mapping class group.

Recall that a "median algebra" is a set, M, together with a a ternary operation, $\mu : M^3 \longrightarrow M$, such that, for all $a, b, c, d, e \in M$,

(M1): $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$,

(M2): $\mu(a, a, b) = a$,

(M3): $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$.

Any finite median algebra can be identified as the vertex set of a finite CAT(0) cube complex. Moreover, any finite subset of a median algebra lies inside a finite subalgebra. In view of this, we make the following definition [3].

Let (Λ, ρ) be a geodesic metric space and $\mu : \Lambda^3 \longrightarrow \Lambda$ be a ternary operation. We say that μ is a "coarse median" if it satisfies the following:

(C1): There are constants, k, $h(0)$, such that for all $a, b, c, a', b', c' \in \Lambda$ we have

$$\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).$$

(C2): There is a function, $h : \mathbb{N} \longrightarrow [0, \infty)$, with the following property. Suppose that $A \subseteq \Lambda$ with $1 \leq |A| \leq p < \infty$, then there is a finite median algebra, (Π, μ_Π) and maps $\pi : A \longrightarrow \Pi$ and $\lambda : \Pi \longrightarrow \Lambda$ such that for all $x, y, z \in \Pi$ we have:

$$\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)$$

and

$$\rho(a, \lambda\pi a) \leq h(p)$$

for all $a \in A$.

The existence of a coarse median on a geodesic space is a quasi-isometry invariant, so we can apply this to finitely generated groups via their Cayley graphs. We can thus define a "coarse median group" as a finitely generated group whose Cayley graph is coarse median. For example, a hyperbolic group is a coarse median group of rank 1. Also, it follows using work of Behrstock and Minsky [2] that a mapping class group is coarse median of finite rank.

From this one can recover various facts [3, 4]. For example the asymptotic cone embeds into a finite product of $\mathbb{R}$-trees [1]. As a result, we recover the rank theorem of Behrstock and Minsky and Hamenstädt, as well as rapid decay, etc. One can also show that the asymptotic cone is bilipschitz equivalent to a CAT(0) space.

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Polyhedral Kähler manifolds

Dmitri Panov

(joint work with Anton Petrunin, Misha Verbitsky)

In this talk I review the theory of Polyhedral Kähler manifolds [2] and state some new results obtained in collaboration with Anton Petrunin and Misha Verbitsky.

Definition. *A polyhedral manifold is a manifold glued from a collection of Euclidean simplexes by identifying their hyperfaces via isometry. A polyhedral manifold M^{2n} is called Polyhedral Kähler if the holonomy of the singular flat metric on it belongs to $U(n) \subset SO(2n)$.*

Note that any orientable polyhedral surface is polyhedral Kähler by definition (since $U(1) = SO(2)$). On the other hand in higher dimensions the condition of being polyhedral Kähler is very restrictive. In particular we have the following conjecture.

Complex conjecture. *Let M^{2n} be a polyhedral Kähler manifold. Then the flat complex structure on the complement to the metric singularities of M^{2n} extends* *to the whole manifold so that the resulting manifold is a normal complex analytic variety with complex singularities in codimension* 3*.*

This conjecture holds trivially for $n = 1$, it was proven in [2] for $n = 2$ and for $n > 2$ this statement is a subject of a joint work in progress [4] with Misha Verbitsky. Note that complex singularities can indeed appear in codimension 3, since the complex hypersurface $x^3 + y^2 + z^2 + t^2 = 0$ is PL diffeomorphic to $\mathbb{R}^6$, it admits a compatible polyhedral Kähler metric and has an isolated singularity at zero.

One of the applications of this *complex* conjecture is a different conjecture.

A polyhedral analogue of Frankel's conjecture. *Consider an orientable manifold M with a polyhedral metric. Assume it is non-negatively curved, i.e., the conical angles along co-dimension* 2 *faces of M are at most 2π. Suppose that the holonomy of the metric on M is irreducible and $b_2(M) > 0$.*

Then M has a natural holomorphic structure with respect to which it is biholomorphic to $\mathbb{C}P^n$ and the original polyhedral metric on M is a singular Kähler metric with respect to this complex structure.

By a theorem of Cheeger [1] any polyhedral manifold satisfying the conditions of this conjecture is polyhedral Kähler. One can show further that this conjecture would follow from the *complex* conjecture combined with an orbi-analogue of Mori's characterization of $\mathbb{C}P^n$.

Let us note that non-negatively curved polyhedral metrics on $\mathbb{C}P^n$ exist for any n, for example such metrics can be obtained by taking symmetric powers of $\mathbb{C}P^1$ with a non-negatively curved polyhedral metric on it. Classifying all such metrics seems to be a very hard problem even in the case of $\mathbb{C}P^2$. Nevertheless Anton Petrunin and I were able to prove in [3] the following theorem.

Theorem. *Take any non-negatively curved polyhedral metric on $\mathbb{C}P^2$ with singularities along a line arrangement. Then the complement to the line arrangement is aspherical.*

As one can see in [2, Theorem 1.12, Corollary 7.8], this class of arrangements is quite nontrivial and include all complex reflection line arrangements. In particular applying results of [2] we get the following corollary.

Corollary. *Suppose we have a line arrangement in $\mathbb{C}P^2$ such that each line intersects $\frac{n+3}{3}$ lines and less than $\frac{2n}{3}$ lines pass through each point. Then the complement to the arrangement of lines is of type $K(\pi, 1)$.*

Arrangements of this type appeared for the first time in the work of Hirzebruch. All such arrangements that are known at this moment are complex reflection arrangements and it is not known if other such arrangements exist.

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Quaternionic Kähler manifolds from physics

Vicente Cortés

(joint work with D. V. Alekseevsky, M. Dyckmanns and T. Mohaupt)

I have presented explicit constructions of quaternionic Kähler manifolds inspired from physics. The talk is based on [1, 2]. We prove that every projective special Kähler manifold $\bar{M}$ gives rise to a one-parameter family of quaternionic Kähler metrics g^c. These metrics arise as one-loop quantum deformations of the Ferrara-Sabharwal metric g^0, which is complete under the assumption that the initial manifold $\bar{M}$ is complete [4]. Complete projective special Kähler manifolds of (real) dimension 6 are constructed in [3]. They define complete quaternionic Kähler metrics of negative scalar curvature of dimension 16, which do all admit an isometric group action of cohomogeneity ranging from 0 to 2.

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Gluing constructions for minimal surfaces and mean curvature self-shrinkers

Nicolaos Kapouleas

I first discussed doubling constructions where given a minimal surface two nearby copies of it are joined by a large number of catenoidal bridges and the resulting surface is corrected to minimality. I discussed doublings of the Clifford torus and the equatorial two-sphere and also work in progress for the round sphere as a self-shrinker and also for minimal hypersurfaces in higher dimensions (with coauthors). I also discussed the difficulties and the conditions for a general doubling construction without symmetries.

I then discussed desingularization constructions either with symmetries or without and I discussed briefly the ideas in the proof of the theorem in the general case.

Finally I discussed some of the questions motivated by the above for minimal surfaces in the round three-sphere.

Mean curvature flow of mean convex hypersurfaces

Robert Haslhofer
(joint work with Bruce Kleiner)

In the last 15 years, White and Huisken-Sinestrari developed a far-reaching structure theory for the mean curvature flow of mean convex hypersurfaces [1, 2, 3, 4, 5, 6]. We recently gave a new treatment of this theory [7], based on the beautiful non-collapsing result of Andrews [8]. Our new proofs are both more elementary and substantially shorter than the original arguments.

Recall that for any mean convex hypersurface $M_0^n \subset \mathbb{R}^{n+1}$ (smooth, closed, embedded), there is a unique weak solution $\{M_t = \partial K_t\}_{t\geq 0}$ of the mean curvature flow starting at M_0. It is characterized by the condition that $\{K_t\}$ is the maximal family of closed sets satisfying the avoidance principle

$$K_{t_0} \cap L_{t_0} = \emptyset \quad \Rightarrow \quad K_t \cap L_t = \emptyset \quad (t \in [t_0, t_1]) \tag{1}$$

for every smooth mean curvature flow $\{L_t\}_{t\in[t_0,t_1]}$. By the main result of Andrews [8] (which we extended to the weak setting via elliptic regularization) we have

Theorem. *There exists a constant $\alpha = \alpha(K_0) > 0$ such that every point $p \in \partial K_t$ admits interior and exterior balls tangent at p of radius at least $\frac{\alpha}{H(p)}$.*

The Andrews condition immediately rules out higher multiplicity planes as potential blowup limits. However, taking a much broader perspective, it turned out that one can actually develop the entire theory based on the Andrews condition. The starting point was the following estimate, which says that curvature control at a single point implies curvature control in a whole parabolic ball.

Theorem (Curvature estimate)**.** *There exist constants $\delta = \delta(\alpha) > 0$ and $C = C(\alpha) < \infty$ such that for any α-Andrews flow K_t in a parabolic ball $P(p,t,r)$:*

$$H(p,t) \leq r^{-1} \qquad \Rightarrow \qquad \sup_{P(p,t,\rho r)} |A| \leq C r^{-1}. \tag{2}$$

To prove this we use comparison and the Andrews condition to show that the flow is, after suitable rescaling, Hausdorff close to a halfspace on a large time interval. Then the local regularity theorem implies the desired curvature bounds.

Theorem (Convexity estimate)**.** *For every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, \alpha) < \infty$ such that if K_t is an α-Andrews flow defined in $P(p,t,\eta H(p,t)^{-1})$ then*

$$\frac{\lambda_1}{H}(p,t) \geq -\varepsilon. \tag{3}$$

In particular, blowup limits of α-Andrews flows are convex.

The proof is very short (one page as opposed to a couple of sophisticated papers): Take a sequence of counterexamples where the infimum of $\frac{\lambda_1}{H}$ is negative. By the local curvature estimate the infimum is actually a minimum. However, by the strict maximum principle, $\frac{\lambda_1}{H}$ can never attain a negative minimum.

Our treatment of the global theory is based on the following global convergence result, which says that after normalizing the mean curvature at a single point we can pass smoothly and globally to a limit.

Theorem (Global convergence)**.** *Every sequence of α-Andrews flows in $P(0,0,\eta_j)$ with $H(0,0)=1$ and $\eta_j\to\infty$ has a smoothly and globally convergent subsequence.*

Roughly, the idea of the proof is as follows: If the global convergence theorem failed, by looking at the first radius where the curvature blows up we could find a nonflat convex cone; this however cannot happen under mean curvature flow.

Theorem (Structure of ancient solutions)**.** *Ancient α-Andrews flows are smooth and convex until they become extinct. In particular, the only self-similarly shrinking ancient α-Andrews flows are the sphere, the cylinders, and the plane.*

For more about ancient solutions we refer to Haslhofer-Hershkovits [9]. By standard stratification the structure theorem immediately implies:

Theorem (Partial regularity)**.** *For every α-Andrews flow $K_t\subset\mathbb{R}^N$, the singular set has parabolic Hausdorff dimension at most $N-2$.*

In fact, using quantitative stratification this can be improved to Minkowski and L^p-estimates, in particular in the k-convex case, see Cheeger-Haslhofer-Naber [10].

Theorem (Cylindrical estimate)**.** *For every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,\alpha,\beta)>0$ such that if K_t is a uniformly k-convex (i.e. $\lambda_1+\ldots+\lambda_k\geq\beta H$ for some $\beta>0$) α-Andrews flow and $\frac{\lambda_1+\ldots+\lambda_{k-1}}{H}(p,t)<\delta$, then K_t is ε-close to a round shrinking cylinder $\mathbb{R}^{k-1}\times S^{n-k+1}$ near (p,t).*

All our estimates are local and universal, i.e. they only depend on the value of the Andrews constant. In a forthcoming paper [11], we will give a new construction of mean curvature flow with surgery based on these estimates.

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Constant mean curvature surfaces in homogeneous manifolds

Benoît Daniel

There are several characterisations of round spheres among constant mean curvature (CMC) surfaces in Euclidean 3-space $\mathbb{R}^3$. In particular, two classical results are Hopf's theorem and Alexandrov's theorem.

- Hopf's theorem [8] states that any immersed CMC surface in $\mathbb{R}^3$ that is *diffeomorphic to a sphere* must be a round sphere. Its proof relies on the fact that the $(2,0)$-part of the second fundamental form of CMC surfaces in $\mathbb{R}^3$ is holomorphic. This theorem also holds in the round 3-sphere $\mathbb{S}^3$ and in hyperbolic 3-space $\mathbb{H}^3$.
- Alexandrov's theorem [4] states that any *compact embedded* CMC surface in $\mathbb{R}^3$ must be a round sphere. Alexandrov's proof uses a moving plane technique and the maximum principle for elliptic partial differential equations. This theorem also holds in $\mathbb{H}^3$ and in a hemisphere of $\mathbb{S}^3$.

We will present some extensions of these two theorems to other ambient homogeneous Riemannian 3-manifolds. Abresch and Rosenberg [1, 2] extended Hopf's theorem to simply connected homogeneous Riemannian 3-manifolds with a 4-dimensional isometry group (i.e., $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, the Heisenberg group Nil_3, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and Berger spheres): they proved that any immersed CMC sphere there is rotational. To do this, they proved that a certain quadratic differential is holomorphic for CMC surfaces. We mention that these ambient manifolds form a two-parameter family $\mathbb{E}(\kappa,\tau)$, and each $\mathbb{E}(\kappa,\tau)$ admits a Riemannian submersion over the simply connected surface of constant curvature κ with bundle curvature τ.

In a joint work with P. Mira [7], we study existence and uniqueness of CMC spheres in the Lie group Sol_3 endowed with a standard left-invariant metric, i.e., the only Thurston geometry that is neither a space-form nor one of the $\mathbb{E}(\kappa,\tau)$ manifolds. Indeed, the isometry group of Sol_3 only has dimension 3.

Together with a work by W. Meeks [9], we obtain that for every $H > 0$ there exists a unique CMC H sphere in Sol_3 (up to translations). The method of Abresch and Rosenberg consisting in finding a holomorphic quadratic differential for CMC surfaces does not work here. Moreover, no explicit CMC spheres are known due to the lack of rotations.

To prove existence, we start from a solution to the isoperimetric problem for a small volume and use the implicit function theorem to deform it. Using nodal domain arguments, we prove that the CMC spheres we obtain have index one and that their Gauss map is a diffeomorphism, from what we deduce curvature estimates. Together with Meeks' height estimates, this yields existence for any $H > 0$.

To prove uniqueness, we introduce a quadratic differential for CMC surfaces that satisfies the *Cauchy-Riemann inequality* [3], a condition weaker than holomorphicity. An important difference with the proof of Abresch and Rosenberg is that here the quadratic differential is *not explicit*: it is defined in terms of the Gauss map of the existing CMC sphere, and the fact that this Gauss map is a diffeomorphism plays a crucial role.

We now mention some related open problems.

The isoperimetric problem in the Heisenberg group Nil_3 is still unsolved. More generally, we do not know a classification of compact stable CMC surfaces in Nil_3. We do not know either a classification of compact embedded CMC surfaces in Nil_3. We conjecture that in both cases these surfaces must be rotational spheres (this would extend respectively Barbosa-do Carmo's theorem [5] and Alexandrov's theorem). For instance, we cannot make the Alexandrov moving plane argument work because there are no "reflections" in Nil_3. Also, since CMC spheres are not totally umbilical, proofs of the Alexandrov theorem relying on the fact that round spheres in $\mathbb{R}^3$ are totally umbilical (like Reilly's one [12]) do not seem obvious to adapt in Nil_3.

Similar questions are still also open in the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and Berger spheres.

In Sol_3, H. Rosenberg proved that compact embedded CMC surfaces must be topological spheres. In particular, all isoperimetric surfaces are spheres. However, it is still an open problem to know if, conversely, all CMC spheres are solutions to the isoperimetric problem (or even stable).

For more details, we refer for instance to [6] (in French) and references therein. See also [10] for possible generalisations of these results to other homogeneous Riemannian 3-manifolds and for other open problems, and the recent work [11] about isoperimetric domains of large volume in simply connected homogeneous Riemannian 3-manifolds and their relation to CMC surfaces.

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Geometrically formal 4-manifolds with nonnegative sectional curvature

Christian Bär

This talk is based on [1]. Throughout the talk, M^n denotes a connected compact oriented smooth n-manifold without boundary. By the classification of surfaces and the Gauß-Bonnet theorem, M^2 possesses a metric of positive curvature if and only if M^2 is diffeomorphic to S^2. If we only demand nonnegative curvature, then T^2 equipped with a flat metric is also possible. In 3 dimensions we also have a good understanding of the possible nonnegatively curved spaces:

Theorem 1 (Hamilton [2, Thm. 1.2])**.** *Let* (M^3, g) *satisfy* $\mathrm{Ric} \geq 0$*. Then one of the following holds:*

(1) M^3 *is diffeomorphic to a spherical spaceform or to* $\mathbb{RP}^3 \sharp \mathbb{RP}^3$*;*
(2) (M^3, g) *is isometric to a twisted product* $S^2 \times_\rho S^1$ *where* S^2 *carries a metric of nonnegative curvature;*
(3) (M^3, g) *is flat.*

In dimension 4, nonnegative sectional curvature is only poorly understood. The only M^4 known to carry a metric with $K > 0$ are S^4 and $\mathbb{CP}^2$. On the other hand, one has only very few obstructions which are not already obstructions to weaker curvature conditions. In particular, it is still unknown whether $S^2 \times S^2$ carries a metric with $K > 0$. This is known as the *Hopf conjecture*. If we want to say something substantial, then we presently have to add further geometric assumptions. In [3, 4] the existence of a 1-parameter family of isometries is assumed. Here we use geometric formality instead.

Definition 1. A Riemannian manifold is called *geometrically formal* if the wedge product of any two harmonic forms is again harmonic.

Now we can prove

Theorem 2 ([1, Thm. A])**.** *Let* (M^4, g) *be geometrically formal such that the sectional curvature satisfies* $K \geq 0$*. Then one of the following holds:*

(1) M *is a rational homology* 4*-sphere with finite fundamental group;*
(2) M *is diffeomorphic to* $\mathbb{CP}^2$*;*
(3) M *is flat;*

(4) *M is isometric to a twisted product $S^2 \times_\rho T^2$ where T^2 carries a flat metric and S^2 carries a metric of nonnegative curvature;*
(5) *M is isometric to a twisted product $\Sigma^3 \times_\rho S^1$, where Σ^3 is isometric to a spherical spaceform or to $\mathbb{RP}^3 \sharp \mathbb{RP}^3$ with a metric satisfying $K \geq 0$;*
(6) *M is isometric to $S^2 \times S^2$ with product metric where both factors carry metrics with nonnegative curvature.*

Conversely, the manifolds in cases (3)–(6) are actually geometrically formal with $K \geq 0$. Cases (1) and (2) also do occur, S^4 and $\mathbb{CP}^2$ with their standard metrics are examples. The classification simplifies if we add the requirement of simple connectedness.

Theorem 3 ([1, Thm. B]). *Let (M^4, g) be simply connected and geometrically formal such that the sectional curvature satisfies $K \geq 0$. Then one of the following holds:*
(1) *M is homeomorphic to S^4;*
(2) *M is diffeomorphic to $\mathbb{CP}^2$;*
(3) *M is isometric to $S^2 \times S^2$ with product metric where both factors carry metrics with nonnegative curvature.*

Kotschick obtained classification results for geometrically formal 4-manifolds with a possibly different metric of nonnegative scalar curvature in [5].

Theorem 4 ([1, Thm. C]). *Let (M^4, g) be geometrically formal such that the sectional curvature satisfies $K > 0$. Then M is homeomorphic to S^4 or diffeomorphic to $\mathbb{CP}^2$.*

Seaman has the following result which applies in the geometrically formal case:

Theorem 5 ([6]). *Let (M^4, g) be such that all harmonic 2-forms have constant length and the sectional curvature satisfies $K > 0$. Then M^4 is homeomorphic to S^4 or to $\mathbb{CP}^2$.*

We can weaken the assumption in Seaman's theorem to the requirement that the length of the harmonic 2-forms is "not too nonconstant".

Theorem 6 ([1, Thm. D]). *Let (M^4, g) be such that the sectional curvature satisfies $K \geq \kappa > 0$ and all harmonic 2-forms ω satisfy $|\nabla|\omega|| \leq \sqrt{8\kappa} \cdot |\omega|$ wherever ω does not vanish. Then M is homeomorphic to S^4 or diffeomorphic to $\mathbb{CP}^2$.*

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