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Mini-Workshop: The Willmore Functional and the Willmore Conjecture

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Abstract. The Willmore functional evaluated on a surface immersed into Euclidean space is given by the L^2-norm of its mean curvature. The interest for studying this functional comes from various directions. First, it arises in applications from biology and physics, where it is used to model surface tension in the Helfrich model for bilipid layers, or in General Relativity where it appears in Hawking's quasi-local mass. Second, the mathematical properties justify consideration of the Willmore functional in its own right. The Willmore functional is one of the most natural extrinsic curvature functionals for immersions. Its critical points solve a fourth order Euler-Lagrange equation, which has all minimal surfaces as solutions.

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Introduction by the Organisers

In recent years there has been substantial progress concerning analytical and geometrical questions related to the Willmore functional. Highlights include the study of surfaces with square integrable second fundamental form, the compactness of $W^{2,2}$-conformal immersions, the regularity of weak solutions of the Willmore equation and the resolution of the longstanding Willmore conjecture.

The aim of this mini-workshop was to bring together people involved in propelling the above mentioned research highlights. In particular, our intention was to make a connection between the experts on minimal surfaces and corresponding min-max techniques that were a crucial ingredient in the proof of the Willmore conjecture, and the experts for the analysis developed for second order curvature functionals such as the Willmore functional.

For this purpose two mini-courses were delivered by Fernando Marques (Min-max theory and the Willmore conjecture) and Tristan Rivière (The variations of the Willmore Lagrangian, a parametric approach). Moreover, every participant gave a talk, with plenty of time left for discussions.

Mini-Workshop: The Willmore Functional and the Willmore Conjecture

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Abstracts

Min-max theory and the Willmore conjecture

Fernando Codá Marques
(joint work with André Neves, Imperial College - UK)

In 1965, T. J. Willmore conjectured that the integral of the square of the mean curvature of any torus immersed in Euclidean three-space is at least $2\pi^2$. In this series of three lectures we will describe a proof of this conjecture that uses the min-max theory of minimal surfaces.

The Willmore conjecture can be reformulated as a question about surfaces in the three-sphere because if $\pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ denotes the stereographic projection and $\Sigma \subset S^3 \setminus \{(0,0,0,1)\}$ is a closed surface, then

$$\int_{\tilde{\Sigma}} \tilde{H}^2 d\tilde{\Sigma} = \int_{\Sigma} (1 + H^2) d\Sigma.$$

Here H and $\tilde{H}$ are the mean curvature functions of $\Sigma \subset S^3$ and $\tilde{\Sigma} = \pi(\Sigma) \subset \mathbb{R}^3$, respectively. The quantity $\mathcal{W}(\Sigma) = \int_\Sigma (1 + H^2)\, d\Sigma$ is then defined to be the *Willmore energy* of $\Sigma \subset S^3$. This energy is specially interesting because it has the remarkable property of being invariant under conformal transformations of S^3. This fact was already known to Blaschke and Thomsen in the 1920s. The Willmore Conjecture has received the attention of many mathematicians since the late 1960s.

Our Main Theorem is:

Theorem A. *Let $\Sigma \subset S^3$ be an embedded closed surface of genus $g \geq 1$. Then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2,$$

and the equality holds if and only if Σ is the Clifford torus up to conformal transformations of S^3.

To each closed surface $\Sigma \subset S^3$, we associate a canonical 5-dimensional family of surfaces in S^3 with area bounded above by the Willmore energy of Σ. This area estimate follows from a calculation of A. Ros, a special case of the Heintze-Karcher inequality. The family is parametrized by the 5-cube I^5, and maps the boundary ∂I^5 into the space of geodesic spheres in a topologically nontrivial way if genus$(\Sigma) \geq 1$. The whole proof revolves around the idea of showing that the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S^3$ can be produced by applying min-max theory for the area functional to the homotopy class of this family. One key point is that, by a result of F. Urbano, the Clifford torus is the only non-totally geodesic minimal surface in S^3 with Morse index at most 5. After ruling out great spheres by a topological argument, the proof of Theorem A then reduces to the following statement about minimal surfaces in the three-sphere, also proven using min-max methods:

Theorem B. *Let $\Sigma \subset S^3$ be an embedded closed minimal surface of genus $g \geq 1$. Then* $\text{area}(\Sigma) \geq 2\pi^2$*, and* $\text{area}(\Sigma) = 2\pi^2$ *if and only if Σ is the Clifford torus up to isometries of S^3.*

The plan is to start with an overview of the argument. We will then proceed to the construction of the canonical family and derivation of its main properties. We will derive a key identity according to which the topological degree of the map that gives the center of the spheres in the boundary of the family is equal to the genus of the original surface. We will talk about the Almgren-Pitts min-max theory for the area functional, and we will explain the topological argument that rules out the possibility of producing great spheres by the min-max process. We will then finish with the proofs of Theorems A and B.

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The variations of Willmore Lagrangian, a parametric approach

Tristan Rivière

(joint work with Yann Bernard, Laura Keller, Paul Laurain and Andrea Mondino)

The course was divided into 3 main parts.

During the first hour we introduced the notion of weak immersions of two dimensional manifolds and we presented some properties of this space, which happens to be a Banach manifold. In particular we established the fact that to each weak immersions corresponds a smooth conformal structure and the attached mapping into the Teichmüller space is smooth. The ultimate goal of this first part was to give an almost weak compactness result for sequences of weak immersions of closed surfaces with uniformly bounded Willmore energy and controled conformal class.

The second part of the course was devoted to the first variations of Willmore energy. After having recalled the classical Willmore equation we explained the incompatibility of the equation with the notion of weak immersions. We then derived a conservative form of the Willmore equation which this time makes sense for weak immersions. The ultimate goal of this second hour was to present a regularity result as well as a strong compactness result for weak Willmore immersions which has been derived from the existence of additional conservation laws for Willmore immersions and their interpretation in the light of *integrabiity by compensation* theory. Combining the two first hours we can in particular deduce the existence of Willmore minimizer under various constrained in the framework of weak immersions (under fixed topology assumption, with prescribed conformal class, with prescribed isoperimetric ratio for spheres and tori in $\mathbb{R}^3$, etc.).

The last part of the lecture was devoted to the blow-up analysis and the proof of "bubble tree" type convergences for sequences of Willmore surfaces with uniformly bounded energy and controlled conformal type. The ultimate goal of this third hour was to present a Willmore-energy quantization result for sequences of Willmore immersions with uniformly bounded Willmore energy and controled conformal class.

Most of the results presented in this mini-course have been obtained by the lecturer in various collaborations together with Yann Bernard, Laura Keller, Paul Laurain and Andrea Mondino.

Scherk's surface and the large-genus limit of the Willmore Problem

Rob Kusner

Let α_g be the infimal area for compact *minimal* surfaces of genus g embedded in S^3, and let β_g be the infimum of the Willmore bending energy $W(\Sigma) := \iint_\Sigma (1+H^2)da$ among *all* compact genus g surfaces $\Sigma \subset S^3$. Clearly $\alpha_g \geq \beta_g$. Lawson's [6] minimal surfaces $\Sigma_g = \xi_{g,1}$ give $8\pi > \alpha_g \geq \beta_g$ (see [3]). Both α_g (see [2]) and β_g (see [8, 4, 1]) are realized by surfaces of genus g. The solution [7] to the Willmore Conjecture [9] gives $\alpha_1 = \beta_1 = 2\pi^2$, realized only by the Clifford torus or any of its images under Möbius transformation; it also shows $\alpha_g \geq \beta_g > 2\pi^2$ for $g > 1$. For genus $g > 1$ it has been conjectured [3] that $\alpha_g = \beta_g = \text{area}(\Sigma_g)$. In the large-genus limit it it is known [5] that $\alpha_g, \beta_g \to 8\pi$ as $g \to \infty$. It can also be shown these smallest-area minimal surfaces subconverge as stationary varifolds to the union of two great two-spheres $S^2 \subset S^3$, and it is conjectured these intersect *orthogonally* along a great circle (and correspondingly for the W-minimizers, up to Möbius transformation). In this talk we explained how a rescaling argument using the Lawson surfaces shows $\beta_g \leq \alpha_g \leq 8\pi - c(\frac{\pi}{2})/g + o(1/g)$ as $g \to \infty$; here $c(\theta) > 0$ is a universal constant representing the "area deficit" (compared with a pair of planes) for the singly-periodic Scherk minimal surface $S_\theta \subset \mathbf{R^3}$ of "wing" angle $\theta \in (0, \frac{\pi}{2}]$. We also showed $c(\theta) \to 0$ as $\theta \to 0$, and discussed why $c(\theta)$ attains its maximum at $\theta = \frac{\pi}{2}$, lending evidence for the above conjectures, at least, asymptotically.

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Two-dimensional curvature functionals with superquadratic growth

Ernst Kuwert
(joint work with Tobias Lamm and Yuxiang Li)

Let Σ be a two-dimensional, closed differentiable manifold and $p > 2$, hence $W^{2,p}(\Sigma, \mathbb{R}^n) \subset C^{1,1-\frac{2}{p}}(\Sigma, \mathbb{R}^n)$ by the Sobolev embedding theorem. On the open subset of immersions $W^{2,p}_{\mathrm{im}}(\Sigma, \mathbb{R}^n)$ we consider the two functionals

$$\begin{aligned} \mathcal{E}^p(f) &= \frac{1}{4}\int_\Sigma (1+|A|^2)^{\frac{p}{2}}\, d\mu_g, \\ \mathcal{W}^p(f) &= \frac{1}{4}\int_\Sigma (1+|H|^2)^{\frac{p}{2}}\, d\mu_g. \end{aligned}$$

Here g denotes the first fundamental form with induced measure μ_g, $A = (D^2 f)^\perp$ the second fundamental form, and H is the mean curvature vector. The main result presented in the talk is:

Theorem *Let $f \in W^{2,p}_{\mathrm{im}}(\Sigma, \mathbb{R}^n)$ be a critical point of $\mathcal{W}^p$ or $\mathcal{E}^p$, where $2 < p < \infty$. Then local graph representations of f are smooth.*

In a graph representation, the Euler-Lagrange equations become fourth order elliptic systems, where the principal term has a double divergence structure. The systems are degenerate, in the sense that in both cases the coefficient of the principal term involves a $(p-2)$-th power of the curvature, which a priori may not be bounded. For the functional $\mathcal{W}^p(f)$, our first step towards regularity is an improvement of the integrability of H. For this we employ an iteration based on a new test function argument. More precisely, we solve the equation $L_g\varphi = |H|^{\lambda-1}H$ for appropriate $\lambda > 1$ and then insert φ as a test function. Here the operator $L_g = \sqrt{\det g}\, g^{\alpha\beta}\partial^2_{\alpha\beta}$ comes up in the principal term of the equation.

Unfortunately, the same strategy does not apply in the case of the functional $\mathcal{E}^p(f)$, since then the corresponding operator is a full Hessian and hence the equation would be overdetermined. Instead we first use a hole-filling argument to show power decay for the L^p integral of the second derivatives, and derive L^2 bounds for the third derivatives by a difference-quotient argument; these steps follow closely the ideas of Morrey [4] and Simon [5]. In the final critical step we adapt a Gehring type lemma due to Bildhauer, Fuchs and Zhong [1] as well as the Moser-Trudinger inequality to get that the solution is of class C^2.

As second issue we addressed the existence of minimizers for the functionals. By the compactness theorem of Langer [2], sequences of closed immersed surfaces $f_k : \Sigma \to \mathbb{R}^n$ with $\mathcal{E}^p(f_k) \leq C$ subconverge weakly to an $f \in W^{2,p}_{\mathrm{im}}(\Sigma, \mathbb{R}^n)$, after suitable reparametrization and translation. In particular, we have existence of a smooth $\mathcal{E}^p$ minimizer in the class of immersions $f : \Sigma \to \mathbb{R}^n$ for $p > 2$. On the other hand, boundedness of $\mathcal{W}^p(f)$ is not sufficient to guarantee the required compactness. This is illustrated by joining two round spheres by a shrinking catenoid neck, showing that the 8π bound in the following result is optimal.

Theorem *Let Σ be a closed surface and $f_k \in W^{2,p}_{\mathrm{im}}(\Sigma, \mathbb{R}^n)$ be a sequence of immersions with $0 \in f_k(\Sigma)$ and*

$$\mathcal{W}^p(f_k) \leq C \quad \text{and} \quad \liminf_{k\to\infty} \frac{1}{4}\int_\Sigma |H_k|^2 \, d\mu_{g_k} < 8\pi.$$

After passing to $f_k \circ \varphi_k$ for appropriate $\varphi_k \in C^\infty(\Sigma, \Sigma)$ and selecting a subsequence, the f_k converge weakly in $W^{2,p}(\Sigma, \mathbb{R}^k)$ to an $f \in W^{2,p}_{\mathrm{im}}(\Sigma, \mathbb{R}^n)$. In particular, the convergence is in $C^{1,\beta}(\Sigma, \mathbb{R}^n)$ for any $\beta < 1 - \frac{2}{p}$ and we have

$$\mathcal{W}^p(f) \leq \liminf_{k\to\infty} \mathcal{W}^p(f_k).$$

For functionals with similar growth conditions the existence of minimizers was proved in [3] in the setting of curvature varifolds.

A classical approach to the construction of harmonic maps, due to Sacks & Uhlenbeck, is by introducing perturbed functionals involving a power $p > 2$ of the gradient. One motivation for our analysis is an analogous approximation for the Willmore functional

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |H|^2 \, d\mu_g = \frac{1}{4}\int_\Sigma |A|^2 \, d\mu_g + \pi\chi(\Sigma). \tag{1}$$

The Willmore functional does not satisfy a Palais-Smale type condition, since it is invariant under the group of Möbius transformations. A suitable version of the Palais-Smale condition is however valid for the functionals $\mathcal{E}^p$ and $\mathcal{W}^p$ with $p > 2$. At the end of talk, we explained a concentration compactness alternative for the limit $p \searrow 0$.

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Regularity theory for a class of critical Cauchy-Riemann-type PDE and its relationship to harmonic maps and conformal immersions in two dimensions

Ben Sharp

We will study a linear first order system, a connection $\overline{\partial}$ problem, on a vector bundle equipped with a connection, over a Riemann surface. We show optimal conditions on the connection forms which allow one to find a holomorphic frame, or in other words to prove the optimal regularity of our solution. The underlying geometric principle, discovered by Koszul-Malgrange [4], is classical and well known; it gives necessary and sufficient conditions for a connection to induce a holomorphic structure on a vector bundle over a complex manifold. Here we explore the limits of this statement when the connection is not smooth and our findings lead to a very short proof of Hélein's regularity theorem for weakly harmonic maps in two dimensions [2] as well as recovering an energy convexity result of Colding-Minicozzi for small energy harmonic maps [1] and an estimate of Lamm and Lin [5] concerning conformally invariant variational problems in two dimensions. The main point of reference is [7].

It is well known that the complex derivative ∂u for harmonic maps $u : \Sigma \to \mathcal{N}$ from a Riemann surface into a closed Riemannian manifold solve a Cauchy Riemann equation

$$\overline{\partial}_{u^*\mathrm{T}\mathcal{N}}(\partial u) = 0 \tag{1}$$

where $\partial u \in \Gamma(u^*\mathrm{T}\mathcal{N} \otimes \wedge^{(1,0)}\mathrm{T}^*_{\mathbb{C}}\Sigma)$ and $\overline{\partial}_{u^*\mathrm{T}\mathcal{N}}$ is the induced covariant Cauchy Riemann operator given by the pulled-back Levi Civita connection on $\mathcal{N}$. Locally, setting $\alpha := \partial u$ it reads

$$\overline{\partial}\alpha^i = -\Gamma^i_{jk}\alpha^j \wedge \overline{\alpha}^k$$

and we see that for weakly harmonic maps $u \in W^{1,2}(\Sigma, \mathcal{N})$ (so $\alpha \in L^2$) we have that $\overline{\partial}\alpha \in L^1$. Hence obtaining higher regularity is an issue from this point since the L^1 theory for singular integrals is not sufficient to induce a bootstrapping argument.

This perspective has already been used in the context of the regularity theory, by Frédéric Hélein [3], however in order to get (1) into the position where a bootstrapping argument can be used, one is first required to consider only targets with a trivial tangent bundle. This can be done by proving that there is a totally geodesic embedding of any such $\mathcal{N}$ into a topological torus, thus enabling us to make this assumption without loss of generality.

In contrast we study such first order equations in a more general form and prove a regularity theorem in this setting, allowing us to side-step the technical issue of trivialising the tangent bundle of $\mathcal{N}$. Specifically we consider a rank m vector bundle over a Riemann surface, equipped with an L^2 connection, respectively (E^m, Σ, ∇), and sections $\alpha \in L^2 \cap \Gamma(E \otimes \wedge^{(1,0)} \mathrm{T}^*_{\mathbb{C}}\Sigma)$ solving

$$\overline{\partial}_{\nabla}(\alpha) = 0. \tag{2}$$

In what follows $D \subset \mathbb{C}$ denotes the unit disc and represents a piece of Σ over which E is trivial. Our main result is the following

Theorem. *Suppose locally the connection forms $\omega^{\overline{z}} \in L^2(D, gl(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ satisfy the following condition: Given the unique forms $\hat{\omega} \in L^2(D, u(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ with $\hat{\omega}^{\overline{z}} = \omega^{\overline{z}}$ (one can always find these) then by a Hodge decomposition*

$$\hat{\omega} = \mathrm{d}a + *\mathrm{d}b$$

for $a, b \in W^{1,2}(D, u(m))$ and $b|_{\partial D} = 0$. We impose that $\nabla b \in L^{2,1}$ - the strongest Lorentz space associated with L^2. Then there exist $\epsilon, K > 0$ such that whenever

$$\|\omega\|_{L^2} + \|\nabla b\|_{L^{2,1}} \leq \epsilon$$

there exists a change of frame $S \in L^\infty \cap W^{1,2}_{loc}(D, Gl(m, \mathbb{C}))$ such that

$$\overline{\partial} S = -\omega^{\overline{z}} S$$

with

$$\|\mathrm{dist}(\mathrm{S}, \mathrm{U(m)})\|_{\mathrm{L}^\infty(\mathrm{D})} \leq \mathrm{K}\epsilon$$

and for any $U \overset{c}{\hookrightarrow} D$ there exists some $C = C(U) < \infty$ such that

$$\|\nabla S\|_{L^2(U)} \leq C \|\omega\|_{L^2}.$$

Moreover on D, α solves

$$\overline{\partial}(S^{-1}\alpha) = 0$$

and we have $\alpha \in (L^\infty \cap W^{1,2})_{loc}$ along with $|\alpha|^2 \in h^1(D)$ the local Hardy space.

There are counter-examples to this theorem if one relaxes the condition on ∇b in the Lorentz space setting, thus the result is sharp in this sense.

As alluded to above, the hypotheses of this theorem are applicable to the study of harmonic maps, and more generally critical points of any quadratic conformally invariant elliptic Lagrangian in two dimensions - though in the latter setting one is required first to apply results of Rivière [6] and Lamm-Lin [5]. Moreover if one considers a (smooth, say) conformal immersion u of a disc into a Riemannian manifold with finite area and bounded Willmore energy then its complex derivative $\alpha = \partial u$ also solves an equation of the form (2). Under the added assumption that the mean curvature $H \in W^{1,2}$ one can apply the theorem above.

We end by listing two fundamental theorems related directly to harmonic maps that we recover from the above theorem, here $B_1 \subset \mathbb{R}^2$ is the unit disc with the flat metric.

Theorem (Hélein)**.** *Suppose $u : B_1 \to \mathcal{N}$ is a weakly harmonic map where $\mathcal{N}$ is a C^l submanifold of $\mathbb{R}^m$ such that the second fundamental form is bounded with* *respect to the induced metric and* $l \geq 2$*. Then for all* $\beta \in (0,1)$ *there exist* $\epsilon = \epsilon(\mathcal{N})$ *and* $C = C(\mathcal{N}, \beta)$ *such that if*

$$\|\nabla u\|_{L^2(B_1)} \leq \epsilon$$

then

$$[\nabla^l u]_{BMO(B_{\frac{1}{2}})} + \|u\|_{C^{l-1,\beta}(B_{\frac{1}{2}})} \leq C\|\nabla u\|_{L^2(B_1)}.$$

We also recover the following Energy convexity theorem in [1], from which local uniqueness of harmonic maps follows easily in two dimensions. The proof can be found in [1, Appendix C] however now we can assume that $\mathcal{N}$ is C^2 with bounded second fundamental form and we do not need to make any assumptions on the tangent bundle.

Theorem (Colding-Minicozzi)**.** *Let* $u, v \in W^{1,2}(B_1, \mathcal{N})$ *and suppose that* u *is weakly harmonic map where* $\mathcal{N}$ *is a* C^2 *submanifold of* $\mathbb{R}^m$ *with bounded sencond fundamental form. Then there exists some* $\epsilon = \epsilon(\mathcal{N})$ *such that if* $u - v \in W_0^{1,2}$ *and*

$$\|\nabla u\|_{L^2(B_1)} \leq \epsilon$$

then

$$\int_{B_1} |\nabla v|^2 - |\nabla u|^2 \geq \frac{1}{2}\int_{B_1} |\nabla(v-u)|^2.$$

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Area comparison in manifolds with a lower bound on the scalar curvature

Mario Micallef

(joint work with Vlad Moraru)

This was a report on a joint work with Vlad Moraru and on some recent developments by Moraru. I presented an area comparison theorem for totally geodesic surfaces in 3-manifolds with a lower bound on the scalar curvature, which is an optimal analogue of a theorem of Heintze, Karcher and Maeda for minimal surfaces in manifolds with non-negative Ricci curvature. The theorem is optimal in the sense that examples by Moraru show that it does not hold in manifolds of dimension greater than or equal to 4. The area comparison theorem provides a unified proof of three splitting & rigidity theorems for 3-manifolds with lower bounds on the scalar curvature that were first proved, independently, by Cai-Galloway (zero case), Bray-Brendle-Neves (positive case) and Nunes (negative case).

Recently, Moraru established a rigidity theorem for manifolds of dimension ≥ 4 with a lower bound on the scalar curvature and which contain an area minimizing hypersurface where area is equal to the lower bound in terms of the σ-constant provided by a theorem of Cai-Galloway.

Dynamical stability and instability of Ricci-flat metrics

Reto Müller

(joint work with R. Haslhofer)

While Willmore surfaces are the critical points of the Willmore energy, an extrinsic curvature functional, this talk is concerned with critical points of intrinsic curvature functionals.

Let M be a compact manifold. A Ricci-flat metric on M is a Riemannian metric with vanishing Ricci curvature. Ricci-flat metrics are fairly hard to construct, and their properties are of great interest. They are the critical points of the Einstein-Hilbert functional, $\mathcal{E}(g) = \int_M R_g dV_g$ and of Perelman's λ-functional [5],

$$\lambda(g) = \inf_{\substack{f \in C^\infty(M) \\ \int_M e^{-f} dV_g = 1}} \int_M \big(R_g + |\nabla f|_g^2\big) e^{-f} dV_g. \tag{1}$$

Obviously, they are also the fixed points of Hamilton's Ricci flow,

$$\partial_t g(t) = -2\mathrm{Rc}_{g(t)}, \tag{2}$$

In this talk, we are concerned with the *stability properties* of Ricci-flat metrics under Ricci flow. This stability problem has previously been studied by Sesum [6] and Haslhofer [3] under additional integrability assumptions, generalizing in turn previous work by Guenther-Isenberg-Knopf [2]. In this talk, we show how the integrability assumption of Sesum and Haslhofer can be removed. More precisely, we prove the following results.

Theorem 1 (Dynamical stability, [4]). *Let $(M, \hat{g})$ be a compact Ricci-flat manifold. If $\hat{g}$ is a local maximizer of λ, then for every $C^{k,\alpha}$-neighborhood $\mathcal{U}$ of $\hat{g}$ ($k \geq 2$), there exists a $C^{k,\alpha}$-neighborhood $\mathcal{V} \subset \mathcal{U}$ such that the Ricci flow starting at any metric in $\mathcal{V}$ exists for all times and converges (modulo diffeomorphisms) to a Ricci-flat metric in $\mathcal{U}$.*

Theorem 2 (Dynamical instability, [4]). *Let $(M, \hat{g})$ be a compact Ricci-flat manifold. If $\hat{g}$ is not a local maximizer of λ, then there exists a nontrivial ancient Ricci flow $\{g(t)\}_{t\in(-\infty,0]}$ that converges (modulo diffeomorphisms) to $\hat{g}$ for $t \to -\infty$.*

Perelman's monotonicity formula for λ implies that the ancient Ricci flow obtained in Theorem 2 must become singular in finite time and hence leaves any $C^{k,\alpha}$-neighborhood of $\hat{g}$.

Theorems 1 and 2 describe the dynamical behavior of the Ricci flow near a given Ricci-flat metric. In fact, they show that dynamical stability and instability are characterized exactly by the local maximizing property of λ, observing whether or not $\lambda \leq 0$ in some $C^{k,\alpha}$-neighborhood of $\hat{g}$ ($k \geq 2$). The converse implications follow immediately from Perelman's monotonicity formula, i.e. if the conclusion of Theorem 1 holds, then $\hat{g}$ it is a local maximizer of λ; if the conclusion of Theorem 2 holds, then $\hat{g}$ is not a local maximizer of λ.

Another related notion is *linear stability*, meaning that all eigenvalues of the Lichnerowicz Laplacian $L_{\hat{g}} = \triangle_{\hat{g}} + 2\mathrm{Rm}_{\hat{g}}$ are nonpositive. If $\hat{g}$ is a local maximizer of λ, then it is linearly stable [1, Thm. 1.1]. If $\hat{g}$ is linearly stable and integrable, then it is a local maximizer of λ, c.f. [3, Thm. A].

In addition to applying to the more general nonintegrable case, the proofs that we give here are substantially shorter than the previous arguments from [6, 3]. Our main technical tool is the following Łojasiewicz-Simon inequality for Perelman's λ-functional, which generalizes [3, Thm. B] to the nonintegrable case.

Theorem 3 (Łojasiewicz-Simon inequality for λ, [4])**.** *Let $(M, \hat{g})$ be a closed Ricci-flat manifold. Then there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U}$ of $\hat{g}$ in the space of metrics on M and a $\theta \in (0, \frac{1}{2}]$, such that*

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g f_g\|_{L^2(M, e^{-f_g} dV_g)} \geq |\lambda(g)|^{1-\theta}\,, \tag{3}$$

for all $g \in \mathcal{U}$, where f_g is the minimizer in (1) *realizing $\lambda(g)$.*

Theorem 3 can be used as a general tool to study stability and convergence problems for the Ricci flow, and might thus be of independent interest. A key step in our proofs of Theorems 1 and 2 is then to modify the Ricci flow by an appropriate family of diffeomorphisms so that we can on the one hand exploit the geometric inequality (3) and on the other hand retain the needed analytic estimates.

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Some applications of $W^{2,2}$ conformal immersions to the Willmore functional

Yuxiang Li

Let (Σ, h) be a Riemann surface without boundary. We define a $W^{2,2}$-conformal immersion from (Σ, h) into $\mathbb{R}^n$ as follows:

Definition 1. *A map $f \in W^{2,2}(\Sigma, h, \mathbb{R}^n)$ is called a conformal immersion, if*

$$df \otimes df = e^{2u} h \quad \text{with} \quad \|u\|_{L^\infty(\Sigma)} < +\infty.$$

We denote the set of all such immersions by $W^{2,2}_{conf}(\Sigma, h, \mathbb{R}^n)$. If $f \in W^{2,2}_{loc}(\Sigma, h, \mathbb{R}^n)$ with $df \otimes df = e^{2u}h$ and $u \in L^\infty_{loc}(\Sigma)$, we say $f \in W^{2,2}_{conf,loc}(\Sigma, h, \mathbb{R}^n)$.

When $f \in W^{2,2}_{conf,loc}(D \setminus \{0\}, \mathbb{R}^n)$, we proved in [K-L] the following:

Theorem 1. [K-L] *Suppose that $f \in W^{2,2}_{conf,loc}(D\setminus\{0\}, \mathbb{R}^n)$ satisfies*

$$\int_D |A_f|^2 \, d\mu_g < \infty \quad \text{and} \quad \mu_g(D) < \infty,$$

where $g_{ij} = e^{2u}\delta_{ij}$ is the induced metric. Then $f \in W^{2,2}(D, \mathbb{R}^n)$ and we have

$$\begin{aligned} u(z) &= \lambda \log|z| + \omega(z) \quad \text{where } \lambda \geq 0, \quad \lambda \in \mathbb{Z}, \quad \omega \in C^0 \cap W^{1,2}(D), \\ -\Delta u &= -2\lambda\pi\delta_0 + K_g e^{2u} \quad \text{in } D. \end{aligned}$$

The density of $f(D_\sigma)$ as varifolds at $f(0)$ is given by $\lambda + 1$ for any small $\sigma > 0$.

Thus, when Σ is closed, a branched $W^{2,2}$-conformal immersion of (Σ, h) can be defined as follows:

Definition 2. *We say $f : (\Sigma, h) \to \mathbb{R}^n$ is a branched conformal immersion or $f \in W^{2,2}_{b,c}(\Sigma, h, \mathbb{R}^n)$, if we can find finite points $p_1, \cdots, p_m \in \Sigma$, such that $f \in W^{2,2}_{conf,loc}(\Sigma \setminus \{p_1, \cdots, p_m\}, h, \mathbb{R}^n)$ with*

$$\mu(f) + \int_\Sigma |A_f|^2 d\mu_f < +\infty.$$

We can check that the Gauss-Bonnet formula and the Helein's convergence theorem [H] still hold for $W^{2,2}$-conformal immersion sequences. As an application, we get in [K-L] the existence of conformally constrained minimizers in any codimension below 8π. This is also obtained independantly in [R] by Rivière.

Further, using blowup analysis, we get the following:

Theorem 2. [C-L] *Suppose that $\{f_k\}$ is a sequence of $W^{2,2}$ branched conformal immersions of closed Riemann surfaces (Σ, h_k) in $\mathbb{R}^n$ and h_k is a smooth metric with constant curvature. If $f_k(\Sigma) \cap B_{R_0} \neq \emptyset$ for a fixed R_0 and*

$$\sup_k \{\mu(f_k) + W(f_k)\} < +\infty,$$

then either $\{f_k\}$ converges to a point, or there is a stratified surface Σ_∞ with $g(\Sigma_\infty) \leq g(\Sigma)$, a map $f_0 \in W^{2,2}_{b,c}(\Sigma_\infty, \mathbb{R}^n)$, such that a subsequence of $\{f_k(\Sigma)\}$ converges to $f_0(\Sigma_\infty)$ in Hausdorff distance with

$$\mu(f_0) = \lim_{k\to+\infty} \mu(f_k) \quad and \quad W(f_0) \leq \lim_{k\to+\infty} W(f_k).$$

For any $\eta \in C_0^\infty(\mathbb{R}^n)$, we have

$$\lim_{k\to+\infty} \int_\Sigma \eta(f_k) d\mu_{f_k} = \int_{\Sigma_\infty} \eta(f_0) d\mu_{f_0}.$$

Moreover, if $y_1, \ldots, y_m \in f_k(\Sigma)$ for all k, then $y_1, \ldots, y_m \in f_0(\Sigma_\infty)$.

As an application, we proved that for any $\epsilon > 0$, we can find an embedded Willmore sphere S, which has at most 5 singularities, such that

$$W(S) \leq 4\pi + \epsilon.$$

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Min-max Theory in Geometry

André Neves

(joint work with Fernando Marques)

Min-max theory was first used in Geometry by Birkhoff in the 20's to show that every sphere admits a close embedded geodesic. Since then the technique was explored to show that every sphere admits three closed embedded geodesics (Lusternick and Shnirelmann), that every manifold of dimension no bigger than eight admits a smooth embedded minimal hypersurface (Pitts and Schoen-Simon for the regularity), and that every 3-sphere admits an embedded minimal sphere (Simon-Smith).

Recently, Fernando and I used this technique to prove the Willmore conjecture and, with Agol, we also used this technique to solve a conjecture regarding two component links with least Mobius energy.

In the end, I mentioned my new result with Fernando Marques, where we show that manifolds with dimension no bigger than eight having a metric of positive Ricci curvature, admit an infinite number of minimal embedded hypersurfaces.

A local rigidity result for the deSitter-Schwarzschild space

Ivaldo Nunes
(joint work with Davi Maximo)

In [6], Schoen and Yau made the important observation that the second variation formula of area provides an interesting interplay between the scalar curvature of an orientable Riemannian three-manifold (M, g) and the topology of an orientable compact stable minimal surface $\Sigma \subset M$. As a consequence, we have that if (M, g) has nonnegative scalar curvature, then either Σ is a two-sphere or a totally geodesic two-torus.

Motivated by the above, Cai and Galloway [2] proved that if (M, g) is a Riemannian three-manifold with nonnegative scalar curvature and Σ is an embedded minimal two-torus which is locally of least area (which is a condition stronger than stability), then Σ is flat and totally geodesic, and M splits isometrically as a product $(-\epsilon, \epsilon) \times \Sigma$. The analogous rigidity result in the case where Σ has either positive constant Gauss curvature or negative constant Gauss curvature were recently proved in [1] and [5], respectively. We note that Micallef and Moraru [4] have found an alternative argument to prove these splitting results.

Our local rigidity result for the deSitter-Schwarzschild space is inspired by the above splitting results. The deSitter-Schwarzschild metrics are complete periodic rotationally symmetric metrics on $\mathbb{R} \times \mathbb{S}^2$ with constant positive scalar curvature, and have $\Sigma_0 = \{0\} \times \mathbb{S}^2$ as a strictly stable minimal two-sphere. They appear as spacelike slices of the deSitter-Schwarzschild spacetime, which is a solution to the vacuum Einstein equation with a positive cosmological constant. The deSitter-Schwarzschild metrics constitute a one-parameter family of metrics $\{g_a\}_{a\in(0,1)}$ and, in our work, we scale each g_a to have scalar cuvature equal to 2.

In [3], we begin by considering the general situation of a two-sided closed surface Σ which is a critical point of the Hawking mass on a Riemannian three-manifold (M, g) with $R \geq 2$. We recall that the Hawking mass of a compact surface $\Sigma \subset (M, g)$, denoted by $m_H(\Sigma)$, is defined as

$$m_H(\Sigma) = \left(\frac{|\Sigma|}{16\pi}\right)^{1/2} \left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\sigma - \frac{\Lambda}{24\pi}|\Sigma|\right),$$

where H is the mean curvature of Σ and $\Lambda = \inf R$. By writing the Euler-Lagrange equation of the Hawking mass, we prove that whenever Σ has nonnegative mean curvature then it must be minimal or umbilic with $R = 2$ and constant Gauss curvature along Σ.

In particular, whenever (M, g) is the deSitter-Schwarzschild space $(\mathbb{R} \times \mathbb{S}^2, g_a)$, the above says that critical points of the Hawking mass are either minimal surfaces of slices $\{r\} \times \mathbb{S}^2$.

The above considerations are evidence that local maximum of the Hawking mass in $(\mathbb{R} \times \mathbb{S}^2, g_a)$ must be slices. In our first result we show that slices are indeed local maxima in the following sense:

Theorem 1. *Let $\Sigma_r = \{r\} \times \mathbb{S}^2$ be a slice of the deSitter-Schwarzschild manifold $(\mathbb{R} \times \mathbb{S}^2, g_a)$. Then there exists an $\epsilon = \epsilon(r) > 0$ such that if $\Sigma \subset \mathbb{R} \times \mathbb{S}^2$ is an embedded two-sphere, which is a normal graph over Σ_r given by $\varphi \in C^2(\Sigma_r)$ with $\|\varphi\|_{C^2(\Sigma_r)} < \epsilon$, one has*

(i) *either $m_H(\Sigma) < m_H(\Sigma_r)$;*

(ii) *or Σ is a slice Σ_s for some s.*

The proof follows by showing that the second variation of the Hawking mass at each slice if strictly negative, unless the variation has constant speed, and using this to argue maximality among surfaces that are graphs with small C^2 norm over the slice.

Our second result is a local rigidity result for the deSitter-Schwarzschild space $(\mathbb{R} \times \mathbb{S}^2, g_a)$ which involves strictly stable minimal surfaces and the Hawking mass. We prove:

Theorem 2. *Let (M, g) be a Riemannian three-manifold with scalar curvature $R \geq 2$. If $\Sigma \subset M$ is an embedded strictly stable minimal two-sphere which locally maximizes the Hawking mass, then the Gauss curvature of Σ is constant equal to $1/a^2$ for some $a \in (0, 1)$ and a neighborhood of Σ in (M, g) is isometric to the deSitter-Schwarzschild metric $((-\epsilon, \epsilon) \times \Sigma, g_a)$ for some $\epsilon > 0$.*

The idea of the proof goes as follows. Let $\lambda_1(\Sigma)$ denote the first eigenvalue of the Jacobi operator of Σ. The first step is to prove an infinitesimal rigidity along Σ which is obtained as follows. Using the fact that Σ is strictly stable we get an upper bound of the form

$$(1 + \lambda_1(\Sigma))|\Sigma| \leq 4\pi. \tag{1}$$

On the other hand, the fact that Σ locally maximizes the Hawking mass implies (1) with opposite sign. Therefore equality is achieved and from it the infinitesimal rigidity is attained.

From this infinitesimal rigidity we next are able to construct a foliation of a neighborhood of Σ by embedded constant mean curvature two-spheres $\{\Sigma(t) \subset M\}_{t \in (-\epsilon, \epsilon)}$, where $\Sigma(0) = \Sigma$. Finally, by using the properties of the foliation $\Sigma(t)$ we obtain, decreasing ϵ if necessary, a monotonicity of the Hawking mass along $\Sigma(t)$. In particular, we get that $m_H(\Sigma(t)) \geq m_H(\Sigma)$ for all $t \in (-\epsilon, \epsilon)$. The rigidity result then follows from this.

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A local regularity theorem for the network flow

Felix Schulze
(joint work with Tom Ilmanen, André Neves)

The network flow is the evolution of a network of curves under curve shortening flow in the plane, where it is allowed that at triple points three curves meet under a 120 degree condition. We present here a local regularity theorem for the network flow, which is similar to the result of B. White, [1], for smooth mean curvature flow.

In the statement of the following theorem k is the curvature of the evolving network. We denote with $\Theta(x,t,r)$ the Gaussian density density ratio of radius r, centered at the space-time point (x,t). By Huisken's monotonicity formula this a increasing function in r and the limit as $r \to 0$ is called the Gaussian density $\Theta(x,t)$. We denote with $\Theta_{S^1} = \sqrt{\frac{2\pi}{e}} > \frac{3}{2}$ the Gaussian density of the centered self-similarly shrinking circle.

Theorem. *Let $(\gamma_t)_{t\in[0,T)}$ be a smooth, regular network flow which reaches the point x_0 at time $t_0 \in (0,T]$. Let $0 < \varepsilon, \eta < 1$. There exist $C = C(\varepsilon,\eta)$ such that if*

$$\Theta(x,t,r) \leq \Theta_{S^1} - \varepsilon$$

for all $(x,t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ and $0 < r < \eta\rho$ for some $\eta > 0$, where $(1+\eta)\rho^2 \leq t_0 < T$. Then

$$|k|^2(x,t) \leq \frac{C}{\sigma^2\rho^2}$$

for $(x,t) \in \big(\gamma_t \cap B_{(1-\sigma)\rho}(x_0)\big) \times (t_0 - (1-\sigma)^2\rho^2, t_0)$ and all $\sigma \in (0,1)$.

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Constrained Willmore Surfaces

Yann Bernard

The seminal work [Ri1] of Tristan Rivière showed that in local conformal coordinates, the Willmore operator may be recast in a *conservative form* (i.e. written as the flat divergence of a suitable quantity). In the first part of the presentation, this fact was derived from "first principles". To do so, one considers an embedded surface $\Sigma \subset \mathbb{R}^3$ with local coordinates $\{x_1, x_2\}$ and described by a three-component function $\vec{\Phi}$. We then consider a variation of the form

$$\delta\vec{\Phi} \;=\; B\vec{n} + A^j\partial_j\vec{\Phi}\,,$$

where $\vec{n}$ is the outward unit normal to Σ, and B and A^j are arbitrary. For any subset $\Sigma_0 \subset \Sigma$, one shows that there holds

$$\delta\int_{\Sigma_0} H^2 d\mathrm{vol}_g \;=\; \int_{\Sigma_0}\Big[B\mathcal{W} + \nabla_j\big(H\nabla^j B - B\nabla^j H + H^2 A^j\big)\Big]d\mathrm{vol}_g\,, \tag{1}$$

where $\mathcal{W}$ is the Willmore operator

$$\mathcal{W} \;:=\; \Delta_g H + 2H(H^2 - K)\,.$$

Specializing to rigid translations (which leave the Willmore energy unchanged) begets a *stress tensor*

$$\vec{T}^j \;:=\; H\nabla^j\vec{n} - \vec{n}\nabla^j\vec{H} + H^2\nabla^j\vec{\Phi}$$

satisfying

$$\nabla_j\vec{T}^j \;=\; -\,\mathcal{W}\,\vec{n}\,. \tag{2}$$

This is the identity originally derived in [Ri1]. Similarly, a rigid rotation in (1) yields a "torque" and an analogue of (2). Finally, one may also consider a dilation to obtain yet another analogue of (2). Namely,

$$\left\{\begin{array}{rcl} \nabla_j\big(\vec{T}^j\times\vec{\Phi} + \vec{H}\times\nabla^j\vec{\Phi}\big) & = & -\,\mathcal{W}\,\vec{n}\times\vec{\Phi} \\ \nabla_j\big(\vec{T}^j\cdot\vec{\Phi}\big) & = & -\,\mathcal{W}\,n\cdot\vec{\Phi}\,. \end{array}\right.$$

We then perform three Hodge decompositions[1]:

$$\left\{\begin{array}{lcl} \vec{T}^j \;=\; \nabla^j\vec{V} + |g|^{-\frac12}\partial_j^{\perp}\vec{L} & ; & \Delta_g\vec{V} \;=\; -\,\mathcal{W}\,\vec{n} \\ |g|^{-\frac12}\vec{L}\times\partial_j^{\perp}\vec{\Phi} - \vec{H}\times\nabla^j\vec{\Phi} \;=\; \nabla^j\vec{X} + |g|^{-\frac12}\partial_j^{\perp}\vec{R} & ; & \Delta_g\vec{X} \;=\; \nabla^j\vec{V}\times\partial_j\vec{\Phi} \\ |g|^{-\frac12}\vec{L}\cdot\partial_j^{\perp}\vec{\Phi} \;=\; \nabla^j Y + |g|^{-\frac12}\partial_j^{\perp}S & ; & \Delta_g Y \;=\; \nabla^j\vec{V}\cdot\partial_j\vec{\Phi}\,, \end{array}\right.$$

for some $\vec{L}$, $\vec{R}$, and S. It can then directly be verified that[2]

$$\left\{\begin{array}{rcl} -\,|g|^{\frac12}\Delta_g S & = & \partial_j\vec{n}\cdot\partial_j^{\perp}\vec{R} + |g|^{\frac12}\nabla_j\big(\vec{n}\cdot\nabla^j\vec{X}\big) \\ -\,|g|^{\frac12}\Delta_g\vec{R} & = & \partial_j\vec{n}\times\partial_j^{\perp}\vec{R} + \partial_j\vec{n}\,\partial_j^{\perp}S + |g|^{\frac12}\nabla_j\big(\vec{n}\times\nabla^j\vec{X} + \vec{n}\nabla^j Y\big) \\ \partial_j\Big(S\,\partial_j^{\perp}\vec{\Phi} + \vec{R}\times\partial_j^{\perp}\vec{\Phi} + |g|^{\frac12}\nabla^j\vec{\Phi}\Big) & = & |g|^{\frac12}\nabla_j\Big(\nabla^j\vec{X}\times\partial_j\vec{\Phi} + \nabla^j Y\,\partial_j\vec{\Phi}\Big)\,. \end{array}\right.$$

---

[1] $\partial_1^{\perp} := -\partial_2$ and $\partial_2^{\perp} = \partial_1$.

[2] the last identity can be obtained through considering an "inversion" in (1).

The second part of the presentation dealt with isolated branch points. We consider a local immersion $\vec{\Phi} : D^2 \setminus \{0\} \to \mathbb{R}^3$ with

$$\vec{\Phi} \in C^0(D^2) \cap C^\infty(D^2 \setminus \{0\}) \qquad \text{and} \qquad \int_{D^2} |\vec{\mathbb{I}}|_g^2 \, d\text{vol}_g < \infty\,,$$

where $\vec{\mathbb{I}}$ is the second fundamental form. According to [KL] and [Ri2], we may assume, after reparametrization if necessary, that $\vec{\Phi}$ is conformal. Furthermore, there exists a finite integer $\theta_0 \in \{1, 2, \dots\}$ such that locally around the singularity at the origin, there holds

$$|\vec{\Phi}|(x) \simeq |x|^{\theta_0} \qquad \text{and} \qquad |\nabla\vec{\Phi}|(x) \simeq |x|^{\theta_0 - 1}\,.$$

We then specialize to the class of *constrained Willmore immersions* satisfying the equation

$$\mathcal{W} \;=\; -\,\mathrm{e}^{-2\lambda}\Re\big(H_0 f\big) \qquad \text{on } D^2 \setminus \{0\}\,, \tag{3}$$

where λ is the conformal parameter, H_0 is the Weingarten operator, and f is some integrable anti-holomorphic function given, in general, independently of the geometric data. Such immersions include as examples Willmore immersions (with $f \equiv 0$) and CMC immersions ($f = \mathrm{e}^{2\lambda}H\overline{H_0}$, anti-holomorphic owing to the Codazzi-Mainardi identity). Constrained Willmore immersions also occur as limits of Palais-Smale sequences for the Willmore functional [BR1]. The constrained Willmore equation (3) is satisfied by the critical points of the Willmore energy restricted to a fixed conformal class [BPP, KS, Ri2, S]. This prompts us in particular to refer to the anti-holomorphic function f in (3) as *Lagrange multiplier.*

In [B], it was observed that

$$\Re(H_0 f)\vec{n} \;=\; \text{div}\big(\mathrm{e}^{-2\lambda}M[f]\nabla^\perp\vec{\Phi}\big) \qquad \text{where} \qquad M[u] := \begin{pmatrix} -\Im(u) & \Re(u) \\ \Re(u) & \Im(u) \end{pmatrix}.$$

Using the aforementioned observation on the operator $\mathcal{W}$, the constrained Willmore equation thus reads

$$\text{div}\Big(\nabla\vec{H} + H^2\nabla\vec{\Phi} - M[\mathrm{e}^{-2\lambda}f - 2H\overline{H_0}]\nabla^\perp\vec{\Phi}\Big) \;=\; \vec{0} \qquad \text{on } D^2 \setminus \{0\}\,.$$

Defining the *first residue*

$$\vec{\beta}_0 \;:=\; \frac{1}{2\pi}\int_{\partial D^2} \vec{\nu}\cdot\Big(\nabla\vec{H} + H^2\nabla\vec{\Phi} - M[\mathrm{e}^{-2\lambda}f - 2H\overline{H_0}]\nabla^\perp\vec{\Phi}\Big)\,,$$

where $\vec{\nu}$ is the outward unit normal to D^2, enables us to infer the existence of $\vec{L}$ satisfying

$$\nabla^\perp\vec{L} \;:=\; \nabla\vec{H} + H^2\nabla\vec{\Phi} - M[\mathrm{e}^{-2\lambda}f - 2H\overline{H_0}]\nabla^\perp\vec{\Phi} - \vec{\beta}_0\nabla\log|x|\,. \tag{4}$$

Proceeding as above, one finds a pair $(S,\vec{R})\in W^{1,2}(D^2)^2$ which satisfy on the whole unit-disk D^2 the system

$$
(5)\qquad \left\{\begin{array}{rcl} -\Delta S &=& \nabla\vec{n}\cdot\nabla^{\perp}\vec{R}+\operatorname{div}\big(\vec{n}\cdot\nabla\vec{X}\big)\\ -\Delta\vec{R} &=& \nabla\vec{n}\times\nabla^{\perp}\vec{R}+\nabla\vec{n}\cdot\nabla^{\perp}S+\operatorname{div}\big(\vec{n}\times\nabla\vec{X}+\vec{n}\nabla Y\big)\\ -\Delta\vec{\Phi} &=& \big(\nabla S-\nabla^{\perp}Y\big)\cdot\nabla^{\perp}\vec{\Phi}+\big(\nabla\vec{R}-\nabla^{\perp}\vec{X}\big)\times\nabla^{\perp}\vec{\Phi}\,, \end{array}\right.
$$

while

$$
\left\{\begin{array}{rcl} \Delta\vec{X} &=& -\vec{\beta}_0\nabla\log|x|\times\nabla\vec{\Phi}\\ \vec{X} &=& \vec{0} \end{array}\right. \;;\qquad \begin{array}{rcll} \Delta Y &=& -\vec{\beta}_0\nabla\log|x|\cdot\nabla\vec{\Phi} & \text{in } D^2\\ Y &=& 0 & \text{on } \partial D^2\,. \end{array}
$$

Calling upon standard Wente-type estimates, one infers from the first two equations in (5) that ∇S and $\nabla\vec{R}$ must in $L^p(D^2)$ for all $p<\infty$. Moreover, using the last equation in (5) along with

$$
-\frac{1}{2}\,\Delta\vec{n}\;=\;\operatorname{div}\big(\vec{H}\times\nabla^{\perp}\vec{\Phi}\big)\;-\;\mathrm{e}^{2\lambda}K\vec{n}
$$

yields that $\nabla\vec{n}$ lies in $BMO(D^2)$. Furthermore, we deduce a first asymptotic behavior for the conformal immersion near the singularity[3]:

Proposition 1. *There exists* $\vec{A}:=\vec{A}_1+i\vec{A}_2\in\mathbb{C}^3$ *with*

$$
\vec{A}_1\cdot\vec{A}_2=0\quad,\quad|\vec{A}_1|=|\vec{A}_2|\neq 0\quad,\quad\vec{n}(0)\cdot\vec{A}=\vec{0}
$$

and

$$
\vec{\Phi}\;=\;\Re\big(\vec{A}\,z^{\theta_0}\big)+O_1(|z|^{\theta_0+1-\epsilon})\qquad\forall\;\epsilon>0\,.
$$

In turn, since the Lagrange multiplier function f is anti-holomophic and integrable, we may write it as $f=a_\mu\overline{z}^{\mu}+[C^\infty_{\text{anti-holo}}]$ for some coefficient $a_\mu\neq 0$ and some integer $\mu\geq -1$. In particular, with $\vec{A}$ as in Proposition 1, we may write

$$
\mathrm{e}^{-2\lambda}f\partial_z\vec{\Phi}\;=\;\partial_{\bar{z}}\vec{F}_\mu+\vec{J}
$$

$$
\text{with}\qquad \vec{F}_\mu\;:=\;\frac{2a_\mu}{\theta_0|\vec{A}|^2}\vec{A}\left\{\begin{array}{lll} 2\log|z| & , & \mu=\theta_0-2\\ (\mu+2-\theta_0)^{-1}\overline{z}^{\,\mu+2-\theta_0} & , & \mu\neq\theta_0-2 \end{array}\right.
$$

and $\vec{J}=\mathrm{O}(|z|^{\mu+2-\theta_0})$.
Note that (4) may be equivalently recast in the form

$$
\partial_{\bar{z}}\big(\vec{H}-\vec{\beta}_0\log|z|+\vec{F}_\mu-i\vec{L}\big)\;=\;-H^2\partial_{\bar{z}}\vec{\Phi}\;-\;2(H\overline{H}_0)\partial_z\vec{\Phi}+\vec{J}\,.
$$

Setting

$$
\partial_{\bar{z}}\vec{Q}\;=\;-H^2\partial_{\bar{z}}\vec{\Phi}\;-\;2(H\overline{H}_0)\partial_z\vec{\Phi}+\vec{J}
$$

gives rise to a meromorphic function

$$
\vec{E}\;:=\;\vec{H}-\vec{\beta}_0\log|z|+\vec{F}_\mu-i\vec{L}-\vec{Q}
$$

with a pole at the origin of order

$$
\max\{0,\theta_0-2-\mu\}\;\leq\;\beta\;\leq\;\theta_0-1\,.
$$

[3]we introduce on D^2 the complex coordinates $z:=x_1+ix_2$ and $\bar{z}:=x_1-ix_2$.

We call the integer β the *second residue* at the origin. It is a decisive quantity:

Theorem 1. *Locally around the singularity, there holds for all* $\epsilon > 0$*:*

$$\vec{\Phi} \;=\; \Re\Big(\vec{A}\,z^{\theta_0} + \sum_{j=1}^{\theta_0-\beta}\vec{B}_j z^{\theta_0+j} \,+\, \vec{C}|z|^{2\theta_0}z^{-\beta}\Big) \,+\, c_0\vec{\beta}_0|z|^{2\theta_0}\big(\log|z|^{2\theta_0}-4\big) \\ +\; O_{\theta_0-\beta+1}\big(|z|^{2\theta_0-\beta+1-\epsilon}\big)\,,$$

where $\vec{A}$ is as in Proposition 1, while $\vec{B}_j$, $\vec{C}$, and $c_0 \neq 0$ are constants.
In particular, we have

$$\vec{\Phi} \in \bigcap_{p<\infty}\left\{\begin{array}{lcl} W^{2,p} & , & \theta_0 = 1 \\ W^{\theta_0+2-\beta,p} & , & \theta_0 \geq 2 \end{array}\right.$$

Furthermore, the mean curvature vector satisfies

$$\vec{H} \;=\; \Re(\vec{C}_\beta\, z^{-\beta}) + \vec{\beta}_0\log|z| + O_{\theta_0-\beta-1}(|z|^{1-\beta-\epsilon}) \qquad \forall\ \epsilon > 0\,,$$

for some constant $\vec{C}_\beta \in \mathbb{C}^3$.

In order to ensure the smoothness of the immersion through the branch point at the origin, it is necessary (although not always sufficient) to demand that both residues β and $\vec{\beta}_0$ vanish. Namely,

Theorem 2. *Suppose that both residues vanish. Then*

(i) *if $\theta_0 < \mu + 2$, the immersion is smooth across the branch point;*
(ii) *if $\theta_0 = \mu + 2$, the immersion is $C^{\theta_0+1,1-\epsilon}$ $\forall\,\epsilon > 0$.*

In the special case when there is no branch point at the origin, it is possible to infer that

Corollary 1. *Suppose the origin is a regular point (i.e. the immersive nature of $\vec{\Phi}$ holds there). Then*

(i) *if the Lagrange multiplier f is regular at the origin ($\mu \geq 0$), the immersion is smooth;*
(ii) *if the Lagrange multiplier f is singular at the origin ($\mu = -1$), the immersion is $C^{2,1-\epsilon}$ $\forall\,\epsilon > 0$.*

Finally, these results apply to specific types of constrained Willmore immersions so as to yield:

Corollary 2. (i) *A CMC immersion has vanishing residues. It is thus smooth across branch points and regular points alike.*
(ii) *A Willmore immersion is smooth across regular points. It is smooth across a branch point if both corresponding residues vanish (see* [BR2]*).*

Amongst others, these results are contained in the article [B].

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A Li-Yau type inequality for free boundary surfaces with respect to the unit ball

Alexander Volkmann

A classical inequality due to Li and Yau [5] states that for a closed immersed 2-surface $F : \Sigma \to \mathbb{R}^n$ the Willmore energy $\mathcal{W}(F)$, given by

$$\mathcal{W}(F) := \frac{1}{4}\int_\Sigma H^2 \, d\mathcal{H}^2_{F^*\delta},$$

can be bounded from below by 4π times the maximum multiplicity of the surface. Here, the mean curvature H is defined to be the trace of the second fundamental form, and $\mathcal{H}^2_{F^*\delta}$ denotes the 2-dimensional Hausdorff measure with respect to $F^*\delta$, the pullback metric of the euclidean metric in $\mathbb{R}^n$ along F.

In [6] Simon used a special test vector field in the first variation identity to prove a monotonicity identity for closed immersed surfaces with square integrable mean curvature, which as a corollary lead to a new proof of the Li-Yau inequality (see also [4]).

In this talk we consider compact *free boundary surfaces with respect to the unit ball B in $\mathbb{R}^n$*, i.e. compact surfaces $\Sigma \subset \mathbb{R}^n$, the boundaries $\partial\Sigma \neq \emptyset$ of which meet the boundary ∂B of the unit ball B orthogonally. More precisely, we consider integer rectifiable 2-varifolds μ in $\mathbb{R}^n$ of compact support $\Sigma := \mathrm{spt}(\mu)$, $\Sigma \cap \partial B \neq \emptyset$, with generalized mean curvature $\vec{H} \in L^2(\mu; \mathbb{R}^n)$ such that

$$\int \mathrm{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu \tag{1}$$

for all vector fields $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ with $X \cdot \gamma = 0$ on ∂B, where $\gamma(x) = x$ denotes the outward unit normal to B (the open unit ball in $\mathbb{R}^n$). Furthermore, we assume that $\mu(\partial B) = 0$.

It follows from the work of Grüter and Jost [3] that μ has bounded first variation $\delta\mu$. Hence, by Lebesgue's decomposition theorem there exists a Radon measure $\sigma = |\delta\mu|\llcorner Z$ ($Z = \{x \in \mathbb{R}^n : D_\mu|\delta\mu|(x) = +\infty\}$) and a vector field $\eta \in L^1(\sigma;\mathbb{R}^3)$ with $|\eta| = 1$ σ-a.e. such that

$$\delta\mu(X) =^{def} \int \mathrm{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu + \int X \cdot \eta \, d\sigma \tag{2}$$

for all $X \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$. It easily follows from (1) that

$$\mathrm{spt}(\sigma) \subset \partial B \quad \text{and} \quad \eta \in \{\pm\gamma\} \ \ \sigma\text{-a.e.}.$$

We shall henceforth refer to such varifolds μ as *free boundary varifolds* (with respect to the unit ball).

In case μ is given by a smooth embedded surface Σ (i.e. $\mu = \mathcal{H}^2\llcorner\Sigma$) η is the outward unit conormal to Σ and $\sigma = \mathcal{H}^1\llcorner\partial\Sigma$, and we say that Σ is a free boundary surface (with respect to the unit ball).

Inspired by the interpretation of Simon's test vector field, a desingularized-cut-off version of $Y(x) = \frac{x-x_0}{|x-x_0|^2}$, as the gradient of the Newtonian potential of $\mathbb{R}^2$ evaluated in $\mathbb{R}^n$, we use a desingularized-cut-off version of the gradient of the Neumann Green's function of the Laplacian with respect to the unit disk in $\mathbb{R}^2$, evaluated in $\mathbb{R}^n$ to plug into equation (1).

From this we obtain a monotonicity identity for these surfaces, which is analogous to Simon's monotonicity identity [6].

Lemma. *For $x_0 \in \mathbb{R}^n$ consider the functions g_{x_0} and $\hat{g}_{x_0}$ given by*

$$g_{x_0}(r) := \frac{\mu(B_r(x_0))}{\pi r^2} + \frac{1}{16\pi}\int_{B_r(x_0)} |\vec{H}|^2 \, d\mu + \frac{1}{2\pi r^2}\int_{B_r(x_0)} \vec{H} \cdot (x - x_0) \, d\mu$$

and

$$\begin{aligned}\hat{g}_{x_0}(r) := {} & g_{\xi(x_0)}(r/|x_0|) \\ & - \frac{1}{\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} (|x - \xi(x_0)|^2 + P_x(x - \xi(x_0)) \cdot x) \, d\mu \\ & - \frac{1}{2\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu \\ & + \frac{1}{2\pi}\int_{\hat{B}_r(x_0)} \vec{H} \cdot x \, d\mu + \frac{\mu(\hat{B}_r(x_0))}{\pi},\end{aligned}$$

for $x_0 \neq 0$, where $\xi(x) := \frac{x}{|x|^2}$, $\hat{B}_r(x_0) = B_{r/|x_0|}(\xi(x_0))$, and

$$\hat{g}_0(r) = -\frac{1}{\pi}\min(r^{-2}, 1)\mu(\mathbb{R}^n) - \frac{\min(r^{-2}, 1)}{2\pi}\int \vec{H} \cdot x \, d\mu.$$

Then for any $0 < \sigma < \rho < \infty$ *we have*

$$
\begin{aligned}
&\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu \\
&+\frac{1}{\pi}\int_{\hat{B}_\rho(x_0)\setminus \hat{B}_\sigma(x_0)} \left|\frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu \qquad (3)\\
&= (g_{x_0}(\rho)+\hat{g}_{x_0}(\rho)) - (g_{x_0}(\sigma)+\hat{g}_{x_0}(\sigma)),
\end{aligned}
$$

where the second integral in (3) *is to be interpreted as* 0 *in case* $x_0 = 0$. *Here* $(x-x_0)^\perp := (x-x_0) - P_x(x-x_0)$, *where* P_x *denotes the orthogonal projection onto* $T_x\mu$, *the approximate tangent space of* μ *at* x. *In particular,* $g+\hat{g}$ *is non-decreasing.*

As a consequence we obtain area bounds, and the existence of the density at *every* point on the surface, that is for every $x_0 \in \mathbb{R}^n$ the following quantity is well defined

$$
\widetilde{\theta}^2(\mu, x_0) := \begin{cases} \lim_{r\searrow 0}\left(\frac{\mu(B_r(x_0))}{\pi r^2} + \frac{\mu(\hat{B}_r(x_0))}{\pi(|x_0|^{-1}r)^2}\right) & ; x_0 \neq 0, \\ \lim_{r\searrow 0}\frac{\mu(B_r(0))}{\pi r^2} & . \end{cases}
$$

As a limiting case of the monotonicity identity we obtain the following inequality.

$$
\widetilde{\theta}^2(\mu,x_0) \leq \frac{1}{8\pi}\int |\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int \vec{H}\cdot x\,d\mu + \frac{\mu(\mathbb{R}^n)}{\pi} = \frac{1}{8\pi}\int |\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int x\cdot\eta\,d\sigma.
$$

This inequality can be seen as a generalization of a sharp isoperimetric inequality for free boundary minimal surfaces with respect to the unit ball in $\mathbb{R}^n$ due to Fraser and Schoen [2, Theorem 5.4 & Corollary 5.5] to not necessarily *minimal* surfaces. In this context we also mention the work of Brendle [1] in which the author had used a similar idea to ours to generalize the Fraser-Schoen inequality to higher-dimensional free boundary minimal surfaces with respect to the unit ball in $\mathbb{R}^n$.

The Willmore energy $\mathcal{W}(F)$ of a smooth immersed compact orientable surface $F : \Sigma \to \mathbb{R}^n$ with boundary $\partial\Sigma$ is given by

$$
\mathcal{W}(F) := \frac{1}{4}\int_\Sigma H^2\,d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma}\kappa_g\,d\mathcal{H}^1_{F^*\delta},
$$

where κ_g denotes the geodesic curvature of $\partial\Sigma$ as a curve in Σ. For free boundary surfaces with respect to the unit ball we have that

$$
\kappa_g = D_\tau\eta\cdot\tau = D_\tau(\eta\cdot x\,x)\cdot\tau = x\cdot\eta, \quad (\tau\in T(\partial\Sigma), |\tau|=1)
$$

hence the Willmore energy may be rewritten as

$$
\mathcal{W}(F) = \frac{1}{4}\int_\Sigma |\vec{H}|^2\,d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma} x\cdot\eta\,d\mathcal{H}^1_{F^*\delta},
$$

which can also be made sense of for free boundary varifolds with respect to the unit ball.

Finally, we have the following Li-Yau type theorem.

Theorem. *For any immersion $F : \Sigma \to \mathbb{R}^n$ of a compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$ and the image varifold $\mu = \theta\mathcal{H}^2 \llcorner F(\Sigma)$, where $\theta(x) = \mathcal{H}^0(F^{-1}(\{x\}))$, we have*

$$\mathcal{H}^0(F^{-1}(\{x, \xi(x)\})) = \widetilde{\theta}^2(\mu, x) \leq \frac{1}{2\pi}\mathcal{W}(F),$$

in particular

$$W(F) \geq 2\pi, \tag{4}$$

and if

$$W(F) < 4\pi,$$

then F is an embedding. Moreover, equality in (4) *implies that F parametrizes a round spherical cap or a flat unit disk.*

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Gradient flow for the Möbius energy

Simon Blatt

In his 1991 paper [4], Jun O'Hara introduced the Möbius energy

$$E(\Gamma) := \int_\Gamma \int_\Gamma \left(\frac{1}{|y-x|^2} - \frac{1}{d_\Gamma(x,y)^2} \right) d\mathcal{H}^1(y) d\mathcal{H}^1(x)$$

for embedded curves $\Gamma \subset \mathbb{R}^3$, where $d_\Gamma(x,y)$ denotes the length of the shorter arc connecting the two points x and y and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. We want to discuss some recent results regarding the negative gradient flow of this energy. We are looking at a smooth family of embedded closed curves Γ_t, $t \in [0, \infty)$ which satisfies the evolution equation

$$\partial_t^\perp \Gamma_t = -\mathcal{H}\Gamma_t \quad \forall t \in [0, T) \tag{1}$$

where $\mathcal{H}\Gamma_t$ is the L^2-gradient of the Möbius energy. Already Freedman, He, and Wang [2] showed that this gradient can be expressed by

$$\mathcal{H}\Gamma := 2 \lim_{\varepsilon \to 0} \int_{\Gamma - B_\varepsilon(x)} \left(2 \frac{P^\perp_{\tau_\Gamma(x)}(y-x)}{|y-x|^2} - \kappa_\Gamma(x) \right) \frac{d\mathcal{H}^1(x)}{|y-x|^2}.$$

In the formula above τ_Γ stands for the unit tangent along Γ and $P^\perp_{\tau_\Gamma(x)} = id - \langle \cdot, \tau_\Gamma(x)\rangle \tau_\Gamma(x)$ denotes the orthogonal projection of $\mathbb{R}^3$ onto the normal space of Γ in x.

Due to the Möbius invariance of this energy and based on numerical experiments, one expects that in general this flow develops singularities after finite or infinite time. In this talk we analyze these singularities by constructing a blowup profile.

The fundamental result is the following: There is an $\varepsilon > 0$ such that either the solution of the gradient flow smoothly exists for all time or there exists a sequence of times t_j, radii $r_j \to 0$ and points $x_j \in \Gamma_{t_j}$ such that

$$\int_{\Gamma_{t_j}\cap B_{r_j}(x_j)} \int_{\Gamma_{t_j}\cap B_{r_j}(x_j)} \frac{|\tau_{\Gamma_{t_j}}(x) - \tau_{\Gamma_{t_j}}(y)|^2}{|x-y|^2} d\mathcal{H}^1(y) d\mathcal{H}^1(x) \geq \varepsilon,$$

i.e. a small quantum of energy concentrates as we approach the singularity. Furthermore, by picking the times t_j and points x_j a bit more carefully one can show that the rescaled curves

$$\tilde{\Gamma}_j := \frac{1}{r_j}\left(\Gamma_{t_j} - x_j\right)$$

satisfy

$$\|\partial_s^k \tilde{\gamma}_j\|_{L^\infty} \leq C_k$$

where $\tilde{\gamma}_j$ is an arc-length parametrization of $\tilde{\Gamma}_j$. Hence, using Arzela-Ascoli's lemma we can choose a subsequence of $\tilde{\Gamma}_j$ converging locally smoothly to a limit curve $\tilde{\Gamma}_\infty$, the *blowup profile*. Due to our construction, this profile will be properly embedded, has finite Möbius energy, cannot be a straight line, and satisfies the equation

$$\mathcal{H}\tilde{\Gamma}_\infty \equiv 0. \qquad (2)$$

In the last part of the talk, we discussed compact and non-compact *planar* solutions of (2) using the following interpretation of this equation which is motivated by the work of He [3]. In contrast to He's approach, we do not explicitly use the Möbius invariance of the energy:

Given two points $x, y \in \Gamma$ there is either a unique circle or a straight line – which we like to think of as a degenerate circle – going through x and y and being tangent to Γ at x. Note that this is the same circle, used to define the integral tangent-point energies. We denote by $\kappa_\Gamma(x,y)$ the curvature vector of this circle in x and set $\kappa_\Gamma(x,y) = 0$ if the tangent on Γ in x is pointing in the direction of y – which is the curvature of the straight line. Since

$$\kappa_\Gamma(x,y) = 2\frac{P^\perp_{\tau(x)}(y-x)}{|y-x|^2},$$

$H\Gamma \equiv 0$ is equivalent to

$$\lim_{\varepsilon\to 0} \int_{\Gamma - B_\varepsilon(x)} \frac{\kappa_\Gamma(x,y) - \kappa_\Gamma(x)}{|y-x|^2} \mathcal{H}^1(y) = 0$$

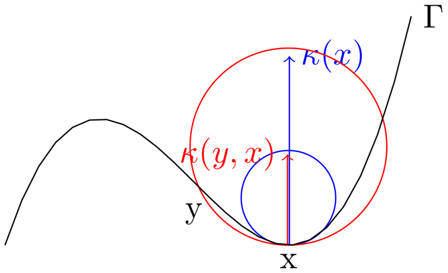

Figure 1. This picture shows the two circles playing a role in the geometric interpretation of the Euler-Lagrange equation of the Möbius energy: The blue circle is the osculating circle at x while the red circle is the circle going through x and y and being tangent to Γ at x.

for all $x \in \Gamma$.

Using this geometric version of the equation, we can prove the following

Theorem 1. *Let $\Gamma \subset \mathbb{R}^2$ be a properly embedded open or closed smooth curve of bounded curvature which satisfies*

$$\tilde{\mathcal{H}}\Gamma := 2 \lim_{\varepsilon \to 0} \int_{\Gamma \cap (B_{\frac{1}{\varepsilon}}(x) - (B_\varepsilon(x))} \left(2 \frac{P^{\perp}_{\tau_\Gamma(x)}(y-x)}{|y-x|^2} - \kappa_\Gamma(x) \right) \frac{d\mathcal{H}^1(x)}{|y-x|^2} = 0.$$

Let furthermore $x \in \Gamma$ be a point in which the curvature of Γ does not vanish, and such that the open ball B_x whose boundary is the osculating circle on Γ satisfies either $B_x \cap \Gamma = \emptyset$ or $\Gamma \subset \overline{B}_x$. Then $\Gamma = \partial B_x$, i.e. Γ agrees with its osculating circle in x.

Since all planar curves except the straight lines have such a point x, we get

Theorem 2. *The only properly embedded open or closed smooth curves $\Gamma \subset \mathbb{R}^2$ which satisfy $\tilde{\mathcal{H}}\Gamma$ are circles and straight lines.*

In [1] it was proven that near local minimizers the flow exists for all time and converges to a local minimizer on the same energy level as time goes to infinity. Combining this with the argument above, we get that the flow for closed planar curves exists for all time. A similar analysis of the asymptotic behavior for planar curves finally leads to

Theorem 3. *If Γ_0 is a planar curve, the solution of* (1) *exists smoothly for all time and converges to a circle as $t \to \infty$.*

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A frame energy for tori immersed in $\mathbb{R}^m$: sharp Willmore-conjecture type lower bound, regularity of critical points and applications

Andrea Mondino

(joint work with Tristan Rivière)

The purpose of the seminar is to present some recent results contained in [15] regarding the Dirichlet energy of moving frames associated to tori immersed in $\mathbb{R}^m$, $m \geq 3$. Moving frames have been played a key role in the modern theory of immersed surfaces starting from the pioneering works of Darboux [5], Goursat [7], Cartan [2], Chern [3]-[4], etc. (note also that in the book of Willmore [25], the theory of surfaces is presented from Cartan's point of view of moving frames, and the recent book of Hélein [8] is devoted to the role of moving frames in modern analysis of submanifolds; see also the recent introductory book of Ivey and Landsberg [10]). Indeed, due to the strong link between moving frames on an *immersed* surface and the conformal structure of the underlying *abstract* surface, the importance of selecting a "best moving frame" in surface theory is comparable to fixing an optimal gauge in physical problems (for instance for the study of Einstein's equations of general relativity it is natural to work in the gauge of the so called harmonic coordinates, for the analysis of Yang-Mills equation it is convenient the so called Coulomb gauge, etc.).

Before going to the description of the main results, we define the objects of our investigation.
Let $\mathbb{T}^2$ be the abstract 2-torus and let $\vec{\Phi} : \mathbb{T}^2 \hookrightarrow \mathbb{R}^m, m \geq 3$, be a smooth immersion (let us start with smooth immersions, then we will move to weak immersions). One denotes with $T\vec{\Phi}(\mathbb{T}^2)$ the tangent bundle to $\vec{\Phi}(\mathbb{T}^2)$, a pair $\vec{e} := (\vec{e}_1, \vec{e}_2) \in \Gamma(T\vec{\Phi}(\mathbb{T}^2)) \times \Gamma(T\vec{\Phi}(\mathbb{T}^2))$ is said *a moving frame* on $\vec{\Phi}$ if, for every $x \in \mathbb{T}^2$, the couple $(\vec{e}_1(x), \vec{e}_2(x))$ is a positive orthonormal basis for $T_x\vec{\Phi}(\mathbb{T}^2)$ (with positive we mean that we fix a priori an orientation of $\vec{\Phi}(\mathbb{T}^2)$ and that the moving frame agrees with it).

Given $\vec{\Phi}$ and $\vec{e}$ as above we define the *frame energy* as the Dirichelet energy of the frame, i.e.

$$\mathcal{F}(\vec{\Phi}, \vec{e}) := \frac{1}{4} \int_{\mathbb{T}^2} |d\vec{e}|^2 \, dvol_g, \tag{1}$$

where d is the exterior differential along $\vec{\Phi}$, $dvol_g$ is the area form given by the immersion $\vec{\Phi}$, and $|d\vec{e}|$ is the length of the exterior differential of the frame which is given in local coordinates by $|d\vec{e}|^2 = \sum_{k=1}^2 |d\vec{e}_k|^2 = \sum_{i,j,k=1}^2 g^{ij} \partial_{x_i} \vec{e}_k \cdot \partial_{x_j} \vec{e}_k$; here $\vec{u} \cdot \vec{v}$ denotes the euclidean scalar product in $\mathbb{R}^m$.

By projecting on the normal and on the tangent spaces and using Gauss Bonnet Theorem, the frame energy decomposes as

$$\mathcal{F}(\vec{\Phi},\vec{e}) = \frac{1}{2}\int_{\mathbb{T}^2} |\vec{e}_1 \cdot d\vec{e}_2|_g^2 \, dvol_g + \int_{\mathbb{T}^2} |\vec{H}|^2 \, dvol_g := \mathcal{F}_T(\vec{\Phi},\vec{e}) + W(\vec{\Phi}) \quad , \tag{2}$$

where $\vec{H}$ is the mean curvature vector and W is the Willmore functional (in our convention $W(\mathbb{S}^2) = 4\pi$) and $\mathcal{F}_T(\vec{\Phi},\vec{e}) := \frac{1}{2}\int_{\mathbb{T}^2} |\vec{e}_1 \cdot d\vec{e}_2|_g^2 \, dvol_g$, called *tangential frame energy*, is the L^2-norm of the covariant derivative of the frame with respect to the Levi-Civita connection.

Let us observe that the frame energy $\mathcal{F}$ is invariant under scaling and under conformal transformations of the pullback metric g, but not under conformal transformations of $\mathbb{R}^m$. Therefore, even if natural on its own, $\mathcal{F}$ can be seen as a more coercive Willmore energy where the extra term $\mathcal{F}_T$ prevents the degenerations caused by the action of the Moebius group of $\mathbb{R}^m$ and the degeneration of the confomal class of the abstract torus. More precisely we have the the following proposition.

Proposition 1. *For every $C > 0$, the metrics induced by the framed immersions in $\mathcal{F}^{-1}([0, C])$ are contained in a compact subset of the moduli space of the torus.*

The proof of Proposition 1 is remarkably elementary and makes use just of the Fenchel-Borsuk lower bound [6]-[1] on the total curvature of a closed curve in $\mathbb{R}^m$. Combining Proposition 1 with the celebrated results of Li-Yau [12] and Montiel-Ros [16] on the Willmore conjecture, we manage to prove the following sharp lower bound (with rigidity) on the frame energy.

Theorem 1. *Let $\vec{\Phi} : \mathbb{T}^2 \hookrightarrow \mathbb{R}^m$ be a smooth immersion of the 2-dimensional torus into the Euclidean $3 \le m$-dimensional space and let $\vec{e} = (\vec{e}_1, \vec{e}_2)$ be any moving frame along $\vec{\Phi}$.*

Then the following lower bound holds:

$$\mathcal{F}(\vec{\Phi},\vec{e}) := \frac{1}{4}\int_{\mathbb{T}^2} |d\vec{e}|^2 \; dvol_g \ge 2\pi^2 \quad . \tag{3}$$

Moreover, if in (3) equality holds then it must be $m \ge 4$, $\vec{\Phi}(\mathbb{T}^2) \subset \mathbb{R}^m$ must be, up to isometries and dilations in $\mathbb{R}^m$, the Clifford torus

$$T_{Cl} := S^1 \times S^1 \subset \mathbb{R}^4 \subset \mathbb{R}^m \quad , \tag{4}$$

and $\vec{e}$ must be, up to a constant rotation on $T(\vec{\Phi}(\mathbb{T}^2))$, the moving frame given by $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi})$, where of course (θ, φ) are natural flat the coordinates on $S^1 \times S^1$.

Remark 1. *Thanks to (2), in* codimension one*, the lower bound (3) follows by the recent proof of the Willmore conjecture by Marques and Neves [13] using min-max principle; the approach here is a more direct energy based consideratton. Indeed from their result non just the frame energy, but the Willmore functional $W(\vec{\Phi})$ is bounded below by $2\pi^2$ for any smooth immersed torus, and $W(\vec{\Phi}) = 2\pi^2$ if and only if $\vec{\Phi}$ is a conformal transformation of the Clifford torus. Curiously, our lower*

bound seems to work better in codimension at least two, where it becomes sharp and rigid; clearly, in codimension one it is not sharp because of the nonexistence of flat immersions of the torus in $\mathbb{R}^3$ *and because of the Marques-Neves proof of the Willmore conjecture.*
Let us also mention that Topping [24, Theorem 6]*, using arguments of integral geometry (very far from our proof), obtained an analogous lower bound on an analogous frame energy for immersed tori in* S^3 *under the assumption that the underlying conformal class of the immersion is a* rectangular *flat torus.*

For variational matters. the framework of smooth immersions has to be relaxed to a weaker notion of immersion introduced by the second author in [19] and denoted by $\mathcal{E}(\mathbb{T}^2, \mathbb{R}^m)$. Let us remark that Proposition 1 and Theorem 1 holds for weak immersions as well. In order to perform the calculus of variations of the frame energy, we establish that $\mathcal{F}$ is differentiable in $\mathcal{E}(\mathbb{T}^2, \mathbb{R}^m)$ and we compute the first variation. As for the Willmore energy (as well as for many important geometric problems as Harmonic maps, CMC surfaces, Yang Mills, Yamabe, etc.) the equation we obtain is *critical*. It is therefore challenging to prove the regularity of critical points of the frame energy.
Inspired by the work of Hélein [8] on CMC surfaces and of the second author on Willmore surfaces [18] (see also [14] for the manifold case and [21] for a comprehensive discussion), in order to study the regularity of the critical points of the frame energy we discover some new hidden conservation laws and we use them in order to deduce an elliptic system involving Jacobian nonlinearities satisfyed by the critical points of $\mathcal{F}$. Thanks to this special form, using the *theory of integrability by compensation* (for a comprehensive treatment see [20]), we are able to show smoothness of the solutions of this critical system. The smoothness of the critical points of the frame energy follows.

Finally we discuss an application of the tools developed here to the study of regular homotopy classes of immersions, more precisely we prove that in each of the two regular homotopy classes of tori immersed into $\mathbb{R}^3$ (for more details see [22],[23],[9] and [17]) there exists a smooth minimizer of the frame energy $\mathcal{F}$; such immersion can be seen as a natural representant of its own class.

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