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Mini-Workshop: Direct and Inverse Spectral Theory of Almost Periodic Operators

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Abstract. This mini-workshop brought together researchers working on direct and inverse spectral theory for Schrödinger operators, Jacobi matrices, and related operators. The talks reported on recent work on these models and related ones, such as the Anderson model.

Mathematics Subject Classification (2010): 47N50, 82B44.

Introduction by the Organisers

The general area of almost periodic Schrödinger operators has seen spectacular progress in recent years. This is partly due to the infusion of new ideas from various areas and also the infusion of new talent in the form of promising junior researchers. The talks presented at this mini-workshop represent many of the recent advances in this area.

Most of these recent advances concern one-dimensional models. Thus, one considers discrete operators

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$ or continuum operators

$$[H\psi](x) = -\psi''(x) + V(x)\psi(x)$$

in $L^2(\mathbb{R})$. Here, the potential V is assumed to be (real-valued and) almost periodic, that is, its translates have compact closure in $\ell^\infty(\mathbb{Z})$ (resp., $L^\infty(\mathbb{R})$). This class of potentials contains periodic potentials, limit-periodic potentials (i.e., uniform limits of periodic potentials), and quasi-periodic potentials (i.e., potentials obtained by continuous sampling along a translation on a finite-dimensional torus).

The operator domain is chosen so that the operator is self-adjoint, and hence the spectral theorem guarantees the existence of spectral measures.

The direct spectral problem consists in going from information about the potential to information about the spectrum and the spectral measures, whereas inverse spectral theory concerns the converse. It is well known that in the discrete case, the study of the inverse problem naturally takes place in a larger class of models, namely the class of Jacobi matrices with almost periodic coefficients. Jacobi matrices act in $\ell^2(\mathbb{Z})$ as follows,

$$[J\psi](n) = a(n)\psi(n+1) + b(n)\psi(n) + a(n-1)\psi(n-1),$$

where $a(n) > 0$ and $b(n) \in \mathbb{R}$. Thus, a discrete Schrödinger operator is a special case of a Jacobi matrix, where the diagonal terms are given by the potential and the off-diagonal terms are constant equal to one.

In the case of almost periodic coefficients, it is natural to consider the operator as a member of a family, the so-called hull. The latter is obtained by considering translates and taking the closure in the uniform topology. This gives a compact space which can be endowed with an Abelian group structure in a natural way. Then, the spectrum and the absolutely continuous spectrum are the same for all members of the family and the singular continuous spectrum and the pure point spectrum and the same for almost all members with respect to Haar measure. The direct spectral problem then consists in identifying the spectrum and the (almost sure) spectral parts associated with a given almost periodic family of operators. Conversely, the inverse spectral problem assumes information about the spectrum and/or the spectral parts and seeks to identify an operator family that has these spectral characteristics (or even all families that do).

Mini-Workshop: Direct and Inverse Spectral Theory of Almost Periodic Operators

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Abstracts

PW spectral theory — the Szegő class and beyond

Jacob Stordal Christiansen

As is well known, there is a one-one correspondence between bounded Jacobi matrices $J = \{a_n, b_n\}_{n=1}^{\infty}$ and probability measures $d\mu$ on $\mathbb{R}$ with compact support. This talk deals with the corresponding inverse spectral theory. Throughout, we fix a Parreau–Widom (PW) set $\mathbf{E}$ and focus only on those J's for which $\sigma_{\text{ess}}(J) = \mathbf{E}$. The notion of PW sets is suitably defined via conformal mappings of the upper half-plane onto comb-like domains of finite total teeth length and it includes all sets that are 'uniformly thick' (e.g., Cantor sets of positive measure). We note that the equilibrium measure $d\mu_{\mathbb{E}}$ is absolutely continuous.

A central object is the set $\mathcal{T}_{\mathbf{E}}$ of all two-sided Jacobi matrices $J' = \{a_n', b_n'\}_{n=-\infty}^{\infty}$ that are reflectionless on $\mathbf{E}$ and have spectrum equal to $\mathbf{E}$. This set is the natural limiting object associated with $\mathbf{E}$ as follows from the Denisov–Rakhmanov–Remling theorem [4]:

Theorem 1. *Suppose $|\mathbf{E}| > 0$ and let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with*

$$\sigma_{\text{ess}}(J) = \Sigma_{\text{ac}}(J) = \mathbf{E}. \tag{1}$$

Then any right limit of J belongs to $\mathcal{T}_{\mathbf{E}}$.

While the above theorem applies to a wide class of Jacobi matrices associated with $\mathbf{E}$, it does not give details on *how* the left-shifts of J approach $\mathcal{T}_{\mathbf{E}}$. In this talk, we shall assume more than (1) but so much the stronger the conclusion will be.

Motivated by the general version of Szegő's theorem proven in [1], we define the Szegő class for $\mathbf{E}$ to be the set of all probability measures $d\mu = f(t)dt + d\mu_{\text{s}}$, with $d\mu_{\text{s}}$ singular to dt, for which

i) the essential support is equal to $\mathbf{E}$,

ii) the absolutely continuous part satisfies the Szegő condition

$$\int_{\mathbf{E}} \log f(t)\, d\mu_{\mathbf{E}}(t) > -\infty,$$

iii) the isolated mass points $\{x_k\}$ outside $\mathbf{E}$, if any, satisfy the condition

$$\sum_k g(x_k) < \infty.$$

Here, g is the potential theoretic Green's function for the domain $\Omega := \overline{\mathbb{C}} \setminus \mathbf{E}$ with pole at ∞. The main result to be presented is valid when the direct Cauchy theorem holds on $\mathbb{C} \setminus \mathbf{E}$ and reads as follows:

Theorem 2. *Suppose $J = \{a_n, b_n\}_{n=1}^{\infty}$ belongs to the Szegő class for* $\mathbf{E}$*. Then there is a unique $J' = \{a'_n, b'_n\}_{n=-\infty}^{\infty}$ in $\mathcal{T}_{\mathbf{E}}$ such that*

$$|a_n - a'_n| + |b_n - b'_n| \to 0 \quad \text{as } n \to \infty. \tag{2}$$

Moreover, $\prod_{n=1}^{\infty}(a_n/a'_n)$ converges conditionally as does $\sum_{n=1}^{\infty}(b_n - b'_n)$.

The direct Cauchy theorem is needed to ensure that the Abel map

$$\mathcal{T}_{\mathbf{E}} \ni J' \longmapsto \chi(J') \in \pi_1(\Omega)^* \tag{3}$$

is a homeomorphism (cf. the seminal paper [5] of Sodin and Yuditskii). As for notation, $\pi_1(\Omega)^*$ is the multiplicative group of unimodular characters on the fundamental group of Ω. Besides, our proof only relies on Theorem 1 and a technical result stating that when passing to a right limit of J, we also have convergence of the associated characters (see [2] for details). Hence, the method of proof is very different from the variational approach of Peherstorfer and Yuditskii [3].

The strength of Theorem 2 lies in providing a unique element $J' \in \mathcal{T}_{\mathbf{E}}$ so that the left-shifts of J are asymptotic to the orbit of J' on $\mathcal{T}_{\mathbf{E}}$. As the proof will reveal, one can single out this unique J' by matching the associated characters. We mention in passing that the direct Cauchy theorem implies that every $J' \in \mathcal{T}_{\mathbf{E}}$ has uniformly almost periodic Jacobi parameters (i.e., almost periodic in the Bohr/Bochner sense).

The latter statement in Theorem 2 is a direct consequence of Szegő asymptotics for the associated orthonormal polynomials. We outline how one can arrive at this by introducing Jost solutions.

Finally, some conjectures and open problems will be mentioned. Suppose $J' = \{a'_n, b'_n\}_{n=-\infty}^{\infty}$ belongs to $\mathcal{T}_{\mathbf{E}}$ and let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be an arbitrary Jacobi matrix.

Conjecture 1. *If*

$$\sum_{n=1}^{\infty} |a_n - a'_n| + |b_n - b'_n| < \infty,$$

then J belongs to the Szegő class for $\mathbf{E}$*.*

Conjecture 2. *If J lies in the Szegő class for $\mathbf{E}$ and $\chi(J) = \chi(J')$, then*

$$\sum_{n=1}^{\infty} (a_n - a'_n)^2 + (b_n - b'_n)^2 < \infty.$$

In other words, the Szegő class is conjectured to lie between the ℓ^1 and ℓ^2 perturbations of points in $\mathcal{T}_{\mathbf{E}}$.

It is a widely open question how to characterize the spectral measures of all ℓ^2 perturbations of $\mathcal{T}_{\mathbf{E}}$. It is even open if ℓ^2 perturbations of points in $\mathcal{T}_{\mathbf{E}}$ preserve the a.c. spectrum.

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Inverse spectral problems for quasi-periodic operators and KdV with quasi-periodic initial data

Michael Goldstein
(joint work with David Damanik)

Consider the Schrödinger equation

$$-\psi''(x) + \varepsilon V(x)\psi(x) = E\psi(x), \qquad x \in \mathbb{R}$$

with quasi-periodic analytic V,

$$V(x) = \sum_{n\in\mathbb{Z}^\nu} c(n)e^{2\pi ixn\omega}, \quad x \in \mathbb{R}$$

$|c(n)|| \le \exp(-\kappa_0|n|)$, $\kappa_0 > 0$, and with Diophantine frequency vector $\omega \in \mathbb{R}^\nu$. Consider the regime of small ε, i.e $0 < \varepsilon < \varepsilon_0(\kappa_0, \omega)$. In the work D.Damanik, M.Goldstein "On the Inverse Spectral Problem for the Quasi-Periodic Schrödinger Equation" Preprint 123*pp.*, to appear in "Publ.Math. IHES" the following result was established. Let (E'_m, E''_m), $m \in \mathbb{Z}^\nu$, be the standard labeled gaps in the spectrum. We prove that $E''_m - E'_m \le 2\varepsilon \exp(-\frac{\kappa_0}{2}|m|)$ for all m. Our main result says that if $E''_m - E'_m \le \varepsilon \exp(-\kappa|m|)$ for all m, with $\varepsilon < \varepsilon_0$, $\kappa > 4\kappa_0$, then the Fourier coefficients of V in fact obey $|c(m)| \le \varepsilon^{1/2} \exp(-\frac{\kappa}{2}|m|)$ for all m.

In the preprint D.Damanik, M.Goldstein "On existence and uniqueness of global solutions of KdV equations with quasi-periodic initial data" the following result was established. Consider the KdV equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

with the initial data

$$u_0(x) = \sum_{n\in\mathbb{Z}^\nu} c(n)e^{2\pi ixn\omega}, \quad x \in \mathbb{R}$$

There exists $\epsilon_0(\kappa_0, \omega) > 0$ such that if $|c(n)| \le \epsilon \exp(-\kappa_0|n|)$, $0 < \epsilon \le \epsilon_0$, then for $0 \le t < \infty$, $x \in \mathbb{R}$, one can define a function

$$u(t, x) = \sum_{n\in\mathbb{Z}^\nu} c(t, n)e^{2\pi ixn\omega}$$

with $|c(t,n)| \le \epsilon^{1/4}\exp(-\frac{\kappa_0}{8}|n|)$, which obeys the KdV equation with the initial condition $u(0,x) = u_0(x)$. The inverse spectral problem estimates on the Fourier coefficients mentioned above plays crucial role in the derivation of this result.

Spectral Properties of Polyharmonic Operators with Quasi-periodic Potentials in Dimension Two

Yulia Karpeshina
(joint work with Roman Shterenberg)

We study an operator

$$H = (-\Delta)^l + V(\mathbf{x}) \tag{1}$$

in two dimensions, where l is an integer, $l \ge 2$, $V(\mathbf{x})$ is a quasi-periodic potential being a trigonometric polynomial:

$$V = \sum_{\mathbf{s}_1,\mathbf{s}_2\in\mathbb{Z}^2,\, 0<|\mathbf{s}_1|+|\mathbf{s}_2|\le Q} V_{\mathbf{s}_1,\mathbf{s}_2} e^{2\pi i\langle \mathbf{s}_1+\alpha\mathbf{s}_2,\mathbf{x}\rangle}, \quad 1\le Q<\infty. \tag{2}$$

We assume that the irrationality measure μ of α is finite: $\mu < \infty$, or in other words, that α is not a Liouville number [1].

We study properties of the spectrum and eigenfunctions of (1) in the high energy region. We prove the following results for the case $d = 2$, $l \ge 2$.

(1) The spectrum of the operator (1) contains a semiaxis.

This is a generalization of a renown Bethe-Sommerfeld conjecture, which states that in the case of a periodic potential, $l = 1$ and $d \ge 2$, the spectrum of (1) contains a semiaxis.

(2) There are generalized eigenfunctions $\Psi_\infty(\mathbf{k},\mathbf{x})$, corresponding to the semi-axis, which are close to plane waves: for every $\mathbf{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\mathbf{k},\mathbf{x})$ of the equation $H\Psi_\infty = \lambda_\infty\Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\mathbf{k},\mathbf{x}) = e^{i\langle\mathbf{k},\mathbf{x}\rangle}\left(1+u_\infty(\mathbf{k},\mathbf{x})\right), \tag{3}$$

$$\|u_\infty\|_{L_\infty(\mathbb{R}^2)} \underset{|\mathbf{k}|\to\infty}{=} O\left(|\mathbf{k}|^{-\gamma_1}\right), \quad \gamma_1>0, \tag{4}$$

where $u_\infty(\mathbf{k},\mathbf{x})$ is a quasi-periodic function, namely a point-wise convergent series of exponentials $e^{i\langle\mathbf{n}+\alpha\mathbf{m},\mathbf{x}\rangle}$, $\mathbf{n},\mathbf{m}\in\mathbb{Z}^2$. The eigenvalue $\lambda_\infty(\mathbf{k})$, corresponding to $\Psi_\infty(\mathbf{k},\mathbf{x})$, is close to $|\mathbf{k}|^{2l}$:

$$\lambda_\infty(\mathbf{k}) \underset{|\mathbf{k}|\to\infty}{=} |\mathbf{k}|^{2l} + O\left(|\mathbf{k}|^{-\gamma_2}\right), \quad \gamma_2>0. \tag{5}$$

The "non-resonant" set $\mathcal{G}_\infty$ of vectors $\mathbf{k}$, for which (3) – (5) hold, is an extensive Cantor type set: $\mathcal{G}_\infty = \cap_{n=1}^\infty \mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^\infty$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appear when n increases, however

[1] Note, that $\mu \ge 2$ for any irrational number α.

holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

(6) $$|\mathcal{G}_\infty \cap \mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1 + O(R^{-\gamma_3})\big), \quad \gamma_3 > 0,$$

where $\mathbf{B}_R$ is the disk of radius R centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\mathbf{k})$,

$$\mathcal{D}_\infty(\lambda) = \{\mathbf{k} \in \mathcal{G}_\infty : \lambda_\infty(\mathbf{k}) = \lambda\},$$

is proven to be a slightly distorted circle with infinite number of holes. It can be described by the formula:

(7) $$\mathcal{D}_\infty(\lambda) = \{\mathbf{k} : \mathbf{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\},$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda \to \infty$:

(8) $$L\big(\mathcal{B}_\infty(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_4/2l}\right), \quad \gamma_4 > 0,$$

here and below $L(\cdot)$ is a length of a curve. The value $\varkappa_\infty(\lambda, \vec{\nu})$ in (7) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2l}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$. It is proven that the deviation is asymptotically small:

(9) $$\varkappa_\infty(\lambda, \vec{\nu}) \underset{\lambda\to\infty}{=} \lambda^{1/2l} + O\left(\lambda^{-\gamma_5}\right), \quad \gamma_5 > 0.$$

(4) The branch of the spectrum corresponding to $\Psi_\infty(\mathbf{k}, \mathbf{x})$ (the semiaxis) is absolutely continuous.

Lyapunov exponents for band lattice quasi-periodic Schrödinger operators

Silvius Klein

(joint work with Pedro Duarte)

Introduction. We consider a general quasi-periodic Schrödinger operator on a band lattice. This model includes all finite range hopping Schrödinger operators both on integer lattices and on band lattices. We prove optimal lower bounds for the (non-negative) Lyapunov exponents associated with this operator under the assumption that the coupling constant is large enough. We use a non-perturbative approach à la M. Herman or E. Sorets and T. Spencer, that involves complexification in the phase variable and exploits the convexity of the maximal Lyapunov exponent as a function of the imaginary variable. Such estimates are correlated (in the classical, integer lattice case) with Anderson localization, Hölder continuity of the integrated density of states or with other spectral properties of the operator. We have work in progress on these types of related problems.

The classical, integer lattice case. Let $H_\lambda(x)$ be the quasi-periodic Schrödinger operator acting on $l^2(\mathbb{Z}, \mathbb{R}) \ni \psi = \{\psi_n\}_n$ by

$$[H_\lambda(x)\,\psi]_n := -\,(\psi_{n+1} + \psi_{n-1} - 2\,\psi_n) + \lambda\, v_n(x)\,\psi_n \tag{1}$$

This type of operator describes the Hamiltonian of a quantum particle on the integer lattice.

It is called *quasi-periodic* because the potential $\mathbb{Z} \ni n \mapsto v_n(x) \in \mathbb{R}$ is a quasi-periodic function, i.e. $v_n(x) = v(x + n\omega)$, where $v\colon \mathbb{T}^d \to \mathbb{R}$ is a sampling function on the torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, $d \geq 1$ and $\omega \in \mathbb{T}^d$ is a rationally independent frequency, i.e. $k \cdot \omega \neq 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$. Moreover, $\lambda > 0$ is a coupling constant measuring the disorder of the system, while the space variable $x \in \mathbb{T}^d$ introduces some randomness into the system. It should be noted, however, that the spectral properties of the operator (1) are not random.

The associated Schrödinger equation for the state ψ and the energy $E \in \mathbb{R}$:

$$[H_\lambda(x)\,\psi]_n = E\,\psi_n \tag{2}$$

is a second order finite differences equation. Solving it formally, we obtain:

$$\begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix} = \begin{bmatrix} \lambda v(x + n\omega) - E & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix} = A^{(n+1)}_{\lambda,E}(x) \cdot \begin{bmatrix} \psi_1 \\ \psi_0 \end{bmatrix}$$

where

$$A_{\lambda,E}(x) = \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix} \text{ and } A^{(n)}_{\lambda,E}(x) = A_{\lambda,E}(x + (n-1)\omega) \cdot \ldots \cdot A_{\lambda,E}(x)$$

are respectivelly the cocycle and the transfer matrices associated with (2).

Since the transfer matrices $A^{(n)}_{\lambda,E}(x)$ solve the Schrödinger equation formally, it is then natural to study their growth as $n \to \infty$. The maximal Lyapunov exponent of (1) is defined as the average exponential rate of growth of the transfer matrices:

$$L_\lambda(E) := \lim_{n\to\infty} \int_{\mathbb{T}^d} \frac{1}{n}\, \log ||A^{(n)}_{\lambda,E}(x)||\, dx \tag{3}$$

The above limit exists due to sub-additivity, and it is non-negative. Whether the Lyapunov exponent is zero or it is bounded away from zero (say uniformly on an interval of energies) has important spectral consequences for the operator (1).

If we assume for instance that on an interval of energies $\mathcal{E} \subset \mathbb{R}$ we have

$$L_\lambda(E) > c \quad \text{for all } E \in \mathcal{E}, \tag{4}$$

then from Ishii-Pastur theorem, there is no absolutely continuous spectrum in $\mathcal{E}$. Moreover, when the sampling function $v(x)$ is real analytic and the frequency vector ω satisfies a generic arithmetic condition, J. Bourgain and M. Goldstein proved Anderson localization for the operator (1), where the rate of exponential decay of the eigenfunctions is given by the lower bound c on the Lyapunov exponents, while M. Goldstein and W. Schlag obtained quantitative continuity properties of the Lyapunov exponent and of the integrated density of states.

The standard result of the form (4) when $d = 1$ is due to M. Herman, who has first introduced the idea of complexifying the space variable in oder to avoid the set $\{x\colon v(x) \approx E\}$. His method applies to sampling functions $v(x)$ which are trigonometric polynomials, and it was later extended to the case of non-constant analytic functions by E. Sorets and T. Spencer. The lower bound on the Lyapunov exponents is of order $\log \lambda$, it holds assuming $\lambda \gg 1$, and it is asymptotically in λ optimal. In a series of papers from 2000 on, J. Bourgain, M. Goldstein, W. Schlag designed a different analytic method (that uses large deviations and the avalanche principle) which is applicable to the case $d > 1$ as well as to other models and types of problems. See [1] and references therein for the results mentioned above.

Band lattice Schrödinger operators. We define a model that covers all quasi-periodic, finite range hopping Schrödinger operators on integer or band lattices.

Let $W(x), R(x), D(x)$ be $l \times l$ real *matrix valued* functions on $\mathbb{T}$. Assume that $R(x)$ and $D(x)$ are symmetric and denote by $W^T(x)$ the transpose of $W(x)$. Given $\omega \in \mathbb{R} \setminus \mathbb{Q}$, for any $n \in \mathbb{N}$ and for any function $M(x)$ denote $M_n(x) := M(x + n\omega)$.

Consider the quasi-periodic Schrödinger (or Jacobi, as referred to by other authors) operator $H_\lambda(x)$ acting on $l^2(\mathbb{Z}, \mathbb{R}^l) \ni \vec{\psi} = \{\vec{\psi}_n\}_{n\in\mathbb{Z}}$ by

$$(5) \quad [H_\lambda(x)\,\vec{\psi}]_n := -(W_{n+1}(x)\,\vec{\psi}_{n+1} + W_n^T(x)\,\vec{\psi}_{n-1} + R_n(x)\,\vec{\psi}_n) + \lambda\, D_n(x)\,\vec{\psi}_n$$

The hopping term is given by a "weighted" Laplacian (i.e. the terms between parenthesis in (5)), where the hopping amplitude is encoded by the quasi-periodic matrix valued functions $W_n(x)$ and $R_n(x)$, while the potential is given by the quasi-periodic matrix valued function $\lambda\, D_n(x)$. The physically more relevant situation is when $D(x)$ is a diagonal matrix, but our result applies to any symmetric matrices.

The associated Schrödinger equation $H_\lambda(x)\,\vec{\psi} = E\,\vec{\psi}$ gives rise to the cocycle:

$$A_{\lambda,E}(x) := \begin{bmatrix} \lambda\, W^{-1}(x+\omega)\,(V(x) - E\cdot I) & -W^{-1}(x+\omega)\cdot W^T(x) \\ I & O \end{bmatrix} \in \mathrm{Mat}_{2l}(\mathbb{R})$$

and to the corresponding transfer matrices

$$A_{\lambda,E}^{(n)}(x) = A_{\lambda,E}(x + (n-1)\,\omega)\cdot \ldots \cdot A_{\lambda,E}(x) \in \mathrm{Mat}_{2l}(\mathbb{R})$$

that solve the Schrödinger equation formally.

The maximal Lyapunov exponent $L_\lambda^{(1)}(E)$ can be defined as in (3), and it measures the average *maximal* exponential rate of growth of the transfer matrices. However, in this higher dimensional case, it is important to control the rate of growth of the transfer matrices along other directions. Oseledets multiplicative ergodic theorem, applied to our setting, defines the Lyapunov exponents

$$L_\lambda^{(1)}(E) \geq L_\lambda^{(2)}(E) \geq \ldots \geq L_\lambda^{(l)}(E) \geq 0$$

as their average exponential rates of growth along invariant subspaces of $\mathbb{R}^l$.

Theorem. *Consider the operator* (5)*, where the matrix valued functions* $W(x)$*,* $R(x)$ *and* $D(x)$ *are analytic. Assume the following generic transversality conditions:*

$$\det[W(x)] \not\equiv 0$$
$$D(x) \text{ has no constant eigenvalues}$$

There are constants λ_0 *and* c*, depending only on some measurements of* $W(x), R(x)$ *and* $D(x)$ *and not on* ω*, so that if* $\lambda > \lambda_0$ *then for all* $1 \le k \le l$ *we have:*

$$L_\lambda^{(k)}(E) \ge \log \lambda - k\, c \quad \text{for all } E \in \mathbb{R}$$

For the proof of this result see [2]. Note that the cocycle $A_{\lambda,E}(x)$ introduced above and the corresponding transfer matrices may have singularities, as we do not assume the function $W(x)$ to be invertible everywhere. However, since $W(x)$ is analytic and non identically singular, there is only a finite number of singularities, which ensures that the Lyapunov exponents are well defined. We "factor out" the singularities and reduce the proof to a statement regarding a more general analytic cocycle with no singularities. Analiticity of the cocycle allows us to complexify the variable x to an annulus of a certain width along the unit circle $\mathbb{T}$. Using the transversality conditions, for every energy we show that there is a circle, arbitrarily close to the unit circle $\mathbb{T}$, along which the behavior of the cocycle is uniformly hyperbolic, hence the maximal Lyapunov exponent corresponding to phases on this circle is bounded away from zero. We transfer this property to the unit circle $\mathbb{T}$ using a convexity argument involving averages of subharmonic functions.

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On the limit set of KdV flow - an extension of Remling's theorem

Shinichi Kotani

For a positive number c set

$$\mathcal{V}_c = \{q;\ q \text{ is a real valued function on } \mathbb{R} \text{ such that } |q(x)| \le c\}$$

endowed with weak convergence. Then, $\mathcal{V}_c$ is compact and admits the shift operation S_t defined by $(S_t q)(\cdot) = q(\cdot + t)$. The right limit set of $\{S_t\}$ for $q \in \mathcal{V}_c$ is defined as

$$\mathcal{R}_q = \left\{ \widetilde{q} \in \mathcal{V}_c;\ \widetilde{q} = \lim_{n\to\infty} S_{t_n} q \text{ for a } \{t_n\} \text{ tending to } \infty \right\}.$$

Remling [6], [7] established a remarkable theorem describing the structure of $\mathcal{R}_q$ in terms of spectral properties of the associated Schrödinger operator

$$H_q = -\frac{d^2}{dx^2} + q.$$

The theorem can be stated by the Weyl-Titchmarsh functions $m_{\pm}(\lambda, q)$ of H_q. Set $H_q^{\pm} = H_q|_{\mathbb{R}_{\pm}}$ with Dirichlet boundary condition at 0. Based on the fundamental results by Breimesser-Pearson [1], [2], Remling proved

(1) For any $\widetilde{q} \in \mathcal{R}_q$, $m_+(\xi + i0, \widetilde{q}) = -\overline{m_-(\xi + i0, \widetilde{q})}$ for a.e. $\xi \in \operatorname{acsp}\left(H_q^+\right)$.

This result is particularly interesting when $\operatorname{acsp}\left(H_q^+\right)$ consists of finite numbers of intervals and $\operatorname{esssp}\left(H_q^+\right) = \operatorname{acsp}\left(H_q^+\right)$, for, in this case (1) makes it possible to describe $\mathcal{R}_q$ completely by θ-functions on the compact Riemannian surface $\overline{\mathbb{C}} \backslash \operatorname{acsp}\left(H_q^+\right)$. The key facts in the proof was the existence of holomorphic transfer matrices $U_q^{\pm}(t, z) \in SL(2, \mathbb{C})$ satisfying the following properties.

(i) $U_q^{\pm}(t, z)$ map $\mathbb{C}_+$ into $\mathbb{C}_+$ as a fractional linear transformation for each $t \geq 0$ and $z \in \mathbb{C}_+$.
(ii) $U_q^{\pm}(t, \xi) \in SL(2, \mathbb{R})$ for real $\xi \in \mathbb{R}$.
(iii) The two identities below are valid

$$\begin{cases} m_{\pm}(z, S_t q) = U_q^{\pm}(t, z) \cdot m_{\pm}(z, q) \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_q^+(t, z) = U_q^-(t, z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}.$$

It is natural to expect that an analogous result to (1) would hold for other flows if they satisfy (i),(ii),(iii) for z on a domain $D \subset \mathbb{C}_+$, and we show that this observation indeed works for the KdV flow. The construction of KdV flow on a space containing functions which are neither periodic nor decaying at $\pm\infty$ is not trivial, and is carried out on the space

$$\Omega_{\lambda_0} = \left\{ \begin{array}{l} q;\ \operatorname{sp}(H_q) \subset [-\lambda_0, \infty) \text{ and } m_+(\xi + i0, q) = -\overline{m_-(\xi + i0, q)} \\ \text{for a.e. } \xi \in [0, \infty) \end{array} \right\},$$

for $\lambda_0 > 0$, which is the closure of the set of all ordinary reflectionless potentials. The detailed structure was investigated by [3], [4]. In the construction of the flow we have employed Segal-Wilson version [8] of Sato's grassmannian method [5]. Suitable transfer matrices associated with each non-linear equation belonging to the KdV hierarchy are introduced by using the linear term of the equation. One of the conclusion of our result is as follows. For any $q \in \Omega_{\lambda_0}$, one can construct a solution $u(t, x)$ to the KdV equation

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}, \qquad u(0, x) = q(x)$$

such that $u(t, \cdot) \in \Omega_{\lambda_0}$ for any $t \in \mathbb{R}$. Then, any limit point $\widetilde{q}$ of $\{u(t, \cdot)\}_{t \geq 0}$ satisfies

$$m_+(\xi + i0, \widetilde{q}) = -\overline{m_-(\xi + i0, \widetilde{q})} \quad \text{for a.e. } \xi \in \operatorname{acsp}\left(H_q^-\right).$$

Especially, if $\operatorname{sp}\left(H_q^-\right) = \operatorname{acsp}\left(H_q^-\right) = [-\lambda_0, \infty)$ holds for an initial function $q \in \Omega_{\lambda_0}$, then the solution u to the KdV equation with initial function q satisfies

$$\lim_{t \to \infty} u(t, x) = -\lambda_0.$$

Moreover, if sp $\left(H_q^-\right) = \text{acsp}\left(H_q^-\right) = I \cup [0,\infty)$ with an interval I in $(-\infty, 0]$ holds, then any right limit function $\widetilde{q}$ of $\{u(t,\cdot)\}_{t\geq 0}$ is described by the Weierstrass elliptic function $\wp(x)$, namely the conoidal wave.

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Kicked quasi-periodic cocycles and coexistence of ac and pp spectrum

Raphaël Krikorian
(joint work with Kristian Bjerklöv)

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, $V : \mathbb{T} \to \mathbb{R}$ be a smooth or real analytic potential and $\alpha \in \mathbb{T}$ an irrational frequency. We consider the Schrödinger cocycle: $(\alpha, S_V) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$, $(\alpha, S_V)(x, y) = (x + \alpha, S_V(x)y)$ with $S_V = \begin{pmatrix} V & -1 \\ 1 & 0 \end{pmatrix}$ and the quasi-periodic discrete 1D Schrödinger operator $H_{\alpha,V} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, $(H_{\alpha,V}u)_n = u_{n+1} + u_{n-1} + V(x + n\alpha)u_n$. We are interested in the regime when V is a "peaky" potential (whence the word "kicked" in the title) and α is a diophantine frequency. More specifically V is of the form

$$V(x) = \frac{K}{1 + 4\lambda \sin^2(\pi x)}$$

where λ and K are big positive constants satisfying $\lambda >> K$. Kristian Bjerklöv observed numerically that if α is the golden mean, there exist values of $E \in \mathbb{R}$ for which the cocycle (α, S_{E-V}) seems to have bounded iterates (meaning that the products $S_{E-V}(\cdot + (n-1)\alpha) \cdots S_{E-V}(\cdot)$ are bounded); this suggests that for these values of E the cocycle (α, S_{E-V}) is conjugated to a constant elliptic cocycle, that is, can be written under the form $S_{E-V}(\cdot) = B_E(\cdot + \alpha)C_E B_E(\cdot)^{-1}$, with $C_E \in SL(2, \mathbb{R})$ and where $B_E : \mathbb{T} \to SL(2, \mathbb{R})$ has some smoothness. We were not able to prove this result but instead a weaker one: for such "peaky" potentials, and α in a set of positive Lebesgue measure, there exists a set of positive Lebesgue measure of E for which the cocycle (α, S_{E-V}) is analytically conjugate to an elliptic system. As a consequence, for these peaky potentials and these frequencies α, the operator $H_{\alpha,V}$ has some a.c. spectrum; on the other hand using a theorem by Bourgain and Goldstein [4], it is possible to see that the operator H_V also has positive Lebesgue measure p.p. spectrum. This is an improvement of a previous work by K. Bjerklöv ([2]). The fact that a.c. and p.p. spectrum could coexist for q.p. discrete 1D Schrödinger operator was already proven by J. Bourgain [3], but for systems with two frequencies or more. Recently A. Avila announced a similar result (in the 1-frequency case) for perturbations of the almost Mathieu potential.

The proof of our result is roughly the following. For α equal to $1/2$ it is not difficult to show that for some values of E, the cocycle $(1/2, S_{E-V})^2$ is of the form $(0, A_E(\cdot))$ where for any value of x, $A_E(x)$ is an elliptic matrix. Using the so-called *cheap trick* introduced by Fayad-Krikorian [6] and Avila-Fayad-Krikorian [1], it is possible to prove that for α very close to $1/2$ but diophantine (with very bad diophantine constant) the situation can be reduced to a (semi-) local one: this means that one can find a conjugacy taking the cocycle to a cocycle close to an elliptic constant, this closeness being related to the diophantine properties of α. Eliasson's theorem [5] or more precisely a version of it with estimate (cf. [6]) can then be applied provided one can prove that the fibered rotation number of the cocycle takes values which are diophantine with respect to α. It can be shown that this is the case for a set of positive measure of E.

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Square-summable q-variations and absolutely continuous spectrum of Jacobi matrices

Milivoje Lukic
(joint work with Yoram Last)

We investigate semi-infinite Jacobi matrices

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & a_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

such that $a_n > 0$, $b_n \in \mathbb{R}$, and

$$\sup_n a_n^{-1} + \sup_n a_n + \sup_n |b_n| < \infty, \tag{1}$$

which obey, for some $q \in \mathbb{N}$, the condition

$$\sum_{n=1}^{\infty} |a_{n+q} - a_n|^2 + \sum_{n=1}^{\infty} |b_{n+q} - b_n|^2 < \infty. \tag{2}$$

The cyclic vector δ_1 corresponds to the canonical spectral measure μ with Lebesgue decomposition $d\mu = f(x)dx + d\mu_{\mathrm{s}}$. We are interested in the essential support of the absolutely continuous spectrum of J, defined as

$$\Sigma_{\mathrm{ac}}(J) = \{x \in \mathbb{R} \mid f(x) > 0\}.$$

This is properly viewed as an equivalence class of sets modulo sets of Lebesgue measure zero.

A two-sided Jacobi matrix $J^{(r)}$ with coefficients $a_n^{(r)} > 0$, $b_n^{(r)} \in \mathbb{R}$ is a right limit of J if there is a sequence $n_j \in \mathbb{Z}$, $n_j \to +\infty$, such that for all $n \in \mathbb{Z}$,

$$\lim_{j\to\infty} a_{n+n_j} = a_n^{(r)}, \qquad \lim_{j\to\infty} b_{n+n_j} = b_n^{(r)}.$$

The condition (2) implies $a_{n+q} - a_n \to 0$ and $b_{n+q} - b_n \to 0$, so every right limit $J^{(r)}$ of J is q-periodic. Thus, its spectrum is a union of q closed intervals on $\mathbb{R}$, called bands, whose interiors are disjoint. The q-interior of the spectrum, denoted $q\text{-int}(\sigma(J^{(r)}))$, is then defined as the union of interiors of the q bands.

We denote the set of right limits of J by $\mathcal{R}$.

Theorem 1 ([6])**.** *Let* (1) *and let* (2) *hold for some* $q \in \mathbb{N}$. *Then*

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) \subset \Sigma_{\mathrm{ac}}(J) \subset \bigcap_{\mathcal{R}} \sigma(J^{(r)}). \tag{3}$$

Moreover, for any closed interval

$$I \subset \bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})),$$

we have

$$\int_I \log f(x)dx > -\infty.$$

The essence of this theorem is in the first inclusion of (3); the second inclusion is a general result of Last–Simon [7, 8]. Note that q-int$(\sigma(J^{(r)}))$ differs from $\sigma(J^{(r)})$ by only a finite set of points, but those points can vary from right limit to right limit. Thus, the intersections in (3) may, in general, differ by a set of positive Lebesgue measure. However, there are two notable cases in which the two inclusions combine to give an equality.

Corollary 2 ([6])**.** *If* (1) *holds and* (2) *holds for* $q = 1$*, then*

$$\Sigma_{\text{ac}}(J) = [\limsup_{n\to\infty}(b_n - 2a_n), \liminf_{n\to\infty}(b_n + 2a_n)].$$

Corollary 3 ([6])**.** *Let* (1) *and let* (2) *hold for some* $q \in \mathbb{N}$*. If* J *converges to an isospectral torus* $\mathcal{T}_{\mathcal{S}}$ *(i.e., if all right limits of* J *have the same spectrum* $\mathcal{S}$*), then* $\Sigma_{\text{ac}}(J) = \mathcal{S}$*.*

Conjecture 9.5 of [1] postulates that the second inclusion of (3) is always an equality. By Corollary 2, this is true for $q = 1$. As we are about to see, it is not always true for $q > 1$ (thus the conjecture of [1] is false).

Theorem 4 ([6])**.** *Let* $q \in \mathbb{N}$*,* $q > 1$*. There exist half-line Jacobi matrices* J_1*,* J_2 *with the properties* (1)*,* (2)*,* $a_n \equiv 1$*, with the same set of right limits* $\mathcal{R}$ *and*

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) = \Sigma_{\text{ac}}(J_1) \subsetneq \bigcap_{\mathcal{R}} \sigma(J^{(r)}),$$
$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) \subsetneq \Sigma_{\text{ac}}(J_2) = \bigcap_{\mathcal{R}} \sigma(J^{(r)}).$$

This shows that, in general, no better statement can be made than (3).

This work continues a string of papers concerned with the property (2) and relies on the methods of those papers. Denisov [3] proved a conjecture of Last [5], that (2) together with $a_n \equiv 1$ and $b_n \to 0$ implies $\Sigma_{\text{ac}}(J) = [-2, 2]$. Kaluzhny–Shamis [4] generalized the method of [3] and proved Theorem 1 under the assumption that a_n, b_n are asymptotically periodic sequences; this can be viewed as a special case of Corollary 3. The method has recently been carried over to orthogonal polynomials on the unit circle (OPUC) and extended beyond asymptotic periodicity by the author in [9].

An analogous result for (continuum) Schrödinger operators has been proved by Denisov [2].

The results for OPUC in [9] mirror closely the results for Jacobi matrices, with one peculiar difference: for OPUC, the two inclusions combine into one equality not just for $q = 1$, but also for $q = 2$. This happens because 2-periodic sequences of Verblunsky coefficients can give rise to closed gaps only at ± 1, so the difference between intersections in the analog of (3) is a finite set.

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Resonant delocalization on the Bethe strip

Mira Shamis

We report on our recent work [11] pertaining to the presence of absolutely continuous spectrum for a class of random operators associated with the product of a rooted regular tree and a finite graph. Our main result extends the work of Aizenman and Warzel [3] who considered the case of a regular rooted tree.

The setting is as follows. Let $\mathcal{T}$ be a regular rooted tree with branching number $K > 1$ (i.e., every vertex has K forward neighbors.) For a fixed $W > 1$ we are interested in random operators $H = H_\lambda$ acting on

$$\ell^2(\mathcal{T} \to \mathbb{R}^W)$$

which are defined by

$$(H\psi)(x) = U(x)\psi(x) + \sum_{y \sim x} \psi(y) \ ,$$

where $U(x)$, $x \in \mathcal{T}$, are independent, identically distributed Hermitian matrices. We focus on the weak disorder setting

$$U(x) = A + \lambda V(x) \ ,$$

where A is fixed $W \times W$ Hermitian matrix, $V(x)$ are independent, identically distributed $W \times W$ random matrices, and $0 < \lambda \ll 1$ is a small parameter (the strength of disorder.)

We remark that random Schroödinger operators on the Bethe strip (the product of $\mathcal{T}$ and a finite graph) are embedded in this framework, but ruled out by the stringent assumption on the distribution of V that we impose below.

Our main result is

Theorem 1. *Assume that $V(x)$ are drawn from the Gaussian Orthogonal Ensemble (GOE). Then, for any $\epsilon > 0$ any open interval*

$$I \subset S_\epsilon = \bigcup_j [\nu_j - (K+1) + \epsilon, \nu_j + (K+1) - \epsilon] \ ,$$

where $\{\nu_j\}_{j=1}^W$ are the eigenvalues of A, almost surely has absolutely continuous spectrum of H in it when λ is sufficiently small.

From the work of Aizenman [1] it is known that the spectrum of H outside $S_{-\epsilon}$ is (almost surely) pure point.

To prove the result in context, note that the set S_0 is exactly the ℓ^1 spectrum of H_0 and it is significantly larger than the ℓ^2 spectrum. For the Bethe lattice ($W = 1, A = 0$) the ℓ^1 spectrum is the interval $[-K-1, K+1]$, whereas the ℓ^2 spectrum is the interval $[-2\sqrt{K}, 2\sqrt{K}]$. The presence of absolutely continuous spectrum in $[-2\sqrt{K}, 2\sqrt{K}]$ was proved by Klein in [6, 7, 8] and later, by different methods by Froese, Hasler and Spitzer [5]. By yet another methods the generalization of this result was proved by Aizenman, Sims and Warzel [2].

Recently, Aizenman and Warzel [3] proved the presence of absolutely continuous spectrum throughout the interval $[-K-1, K+1]$ (the precise formulation is parallel to Theorem 1.) The appearance of absolutely continuous spectrum well outside the ℓ^2 spectrum of the unperturbed operator is due to a mechanism put forth in [3] under the name "resonant delocalization". Our result is an extension of their work to the case of matrix-valued potential (corresponding to a graph with loops). The main new technical issue that we had to address is the significant difference between the fastest and the slowest Lyapunov exponents.

In the smaller set

$$S_\epsilon^- = \bigcap_j [\nu_j - 2\sqrt{K} + \epsilon, \nu_j + 2\sqrt{K} - \epsilon] \ ,$$

in the recent work, Klein and Sadel [9, 10] proved (under weaker assumptions on the potential – in particular, it may be diagonal) that the spectrum of the perturbed operator is almost surely purely absolutely continuous. A special case of this result for $W = K = 2$, was treated by Froese, Halasan and Hasler [4].

Compared to this result we replaced the intersection with the union (namely, the fastest Lyapunov exponent with the slowest one) and the ℓ^2 spectrum of the unperturbed operator with the ℓ^1 spectrum. The price is that we have more restrictive assumptions on the potential and we only show the existence of the absolutely continuous spectrum rather than its purity.

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Fluctuation of the Green function

Sasha Sodin

The goal of the present talk is to advertise a class of problems recently discussed in the physical literature, in particular, by Ortuño, Prior, and Somozo, and more recently by le Doussal. The setting is as follows.

Let H be an ergodic Schrd̈ingier operator acting on $\ell^2(\mathbb{Z}^d)$:

$$(H\psi)(x) = \sum_{y\sim x}(\psi(x) - \psi(y)) + V(x)\psi(x) \ .$$

We mainly focus on the case when the potential entries $V(x)$ are independent, identically distributed, and the dimension is $d = 2$, but to connect to the subject of the conference we shall also discuss general ergodic potentials in dimension $d = 1$.

For $\lambda \in \mathbb{C}$ outside the spectrum of H, we consider the Green function

$$G_\lambda(x, y) = (H - \lambda)^{-1}(x, y) \ .$$

For λ in the spectrum of H, we consider the boundary value $G_{\lambda+i0}(x, y)$. Assume that the Green function decays exponentially: for every $x \in \mathbb{Z}^d$

$$\liminf_y \frac{\log|G_\lambda(x, y)|}{\|x - y\|} < 0 \tag{1}$$

almost surely. According to the Combes–Thomas estimate, (1) holds for λ outside the spectrum of H. The range of energies λ in the spectrum for which (1) holds (for the boundary value of G) is roughly speaking the region of Anderson localisation.

In either case, we ask the following: how does the typical order of fluctuations of $\log |G_\lambda(x, y)|$ grow with $\|x - y\|$? One may interpret the typical order of fluctuations as $\sqrt{\operatorname{Var} \log |G_\lambda(x, y)|}$, or perhaps as the difference between the 75-th and the 25-th percentile.

One-dimensional operators. In dimension $d = 1$, the following seems plausible: under the condition (1) and under mild assumptions on the potential, the growth of fluctuations is similar to that of

$$\sum_{u=x}^{y} \phi(V(u)) \tag{2}$$

for a suitable test function ϕ. We remark that typically the growth of fluctuations is not very sensitive to the choice of test function ϕ. Thus for independent identically distributed V the variance of (2) grows linearly in $|x - y|$. In the almost periodic case the growth is slower, and is related to the behaviour of Weyl sums.

The motivation comes from the random walk representation

$$G_\lambda(x, y) = \sum \prod_j \frac{1}{2d + V(u_j) - \lambda} \ , \tag{3}$$

where the sum is over paths (u_j) from x to y. The representation is valid in any dimension, provided that the right-hand side is convergent (which is the case for λ outside a disc about the spectrum). In the region of validity of (3) in $d = 1$, one may argue that the main contribution comes from the straight path connecting x to y. A more convincing plausible argument is based on renormalisation group ideas; see below.

In the random i.i.d. case the exponential decay (1) is known to hold for the boundary value of G throughout the spectrum of H (this follows from the work of Furstenberg on the products of random matrices). The linear growth

$$\operatorname{Var} \log |G_\lambda(x, y)| \asymp |x - y|$$

follows from the work of Furstenberg and Kesten. Thus the belief vaguely stated above is correct in the i.i.d. case.

Michael Goldstein kindly informed us that a proof of the belief for general ergodic potentials in one dimension (satisfying mild assumptions), based on multi-scale analysis, is known to experts.

Higher dimension. From this point we focus on $d \geq 2$ and assume that the potential is random i.i.d. The analogue of (2) in high dimension is (site) first passage percolation, which is a random metric on $\mathbb{Z}^d$ constructed as follows:

$$\rho(x, y) = \inf \sum W(u_j) \ ,$$

where the infimum is over paths (u_j) from x to y, and W's are random weights on $\mathbb{Z}^d$. First passage percolation is well defined when $W \geq 0$ (for simplicity we may focus on the case $W > 0$). Alternatively, one may restrict the infimum to directed case, then the model makes sense for any W. A plausible connection between $\log |G_\lambda|$ and ρ comes from the random walk representation (3). In particular, for real λ below the spectrum of H, $\log G_\lambda$ may be seen as a positive-temperature version of ρ. Moreover, one can construct a random metric

$$\rho_\lambda(x, y) = \log \frac{\sqrt{G_\lambda(x, x) G_\lambda(y, y)}}{G_\lambda(x, y)}$$

which shares the order of fluctuations with $\log G_\lambda$ and from which ρ can be recovered by taking a suitable limit.

Le Doussal argues that, in dimension $d = 2$, the fluctuations of $\log |G_\lambda(x, y)|$ are of the same order as those of ρ. The latter are believed to be of order $\|x - y\|^{1/3}$ (again, in dimension 2). We remark that in dimension $d = 2$ the condition (1) is conjectured to hold throughout the spectrum of H.[1] One of the plausible arguments of le Doussal, going back to ideas of Polyakov, states that (in dimension $d = 1, 2$) renormalisation group drives the effective temperature to 0 (which corresponds to first passage percolation). This argument is far from being rigorous; the application of the argument to models with sign-indefinite weights such as (3) is also subtle. In higher dimension, the connection to first-passage percolation may fail for some ranges of parameters (cf. the work of Imbrie and Spencer).

Rigorous results are scarce. The order of fluctuations for first passage percolation is by itself a major open problem. In dimension $d = 2$, there is a lower bound

$$\mathrm{Var} \rho(x, y) \geq \frac{1}{C} \log(\|x - y\| + 1) ,$$

due to Newman and Piza. Piza proved an analogue of this bound for certain positive-temperature models. His arguments can be adapted to show that

$$\mathrm{Var} \log |G_\lambda(x, y)| \geq \frac{1}{C} \log(\|x - y\| + 1)$$

for real λ below the spectrum of H.

Kesten showed that (in any dimension)

$$\mathrm{Var} \rho(x, y) \leq C \|x - y\| .$$

Benjamini, Kalai, and Schramm proved the slightly stronger bound

$$\mathrm{Var} \rho(x, y) \leq C \frac{\|x - y\|}{\log(\|x - y\| + 2)|}$$

in dimension $d \geq 2$. They considered the case of random signs $W = \pm 1$; their result was generalised by Benaïm and Rossignol; recently (2013), Damron, Hanson, and Sosoe obtained a further generalisation requiring only the assumption $\mathbb{E} W^2 \log_+ W^2 < \infty$.

[1] rigorously, this is now known for strong disorder and at spectral edges, due to the work of Fröhlich and Spencer

The arguments of Benjamini, Kalai and Schramm (and maybe also of Damron, Hanson, and Sosoe) can be adapted to handle $\log|G_\lambda|$ for real λ outside the spectrum of H.

Embarassingly, the speaker is not aware of any bounds for *complex* λ outside the spectrum beyond the trivial bound

$$\frac{1}{C} \leq \operatorname{Var} \log |G_\lambda(x,y)| \leq C\|x-y\|^2 \ .$$

The more interesting question about the boundary values in the region of Anderson localisation (or even the region of *proved* Anderson localisation) seems even more open.

Separation of Eigenvalues for Quasiperiodic Jacobi Matrices

Mircea Voda

(joint work with Ilia Binder)

The goal of this talk was to discuss the main result from [2]. First we establish the setting of our work, which is similar to [3], [4], [5]. We consider quasiperiodic Jacobi operators on $l^2(\mathbb{Z})$ of the form

$$[H(x,\omega)\psi]_k = -b(x+(k+1)\omega)\psi_{k+1} - \overline{b(x+k\omega)}\psi_{k-1} + a(x+k\omega)\psi_k,$$

where $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\omega \in \mathbb{T}_{c,\alpha}$, $c \ll 1$, $\alpha > 1$, with

$$\mathbb{T}_{c,\alpha} = \left\{ \omega \in (0,1) : \ \|n\omega\| \geq \frac{c}{n(\log n)^\alpha} \right\}.$$

The functions $a : \mathbb{T} \to \mathbb{R}$ and $b : \mathbb{T} \to \mathbb{C}$ are assumed to be real-analytic.

Let $L(\omega, E)$ be the Lyapunov exponent of the cocycle $(x,v) \mapsto (x+\omega, A^E(x)v)$, with

$$A^E(x) = \frac{1}{b(x+\omega)} \begin{bmatrix} a(x)-E & -\overline{b(x)} \\ b(x+\omega) & 0 \end{bmatrix}.$$

We work in the positive Lyapunov exponent regime. Namely, from now on we only deal with (ω, E) in some rectangle $\Omega^0 \times \mathcal{E}^0$ on which $L(\omega,E) \geq \gamma > 0$.

We restrict our attention to the matrices $H^{(N)}(x,\omega)$ corresponding to the Dirichlet problem on the interval $[0,N]$. We use $E_j^{(N)}(x,\omega)$ and $\psi_j^{(N)}(x,\omega)$ to denote the eigenvalues and a choice of l^2-normalized eigenvectors for $H^{(N)}(x,\omega)$.

Since the inverse spectral theory for the periodic case is well-understood, it is natural to try to understand how the positive Lyapunov exponent regime plays out with the periodic approximation of the frequency via the standard convergent of its continued fraction. For example, we want understand the relation between $H^{(N)}(x,\omega)$ and $H^{(N)}(x,\omega_s)$, when $N = q_s$, where $\omega_s = p_s/q_s$ is a convergent of ω. In particular we want to figure out if we can say that the corresponding eigenvalues and eigenvectors are close. We know that $|\omega - \omega_s| \leq (q_s q_{s+1})^{-1}$, and it is not hard to see that $|E_j^{(N)}(x,\omega) - E_j^{(N)}(x,\omega_s)| \leq C(q_{s+1})^{-1}$. It is not immediately clear what can be said about the eigenvectors. This is where the separation of eigenvalues comes into play. It is possible to see that if $E_j^{(N)}(x,\omega)$ is separated from the other eigenvalues of $H^{(N)}(x,\omega)$ by $CN^{-1}(\log N)^{-p}$, $p<1$, then $|\psi_j^{(N)}(x,\omega)-\psi_j^{(N)}(x,\omega_s)|\le C(\log N)^{p-1}$. Furthermore, if $\psi_j^{(N)}(x,\omega)$ is localized then $\psi_j^{(N)}(x,\omega_s)$ is also localized, with the same localization center. However, obtaining such a good separation of eigenvalues is a delicate matter. Prior to our work, the best known separation was by $\exp(-(\log N)^A)$, $A\gg 1$, due to Goldstein and Schlag [5]. We were able to improve this estimate and bring it to the same order of magnitude as the desired estimate. We give a rough statement of our result.

Theorem. *Fix $p>15$. For $N\ge N_0$, $\omega\notin\Omega_N$, $E_j^{(N)}(x,\omega)\notin\mathcal{E}_{N,\omega}$ we have*

$$|E_j^{(N)}(x,\omega)-E_j^{(N)}(x,\omega_s)|\ge\frac{1}{N(\log N)^p},$$

for all $k\ne j$.

The improvement of the separation comes at the cost of having to eliminate rather large sets of frequencies and energies. More precisely, we have

$$\mathrm{mes}(\Omega_N),\mathrm{mes}(\mathcal{E}_{N,\omega})\le(\log\log N)^{-c}.$$

Nonetheless, our result (at least in the Schrödinger case) applies to the bulk of the spectrum, since in [5] it was shown that

$$\mathrm{mes}(\cup_{x\in\mathbb{T}}\ \mathrm{spec}(H^{(N)}(x,\omega)\cap\mathcal{E}^0))\to\mathrm{mes}(\mathrm{spec}(H(x,\omega)\cap\mathcal{E}^0))>0.$$

From [7] it is known that the eigenvalues can get as close as $\exp(-cN)$. This justifies the removal of energies for obtaining any separation better than $\exp(-cN)$.

The most important basic tool for our work is a large deviation estimate for the determinant of the Jacobi matrix. Namely, we have

$$\mathrm{mes}\{x\in\mathbb{T}:\ \log|\det(H^{(N)}(x,\omega)-E)|>H(\log N)^C\}\le C\exp(-H).$$

Such an estimate was established in [4] for the Schrödinger case and in [1] for the Jacobi case. Typically, the main difficulty in dealing with results for the more general Jacobi case come from the singularities of the cocycle at the zeros of b. See, for example, [6] and the references therein.

Ultimately, the large deviation estimate for the determinant allows us to establish finite scale elimination of resonances and localization as in [5]. The elimination of resonances result roughly says that for $N\ge N_0$, $l=[(\log N)^A]$, $A\gg 1$, $\omega\notin\Omega_N$, $E_j^{(l)}(x,\omega)\notin\mathcal{E}_{N,\omega}$, $Q_N\le|m|\le N$ we have

$$|E_j^{(l)}(x,\omega)-E_k^{(l)}(x+m\omega,\omega)|\ge\sigma_N.$$

Such an elimination result implies exponential localization at scale N with a localization window of size CQ_N. In [4] it was shown that this further implies separation of eigenvalues by $\exp(-CQ_N)$. It can be seen that we can only have $Q_N\gg l$, so by this approach the best separation is by $\exp(-(\log N)^A)$. To circumvent this we showed that in fact we can obtain separation of eigenvalues at scale N by $\sigma_N\gg\exp(-CQ_N)$. In other words, we reduced the problem of improving separation to the problem of improving the elimination of resonances. This latter problem is technically delicate but the main ideas are the same as for the trivial case when $b \equiv 0$. We note that in this degenerate case one can obtain elimination of resonances by CN^{-1}.

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On Effective Localization Length for the Anderson Model on a Strip

Mircea Voda

(joint work with Ilia Binder, Michael Goldstein)

The results presented in this talk are part of work in progress. We consider random operators on the strip $\mathbb{Z}_W = \mathbb{Z} \times \{1, \ldots, W\}$ defined by

$$(H\psi)_n = -\psi_{n-1} - \psi_{n+1} + S_n\psi_n,$$

where $\psi \in l^2(\mathbb{Z}, \mathbb{C}^W) \equiv l^2(\mathbb{Z}_W)$, $S_n = S + \operatorname{diag}(V_{(n,1)}, \ldots, V_{(n,W)})$, with S a Hermitian matrix and V_i, $i \in \mathbb{Z}_W$ i.i.d. random variables. Such operators are studied with various restrictions on the distribution of the random variables V_i.

The one-dimensional case, when $W = 1$, is well understood. In particular, localization results go back to Kunz and Souillard [6], and the most general result was given by Carmona, Klein, and Martinelli [2]. For the general quasi-one-dimensional case localization was proven by Klein, Lacroix, and Speis [5], extending the work from [2]. However, this result is not effective. Exponential localization of eigenvectors is obtained, but without an explicit estimate for the decay rate. The decay rate is given by the lowest nonnegative Lyapunov exponent γ_W^E (a definition was given in the talk of Silvius Klein). It followed from the work of Goldsheid and Margulis [4] that $\gamma_E^W > 0$ with probability one (for all energies E), but without an explicit lower bound. Explicit lower bounds were given later by Schulz-Baldes [8], but only for small disorder. Very recently Bourgain [1] showed that $\gamma_W^E \geq \exp(-CW(\log W)^4)$, provided that V_i have bounded density. Our goal is to obtain a similar result, by a different method than [1]. Our approach is to first provide explicit lower bounds for the fluctuations of the Green's function. This idea has been previously used in related work on random band matrices by Schenker [7].

For $\Lambda = [a, b] \times [1, W] \subset \mathbb{Z}_W$ let P_Λ be the projection onto the subspace of vectors in $l^2(\mathbb{Z}_W)$ that are zero outside Λ. We consider the finite volume restrictions of H defined by $H_\Lambda = P_\Lambda H P_\Lambda$ (corresponding to Dirichlet boundary conditions) and we let $G_\Lambda^E = (H_\Lambda - E)^{-1}$. A key part of the multi-scale analysis approach to localization (see, for example, [3] and [9]) is to establish off-diagonal decay for the entries of the resolvent. Namely, we want that with high probability we have an estimate of the form $|G_\Lambda^E(i, j)| \leq \exp(-m|i-j|)$. The quantity $1/m$ is called localization length and it measures how far from the diagonal we need to be before we start seeing decay. Usually, m is approximately γ_W^E. We claim that such estimates follow from fluctuation estimates of the form $\mathrm{Var}(\log |G_\Lambda^E(i, j)|) \geq C|i - j|$ with m of the same order of magnitude as $1/C$. This idea appears in [7], but it is not implemented in this form there. However, we think that such a claim is relatively straightforward to prove.

So, the main purpose of our work is to provide an explicit lower bound for the fluctuations of the Green's function. For this we need to impose some restrictions on the random potentials. First we assume that V_i have density function v bounded by a constant A_0. Next we assume that $\mathbb{P}(|V_i| \geq T) \leq A_1/T$, for $T \geq 1$. This is just a condition that guarantees the existence of the second moment of $\log |G_\Lambda^E(i, j)|$. Finally, and most restrictively, we assume that $\inf_{|x| \leq T} v(x) \geq T^{-A_2}$, for $T \geq 1$. Under these assumptions we have the following result.

Theorem. *There exists a constant* $C_0 = C_0(A_0, A_1, A_2, |E|, \|S\|)$ *such that if* $G_\Lambda^E(i, j)$ *is not identically zero then*

$$\mathrm{Var}(\log |G_\Lambda^E(i, j)|) \geq \exp(-C_0 W^3)|i - j|,$$

for any $i, j \in \Lambda$ *with* $|i - j| > W$.

The idea of the proof is to exploit the fact that $G_\Lambda^E(i, j)$ is a rational function in terms of the potentials, with the denominator and the numerator having different degrees. Ultimately, we reduce the problem to the study of the variance of logarithmic potentials of the form $u_\mu(x) = \int_{\mathbb{C}} \log |x - \zeta| \, d\mu(\zeta)$. The key fact for the proof is that if $\mu(|\zeta| \geq R) = 0$ then

$$|\mathrm{Var}_{[0,M]}(u_\mu) - 1| \leq C(R/M)^{1/5},$$

where the variance is with respect to the uniform distribution on $[0, M]$.

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Smooth quasi-periodic Schrödinger operators and L. Young's method

Yiqian Wang
(joint work with Zhenghe Zhang)

The first result on the positive lower bound of Lyapunov exponent for smooth quasi-periodic Schrödinger operators is due to Sinai[7], who studied a class of C^2 Schrödinger operators with Morse-type potentials. Since then, a series of work have been obtained for smooth cases. For the details, one can refer to [1]–[7].

Inspired by Benedicks-Carleson's method, L. Young invented a new idea to estimate lower bound of Lyapunov exponent for a wide class of $SL(2,R)$-cocycles.

More precisely, she considered a family of C^1 quasi-periodic $SL(2,R)$-cocycles:

$$A^n(x,t) = A(T^{n-1}x,t)\cdots A(x,t), \quad x \in S^1,\ t \in [0,1] \tag{1}$$

satisfying

$$\text{(i) } c\lambda < \|A(x,t)\| < C\lambda; \qquad \text{(ii) } \|\frac{\partial A(x,t)}{\partial x}\| < C\lambda$$

with $Tx = x+\omega$, c, $C>0$ two constants and λ sufficiently large and ω satisfies Brjuno condition and $\omega = \lim_{n\to\infty}\frac{p_n}{q_n}$.

Denote the most contracted and the most expanded direction of a matrix A by $\overline{s(A)}$ and $\overline{s'(A)}$, respectively.

Let μ be a large number, and let $n \geq 10$. We say that the sequence of matrices $\{A_0, A_1, ..., A_{n-1}\}$ is μ-hyperbolic if

$$\begin{array}{ll} \text{iii} \quad \|A_i\| \leq \mu^{3/2} \quad \forall i, & \text{(iv)} \quad \|A^i\| \geq \mu^i \quad \forall i, \\ \text{(v)} \quad |A_i(s(A^n))| \leq \mu^{-1} \quad \text{for } i=1,2,3, & \end{array}$$

and (iii)-(v) hold if $A_0, ..., A_{n-1}$ is replaced by $\{A_{n-1}^{-1}, ..., A_0^{-1}\}$.

We say $A^n(x,t)$ satisfies proposition (P_i) if the following hold true:

there is a set $C^{(i)} := \{c_1^{(i)}, ..., c_k^{(i)}\}$, *which we regard as the* i*th approximation to the critical set* $C^{(\infty)} := \{c_1^{(\infty)}, ..., c_k^{(\infty)}\}$. *Centered at each* $c_j^{(i)}$ *is an interval* $I_{i,j}$

of length $\frac{1}{q_i^2}$*. For* $x \in I_i := \bigcup_j I_{i,j}$*, the first return time to* I_i*, denoted* $r_i^+(x)$*, is* $\geq q_i$ *and the sequence of matrices* $\{A(x,t), ..., A(T^{r_i^+(x)-1}(x,t))\}$ *is* λ_i*-hyperbolic for for some* $\lambda_i > \lambda^{1-\epsilon_0}$*. The same is true for backward iterates. Moreover, if* $r_{i,j}^{\pm} = min_{x \in I_{i,j}} r_i^{\pm}(x)$*, then on each* $I_{i,j}$*, the functions* $s_i(x,t) := \overline{s(A^{r_{i,j}^+}(x,t))}$ *and* $s_i'(x,t) =: \overline{s(A^{r_{i,j}^-}(x,t))}$ *satisfy* **the non-tangential intersection property**.

We say $A^n(x,t)$ satisfies **Non-resonance assumption between critical points** for some $i \geq N$ if

$$|T^j c_k^{(i)} - c_l^{(i)}| \geq \frac{1}{q_i^2}, \qquad j \leq q_i.$$

With initial non-degenerate intersection assumption and the non-resonance assumption for i, Young[9] obtained that

Proposition *There exist positive constants* $\lambda_0 \gg N \gg 1$ *depending on* ω *such that if for* $\lambda > \lambda_0$*,* ($\mathrm{P_i}$) *holds true. Then with non-resonance assumption for* i*,* ($\mathrm{P_{i+1}}$) *also holds true.*

Corollary *There exists a positive (but not full) measure subset of the parameter space* $[0,1]$ *such that for each* t *in this subset, the cocycles* $A^n(x,t)$ *are nonuniformly hyperbolic with* $LE(t) \geq (1-\epsilon_0)\log\lambda$*, where we denote the Lyapunov exponent of* $A^n(x,t)$ *by* $LE(t)$ *and* $\epsilon_0 \to 0$ *as* $\lambda \to \infty$*. Moreover, for each* t *in this subset, the corresponding system is nonuniformly hyperbolic.*

Remark 1. In the Proposition, the conclusion holds true only for parameters in a positive measure subset due to the following reason. For each i, there is some parameters of positive measure such that the non-resonance assumption does not hold true.

Remark 2. Obviously, for Schrödinger operator with potential λv and energy $E \in [\lambda \cdot \min v, \lambda \cdot \max v]$, the conditions (i) and (ii) can not be satisfied. Thus Young's method is not directly applicable for Schrödinger cases.

From the above remarks, it seems difficult to study Schrödinger operators with Young's method although it is powerful. In 2012, Z. Zhang [10] provided a trick such that Young's method is applicable for Schrödinger cases. In fact, he constructed the conjugation $A^{(t,\lambda)} = TA^{(E-\lambda v))}T^{-1}$, where

$$T = \begin{pmatrix} \sqrt{\lambda}^{-1} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

With such a conjugation, Schrödinger cocycle with potential λv and $\lambda \gg 1$ is changed into Young's cocycle. With such a trick, among many other results Zhang obtained some partial result on positive lower bound of Lyapunov exponents for Schrödinger operators. Similar idea was also obtained by Avila, which combining with Young's method, was used in [8] to show that for in the space of smooth Schrödinger cocycles, the Lyapunov exponent is not necessarily continuous.

Thus for the application of Young's method in Schrödinger operators, we have to solve the difficulty of resonance between "critical points". Thus we need to prove some higher order non-degenerated intersection property. For this purpose, we need to ensure that the numbers of critical points for all induction steps have a finite upper bound. Since otherwise, as critical points become more and more, the order of derivatives we need to estimate becomes higher and higher. Thus the computation is more and more difficult. Moreover, in this situation, the cocycle has to be C^∞, which is too strong.

In the case that potentials of Schrödinger operators satisfy the conditions of Sinai [Sinai], we found that the upper bound for the number of critical points is 2. Then we can prove that critical points in each induction step are C^2-non-degenerate, which implies the following result:

Theorem 1 (Positive lower bound for Lyapunov exponent)
Let ω be irrational and satisfying the following Diophantine condition

$$|\omega - \frac{p}{q}| \geq \frac{\gamma}{|q|^\tau}, \quad p, q \in \mathbb{Z},\ q \neq 0,\ \tau > 2.$$

Assume $v \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ is of Morse-type function, i.e., $\frac{dv}{dx}$ vanish only at two extremal points and both of them are non-degenerated. Consider the Schrödinger cocycle $A^n_{E,\lambda}(x)$ with potential v and coupling constant λ. Then $\forall \epsilon > 0$, $\exists \lambda_0 = \lambda_0(\omega, v) > 0$ such that $L(E,\lambda) > (1-\epsilon)\log\lambda,\ \forall (E,\lambda) \in \mathbb{R} \times [\lambda_0, \infty)$.

The following result for the above Schrödinger operator also holds true:

Theorem 2 (Large Deviation Theorem)
There exists constants $n_0 \geq N$, $0 < \kappa \ll 1$ independent of λ such that for any $n \geq n_0$, the following estimate holds true:

$$(2) \quad mes\Big[x \Big| \, |\frac{1}{i}\log\|A^i_{E,\lambda}(x)\| - \log\lambda| > \kappa\log\lambda\Big] < e^{-\kappa^2 i} \quad \text{for any } i \in [q_n, q_{n+1}].$$

Remark 3 From the last statement in the Corollary of Young, we can see that resonance between critical points is the unique route for the appearance of uniform hyperbolicity when energy E lies in $[\min v, \quad \max v]$, our method may be used to study whether cantor spectrum exists or not for smooth situation.

Remark 4 It is not clear whether our method can be generalized to the situation that Sinai's condition is invalid, i.e., the situation that v has more degenerate points.

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Kotani-Last problem and Hardy spaces on surfaces of Widom type

Peter Yuditskii

(joint work with Alexander Volberg)

In this talk we show that it is not true that absolutely continuous spectrum forces reflectionless Jacobi matrices to be almost periodic. Moreover this is not an example but a small spectral theory saying that under certain simple assumptions on the spectrum either all such matrices are almost periodic or all of them are not almost periodic. For distinguishing between these two cases there is a “switch” that says exactly when which case happens. The switch is called Direct Cauchy Theorem. Our paper [4] builds a spectral side of the theory that complements nicely an example of Artur Avila working on the potential side [1].

By Kotani theory, see e.g. [2], ergodic matrices with a.c. spectrum are *reflectionless* and we start with the definition of this class.

Let J be a two-sided matrix

$$Je_n = p_n e_{n-1} + q_n e_n + p_{n+1} e_{n+1}. \tag{1}$$

where e_n's are the vectors of the standard basis in $\ell^2 = \ell^2(\mathbb{Z})$. Let $\ell^2_+ = \text{span}_{n\geq 0}\{e_n\}$ and $\ell^2_- = \text{span}_{n\leq -1}\{e_n\}$. We define

$$J_\pm = P_{\ell^2_\pm} J|\ell^2_\pm,$$

and corresponding resolvent functions

$$r_\pm(z) = \langle (J_\pm - z)^{-1} e_{\frac{-1\pm 1}{2}}, e_{\frac{-1\pm 1}{2}}\rangle. \tag{2}$$

For a compact $E = [b_0, a_0] \setminus \cup_{j\geq 1}(a_j, b_j)$ of positive Lebesgue measure, $|E| > 0$, we say that $J \in J(E)$ if the spectrum of J is E and

$$\frac{1}{r_+(x+i0)} = \overline{p_0^2 r_-(x+i0)} \text{ for almost all } x \in E. \tag{3}$$

Notice that the property (3) is shift invariant, that is, $S^{-1}JS \in J(E)$ as soon as $J \in J(E)$. $J(E)$ is a compact in the sense of pointwise convergence of coefficients sequences.

We are especially interested in the case when every element of $J(E)$ has *absolutely continuous* spectrum. For instance, if E is a system of non-degenerated intervals with a unique accumulation point, say 0, then E is a subject for the condition $\int_E \frac{dx}{|x|} = \infty$. In general the structure of corresponding sets E was studied quite completely by Poltoratski and Remling [3].

To be able to use freely Complex Analysis we assume, in addition, that the resolvent domain $\Omega = \bar{\mathbb{C}} \setminus E$ is of Widom type. One can give a *parametric description of regular Widom domains* by means of conformal mappings on the so called comb-domains. Consider two sequences $\{\omega_k\}$, $\omega_k \in (0,1)$, $\omega_k \neq \omega_j$ if $k \neq j$, and $\{h_k\}$, $h_k > 0$, $\sum h_k < \infty$. Let Π be a region obtain from the half-strip

$$\{z = x + iy : \ -\pi < x < 0, \ y > 0\}$$

by removing vertical intervals $\{-\pi\omega_k + iy : 0 < y \leq h_k\}$. Let θ be the conformal map from the upper half-plane to Π such that

$$\theta(b_0) = -\pi, \ \theta(a_0) = 0, \ \theta(\infty) = \infty.$$

Notice that it has a continuous extension to the closed half-plane. The set E corresponds to the base of the comb, $E = \theta^{-1}([-\pi, 0])$, and the gaps correspond to the slits.

We say that the Direct Cauchy Theorem (DCT) holds in a Widom domain Ω if

$$\frac{1}{2\pi i} \oint_{\partial\Omega} \frac{F(x)}{x - z} dx = F(z), \quad z \in \Omega, \tag{4}$$

for all F of the Smirnov class functions in Ω (a ratio of two bounded functions on the universal covering $\mathbb{D}$ with an outer denominator) such that

$$\oint_{\partial\Omega} |F(x)dx| < \infty \text{ and } F(\infty) = 0.$$

Main Theorem. Let E be such that

(i) Every $J \in J(E)$ has absolutely continuous spectrum;
(ii) The domain Ω is of Widom type;
(iii) The frequencies $\{\omega_j\}$ are independent.

Then on the compact $\Xi := J(E)$ there is an unique shift invariant measure dJ, $dJ = d(S^{-1}JS)$, such that the measurable functions, see (1),

$$\mathcal{P}(J) = p_0 = \langle Je_0, e_{-1} \rangle, \quad \mathcal{Q}(J) = q_0 = \langle Je_0, e_0 \rangle, \quad J \in J(E), \tag{5}$$

define an ergodic family $\{J\}_{J \in \Xi}$.

Moreover, all elements of the family are either almost periodic or they all are not, depending on the following analytic property of the domain Ω: either DCT holds true or it fails.

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Szegő theorem for finite and infinite gap OPUC

Maxim Zinchenko

It is known [1] that probability measures $d\mu$ on the unit circle are in one-to-one correspondence with sequences of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}$ that arise as recursion coefficients of the corresponding orthogonal polynomials. This relation is highly nonlinear and may be viewed as a nonlinear analog of the Fourier transform [4]. A classical result of Szegő [3] provides a qualitative and quantitative insight into this nonlinear relation and in some sense is a nonlinear version of Parseval's identity. In modern language [1], Szegő's theorem asserts that for any probability measure $d\mu = w\,dm + d\mu_s$ (here dm denotes the normalized Lebesgue measure on the unit circle and $d\mu_s$ the singular component of the measure $d\mu$) and the corresponding Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ the following relation holds

$$\log \prod_{n=0}^{\infty} (1 - |\alpha_n|^2) = \int_{\partial\mathbb{D}} \log w(\zeta)\, dm(\zeta),$$

where both sides are allowed to be infinite. In particular, the qualitative version of Szegő's theorem is give by the following equivalence relation

$$\{\alpha_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0) \iff \log w \in L^1(dm).$$

Thus, the classical Szegő's theorem provides a nontrivial result only in the case of probability measures supported on the entire unit circle.

In an on going work [6], we propose an extension of the above result to probability measures with derived set E (i.e., support without isolated points) given by the unit circle minus finite or infinite number of open arcs (a_j, b_j), which have disjoint closures,

$$E = \partial\mathbb{D} \setminus \bigcup_{j=0}^{\ell} (a_j, b_j), \quad \ell \in \mathbb{N} \cup \{\infty\}.$$

Let $\operatorname{Cap}(E)$ denote the logarithmic capacity of the set E. Then in the finite gap setting (i.e., $\ell < \infty$) we have the following extension of the classical Szegő theorem: Any two of the following conditions $(a)-(c)$ imply the third,

$$(a) \qquad \sum_{\lambda\in\operatorname{supp}(d\mu)\setminus E} \operatorname{dist}(\lambda, E)^{1/2} < \infty,$$

$$(b) \qquad \int_E \frac{\log w(\zeta)\, dm(\zeta)}{\operatorname{dist}(\zeta, \partial\mathbb{D}\setminus E)^{1/2}} > -\infty,$$

$$(c) \qquad \begin{cases} \liminf_{n\to\infty} \left(\dfrac{\rho_0\cdots\rho_{n-1}}{\operatorname{Cap}(E)^n}\right) > 0, \\ \limsup_{n\to\infty} \left(\dfrac{\rho_0\cdots\rho_{n-1}}{\operatorname{Cap}(E)^n}\right) < \infty. \end{cases}$$

In the infinite gap setting we assume that either E is homogeneous, that is, for some $\delta > 0$

$$m[E \cap (\zeta e^{-i\varepsilon}, \zeta e^{i\varepsilon})] \geq \varepsilon\delta \text{ for all } \zeta \in E \text{ and } \varepsilon \in (0, \pi],$$

or more generally that $\hat{\mathbb{C}}\setminus E$ is a Widom domain with Direct Cauchy Theorem [5]. Let $G_E(z)$ denote the potential theoretic Green's function and $d\nu_E = \nu_E\, dm$ the equilibrium measure associated with the set E, [2]. Then assuming that

$$\sum_{\lambda\in\operatorname{supp}(d\mu)\setminus E} G_E(\lambda) < \infty$$

we obtain the following equivalence

$$\int_E \log w(\zeta)\, d\nu_E(\zeta) > -\infty \quad \Leftrightarrow \quad \limsup_{n\to\infty} \left(\frac{\rho_0\cdots\rho_{n-1}}{\operatorname{Cap}(E)^n}\right) > 0.$$

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