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Lattice Differential Equations

Organised by
Guillaume James, Grenoble
Dmitry Pelinovsky, Hamilton
Zoi Rapti, Urbana
Guido Schneider, Stuttgart

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Abstract. The workshop focused on recent advances in the analysis of lattice differential equations such as discrete Klein-Gordon and nonlinear Schrödinger equations as well as the Fermi-Pasta-Ulam lattice. Lattice differential equations play an important role in emergent directions of modern science. These equations are fascinating subjects for mathematicians because they exhibit phenomena, which are not encountered in classical partial differential equations, on one hand, but they may present toy problems for understanding more complicated Hamiltonian differential equations, on the other hand.

Mathematics Subject Classification (2010): 34K31, 34A33.

Introduction by the Organisers

The workshop was well attended with 24 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds interested in lattice differential equations such as discrete Klein-Gordon and nonlinear Schrödinger equations as well as the Fermi-Pasta-Ulam lattice. Nonlinear differential-difference equations for the dynamics of localized excitations in lattices have been recently studied in a number of physical contexts. The novelty and significant differences of these models from the classical nonlinear partial differential equations make the topic of lattice dynamics fascinating for researchers. New phenomena can arise in lattice differential equations such as bifurcation of large-amplitude breathers from infinity, multiple resonances with linear waves in polyatomic models, and propagation failure for non-smooth nonlinear potentials. At the same time, mathematical analysis of differential-difference equations may be simplified in some problems because the difference operators are bounded, strong solutions of initial-value problems exist globally in time, and the system of differential equations can be uncoupled in the anti-continuum limit. Recent applications of the lattice differential equations are important for inter-disciplinary studies between mathematics, physics, biology, and electrical engineering. These applications call mathematicians to contribute to the analysis of lattice differential equations. Among other applications, we singled out the following particular topics of increasing interest.

- Fermi-Pasta-Ulam problems with non-analytic potentials are considered in the context of granular crystals, where particles interact according to Hertzian forces. Understanding and controlling localization in granular crystals may lead to new engineering devices related to energy filters.
- Discrete Klein-Gordon lattices are used in the Peyrard-Bishop model of the base pairs in DNA. The Morse potential is confining for negative displacements and bounded for positive displacements. Intrinsic localized modes bifurcate with large amplitudes in these models and may be relevant for the analysis of global dynamics of base pairs in DNA.
- Nanophotonics, engineering of photo-refractive crystals, and trapping of atomic Bose-Einstein condensates in optical lattices, all rely on modeling of the discrete nonlinear Schrödinger equation in the space of one, two, and three dimensions. Recent works on discrete vortices and Gibbs measure for phase transitions to solitons have stimulated further studies of dynamics of localization at large energies in these models.

Mathematical studies of lattice differential equations are developed by using various popular approaches, like bifurcation theory, dynamical systems methods, applied harmonic analysis, perturbation theory, numerical simulations, KAM theory, and symplectic geometry. The intensity and extent of recent developments and the increasing amount of interesting problems arising from applications render especially timely a meeting that will bring specialists in the relevant areas together. The workshop focused on aspects related to the dynamics of the above lattice differential equations and on connections with relevant applications. In particular, the workshop covered the following concentration areas.

(1) Resonances in lattice equations and macroscopic analysis;
(2) Spatially localized oscillations in nonlinear lattices, applications to DNA dynamics;
(3) Travelling waves in lattices with limited smoothness, applications to granular crystals;
(4) Orbital and asymptotic stability of solitons in lattices; and
(5) Vortices in multi- dimensional lattices and justification of variational approximations.

Workshop: Lattice Differential Equations

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Quasi-periodic breathers in Hamiltonian networks

Yingfei Yi

A Hamiltonian network (lattice, chain) models the interaction of an array of particles in a lattice. In many physical situations, the total energy of such a network has the form

$$H = \sum_{k \in Z^n} (\frac{p_k^2}{2} + V_k(q_k)) + \varepsilon \sum_{k \in Z^n} W_k(\{q_j\}),$$

where for each k, $p_k = \dot{q}_k$, V_k is the onsite potential, W_k is the coupling or contact potential between the kth particle and those on other sites, and $\varepsilon > 0$ is the coupling strength which is usually taken as a small parameter to reflect the anti-continuum limiting nature of the network. The simplest type of coupling among sites is the one with the nearest-neighbor coupling potential

$$W_k(\{q_j\}) = (q_{k+1} - q_k)^l, \quad l \geq 2,$$

in which $l = 2$ models linear coupling of Klein-Gordon type and $l \geq 3$ models nonlinear couplings. But in many biological and physical settings, short-range couplings among sites are not the only mechanisms by which the state at one location affects the state elsewhere. For instance, in many models of mean field theory and solid state physics long-range coupling potentials of the form

$$W_k(\{q_j\}) = \sum_{m \neq k} C_{k,m}(q_k - q_m)^l, \quad l \geq 2$$

are often considered in which either $C_{k,m} = 1/|k-m|^\alpha$ or $\exp(-|k-m|^\alpha)$, $\alpha \geq 1$, representing power-law or exponential interactions, respectively.

With harmonic or anharmonic onsite potential, each un-coupled oscillator oscillates periodically around an equilibrium position. Depending on the nature of couplings, one or more sites can be excited to yield oscillations for the coupled Hamiltonian network which can be time periodic or quasi-periodic. In fact, among the important coherent structures (organized, localized, and repeated patterns) in a Hamiltonian network, of particular physical interests are the so-called *breathers* or *quasi-periodic breathers*, often referred to as dynamical solitons or intrinsic localized modes in physics, which are self-localized, time periodic or quasi-periodic solutions whose amplitudes decay to 0 as $|n| \to \infty$. Although breathers were first discovered in Hamiltonian PDEs like sine-Gordon, modified KdV, and nonlinear Schrödinger equations, they appear to be rare and non-robust objects in Hamiltonian PDEs. To the contrary, numerical study in many physical models seems to show that the existence of breathers and quasi-periodic breathers is a general phenomenon in Hamiltonian networks, suggesting an important mechanism of localizations due to the discreteness and the nature of nonlinearities rather than disorder.

For Klein-Gordon like lattices involving linear coupling of identical anharmonic oscillators, the existence of breathers was rigorously proved using continuation method in [8] with respect to the short-range potential and in [1] with respect to the long-range, power-law potential. Motivated by experimental findings in cantilever arrays, coupled optical waveguide, DNA breathing, granular crystals, Josephson junction arrays etc, there have been many recent studies and numerical simulations towards the existence of breathers, their stabilities and bifurcations in much broader classes of Hamiltonian lattices.

In contrast to the case of breathers, there have not been much study toward quasi-periodic breathers in Hamiltonian networks. When considering small amplitude quasi-periodic breathers, a nonlinear coupling potential often need to be adopted in order to avoid possible continuous spectrum at the equilibrium. Indeed, this is the case with the short-range cubic potential for the existence of quasi-periodic and almost periodic breathers ([3, 9, 10]). For the case with long-range, cubic, either power-law or exponential coupling potentials, the existence of quasi-periodic breathers oscillating on any finite number of excited sites is shown in [4, 5]. However, quasi-periodic breathers may still exist under short-range linear couplings when the network either possesses certain symmetry ([2, 7]) or admits a dense set of eigenvalues at the equilibrium ([6]).

Due to the multi-frequency nature of the problem, the study of quasi-periodic breathers naturally involves normal form reduction and KAM iteration technique. More precisely, for fixed N sites of the lattice to be excited, one can parameterize the uncoupled invariant N-tori sitting on these sites by ξ in a region $\mathcal{O} \subset R^N$ and introduce action-angle variable $(I, \theta) \in R^N \times T^N$ for I lying in a neighborhood of the origin. The remaining sites are re-coordinated as the normal variable $z = (z_k) \in l^1$ near the origin. Then after some rescalings, the network is converted into a normal form

$$H = e(\xi) + \langle \omega(\xi), I \rangle + \frac{1}{2} \sum_k \Omega_k(\xi) z_k \bar{z}_k + \varepsilon P(\theta, I, z, \bar{z}, \xi, \varepsilon),$$

where $\omega(\xi)$ denotes the frequency vectors of the unperturbed N-tori. A successful KAM iteration technique is to introduce a sequence of convergent canonical transformations on nested frequency and phase domains so that the angular-dependency of the perturbation is pushed into a higher order after each iteration step and finally vanishes, and the common frequency domain $\mathcal{O}_\infty$, usually a Cantor set, has almost full Lebsgue measure relative to $\mathcal{O}$. As results, each $\xi \in \mathcal{O}_\infty$ then corresponds to a quasi-periodic breather with N-frequencies and amplitude proportional to $\sqrt{|\xi|}$. Depending on the form of contact potential, the truncation orders of the normal variable during KAM iterations also reflect the nature of localization of the quasi-periodic breather, for instance, being super-exponential when the interaction is short-ranged, exponential when the interaction is exponentially long-ranged, and algebraic when the interaction is algebraically long-ranged.

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Periodic Toda chains with many particles

Thomas Kappeler
(joint work with D. Bambusi, T. Paul)

Our aim is to study the asymptotics of periodic Toda chains with a large number of particles of equal mass for initial data close to the equilibrium. To state our first result, assume that β, α are in the space $C^2_0(\mathbb{T}) \equiv C^2_0(\mathbb{T}, \mathbb{R})$ of one periodic, real valued functions of class C^2 and mean 0. For any $N \geq 3$, we then introduce the periodic Toda chain with N particles, defined in terms of Flaschka coordinates by $b^N_n = \frac{1}{4N^2}\beta\big(\frac{n}{N}\big)$ and $a^N_n = 1 + \frac{1}{4N^2}\alpha\big(\frac{n}{N}\big)$. Denote by ω^N_n and I^N_n, $0 < n < N$, the frequencies and actions of (b^N, a^N). Further introduce the two potentials $q_\pm(x) = -2\alpha(2x) \mp \beta(2x)$. Note that they have period $\frac{1}{2}$ and are of class C^2. Let $I^\pm_j$ and $\omega^\pm_j$, $j \geq 1$, be the corresponding KdV actions and frequencies and define the following sequence of KdV Hamiltonians and corresponding frequencies

$$\mathcal{H}^N_{KdV} := \frac{1}{2N}\mathcal{H}_1 - \frac{1}{24}\frac{1}{(2N)^3}\mathcal{H}_2 \quad \text{and} \quad \partial_{I^\pm_n}\mathcal{H}^N_{KdV}(q_\pm) = \frac{2\pi n}{N} - \frac{1}{24}\frac{1}{(2N)^3}\omega^\pm_n.$$

where $\mathcal{H}_1(q) = \frac{1}{2}\int_0^{\frac{1}{2}} q^2 dx$ and $\mathcal{H}_2(q) = \int_0^{\frac{1}{2}} \big(\frac{1}{2}(\partial_x q)^2 + q^3\big)dx$. Finally consider the edge defining functions $F : \mathbb{N} \to \mathbb{R}_{\geq 1}$ satisfying

$$(F) \qquad \lim_{N\to\infty} F(N) = \infty; \qquad F \text{ increasing}; \qquad F(N) \leq N^\eta \text{ with } \eta > 0.$$

Theorem 1. *([3]) Let $F : \mathbb{N} \to \mathbb{R}_{\geq 1}$ satisfy (F) with $\eta \leq 1/3$ and $M = [F(N)]$. Then the asymptotics of the frequencies $(\omega^N_n)_{0<n<N}$ of (b^N, a^N) are as follows: at*

the left and right edges one has for $1 \leq n \leq F(M)$

$$\omega_n^N = \partial_{I_n^-} \mathcal{H}^N_{KdV} + O\left(\frac{1}{N^3}\big(\frac{n^2 F(M)}{M^{1/2}} + \frac{1}{F(M)^{5/2}}\big)\right) \tag{1}$$

$$\omega_{N-n}^N = \partial_{I_n^+} \mathcal{H}^N_{KdV} + O\left(\frac{1}{N^3}\big(\frac{n^2 F(M)}{M^{1/2}} + \frac{1}{F(M)^{5/2}}\big)\right) \tag{2}$$

whereas in the bulk, $M < n < N - M$*,* $\omega_n^N = 2\sin\frac{\pi n}{N}\Big(1 + O\big(\frac{\log M}{M^2}\big)\Big)$*. Finally, for* $F(M) < n \leq M$*,* $0 < \omega_n^N = \frac{2\pi n}{N} + O\left(\frac{n^3}{N^3}\right)$ *and* $0 < \omega_{N-n}^N = \frac{2\pi n}{N} + O\left(\frac{n^3}{N^3}\right)$*. These estimates hold uniformly in* $0 < n < N$ *and uniformly on bounded subsets of functions* α, β *in* $C_0^2(\mathbb{T})$*.*

To state the asymptotics of the actions I_n^N, $0 < n < N$, recall that the periodic eigenvalues of $-\partial_x^2 + q_\pm$, considered with periodic boundary conditions on the interval $[0,1]$, satisfy, when listed in increasing order and with their multiplicities, $\lambda_0^\pm < \lambda_1^\pm \leq \lambda_2^\pm < \cdots$. As $q_\pm$ are of class C^2, one has $\sum_{n=1}^\infty n^4(\gamma_n^\pm)^2 < \infty$ where $\gamma_n^\pm := \lambda_{2n}^\pm - \lambda_{2n-1}^\pm$.

Theorem 2. *([3]) Let* $F : \mathbb{N} \to \mathbb{R}_{\geq 1}$ *satisfy (F) with* $\eta < 1/2$ *and* $M = [F(N)]$*. Then the asymptotics of the actions* $I^N = (I_n^N)_{0<n<N}$ *of* (b^N, a^N) *are as follows: at the left and right edges one has for* $1 \leq n \leq F(M)$

$$8N^2 I_n^N = I_n^- + O\Big(\frac{M^2}{N}\frac{F(M)}{M^{1/2}} + \frac{M^3}{N^{3/2}} + \gamma_n^-\big(\frac{F(M)}{M^{1/2}} + \frac{M}{N^{1/2}}\big)\Big)$$

$$8N^2 I_{N-n}^N = I_n^+ + O\Big(\frac{M^2}{N}\frac{F(M)}{M^{1/2}} + \frac{M^3}{N^{3/2}} + \gamma_n^+\big(\frac{F(M)}{M^{1/2}} + \frac{M}{N^{1/2}}\big)\Big)$$

whereas in the bulk, $M < n \leq N/2$*,* $I_n^N, I_{N-n}^N = O\Big(\frac{1}{nM^2}\frac{1}{N^2}\Big)$*. Finally for* $F(M) < n \leq M$*,* $I_n^N = O\Big(\frac{1}{n}\big((\gamma_n^-)^2 + \frac{M^4}{N^2}\big)\frac{1}{N^2}\Big)$ *and* $I_{N-n}^N = O\Big(\frac{1}{n}\big((\gamma_n^+)^2 + \frac{M^4}{N^2}\big)\frac{1}{N^2}\Big)$*. These estimates hold uniformly in* $0 < n < N$ *and uniformly on bounded subsets of functions* α, β *in* $C_0^2(\mathbb{T})$*.*

The implications of Theorem 1 and Theorem 2 on the approximation of the Toda chain with initial data (b^N, a^N) by KdV type solutions are the following ones. For simplicity we assume that $F(N) = N^\eta$ with $0 < \eta \leq 1/3$. Then $M = [N^\eta]$ and $F(M) \sim N^{\eta^2}$. In Birkhoff coordinates, the n'th component $(x_n^N(t), y_n^N(t))$ of the solution, $0 < n < N$, is of the form

$$(x_n^N(t), y_n^N(t)) = \sqrt{2I_n^N}(\cos(\theta_n^N + t\omega_n^N), \sin(\theta_n^N + t\omega_n^N)) \quad \text{if} \quad I_n^N \neq 0$$

and zero otherwise. Here θ_n^N is the n'th angle coordinate, determined by the initial data. First note that according to Theorem 2, the size of the components of the solution in the bulk is small, i.e., $\sum_{F(M)<n<N-1-F(M)} I_n^N$ is $O\Big(\frac{1}{N^2}\big(\frac{1}{N^{5\eta^2}} + \frac{\log N}{N^{2\eta}}\big)\Big)$. Hence for N sufficiently large, the solution of the Toda chain with initial data (b^N, a^N) can be viewed as a small perturbation of a long wave obtaind by setting

$(x_n^N(t), y_n^N(t))$ with $F(M) < n < N-1-F(M)$ to zero. Secondly we note that it follows from Theorem 1 that on a time interval of size larger than N^3, these long waves are approximated by two KdV type solutions. More precisely, in the case where $0 < \eta < \frac{1}{11}$, one has for any $1 \leq n \leq F(M)$ by (1) and (2),

$$\omega_n^N - \partial_{I_n^-}\mathcal{H}_{KdV}^N, \quad \omega_{N-n}^N - \partial_{I_n^+}\mathcal{H}_{KdV}^N = O(\frac{1}{N^3 N^{5\eta^2/2}}).$$

Hence the approximation of the solutions considered above is valid on a longer time interval than the one obtained in [4] (cf also [5]). The asymptotics of the Toda frequencies up to order 5 – in principle possible – conjecturely involves in (1)-(2) the third Hamiltonian of the KdV hierarchy.

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Forced Travelling Waves in Nonlinear Lattices

Michal Fečkan

(joint work with Josef Diblík, Michal Pospíšil, Vassilis M. Rothos and Hadi Susanto)

The propagation of electromagnetic waves in metamaterials has been discussed and modelled by several types of nonlinear equations, such as a nonlinear Klein-Gordon equation [9], coupled short-pulse equation [10], higher-order nonlinear Schrödinger equations [11] and coupled Klein-Gordon equations [7, 8]. We investigate [1]

$$\lambda\ddot{q}_{n+1} - \ddot{q}_n + \lambda\ddot{q}_{n-1} = \gamma\dot{q}_n + \varphi(q_n) + f\cos(\omega t + pn),$$

where as $\gamma \geq 0$ is the dissipative loss of the medium, $\lambda \in \mathbb{R}$ is the coupling constant between nearest neighbor resonators, $\omega > 0$ is the external driving frequency, $f \neq 0$ is the amplitude of the external force, $p \neq 0$ is the wavenumber of the travelling wave field and φ is the nonlinearity of the magnetic material, which is normally assumed to be of Kerr-type [8]. Seeking for waves travelling in the same direction as the external drive, we take $q_n(t) = U(z)$, $z = \omega t + pn$ with $U(z+\pi) = -U(z)$ to obtain

$$\omega^2(\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) = \gamma\omega U'(z) + \varphi(U(z)) + f\cos z.$$

Note that $p \in \mathbb{R}\backslash\{0\}$. Considering the case $p = -\pi$, one will obtain alternating charges between the nearest-neighbor resonators as $f\cos(\omega t + pn) = (-1)^n f\cos\omega t$.

First we are looking for a periodic travelling wave when its amplitude is limited by the magnitude of the forcing. Then we suppose that an unperturbed system has a periodic solution, and we use the subharmonic Melnikov bifurcation method to find conditions under which this periodic solution persists. Homoclinic Melnikov bifurcation method is used as well. We obtain large solutions under small perturbations but also we obtain small solutions under small perturbations. The main difference is the resonance property. Next we solve the governing equations numerically, in particular for the asymptotic waves.

Similar results are derived for:

1. Damped FPU lattice forced by a travelling wave field (see [2, 5, 6]):

$$\begin{aligned}\ddot{u}_n = \alpha(u_{n+1} + u_{n-1} - 2u_n) + \beta(u_{n+1} - u_n)^3 + \beta(u_{n-1} - u_n)^3 \\ -\gamma\dot{u}_n + f\cos(\omega t + pn)\,,\end{aligned}$$

where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters.

2. Nonlinear magneto-inductive lattice [3, 12] forced by a travelling wave field

$$\begin{aligned}\frac{d^2}{dt^2}(u_n - \lambda u_{n-1} - \lambda u_{n+1}) + \gamma\frac{d}{dt}u_n + u_n \\ +\frac{d^2}{dt^2}(u_n^2 - \lambda u_{n-1}^2 - \lambda u_{n+1}^2) + \gamma\frac{d}{dt}u_n^2 - h(\omega t + pn) = 0\end{aligned}$$

where $\gamma \geq 0$, $\lambda, \omega > 0$, $p \neq 0$ are parameters.

3. Forced FPU lattice maps with local interactions. We consider a discrete version of the following 1-dimensional damped FPU lattice forced by a travelling wave field:

$$\begin{aligned}\ddot{u}_n = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ -\gamma\dot{u}_n + f\cos(\omega t + pn),\end{aligned}$$

where $\alpha > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters and φ_1, φ_2 are odd analytic functions with radius of convergence ρ_1, ρ_2, respectively, such that

$$\mathrm{D}\varphi_{1,2}(0) = 0.$$

We substitute the differentiation by the symmetric difference, i.e.[4]

$$\dot{u}_n(t) \to u_n(t + 1/2) - u_n(t - 1/2), \qquad \ddot{u}_n(t) \to u_n(t + 1) - 2u_n(t) + u_n(t - 1).$$

So we study the travelling waves of the system

$$\begin{aligned}u_n(t + 1) - 2u_n(t) + u_n(t - 1) = \alpha(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) \\ +\varphi_1(u_{n+1}(t) - u_n(t)) + \varphi_2(u_{n-1}(t) - u_n(t)) \\ -\gamma(u_n(t + 1/2) - u_n(t - 1/2)) + f\cos(\omega t + pn).\end{aligned}$$

Putting $u_n(t) = U(\omega t + pn)$ for $U(z + \pi) = -U(z)$, we get

$$\begin{aligned}U(z + \omega) - 2U(z) + U(z - \omega) = \alpha(U(z + p) - 2U(z) + U(z - p)) \\ +\varphi_1(U(z + p) - U(z)) + \varphi_2(U(z - p) - U(z)) \\ -\gamma(U(z + \omega/2) - U(z - \omega/2)) + f\cos z\end{aligned}$$

with $z = \omega t + pn$.

4. Forced lattices with nonlocal interactions like [2]

$$\ddot{u}_n = \sum_{j\in\mathbb{Z}} \alpha_j(u_{n+j}-u_n) + \sum_{j\in\mathbb{Z}} \beta_j(u_{n+j}-u_n)^3 - \gamma\dot{u}_n + f\cos(\omega t + pn)\,,$$

where $\alpha_j = \alpha_{-j} > 0$, $\alpha_0 = 0$, $\beta_j = \beta_{-j} > 0$, $\beta_0 = 0$ for any $j \in \mathbb{Z}$ and $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters. Moreover, we suppose

$$\sum_{j\in\mathbb{N}} \alpha_j < \infty\,, \quad \sum_{j\in\mathbb{N}} \beta_j < \infty\,.$$

Putting $u_n(t) = U(\omega t + pn)$ for $U \in Y$, we obtain

$$\omega^2 U''(z) = \sum_{j\in\mathbb{N}} \alpha_j(U(z+pj) + U(z-pj) - 2U(z)) - \gamma\omega U'(z)$$
$$+ \sum_{j\in\mathbb{N}} \beta_j((U(z+pj) - U(z))^3 + (U(z-pj) - U(z))^3) + f\cos z\,.$$

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Interface dynamics in discrete forward-backward diffusion equations

Michael Herrmann
(joint work with Michael Helmers, University of Bonn)

The diffusive lattice ODE

$$\dot{u}_j = p_{j+1} - 2p_j + p_{j-1}, \qquad p_j = \Phi'(u_j) \tag{1}$$

with bistable nonlinearity Φ' can be regarded as a *microscopic regularization* of the ill-posed PDE

$$\partial_\tau U = \partial_\xi^2 P, \qquad P = \Phi'(U), \tag{2}$$

provided that the macroscopic variables are introduced by the *hyperbolic scaling*. The latter reads

$$\tau = \varepsilon^2 t, \qquad \xi = \varepsilon j, \qquad u_j(t) = U(\varepsilon^2 t, \, \varepsilon j)$$

with $\varepsilon > 0$ being the small scaling parameter. Other notably regularizations of (2) are the *Cahn-Hillard model* and the *viscous approximation*, which add $-\varepsilon^2 \partial_\xi^4 U$ and $+\varepsilon^2 \partial_\xi^2 \partial_\tau U$, respectively, to the right hand side of $(2)_1$.

Hysteretic interface motion. A key dynamical feature of any regularization of (2) are *phase interfaces*. These curves separate space-time regions in which U attains values in different *phases*, that means in either one of the two connected components of $\{u : \Phi''(u) > 0\}$; see figures 1 and 2 for illustration.

Heuristic arguments as well as numerical simulations of (1), see §2 in [3], indicate that the effective lattice dynamics for $\varepsilon \to 0$ can – for a wide class of initial data – be described by a hysteretic free boundary problem. In the case of a single interface located at $\xi_*(\tau)$, this model combines *bulk diffusion*

$$\partial_\tau U = \partial_\xi^2 P \qquad \text{for} \qquad \xi \neq \xi_*(\tau)$$

with the *Stefan condition*

$$\frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| + |[\partial_\xi P]| = |[P]| = 0$$

and the *hysteretic flow rule*

$$\frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| < 0 \quad \frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| > 0 \qquad \Longrightarrow \qquad P = p_*,$$

where $|[\cdot]|$ denotes the jump across the interface. The same equations can – at least on a formal level – also be derived from the viscous approximation, see [1]. The sharp interface limit of the Cahn-Hilliard equation, however, is different as it replaces the hysteric flow rule by $P = p_{\mathrm{mx}}$, where p_{mx} represents the Maxwell line.

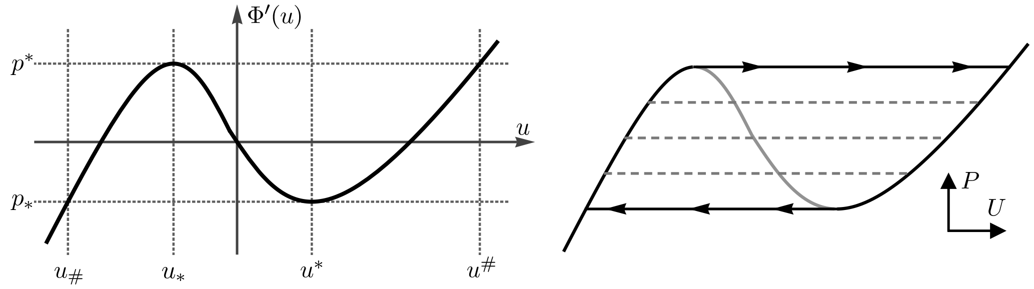

Figure 1. *Left panel.* Bistable derivative Φ' of a general double-well potential Φ. *Right panel.* The hysteresis loop for phase interfaces: solid and dashed lines represent moving and standing interfaces, respectively; there arrows indicate the temporal jump when U undergoes a phase transition at fixed position ξ.

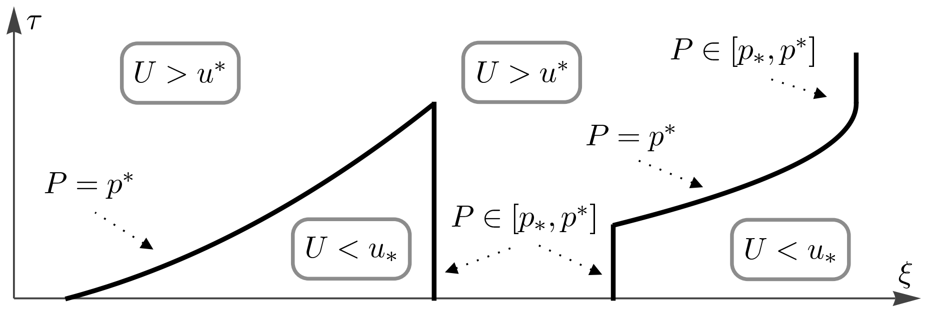

Figure 2. Cartoon of three phase interfaces: The first one (moving) and the second one (standing) annihilate each other in a collision. The third interface illustrates both pinning and depinning.

Rigorous analysis for a special case. Due to the existence of multiple time scales, the rigorous justification of the above limit model is – in the case of moving interfaces – currently out of reach; for standing interfaces, see [2]. For the piecewise quadratic double-well potential

$$\Phi(u) = \tfrac{1}{2}\min\left\{(1-u)^2,\, (1+u)^2\right\}, \qquad \Phi'(u) = u - \operatorname{sgn} u, \tag{3}$$

however, the analysis of (1) is considerably simpler and allows us to prove the following results:

(1) *Existence of single-interface solutions*: A certain class of microscopic single-interface states is invariant under the flow of (1)+(3). In particular, the lattice dynamics generates a single phase interface which moves since the lattice data u_j undergo a phase transition (crossing of the spinodal state $u = 0$) one after another.
(2) *Justification of the limit model*: For macroscopic single-interface data, the lattice solutions converge as $\varepsilon \to 0$ to the unique solution of the hysteretic free boundary problem.

The first thesis is a direct consequence of elementary ODE arguments and implies the representation formula

$$p_j(t) = \sum_{i\in\mathbb{Z}} g_{j-i}(t)p_i(0) - 2\sum_{k\geq 1} \chi_{(t_k^*,\,\infty)}(t)g_{j-k}(t-t_k^*). \tag{4}$$

Here, g abbreviates the discrete heat kernel and χ_I denotes the indicator function of the interval I. Moreover, t_k^* is the k^{th} phase transition time, which is, however, not given a priori but depends nonlinearly on the whole solution p via $\lim_{t\nearrow t_k^*} p_k(t) = +1$.

The representation formula (4) is crucial for passing to the limit $\varepsilon \to 0$ as it allows us to exploit the temporal and spation decay properties of the discrete heat kernel. In particular, assuming that the initial data are sufficiently nice we can derive upper bounds for the macroscopic interface speed as well as macroscopic compactness results for the scaled lattice data in the space of Hölder continuous functions. The convergence result then follows by combining standard arguments with a direct justification of the hysteretic flow rule and a uniqueness result from [4]. The details can be found in §3 of [3].

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Evolutions of Probability Measures on Collision Trees as a Tool for Micro-macro Transitions in discrete Systems

Karsten Matthies

(joint work with Florian Theil)

A crucial question in the understanding of many particle system is the validity of effective continuum descriptions. The talk reported on a method developed in [1, 2] to rigorously show the validity of Boltzmann-type descriptions for the deterministic dynamics of n hard spheres of diameter a with random initial data. The particles have random initial conditions

$$(u^0, v^0) = (z_1, \ldots, z_n) \in (\mathbb{T}^d \times \mathbb{R}^d)^n, \tag{1}$$

with $z_1, \ldots, z_n$ independent, identically distributed according to f_0. The diameter a is determined by the Boltzmann-Grad scaling $na^{d-1} = 1$. They evolve by force-free Newtonian dynamics, i.e. according to the differential equations

$$\begin{aligned} \dot{u}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= 0 \end{aligned} \tag{2}$$

The scattering state $\beta^i(t)$ is for each particle $i = 1, \ldots, n$ and time t defined as 1 for unscattered, 0 for scattered and removed. This means in particular, that particles are removed if they overlap at time $t = 0$. We compare the multi-body evolution with the single-body description $f : U \times \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$

$$\begin{aligned} \partial_t f + v \cdot \nabla_u f &= -Q_-[f, f], \\ f(u, v, 0) &= f_0(u, v), \end{aligned} \tag{3}$$

where

$$Q_-[f, g](u, v) = L[g](u, v)\, f(u, v) = \left(\int_{\mathbb{R}^d} g(u, v')\, \kappa_d\, |v - v'|\, \mathrm{d}v' \right) f(u, v) \tag{4}$$

is the loss term and κ_d is the volume of the $(d-1)$ dimensional unit ball. We assume finiteness of moments up to order two and initial data that are L^∞ in space.

Then the density of the unscattered particles converges in probability to a solution of the Boltzmann equation i.e. that for all $\epsilon > 0$ and all open $A \subset U \times \mathbb{R}^d$ uniformly for $t \in [0, T]$

(5)
$$\lim_{a \to 0} \mathrm{Prob}_a \left(\left| \frac{1}{n} \# \left\{ i \;\middle|\; (u_i(t), v_i(t)) \in A,\, \beta_i^{(a)}(t) = 1 \right\} - \int_A f_t(u, v) \mathrm{d}u\, \mathrm{d}v \right| > \epsilon \right) = 0,$$

where f_t is the unique mild solution of (3).

We characterize the many-particle flow by collision trees which encode possible collisions. The convergence of the many-particle dynamics to the Boltzmann dynamics is achieved via the convergence of associated probability measures on collision trees. Related to (3) we construct an time dependent idealised probabilty measure P_t ignoring any correlations inside trees. These probability measures satisfy nonlinear Kolmogorov equations, which are shown to be well-posed by semigroup methods. On the other hand there is a time dependent empiric probability measure $\hat{P}_t^a$ associated to (2). For a large class of good trees $\mathcal{G}(a)$, another Kolomogorov equation is derived, which allows Gronwall estimates to compare with the idealised probability P_t. Convergence is then achieved by showing that P_t and $\hat{P}_t^a$ are close in $\mathcal{G}(a)$ as well as $\lim_{a \to 0} P_t(\mathcal{G}(a)) = 1$.

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Asymptotic stability of breathers in some Hamiltonian networks of weakly coupled oscillators

Dario Bambusi

Consider the lattice with Hamiltonian

$$H := \sum_{k\in\mathbb{Z}} \left[\frac{p_k^2 + q_k^2}{2} + V(q_k) \right] + \frac{\epsilon}{2} \sum_{k\in\mathbb{Z}} (q_{k+1} - q_k)^2 \ , \tag{1}$$

where V is an analytic function having a zero of order higher then 2 at the origin. In 1994 MacKay and Aubry [MA94] proved that if ϵ is small enough, then there exist periodic solutions which are exponentially localized in space (breathers).

In the talk I presented the result of [Bam13] in which it is proved that breathers are asymptotically stable, at least if the first Melnikov condition holds and the nonlinear part of the on site potential fulfills $V(q) = O(|q|^8)$ as $q \to 0$. More precisely it is proved that if the initial datum is close in the energy norm to a breather, then the distance of the solution from the breathers, as a function of time, is small as an element of $L^q_t(\mathbb{R}, \ell^r)$. As usual (q, r) are admissible pairs. In particular it is obtained that the breather is stable in the energy norm.

The main idea, following MacKay and Aubry, is to consider first the system with $\epsilon = 0$, which consists of infinitely many decoupled anharmonic oscillators. Consider now the particular solution consisting of one particle (say the zero-th one) oscillating with large amplitude and all the others at rest. Due to the presence of a nonlinear potential, the frequency ω of such nonlinear oscillations depends on the amplitude and thus ω is typically nonoresonant with the frequency $\omega_0 = 1$ of the linear system. If the standard first Melnikov's condition holds, namely if $|\omega - \omega_0/n| \geq C > 0$, $\forall n \in \mathbb{Z}$, then Poincaré's theory applies and the solution can be continued to a localized oscillation of the complete system ($\epsilon \neq 0$), namely the Breather.

A formal computation [Bam98] shows that, at first order in ϵ, the expansion of the system at the breather is given again by the Hamiltonian (1) in which however $q_0 \equiv 0$. On the other hand it is known [KKK06] that, provided V has a zero of sufficiently high order at the origin, then all small amplitude solutions of (1) tend to zero as $t \to \infty$. So, one can expect the breather to be asymptotically stable.

In order to actually achieve the proof of asymptotic stability, I use normal form theory in order to construct the breather and to get the expansion of the Hamiltonian at the breather. Then I use such an expression to show that higher order correction the expansion of the Hamiltnian at the breather do not destroy asymptotic stability.

The first step goes as follows: consider first the system with $\epsilon = 0$ and introduce action angle coordinates (I, α) for the zero-th oscillator, thus one is reduced to a perturbation of a Hamiltonian of the form

$$\mathfrak{h}_0(I) + \sum_{k\neq 0} \frac{p_k^2 + q_k^2}{2} \ , \tag{2}$$

with $\mathfrak{h}_0$ a suitable function. If the perturbation does not contain terms linear in (p, q) then the manifold $p = q = 0$ is invariant. So the idea is to iteratively eliminate from the perturbation the terms linear in such variables. Furthermore, it also useful to eliminate the terms of order zero in p, q, which depend on the angle α conjugated to I. This is obtained by extending to the infinite dimensional case the methods introduced by Giorgilli in [Gio12] to prove the convergence of the normal form in the case of Lyapunov periodic orbits.

The dispersive step is more standard and consists of a variant of the theory of [KPS09], which in turn is based on the previous results [SK05], [KKK06], [PS08] and on ideas by [Miz08]. The only difference with respect to such works rests in the fact that in our case the dispersion is of order ϵ so one has to get the dependence on ϵ of all the constants.

I think it would be interesting to try to weaken the assumption on the order of the zero of the nonlinearity by exploiting the techniques developed in [MP10, MP12] or by studying higher dimensional lattices.

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Breathers in oscillator chains with Hertzian contact interactions

Guillaume James
(joint work with B. Bidégaray-Fesquet, E. Dumas, P.G. Kevrekidis, J. Cuevas, Y. Starosvetsky)

Understanding the dynamics of granular media is a fundamental issue for the modeling of natural phenomena such as mudslides and in technological applications like the design of shock absorbers or ballasted railway tracks. Granular media can display a highly nonlinear behavior, illustrated e.g. by the propagation of solitary waves along bead alignments [8]. One of the important factors that influence wave propagation is the nature of elastic interactions between grains. According to Hertz's theory, the repulsive force f between two identical initially tangent spherical beads compressed with a small relative displacement δ is $f(\delta) = k\,\delta^{\alpha}$ at leading order in δ, where k depends on the ball radius and material properties and $\alpha = 3/2$. The Hertz contact force has several properties that make the analytical study of wave propagation in chains of beads more difficult than in classical systems of interacting particles : the dependency of $f(\delta)$ on $\delta \approx 0$ is fully nonlinear for $\alpha > 1$, the stress-strain relation at the contact point has a limited smoothness ($f''(0)$ is not defined for $\alpha < 2$), and no force is present when beads are not in contact.

The simplest model in which these difficulties show up consists of a line of beads, in contact with their neighbors at a single point when the chain is at rest. For an infinite chain of identical beads, the dynamical equations read in dimensionless form

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}, \tag{1}$$

where $x_n(t) \in \mathbb{R}$ is the displacement of the nth bead from a reference position and the interaction potential V accounts for Hertzian contact forces. It takes the form

$$V(x) = \frac{1}{1+\alpha}\,|x|^{1+\alpha}\,H(-x), \tag{2}$$

where H denotes the Heaviside function vanishing on $\mathbb{R}^-$ and equal to unity on $\mathbb{R}^+$ and $\alpha > 1$ a fixed constant. System (1)-(2) supports solitary waves with unusual properties (such as a doubly-exponential spatial decay), whose existence was first predicted by Nesterenko [8] and rigorously established by Friesecke and Wattis [2] (see also [6] for more references and a study of solitary waves near the limit $\alpha \to 1^+$). On the other hand, system (1)-(2) and spatially inhomogeneous generalizations thereof do not support breather solutions (time-periodic and spatially localized) due to the purely repulsive character of the Hertzian potential [5].

A model reminiscent of (1) and supporting breather solutions is *Newton's cradle*, a classical mechanical system consisting of a chain of beads suspended from a bar by inelastic strings. All beads are identical and behave like linear pendula in the absence of contact with nearest neighbors, but mechanical constraints between touching beads introduce geometric nonlinearities. A simplified model for

Newton's cradle reads in dimensionless form

$$\frac{d^2x_n}{dt^2} + x_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}. \tag{3}$$

Time-periodic breather solutions of (2)-(3) have been computed in [5] (using the Gauss-Newton method and path-following) and display a super-exponential spatial localization (see [4] for a decay estimate). Dynamical simulations of [5] indicate that extremely small perturbations of the static breathers can lead to their translational motion, generating a so-called traveling breather. This phenomenon is linked with an extremely small difference between the energies of site- and bond-centered breathers having the same frequency (the so-called approximate Peierls-Nabarro barrier). Another manifestation of the high breather mobility is the systematic formation of a travelling breather after an impact at one end of the oscillator chain.

For small amplitude oscillations, it has been recently argued that such dynamical phenomena can be captured by the discrete p-Schrödinger (DpS) equation introduced in [3] and taking the form

$$2i\tau_0 \dot{A}_n = (A_{n+1} - A_n)\,|A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1})\,|A_n - A_{n-1}|^{\alpha-1}, \tag{4}$$

with some time constant τ_0 depending on α. The most standard model reminiscent of this family of equations is the so-called discrete nonlinear Schrödinger (DNLS) equation. However, the DpS equation is fundamentally different in that it contains a fully nonlinear inter-site coupling term, corresponding to a discrete p-Laplacian with $p = \alpha + 1$. Equation (4) leads to energy localization for a large class of initial data. Indeed, it was proved in [1] that the nontrivial solutions of (4) associated with localized (square-summable) initial conditions do not completely disperse, i.e. satisfy $\inf_{\tau\in\mathbb{R}} \|A(\tau)\|_\infty > 0$. Moreover, equation (4) admits time-periodic breather solutions with super-exponential spatial decay [7]. More quantitatively, static breather solutions to (2)-(3) were numerically computed in [5] and compared to approximate solutions of the form

$$x_n^{\mathrm{app}}(t) = 2\,\varepsilon\,\mathrm{Re}\,[\,A_n(\varepsilon^{\alpha-1}t)\,e^{it}\,], \tag{5}$$

where $\varepsilon \ll 1$ and A_n denotes a breather solution to the DpS equation [7], which depends on the slow time variable $\tau = \varepsilon^{\alpha-1}t$. For small amplitudes, the Ansatz (5) was found to approximate breather solutions to (2)-(3) with good accuracy [5]. Moreover, a small amplitude velocity perturbation at the boundary of a semi-infinite chain (2)-(3) generates a traveling breather whose profile is qualitatively close to (5), where A_n corresponds to a traveling breather solution of the DpS equation [9, 5].

The connection between the DpS equation and Newton's cradle was subsequently justified in [1]. Given a solution of (4), for initial conditions in (2)-(3) of the form $x_n(0) = 2\,\epsilon\,\mathrm{Re}\,[\,A_n(0)\,] + O(\epsilon^\alpha)$, $\dot{x}_n(0) = -2\,\epsilon\,\mathrm{Im}\,[\,A_n(0)\,] + O(\epsilon^\alpha)$ with $\epsilon \approx 0$, it was proved that the exact solution of (2)-(3) remains $O(\epsilon^\alpha)$-close to approximation (5) at least up to times $t = O(\epsilon^{1-\alpha})$. As an application, from the breather existence theorem proved in [7] for the DpS equation, the above error bounds imply the existence of stable small amplitude "long-lived" breather solutions to equation (3), which remain close to time-periodic and spatially localized oscillations over long times.

Possible extensions of these works concern the generalization and justification of the DpS equation when small spatial inhomogeneities are present in the original lattice (3), as well as the addition of dissipative terms in (3) and (4). Considering these effects is particularly important from a physical point of view when system (3) describes a granular chain [5]. Other open problems concern the qualitative analysis of the DpS equation. In particular, the existence of exact traveling breather solutions of (4) is an open problem, and would imply (in the case of small amplitude waves) the existence of similar excitations in Newton's cradle on long time scales. More generally, understanding in system (4) the complex mechanisms of fully nonlinear energy propagation from a localized disturbance is a challenging open problem [5]. This would allow in particular to analyze the propagation of nonlinear acoustic waves after an impact in granular chains with local potentials, thanks to the connection established between (4) and (3).

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Gaussian solitary waves in granular chains

Dmitry Pelinovsky
(joint work with Guillaume James)

We consider a class of fully-nonlinear Fermi-Pasta-Ulam (FPU) lattices,

$$\frac{du_n}{dt} = p_n - p_{n-1}, \quad \frac{dp_n}{dt} = V'(u_{n+1}) - V'(u_n), \quad n \in \mathbb{Z}. \tag{1}$$

consisting of a chain of particles coupled by fractional power nonlinearities

$$V(x) = \frac{1}{1+\alpha} |x|^{1+\alpha} H(-x) \tag{2}$$

with $\alpha > 1$ and H being the Heaviside step function. This class of systems incorporates a classical Hertzian model describing acoustic wave propagation in chains of touching beads in the absence of precompression.

We analyze the propagation of localized waves when α is close to unity. Solutions varying slowly in space and time are searched with an appropriate scaling, and two asymptotic models of the chain of particles are derived consistently. The first one is a logarithmic KdV equation,

$$\partial_\tau v + \partial_\xi^3 v + \partial_\xi (v \ln |v|) = 0, \tag{3}$$

where $\epsilon := \sqrt{\alpha - 1} \approx 0$ and the asymptotic correspondence of the solution is $u_n(t) = v(\xi, \tau) +$ *higher order terms*, with $\xi := 2\sqrt{3}\,\epsilon(n-t)$, $\tau := \sqrt{3}\,\epsilon^3 t$. We show that the logarithmic KdV equation possesses linearly orbitally stable Gaussian solitary wave solutions.

The second model consists of a generalized KdV equation with Hölder-continuous fractional power nonlinearity

$$\partial_\tau v + \partial_\xi^3 v + \frac{\alpha}{\alpha - 1} \partial_\xi (v - v\,|v|^{\frac{1}{\alpha} - 1}) = 0, \tag{4}$$

and this model is established in the same asymptotic limit as the logarithmic KdV equation. We show that the generalized KdV equation admits compacton solutions, i.e. solitary waves with compact support.

When $\alpha \to 1^+$, we numerically establish the asymptotically Gaussian shape of exact FPU solitary waves with near-sonic speed, and analytically check the pointwise convergence of compactons towards the limiting Gaussian profile. The existence theorems for existence and double-exponential decay of tails of the exact FPU solitary waves were proved in the series of papers [1, 2, 3], but the stability properties of these solitary waves remain an open problem up to the date.

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Nonlinear Waves in Granular Crystals

Panayotis G. Kevrekidis

In this presentation, we explore the dynamics of some prototypical nonlinear FPU chains of beads interacting through Hertzian contacts [1, 2]. The relevant dynamical equation reads (upon suitable rescalings):

$$\ddot{u}_n = [\delta_0 + u_{n-1} - u_n]_+^p - [\delta_0 + u_n - u_{n+1}]_+^p. \tag{1}$$

for the case of "monomers" (i.e., one type of beads), whereas e.g. in the case of dimers the equations become:

$$m_1 \ddot{u}_j = (\delta_0 + w_j - u_j)_+^p - (\delta_0 + u_j - w_{j-1})_+^p\,, \tag{2}$$

$$m_2 \ddot{w}_j = (\delta_0 + u_{j+1} - w_j)_+^p - (\delta_0 + w_j - u_j)_+^p\,, \tag{3}$$

The prototypical case of experimental interest is that of spherical contacts with $p = 3/2$ for spheres, although other values of the exponent p are also experimentally accessible. In these equations, u_j (and w_j) denote the displacement of the bead centers from their equilibrium positions, m_j denote the masses (of different materials), while δ_0 denotes the so-called precompression, i.e., a displacement induced on each bead by an applied stress at the boundaries of the chain.

One of the principal features of the model is its tunability from the weakly nonlinear case of strains satisfying $u_n - u_{n-1} \ll \delta_0$ (where a Taylor expansion around δ_0 can convert it to an FPU model with both α and β terms) to the highly nonlinear case of $\delta_0 = 0$. Notice also the critical additional element of the nonlinearity, namely its "tensionless" nature i.e., the parenthesis in Eqs. (1)-(3) is only evaluated when the argument is positive; physically this translates in that the force is acting only when the beads are in contact.

Using the so-called strain variables $r_n = u_{n-1} - u_n$, we can convert the monomer model to the following form:

$$\ddot{r}_n = [\delta_0 + r_{n+1}]_+^p - 2[\delta_0 + r_n]_+^p + [\delta_0 + r_{n-1}]_+^p, \tag{4}$$

For this model, in a recent work [3], we establish the following possibilities for nonlinear wave excitations that could arise within the 1d setting.

1. Fix $\delta_0 > 0$ and let $p \in \mathbb{R}^+$ and let $A \in C\left([0, T_0]\,, H^4\right)$ with

$$\sup_{T\in[0,T_0]} \sup_{X\in\mathbb{R}} |A(X,T)| \le \delta_0/2 \qquad \text{and} \qquad \sup_{T\in[0,T_0]} \|A(\cdot,T)\|_{H^4} \le C_1$$

be a solution of the partial differential equation (PDE)

$$\partial_T^2 A = \partial_X^2((\delta_0 + A)^p). \tag{5}$$

Then for $C_1 > 0$ sufficiently small there exists $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions $(r_n(t))_{n\in\mathbb{Z}}$ of (4) satisfying

$$\sup_{t\in[0,T_0/\varepsilon]} \sup_{n\in\mathbb{Z}} |r_n(t) - A(\varepsilon n, \varepsilon t)| < C\varepsilon^{3/2}.$$

2. Let $A \in C([0, T_0]\,, H^6)$ be a solution of the well-known Korteweg-de Vries (KdV) equation $\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X (A^2)$ with suitable chosen coefficients $\nu_1, \nu_2 \in$

$\mathbb{R}$. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $(r_n)_{n\in\mathbb{Z}}$ of (4) with

$$\sup_{t\in[0,T_0/\varepsilon^3]} \sup_{n\in\mathbb{N}} |r_n(t) - \psi_n(t)| \le C\varepsilon^{5/2},$$

where

$$\psi_n(t) = \varepsilon^2 A\left(\varepsilon(n - \omega_1'(0)t), \varepsilon^3 t\right)$$

with $\omega_1(k) = \omega(k)^2 b_1$.

3. Let $A \in C([0, T_0], H^{19})$ be a solution of the Nonlinear Schrödinger (NLS) equation $\partial_T A = i\nu_1 \partial_X A + i\nu_2 A |A|^2$ with suitable chosen coefficients $\nu_1, \nu_2 \in \mathbb{R}$. Then there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $(r_n)_{n\in\mathbb{Z}}$ of (4) with

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{n\in\mathbb{N}} |r_n(t) - \psi_n(t)| \le C\varepsilon^{3/2}$$

where

$$\psi_n(t) = \varepsilon A\left(\varepsilon(n - \omega_1'(k_0)t), \varepsilon^2 t\right) e^{i(k_0 n - \omega_0 t)} + \text{c.c.}$$

with $\omega_1(k) = \omega(k)^2 b_1$

In **2**. and **3**., $\omega^2(k) = 2(1 - \cos(k))$ and $b_1 = p\delta_0^{p-1}$. These results suggest the following possibilities for the 1d granular crystal. The first one indicates that in the case of finite precompression, if we consider long wavelength excitations (of spatial scale $1/\varepsilon$) over long times (of order $1/\varepsilon$), but of amplitude comparable to δ_0, then these excitations will follow the dynamics of a p-system of the form:

$$\left.\begin{aligned} \partial_T A - \partial_X v &= 0 \\ \partial_T v - \partial_X[(\delta_0 + A)^p] &= 0. \end{aligned}\right\} \tag{6}$$

Such a p-system has eigenvalues $\lambda_\pm(U) = \pm\sqrt{p(\delta_0 + A)^{p-1}}$ which suggests that larger amplitudes will propagate with larger speeds and hence that pulse-like initial data in the strains will lead to the formation of (oscillatory, due to the underlying Hamiltonian structure of the model) shock waves. This is a particularly intriguing observation that, to the best of our knowledge, has not been explored in much detail experimentally or theoretically so far in granular crystals and certainly merits further investigation.

The second one is perhaps the most well-established possibility in the realm of granular crystals and more generally in FPU chains [4] and concerns the propagation of small amplitude ($\propto \varepsilon^2$), long wavelength (again of size $1/\varepsilon$) traveling waves in the granular crystal. Traveling waves in this system have been studied rather extensively [1, 4, 5], not only in the case of the exponentially decaying, supersonic waves of the problem in the presence of precompression (as above), but also even in the case of the so-called "sonic vacuum" of $\delta_0 = 0$. In the latter case, the linear KdV type term of the previous limit is absent and different techniques need to be employed to identify the traveling waves and to understand their decay properties. In this $\delta_0 = 0$ case, the pioneering work of [6] used $r_n = \phi(x) \equiv \phi(n - ct)$ to obtain the advance-delay equation:

$$c^2\phi''(x) = (\delta_0 + \phi(x-1))^p + (\delta_0 + \phi(x+1))^p - 2(\delta_0 + \phi(x))^p \tag{7}$$

Fixing (without loss of generality) the speed $c = 1$, and using a Fourier transform, denoted by hat, one obtains $\hat{\phi}(k) = \left(\frac{4\sin^2(\frac{k}{2})}{k^2}\right)(\delta_0 \hat{+} \phi)^p(k)$, which upon use of the convolution theorem and Fourier transforming back to real space yields:

$$\phi(x) = \frac{(\Lambda \star (\delta_0 + \phi)^p)(x)}{\int \Lambda \star (\delta_0 + \phi)^p dx}. \tag{8}$$

where $\Lambda(x) = (1 - |x|)_+$ and $\star$ denotes the convolution. Notice that here, we have examined the problem in its full generality (i.e., with $\delta_0 \neq 0$), although the considerations of [6] were put forth only for $\delta_0 = 0$. Now provided that one can prove the existence of a bell-shaped traveling wave (a fact that was variationally proved in our recent work of [7]), the decay of the wave is given as follows. For $\delta_0 = 0$ and $x > 0$,

$$\phi(x+1) = \int_{-1}^{1} \Lambda(y)\phi^p(x+1-y)dy \leq \phi^p(x) \Rightarrow \phi(x+n) \leq \phi(x)^{p^n}, \tag{9}$$

i.e., the wave decays extremely fast, with a doubly exponential decay law (but is not genuinely compact as is often incorrectly suggested in the bibliography). On the other hand, in the presence of precompression, we obtain:

$$\phi(x+1) \sim \delta_0^{p-1}\phi(x) \Rightarrow \phi(x) \sim \delta_0^{p-1}\exp(n \log r(x_0)), \tag{10}$$

i.e., the wave decays exponentially. These features have also been confirmed numerically in the recent studies of [6, 7, 5].

Lastly, the third result suggests a more exotic possibility, namely that when seeking small amplitude ($\propto \varepsilon$), long-wavelength *breathing* excitations, one can derive an effective NLS equation, which in the monomer granular crystal possesses dark (i.e., tanh-like) breather solutions. These are exponentially localized (staggered i.e., $\propto (-1)^n$), time periodic states that have recently not only been theoretically proposed [8], but even been experimentally realized [9] as a result of destructive interference of two out-of-phase signals induced at the crystal boundaries. Clearly among these recently predicted and observed structures, there are numerous advances still lying ahead in this nonlinear wave subject of intense current interest.

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Multi-breather stability in Klein-Gordon chains: beyond nearest neighbor interactions, nonstandard phase shifts and bifurcations

Zoi Rapti

Spatially localized excitations in discrete nonlinear dynamical systems have received a lot of attention since the work of Sievers and Takeno [5]. For Klein-Gordon chains, **breathers** (excitations that are localized in space and time periodic) have been found numerically and shown to exist analytically [4].

Here, we study the linear stability of multi-breather solutions in an extended Klein-Gordon chain with a Morse onsite potential V, where oscillators are coupled linearly not only with their nearest neighbor, but also with oscillators that are up to 3 neighbors away. The equations of motion read

$$\ddot{y}_n + V'(y_n) + \epsilon \sum_{m=1}^{N} C_{nm} y_m = 0, \tag{1}$$

where y_n are functions of time, $\dot{}$ denotes the derivative with respect to time, the Morse onsite potential is $V(y_n) = \frac{1}{2}(e^{-y_n} - 1)^2$, ϵ is the small coupling parameter, and

$$C_{nm} = 0, \text{ if } |n-m| > 3, \; n, m = 1, ..., N$$

– with N being the number of oscillators in the chain – is the coupling constant between oscillators m and n. We assume that y_n are periodic functions with period T and frequency ω. The linear stability properties of these solutions can be obtained by studying the Newton operator $\mathcal{N}_\epsilon$ defined by

$$\mathcal{N}_\epsilon \xi \equiv \ddot{\xi} + V''(y) \cdot \xi + \epsilon C \xi = E\xi, \tag{2}$$

where ξ is a small perturbation of y and $\cdot$ is the list product, namely $f(y) \cdot \xi$ is the vector with elements $f(y_n)\xi_n$. The method developed by Aubry in 1997 [2] is used, which focuses on the study of the Newton operator of the system with the extra parameter E. This auxiliary parameter is used to reformulate the Krein theory for the Floquet operator in terms of the band structure of the Newton operator. It is known that periodic solutions exist for the uncoupled oscillators. As the coupling ϵ is turned on (but still small), one is interested to determine whether the periodic solutions that persist are stable. Linear stability corresponds to the existence of $2N$ bands that cross $E = 0$, where N is the length of the chain. As crossings are lost, the Floquet multipliers leave the unit circle, which brings on instability.

For multi-breathers, the Newton operator properties are analyzed in the framework of the perturbation theory developed in [1]. Namely, for the linear Newton operator of the unperturbed problem (zero coupling between the oscillators) there exists a basis of orthonormal eigenvectors corresponding to an eigenvalue E. For small coupling ϵ, the eigenvalues of the perturbed operator are given in terms of E, ϵ and the eigenvalues Λ of a perturbation matrix Q, which depends on the coupling between the oscillators and the phase shifts between the excited oscillators.

The elements of Q are

$$\begin{aligned} Q_{nm} &= 0, \text{ if oscillator } n \text{ or } m \text{ is at rest } (\sigma_n\sigma_m = 0); \\ Q_{nm} &= C_{nm}, \text{ if } \sigma_n\sigma_m = 1,\ n \neq m; \\ Q_{nm} &= -\gamma\, C_{nm}, \text{ if } \sigma_n\sigma_m = -1,\ n \neq m. \end{aligned}$$

The diagonal elements are

$$Q_{nn} = -\sum_{m\neq n} Q_{nm}.$$

The convention for the code describing the multibreather is the following: $\sigma = 0$ represents an oscillator at rest $y_n(t) \equiv 0$, $\sigma = 1$ represents an excited oscillator $y_n(t) = y^0(t)$, and $\sigma = -1$ represents an excited oscillator with a phase difference of π, with the previous ones $y_n(t) = y^0(t + \frac{T}{2})$.

The Morse potential is non-symmetric, so for excited oscillators that are out of phase, the perturbation matrix will depend on the symmetry coefficient γ [1] defined as follows

$$\gamma = -\frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} \dot{y}^0(t)\dot{y}^0(t+\frac{T}{2})dt}{\int_{-\frac{T}{2}}^{\frac{T}{2}} (\dot{y}^0(t))^2 dt}. \tag{3}$$

It was shown in [1] that $\gamma = \omega$ for the Morse potential and $\gamma = 1$ for a symmetric potential.

For multibreather configurations where oscillators can be at rest, excited and in-phase or excited and out of phase, it is shown that the numbers of negative and positive eigenvalues Λ in the perturbation matrix, are related to the number of oscillators in- and out of phase. The result follows by Sturm-type arguments in the case of nearest-neighbor interactions. For interactions with further neighbors, stability results can be obtained for the case when all oscillators are in- or out of phase. The perturbation matrix eigenvalues can be characterized using Gerschgorin's theorem [6].

Also, the critical coupling constants k for the next-nearest interactions were explored and their dependence on the multibreather frequency ω was found. The coupling parameter is defined as critical when a second zero eigenvalue occurs in the perturbation matrix. For the case of interactions with neighbors beyond the nearest one, it was shown in [3] that new non-standard phase-shifts ($\neq \pi$) exist. Using a generalization of the symmetry coefficient γ, a pitchfork bifurcation was found in the dependence of the shift on the critical coupling parameter k [6].

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Multibreathers in Klein-Gordon chains with interactions beyond the nearest neighbor

Vassilis Koukouloyannis
(joint work with Panayotis Kevrekidis and Zoi Rapti)

In this work we study the effect of long, but finite, range interactions in the existence and the linear stability of multibreathers.

The **classical** Klein-Gordon (KG) chain with **nearest-neighbor** (NN) interactions is described by the Hamiltonian

$$H = H_0 + \varepsilon H_1 = \sum_{i=-\infty}^{\infty} \left[\frac{1}{2}p_i^2 + V(x_i)\right] + \frac{\varepsilon}{2}\sum_{i=-\infty}^{\infty}(x_i - x_{i-1})^2 . \tag{1}$$

In the uncoupled limit $\varepsilon = 0$ we consider $n+1$ "central" oscillators moving in periodic orbits with the same frequency ω and arbitrary phases. In this limit the motion of these oscillators is described by $w_i = \omega_i t + w_{i0}$, $J_i = \text{const.}$, where w, J are the action angle variables of a single oscillator. In order for this trivially space-localized and periodic motion to be continued for $\varepsilon \neq 0$ to provide a multibreather, the phase differences $\phi_i = w_{i+1} - w_i$ between the successive oscillators must satisfy the persistence conditions, which for the KG chain with NN interactions are

$$M(\phi) \equiv \sum_{m=1}^{\infty} m A_m^2 \sin(m\phi_i) = 0, \quad i = 1 \ldots n. \tag{2}$$

In [1] it is shown that eqs (2) have only the $\phi_i = 0, \pi$ solutions which correspond to the *standard configurations*.
The linear stability of the multibreathers is defined by the corresponding *characteristic exponents* which in leading order of approximation are given by

$$\sigma_i = \pm\sqrt{-\varepsilon \frac{\partial \omega}{\partial J}\chi_z}, \tag{3}$$

where χ_z are the eigenvalues of

$$\mathbf{Z} = \frac{\partial^2 \lambda H_1\rangle}{\partial \phi_i \partial \phi_j} \cdot \mathbf{L}, \tag{4}$$

with $\frac{\partial^2 \lambda H_1\rangle}{\partial\phi_i \partial\phi_j} = \mathrm{diag}(f_1 \dots f_n)$, $f_i = \frac{1}{2}\sum_{m\geq 1} m^2 A_m^2 \cos(m\phi_i)$ and

$$\mathbf{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & \\ -1 & 2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -1 & 2 & -1 \\ & 0 & 0 & -1 & 2 \end{pmatrix}.$$

In order for a specific configuration to be stable all the corresponding exponents must be purely imaginary. So, the linear stability is determined by the sign of the product $P \equiv \varepsilon\frac{\partial\omega}{\partial J}$ and the sign of χ_z. Finally, [2] we can say that
In systems of the form (1), if $P < 0$ the only configuration which leads to linearly stable multibreathers, for $|\varepsilon|$ small enough, is the one with $\phi_i = \pi \quad \forall i = 1 \dots n$ (anti-phase multibreather). Moreover, for unstable configurations, their number of unstable eigenvalues will be precisely equal to the number of nearest neighbors which are in- phase between them. In the $P > 0$ case, the stability results are reversed.

This picture changes when **long-range interactions** are introduced. We consider interactions of oscillators within a **finite** range r of the same kind as in (1) but with various coupling constants ε_i. The Hamiltonian of the system in this case becomes

$$H = H_0 + \varepsilon H_1 = \sum_{i=-\infty}^{\infty}[\frac{p_i^2}{2} + V(x_i)] + \frac{\varepsilon}{2}\sum_{i=-\infty}^{\infty}\sum_{j=1}^{r} k_j\,(x_i - x_{i+j})^2 \tag{5}$$

where $k_j = \varepsilon_j/\varepsilon_1$. The corresponding persistence conditions are [3]

$$\sum_{p=1}^{r}\sum_{s=z_1}^{z_2} k_p M(\sum_{l=0}^{p-1}\phi_{s+l}) = 0, \tag{6}$$

where $z_1 = \max(1, i-p+1)$ and $z_2 = \begin{cases} i & \text{for} \quad i+p-1 \leqslant n \\ n-p+1 & \text{for} \quad i+p-1 > n \end{cases}$.
Again the characteristic exponents are given by (3) and $\mathbf{Z}$ is given by (4) but in this case it is

$$\frac{\partial^2 \lambda H_1\rangle}{\partial\phi_i \partial\phi_j} = \begin{cases} 0 & \text{if} \quad d > r \\ \sum_{p=d}^{r}\sum_{s=z_1}^{z_2} k_p f(\sum_{l=0}^{p-1}\phi_{s+l}) & \text{if} \quad d \leqslant r \end{cases}, \tag{7}$$

with $\quad z_1 = \max(1, i-p+d), \quad z_2 = \begin{cases} i & \text{for} \quad i+p-1 \leqslant n \\ n-p+1 & \text{for} \quad i+p-1 > n \end{cases}$.
Since the relations (6) and (7) are cumbersome to handle we use an example in order to present our case. Consider a 4**-site (**$n = 3$**) breather with range** $r = 3$.

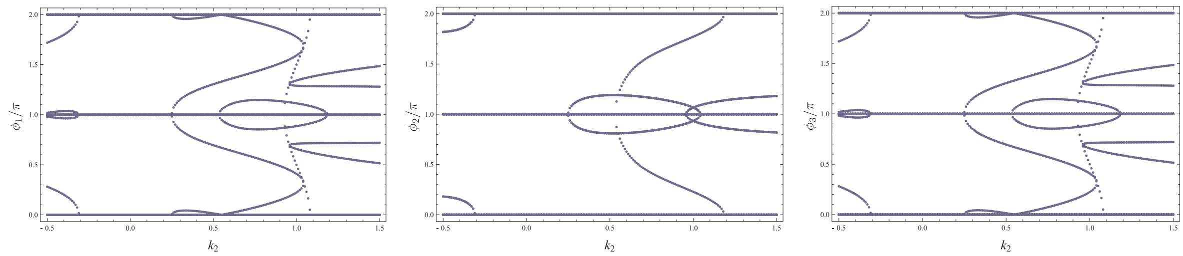
The persistence conditions (6) become

$$M(\phi_1) + k_1\, M(\phi_1 + \phi_2) + k_2\, M(\phi_1 + \phi_2 + \phi_3) = 0$$

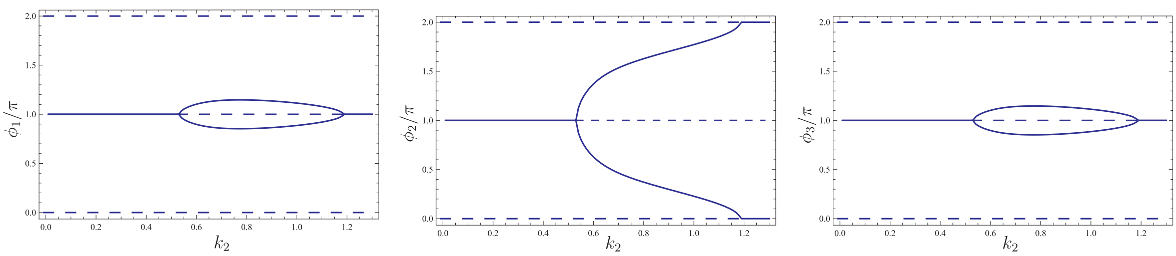
$$M(\phi_2) + k_1\, [M(\phi_1 + \phi_2) + M(\phi_2 + \phi_3)] + k_2\, M(\phi_1 + \phi_2 + \phi_3) = 0$$

$$M(\phi_3) + k_1\, M(\phi_2 + \phi_3) + k_2\, M(\phi_1 + \phi_2 + \phi_3) = 0$$

In this case, in addition to the standard $\phi_i = 0, \pi$ configurations, *phase-shift* configurations with $\phi_i \neq 0, \pi$ can be supported as well. All the permitted configurations of the phase differences ϕ_i's for $k_3 = 0.3$ are shown in the following figure

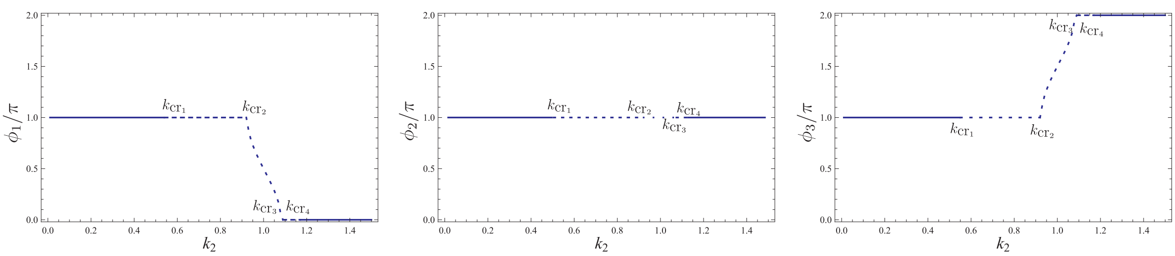


In the above, many families of configurations, for varying k_2, are shown which are not easily distinguishable. So, we present some characteristic ones. In the figures bellow a family with $\phi_1 = \phi_3$ is depicted.

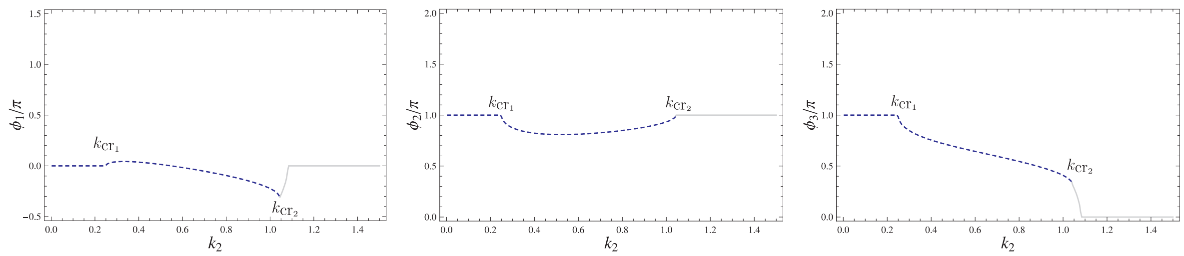


Here we see that the system undergoes a supercritical pitchfork bifurcation which destabilizes the, until then, stable configuration $\phi_i = \pi$. The family collides with the $\phi_1 = \phi_3 = \pi$, $\phi_2 = 0$ through a subcritical pitchfork bifurcation stabilizing the later.

Another pitchfork of the $\phi_i = \pi$ family occurs for larger values of k_2 which collides with the $\phi_1 = \phi_3 = 0, \phi_2 = \pi$ family through a subcritical pitchfork.



In the above case, the solution has the $\phi_3 = 2\pi - \phi_1$ symmetry. There are also families with no particular symmetry like the one bellow.



In this figure we can see a pitchfork bifurcation of the $\phi_1 = 0, \phi_2 = \phi_3 = \pi$ family until it collides with the family described earlier.

Each time a solution family undergoes a bifurcation, its stability changes. Consequently besides the numerous possible phase-shift configurations such a system can support, various new stability scenaria unfold which cannot be determined as easily as in the KG chain with NN interactions.

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Effects of $\mathcal{PT}$-symmetry in Nonlinear Klein-Gordon Models and Their Stationary Solitary Waves

Aslihan Demirkaya

(joint work with D. J. Frantzeskakis, P. G. Kevrekidis, A. Saxena, A. Stefanov)

In the present work, we introduce some basic principles of $\mathcal{PT}$-symmetric Klein-Gordon nonlinear field theories. The proposed model

$$u_{tt} - u_{xx} = f(u) + \gamma(x) u_t. \tag{1}$$

possesses gain and loss balancing through an odd function $\gamma(x)$, and this feature is studied for the sine-Gordon (sG) and the ϕ^4 models [1, 2], arguably two of the most famous field theoretic examples of nonlinear Klein-Gordon models. The gain/loss is introduced in a way that does not affect the steady states of the problem. Among such states, we select to study one of the prototypical kind of relevance to the field theory, namely the kink-like instanton structure, i.e., the heteroclinic orbit that connects two of the stable uniform states of the system. In particular, for the sine-Gordon field theory, $f(u) = -\sin(u)$, and the prototypical kink solution that we consider is of the form $u(x) = 4\arctan(e^x)$. On the other hand, for the ϕ^4 model, $f(u) = 2(u - u^3)$, and the relevant kink assumes the form $u(x) = \tanh(x)$. While the D'Alembertian (linear) operator $\partial_t^2 - \partial_x^2$ preserves the $\mathcal{PT}$ symmetry, this is not generically true in the case of the gain/loss term $\gamma(x)u_t$. In order this gain/loss term to preserve the $\mathcal{PT}$ symmetry, we use a localized gain/loss profile, namely $\gamma(x) = \epsilon x e^{-x^2/2}$, where ϵ characterizes the strength of the relevant perturbation.

As a vehicle for our studies, we use the discrete analogue of the PDE, namely the nonlinear dynamical lattice which constitutes its discretization:

$$\ddot{u}_n = \frac{1}{\Delta x^2}(u_{n+1} + u_{n-1} - 2u_n) + f(u_n) + \gamma_n \dot{u}_n. \tag{2}$$

This is of interest in its own right, not only due to the fundamental mathematical and physical features that arise therein (radiation, internal modes, Peierls-Nabarro barriers, etc. [5, 6]), but also because the discrete model arises in experimentally relevant mechanical and electrical systems [7, 8, 3, 4], including $\mathcal{PT}$-symmetric ones.

We argue that the stability of the steady states should *not* be affected for suitably, symmetrically placed kinks between the gain/loss region. In the discrete problem, similar symmetric placement of the kink leads to a vanishing effect both for the even site case of inter-site centered kinks and for the odd site case of site-centered kinks. However, asymmetric placement of the kink with respect to the gain/loss interface may generate a destabilization effect. A perturbative analytical expression is provided that allows to evaluate whether such a destabilization may occur. Furthermore, we provide a rigorous estimate that bounds the growth rate of any potential instability by an explicit numerical factor multiplying the order ϵ of the $\mathcal{PT}$-symmetric perturbation. In our numerical computations, it is typically found that the kink is stabilized when moving towards the lossy side, while it is destabilized when moving towards the gain side. In addition, interesting finite size effects arise that disappear in the infinite lattice size limit.

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An extensive adiabatic invariant for the Klein-Gordon model in the thermodynamic limit

Simone Paleari and Tiziano Penati
(joint work with Antonio M. Giorgilli)

In the quest for a mathematically rigorous foundation of Statistical Physics in general, and Statistical Mechanics in particular, despite many efforts and recent successes, a lot of work is still to be done. More specifically, if one considers an Hamiltonian system, instead of some ad hoc model, for the microscopic description of large systems, the behaviour over different long time scales is often still a challenge. One of the possible, and natural strategies, is to apply the techniques and results of Hamiltonian perturbation theory to large systems, with particular attention to the thermodynamic limit, i.e. when the number of degrees of freedom grows very large, at fixed, non vanishing, specific energy. The present report (based on paper [10]) is concerned with the existence of an adiabatic invariant for an arbitrarily large one dimensional Klein-Gordon chain, with estimates uniform in the size of the system.

It is well known that results like the KAM and the Nekhoroshev theorems stated for finite dimensional systems (see e.g. [11, 14, 1, 15, 16]) appear to be somewhat useless as the number N of degrees of freedom of the system system grows, for the estimated dependence on N of the constants involved is usually very bad, and in particular the (specific) energy thresholds do vanish in the limit $N \to \infty$. Extensions in the infinite dimensional case have been made (see, e.g., [5, 12, 3, 17] for the case of partial differential equations, or [2, 4] for the case of lattices), but always for finite energy, i.e., for zero specific energy. Our aim is precisely to remove such a limitation, producing a long time estimate for positive *specific* energy.

In this direction, it is worth to mention however that a first theoretical result at finite specific energy, hence with estimates uniform in N, and on an average time scale can be found in [6]: the model considered by the author is a system of coupled rotators. We consider instead the Hamiltonian

$$H(x,y) = \frac{1}{2}\sum_{j=1}^{N}\left[\left(y_j^2 + x_j^2\right) + a(x_j - x_{j-1})^2 + \frac{1}{2}x_j^4\right] = H_0 + H_1 \ , \tag{1}$$

of a Klein-Gordon chain with N degrees of freedom, periodic boundary conditions $x_0 = x_N$, and coupling constant a; the H_0 term represents the quadratic part and H_1 the quartic one.

A previous investigation of a similar model has been made in [9]. In that paper a first order (in the sense of perturbation theory) adiabatic invariant has been analytically constructed. Moreover, by numerical investigation it has been shown that the adiabatic invariance persists for times much longer than those predicted by the first order theoretical analysis. Thus, the model appeared to be worth of further theoretical investigation. A very recent breakthrough in this direction is represented by the paper [7], which provides higher order perturbation estimates complemented with probabilistic techniques, thus producing a control of the long time evolution in the thermodynamic limit.

The basic idea of both the quoted works [7, 9] is to avoid the usual procedure of introducing normal modes for the quadratic part of the Hamiltonian (1), thus considering the model as a set of identical harmonic oscillators with a coupling which includes a small quadratic term describing a nearest neighbours interaction controlled by the small parameter a.

In paper [10] we construct an extensive adiabatic invariant as follows. First, as in [9], we exploit a transformation of the quadratic part of the Hamiltonian into the sum of two terms in involution, $H_0 = H_\Omega + Z_0$, where Z_0 includes all the resonant coupling terms. The relevant fact is that the transformation preserves the extensive nature of the system and produces new coordinates which are each exponentially localized around the corresponding original ones. As a subsequent step, the perturbation process is performed at higher orders, starting from H_Ω; following the procedure in [8], we produce an adiabatic invariant Φ which still preserves both the extensive nature of the system and the exponential decay of the interaction with the distance. Furthermore we produce estimates which are uniform in the number N of degrees of freedom.

We stress that our model contains two independent perturbation parameters, namely: (i) the coupling parameter a, and (ii) the specific energy ϵ. This is a point that deserves particular consideration. We pay special attention in keeping these two parameters separated, so that we can deal with the physically sound hypothesis that the coupling parameter a is fixed, and the inverse temperature β can be taken arbitrarily large.

A second relevant point, as already stressed in [7, 9], is concerned with the question how to assess the adiabatic invariance of our quantity. The delicate point is again related to the thermodynamic limit, which was indeed a major obstacle in tackling the problem with perturbation methods, but can be dealt with using a statistical approach. In a simplified description (see, e.g., [13]) we can say that, as the number of degrees of freedom grows, all the extensive functions appear as essentially constant over the energy surface, in the sense that for increasing N their densities approach a delta function centered around their average value. Clearly an almost constant function is also approximately constant along an orbit, which seems not to give a meaningful information. The idea is thus to compare the dynamical fluctuation with the statistical deviation of the function over the phase space, using the Gibbs measure. A function defined on the phase space will be considered reasonably conserved if its fluctuation along the orbit is significantly smaller than its Gibbs variance, for a large set of initial data. We are able to show that, in the physically sound assumption of fixed coupling constant a, as the specific energy ϵ goes to zero, for a large (asymptotically full) Gibbs measure of initial data, and for time scaling as inverse powers of ϵ, the time variance of our quantity is smaller than the corresponding Gibbs variance, their ratio vanishing as a power of ϵ.

We come now to a formal presentation of the results in a somehow simplified form. Whenever it will be useful we shall denote by z all the coordinates and momenta (x, y), and by $H(z, a)$ the Hamiltonian so as to bring into evidence the dependence on the coupling constant a. We denote here by dz the $2N$ dimensional Lebesgue measure in the phase space $\mathcal{M} := \mathbb{R}^{2N}$, by dm the Gibbs measure and by Z the corresponding partition function, namely

$$dm(\beta, a) := \frac{e^{-\beta H(z,a)}}{Z(\beta, a)} dz \ , \qquad Z(\beta, a) := \int_{\mathcal{M}} e^{-\beta H(z,a)} dz \ ;$$

for every function $X : \mathcal{M} \to \mathbb{R}$ we denote its phase average and its variance respectively by

$$\langle X \rangle := \int_{\mathcal{M}} X dm(\beta, a) \ , \qquad \sigma^2[X] := \left\langle X^2 \right\rangle - \langle X \rangle^2 \ .$$

For every measurable set $A \in \mathcal{M}$, we will denote $m(A) := \int_A dm(\beta, a)$.

We recall that, for β large and a small, β is roughly the inverse of the average specific energy $\frac{1}{\beta} \sim \frac{\langle H \rangle}{N}$.

We also need to define the time average and the time variance, evaluated along the time evolution. Denoting by ϕ^t the Hamiltonian flow, these quantities are naturally defined as

$$\overline{X}(z, t) := \frac{1}{t} \int_0^t (X \circ \phi^s)(z) ds \ , \qquad \sigma_t^2[X] := \overline{X^2} - \overline{X}^2 \ .$$

We state here a particular version of the main result of the paper [10] giving, for fixed coupling constant, and small specific energies, a control for time scales growing as a power of β.

Theorem 1. *There exist positive constants a^*, β_0, β_1, C_1 and C_2 such that, for all $0 < a < a^*$, given the integer $r := \left\lfloor C_1 \sqrt{\frac{1+2a}{a}} \right\rfloor$, there exists an extensive polynomial $\Phi : \mathcal{M} \to \mathbb{R}$ of degree $2r + 2$, such that, for all $\beta > \max\{\beta_0, \beta_1 r^6\}$ one has*

$$m\Big(z \in \mathbb{R}^{2N} \ : \ \big|\Phi(\phi^t(z)) - \Phi(z)\big| \geq \delta\sigma[\Phi]\Big) \leq \frac{12 C_2}{\delta^2} \left(\frac{t}{\overline{\overline{t}}}\right)^2 ,$$

$$m\Big(z \in \mathbb{R}^{2N} \ : \ \Big|\overline{\Phi(\phi^t(z))} - \Phi(z)\Big| \geq \delta\sigma[\Phi]\Big) \leq \frac{3 C_2}{\delta^2} \left(\frac{t}{\overline{\overline{t}}}\right)^2 ,$$

$$m\left(z \in \mathbb{R}^{2N} \ : \ \sigma_t^2[\Phi] \geq \frac{\sigma^2[\Phi]}{\sqrt{\beta}}\right) \leq \frac{4 C_2}{\beta} \left(\frac{t}{\overline{\overline{t}}}\right)^2 , \qquad \overline{\overline{t}} = \beta^{r/2} \ .$$

Remark 1. *According to the result stated above, given a system with Hamiltonian (1) having a sufficiently small, and fixed, coupling constant a, there exists an almost conserved quantity in a* weak *sense: for any given time interval, its time average is close to its initial value and its time variance is smaller than its phase variance for a set of initial data of large Gibbs measure; this holds over long times scaling with $\epsilon^{-C/\sqrt{a}}$, for small enough average specific energy ϵ. Actually, given the* *relation between β, r and a, the minimal time scale (corresponding to the maximal specific energy allowed) is of order r^r, i.e. $(1/\sqrt{a})^{1/\sqrt{a}}$.*

We may state another result, where the time scale is a stretched exponential in β. The price to be paid, again in the hypothesis of a fixed coupling constant, is that the specific energy must be bounded both from below and from above; otherwise it is necessary to let the coupling vanish as the specific energy goes to zero, as in [7].

Theorem 2. *There exist positive constants a^*, β_*, C^*, C_1, C_2 and C_3 such that, for all $\beta \geq \beta_*$ and $0 < a < a^*$ satisfying*

$$\sqrt{a}\,\sqrt[3]{\beta} \leq C^* \ ,$$

given the integer $r := \left\lfloor C_1 \sqrt[3]{\beta} \right\rfloor$, there exist an extensive polynomial $\Phi : \mathcal{M} \to \mathbb{R}$ of degree $2r+2$, such that one has

$$m\left(z \in \mathbb{R}^{2N} \ : \ \sigma_t^2[\Phi] \geq \frac{\sigma^2[\Phi]}{\sqrt{\beta}} \right) \leq \frac{C_2}{\beta}\left(\frac{t}{\bar{\bar{t}}}\right)^2 \ , \qquad \bar{\bar{t}} = e^{C_3 \sqrt[3]{\beta}}$$

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Optical solitons in nematic liquid crystals

Panayotis Panayotaros
(joint work with Tim Marchant)

We study some basic properties of solitary waves of the Hartree-type nonlinear Schrödinger (NLS) equation

$$iu_t + \frac{1}{2}D\Delta u + 2AG(|u|^2)u = 0 \tag{1}$$

in $\mathbb{R}^2$, where

$$G(|u|^2)(x) = \frac{A}{\nu}\int_{\mathbb{R}^2} K_0(m|x-y|)|u(y)|^2 d^2y, \tag{2}$$

and K_0 is a modified Bessel function, or (up to constants) the Bessel potential in $\mathbb{R}^2$. D, A, ν, m are positive constants. The equation models the propagation of laser light through nematic liquid crystals, and was originally proposed by Conti, Peccianti, and Assanto [CPA03], who also observed experimentaly stable optical solitons in that medium. Several subsequent works compared approximate analytical and numerical solutions of this model to experiments, see [YZK05], [MSXK09], [CMMSW08], although the equation and some of its properties had been studied earlier in a different physical context, see [T86], and in the work [GV80] on NLS equations with general Hartree nonlinearities.

The radial kernel K_0 has a logarithmic singularity at the origin

$$K_0(r) = \frac{1}{2\pi}\left(-\log r + O(1)\right), \quad \text{as} \quad r \to 0, \tag{3}$$

and decays exponentially at infinity. The kernel K_0 is then integrable in $\mathbb{R}^2$, and turns out to have a regularizing effect. In particular, the initial value problem with $u(0) \in H^1$ has a unique solution $u \in C^0(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; H^{-1})$. Moreover, $||u(t)||_{H^1} \leq C_0$ for some $C_0 > 0$, for all $t \in \mathbb{R}$. Here and in what follows $L^p = L^p(\mathbb{R}^2; \mathbb{C})$, $H^s = H^s(\mathbb{R}^2; \mathbb{C})$. The long-time existence argument is based on energy conservation, and a Gagliardo-Nirenberg inequality, see [GV80], [T86]. From the physical point of view, these works show that a suitable nonlocal optical medium prevents the finite time blow-up behavior modeled by the cubic NLS in $\mathbb{R}^2$, see especially [W83] for H^1 initial conditions.

In the work summarized below we show the existence, regularity, and radial symmetry of energy minimizing soliton solutions of (1), (2) in $\mathbb{R}^2$. We also give analytically lower bounds for the L^2-norm of energy minimizing solitons of negative energy. These results are also compared with numerical computations of radial solitons. More details can be found in [PM13].

By soliton solutions we mean solutions of (1) of the form $u(x,t) = e^{-i\omega t}\psi(x)$. To find such ψ we minimize the Hamiltonian functional H, defined as

$$H = \int_{\mathbb{R}^2} \left(\frac{D}{2}|\nabla u|^2 - A|u|^2 G(|u|^2) \right), \tag{4}$$

over H^1 functions of constant L^2 norm.

Let $\lambda > 0$ and define

$$I_\lambda = \inf\{H(u) : \ u \in H^1, \ ||u||^2_{L^2} = \lambda\}. \tag{5}$$

A $u_* \in H^1$, $||u_*||^2_{L^2} = \lambda$, satisfying $H(u_*) = I_\lambda$ is referred to as a *minimizer* or ground state of H at norm(-squared) λ. The set of minimizers of H at λ is denoted as M_λ.

Theorem 1 There exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ we have: (i) The set M_λ is nonempty. (ii) Any $u(x,t) = e^{-i\omega t}\psi_*(x)$, $\psi_* \in M_\lambda$ is a C^2 solution of equation (1), (2), and ψ_* can be chosen to be real-valued, i.e. there exists $\phi \in \mathbb{R}$ (independent of x) such that $e^{i\phi}\psi_*(x)$ is real-valued. (iii) For any real $\psi_* \in M_\lambda$ we can choose $y \in \mathbb{R}^2$ so that $\psi_*(x-y)$ is positive, radial, and strictly decreasing.

The proof of (i) uses the concentration-compactness lemma, see [L84a], [L84b], where there are already results on related Hartree-type functionals with kernels satisfying the integrability properties of K_0. The C^2 regularity of (ii) follows from the differentiability of the functional H, and elliptic regularity arguments. Part (iii) uses rearrangement arguments, and relies on the fact that K_0 is radial, positive, and decreasing.

The condition $\lambda > \lambda_0$ is imposed so that we can show convergence of the minimizing sequence, following the concentration-compacteness lemma. This requires that we find $\psi \in H^1$, $||\psi||^2_{L^2} = \lambda$, satisfying $H(\psi) < 0$, and we see that for λ sufficiently large such states exist. λ_0 is not known, but we show that it cannot be arbitrarily small. We have at present two estimates.

Proposition 2 Let $v \in H^1$ satisfy $H(v) < 0$. Then $||v||_{L^2} > \lambda_* = \max\{\lambda_1, \lambda_2\}$, where

$$\lambda_1 = \frac{D\nu}{2A^2}\frac{m^2}{(\kappa_{2,4,\frac{1}{2}})^4}, \quad \lambda_2 = \frac{D\nu}{2A^2}\frac{m^2}{(c_{2,2})^{\frac{1}{2}}(\kappa_{2,4,\frac{1}{2}})^4||K_0||_{L^1}}, \tag{6}$$

where the constants $\kappa_{2,4,\frac{1}{2}}$, and $c_{2,2}$ appear in Gagliardo-Nirenberg, and Hardy-Littlewood inequalities respectively.

Proposition 2 is obtained by examining the ratio between the product of the L^2 norms of v, ∇v, and the quartic part V of H of test functions $v \in H^1$. This ratio appears in [W83], and [W99]. The bounds λ_1, λ_2 follow from alternative upper estimates of V that lead to a cancellation of the test functions from this ratio. The above imply non-existence of negative energy solitons of small L^2-norm. A more direct proof of the existence of an L^2-norm threshold for solitons is given by the following decay result.

Theorem 3 Consider the solution u of the initial value problem (1), (2) with initial condition $u(0) \in H^1$. There exists a λ_3 such that $||u(0)||_{L^2} \leq \lambda_3$ implies that $||u(t)||_{L^4} \to 0$ as $t \to +\infty$.

This proof uses the Strichartz estimates for the free Schrödinger evolution, and is similar to the decay proof of [CW89] for the cubic power NLS on the plane. A similar combination of absence of blow-up and decay for small solutions was also seen in the discrete cubic NLS in the two dimensional integer lattice, see [SK05].

Our work includes computations of numerical soliton profiles. To obtain these we look for solutions of (1) of the form $u(x,t) = e^{-i\omega t}\psi(r)$, $r = |x|$. This radial Ansatz leads to a pair of coupled second order ODEs for $\psi(r)$; note that we compute $\theta = G(|u|^2)$ numerically by solving an elliptic equation. The equations are discretized and solved in a finite interval $r \in [0, R]$ with Dirichlet boundary conditions at R, using the approach of [LMS13]. The numerical study exhibits profiles with the properties expected by Theorem 1 and gives indirect evidence for the existence of a power threshold for negative energy solitons in Proposition 2. The finite domain evolution is not expected to have the decay of Theorem 3, in fact we prove that the analogue of (1), (2) should have solutions with arbitrarily small L^2-norm, see [PM13].

We would like to have a better understanding of the L^2-norm threshold for the existence of solitons, e.g. determine which of the estimates above is sharper. It was suggested during the workshop that the threshold could be independent of some of the parameters, and can be related to the cubic NLS threshold. This remark can be first checked numerically.

There is still incomplete information on possible bifurcations in the disc problem, see [FKM97], [W99], [SK05] for related questions in the discrete NLS with power nonlinearity. The present problem also motivates a study of the discretized radial system with nonlocal nonlinearities.

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Homoclinic snaking in lattices

Jonathan Dawes

The study of localised states in nonlinear dissipative systems in recent years has been motivated by a wealth of experimental and numerical results. Mathematical approaches to these fully nonlinear problems, in which localised states arise near pattern forming (sometimes also called 'Turing') instabilities have been developed both in continuum systems [2, 4] and in spatially discrete models [1, 3]. The localised states have been referred to by a number of different names, including 'light bullets' and 'dissipative solitons' in different contexts. In the continuum case, the classic model equation is the Swift–Hohenberg PDE. However, discrete lattice models, rather than continuum models, are appropriate in many physical and biological applications [8, 7]. The discrete nature of the problem provides the pinning effect that stabilises localised states of 'odd' and 'even' parities. For a odd on-site nonlinearity these two families are also described as 'site-centred' and 'bond-centred' states.

One model lattice dynamical system is the discrete bistable nonlinear Schrödinger equation

$$\mathrm{i}\dot{\psi}_n + C\Delta\psi_n + s\psi_n|\psi_n|^2 - \psi_n|\psi_n|^4 = 0, \tag{1}$$

where Δ denotes a discrete form of the Laplacian, for an array of complex fields $\psi_n(t)$ on a lattice whose sites are indexed by the (multiple) suffix $n \in \mathbb{Z}^k$. Substituting the ansatz $\psi_n = u_n \exp(-\mathrm{i}\mu t)$ where u_n is a real stationary field, we see that u_n must satisfy the real difference equation

$$\mu u_n + C\Delta u_n + s u_n^3 - u_n^5 = 0.$$

Hence the existence of single-frequency time-periodic solutions of (1) corresponds, in one dimension, to the existence of equilibria for the discrete Allen–Cahn equation

$$\dot{u}_n = C(u_{n+1} + u_{n-1} - 2u_n) + \mu u_n + 2u_n^3 - u_n^5, \tag{2}$$

where u_n, $n \in \mathbb{Z}$, is a real scalar variable defined at each lattice site, C is the strength of linear coupling between adjacent sites, and $-1 < \mu < 0$ is the primary (real) bifurcation parameter. Equilibrium states of (2) display homoclinic snaking: families of localised equilibria lie on curves that 'snake' backwards and forwards in parameter space. The resulting bifurcation diagram is qualitatively identical to that derived in the PDE case for the Swift–Hohenberg equation.

Current work considers extensions of homoclinic snaking behaviour to a square lattice in two spatial dimensions, i.e. the coupled ODEs

$$\dot{u}_{n\,m} = C^{+}\Delta^{+}u_{n\,m} + C^{\times}\Delta^{\times}u_{n\,m} + \mu u_{n\,m} + 2u_{n\,m}^{3} - u_{n\,m}^{5},$$

where the scalar variables $u_{nm}(t)$ are arranged on a square lattice $(n,m) \in \mathbb{Z}^2$ and there are two natural nearest-neighbour difference operators:

$$\Delta^{+}u_{n\,m} := u_{n+1\,m} + u_{n-1\,m} + u_{n\,m+1} + u_{n\,m-1} - 4u_{n\,m},$$
$$\Delta^{\times}u_{n\,m} := u_{n+1\,m+1} + u_{n-1\,m-1} + u_{n-1\,m+1} + u_{n+1\,m-1} - 4u_{n\,m}.$$

The two coupling parameters C^{+} and $C^{\times}$ are the coefficients of nearest-neighbor (NN) and next-nearest-neighbor (NNN) coupling. The detailed description of the existence and bifurcations that arise as C^{+} and $C^{\times}$ are varied remains largely open: a few remarks are contained in [6]. It is clear that the snaking curve collides, and then separates from, many separate isolas that also exist. These events occur over small intervals in C^{+} and $C^{\times}$ and lead to 'switchbacks' at which the solution norm $\left(\sum_{n,m} u_{n\,m}^2\right)^{1/2}$ decreases, and the localised state shrinks, before continuing to grow outwards again and having increasing norm. At small coupling strengths the behaviour should be amenable to analysis through the dynamics of the cells on the boundary of the localised state: since these cells form a quasi-one-dimensional set, there is further hope that the snaking dynamics in 2D can be related to other one-dimensional problems, for example the dynamics of one-dimensional fronts aligned a various angles to the lattice vectors [5], or fronts containing isolated defects.

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On the validity of the NLS approximation for the FPU system

GUIDO SCHNEIDER

The Fermi-Pasta-Ulam (FPU) system, cf. [4],

$$\partial_t^2 q_j = W'(q_{j+1}(t) - q_j(t)) - W'(q_j(t) - q_{j-1}(t)) \ , \quad j \in \mathbb{Z} \ , \tag{1}$$

describes an infinite chain of masses coupled by nonlinear springs which are described by the potential function $W : \mathbb{R} \to \mathbb{R}$ where for notational simplicity $W'(u) = u + u^2$ in the following. Our goal is to prove rigorously that the evolution of a slowly varying envelope of small amplitude of an underlying oscillating wave packet in the FPU system can be described approximately by the NLS equation. In order to do so we rewrite (1) in terms of the difference variables $u(j,t) = q_{j+1}(t) - q_j(t)$, so that (1) becomes

$$\partial_t^2 u(j,t) = W'(u(j+1,t)) - 2W'(u(j,t)) + W'(u(j-1,t)) \ , \quad j \in \mathbb{Z} \ . \tag{2}$$

In order to derive the NLS equation we make the ansatz

$$u(j,t) = \varepsilon A(\varepsilon(j + c_g t), \varepsilon^2 t) e^{i(k_0 j + \omega_0 t)} + \text{c.c.} + \mathcal{O}(\varepsilon^2), \tag{3}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, where the wave numbers $k = k_0$ and $\omega = \omega_0$ of the underlying oscillating carrier wave satisfy the linear dispersion relation

$$\omega^2 = -a_1(e^{ik} - 2 + e^{-ik}) = 2a_1(1 - \cos k). \tag{4}$$

and where the negative linear group velocity c_g is given by $c_g = \omega'(k_0)$. We find that the complex-valued amplitude function A satisfies in lowest order the Nonlinear Schrödinger equation

$$\partial_T A(X,T) = -i\omega''(k_0)\partial_X^2 A(X,T)/2 + i\gamma A(X,T)|A(X,T)|^2, \tag{5}$$

with a real-valued coefficient γ. For smooth $W : \mathbb{R} \to \mathbb{R}$ our approximation result is as follows.

Theorem. *Fix $s_A \geq 19$ and let $A \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ be a solution of the NLS equation (5). Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of the FPU system (2) satisfying*

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{j \in \mathbb{Z}} |u(j,t) - (\varepsilon A(\varepsilon(j + c_g t), \varepsilon^2 t) e^{i(k_0 j + \omega_0 t)} + \text{c.c.})| \leq C\varepsilon^2.$$

The proof is non-trivial in case of non-vanishing quadratic terms in W'. These quadratic terms can be eliminated by a near identity change of variables if the non-resonance condition

$$\inf_{j_1 . j_2, j_3 \in \{1,2\}} \inf_{k \in \mathbb{R}} |\omega_{j_1}(k) - \omega_{j_2}(k - k_0) - \omega_{j_3}(k_0)| \geq C > 0 \tag{6}$$

is satisfied. Herein, $\omega_{j_1}, \ldots, \omega_{j_3}$ stand for one of the two solutions which are defined by the dispersion relation (4) of the FPU problem. It is easy to check that the non resonance condition is not satisfied for the dispersion relation (4) of the FPU system due to $\omega_{1,2}(0) = 0$. There are exactly two zeroes of

$$\inf_{j_1.j_2,j_3\in\{1,2\}} |\omega_{j_1}(k) - \omega_{j_2}(k-k_0) - \omega_{j_3}(k_0)|,$$

in $(-\pi,\pi]$, namely $k = 0$ and $k = k_0$. The answer to overcome the problem with the resonances in $k = 0$ and $k = k_0$ can be found in the paper [3] about a similar problem in the continuous case. The easy connection is Fourier transform. The FPU system (2) can be transfered into Fourier space by

$$\hat{u}(k,t) = \mathcal{F}(u)(k,t) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} u(j,t)e^{-ikj}. \tag{7}$$

The FPU system in Fourier space is given by

$$\partial_t^2 \hat{u}(k,t) = -\omega^2(k)\hat{u}(k,t) - \omega^2(k)\int_{-\pi}^{\pi} \hat{u}(k-m,t)\hat{u}(m,t)dm \tag{8}$$

where in the convolution integral the 2π-periodicity of $\hat{u}$ has to be used. This representation formula allows us to transfer the ideas of [3], where the NLS equation has been justified for a Boussinesq equation. The major difference to the continuous case is that for lattice equations, like the FPU system, the Fourier transformed system is 2π-periodic, whereas in the continuous case the Fourier transformed system lives on the complete real axis. This idea in principle also allows us to transfer other approximation results almost line for line from the continuous case to the lattice situation. Another example is the generalization of the validity of the KdV approximation for the FPU lattice [5] to poly-atomic FPU lattices in [1] which is based on [2].

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On the validity of the KdV equation for a spacial periodic Boussinesq model with non small constrast

Roman Bauer
(joint work with Guido Schneider)

There exists a zoo of amplitude equations which can be derived via multiple scaling analysis in the long wave limit for various dispersive wave systems possessing conserved quantities. Among these amplitude equations there are only two which are independent of the small perturbation parameter, namely the KdV equation and the Whitham system. In this short talk we discuss the validity of the KdV approximation for a spatially periodic Boussinesq model with non small contrast.

The KdV equation occurs as an approximation equation in the description of small temporally and spatially modulations of long waves in various dispersive wave systems. Examples are the water wave problem or the equations from plasma physics, cf. [CeSa98]. For the Boussinesq equation

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \partial_x^4 u(x,t) + \partial_x^2 (u(x,t)^2)$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$ with the ansatz

$$u(x,t) = \varepsilon^2 A(X,T),$$

where $X = \varepsilon(x-t)$, $T = \varepsilon^3 t$, $A(X,T) \in \mathbb{R}$, and $0 < \varepsilon \ll 1$ a small perturbation parameter, the KdV equation

$$\partial_T A = -\frac{1}{2}\partial_X^3 A + \frac{1}{2}\partial_X(A^2)$$

can be derived by equating the coefficients in front of ε^6 to zero. There exist various justification results. Estimates that the formal KdV approximation and true solutions of the various formulations of the water wave problem stay close together over the natural KdV time scale have been shown for instance in [Cra85, SW00, SW02, BCL05, Du12]. Such results are a non trivial task since solutions of order $\mathcal{O}(\varepsilon^2)$ have shown to be existent on an $\mathcal{O}(1/\varepsilon^3)$ time scale.

The system we consider is

$$\partial_t^2 u(t,x) = \partial_x(a\partial_x)u(t,x) - \partial_x^2(b\partial_x^2)u + \partial_x(a\partial_x)u^2(t,x) \tag{1}$$

where $u(t,x) \in \mathbb{R}$ and a and b are strictly positive periodic coefficient functions. For this equation we derive the KdV equation and prove an approximation result which is formulated in the following

Theorem: Fix $s > 1/2$. Then there exist constants μ_1 and μ_2 such that for a solution $A \in C([0,T_0], H^{s+4})$ to the KdV equation

$$\partial_T A = -\mu_1 \partial_X^3 A + \mu_2 \partial_X(A^2) \tag{2}$$

there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$ we have solutions $u \in C^0([0,T_0/\epsilon^3], H^s(\mathbb{R}))$ of 1 with

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|u(\cdot,t) - \varepsilon^2 A(\varepsilon(\cdot - t), \varepsilon^3 t)\|_{C^0} \le C\varepsilon^{7/2}.$$

It guarantees that the KdV equation makes correct predictions about the dynamics of the spatially periodic Boussinesq model 1 over the natural KdV time scale.

The proof is based on the Floquet-/Bloch transformation

$$\mathcal{B}u(x,\ell) := \sum_{k\in\mathbb{Z}} e^{ikx}\mathcal{F}u(k+\ell), \quad \mathcal{B}^{-1}\tilde{u}(x) = \int_{-1/2}^{1/2} e^{i\ell x}\tilde{u}(x,\ell)d\ell$$

under wich (1) transforms to the problem

$$\partial_t^2\tilde{u}(t,x,\ell) = \partial(a\partial)\tilde{u}(t,x,\ell) - \partial^2(b\partial^2)\tilde{u}(t,x,\ell) + \partial(a\partial)(\tilde{u}\star\tilde{u})(t,x,\ell) \tag{3}$$

where $\tilde{u}(t,\cdot,\ell)$ is a periodic function for fixed $\ell \in (-1/2, 1/2)$ and where we define $\partial := \partial_x + i\ell$. The convolution term is given by

$$(\tilde{u}\star\tilde{u})(x,\ell) = \int_{-1/2}^{1/2} \tilde{u}(x,\ell-m)\tilde{u}(x,m)\,d\ell.$$

The linearity is a positive semi definite elliptic operator on $L^2(S^1)$ and provides a series of non negative eigentvalues $\{-\omega_j(\ell)^2\}_{j\in\mathbb{N}}$ where the corresponding eigenfunctions $\tilde{f}_j(\ell)$ form a complete orthonormal system so we can use the decomposition $\tilde{u}(t,x,\ell) = \sum_{j\in\mathbb{N}} \tilde{u}_j(t,\ell)\tilde{f}_j(\ell)(x)$. Be ω_1 the curve possesing a zero for $\ell = 0$. Then for the corresponding coefficient $\tilde{u}_1(t,\ell)$ using the ansatz

$$\epsilon^2\tilde{\psi}_1(t,\ell) = \epsilon\hat{A}(\epsilon^{-1}\ell, \epsilon^3 t)e^{i\omega_1'(0)\ell t}$$

for A the KdV equation (2) can be derived. By adding additional terms to the approximation one can achieve that the so called residuum

$$Res(\epsilon^2\psi) := -\partial_t^2\epsilon^2\psi + \partial_x(a\partial_x)\epsilon^2\psi - \partial_x^2(b\partial_x^2)\epsilon^2\psi + \partial_x(c\partial_x)(\epsilon^2\psi)^2$$

suffices $||\partial_x^{-1}Res(\epsilon^2\psi)||_{L^2} \leq C\epsilon^{13/2}$ and $||Res(\epsilon^2\psi)||_{L^2} \leq C\epsilon^{15/2}$. For the error $\epsilon^{7/2}R := u - \epsilon^2\psi$ we find the equation

$$\partial_t^2 R = \partial_x B^2\partial_x R + \epsilon^{7/2}\partial_x(c\partial_x)R^2 + \epsilon^{-7/2}Res(\epsilon^2\psi)$$

where $B^2 := a + 2c\epsilon^2\psi - \partial_x(b\partial_x)$ for fixed t is a positive definite selfadjoint operator which corresponding quadratic form is equivalent to the squared H^1-norm. Using B we can define the energy

$$E(t) = \frac{1}{2}(||B^{-1}\partial_x^{-1}\partial_t R||_{L^2}^2|| + ||\partial_t R||_{L^2}^2 + ||R||_{L^2}^2 + ||B\partial_x R||_{L^2}^2) \tag{4}$$

which is an upper bound for the squared H^2-norm of the error term R. It is easy to show that $\partial_t E(t) \leq \epsilon^3 F(E(t))$ where F is a continouos mapping such that Gronwall's inequality can be applied. Hence E is bounded on $[0, T_0/\epsilon^3]$ and our approximation result follows.

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Snaking of patterns over patterns in the 2D Schnakenberg system

Hannes Uecker

(joint work with Daniel Wetzel)

Homoclinic snaking refers to the back and forth oscillation in parameter space of a branch of stationary localized patterns for some pattern forming partial differential equation, see, e.g., [1] and the references therein. It also occurs for lattice differential equations, see, e.g., [2].

The talk summarizes recent results [3] on localized stationary patterns in some 2D Reaction–Diffusion model systems, namely the 2D Schnakenberg model

$$\partial_t U = D\Delta U + N(U,\lambda), \qquad N(U,\lambda) = \begin{pmatrix} -u+u^2v \\ \lambda - u^2 v\end{pmatrix}, \tag{1}$$

with $U=(u,v)(t,x,y)\in\mathbb{R}^2$, $(x,y)\in\Omega\subset\mathbb{R}^2$, Neumann boundary conditions in case of boundaries, diffusion matrix $D=\left(\begin{smallmatrix}1&0\\0&d\end{smallmatrix}\right)$, d fixed to $d=60$, and bifurcation parameter $\lambda\in\mathbb{R}_+$. This system has the homogeneous branch $w=(\lambda,1/\lambda)$ of stationary solutions, which becomes Turing unstable as λ is decreased below $\lambda_c = \sqrt{d}\sqrt{3-\sqrt{8}}$, with critical wave vectors $\mathbf{k}$ with $|\mathbf{k}| = k_c = \sqrt{\sqrt{2}-1}$. In 2D, the most prominent Turing patterns near bifurcation are stripes ($A{\neq}0, B{=}0$) and hexagons ($A{=}B{\neq}0$), which modulo rotational invariance can be expanded as

$$U = w + 2\left(A\cos(k_c x) + 2B\cos\left(\frac{k_c}{2}x\right)\cos\left(\frac{k_c}{2}\sqrt{3}y\right)\right)\Phi + \text{h.o.t.},$$

with $\Phi\in\mathbb{R}^2$ a constant vector, and where h.o.t. refers to higher order terms in A, B and $\lambda-\lambda_c$.

We use the bifurcation and continuation software `pde2path` [4] to numerically calculate stationary patterns for (1), and their stability. First we use domains of type $\Omega = (-l_x,l_x)\times(-l_y,l_y)$, $l_x = 2l_1\pi/k_c$, $l_y = 2l_2\pi/(\sqrt{3}k_c)$, $l_1,l_2\in\mathbb{N}$, which are chosen to accomodate the basic stripe and hexagon patterns. See Fig. 1(a) for the "basic" bifurcation diagram with $l_1=l_2=2$, which we call a 2×2 domain as in both directions 2 hexagons "fit".

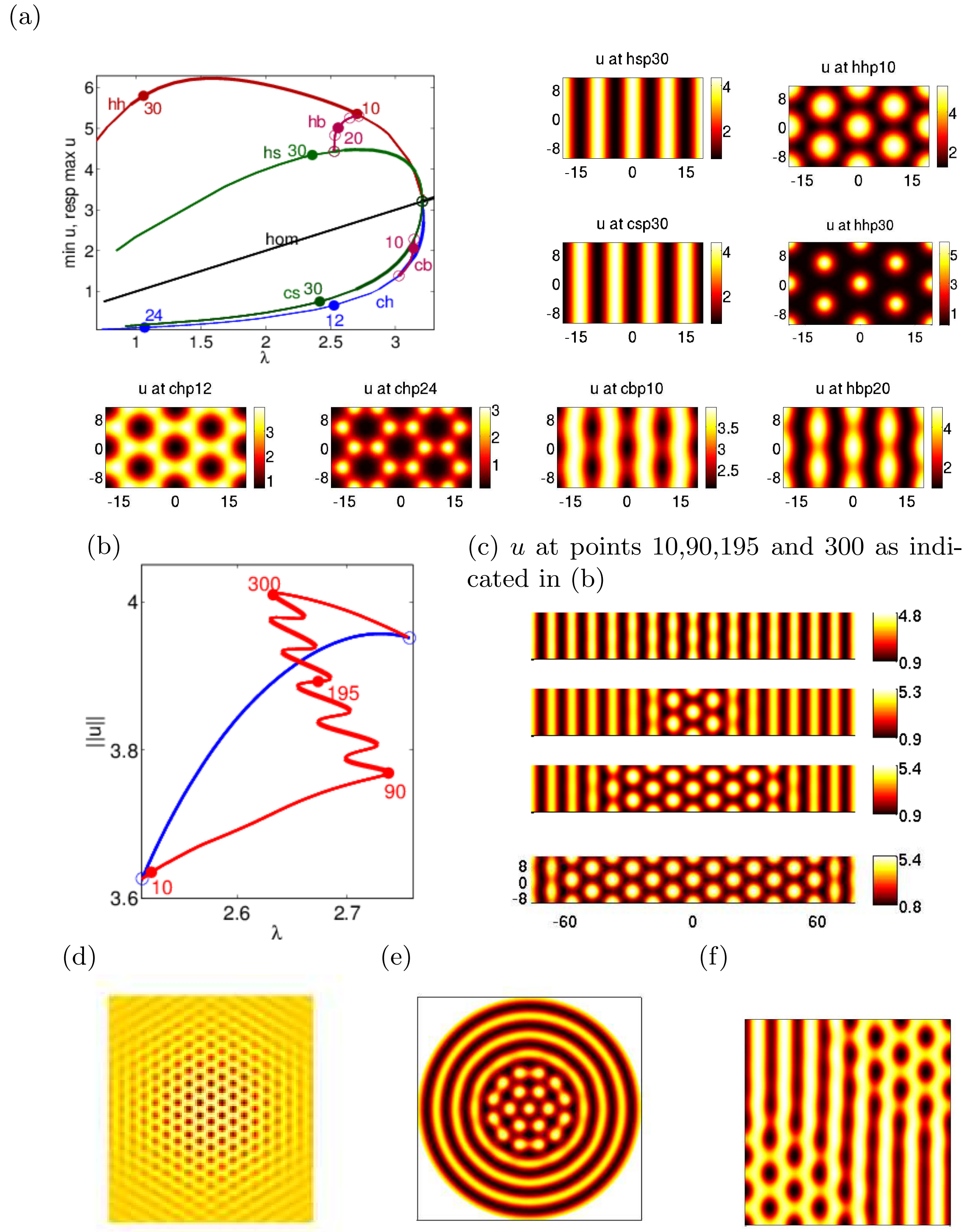

FIGURE 1. (a) "Basic" Bifurcation diagram and example solutions of (1) obtained with `pde2path` over the "2×2"–domain. (b), (c) Bifurcation of a snaking branch of hexagons over stripes from the `hb` mixed mode branch (blue) over the 8×2 domain, $\|u\| = \|u\|_{L^8} := \left(\frac{1}{|\Omega|}\int_\Omega |u(x,y)|^8 \mathrm{d}(x,y)\right)^{1/8}$. (d)-(f) u for fully localized spot patterns over homogeneous background, and over radial and straight stripes. See [3] for details.

It turns out that there are a number of bistable ranges between different patterns, for instance a bistable range between "hot stripes" `hs` and "hot hexagons" `hh` for λ between $\lambda_s^e \approx 2.51$ and $\lambda_{hh}^b \approx 2.73$, and there are mixed mode branches (e.g., `hb`) connecting patterns (`hs`) at start ($\lambda = \lambda_s^e$) and end (`hh`, $\lambda = \lambda_{hh}^b$) of the respective bistable range. Typically we can then expect localizes patterns. For (1) these can for instance be found via bifurcation off the mixed mode branch `hb`, leading to a snaking branch of hexagons over stripes, see Fig. 1(b),(c) for an example. These localized patterns with a planar interface can be approximately described by a Ginzburg–Landau reduction. Additionally, we describe some snaking branches of fully localized patterns patterns, see Fig. 1(d)–(f) for some examples.

Our results are mostly numerical, and are rather first steps towards the understanding of large classes of new patterns. At the end we list a number of Open Problems for further numerical and analytical investigation.

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Nonequilibrium chaos of nonlinear disordered waves

Sergej Flach

(joint work with Ch. Skokos, I. Gkolias)

In one and two dimensions, wave packets of noninteracting particles subject to a random potential on a lattice do not propagate due to exponential Anderson localization of the corresponding eigenstates [1, 2]. Anderson localization is a highly intriguing wave phenomenon, and has been recently probed in experiments on ultracold atomic gases in optical potentials [3, 4]. The wave localization effect is completely relying on keeping the phase coherence of participating waves. The presence of interaction between the particles may change this picture qualitatively. For many weakly interacting particles this is often taken into account by adding nonlinear terms to the linear wave equations of the noninteracting particles [5].

Numerical simulations of wave packets propagating in random lattice potentials showed the destruction of localization and a subdiffusive growth of the second moment of the wave packet in time as t^γ [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In particular, it was predicted that, for asymptotically large t, the coefficient γ should converge to $1/(1+\sigma d)$ in the so-called 'weak chaos' regime, where d is the dimension of the lattice, and $\sigma+1$ is the exponent of the nonlinear term in the wave equation (note that usual two-body interactions yield cubic nonlinear terms with $\sigma = 2$)

[9, 10, 17]. A transient 'strong chaos' regime was also predicted [17] and observed [13, 14] where $\gamma = 2/(2+\sigma d)$. Remarkably the onset of subdiffusive spreading was also experimentally detected for interacting ultracold atoms in optical potentials [18].

The main dynamical origin of the observed subdiffusion is believed to be deterministic chaos. Indeed, assume that a wave packet which is exciting a few lattice sites (or more) is not spreading. Then the dynamics of this trapped excitation can be described within a Hamiltonian system with a finite number of degrees of freedom, which is nonlinear, and typically not integrable [19]. Excluding very weak nonlinearities and the Kolmogorov-Arnold-Moser (KAM) regime, the dynamics should be chaotic. Since deterministic chaos deteriorates correlations and leads to a decoherence of the wave phases, the main ingredient of Anderson localization is lost, and the wave packet can not be anymore localized [20]. Therefore one contradicts the initial assumption of persistence of localization. The wave packet can not keep its localization, but will spread. From this it also follows, that for weak enough nonlinearity the initially excited wave packet is in a KAM regime, where there is a finite probability to start on a chaotic trajectory, but with a finite complementary probability to end up on a regular trajectory which enjoys phase coherence and localization [19]. Indeed it was recently shown that the probability for chaos is practically equal to one above a nonlinearity threshold, and less than one below that threshold, tending towards zero in the limit of vanishing nonlinearity, thus restoring Anderson localization in this probabilistic sense [21]. The microscopic origin of chaos is expected to be hidden in the chaotic seeds of nonlinear resonances, which will fluctuate in time and space as the wave packet spreads [22].

To summarize, two main assumptions - the deterministic chaoticity of the wave packet dynamics, and the space-time fluctuation of the chaotic seeds - have to be confirmed to provide solid grounds for subdiffusion theories. A study of the interrelation between the appearance of chaos for nonlinear waves in the disordered Schrödinger equation was performed by Mulansky et al. [23]. Various relations between mode energies and capabilities to reach thermal equilibrium were studied for small systems, but not for the nonequilibrium wave packet spreading situation. In [24] it has been shown that chaoticity does not necessarily imply thermalization for small-size disordered lattices. Michaely and Fishman [25] recently studied the temporal and frequency characteristics of effective nonlinear forces inside the wave packet, and concluded that these forces show sufficient randomness to qualify as effective noise terms - which is thought to be a result of deterministic chaos (but not a direct proof of its existence). Similarly Vermersch and Garreau [26] measured a spectral entropy, which however is only a very rough indicator for chaos. They also attempted to measure Lyapunov exponents, but only on short times. Moreover, all these attempts do not account for the *temporal dependence* of chaos strength. Indeed, the more the wave packet spreads, the weaker the chaos should become, since densities decrease. This is also reflected in the fact that the packets spread subdiffusively. Therefore we need a temporal resolution of the chaos indicators. That is what we will present in this work.

The spreading of wave packets was numerically studied in a number of classes of wave equations. These equations share a surprising universality in that only the dimensionality of the lattice and the nonlinearity power σ influence the value of the exponent γ. Therefore we choose a chain of coupled anharmonic oscillators with random harmonic frequencies which belongs to the class of quartic Klein-Gordon (KG) lattices. This model is dynamically very similar to nonlinear Schrödinger equations with random potentials for small densities [9, 10, 13, 14, 15, 21, 27]. The Hamiltonian of the quartic KG chain of coupled anharmonic oscillators with coordinates u_l and momenta p_l is

$$\mathcal{H}_K = \sum_l \frac{p_l^2}{2} + \frac{\tilde{\epsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}(u_{l+1} - u_l)^2 \ . \tag{1}$$

The equations of motion are $\ddot{u}_l = -\partial \mathcal{H}_K / \partial u_l$, and $\tilde{\epsilon}_l$ are chosen uniformly from the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. The value of $\mathcal{H}_K$ serves as a control parameter of the system's nonlinearity. In the absence of the quartic term in (1) the ansatz $u_l(t) = A_l \mathrm{e}^{i\omega t}$ yields the linear eigenvalue problem $\lambda A_l = \epsilon_l A_l - (A_{l+1} + A_{l-1})$ with $\epsilon_l = W(\tilde{\epsilon}_l - 1)$ and $\lambda = W(\omega^2 - 1) - 2$. This eigenvalue problem corresponds precisely to the well known Anderson localization in a one-dimensional chain with diagonal disorder [1, 2].

We analyze normalized energy distributions $\epsilon_l \geq 0$ using the second moment $m_2 = \sum_l (l - \bar{l})^2 \epsilon_l$, which quantifies the wave packet's degree of spreading and the participation number $P = 1/\sum_l \epsilon_l^2$, which measures the number of the strongest excited sites in ϵ_l. Here $\bar{l} = \sum_l l \epsilon_l$ and $\epsilon_l \equiv h_l / \mathcal{H}_K$ with $h_l = p_l^2/2 + \tilde{\epsilon}_l u_l^2/2 + u_l^4/4 + (u_{l+1} - u_l)^2/4W$. During the wave packet evolution we further estimate the maximum Lyapunov exponent (mLE) Λ_1 as the limit for $t \to \infty$ of the quantity $\Lambda(t) = t^{-1} \ln(\|\vec{v}(t)\| / \|\vec{v}(0)\|)$, often called *finite time mLE* [28, 29, 30]. $\vec{v}(0)$, $\vec{v}(t)$ are deviation vectors from a given trajectory, at times $t = 0$ and $t > 0$ respectively, and $\|\cdot\|$ denotes the usual vector norm. $\Lambda(t)$ is a widely used chaos indicator. It tends to zero in the case of regular motion as $\Lambda^r(t) \sim t^{-1}$ [31, 30], while it tends to nonzero values for chaotic motion. Its inverse, T_L, is the characteristic timescale of the studied dynamical system, the so-called Lyapunov time. It quantifies the time needed for the system to become chaotic. The vector $\vec{v}(t)$ has as coordinates small deviations from the studied trajectory in positions and momenta ($v_i = \delta u_i$, $v_{i+N} = \delta p_i$, $1 \leq i \leq N$, N being the total number of lattice sites). Its time evolution is governed by the so-called variational equations. In our study we also compute normalized deviation vector distributions (DVDs) $w_l = (v_l^2 + v_{l+N}^2) / \sum_l (v_l^2 + v_{l+N}^2)$.

We use the symplectic integrator SABA$_2$ with corrector [32, 10] for the integration of the equations of motion, and its extension according to the so-called tangent map method [33, 34, 35] for the integration of the variational equations. We considered lattices with $N = 1000$ to $N = 2000$ sites in our computations, in order to exclude finite-size effects in the evolution of the wave packets, and an integration time step $\tau = 0.2$, which kept the relative energy error always less than 10^{-4}.

The numerical simulation of a single site excitation in the absence of nonlinear terms (orange curve) corresponds to regular motion and Anderson localization is observed. At variance to the t^{-1} decay for the regular nonchaotic trajectory, the observed decay for the 'weak chaos' orbit is much weaker and well fitted with $\Lambda \sim t^{-1/4}$.

We substantiate the findings by averaging $\log_{10} \Lambda$ over 50 realizations of disorder and extending to two more 'weak chaos' parameter cases with initial energy density $\epsilon = 0.01$ distributed evenly among a block of 21 central sites for $W = 4$ (case II) and 37 central sites for $W = 3$ (case III). All cases show convergence towards $\Lambda \sim t^{-1/4}$. We further differentiate the numerically obtained following the approach used in [13, 14], estimate their slope $\alpha_L = \frac{d(\log_{10} \Lambda(t))}{d \log_{10} t}$, and find a result which underpins the above findings. Therefore

$$\Lambda(t) \sim t^{-1/4} \gg \frac{1}{t} \, . \tag{2}$$

We computed nonequilibrium chaos indicators of the spreading of wave packets in disordered lattices (see also [37]). For the first time we find that chaos not only exists, but also persists. Using a set of observables we are able to show that the slowing down of chaos does not cross over into regular dynamics, and is at all times fast enough to allow for a thermalization of the wave packet. Moreover the monitoring of the spatio-temporal dynamics of hot chaotic spots yields an increase in their spatial fluctuations - in accord with previous unproven assumptions.

The mLE decreases when the wave packet spreads, since the energy density decreases as well. Nevertheless, the mLE follows a completely different power law as compared to the case of regular motion. This signals the lack of sticking to regular structures in phase space, as conjectured recently [38, 39].

We also studied the spatial evolution of the deviation vector associated with the mLE. The corresponding distributions remain localized with a very pointy profile. This observation supports theoretical assumptions that in the 'weak chaos' regime only few nonlinear resonances appear at a time. The mean position of these distributions performs random oscillations, whose amplitude increases as the wave packet spreads. These oscillations result in a homogeneity of chaos inside the packet, i.e. in thermalization.

All these findings clearly show that nonlinear wave packets spread in random potentials due to deterministic chaos and dephasing. Moreover, wave packets first thermalize, and only later perform subdiffusive spreading. That is a basic prerequisite for the existing theoretical description of energy spreading in disordered nonlinear lattices, and the applicability of nonlinear diffusion equations [27, 40, 41, 42].

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Stability of line solitons for the KP-II equation in $\mathbb{R}^2$

TETSU MIZUMACHI

The KP-II equation

$$\text{(KP)} \qquad \partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\sigma\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2$$

is a 2-dimensional generalization of the KdV equation that takes slow variations in the transversal direction into account. If $\sigma = 1$, then (KP) is called KP-II and it describes the motion of shallow water waves with weak surface tension.

The KdV equation has a two parameter family of solitary wave solutions

$$\{\varphi_c(x - 2ct - \gamma) \mid c > 0\,, \gamma \in \mathbb{R}\}\,, \quad \varphi_c(x) = c\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}\,x\right).$$

The KP-II equation is supposed to explain stability of these solitary wave solutions to the transversal perturbations (see [7]). When $\varphi_c(x - 2ct)$ is considered as a solution of the KP equations, we call it a *line soliton*. The KP-II equation is known to be well-posed in $L^2(\mathbb{R}^2)$ on the background of line solitons ([11]).

The KP equation (KP) is an integrable system as well as the KdV equation and have conserved quantities such as

$$\int_{\mathbb{R}^2} u(t,x,y)^2\,dxdy \quad \text{(momentum)},$$

$$\frac{1}{2}\int \left(u_x^2(t,x,y) - 3\sigma(\partial_x^{-1}\partial_y u(t,x,y))^2 - 2u^3(t,x,y)\right)dxdy \quad \text{(Hamiltonian)}.$$

For the KP-I equation, that is (KP) with $\sigma = -1$, the first two terms of the Hamiltonian have the same sign and it was shown by [6, 14] that there exists a stable ground state for the KP-I equation, whereas the KP-II equation has no traveling wave solutions that belong to $L^2(\mathbb{R}^2)$ ([5]).

On the other hand, it is known that the line solitons of the KP-I equation are unstable ([16]) and line solitons of the KP-II equation are linearly stable ([1, 2]). Because the dominant quadratic part of the Hamiltonian of the KP-II equation is indefinite, it is more natural to explain stability of line solitons by using propagation estimates such as [13] rather than by using variational arguments such as [3].

The main difference between stability analysis of the KdV 1-soliton or the stability of line soliton with the y-periodic boundary condition is that $\varphi_c(x - 2ct)$ does not have the finite L^2-mass because it is not localized in the y-direction. As a consequence, the linearized operator of the KP-II equation around the line soliton has a family of continuous eigenvalues converging to 0 in exponentially weighted space, whereas 0 is an isolated eigenvalue of the linearized KdV operator around a solitary wave in exponentially weighted space.

Since we have continuous spectrum converging to 0, the modulations of the speed and the phase shift of line solitons for the KP-II equation cannot be described by ODEs as KdV ([13]) or the KP-II equation posed on $\mathbb{R}_x \times \mathbb{R}_y/(2\pi\mathbb{Z})$ ([10]).

I find that for the KP-II equation posed on $\mathbb{R}^2$, modulation of the speed parameter $c(t,y)$ and the phase shift $x(t,y)$ cannot be uniform in y and their long time behavior is described by the Burgers equation. Note that similar modulation equations have formally derived for a two spatial dimensional Boussinesq model ([12]).

My results are the following ([9]).

Theorem 3. *Let $c_0 > 0$ and $a \in (0, \sqrt{c_0/2})$. Then there exist positive constants ε_0 and C satisfying the following: if $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$, $v_0 \in H^1(\mathbb{R}^2)$ and $\varepsilon := \|e^{ax}v_0\|_{L^2(\mathbb{R}^2)} + \|e^{ax}v_0\|_{L^1_yL^2_x} + \|v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0$, then there exist C^1-functions $c(t,y)$ and $x(t,y)$ such that for $t \geq 0$,*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon\,, \tag{1}$$

$$\sup_{y\in\mathbb{R}}(|c(t,y) - c_0| + |x_y(t,y)|) \leq C\varepsilon(1+t)^{-1/2}\,, \tag{2}$$

$$\|x_t(t,\cdot) - 2c(t,\cdot)\|_{L^2} \leq C\varepsilon(1+t)^{-3/4}\,, \tag{3}$$

$$\left\|e^{ax}(u(t, x + x(t,y), y) - \varphi_{c(t,y)}(x))\right\|_{L^2} \leq C\varepsilon(1+t)^{-3/4}\,. \tag{4}$$

We find that $c(t,y)$ and $\partial_y x(t,y)$ behave like a self-similar solution of the Burgers equation around $y = \pm\sqrt{8c_0}t$.

Theorem 4. *Let $c_0 = 2$ and let v_0 and ε be the same as in Theorem 3. Then for any $R > 0$,*

$$\left\|\begin{pmatrix} c(t,\cdot) \\ x_y(t,\cdot)\end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1\end{pmatrix}\begin{pmatrix} u_B^+(t, y+4t) \\ u_B^-(t, y-4t)\end{pmatrix}\right\|_{L^2(|y\pm 4t|\leq R\sqrt{t})} = o(t^{-1/4})$$

as $t \to \infty$, where $u_B^\pm$ are self similar solutions of the Burgers equation

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

such that

$$u_B^\pm(t,y) = \frac{\pm m_\pm H_{2t}(y)}{2\left(1 + m_\pm \int_0^y H_{2t}(y_1)\,dy_1\right)}\,, \quad H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t}\,,$$

and that $m_\pm$ are constants satisfying

$$\int_{\mathbb{R}} u_B^\pm(t,y)\,dy = \frac{1}{4}\int_{\mathbb{R}} c(0,y)\,dy + O(\varepsilon^2)\,.$$

The KP-II equation (KP) is invariant under a change of variables

$$x \mapsto x + ky - 3k^2t + \gamma \quad \text{and} \quad y \mapsto y - 6kt \quad \text{for any } k,\, \gamma \in \mathbb{R}, \tag{5}$$

and has a 3-parameter family of line soliton solutions

$$\mathcal{A} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0\,,\ k\,, \gamma \in \mathbb{R}\}\,.$$

Thanks to propagations of the local phase shifts along the crest of line solitons, the set $\mathcal{A}$ is not stable in $L^2(\mathbb{R}^2)$.

Theorem 5. *Let $c_0 > 0$. There exists a positive constant C such that for any $\varepsilon > 0$, there exists a solution of* (KP) *such that $\|u(0,x,y) - \varphi_{c_0}(x)\|_{L^2} < \varepsilon$ and*

$$\liminf_{t\to\infty} t^{-1/4} \inf_{v\in\mathcal{A}} \|u(t,x,y) - v\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon\,.$$

Similar problems have already been studied for nonlinear heat equations ([15, 8]) and also for kink solutions to a wave equation ([4]). However, we need some correction of the phase shift for the KP-II equation, which does not appear in these former results because the transversal dimension or the power of nonlinear terms are higher than our problem.

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Scattering of small solutions of cubic NLS

Atanas Stefanov
(joint work with Vladimir Georgiev)

The Cauchy problem for the Schrödinger equation, subject to a decaying potential V is

$$\left| \begin{array}{l} iu_t - \partial_x^2 u + V(x)u = F(u), \quad (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ u(0,x) = f(x). \end{array} \right. \tag{1}$$

The nonlinearity is most often taken to be in the form $F(u) = \pm |u|^{2p} u$.

Various authors have considered the question for scattering of solutions to (1) for $p \geq 1$. Namely, if the initial data f is small[1] does it follow that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_Y = 0?$$

Our motivation comes from a model problem for asymptotic stability for standing waves of the Schrödinger equation. Therein, one is led to show the scattering of the radiation portion of the solution. This problem takes a form similar to (1), thus (1) is a toy model for this more complicated question.

Back to (1), and for the case $p = 1$ and $V = 0$, the case is especially well-studied, see [5], [4] (see also [6] for similar result for the Klein-Gordon equation). In these papers, the authors have shown, among other things that the solutions behave essentially like the free solution, in that the decay estimate

$$\|u(t,\cdot)\|_{L^\infty} \leq Ct^{-1/2} \tag{2}$$

for large t. The situation is quite a bit more complicated, when one adds a potential V. This is the subject of the recent paper [1]. In this paper, the authors prove (after imposing some additional spectral assumption on the Schrödinger operator $-\partial_x^2 + V$ and decay assumptions on V) that if the initial data f is small in the space $L^2_{\frac{1}{2}+} \cap H^{\frac{1}{2}+}$, then the standard decay estimate (3) holds again, provided $p > 1$! Thus the important cubic case remains open.

In this work, we aim on studying the resonance phenomena occurring in the cubic case, again in the presence of a decaying potential, so that $\sigma[-\partial_{xx} + V] = [0,\infty)$. Similar to the approach in [3], we cut out the *pieces of the nonlinearity that create resonances.* More precisely, we work with the following

$$\left| \begin{array}{l} iu_t - u_{xx} + V(x)u(x) = \pm Q^\epsilon(u,\bar{u},u), \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^1 \\ u(0,x) = f(x). \end{array} \right. \tag{3}$$

where the trilinear form $Q^\epsilon(u,v,w)$ is given by

$$\frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \hat{u}(\xi_1)\hat{v}(\xi_2)\hat{w}(\xi_3)\left(1-\psi\left(\frac{\xi_1+\xi_2}{\epsilon}\right)\right)\left(1-\psi\left(\frac{\xi_2+\xi_3}{\epsilon}\right)\right)e^{i(\xi_1+\xi_2+\xi_3)x}d\xi.$$

for some smooth even cutoff function ψ which is equal to one for $|\xi| < 1$ and zero in $|\xi| > 2$. The algebraic identity

$$(\tau_1+\tau_2+\tau_3)-(\xi_1+\xi_2+\xi_3)^2 = (\tau_1-\xi_1^2)+(\tau_3-\xi_3^2)+(\tau_2+\xi_2^2)-2(\xi_1+\xi_2)(\xi_2+\xi_3),$$

implies that for the operator

$$T(u_1,u_2,u_3) = \frac{c}{(2\pi)^3}\int_{\mathbf{R}^3} \sigma(\xi)\left(\prod_{j=1}^3 \widehat{v_j}(\xi_j)\right)e^{ix\cdot(\xi_1+\xi_2+\xi_3)}d\xi.$$

$$\sigma(\xi) = \frac{(1-\psi(\frac{\xi_1+\xi_2}{\epsilon}))(1-\psi(\frac{\xi_2+\xi_3}{\epsilon}))}{(\xi_1+\xi_2)(\xi_2+\xi_3))}$$

[1]in appropriate sense $f : \|f\|_X << 1$

we have

$$\begin{aligned}(i\partial_t - \partial_{xx})T(v,\overline{v},v) &= Q(v,\overline{v},v) + 2T(-Vv \pm Q(v,\overline{v},v),\overline{v},v) + \\ &+ T(v,\overline{-Vv \pm Q(v,\overline{v},v)},v)\end{aligned}$$

As a result, the near identity transformation $u = z + T(u,\bar{u},u)$ turns (3) into an essentially quintic equation with small data and the small scattering result follows from standard estimates for the semigroup $e^{it(-\partial_{xx}+V)}$ in either decay or Strichartz norms.

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Reporter: Roman Bauer

Participants

Prof. Dr. Dario Bambusi
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
20133 Milano
ITALY

Roman Bauer
Institut f. Analysis, Dynamik &
Modellierung
Fakultät f. Mathematik & Physik
Universität Stuttgart
70569 Stuttgart
GERMANY

Dr. Jonathan Dawes
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Dr. Aslihan Demirkaya-Ozkaya
Department of Mathematics
University of Hartford
Dana Hall, Room 220
200 Bloomfield Avenue
West Hartford CT 06117
UNITED STATES

Prof. Dr. Michal Feckan
Department of Math. Analysis
Comenius University
Mlynska dolina
84215 Bratislava
SLOVAKIA

Dr. Sergej Flach
Centre for Theoretical Chemistry &
Physics
New Zealand Institute for Advanced
Study
Massey University
Private Bag 102-904
Auckland
NEW ZEALAND

Prof. Dr. Michael Herrmann
Fachbereich Mathematik - FB 9
Universität des Saarlandes
Gebäude E2 4
66041 Saarbrücken
GERMANY

Dr. Aaron H. Hoffman
Olin College
Olin Way
Needham, MA 02492-1200
UNITED STATES

Prof. Dr. Guillaume James
Laboratoire Jean Kuntzmann
Université Joseph Fourier
Tour IRMA
B.P. 53
38041 Grenoble Cedex
FRANCE

Prof. Dr. Thomas Kappeler
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
8057 Zürich
SWITZERLAND

Prof. Dr. Panos Kevrekidis
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst, MA 01003-9305
UNITED STATES

Prof. Dr. Vassilis Koukouloyannis
Department of Physics
Aristoteles University of Thessaloniki
Thessaloniki 54124
GREECE

Dr. Karsten Matthies
Department of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Tetsu Mizumachi
Faculty of Mathematics
Kyushu University
744, Motooka, Nishi-ku
Fukuoka 819-0395
JAPAN

Prof. Dr. Simone Paleari
Dipartimento di Matematica
Universita di Milano
Via C. Saldini, 50
20133 Milano
ITALY

Prof. Dr. Panos Panayotaros
Depto. Matemáticas y Mecánica, IIMAS
Univ. Nacional Autonoma de Mexico
Apartado Postal 20-726
Mexico D. F. 01000
MEXICO

Prof. Dr. Dmitry Pelinovsky
Dept. of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton, Ont. L8S 4K1
CANADA

Dr. Tiziano Penati
Dipartimento di Matematica
Universita di Milano
Via Saldini 50
20133 Milano
ITALY

Prof. Dr. Zoi Rapti
Department of Mathematics
University of Illinois at
Urbana-Champaign
1409 West Green Street
Urbana IL 61801
UNITED STATES

Prof. Dr. Guido Schneider
Institut f. Analysis, Dynamik &
Modellierung
Fakultät f. Mathematik & Physik
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Atanas Stefanov
Department of Mathematics
University of Kansas
405 Snow Hall
Lawrence, KS 66045-7567
UNITED STATES

Prof. Dr. Hannes Uecker
Institut fuer Mathematik
Carl v. Ossietzky-Universität Oldenburg
Fakultät V: Mathematik &
Naturwissensch.
26111 Oldenburg
GERMANY

Prof. Dr. Yingfei Yi
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Dominik Zimmermann
Universität Stuttgart
Fachbereich Mathematik
Institut f. Analysis, Dynamik &
Modellierung
Pfaffenwaldring 57
70569 Stuttgart
GERMANY