Arbeitsgemeinschaft: Sofic Entropy

Organised by
Lewis Bowen, Austin
David Kerr, College Station

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ABSTRACT. The notion of soficity for a group is a weak type of finite approximation property that simultaneously generalizes both amenability and residual finiteness. In 2008 L. Bowen discovered how it can be used to significantly broaden the scope of the classical theory of dynamical entropy beyond the setting of amenable acting groups. This Arbeitsgemeinschaft aimed to provide a comprehensive picture of the subject of sofic entropy as it has developed over the last five years.

Introduction by the Organisers

The Arbeitsgemeinschaft *Sofic Entropy* was organized by Lewis Bowen (Austin) and David Kerr (College Station) and held from October 6 to 11, 2013. There were more than 40 participants, a large proportion of which were graduate students and postdocs. Many participants came with expertise in closely related subjects like infinite group theory, operator algebras, and graph theory, while others represented areas such as number theory and coarse geometry, making for a lively mixture of backgrounds and interests. There was a total of 18 lectures, each one hour in length.

The meeting aimed to address the main concepts and results in the theory of sofic entropy as it has developed starting from the seminal work of Bowen five years ago. The concept of entropy was introduced into ergodic theory by Kolmogorov in the late 1950s with motivation from Shannon's information theory. An analogous theory of topological entropy was initiated by Adler, Konheim, and McAndrew in the early 1960s, and the two entropies, measure and topological, are related by a variational principle. These classical approaches to dynamical entropy

involve averaging across partial orbits and thus are ultimately suited to actions of amenable groups, for which much of the basic theory was developed by Ornstein and Weiss. Ornstein and Weiss showed in particular that entropy is a complete invariant for Bernoulli actions of a countably infinite amenable group, extending a celebrated result of Ornstein for single Bernoulli shifts.

In the broader realm of sofic acting groups, Bowen showed that one could produce an entropy invariant by externalizing the averaging in the classical amenable definition of entropy to a finite set on which the group acts in an approximate sense, according to the definition of soficity. This led to an entropy classification for Bernoulli actions of all countably infinite nontorsion sofic groups. A more general definition of measure entropy and a corresponding notion of sofic topological entropy were subsequently introduced by Kerr and Li, who also established a variational principle relating the two. Sofic entropy has been applied for example to the study of algebraic actions and questions surrounding the Fuglede-Kadison determinant in group von Neumann algebras, and it has also inspired the development of sofic dimension for groups and equivalence relations. These topics were all covered in the lectures, as well as Gottschalk's surjunctivity problem, the f-invariant, sofic mean dimension, the computation of sofic entropy for algebraic actions, combinatorial independence, Li-Yorke chaos, and entropy in the framework of groupoids.

Arbeitsgemeinschaft: Sofic Entropy

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Abstracts

Introduction to sofic groups

Andreas Thom

In this talk I gave an overview about the class of sofic groups. Most of the results and open problems that I mentioned are explained in great detail in the excellent survey by Vladimir Pestov [1].

References

[1] V. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic **14** (2008), 449–480.

Entropy for single automorphisms

Martha Łącka

The talk was a refresher on classical entropy theory for actions of a single map. We followed mainly the material contained in [3] and [1].

Definition 1. Given a group action G on X, we say that a pseudometric d on X is dynamically generating if for two distinct points x and y in X there is an element s in G such that $d(sx, sy) > 0$.

Let X be a compact Hausdorff space and let $T : X \to X$ be a homeomorphism. Let d be a dynamically generating pseudometric on X (we assume that $\mathbb{Z}$ acts on X in a usual way). Let d_n denote the n-th Bowen pseudometric defined in the following way:

$$d_n(x, y) = \max_{0 \leq s \leq n-1} d(sx, sy).$$

Definition 2. A subset E of the space X is (n, ε)-separated with respect to d if for any two distinct points x and y in E the following holds: $d_n(x, y) \geq \varepsilon$.

Let $s(n, \varepsilon)$ denote the maximal cardinality of a (n, ε)-separated subset of X.

Definition 3. The topological entropy of f is defined as

$$h_s(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

Definition 4. The subset F of the space X is (n, ε)-spanning if for any point x in X there is a point y in F such that $d_n(x, y) < \varepsilon$.

Let $r(n, \varepsilon)$ denote the minimal cardinality of a (n, ε)-spanning set. Let us put:

$$h_r(T) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon).$$

Now, let $\mathcal{U}$ be an open cover of X and let:

$$\mathcal{U}^n := \left\{ \bigcap_{i=0}^{n-1} T^{-i} U_i \quad | \quad U_i \in \mathcal{U} \right\}.$$

Let $N(\mathcal{U}, n)$ denote the minimal cardinality of a subcover of $\mathcal{U}^n$. Let us define:

$$h_c(T, \mathcal{U}) := \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}, n) \text{ and } h_c(T) := \sup_{\mathcal{U}} h_c(\mathcal{U}).$$

Theorem 5. *For any T the values of $h_r(T)$, $h_s(T)$ and $h_c(T)$ are equal. In particular, the topological entropy $h(T)$ does not depend on the choice of the dynamically generating pseudometric d.*

Let us list some properties of topological entropy:

(1) $h(T) \geq 0$, $h(I) = 0$, where I is an identity map;
(2) in the definition of $h(T)$ one can take the supremum over finite open covers;
(3) if Y is a closed subset of X and $TY = Y$, then $h(T_{|Y}) \leq h(T)$;
(4) $h(T^m) = m h(T)$.

Let us consider two examples:

(1) Let d be a pseudometric on $\{1, \ldots, k\}^{\mathbb{Z}}$ defined in the following way:

$$d(\{a_n\}, \{b_n\}) = \begin{cases} 0 & \text{if} & a_0 = b_0, \\ 1 & \text{if} & a_0 \neq b_0 \end{cases}$$

This pseudometric is dynamically generating for the shift map σ. Fix $\varepsilon < \frac{1}{2}$. Then, the maximal cardinality of the (n, ε)-separated set with respect to d is equal to the number of n-th cylinders and so: $h(\sigma) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log k^n = \log k$.

(2) The entropy of the hyperbolic toral automorphism A is given by $h(A) = \log |\lambda_1|$, where λ_1 is the eigenvalue of A with modulus greater than 1.

Now, let $\xi = \{A_1, \ldots, A_k\}$ and $\eta = \{B_1, \ldots, B_p\}$ be two finite and measureable partitions of a Borel probability space $(X, \mathcal{B}, \mu)$.

Definition 6. The measure entropy of ξ (denoted by $H(\xi)$) is equal to

$$-\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

If $\xi = \{A_1, \ldots, A_k\}$ then $H(\xi) \leq \log k$ and $H(\xi) = \log k$ only when $\mu(A_i) = \frac{1}{k}$ for all $i = 1, \ldots, k$.

Definition 7. Conditional entropy of ξ given η is the following:

$$H(\xi, \eta) = -\sum_{i,j} \mu(A_i \cap C_j) \log \frac{\mu(A_i \cap C_j)}{\mu(C_j)}.$$

Definition 8. Suppose $T : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$. If $\mathcal{A}$ is a finite sub-σ-algebra of $\mathcal{B}$ then

$$h(T, \xi(\mathcal{A})) = h(T, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i} \mathcal{A})$$

is called the entropy of T with respect to $\mathcal{A}$.

Definition 9. If $T : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$ then $h(T) = \sup h(T, \xi)$, where the supremum is taken over all finite partitions of $(X, \mathcal{B}, m)$, is called the entropy of T.

Theorem 10. *Let $(X, \mathcal{B}, m)$ be a probability space. If $\mathcal{A}, \mathcal{C}, \mathcal{D}$ are finite subalgebras of $\mathcal{B}$ then:*

(1) $H(\mathcal{A}|\mathcal{C}) \geq 0$,
(2) *if* $\mathcal{A} \overset{\circ}{=} \mathcal{D}$ *then* $H(\mathcal{A}|\mathcal{C}) = H(\mathcal{D}|\mathcal{C})$,
(3) *if* $\mathcal{C} \overset{\circ}{=} \mathcal{D}$ *then* $H(\mathcal{A}|\mathcal{C}) = H(\mathcal{A}|\mathcal{D})$,
(4) $H(\mathcal{A} \vee \mathcal{C}|\mathcal{D}) = H(\mathcal{A}|\mathcal{D}) + H(\mathcal{C}|\mathcal{A} \vee \mathcal{D})$,
(5) $H(\mathcal{A} \vee \mathcal{C}) = H(\mathcal{A}) + H(\mathcal{C}|\mathcal{A})$,
(6) $\mathcal{A} \subset \mathcal{C}$ *implies that* $H(\mathcal{A}|D) \leq H(\mathcal{C}|\mathcal{D})$ *and* $H(\mathcal{A}) \leq H(\mathcal{C})$,
(7) $\mathcal{C} \subset \mathcal{D}$ *implies that* $H(\mathcal{A}|\mathcal{C}) \geq H(\mathcal{A}|\mathcal{D})$,
(8) $H(\mathcal{A}) \geq H(\mathcal{A}|\mathcal{D})$,
(9) $H(\mathcal{A} \vee C|\mathcal{D}) \leq H(\mathcal{A}|\mathcal{D}) + H(\mathcal{C}|\mathcal{D})$,
(10) *if T is measure-preserving then* $H(T^{-1}\mathcal{A}|T^{-1}\mathcal{C}) = H(\mathcal{A}|\mathcal{C})$ *and* $H(T^{-1}\mathcal{A}) = H(\mathcal{A})$.
(11) *if $\mu_i \in M(X)$, $1 \leq i \leq n$, and $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$ then*

$$H_{\sum_{i=1}^{n} p_i \mu_i}(\xi) \geq \sum_{i=1}^{n} p_i H_{\mu_i}(\xi)$$

for any finite partition ξ of $(X, \mathcal{B}(X))$.

Theorem 11. *Suppose $\mathcal{A}, \mathcal{C}$ are finite subalgebras of $\mathcal{B}$ and T is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$. Then:*

(1) $h(T, \mathcal{A}) \geq H(\mathcal{A})$,
(2) $h(T, \mathcal{A} \vee \mathcal{C}) \leq h(T, \mathcal{A}) + h(T, \mathcal{C})$,
(3) $\mathcal{A} \subset \mathcal{C}$ *implies that* $h(T, \mathcal{A}) \leq h(T, \mathcal{C})$,
(4) $h(T, \mathcal{A}) \leq h(T, \mathcal{C}) + H(\mathcal{A}|\mathcal{C})$,
(5) $h(T, T^{-1}\mathcal{A}) = h(T, \mathcal{A})$,
(6) *if $k \geq 1$, then* $h(T, \mathcal{A}) = h(T, \vee_{i=0}^{k-1} T^{-i} \mathcal{A})$,
(7) *if T is invertible and $k \geq 1$ then* $h(T, \mathcal{A}) = h(T, \vee_{i=-k}^{k} T^{i} \mathcal{A})$,
(8) $h(T) \geq 0$, $h(T)$ *could be infinite.*

Let μ be a Borel probability T-invariant ergodic measure. For $\varepsilon > 0$, $\delta \in (0, 1)$, let us denote by $N_T(n, \varepsilon, \delta)$ the minimal number of ε−balls in the d_m-metric which cover the set of measure more than or equal to $1 - \delta$. Katok proved the following theorem:

Theorem 12. *For any $\delta \in (0,1)$:*

$$h_\mu(T) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\ln N_T(n, \varepsilon, \delta)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\ln N_T(n, \varepsilon, \delta)}{n}.$$

The following theorem enables us to restrict our attention to generating partitions.

Theorem 13 (Kolmogorov-Sinai). *Let T be an invertible measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$ and let $\mathcal{A}$ be a finite sub-algebra of $\mathcal{B}$ such that $\vee_{n=-\infty}^{\infty} T^n \mathcal{A} \overset{\circ}{=} \mathcal{B}$. Then $h(T) = h(T, \mathcal{A})$.*

Using the above theorem we can prove that:

Theorem 14. *The two-sided $(p_0, \ldots, p_{k-1})$-shift has entropy $-\sum_{i=0}^{k-1} p_i \log p_i$.*

The relation between topological entropy and measure-theoretic entropy is established by the Variational Principle.

Theorem 15 (Variational Principle). *Let $T : X \to X$ be a continuous map of a compact metric space X. Then $h(T) = \sup\{h_\mu(T) \mid \mu \in M(X, T)\}$.*

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Factors of Bernoulli shifts

BRANDON SEWARD

If G is a countable amenable group, then a well known property of entropy states that if $G \curvearrowright (X, \mu)$ factors onto $G \curvearrowright (Y, \nu)$, then $G \curvearrowright (X, \mu)$ has greater entropy than $G \curvearrowright (Y, \nu)$. In particular, since the Bernoulli shift $G \curvearrowright (n^G, u_n^G)$ has entropy $\log(n)$, where u_n is the uniform probability measure on $\{1, 2, \ldots, n\}$, it follows that for a countable amenable group G the Bernoulli shift $G \curvearrowright (n^G, u_n^G)$ cannot factor onto $G \curvearrowright (k^G, u_k^G)$ if $k > n$. In 1987, Ornstein and Weiss [4] discovered the seemingly bizarre fact that for the rank two free group F_2, the Bernoulli shift $F_2 \curvearrowright (2^{F_2}, u_2^{F_2})$ factors onto the larger Bernoulli shift $F_2 \curvearrowright (4^{F_2}, u_4^{F_2})$. This example convinced many people that there could not exist an entropy theory for actions of non-amenable groups. Many years later in 2005, Ball [1] greatly expanded upon the Ornstein–Weiss example by proving that for every countable non-amenable group G there is $n \in \mathbb{N}$ so that $G \curvearrowright (n^G, u_n^G)$ factors onto $G \curvearrowright ([0,1]^G, \lambda^G)$, where λ is Lebesgue measure. In particular, it follows that $G \curvearrowright (n^G, u_n^G)$ factors onto all other Bernoulli shifts over G. Most recently, Bowen [2] improved upon the Ornstein–Weiss example to prove that if G contains F_2 as a subgroup then in fact all Bernoulli shifts over G factor onto one-another. It is unknown whether this is true for all countable non-amenable groups. Furthermore, from Bowen's proof one

can show that Ball's theorem follows from the Gaboriau–Lyons theorem [3]. The Gaboriau–Lyons theorem roughly says that for sufficiently large Bernoulli shifts over a countable non-amenable group, it appears in a certain measurable sense that the group contains F_2 as a subgroup.

References

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Sofic topological entropy

Joav Orovitz

The goal of this lecture is to introduce a notion of topological entropy in the setting of a sofic group acting on a compact metrizable space. We follow the approach of [1] to define the numerical invariant

$$h_{\Sigma,p}(\rho)$$

where $\Sigma = \{\sigma_i : G \to S_{n_i}\}_{i=1}^{\infty}$ is a sofic approximation sequence of a sofic group G, $p \in \{2, \infty\}$ and ρ is a dynamically generating, continuous pseudometric on a compact, metrizable, topological X on which G acts by homeomorphisms. In vague terms, this is done by fixing $\varepsilon, \delta > 0$ and a finite set $F \subseteq G$ and counting how many ε-different (ρ, F, δ)-equivariant embeddings one can find of the sofic model space $\{1, \ldots, n_i\}$ into X. Then one takes the natural logarithm and normalizes by the size of the model space and takes a suitable limit over the set of parameters. A vastly oversimplified but effective way to think of this notion as a generalization of classical topological entropy for actions of amenable groups is to think of each such embedding of the model space as an approximate partial orbit of a single point under a set coming from a Følner sequence.

To illustrate, consider the case where $G = \mathbb{Z}$ and σ_i is the natural action of $\mathbb{Z}$ on $\mathbb{Z}/i\mathbb{Z}$ which can be identified with the set $\{1, \ldots, i\}$. Then an embedding of $\{1, \ldots, i\}$ into X which is exactly equivariant with respect to group elements coming from $F = \{0, \ldots, i-1\} \subseteq G$ is precisely the partial orbit of a point $x \in X$ under the elements of F.

Of course sofic groups can be far more complicated, and even if $G = \mathbb{Z}$ the sofic approximation sequence need not be so nice. It turns out however (as proved in lecture 7) that if G is amenable then the sofic topological entropy coincides with the classical topological entropy, regardless of the sofic approximation sequence used. The main idea of the proof of this fact basically amounts to showing that even though we can't expect each (ρ, F, δ)-equivariant embedding to correspond

to an approximate partial orbit of a point under a Følner set, it still admits an approximate decomposition into such approximate partial orbits.

Next, it is shown that $h_{\Sigma,p}(\rho)$ does not depend on p or on ρ (though it can apparently depend on Σ) and the common value is denoted $h_\Sigma(X,G)$. A new phenomenon appears here that does not appear in the analogous proof in the amenable case. This is that for two continuous pseudometrics ρ and ρ' on X we need both to be dynamically generating to show one inequality. This is because we now have two (sets of) parameters instead of one. With (F, δ) we control to what extent the embeddings of our model spaces resemble partial orbits and with ε we control the scale at which we differentiate between these embeddings. We need ρ' to be dynamically generating if we want embeddings of a model space that are increasingly ρ'-equivariant to eventually become approximately ρ-equivariant. Whereas we need ρ to be dynamically generating if we want two embeddings that are ρ-close to be ρ'-close. Both of these are needed in order to show the inequality

$$h_{\Sigma,2}(\rho) \leq h_{\Sigma,2}(\rho').$$

Suppose that a group G acts on a compact metrizable space X by homeomorphisms and that ρ, ρ' are continuous pseudometrics on X such that ρ is dynamically generating. Recall that if ρ' is not dynamically generating then G acts naturally on $Y = X/\sim$ where $\sim$ is the equivalence relation on X defined by $x \sim y$ if $\rho'(gx, gy) = 0$ for all $g \in G$. Since the action of G on Y is a factor of the action of G on X, one might expect that $h(\rho') \leq h(\rho)$ for any reasonable notion of entropy. As the previous paragraph suggests, this is not the case for sofic entropy, and we finish by demonstrating this fact.

In lecture 3 it is shown that for $G = \mathbb{F}_2$ the shift action of G on the space $\{0,1\}^G$ factors onto the shift on $\{0,1,2,3\}^G$. We show that for any $k \in \mathcal{N}$, for any sofic group G, and for any sofic approximation sequence Σ, the action of G on $X = \{0, \ldots, k-1\}^G$ has entropy $h_\Sigma(X,G) = \log k$. This is done by considering the generating pseudometric ρ defined by

$$\rho(x,y) = \begin{cases} 0, & x(e) = y(e) \\ 1, & x(e) \neq y(e) \end{cases}.$$

References

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Gottschalk's surjunctivity conjecture

MARTINO LUPINI

Suppose that Γ is a countable discrete group and A is a finite set. Denote by A^Γ the set of Γ-sequences of elements of A. The (left) Bernoulli shift of Γ with alphabet A is the action of Γ on the set A^Γ defined by

$$\gamma \cdot (a_\rho)_{\rho \in \Gamma} = (a_{\gamma^{-1}\rho})_{\rho \in \Gamma}.$$

A function $f : A^\Gamma \to A^\Gamma$ is Γ-equivariant if

$$f(\gamma \cdot x) = \gamma \cdot f(x)$$

for every $x \in A^\Gamma$ and $\gamma \in \Gamma$. The following notion has been introduced by Gottschalk in [3]: a countable discrete group Γ is *surjunctive* if for every finite set A every continuous injective equivariant function $f : A^\Gamma \to A^\Gamma$ is surjective. The class of surjunctive groups is closed with respect to taking subgroups and direct unions, see [8, Lemma 1.1 and Lemma 1.2]. To this day no example of a group which is not surjunctive is known. The problem of determining whether every group is surjunctive was suggested by Gottschalk in [3], and it is commonly known as *Gottschalk's surjunctivity problem.*

Gromov showed in [4] that the groups that will be later named *sofic* in [8] are surjunctive. The attempt to find a common generalization of the known proof of surjunctivity for amenable and residually finite groups was in fact one of the main motivations for the introduction of the notion of sofic group in [4, 8]. Later an alternative proof of surjunctivity for sofic groups was provided by Kerr and Li in [6] by means of the notion of topological entropy for actions of a sofic group on a compact Hausdorff space. More precisely they show that the Bernoulli shift of Γ with alphabet A has entropy $\log |A|$ (with respect to any sofic approximation), while any *proper* subshift has strictly smaller entropy [6, Theorem 4.12]. By the conjugation invariance of entropy, this is enough to conclude that Γ is surjunctive. The details of the proof can be found in the original paper [6] as well as in Section II.11 of the survey [2].

The importance of Gottschalk's surjunctivity problem, beside its intrinsic interest, is due to its relation with another long-standing open problem in group theory known as *Kaplansky's direct finiteness conjecture.* Suppose that Γ is a countable discrete group and K is a field. The group algebra $K\Gamma$ is the algebra over K of formal finite linear combinations of elements of Γ with coefficients in K, with pointwise sum and convolution product. The algebra $K\Gamma$ is *directly finite* if every left invertible element is also right invertible or, equivalently, $ba = 1$ whenever a,b are elements of $K\Gamma$ such that $ab = 1$. It is a well known fact that the *complex* group algebra $\mathbb{C}\Gamma$ is directly finite (this can be proved using von Neumann algebras techniques, see for example [1, Theorem 2.1]). It has been conjectured by Kaplansky in [5] that the group algebra $K\Gamma$ is directly finite for every field K. This is in fact equivalent to the assertion that $K\Gamma$ is directly finite for every *finite* field K (this follows from the observation that every field embeds into an ultraproduct of finite fields, cf. [7, Observation 2.1]). The latter reformulation in turn allows one to deduce that every surjunctive group satisfies Kaplansky's direct finiteness conjecture. In fact if Γ is a finite field one can consider the Bernoulli shift $\Gamma \curvearrowright K^\Gamma$ with alphabet K, where the Bernoulli space contains a canonical copy of the group algebra $K\Gamma$. The *right* Bernoulli shift $K^\Gamma \curvearrowleft \Gamma$ defined by

$$(a_\rho)_{\rho \in \Gamma} \cdot \gamma = (a_{\rho\gamma})_{\rho \in \Gamma}$$

lifts by linearity to an action of $K\Gamma$ on K^Γ extending the multiplication in $K\Gamma$ and commuting with the left Bernoulli shift. It is not hard to check now that if a

a right invertible element of $K\Gamma$, then the continuous equivariant function

$$x \mapsto x \cdot a$$

is injective. By surjunctivity of Γ such a function must be surjective, which readily implies that a is also left invertible, concluding the proof that $K\Gamma$ is directly finite.

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Sofic measure entropy
Jianchao Wu

The main goal of the talk was to introduce sofic measure entropy. The concept, a surprising generalization of the Kolmogorov-Sinai entropy for amenable groups, was first invented by Bowen [1]. Later Kerr and Li [3] removed a generator assumption in Bowen's definition by reformulating it with the language of operator algebras, and linked sofic measure entropy with sofic topological entropy by proving the variational principal. Earlier this year, Kerr [2] gave an elementary definition in the spirit of Kolmogorov-Sinai and Bowen. By working with *two* finite partitions playing different roles, he was able to do without any choice of generators, and as a byproduct, obtain an analogue of the Kolmogorov-Sinai theorem. It is this third definition that we focused on for the talk.

The limitation of the Kolmogorov-Sinai entropy lies in its reliance on a Følner sequence in order to compute the average rate at which the entropy increases as a finite partition is refined by joining more and more of its shifts. The need for taking the average is due to the "unequivariant" nature of the core of this definition, namely the entropy formula for a finite partition. To go beyond amenable groups, one needs to replace such a formula with something that has equivariance built into it. For this purpose, we revisit how entropy arises in statistical mechanics.

Consider a large number d of independent finite-valued random variables with the same distribution κ given by weights $p_1, \cdots, p_n$ (positive real numbers such

that $\sum_{i=1}^{n} p_i = 1$). Combined, they amount to a random variable with values in $\Lambda = \{1, \cdots, n\}^d$. The law of large numbers tells us that, for any $\delta > 0$,

$$\mathrm{Map}(\kappa, \delta, d) := \{\lambda \in \Lambda \mid \sum_{i=1}^{n} \left| \frac{\lambda^{-1}(i)}{d} - p_i \right| < \delta\},$$

the set of all outcomes that give δ-good numerical approximations of the probability distribution $(p_1, \cdots, p_n)$, has probability tending to 1 as $d \to \infty$. On the other hand, when one counts the cardinality of this set, one finds:

Proposition. *Let H_κ denote the entropy for the probability distribution κ. Then*

$$\inf_{\delta > 0} \lim_{d \to \infty} \frac{1}{d} \log |Map(\kappa, \delta, d)| = H_\kappa.$$

The proof is a simple estimation argument that employs the multinomial expansion formula and Stirling's approximation formula.

Now let G be a discrete group acting on a probability measure space $(X, \mathcal{B}, \mu)$ by measure-preserving transformations. Since the last proposition relates the entropy for a finite partition to counting the number of maps that satisfy a certain approximation condition, a natural way to weave the group action into it is simply to impose a certain kind of equivariance condition on the maps to be counted. This idea was first applied by Bowen to the case where G is an (infinite) residually finite group and (X, G) has a generating finite partition α, whereby fixing a decreasing sequence of finite-index subgroups $G_i < G$ that tend to the trivial group, one puts $d_i = |G/G_i|$ and, for any finite $F \subset G$, $\alpha_F := \vee_{g \in F} g \cdot \alpha$, and then counts the cardinality of $\mathrm{Map}(\alpha_F, \delta, G/G_i)_F$, comprised of maps $(\lambda : G/G_i \to \alpha_F) \in \mathrm{Map}(\alpha_F, \delta, d_i)$ that are equivariant in the sense that $g \cdot \lambda^{-1}(A) = \lambda^{-1}(g \cdot A)$ for all $g \in F$ and $A \in \alpha$. Then $\inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{d_i} \log |\mathrm{Map}(\alpha_F, \delta, G/G_i)_F|$ gives a suitable notion of "average rate of entropy" for α_F, and taking the infimum over all finite $F \subset G$ yields the entropy of the dynamical system.

For sofic groups, one simply relaxes the equivariance condition to a condition of almost equivariance. From this point on, G denotes a sofic group with a (fixed) sofic approximation $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i=1}^{\infty}$ such that $d_i \to \infty$ as $i \to \infty$. For any finite partition α of X, any finite $F \subset G$ and any $\delta > 0$, we define the set $\mathrm{Map}_\mu(\alpha, F, \delta, \sigma_i)$ to be made up of maps $\lambda : \{1, \cdots, d_i\} \to \alpha_F$ such that

$$(1) \qquad \sum_{A \in \alpha_F} \left| \frac{\lambda^{-1}(A)}{d_i} - \mu(A) \right| < \delta \text{ and}$$

$$(2) \qquad \sum_{A \in \alpha} \frac{\left| \left(\sigma_i(g) \cdot \lambda^{-1}(A) \right) \triangle \left(\lambda^{-1}(g \cdot A) \right) \right|}{d_i} < \delta \text{ for all } g \in F.$$

Note that in [2], the set $\mathrm{Hom}_\mu(\alpha, F, \delta, \sigma_i)$, made up of homomorphisms between the Boolean algebras of α_F and $\{1, \cdots, d_i\}$, is used instead, which, by Stone duality, is equivalent to our presentation.

How does the size of $\mathrm{Map}_\mu(\alpha, F, \delta, \sigma_i)$ change with the variables? It is immediate from the definition that it decreases if one enlarges F or decreases δ. But it is not clear how it changes if one refines α, because on the one hand, α_F, the range of the maps, enlarges, while on the other hand, the conditions for the maps are more strict. Indeed it can go both ways, and this suggests we cannot simply take supremum or infimum over all finite partitions to obtain the correct entropy.

The novel idea in [2] is that one should use two finite partitions, one controlling the range of the maps, and the other implementing the approximation restrictions (1) and (2). More precisely, one takes a pair of finite partitions (ξ, α), with α a refinement of ξ, and then defines an equivalence relation $\sim_\xi$ on $\mathrm{Map}_\mu(\alpha, F, \delta, \sigma)$ by identifying $\lambda, \lambda' : \{1, \cdots, d_i\} \to \alpha_F$ iff $\lambda^{-1}(A) = \lambda'^{-1}(A)$ for any $A \in \xi_F$, i.e. they are indistinguishable as maps $\{1, \cdots, d_i\} \to \xi_F$. Then the cardinality of $\mathrm{Map}_\mu(\alpha, F, \delta, \sigma)/ \sim_\xi$ increases if one refines ξ but decreases if one refines α.

Definition. Given a dynamical system $(X, \mathcal{B}, \mu, G)$, the *sofic measure entropy* $h_{\Sigma,\mu}(X, G)$ with respect to the sofic approximation Σ is defined as

$$(3) \qquad h_{\Sigma,\mu}(X, G) := \sup_\xi \inf_{\alpha, F, \delta} \limsup_{i\to\infty} \frac{1}{d_i} \log \left| \mathrm{Map}_\mu(\alpha, F, \delta, \sigma)/ \sim_\xi \right|$$

where ξ is taken over all finite partitions of X, α over all finite refinement of ξ, F over all finite subsets of G, and δ over all positive numbers.

For a subalgebra $\mathcal{S}$ of the σ-algebra $\mathcal{B}$, one may also define the entropy $h_{\Sigma,\mu}(\mathcal{S}, G)$ by the same formula as above but with the extra requirement that $\xi, \alpha \subset \mathcal{S}$. This allows us to formulate an analogue of Kolmogorov-Sinai theorem which is a powerful tool for computations of entropy. Recall that $\mathcal{S}$ is *generating* for $(X, \mathcal{B}, \mu, G)$ if $\mathcal{S}$ together with all its shifts by elements of G generates $\mathcal{B}$ up to measure-zero sets.

Theorem. *If $\mathcal{S}$ is a generating subalgebra of $\mathcal{B}$, then $h_{\Sigma,\mu}(\mathcal{S}, G) = h_{\Sigma,\mu}(X, G)$.*

Please refer to [2] for a proof, which involves a back and forth argument, common when comparing two generating partitions. As an example, we consider the computation of sofic entropy for a Bernoulli shift, where one can take $\mathcal{S}$ to consist of all cylinder sets over $e \in G$, which is isomorphic to the σ-algebra of $\mathcal{B}$, cf. the succeeding talk "entropy of Bernoulli actions".

Finally we remark that the use of two partitions ξ and α in the definition accounts for and elucidates an important feature of the sofic entropy: it may increase under taking factors of a dynamical system, unlike the case for Kolmogorov-Sinai entropy. In the famous example given by Ornstein and Weiss [4], the Bernoulli 2-shift $(\mathbb{Z}_2^{\mathbb{F}_2}, \mathcal{P}(\mathbb{Z}_2)^{\otimes \mathbb{F}_2}, \nu^{\mathbb{F}_2}, \mathbb{F}_2)$ factors onto the Bernoulli 4-shift $((\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\mathbb{F}_2}, \mathcal{P}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\otimes \mathbb{F}_2}, (\nu \times \nu)^{\mathbb{F}_2}, \mathbb{F}_2)$ (cf. the preceding talk "factors of Bernoulli shifts"). Thus the σ-algebra $\mathcal{P}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\otimes \mathbb{F}_2}$ embeds as a subalgebra of $\mathcal{P}(\mathbb{Z}_2)^{\otimes \mathbb{F}_2}$ via taking preimages. Let ξ be a generating partition of $\mathcal{P}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\otimes \mathbb{F}_2}$. If one also requires α to be in this subalgebra, (3) yields the entropy of the 4-shift, $\log 4$, while if one allows α to run over all measurable finite partitions of the 2-shift, the infimum in (3) makes the entropy *decrease* to $\log 2$, the entropy of the 2-shift.

Had we not distinguished ξ and α but naively tried to take a supremum (respectively, infimum) of $\inf\limits_{F,\delta} \limsup\limits_{i\to\infty} \frac{1}{d_i} \log \left| \mathrm{Map}_\mu(\alpha, F, \delta, \sigma) \right|$ over all finite partitions α, one would get, for this example, the unrevealing $+\infty$ (respectively, 0), useless for the classification of Bernoulli shifts.

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Comparing amenable and sofic entropy

ALBRECHT BREHM

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Throughout, let G be a countable group endowed with the discrete topology and X be a compact metrizable space. Assume that $G \curvearrowright X$ continuously.

In [2] Kerr and Li have proved that the sofic measure entropy and the sofic topological entropy coincide with their classical counterparts. By applying both variation principles, which we know to hold in the sofic and in the amenable setting, we can transport the assertion from the world of measure entropies to the world of topological entropies immediately. So it is enough to prove the assertion in the measurable case. Nevertheless we direct our focus on the topological setup because the technical part is less complicated than in the measurable setup and the essential methods can be seen as well as in the measurable case.

Theorem ([2]). *Let G be an amenable countable discrete group. Then the sofic topological entropy of the group action above coincides with the amenable topological entropy of this action. In particular in this case the sofic topological entropy does not depend on the choice of the sofic approximation sequence.*

The main tool in order to prove this theorem is the so-called Rokhlin lemma.

Theorem (Rokhlin lemma [2]). *Let G be a countable discrete group. Let $0 \leq \tau < 1$, and $0 < \eta < 1$. Then there are an $l \in \mathbb{N}$ and η', η'', such that whenever $e \in F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l$ are finite subsets of G with $|(F_{k-1}^{-1} F_k) \setminus F_k| \leq \eta'|F_k|$ for $k = 2, \ldots, l$, there exists a finite set $F \subseteq G$ containing e such that for every $d \in \mathbb{N}$, every map $\sigma \colon G \to \mathrm{Sym}(d)$ with a set $B \subseteq \{1, \ldots, d\}$ satisfying $|B| \geq (1 - \eta'')d$ and*

$$\sigma_{st}(a) = \sigma_s \sigma_t(a), \quad \sigma_s(a) \neq \sigma_{s'}(a)$$

for all $a \in B$ and $s, t, s' \in F$ with $s \neq s'$, and any set $V \subseteq \{1, \ldots, d\}$ with $|V| \geq (1 - \tau)d$, there exist $C_1, \ldots, C_l \subseteq V$ such that

(1) *for every $k = 1, \ldots, l$ and $c \in C_k$ the map $s \mapsto \sigma_s(c)$ from F_k to $\sigma(F_k)c$ is bijective,*

(2) *the sets $\sigma(F_1)C_1, \ldots, \sigma(F_l)C_l$ are pairwise disjoint and the family $\bigcup_{k=1}^{l} \{\sigma(F_k)c : c \in C_k\}$ is η-disjoint and $(1 - \tau - \eta)$-covers $\{1, \ldots, d\}$.*

For the definitions of the notions of δ-disjointness and δ-covering we refer to [2] section 4. In the context above you can think of almost disjointness and almost covering.

When G is amenable the Rokhlin lemma enables us to decompose the finite set $\{1, \ldots, d\}$ for a good enough sofic approximation into partial orbits of Følner sets of G in an approximate way. If we remember in the definitions of the pseudometrics

$$\varrho_\infty(\varphi, \psi) = \max_{a=1}^{d} \varrho(\varphi(a), \psi(a))$$

and

$$\varrho_F(x, y) = \max_{s \in F}(x, y) = \max_{s \in F} \varrho(sx, sy)$$

we can see that the restriction of an almost equivariant map on such an orbit of $\{1, \ldots, d\}$ can be understood as an orbit of a Følner set in the topological space X. By recalling the definitions of sofic topological entropy and the metric definition of the classical topological entropy we obtain that the sofic topological entropy is at most the amenable topological entropy.

For the converse direction it suffices to construct an ε'-separated subset of the almost equivariant maps from a (ϱ_F, ε)-separated subset of X for some Følner set F. Therefore we want to construct almost equivariant maps with values in a (ϱ_F, ε)-separated subset of X. Let us remark that we have taken for simplification only one set F but this is not precise. The construction is best done by taking an almost decomposition of the set $\{1, \ldots, d\}$ into genuinely disjoint partial orbits of Følner sets for a sufficiently good sofic approximation and defining a map on the generators of these partial orbits. That the resulting map is well-defined is guaranteed by the genuine disjointness of the partial orbits. This is the reason why we have to use a slightly modified version of the Rokhlin lemma which holds for an amenable group action and can be found in [2] Corollary 4.6. The map which we obtain by this procedure can be arbitrarily continued on the remaining part by adding a negligible error and we are done.

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Entropy of Bernoulli actions

PAVLO MISHCHENKO

Following [2], we present computations of the entropy of a Bernoulli action $G \curvearrowright (Y, \nu)^G$, where G is a countable sofic group, (Y, ν) is a probability space. We show that the sofic entropy of a Bernoulli action is equal to the Shannon entropy of the base space (Y, ν). As a corollary we have that it does not depend on the sofic approximation of G. The concept of sofic measure entropy was defined in Lecture 6, and we systematically use definitions from that lecture. Shannon entropy is defined as follows: if there is a finite or countable set $Y' \subset Y$ such that $\nu(Y') = 1$ then

$$H(\nu) = - \sum_{y \in Y'} \nu(\{y\}) \log \nu(\{y\}),$$

otherwise, $H(\nu) = +\infty$. The proof of the statement follows the section 4 of the paper [2]. We begin with establishing the following basic property of the sofic entropy (Lemma 4.1. from [2]):

Let (X, μ) be a probability space and $G \curvearrowright X$ a measure-preserving action. Let ξ and α be finite measurable partitions of X satisfying $\alpha \geq \xi$. Then we have

$$(1) \qquad H_\mu(\xi) \geq h_{\Sigma,\mu}^\xi(\alpha) \geq h_{\Sigma,\mu}^\alpha(\alpha) - H_\mu(\alpha|\xi).$$

The proof of both inequalities is rather technical – roughly speaking, we apply a combinatorial argument for counting the number of homomorphisms arising in the definition of the sofic entropy. Then we use Stirling's formula and continuity property of $H(\cdot)$ to finish the proof. Next, the main result is established:

$$h_{\Sigma,\nu^G}(Y^G, G) = H(\nu).$$

We use the following property of the Shannon entropy: $H(\nu)$ is equal to the supremum of $H_\nu(\alpha)$ over all measurable partitions α of Y. Define C to be the σ-algebra consisting of those measurable subsets of Y^G which are cylinder sets over e. Then C is generating. In Lecture 6 it was shown that in this case sofic entropy of the action is equal to $h_{\Sigma,\nu^G}(C)$. The inequality $h_{\Sigma,\nu^G}(C) \leq H(\nu)$ follows from (1). Again, by (1) we see that it is sufficient to prove $h_{\Sigma,\mu}^\alpha(\alpha) \geq H_{\nu^G}(\alpha)$. The proof of this result is essentially contained in [1]. We construct the family of homomorphisms φ_γ, parameterized by $\gamma \in \{1, ..., n\}^d$, which automatically satisfies the first condition on φ in the definition of sofic entropy ($\alpha = \{A_1, ..., A_n\}$, d stands for the size of symmetric group to which G is mapped by some approximation map). Then the measure κ on $\{1, ..., n\}$ is defined by $\kappa(\{i\}) = \nu^G(A_i)$, and we view $\{1, ..., n\}^d$ as a probability space with the product measure κ^d. After this definition we prove that the constructed homomorphisms satisfy the second condition on φ with a high probability. We estimate their number using the law of large numbers, establishing the desired inequality by definition of $h_{\Sigma,\nu^G}(C)$.

One important corollary is: if $H(\nu)$ is infinite, then the action $G \curvearrowright (Y, \nu)^G$ has no finite generating partition.

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Isomorphism of Bernoulli shifts

Ben Hayes

Let Γ be a countable discrete group. We know from [1] that if Γ is sofic, if $(X, \mu), (Y, \nu)$ are standard probability spaces, and the Bernoulli actions $\Gamma \curvearrowright (X, \mu)^\Gamma, \Gamma \curvearrowright (Y, \nu)^\Gamma$ are isomorphic, then the entropy of (X, μ) is the entropy of (Y, ν). A natural question is if the converse is true: i.e. if (X, μ) and (Y, ν) have equal entropy are the Bernoulli shifts $\Gamma \curvearrowright (X, \mu)^\Gamma, \Gamma \curvearrowright (Y, \nu)^\Gamma$ isomorphic? We do not need an invariant such as entropy to show that two actions *are* isomorphic, so it is reasonable to ask whether the converse is true for any group Γ, or to what extent it is true for arbitrary Γ. Therefore soficity will play no role in this talk.

Ornstein proved the converse for $\Gamma = \mathbb{Z}$ in 1970 in [3],[4]. Because of this we shall make the following definition.

Definition. A countable discrete group Γ is said to be *Ornstein* if whenever $(X, \mu), (Y, \nu)$ are standard probability spaces with equal entropy, then the Bernoulli shifts $\Gamma \curvearrowright (X, \mu)^\Gamma, \Gamma \curvearrowright (Y, \nu)^\Gamma$ are isomorphic.

Ornstein's Theorem was later generalized by Ornstein and Weiss in [5] to show that every infinite amenable group is Ornstein. Essential to their proof is the Rohklin Lemma. As the Rohklin Lemma is something very special to amenable groups, it is not clear how to generalize their approach to larger classes of groups.

However, Stepin in [6] made the following observation: if Γ contains an Ornstein subgroup, then Γ is Ornstein. This can be rephrased in modern language in terms of coinduction.

If $\Lambda \subseteq \Gamma$ are countable discrete groups, and $\Lambda \curvearrowright (X, \mu)$ is probability measure preserving, we can define a new space

$$Y = \{f \colon \Gamma \to X : f(\lambda g) = \lambda^{-1} f(g), \lambda \in \Lambda, g \in \Gamma\}.$$

Choosing coset representatives gives an isomorphism $Y \cong X^{\Gamma/\Lambda}$, and it is easy to see that the pushforward of $\mu^{\Gamma/\Lambda}$, denoted ν, does not depend on the choice of coset representatives. Additionally Γ acts on (Y, ν) by measure-preserving transformations by

$$(gf)(x) = f(g^{-1}x).$$

We call $\Gamma \curvearrowright (Y, \nu)$ the coinduction of $\Lambda \curvearrowright (X, \mu)$. The coinduction of $\Lambda \curvearrowright (X, \mu)^\Lambda$ is $\Gamma \curvearrowright (X, \mu)^\Gamma$. However, it is important to realize that the proof of this fact relies on two structures of a Bernoulli action: we use the subgroup structure of Λ to construct such an isomorphism, but we need a system of coset representatives in order to check that the isomorphism is actually measure preserving.

Lewis Bowen in [2] showed that, ignoring two atom spaces, every group is Ornstein. We state this precisely as follows.

Definition. A countable discrete group Γ is *almost Ornstein* if whenever (X, μ), (Y, ν) are standard probability spaces neither of which is a two atom space, and the entropy of (X, μ) is the entropy of (Y, ν) then $\Gamma \curvearrowright (X, \mu)^{\Gamma}$ is isomorphic to $\Gamma \curvearrowright (Y, \nu)^{\Gamma}$.

Then the theorem of Bowen is the following:

Theorem. *Every countable discrete infinite group is almost Ornstein.*

The proof follows the coinduction machinery as much as one can. First we find a measure space (L, λ) which is a factor of both $(X, \mu), (Y, \nu)$ and which does not have zero entropy (for technical reasons, this is not always possible but it is possible often enough to still prove the theorem). Using standard facts in orbit equivalence, we can find an ergodic U in the full group of $\Gamma \curvearrowright (L, \lambda)^{\Gamma}$ this will be our ersatz "subgroup" isomorphic to $\mathbb{Z}$. Using the factor maps

$$(X, \mu)^{\Gamma} \to (L, \lambda)^{\Gamma}$$

$$(Y, \nu)^{\Gamma} \to (L, \lambda)^{\Gamma}$$

we will build actions of $\mathbb{Z}$ on $(X, \mu)^{\Gamma}(Y, \nu)^{\Gamma}$ which are "Bernoulli relative to U" (defined precisely in [2]). This will be the Bernoulli-type structure we need to copy the proof of the coinduction of a Bernoulli shift is Bernoulli. We will apply a result of Thouvenot in [7] to get an isomorphism of these spaces which respects the factor U. From here, the proof gets more complicated as one needs to use disintegration of measure and the fact that U is the full group of Γ to build an analogue of coset representatives.

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The variational principle

VADIM ALEKSEEV

The variational principle for continuous actions of a group Γ on a compact metrizable space X asserts that the topological entropy of such an action is equal to the supremum of the measure entropies of invariant measures:

$$h(\Gamma \curvearrowright X) = \sup_{\mu \in M(X)^\Gamma} h_\mu(\Gamma \curvearrowright (X, \mu)).$$

Intuitively, it means that some invariant measures might not quite detect topological complexity of the dynamics, but if the measure is concentrated on the subsets where the dynamics is sufficiently complicated, it can completely detect topological complexity (measured in terms of entropy). The variational principle for actions of $\mathbb{Z}$ was proved by Dinaburg [1, 2] and Goodman [3]. In the talk we presented the variational principle for actions of sofic groups based on the exposition in [4], but adapting the proof to the definitions of sofic entropy used in previous talks. The inequality $h_\mu(\Gamma \curvearrowright (X, \mu)) \leqslant h(\Gamma \curvearrowright X)$ is quite straightforward, and the strategy to prove equality is to construct measures with sufficiently large entropy. In the case of integer actions this is done by starting with Dirac measures along a piece of the orbit, moving them around and taking weak* limits, but in the sofic case the technique is slightly different: one just picks measures which approximate equidistributed measures along models of the orbit used in the definition of topological entropy out of an "almost dense" finite set in the space of probability measures. The fact that counting these models gives topological entropy is used to show that their measure entropy is large, and equidistribution along models of the orbit yields that they are almost invariant. The weak* limit point of such measures then necessarily approximates the topological entropy on a given scale.

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Entropy of principal algebraic actions

BINGBING LIANG

The computation of entropy for principal algebraic actions in terms of periodic points has been well-developed in the amenable case ([10], [9], [2], [3], [7], [8]), and it is still true when passing to the sofic case ([1] [4]). Here one would use the newest definition of sofic entropies in ([5], [6]) to compute the sofic entropy of a

principal algebraic action by applying the variational principle for sofic actions in [4] .

Let G be a countable residually finite group. Then G admits a sequence of finite index normal subgroups $\{G_n\}_{n\in\mathbb{N}}$ with $\limsup_{n\to\infty} G_n = \{e\}$. This sequence induces a sofic approximation sequence by left translation: $\sum = \{\sigma_n : G \to \mathrm{Sym}(G/G_n)\}_{n\in\mathbb{N}}$. Denote by $\mathcal{L}G$ the left group von Neumann algebra of G. The Fuglede-Kadison determinant of an invertible element f of $\mathcal{L}G$ is defined by

$$\det\nolimits_{\mathcal{L}G} f = \exp \mathrm{tr}_{\mathcal{L}G}\log(f^*f)^{1/2},$$

where the canonical trace $\mathrm{tr}_{\mathcal{L}G}$ on $\mathcal{L}G$ is given by $\mathrm{tr}_{\mathcal{L}G} f =< f\delta_e, \delta_e >$.

Given $f \in \mathbb{Z}G$, the $\mathbb{Z}G$ module structure of $\mathbb{Z}G/\mathbb{Z}Gf$ induces an action of G on the Pontryagin dual $X_f := \widehat{\mathbb{Z}G/\mathbb{Z}Gf} = \{x \in (\mathbb{R}/\mathbb{Z})^G : fx = 0\}$, which is called a *principal algebraic action*. It admits a dynamically generating continuous pseudometric ρ on X_f defined by

$$\rho(x, y) = \theta(x_e, y_e), \ x, y \in (\mathbb{R}/\mathbb{Z})^G,$$

where $\theta(s + \mathbb{Z}, t + \mathbb{Z}) = \min_{m\in\mathbb{Z}}|s - t - m|$.

Denote by $C^*(G)$ the full group C^*-algebra of G. Combining Theorem 7.3 in [4] with Theorem 3.2 in [5], one has an approximation formula for the Fuglede-Kadison determinant in terms of the fixed points of G_n in X_f.

Lemma. *If $f \in \mathbb{Z}G$ is invertible in $C^*(G)$, then*

$$\log \det\nolimits_{\mathcal{L}G} f = \lim_{n\to\infty} \frac{1}{|G/G_n|}\log|\mathrm{Fix}_{G_n}(X_f)|.$$

Given a continuous sofic action of G on a compact metrizable space X, the sofic version of the variational principle is as follows ([4], Theorem 6.1):

Theorem. *Given a sofic action $\alpha : G \curvearrowright X$ and a sofic approximation $\Sigma = \{\sigma_n : G \to \mathrm{Sym}(d_n)\}_{n\in\mathbb{N}}$, then*

$$h_\Sigma(X, G) = \sup_{\mu\in M_G(X)} h_{\Sigma,\mu}(X, G).$$

With the help of the above two results, one can compute the sofic entropy of a principal algebraic action ([4], Theorem 7.1).

Theorem. *If $f \in \mathbb{Z}G$ is invertible in $C^*(G)$, then*

$$h_\Sigma(X_f, G) = \log \det\nolimits_{\mathcal{L}G} f.$$

Applying the above approximation formula and variational principle, it reduces to showing the following two inequalities (using Definition 2.2 in [5] and Definition 3.2 in [6]):

1) $h_{\Sigma,\infty}(X_f, G) \geq \lim_{n\to\infty} \frac{1}{|G/G_n|}\log|\mathrm{Fix}_{G_n}(X_f)|$, where

$$h_{\Sigma,\infty}(X, G) = \sup_{\epsilon>0} \inf_{F,\delta} \limsup_{n\to\infty} \frac{1}{d_n} N_\epsilon(\mathrm{Map}(\rho, F, \delta, \sigma_n), \rho_\infty),$$

$$\mathrm{Map}(\rho, F, \delta, \sigma_n) = \{\varphi : \{1, 2, \ldots, d_n\} \to X|\ \rho_2(\varphi\sigma_s, \alpha_s\varphi) < \delta, \forall s \in F\};$$

2) $\forall\, \mu \in M_G(X_f)$,

$$h_{\Sigma,\mu,2}(X_f, G) \leq \lim_{n\to\infty} \frac{1}{|G/G_n|}\log|\mathrm{Fix}_{G_n}(X_f)|,$$

where

$$h_{\Sigma,\mu,2}(X, G) = \sup_{\epsilon>0}\ \inf_{F,L,\delta}\ \limsup_{n\to\infty} \frac{1}{d_n}N_\epsilon(\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma_n), \rho_2),$$

$\mathrm{Map}_\mu(\rho, F, L, \delta, \sigma_n) =$
$\{\varphi : \{1, 2, \ldots, d_n\} \to X|\ \rho_2(\varphi\sigma_s, \alpha_s\varphi) < \delta, \forall s \in F \text{ and } |\varphi_*\zeta(g) - \mu(g)| < \delta, \forall g \in L\},$

$L \subseteq C(X)$ is a finite subset.

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Bernoulli shifts over free groups of finite rank and the f-invariant
Zoran Šunić

In the first section we present the definition, due to L. Bowen [1], of the f-invariant of an action of a free group of finite rank on a probability space by measure preserving transformations. In the second section we indicate how this invariant was used by Bowen to provide a negative answer to the question of Ornstein and Weiss [2], who asked if all Bernoulli shifts over a countable nonamenable group are measurably conjugate. Implicit in Bowen's work is a new characterization of free groups of finite rank, which we bring to view in the last section.

1. Definition of the f-invariant for free groups of finite rank

Let $G = F(s_1, \ldots, s_r)$ be a free group of rank r, $r \geq 1$, acting on the probability space (X, μ) by measure preserving transformations. A finite partition of X is measurable if each member of the partition is measurable. Consider the space $\mathcal{P}_g$ of all finite, measurable, generating partitions of X, up to measure 0 (two partitions α and β are identified if for every $A \in \alpha$ there exists $B \in \beta$ such that $\mu(A \triangle B) = 0$, where $A \triangle B$ is the symmetric difference of A and B). A partition α is generating if the smallest G-invariant σ-algebra containing α is, up to sets of measure 0, the σ-algebra of measurable sets in X. The partition α refines the partition β if for every $A \in \alpha$ there exists $B \in \beta$ such that $\mu(A \setminus B) = 0$. For two partitions α and β let $\alpha \vee \beta$ be the smallest partition that refines both α and β, i.e., $\alpha \vee \beta = \{A \cap B \mid A \in \alpha, \ B \in \beta\}$ and, for any finite subset $F \subseteq G$, let $\alpha^F = \bigvee_{f \in F} f\alpha$. In particular, when $F = B_e(n)$ is the closed ball of radius n centered at e in G, we write $\alpha^n = \bigvee_{g \in B_e(n)} g\alpha$. For a partition $\alpha \in \mathcal{P}_g$, define

$$H(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A),$$

$$F(\alpha) = (1 - 2r)H(\alpha) + \sum_{i=1}^{r} H(\alpha \vee s_i \alpha),$$

$$f(\alpha) = \inf_{n \geq 0} F(\alpha^n).$$

Theorem 1.1 ([1]). *If α and β are two finite, measurable, generating partitions in $\mathcal{P}_g$, then $f(\alpha) = f(\beta)$.*

Therefore $f(\alpha)$ is a measure conjugacy invariant of the measure preserving action $G \curvearrowright (X, \mu)$. We denote it by $f(G, X, \mu)$, and call it the f-invariant of the triple (G, X, μ). Note that, when $r = 1$, $f(G, X, \mu)$ is the Kolmogorov entropy.

The proof of Theorem 1.1 is based on the following two statements

(1) f is constant on the subspace $\mathcal{P}_{EQ(\alpha)}$, and
(2) $\mathcal{P}_{EQ(\alpha)}$ is dense in $\mathcal{P}_g$,

where $\mathcal{P}_{EQ(\alpha)}$ is the space of partitions that are combinatorially equivalent to α (β is combinatorially equivalent to α if and only if there exists nonnegative integers m and ℓ such that α^m refines β and β^ℓ refines α), and the density is considered with respect to the Rokhlin metric on $\mathcal{P}_g$, given by $d(\alpha, \beta) = 2H(\alpha \vee \beta) - H(\alpha) - H(\beta)$.

2. Bernoulli shifts over free groups of finite rank

Given an action of the free group $G = F(s_1, \ldots, s_r)$ of rank r, $r \geq 1$, on a standard probability space (Y, ν) by measure preserving transformations, an action of G on the space $(X, \mu) = (Y^G, \nu^G)$ with product Borel structure by measure preserving transformations is defined by $g(x_h) = x_{g^{-1}h}$, for $g, h \in G$ and $x \in Y^G$. The triple (G, Y^G, ν^G) is called the Bernoulli shift over G with base space (Y, ν).

Theorem 2.1 ([1]). *If the free group G of rank r acts on a finite probability space (Y, ν) by measure preserving transformations, then the f-invariant of the Bernoulli shift (G, Y^G, ν^G) is equal to the entropy of the base space, i.e.*

$$f(G, Y^G, \nu^G) = H(Y, \nu) = -\sum_{y \in Y} \mu(y) \log \mu(y).$$

Sketch of the proof. Consider the finite, measurable, generating partition $\alpha = \{A_y \mid y \in Y\}$ of $X = Y^G$, where $A_y = \{x \in Y^G \mid x_e = y\}$, for $y \in Y$, is the set of functions $x : G \to Y$ that have value y at the identity. Note that $H(\alpha) = H(Y, \nu)$. The Bernoulli independence condition implies that, for distinct elements $g_1, \ldots, g_m \in G$, $H(\bigvee_{i=1}^{m} g_i \alpha) = \sum_{i=1}^{m} H(g_i \alpha) = m H(\alpha)$. Therefore, for all $n \geq 0$,

$$F(\alpha^n) = (1 - 2r)H(\alpha^n) + \frac{1}{2}\sum_{s \in S^{\pm}} H(\alpha^k \cup s\alpha^k) =$$

$$= \left((1 - 2r)|B_e(n)| + \frac{1}{2}\sum_{s \in S^{\pm}} |B_e(n) \cup B_s(n)| \right) H(\alpha),$$

where $B_e(n)$ and $B_s(n)$ are the closed balls of radius n in G, centered at the identity and at s, respectively, with respect to the generating set $S = \{s_1, \ldots, s_r\}$. However, in the free group G of rank r, the expression $(1 - 2r)|B_e(n)| + \frac{1}{2}\sum_{s \in S^{\pm}} |B_e(n) \cup B_s(n)|$ is equal to 1, for all $n \geq 0$, and it follows that $F(\alpha^n) = H(\alpha) = H(Y, \nu)$. $\square$

As a corollary, Bowen obtains the following (negative) answer to the question of Ornstein and Weiss [2] (note that the backward direction of this result was already known and it is due to Stepin [3].)

Theorem 2.2 ([1]). *Let (Y_1, ν_1) and (Y_2, ν_2) be standard probability spaces with finite entropies and let the free group G of finite rank act on each by measure preserving transformations. Then the Bernoulli shifts (G, Y_1^G, ν_1^G) and (G, Y_2^G, ν_2^G) are measurably conjugate if and only if the base entropies $H(Y_1, \nu_1)$ and $H(Y_2, \nu_2)$ are equal.*

3. Why not other groups?

Even though perfectly aware that the definition makes sense for actions of any finitely generated group $G = \langle s_1, \ldots, s_r \rangle$ with generating set of size r, Bowen was careful to define the f-invariant only for actions of free groups. One reason is that if G is infinite and not free over $S = \{s_1, \ldots, s_r\}$, the f-invariant of the Bernoulli shift (G, Y^G, ν^G) is negative infinity, regardless of the action and, as long as ν is not supported on a single point, regardless of the finite base space (Y, ν). This claim follows immediately from the sketch of the proof of Theorem 2.1 and the following observation (which characterizes free groups of rank r).

Proposition 3.1. *Let G be an infinite group, $S = \{s_1, \ldots, s_r\} \subseteq G$ and $G = \langle s_1, \ldots, s_r \rangle$. The sequence $\{a_n\}_{n=0}^{\infty}$, given by*

$$a_n = (1 - 2r)|B_e(n)| + \frac{1}{2} \sum_{s \in S^{\pm}} |B_e(n) \cup B_s(n)|$$

is bounded below if and only if G is free of rank r.

Proof. Assume that the sequence is bounded below by b. Then, for $n \geq 1$,

$$b \leq a_n = (1 - 2r)|B_e(n)| + \frac{1}{2} \sum_{s \in S^{\pm}} |B_e(n) \cup B_s(n)| =$$

$$= |B_e(n)| + \frac{1}{2} \sum_{s \in S^{\pm}} \Big(|B_e(n) \cup B_s(n)| - 2|B_e(n)| \Big) =$$

$$= \gamma(n) - \frac{1}{2} \sum_{s \in S^{\pm}} |B_e(n) \cap B_s(n)| =$$

$$= \gamma(n) - \frac{1}{2} \sum_{s \in S^{\pm}} |B_e(n-1) \cup (S_e(n) \cap B_s(n))| =$$

$$= \gamma(n) - r\gamma(n-1) - \frac{1}{2} \sum_{s \in S^{\pm}} |S_e(n) \cap B_s(n)| \leq$$

$$\leq \gamma(n) - r\gamma(n-1) - \frac{1}{2}\sigma(n) =$$

$$= \frac{1}{2}\Big(\gamma(n) - (2r - 1)\gamma(n-1) \Big),$$

where $S_e(n)$ is the sphere of radius n centered at the identity, $\gamma(n) = |B_e(n)|$ is the size of the closed ball of radius n, and $\sigma(n) = |S_e(n)|$ is the size of the sphere of radius n in G, with respect to the generating set S. Therefore, for $n \geq 1$,

$$\frac{2b}{\gamma(n-1)} + (2r - 1) \leq \frac{\gamma(n)}{\gamma(n-1)},$$

and by passing to the limit (note that $\gamma(n-1)$ is unbounded, since G is infinite)

$$\lim_{n \to \infty} \frac{\gamma(n)}{\gamma(n-1)} \geq 2r - 1.$$

This shows that the growth exponent of G is $2r-1$ and, for a group with generating set of size r, this is only possible for the free group of rank r. $\qquad\square$

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Markov chains over free groups

Phu Chung

Let $G = \langle s_1, ..., s_r \rangle$ be the free group on r generators. Let α be an action of G on a probability space $(X, \mathcal{B}, \mu)$ by measure preserving transformations. For $P \in \mathcal{B}$ and a sub-σ-algebra $\mathcal{F} \subset \mathcal{B}$, let $E(P|\mathcal{F})$ be the conditional expectation of the characteristic function χ_P of P with respect to $\mathcal{F}$. For a measurable partition $\mathcal{P}$ of X, the entropy of $\mathcal{P}$ conditioned on $\mathcal{F}$ is

$$H(\mathcal{P}|\mathcal{F}) := \int_X -\log(E(P_x|\mathcal{F}))d\mu(x),$$

where P_x is the atom of $\mathcal{P}$ containing x. We define

$$F(\alpha, \mathcal{P}|\mathcal{F}) := (1 - 2r)H_\mu(\mathcal{P}|\mathcal{F}) + \sum_{i=1}^{r} H_\mu(\mathcal{P} \vee s_i^{-1}\mathcal{P}|\mathcal{F} \vee s_i^{-1}\mathcal{F}),$$

$$f(\alpha, \mathcal{P}) := \inf_{n \in \mathbb{N}} F(\alpha, \bigvee_{s \in B(e,n)} s^{-1}\mathcal{P}| \bigvee_{s \in B(e,n)} s^{-1}\mathcal{F}),$$

where $B(e, n)$ is the ball of radius n centered at the identity element.

A partition $\mathcal{P}$ is called a generating partition if the smallest G-invariant σ-algebra containing $\mathcal{P}$ equals $\mathcal{B}$ up to sets of measure zero. Then

Theorem. [1] *Let α be an action of G on a probability space $(X, \mathcal{B}, \mu)$ by measure preserving transformations. If $\mathcal{P}$ and $\mathcal{Q}$ are generating partitions with $H(\mathcal{P}) + H(\mathcal{Q}) < \infty$ and $\mathcal{F} \subset \mathcal{B}$ is any α-invariant σ-algebra then $f(\alpha, \mathcal{P}|\mathcal{F}) = f(\alpha, \mathcal{Q}|\mathcal{F})$ and this common quantity is called the relative f-invariant with respect to $\mathcal{F}$ of the action.*

Definition. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions. We say that $\mathcal{Q}$ is a coarsening of $\mathcal{P}$ if for every atom $P \in \mathcal{P}$ there exists an atom $Q \in \mathcal{Q}$ such that $\mu(P \setminus Q) = 0$.

Now we investigate Markov processes of systems. Put $S = \{s_1, ..., s_r, s_1^{-1}, ..., s_r^{-1}\}$. The left Cayley graph G_L is defined as follows. Its vertex set is G and for every $g \in G$ and $s \in S$ there is a directed edge from g to sg, there are no other edges.

A *G-process* is a quadruple $(G, X, \mu, \mathcal{P})$, where $\mathcal{P}$ is a partition of X.

Definition. For all $g_1, g_2 \in G$ we denote by $\text{Past}(g_1; g_2) \subset G$ the set of all $g \in G$ such that every path in the left Cayley graph G_L from g to g_1 passes through g_2.

Definition. A *G*-process $(G, X, \mu, \mathcal{P})$ is a *Markov process* if for every $s \in S, g \in G$ and every $P \in \mathcal{P}$,

$$E\big((sg)^{-1}P| \bigvee_{f \in \text{Past}(sg;g)} f^{-1}\mathcal{P}\big)(x) = E\big((sg)^{-1}P|g^{-1}\mathcal{P}\big)(x),$$

for μ-almost every $x \in X$.

Then we can calculate explicitly the relative f-invariant if the process is Markov.

Theorem. *If $(G, X, \mu, \mathcal{P})$ is a Markov process and $\mathcal{Q}$ is a coarsening of $\mathcal{P}$ then $f(\mathcal{P}|\mathcal{Q}^G) = F(\mathcal{P}|\mathcal{Q}^G)$ where $\mathcal{Q}^G$ is the smallest G-invariant σ-algebra containing $\mathcal{Q}$.*

The standard examples of Markov processes are from transition matrices and symbolic dynamics.

Definition. Let K be a finite or countable infinite set. An invariant transition system for (G, S) is a collection of $K \times K$ matrices $\{P^s\}_{s \in S}$ and a $1 \times K$ vector π satisfying

    (1) $0 \le P_{ij}^s \le 1$ for all i, j, s;
    (2) For each $i, s, \sum_{j \in K} P_{ij}^s = 1$;
    (3) $\pi P^s = \pi$ for all $s \in S$;
    (4) For all $s \in S, i, j \in K, \pi_i P_{ij}^{s^{-1}} = \pi_j P_{ji}^s$.

Definition. For any measure μ and any Borel sets $A, B \subset X$ with $\mu(B) > 0$ define $\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}$.

The canonical action α of G on K^G is defined by $\alpha_g x(f) = x(fg)$ for all $x \in K^G, f, g \in G$. For each $k \in K$, let $A_k = \{y \in K^G : y(e) = k\}$ then $\mathcal{P} = \{A_k | k \in K\}$ is the canonical partition of K^G.

Let $(\{P^s\}_{s \in S}, \pi)$ be an invariant transition system and μ be the probability measure on K^G satisfying

    (1) for all $k \in K, \mu(A_k) = \pi_k$;
    (2) let $g \in G$ and $s \in S$ be such that $|sg| = |g| + 1$. Let $f_1, ..., f_n \in \mathrm{Past}(sg; g) \setminus \{g\}$. Then for any $k, k_0, ..., k_n \in K$,

$$\mu\big((sg)^{-1}A_k|g^{-1}A_{k_0} \cap \bigcap_{i=1}^n f_i^{-1}A_{k_i}\big) = \mu\big((sg)^{-1}A_k|g^{-1}A_{k_0}\big) = P_{k_0,k}^s.$$

Then the process $(\alpha, K^G, \mu, \mathcal{P})$ induced by an invariant transition system $(\{P^s\}_{s \in S}, \pi)$ is Markov and its f-invariant is

$$(2r - 1) \sum_{i \in K} \pi_i \log(\pi_i) - \sum_{s \in S_+} \sum_{i,j \in K} \pi_i P_{ij}^s \log(\pi_i P_{ij}^s),$$

where $S_+ = \{s_1, ..., s_r\}$.

Three examples of Markov chains over free groups, one related to Wired Spanning Forest, to perfect matchings, and the last one with negative f-invariant are illustrated.

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Combinatorial independence and sofic entropy

Dominik Kwietniak

The main aim of this lecture was to explain how positive topological entropy can be characterized in terms of combinatorial independence via a positive density condition.

Inspired by results of the local theory of entropy (see [1]) Kerr and Li developed in [2, 3] a general theory of independence in dynamics of actions of amenable groups. Their study revealed a deep connection between independence, entropy and weak mixing. Emerging theory of entropy of sofic group actions motivates the question of the meaning of independence in this broader context. Kerr and Li in [4] extended the notion of independence to the framework of actions of sofic groups. They define Σ-IE-tuples for a continuous action $G \curvearrowright X$ of a countable discrete sofic group G on a compact Hausdorff space X and a fixed sofic approximation net $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$. To achieve this they externalize the positive independence density condition, which is easy to define in the amenable case, to the finite sets $\{1, \ldots, d_i\}$ which are targets of sofic approximation maps Σ_i. Such defined Σ-IE-tuples have similar properties to IE-tuples for actions of discrete countable amenable groups defined in [2]. In particular, the topological entropy $h_\Sigma(X, G)$ of (X, G) with respect to Σ is positive if and only if there is a nondiagonal IE-pair. Moreover, in the amenable case, Σ-IE-tuples are the same thing as IE-tuples.

Below we only sketch some results of Kerr and Li. Note also that we are not able to introduce and explain all the notation we use. For the proofs and details we refer the reader to [4].

1. Orbit IE-tuples

Let G be a discrete group. Assume that $G \curvearrowright X$ is a continuous G-action on a compact Hausdorff space X. One can define a notion of independence in this very general setting.

Definition 1.1. A set $F \subseteq G$ is an *independence set* for a k-tuple $\mathbf{A} = (A_1, \ldots, A_k)$ of subsets of X (that is, $A_j \subseteq X$ for $j = 1, \ldots, k$) if for every finite set $J \subseteq F$ and every function $\omega : J \to \{1, \ldots, k\}$ we have

$$\bigcap_{s \in J} s^{-1} A_{\omega(s)} \neq \emptyset.$$

We denote the family of all independence sets of a tuple $\mathbf{A} = (A_1, \ldots, A_k)$ by $\mathrm{Ind}(\mathbf{A})$.

Definition 1.2. The *independence density* of $\mathbf{A} = (A_1, \ldots, A_k)$ (over G) is the number

$\sup\{q \geq 0 : \text{for any finite } F \subseteq G \text{ there is } J \subseteq F \text{ s.t. } J \in \mathrm{Ind}(\mathbf{A}) \text{ and } |J| \geq q|F|\}.$

Definition 1.3. We say that a k-tuple $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is an *orbit IE-tuple* (of length k) if for every basic open neighborhood $U_1 \times \ldots \times U_k$ of $\mathbf{x}$ in X^k the k-tuple $\mathbf{U} = (U_1, \ldots, U_k)$ has positive independence density. We write $\mathrm{IE}_k(X, G)$

to denote the set of orbit IE-tuples of length k, and we call elements of $\mathrm{IE}_2(X, G)$ *orbit IE-pairs*.

Remark 1.4. Let X and Y be nonempty sets. Throughout this note we will often identify $((x_1, \ldots, x_k), (y_1, \ldots, y_k)) \in X^k \times Y^k$ with $((x_1, y_1), \ldots, (x_k, y_k)) \in (X \times Y)^k$.

Theorem 1.5 (Theorem 3.3. of [4]). *Let $k \in \mathbb{N}$. If G acts continuously on compact Hausdorff spaces X and Y, then $\mathrm{IE}_k(X \times Y, G) = \mathrm{IE}_k(X, G) \times \mathrm{IE}_k(Y, G)$, with the equality understood as in Remark 1.4.*

2. Sofic IE-tuples

From now on G denotes a countable discrete sofic group and $\Sigma = \{\sigma_i \colon G \to \mathrm{Sym}(d_i)\}$ is a sofic approximation net indexed by some set $\mathcal{I}$. Let $G \curvearrowright X$ be a continuous action on a compact Hausdorff space X equipped with a continuous dynamically generating pseudometric ρ.

Definition 2.1. Let $F \subseteq G$ be a nonempty finite set, $\delta > 0$, $d \in \mathbb{N}$, and let $\sigma \colon G \to \{1, \ldots, d\}$ be a function. A set $J \subset \{1, \ldots, d\}$ is a $(\rho, F, \delta, \sigma)$-*independence set* for a tuple $\mathbf{A} = (A_1, \ldots, A_k)$ if every function $\omega \colon J \to \{1, \ldots, k\}$ there is $\phi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ such that $\phi(a) \in A_{\omega(a)}$ for every $a \in J$.

Definition 2.2. Let $\mathfrak{F}$ be a free filter on $\mathcal{I}$. We say that a k-tuple $\mathbf{A} = (A_1, \ldots, A_k)$ has *positive upper independence density over* Σ (with respect to $\mathfrak{F}$) if there exists a $q > 0$ such that for every nonempty finite set $F \subseteq G$ and $\delta > 0$ we can find a cofinal set $I' \subseteq \mathcal{I}$ (a set $I' \in \mathfrak{F}$) for which the following holds: for every $i \in I'$ there exists a set $J_i \subset \{1, \ldots, d_i\}$ which is a $(\rho, F, \delta, \sigma)$-independence set for $\mathbf{A}$ with $|J_i| \geq q d_i$.

Remark 2.3. The above definition does not depend on the choice of pseudometric.

Remark 2.4. Definition 2.2 has two variants: a weaker one in which we demand that I' be a cofinal subset of $\mathcal{I}$, and a stronger one in which cofinality is replaced by a requirement that I' is a member of a fixed free ultrafilter $\mathfrak{F}$ on $\mathcal{I}$. These notions have similar properties, but in order to have a product formula like in Theorem 1.5 we need the stronger one. Both variants may be used to define Σ-IE-tuples below.

Definition 2.5. We say that a k-tuple $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$ is a Σ-*IE-tuple* if for every basic open neighborhood $U_1 \times \ldots \times U_k$ of $\mathbf{x}$ in X^k the k-tuple $\mathbf{U} = (U_1, \ldots, U_k)$ has positive upper independence density over Σ. We write $\mathrm{IE}_k^\Sigma(X, G)$ to denote the set of Σ-IE-tuples in X^k, and we call elements of $\mathrm{IE}_2(X, G)$ Σ-*IE-pairs*.

One defines an *approximation net* Σ^u for G and calls Σ^u-IE-tuples the *sofic IE-tuples*. It turns out that if $\mathbf{x} \in X^k$ is a Σ-IE-tuple for some Σ, then $\mathbf{X}$ is also a sofic IE-tuple, and every sofic IE-tuple is also an orbit IE-tuple. Moreover, all these notions coincide with IE-tuples provided G is amenable.

Remark 2.6. We define Σ-IE-tuples, orbit IE-tuples and sofic IE-tuples of *sets* in the same way as for points.

3. Basic properties of IE-tuples

Theorem 3.1. *Let* $k \in \mathbb{N}$. *Assume we are given continuous actions* $G \curvearrowright X$ *and* $G \curvearrowright Y$ *of a countable discrete sofic group* G *on compact Hausdorff spaces* X *and* Y. *Let* Σ *be a sofic approximation net for* G *and* $k \in \mathbb{N}$. *Then the following are true:*

1. $h_\Sigma(X, G) \geq 0$ *if and only if* X *as 1-tuple has positive upper independence density over* Σ.
2. *If* $\mathbf{A} = (A_1, \ldots, A_k)$ *is a tuple of closed subsets of* X *with positive upper independence density over* Σ, *then there exists a* Σ-IE-tuple $(x_1, \ldots, x_k)$ *with* $x_j \in A_j$ *for* $j = 1, \ldots, k$.
3. $\mathrm{IE}_1^\Sigma(X, G) \neq \emptyset$ *if and only if* $h_\Sigma(X, G) \geq 0$.
4. $\mathrm{IE}_2^\Sigma(X, G) \setminus \Delta_2(X) \neq \emptyset$ *if and only if* $h_\Sigma(X, G) > 0$, *where* $\Delta_2(X)$ *denotes the diagonal in* X^2.
5. $\mathrm{IE}_k^\Sigma(X, G)$ *is a closed subset of* X^k *which is invariant under the product action.*
6. *If* $\pi \colon (X, G) \to (Y, G)$ *is a factor map, then* $(\pi \times \ldots \times \pi)(\mathrm{IE}_k^\Sigma(X, G)) \subset \mathrm{IE}_k^\Sigma(Y, G)$.
7. *If* Z *is a closed* G-*invariant subset of* X, *then* $\mathrm{IE}_k^\Sigma(Z, G) \subset \mathrm{IE}_k^\Sigma(X, G)$.
8. $\mathrm{IE}_k(X \times Y, G) \subseteq \mathrm{IE}_k(X, G) \times \mathrm{IE}_k(Y, G)$, *with the identification described in Remark 1.4.*

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Entropy, Li-Yorke chaos and distality

Jeremias Epperlein

In this talk we presented a proof by David Kerr and Hanfeng Li from [5], showing that positive topological entropy implies Li-Yorke chaos for the action of a sofic group on a compact metric space.

Let G be a discrete sofic group with sofic approximation sequence Σ acting continuously on a compact metric space (X, ρ). One way to investigate the dynamics of this action is to look at pairs of orbits and their behaviour relative to each other.

In order to do so, we partition the nondiagonal pairs of points $(x, y) \in X \times X, x \neq y$ into three sets:

- *asymptotic* pairs with $\limsup\limits_{G \ni s \to \infty} \rho(sx, sy) = 0$,
- *distal* pairs with $\liminf\limits_{G \ni s \to \infty} \rho(sx, sy) > 0$,
- *Li-Yorke* pairs with $\liminf\limits_{G \ni s \to \infty} \rho(sx, sy) = 0$ and $\limsup\limits_{G \ni s \to \infty} \rho(sx, sy) > 0$.

A set $Y \subseteq X$ is called *scrambled* if every nondiagonal pair in Y is Li-Yorke. Finally the action (X, G) is called *Li-Yorke chaotic* if X contains an uncountable scrambled set.

This definition goes back to the seminal paper of Li and Yorke [6] on interval transformations. There is a rich relation between Li-Yorke chaos and other definitions of chaoticity, Blanchard et al. give a good overview in [2]. In that paper the authors show in particular that positive topological entropy implies Li-Yorke chaos for integer actions. Despite being a purely topological statement, their proof heavily used ergodic theory and the structure theory of measurable dynamical systems.

The proof by Kerr and Li presented in this talk, however, only uses combinatorial and topological methods. As was shown in the talk by Dominik Kwietniak on combinatorial independence, positive topological entropy implies the existence of a $\Sigma - IE$ pair. Thus we are able to localize the complex behaviour of the action implied by the positive entropy condition. Starting from a Σ-IE pair one constructs, by combinatorial means, larger and larger families of sets in X with positive upper independence density. Induction then proves the following lemma [5, Lemma 8.3].

Lemma. *Let $k \geq 2$ and let $\mathcal{A} = (A_1, \ldots, A_k)$ be a tuple of closed subsets of X with positive upper independence density over Σ. For each $j = 1, \ldots, k$ let U_j be an open set containing A_j. Let E be a finite subset of G. Then there exist $s_1, \ldots, s_m \in G \setminus E$ with $s_i^{-1} s_j \notin E$ for distinct $i, j = 1, \ldots, m$ such that the tuple*

$$(A_i \cap \bigcap_{\ell=1}^{m} s_\ell^{-1} \overline{U_{\omega(\ell)}})_{i=1,\ldots,k, \omega \in \{1,\ldots,k\}^m}$$

has positive upper independence density over Σ.

With these preparations one constructs a sequence of nested sets, where one has good control over the dynamics and whose intersection forms a Cantor space, and gets the following theorem [5, Theorem 8.3].

Theorem. *Let $k \geq 2$ and $\mathcal{A} = (A_1, \ldots, A_k)$ be a tuple of closed subsets of X with positive upper independence density over Σ. Then there exists for each $j = 1, \ldots, k$ a Cantor set $Z_j \subseteq A_j$ such that their union $Z = \bigcup_{j=1}^{k} Z_j$ fulfills*

(1) *every nonempty finite tuple of points in Z is a Σ-IE-tuple,*

(2) $\forall m \in \mathbb{N}, \forall y_1, \ldots, y_m \in Z, y_1', \ldots, y_m' \in Z$ with $y_i \neq y_j$ for distinct $i, j = 1, \ldots, m$ we have

$$\liminf_{G \ni s \to \infty} \max_{i=1,\ldots,m} \rho(sy_i, y_i') = 0.$$

In particular, every nondiagonal pair in Z^2 is a Li-Yorke pair.

For integer actions Blanchard, Host and Ruette [3] also showed that positive entropy implies the existence of an uncountable set of asymptotic pairs. This seems to be open in the sofic case.

An important class of group actions having no Li-Yorke pairs, is given by *distal actions*, i.e. actions for which each pair of distinct points is distal. Thus the theorem above implies the following corollary [5, Corollary 8.5].

Corollary. *Each distal action has sofic topological entropy 0 or $-\infty$.*

Distal systems play an important role in the structure theory of topological dynamical systems (see [1] and [4]). Although we know that distal actions always have an invariant Borel probability measure, it is open if there is a distal action with sofic topological entropy $-\infty$.

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Sofic dimension

YONGLE JIANG

Sofic dimension for a probability measure preserving (pmp) equivalence relation was introduced in [2] as an analogy of the free entropy dimension introduced by Voiculescu [7][8]. Roughly speaking, it is defined by counting the number of sofic models on finite sets to within a given precision.

Note that sofic dimension is also defined in [3] for measurable discrete groupoids, which includes also (a) probability-measure-preserving actions of groups and (b) countable discrete groups. To keep it simple, we focus on the pmp equivalence relations.

Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action of a countable discrete group, denote by R the equivalence relation generated by this action, and

by $[R], [[R]]$ the full group and pseudo full group of R respectively; especially, when d is an integer, denote by $[d], [[d]]$ the full group and pseudo full group for the permutation action $S_d \curvearrowright \{1, \cdots, d\}$. Note that $[d] \cong S_d$.

Given F a finite subset of $[[R]]$, $n \in \mathbb{N}$, and $\delta > 0$, we define $SA(F, n, \delta, d)$ to be the set of all unital maps $\phi : [[R]] \to [[d]]$ which are (F, n, δ)-multiplicative and (F, n, δ)-trace-preserving. We write $NSA(F, n, \delta, d)$ for the number of distinct restrictions of elements of $SA(F, n, \delta, d)$ to the set F.

Definition 1. (Definition of sofic dimension)

$$
\begin{array}{rcl}
s(F, n, \delta) & = & \limsup\limits_{d \to \infty} \dfrac{1}{d \log d} \log NSA(F, n, \delta, d), \\[2mm]
s(F, n) & = & \inf\limits_{\delta > 0} s(F, n, \delta), \\[2mm]
s(F) & = & \inf\limits_{n \in \mathbb{N}} s(F, n).
\end{array}
$$

Then, after introducing the notion of dynamical generating set for R, the first main theorem is the following ([2], Theorem 4.1).

Theorem 2. *Let R be a pmp equivalence and let E and F be finite dynamical generating sets. Then $s(E) = s(F)$.*

So we have the following definition.

Definition 3. Let R be a pmp equivalence relation on (X, μ). Assume R is dynamically finitely generated and let F be a finite dynamical generating set. Then we set $s(R) := s(F)$ and call this value the sofic dimension of R.

Then, considering the computation of $s(R)$ for a given pmp equivalence relation R, the first and the most generic case is the following:

Proposition 4. $s(R) = 1 - \frac{1}{d}$ *where R is the equivalence relation for the permutation action of S_d on $X = \{1, \cdots, d\}$.*

In general, by using the Connes-Feldman-Weiss theorem [1], we have the following theorem ([2], Corollary 5.2).

Theorem 5. $s(R) = 1 - \mu(D)$, *where R is any amenable equivalence relation and D is the fundamental domain of the finite components of R.*

Then, under a mild technical assumption called "s-regularity", the following additivity formula for amalgamated free products holds ([2], Theorem 1.2).

Theorem 6. *Assume that the pmp equivalence relation R is an amalgamated free product of the form $R = R_1 *_{R_3} R_2$, where the finitely generated relations R_1 and R_2 are s-regular and R_3 is amenable. Then R is s-regular and*

$$
s(R) = s(R_1) + s(R_2) - s(R_3).
$$

Finally, we mention the relation between $s(R)$ and $cost(R)$. ([2], Proposition 4.5, Corollary 7.5)

Proposition 7. *Let R be a finitely generated ergodic pmp equivalence relation on (X, μ). Then $s(R) \leq cost(R)$.*

Proposition 8. *Let R be an ergodic finitely generated pmp equivalence relation. If R is treeable, it is s-regular and s(R)=cost(R) (in particular, R is sofic).*

For background on pmp equivalence relation theory and cost theory, see [4], [6]. For the application of free entropy to von Neumann algebras, see [5].

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Sofic mean dimension

GÁBOR SZABÓ

INTRODUCTION

Mean dimension was introduced by Gromov about a decade ago as an analogue of dimension for dynamical systems (see [1]), and was studied systematically by Lindenstrauss and Weiss for continuous actions of countable amenable groups on compact metrizable spaces, see [3]. Among other beautiful results, Lindenstrauss and Weiss used mean dimension to show that there exists a minimal action of $\mathbb{Z}$ on some compact metrizable space which can not be equivariantly embedded into $[0,1]^{\mathbb{Z}}$ equipped with the shift action.

The goal of this lecture is to extend mean dimension to continuous actions of countable sofic groups on compact metrizable spaces following a paper of Hanfeng Li, see [2]. We compute the sofic mean dimension for some examples, such as certain Bernoulli actions and actions with the small boundary property.

1. Sofic mean dimension

The definition of sofic mean dimension is designed as a dynamical analogue of covering dimension. Every analogue of covering dimension reads roughly as follows:

'$\dim'(X) \leq d$ iff every open cover has *order* $\mathcal{D} \leq d$.

Additionally, every appropriate notion of order has to respect the natural refinement structure on open covers, i.e. $\mathcal{U} \prec \mathcal{V}$ implies $\mathcal{D}(\mathcal{U}) \leq \mathcal{D}(\mathcal{V})$. Thus, the precise notion of an order for an open cover is the key to such a definition. For ordinary covering dimension, $\mathcal{D}$ denotes the minimal coloring number of any refinement. That is, $\mathcal{D}$ is defined via

$$\mathcal{D}(\mathcal{U}) + 1 = \min_{\mathcal{U} \prec \mathcal{V}} \sup_{x \in X} \sum_{V \in \mathcal{V}} \chi_V(x).$$

Notation.
- From now on, (X, ρ) denotes a compact metric space. G denotes a countable sofic group with a chosen sofic approximation Σ. Let $\alpha : G \curvearrowright X$ denote a continuous action.
- If $\mathcal{U}$ is an open cover of X and $d \in \mathbb{N}$, denote by $\mathcal{U}^d$ the corresponding open cover $\{U_1 \times \cdots \times U_d \mid U_i \in \mathcal{U}\}$ of X^d. Since $\mathrm{Map}(\rho, F, \delta, \sigma)$ is identified with a closed subset of X^d, we may define

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) = \mathcal{D}(\mathcal{U}^d|_{\mathrm{Map}(\rho, F, \delta, \sigma)}).$$

  Note that in case of $\mathrm{Map}(\rho, F, \delta, \sigma) = \emptyset$, we set this value to be $-\infty$.
- Let $F, F' \Subset G$ and $\delta, \delta' > 0$. Define

$$(F, \delta) \leq (F', \delta') \quad :\Longleftrightarrow \quad F \subset F' \text{ and } \delta \geq \delta'.$$

  Then the set of all such pairs (F, δ) forms a directed set.

Definition 1.1. Let now ρ be a (compatible) metric on X. Let $\mathcal{U}$ be an open cover of X, $F \Subset G$ and $\delta > 0$. Consider

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) = \limsup_{i \to \infty} \frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma_i)}{d_i},$$

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho) = \lim_{(F, \delta) \to \infty} \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta).$$

Note that the limit does exist, since the net

$$(F, \delta) \longmapsto \mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) = \mathcal{D}(\mathcal{U}^d|_{\mathrm{Map}(\rho, F, \delta, \sigma)})$$

is decreasing. Finally, set

$$\mathrm{mdim}_\Sigma(X, \alpha) = \mathrm{mdim}_\Sigma(X, \alpha, \rho) = \sup_{\mathcal{U}} \mathcal{D}_\Sigma(\mathcal{U}, \rho).$$

Proposition 1.2. *Let $\alpha : G \curvearrowright X$ be an action and $Y \subset X$ a nonempty, α-invariant closed subset. Then $\mathrm{mdim}_\Sigma(Y, \alpha) \leq \mathrm{mdim}_\Sigma(X, \alpha)$.*

Proposition 1.3. *Let X_n be a sequence of compact metric spaces with a sequence of group actions $\alpha_n : G \curvearrowright X_n$. Consider the product action*

$$\alpha : G \curvearrowright X := \prod_{n \in \mathbb{N}} X_n \quad via \quad \alpha((x_n)_n) = (\alpha_n(x_n))_n.$$

Then we have $\operatorname{mdim}_\Sigma(X, \alpha) \leq \sum_{n=1}^\infty \operatorname{mdim}_\Sigma(X_n, \alpha_n).$

2. Mean dimension of Bernoulli shifts

In contrast to entropy, mean dimension becomes particularly interesting for higher-dimensional Bernoulli shifts. For a fixed group G, these are dynamical systems of the form $\beta : G \curvearrowright Z^G$ via $\beta^g((x_h)_{h \in G}) = (x_{g^{-1}h})_{h \in G}$ for some compact metric space Z. Let G be sofic.

Theorem 2.1 (general case)**.** *For a dynamical system as above, we have*

$$\operatorname{mdim}_\Sigma(Z^G, \beta) \leq \dim(Z).$$

Theorem 2.2 (cubic case)**.** *In case that Z is some cube, we have*

$$\operatorname{mdim}_\Sigma(([0,1]^n)^G, \beta) = n \quad \text{for all } n \in \mathbb{N}.$$

3. The small boundary property

Following the definition of covering dimension, a zero-dimensional space is defined as a space whose open covers admit a refining clopen partition. An alternative definition (following the so-called inductive dimension) is that the space has a base for the topology consisting of open sets with empty boundaries. The small boundary property is meant to be a dynamical analogue of this definition, i.e. "empty" is replaced by a dynamical version of smallness.

Definition 3.1. A dynamical system $\alpha : G \curvearrowright X$ has the SBP if X has a base for the topology consisting of open sets U such that

$$\mu(\partial U) = 0 \quad \text{for all } \mu \in \mathcal{M}_\alpha(X).$$

Example 3.2. • If a system has less than $2^{\aleph_0}$ invariant ergodic measures, it has the SBP.
• If G acts freely and $\dim(X) < \infty$, then (X, α, G) has the SBP.

Theorem 3.3. *If (X, α, G) has the SBP, then $\operatorname{mdim}_\Sigma(X, \alpha) \leq 0$.*

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Measured equivalence relations and entropy theory

Alessandro Carderi

A sequence of finite subsets S_i of a countable group Γ *spreads* if any element $\gamma \in \Gamma$ that is not the identity belongs to at most finitely many subsets $S_i S_i^{-1}$. For instance, a sequence of subsets S_i of the integer group $\mathbb{Z}$ spreads if the gaps between consecutive elements in S_i tend to infinity. An action of the countable amenable group Γ on the probability space (X, μ) has *completely positive entropy* if for every finite measurable partition of X the entropy $h(\Gamma, P)$ is positive.

Theorem (Rudolph and Weiss, [7]). *Consider an action of the countable amenable group Γ on the probability space (X, μ), let P be a finite measurable partition of X and let S_i be a sequence of finite subsets of Γ which spreads. Then*

$$\lim_{i \to \infty} \frac{1}{|S_i|} H \left(\bigvee_{\gamma \in S_i} \gamma^{-1} P \right) = H(P).$$

This theorem shows that completely positive entropy actions have very strong mixing properties.

The theorem was already known for actions of $\mathbb{Z}$ where the spread condition is easier to understand. The proof of the theorem in the amenable case is done by a reduction to the case of actions of $\mathbb{Z}$, using the theory of *orbit equivalence*. We recall that two actions of two groups on a probability space are said to be *orbit equivalent* if the induced orbit equivalence relations are isomorphic as measured equivalence relations, see for example [6]. A fundamental result [2] states that all probability measure preserving actions of an amenable group are orbit equivalent to an action of the integer group.

Entropy is, however, not an invariant of orbit equivalence, for instance Dye proved in [4] that all actions of the integer group are orbit equivalent (and hence by [2], all actions of all amenable groups are). What they used is the *relative entropy* of the action with respect to a factor, or equivalently, a Γ-invariant sub-σ-algebra. Given an action of the amenable group Γ on the probability space (X, μ) and given a Γ-invariant sub-σ-algebra $\mathcal{A}$ of the measure algebra of (X, μ), for every measurable finite partition P of X, we define

$$h(\Gamma, P|\mathcal{A}) := \inf \left\{ h(\Gamma, P \vee Q) - h(\Gamma, Q) \ : \ Q \text{ is } \mathcal{A}\text{-measurable} \right\}.$$

The relative entropy is invariant under a special kind of orbit equivalence: the orbit equivalence that preserves the sub-σ-algebra $\mathcal{A}$. Using this invariance it is possible to reduce the theorem to the case of $\mathbb{Z}$-actions where the completely

positive entropy condition is replaced by a relative one and the spread sequence is replaced by a *random* spread sequence. Anyway, this problem can be solved using standard techniques of entropy for actions of $\mathbb{Z}$.

Shortly after, Danilenko in [3] realized that the definition of relative entropy is equivalent to a definition of entropy for *actions* of equivalence relations. In this way he was able to simplify the proof of Rudolph-Weiss' theorem using methods from the theory of orbit equivalence and measured equivalence relations. This definition has been generalized by Bowen to sofic equivalence relations, [1].

Sofic equivalence relations were defined by Elek and Lippner in [5] using the idea of *labelled Benjamini-Schramm convergence*: an equivalence relation generated by the group Γ on the probability space (X, μ) is *sofic* if it is a limit of finite graphs which have edges labelled by the acting group Γ and vertices labelled by the space X. In a hand-written and unpublished note, Ozawa proposed a new equivalent definition: an equivalence relation is sofic if its *full pseudo-group* is sofic as a pseudo-group. That is, an equivalence relation is sofic if every finite collection of partially defined measure preserving morphisms of the space whose graph is contained in the equivalence relation is well-approximable by partially defined bijections of finite sets. See for example [1] for the formal definition.

Bowen's definition of sofic entropy of actions of sofic equivalence relations uses Ozawa's definition of soficity. Like the sofic entropy for groups, the sofic entropy of an equivalence relation counts the exponential growth of finite approximation where the finite subsets of the group are replaced by finite subsets of the full pseudo-group and the sofic approximation is replaced by Ozawa's sofic approximation of the full pseudo-group.

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A new approach to sofic entropy

MIKLÓS ABÉRT

Let $\kappa = (p_1, p_2, \ldots)$ be a distribution. The Shannon entropy of κ is defined as

$$H(\kappa) = -\sum_i \mu(p_i) \log p_i.$$

The meaning of Shannon entropy is 'the amount of randomness needed to generate κ'.

Let Γ be a countably infinite group and consider an invariant random coloring of Γ with k colors. Of course the entropy of the whole process is in general infinite, so one would like to define the entropy *per site* for these processes in a meaningful way. It is very much not clear how to do this for general countable groups.

However, in the case when Γ is a sofic group (that is, its Cayley diagrams can be approximated by finite graphs) Lewis Bowen recently initiated a new entropy theory, which was then further advanced by David Kerr, Hanfeng Li and others. In short, Bowen suggested to mimic the infinite process by certain vertex colorings of the finite graphs (called good configurations) and then count the normalized logarithmic limit of the number of good configurations. This approach goes back to statistical physics. Note that I am at least 3 quantors apart from an actual definition here.

Together with Benjy Weiss, we are suggesting a somewhat different approach: we suggest to mimic the infinite coloring process with a random coloring of the finite graph and then instead of counting configurations, compute the normalized maximal Shannon entropies of these finite processes.

This approach allows us to use some of the machinery that has been invented to handle entropy of measure preserving actions of amenable groups. As an application, I presented a simple and transparent proof for the following theorem of Lewis Bowen [1].

Theorem (Bowen). *Let Γ be a sofic group and let κ_1 and κ_2 be two finite distributions. Assume that the Bernoulli actions κ_1^Γ and κ_2^Γ are isomorphic as measure preserving Γ-actions. Then κ_1 and κ_2 have the same Shannon entropy.*

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Reporter: Ana Khukhro

Participants

Prof. Dr. Miklos Abert
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Dr. Vadim Alekseev
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Andrey Alpeev
St. Petersburg State University
Chebyshev Laboratory
14th Line 29B, Vasilievsky Island
199 178 St. Petersburg
RUSSIAN FEDERATION

Dr. Hiroshi Ando
Erwin Schrödinger International
Institute for Mathematical Physics
Boltzmanngasse 17
1090 Wien
AUSTRIA

Agnes Backhausz
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Lewis Bowen
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712
UNITED STATES

Albrecht Brehm
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Alessandro Carderi
Mathématiques
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Nhan-Phu Chung
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. Anton Deitmar
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Philip Dowerk
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04103 Leipzig
GERMANY

Jeremias Epperlein
Abteilung Mathematik
Technische Universität Dresden
01062 Dresden
GERMANY

Dr. Martin Finn-Sell
Mathematisches Institut
Georg-August-Universität Göttingen
37073 Göttingen
GERMANY

Dr. Dan Fulea
Lindenhofstraße 86
68163 Mannheim
GERMANY

Christoph Gamm
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig
GERMANY

Prof. Dr. Roland Gunesch
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

Benjamin Hayes
Department of Mathematics
UCLA
405 Hilgard Ave.
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Huichi Huang
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Yongle Jiang
Department of Mathematics
North Campus of SUNY at Buffalo
Room 129, Mathematics Building
Buffalo NY 14260-2900
UNITED STATES

Dr. Kate Juschenko
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston IL 60208
UNITED STATES

Dr. David Kerr
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Dr. Ana Khukhro
Université Blaise Pascal
Laboratoire de Mathématiques
UMR 6620 - CNRS
Campus des Cézeaux
P.O. Box 80026
63171 Aubiere
FRANCE

Philipp Kunde
Fachbereich Mathematik
Universität Hamburg
20146 Hamburg
GERMANY

Dr. Dominik Kwietniak
Jagiellonian University
Institute of Mathematics
Lojasiewicza 6
30-348 Krakow
POLAND

Martha Lacka
ul. Sciegiennego 57/16
30-830 Krakow
POLAND

Dr. Francois Le Maitre
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Bingbing Liang
Department of Mathematics
SUNY at Buffalo
244 Math. Bldg.
Buffalo NY 14260-2900
UNITED STATES

Prof. Dr. Fernando Lledo
Departamento de Matematicas
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganes, Madrid
SPAIN

Martino Lupini
Dept. of Mathematics & Statistics
York University
4700 Keele Street
Toronto Ont. M3J 1P3
CANADA

Dr. Pavlo Mishchenko
pr. Marshala Zhukova 13/ 98
Donetsk 83071
UKRAINE

Frank Monheim
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Joav Orovitz
Dept. of Mathematics & Computer
Science
Ben Gurion University of the Negev
PO BOX 653
84105 Beer Sheva
ISRAEL

Dr. Karoly Podoski
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Felix Pogorzelski
Friedrich-Schiller-Universität Jena
Institut für Mathematik
Ernst-Abbe-Platz 2
07743 Jena
GERMANY

**Prof. Dr. Jan-Christoph
Schlage-Puchta**
Institut für Mathematik
Universität Rostock
18057 Rostock
GERMANY

Dr. Brandon Seward
Department of Mathematics
University of Michigan
East Hall
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Zoran Sunic
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Gabor Szabo
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

László Márton Tóth
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Balazs Szegedy
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Dr. Robin Tucker-Drob
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
UNITED STATES

Nora Gabriella Szoke
Institute of Mathematics
Eötvös Lorand University
ELTE TTK
Pazmany Peter setany 1/C
1117 Budapest
HUNGARY

Dr. Jianchao Wu
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Andreas B. Thom
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04103 Leipzig
GERMANY