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Set Theory

Organised by
Ilijas Farah, Toronto
Sy-David Friedman, Vienna
Menachem Magidor, Jerusalem
W. Hugh Woodin, Berkeley

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Abstract. This stimulating workshop featured a broad selection of some of the most important recent developments in combinatorial set theory, the theory and applications of forcing, large cardinal theory and descriptive set theory.

Mathematics Subject Classification (2010): 03E.

Introduction by the Organisers

This set theory workshop was unusually broad, covering a wide range of topics in the theory of forcing, large cardinals, combinatorial set theory and descriptive set theory. We had ten 50-minute talks and twenty 30-minute talks, making no attempt to organise them into themes, but rather to mix them randomly; this gave us a good opportunity to learn about areas of the field other than our own, providing a good overview of the current state of the subject. The schedule was sufficiently relaxed to allow plenty of time for informal discussions, which are of course essential to the success of an Oberwolfach meeting.

In combinatorial set theory we heard from Todorcevic, Dobrinen, Solecki and Rinot on deep new developments in Ramsey theory and the basis problem for generalised gaps. Rinot's results constitute a major advance in partition theory and were the topic of an additional evening session in which he presented more details of his proof. We heard nine talks about forcing; among the highlights were Moore's recent advances on how to iterate without adding reals, Spinas' work on the additivity of the Silver ideal, Fischer-Toernquist's work using template iterations to study maximal cofinitary groups and Brendle-Mejia's study of Rothberger gaps. Large cardinal theory was also well-represented in talks by Sargsyan on the core model induction, Zeman on self-iterability of core models, Gitik on mixing collapses into short-extender forcings as well as in talks making use of large cardinal forcing by Dzamonja, Sinapova and Cummings. A very strong component of the meeting was represented by the new wave of young descriptive set-theorists, including Tserunyan, Sabok, Lupini (the only student at the meeting), Motto Ros, Melleray, Conley and Toernquist. They presented deep and fascinating talks about automatic continuity, Borel complexity in the context of C* algebras, Borel and measurable graph colourings, the nonexistence of analytic families which are maximal in various senses as well as an important generalisation of the van der Corput lemma.

The workshop as a whole demonstrated the increasing breadth of the field as well as its steady progress towards resolving problems that have been of constant interest and under investigation for decades.

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Workshop: Set Theory

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Abstracts

Rothberger gaps in analytic quotients

Jörg Brendle
(joint work with Diego Mejía)

This is joint work with Diego Mejía [1].

The study of gaps in the quotient Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ has a long and rich history, harking back to Hausdorff's construction of an (ω_1, ω_1)-gap and to Rothberger's later $(\omega, \mathfrak{b})$-gap. (Here, $\mathfrak{b}$ denotes the *(un)bounding number*, that is, the least size of an unbounded family in the preorder $\leqslant^*$ on ω^ω, defined by $f \leqslant^* g$ iff $f(i) \leqslant g(i)$ holds for all but finitely many i.) In fact, Rothberger proved that $\mathfrak{b}$ is the least cardinal κ such that there are (ω, κ)-gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$. It is well-known that these are the only two types of gaps that exist in ZFC; for example, both the continuum hypothesis CH and the proper forcing axiom PFA imply that any gap in $\mathcal{P}(\omega)/\mathrm{Fin}$ is either of type (ω_1, ω_1) or of type $(\omega, \mathfrak{b})$.

We shall consider gaps in more general quotients of the form $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is a definable ideal on the natural numbers. Let us first fix some notation: two families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\omega)$ are *$\mathcal{I}$-orthogonal* if $A \cap B$ belongs to $\mathcal{I}$ for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$. The pair $(\mathcal{A}, \mathcal{B})$ is a *gap* (in $\mathcal{P}(\omega)/\mathcal{I}$) if $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{I}$-orthogonal and there is no interpolation C, i.e., no $C \subset \omega$ such that $A \backslash C \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $B \cap C \in \mathcal{I}$ for all $B \in \mathcal{B}$. A gap $(\mathcal{A}, \mathcal{B})$ is called a (*linear*) *(κ, λ)-gap* if $\mathcal{A}$ and $\mathcal{B}$ are well-ordered by inclusion modulo $\mathcal{I}$, of regular order type κ and λ, respectively. (This means that $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ with $A_\alpha \backslash A_\beta \in \mathcal{I}$ for $\alpha < \beta$, and similarly for $\mathcal{B}$.) If both κ and λ are uncountable, $(\mathcal{A}, \mathcal{B})$ is a Hausdorff gap, and if at least (= exactly) one is countable, it is a Rothberger gap.

An important general embedding theorem for quotients due to Todorčević [3] says that for a large class of ideals $\mathcal{I}$, including the F_σ ideals and the analytic P-ideals, the gap spectrum of $\mathcal{P}(\omega)/\mathcal{I}$ includes the one of $\mathcal{P}(\omega)/\mathrm{Fin}$, that is, every type of gap that exists in $\mathcal{P}(\omega)/\mathrm{Fin}$ also exists in $\mathcal{P}(\omega)/\mathcal{I}$. In particular, all such quotients have (ω_1, ω_1)-gaps and $(\omega, \mathfrak{b})$-gaps. He also addressed the general problem of determining the gap spectrum of such quotients [3, Problem 2]. We shall look at this problem for Rothberger gaps in quotients by a special class of F_σ ideals, the fragmented ideals.

To simplify the discussion, define the *Rothberger number* $\mathfrak{b}(\mathcal{I})$ for an ideal $\mathcal{I}$ on ω as the minimal cardinal κ such that there exists an (ω, κ)-gap in $\mathcal{P}(\omega)/\mathcal{I}$. Rothberger's result mentioned above says that $\mathfrak{b}(\mathrm{Fin}) = \mathfrak{b}$ while Todorčević's theorem implies that $\mathfrak{b}(\mathcal{I}) \leqslant \mathfrak{b}$ when $\mathcal{I}$ is either an analytic P-ideal *or* an F_σ ideal. By Solecki's characterization [2] of analytic P-ideals as ideals of the form $\mathrm{Exh}(\varphi)$ where φ is a lower semicontinuous submeasure on $\mathcal{P}(\omega)$, it then follows that $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$ for such ideals.

An ideal $\mathcal{I}$ is *fragmented* if there are a partition $(a_i : i \in \omega)$ of ω into finite sets and submeasures φ_i on the a_i such that a set I belongs to $\mathcal{I}$ iff there is a k such that $\varphi_i(a_i \cap I) \leqslant k$ for all i. Clearly such ideals are F_σ. $\mathcal{I}$ is *gradually* *fragmented* if it is fragmented and for all k there is m such that for all ℓ, all but finitely many i, and all $B \subseteq \mathcal{P}(a_i)$ with $|B| \leqslant \ell$ and $\varphi_i(b) \leqslant k$ for all $b \in B$, we have $\varphi_i(\bigcup B) \leqslant m$. Typical examples of fragmented ideals are:

- the ideal $\mathcal{ED}_{\text{fin}}$ whose underlying set consists of the ordered pairs below the identity function and which is generated by (graphs of) functions (more explicitly, letting $\Delta = \{(i,j) : j \leqslant i\} \subset \omega \times \omega$, $I \subset \Delta$ belongs to $\mathcal{ED}_{\text{fin}}$ iff there is a k such that $|\{(i,j) \in I; j \leqslant i\}| \leqslant k$ for all i); this ideal is not gradually fragmented;
- the *linear growth ideal* $\mathcal{L}$ defined by stipulating that I belongs to $\mathcal{L}$ iff there is a k such that $I \cap [2^i, 2^{i+1})| < (i+1) \cdot k$ for all i; this ideal is not gradually fragmented;
- the *polynomial growth ideal* $\mathcal{P}$ defined by stipulating that I belongs to $\mathcal{P}$ iff there is a k such that $I \cap [2^i, 2^{i+1})| < (i+1)^k$ for all i; this ideal is gradually fragmented.

A couple of years ago, we observed, using an argument with eventually different functions:

Theorem 1. $\mathfrak{b}(\mathcal{ED}_{\mathit{fin}}) = \aleph_1$.

Hrušák then conjectured that this was true for all fragmented not gradually fragmented ideals. While this is still open, an argument different from the one of the proof of Theorem 1, involving independent families of functions, shows that it holds for a large class of such ideals, including for example $\mathcal{L}$:

Theorem 2. *If $\mathcal{I}$ is fragmented but not gradually fragmented, and all φ_i are* uniform *submeasures (i.e., the measure only depends on the size of the set), then $\mathfrak{b}(\mathcal{I}) = \aleph_1$.*

On the other hand, for gradually fragmented ideals it is consistent that there are no (ω_1, ω)-Rothberger gaps; in fact, adding a generic slalom localizing all ground model functions necessarily destroys any such gap:

Theorem 3. *If $\mathcal{I}$ is gradually fragmented, then*

(1) $\mathsf{add}(\mathcal{N}) \leqslant \mathfrak{b}(\mathcal{I}) \leqslant \mathfrak{b}$*;*
(2) *if $\mathcal{I}$ is nowhere tall, then $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$;*
(3) *if $\mathcal{I}$ is somewhere tall, then $\mathfrak{b}(\mathcal{I}) < \mathfrak{b}$ is consistent;*
(4) *for a large class of gradually fragmented ideals $\mathcal{I}$ including $\mathcal{P}$, $\mathsf{add}(\mathcal{N}) < \mathfrak{b}(\mathcal{I})$ is consistent.*

Finally, there may be (simultaneously) many fragmented ideals with distinct Rothberger number:

Theorem 4. *It is consistent that there are $\aleph_1$ many gradually fragmented ideals whose Rothberger numbers are all distinct. Assuming the consistency of an inaccessible cardinal, the same is true with ω_1 replaced by $\mathfrak{c}$.*

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Measurable analogs of Brooks's theorem for graph colorings.

Clinton Conley
(joint work with Andrew Marks and Robin Tucker-Drob)

In finite combinatorics, a classical theorem of Brooks asserts that, aside from cliques and odd cycles, every connected graph with degree bounded by d may be d-colored. We discuss the measure-theoretic analog of this for Borel graphs on standard probability spaces (where it holds for d at least 3) and connect it to the ability to find one-ended subforests of such graphs, resembling wired spanning forest constructions from probability. This is joint work with Andrew Marks and Robin Tucker-Drob.

Inverse limit reflection and generalized descriptive set theory

Scott Cramer

In this talk we introduce an axiom called Inverse Limit Reflection (ILR), which we argue is analogous to the Axiom of Determinacy (AD) for $L(\mathbb{R})$, but in the context of $L(V_{\lambda+1})$. That is, ILR is in a sense a fundamental regularity property for the context of $L(V_{\lambda+1})$ just as AD is a fundamental regularity property for the context of $L(\mathbb{R})$. Previously, H. Woodin investigated the large cardinal I_0 as an analogue of AD for $L(V_{\lambda+1})$, proving a number of similar structural results for $L(V_{\lambda+1})$, but we argue that ILR is in fact more analogous, as it outright implies the λ-splitting perfect set property, just as AD outright implies the perfect set property.

The structure $L(V_{\lambda+1})$ was first investigated by Woodin in order to show that $AD^{L(\mathbb{R})}$ follows from large cardinals. In particular he defined the axiom I_0, which states that there is a (nontrivial) elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\text{crit}\,(j) < \lambda$. He then showed that the structure of $L(V_{\lambda+1})$, assuming I_0 holds, is remarkably similar to the structure of $L(\mathbb{R})$ assuming that $AD^{L(\mathbb{R})}$. In particular he showed the following.

Theorem 1 (Woodin [5])**.** *Assume I_0 holds at λ. Then the following hold in $L(V_{\lambda+1})$.*

(1) λ^+ *is measurable.*
(2) Θ *is a limit of measurable cardinals.*

This type of result suggested that I_0 was analogous to AD in the context of $L(V_{\lambda+1})$.

We continued this line of research, showing the following theorems.

Theorem 2 ([2])**.** *Assume I_0 at λ. Then there are no disjoint stationary subsets T_1, T_2 of S_ω such that $T_1, T_2 \in L(V_{\lambda+1})$.*

Theorem 3 ([1])**.** *Assume I_0 at λ. Then every subset $X \subseteq (V_{\lambda+1})m$ such that $X \in L(V_{\lambda+1})$ has the λ-splitting perfect set property. That is either $|X| \leqslant \lambda$ or X contains a λ-splitting perfect set and hence $|X| = 2^\lambda$.*

The proofs rely on a tool called inverse limits which arises out of techniques for reflecting large cardinals at this level. In particular R. Laver proved (see [3], [4]) the following reflection result using inverse limits.

Theorem 4 (Laver)**.** *Assume there exists $j : L_{\lambda^++\omega+1}(V_{\lambda+1}) \to L_{\lambda^++\omega+1}(V_{\lambda+1})$ elementary with crit$(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ such that there is an elementary embedding $k : L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$ with crit$(k) < \bar{\lambda}$.*

We extended this result using inverse limits, to show the following.

Theorem 5 ([1])**.** *Assume there exists an elementary embedding*

$$j : L(V^{\#}_{\lambda+1}) \to L(V^{\#}_{\lambda+1})$$

with crit$(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding

$$k : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

with crit$(k) < \bar{\lambda}$.

From the techniques involved in this reflection theorem, we derived the axiom ILR and showed the following.

Theorem 6 ([2])**.** *ILR implies the λ-splitting perfect set property.*

This property distinguishes ILR from I_0, as the natural generalization $I_0(X)$ of I_0 to models of the form $L(X, V_{\lambda+1})$, where $X \subseteq V_{\lambda+1}$, does not have this property. Namely, assuming I_0, it is consistent that there is an $X \subseteq V_{\lambda+1}$ such that $I_0(X)$ holds, but $L(X, V_{\lambda+1})$ does not satisfy the λ-splitting perfect set property. So in this sense ILR is a better analog of AD than I_0, in so far as it plays the role of a fundamental regularity property.

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Universal graphs at successors of singulars

James Cummings

If $2^\kappa = \kappa^+$, then there is a universal graph on κ^+ regardless of the value of 2^{κ^+}. But for $2^\kappa > \kappa^+$ the situation is more mysterious. Say that u_λ is the least size of a family of graphs on λ which is jointly universal for graphs on λ. Then the following hold.

Theorem 1. *(C. Džamonja, Morgan) It is possible to prove from large cardinals the consistency of "$2^{\aleph_\omega} > \aleph_{\omega+2}$ with $\aleph_\omega$ strong limit and $u_{\aleph_\omega} \leqslant \aleph_{\omega+2}$".*

Theorem 2. *(C. Džamonja, Magidor, Morgan, Shelah) It is possible to prove from large cardinals the consistency of "there exists a singular strong limit cardinal κ of cofinality ω_1 such that $2^\kappa > \kappa^{++}$ and $u_{\kappa^+} \leqslant \kappa^{++}$".*

The proof uses a Prikry type forcing.

Progress in Ramsey theory

Natasha Dobrinen

A triple $(\mathcal{R}, \leqslant, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ which has the property of Baire is Ramsey, and every meager subset is Ramsey null. Such spaces are abstractions of the Ellentuck space. (See [8] for a proper introduction to topological Ramsey spaces.) In [7], Todorcevic proved that every Ramsey ultrafilter is Tukey minimal among nonprincipal ultrafilters. The proof uses the Pudlák-Rödl Theorem canonizing equivalence relations on barriers of the Ellentuck space (which in turn generalizes the Erdös-Rado Theorem for canonical equivalence relations on $[\omega]^k$, for $k < \omega$) along with the existence of continuous cofinal maps for p-points proved in [3]. In work with Todorcevic in [4] and [5], new Ramsey-classification theorems were obtained for fronts on a new class of topological Ramsey spaces extracted from forcings of Laflamme in [6]. These theorems generalize the Pudlák-Rödl Theorem. These Ramsey-classification theorems were applied to decode the Tukey structure of the associated ultrafilters in terms of their Rudin-Keisler structures, these being rapid p-points satisfying some partition properties. These results motivated much of the research outlined below.

Recall the following definitions. For ultrafilters $\mathcal{U}, \mathcal{V}$, $\mathcal{V} \leqslant_T \mathcal{U}$ if and only if there is a map $f : \mathcal{U} \to \mathcal{V}$ such that the f-image of every filter base for $\mathcal{U}$ is a filter base for $\mathcal{V}$. Such an f is called a *cofinal map*. The set of all ultrafilters Tukey equivalent to $\mathcal{U}$ is called the *Tukey type* of $\mathcal{U}$. $\mathcal{V} \leqslant_{RK} \mathcal{U}$ if and only if there is a map $g : \omega \to \omega$ such that $\mathcal{V} = \{X \subseteq \omega : g^{-1}(X) \in \mathcal{U}\}$. Rudin-Keisler reducibility implies Tukey reducibility, but not vice versa. We shall call a collection of Tukey types of ultrafilters an *initial Tukey structure* if it is closed under Tukey reducibility.

The work in [4] and [5] showed, among other things, that for each countable successor ordinal α, there is a rapid p-point which has initial Tukey structure below it exactly that of a decreasing chain of ordered type α. Moreover, the equivalence relations within the Tukey types are completely understood as Fubini iterates of certain p-points. This raised the following question.

Question What are the possible initial structures in the Tukey types ultrafilters?

In [2], we have set forth a general construction scheme for new topological Ramsey spaces. These spaces are built using products of structures from Fraïssé classes of finite ordered relational structures. A main tool in this paper is a new canonization theorem for equivalence relations on finite products of structures (Theorem 1).

Definition 1. *Let $\mathcal{K}_j$, $j \in J < \omega$ be Fraïssé classes of finite ordered relational structures with the Ramsey property. For each $j \in J$, let $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathcal{K}_j$ such that $\boldsymbol{A}_j \leqslant \boldsymbol{B}_j$. Given a subset $I_j \subseteq \|\boldsymbol{A}_j\|$ and $\boldsymbol{X}_j, \boldsymbol{Y}_j \in \binom{\boldsymbol{B}_j}{\boldsymbol{A}_j}$, we write $|\boldsymbol{X}_j|\, E_{I_j}\, |\boldsymbol{Y}_j|$ if and only if for all $i \in I_j$, $x^i_j = y^i_j$, where $\{x^i_j : i < \|\boldsymbol{X}_j\|\}$ is the $<$-increasing enumeration of the universe of $\boldsymbol{X}_j$.*

An equivalence relation E on $\binom{(\boldsymbol{B}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ is canonical *if and only if for each $j \in J$, there is a set $I_j \subseteq \|\boldsymbol{A}_j\|$ such that for all $(\boldsymbol{X}_j)_{j\in J}, (\boldsymbol{Y}_j)_{j\in J} \in \binom{(\boldsymbol{B}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$,*

$$(\boldsymbol{X}_j)_{j\in J}\, E\, (\boldsymbol{Y}_j)_{j\in J} \longleftrightarrow \forall j \in J,\ |\boldsymbol{X}_j|\, E_{I_j}\, |\boldsymbol{Y}_j|. \tag{1}$$

Theorem 1 (Dobrinen, [2])**.** *Let $\mathcal{K}_j$, $j \in J < \omega$, be Fraïssé classes of ordered relational structures with the Ramsey property. For each $j \in J$, let $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathcal{K}_j$ such that $\boldsymbol{A}_j \leqslant \boldsymbol{B}_j$. Then for each $j \in J$, there is a $\boldsymbol{C}_j \in \mathcal{K}_j$ such that for each equivalence relation E on $\binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$, there is a sequence $(\boldsymbol{B}'_j)_{j\in J} \in \binom{(\boldsymbol{C}_j)_{j\in J}}{(\boldsymbol{B}_j)_{j\in J}}$ such that E restricted to $\binom{(\boldsymbol{B}'_j)_{j\in J}}{(\boldsymbol{A}_j)_{j\in J}}$ is canonical.*

Thus, the Finite Erdös-Rado Theorem generalizes to products and moreover to products of certain Fraïssé classes. These are applied to show the following.

Theorem 2 (Dobrinen, Mijares, Trujillo, [2])**.** *CH or MA imply that for each $n < \omega$, there are rapid p-points which have initial Tukey structure exactly that of $\mathcal{P}(n)$. Further, there are ultrafilters such that the structure of the Tukey types of all p-points Tukey reducible to it is exactly $[\omega]^{<\omega}$.*

Interestingly, we also find that the Rudin-Keisler structure below an ultrafilter generated by our scheme is exactly the same as the embedding structure for the products of the Fraïssé classes used to generate the topological Ramsey space. Many ultrafilters with partition properties are seen to arise from topological Ramsey spaces constructed by our methods. In particular the norm-defined ultrafilters of Baumgartner and Taylor in [1] fit into the scheme in [2]. A broader construction scheme for topological Ramsey spaces is currently being developed.

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Combinatorial versions of SCH

Mirna Džamonja

It is commonly accepted that the combinatorics of singular cardinals is in some sense strange and difficult, unrepresentative. We are interested in singular cardinals and their successors and we put forward an antithesis that they are in fact nicer than the successors of regulars.

Some well known nice properties of singular cardinals are singular cardinal compactness, and the fact that in the absence of large cardinals $\square_\kappa$ holds for all singular κ. One needs a Woodin cardinal to kill $\square^*_\kappa$. A celebrated hypothesis is SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. SCH is not true in ZFC + large cardinals (Silver, Magidor), its failure is equiconsistent with $\exists \kappa\, o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen), above a supercompact (Solovay) and in models of various forcing axioms, such as MM (Foreman, Magidor, Shelah) down to MRP (Viale). However a version of SCH holds just in ZFC (Shelah):

$$(\forall n < \omega) 2^{\aleph_n} < \aleph_\omega \implies 2^{\aleph_\omega} < \aleph_{\omega_4}.$$

We present some newer results showing the ZFC feel of the singular cardinal combinatorics. The first topic is on the universality of graphs.

Let κ be a cardinal $\geqslant \aleph_1$. Consider the embeddings $f : G \to H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leqslant H$ if there is such an embedding. We are interested in the smallest size of a dominating family in the resulting structure, call this u_κ.

If GCH holds then $u_\kappa = 1$ for all κ, while for κ the successor of a regular Cohen forcing gives the consistency of $u_\kappa = 2^\kappa > \kappa^+$. For κ the successor of a regular it is consistent to have $u_\kappa < 2^\kappa > \kappa^+$ (Mekler, Shelah for $\aleph_1$, Dž. + Shelah in general). The following was obtained by Dž. and Shelah for $\lambda = \aleph_0$ (2005) and by Cummings, Dž., Magidor, Morgan and Shelah (recent) in general:

Theorem 1. *If κ is a supercompact cardinal, $\lambda < \kappa$ is a regular cardinal and $\theta \geqslant \kappa^{+3}$ is a cardinal with* $\mathrm{cf}(\theta) \geqslant \kappa^{++}$, *then there is a forcing extension in which* $\mathrm{cf}(\kappa) = \lambda$, $2^\kappa = 2^{\kappa^+} = \theta \geqslant \kappa^{+3}$ *and* $u_{\kappa^+} \leqslant \kappa^{+2}$.

To prove this, iterate a forcing which blows up the power of κ, builds the future universal graphs and controls the names in Radin forcing of graphs on κ^+. Radin forcing with respect to what, subsets of κ are being added all the time? Well, a measure sequence is being constructed as we go. The universal family is obtained using a cofinal sequence in λ. We also have more recent work where κ can be made to be $\aleph_\omega$ (Cummings, Dž. and Morgan).

The point of this talk is that *for all we know* u_{κ^+} might be κ^+ (so 1) in our model! In fact:

Question Can we have κ singular and $u_{\kappa^+} = 2^{\kappa^+} > \kappa^{++}$?

The proof with the Cohen subsets does not seem to generalize in any sense! *For all we know* u_{κ^+} might be κ^+ (so 1) in *every* model, i.e. in ZFC! This brings us to ask if one has any universality results just in ZFC. We recall another universality problem where we actually obtain such a situation.

Consider trees of size κ without a branch of size κ, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$. This says that branches go into branches, but f is not necessarily 1-1. The (non-existing) κ-branch is universal. These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen). The resulting structure $\mathcal{T}_\kappa$ has been studied extensively, for example at $\aleph_1$ (see e.g. Todorčević- Väänänen). Interestingly, using Todorčević's σ-operator, we can see that the universality number at $\aleph_1$ under GCH is the maximal possible, $2^{\aleph_1} = \aleph_2$. (GCH always gives universality number 1 if the structure is first order).

Open Question Is there a model of set theory where $u_{\mathcal{T}_{\aleph_1}} = 1$?
However, the singulars are easy in this context.

Theorem 2. *(Dž. + Väänänen 2010) Let κ be a strong limit singular of cofinality ω. Then $u_{\mathcal{T}_\kappa} = \kappa^+$.*

This is the only ZFC universality result on the uncountable that I know. It is sort of a combinatorial version of SCH.

Conclusions

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically. Not everything can be a ZFC result, of course we know that SCH can fail (Magidor), and GCH can fail everywhere (Foreman and Woodin). There are combinatorial results which show that some cardinal invariants can be made as high as possible, for example Cummings and Shelah

(1995) show that it is consistent modulo large cardinals to have that every infinite Boolean algebra $\mathfrak{B}$ has $2^{|\mathfrak{B}|}$ subalgebras. We need a systematic study, involving also development - if possible of forcing axioms.

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods. The combinatorics at the singulars shows that to some extent we catch our tail at singular cardinals. ZFC is capable of telling us the truth *asymptotically*.

This is very pleasing and stands as a good answer, at least to me, to "what is the relevance of set theory in mathematics? Why work in ZFC and not in some other alternative consistent system? ".

Template iterations and maximal cofinitary groups

Vera Fischer

(joint work with Asger Törnquist)

The template iteration technique, which we consider, was introduced by S. Shelah in his work on the consistency of $\aleph_2 \leqslant \mathfrak{d} < \mathfrak{a}$, see [6]. The technique was further developed by J. Brendle and used to establish the consistency of the minimal size of a maximal almost disjointness number being of countable cofinality (see [1]). In a broad sense, this template iteration can be though of as a forcing construction, which on one side has characteristics of a "product-like" forcing, and on the other hand, characteristics of finite support iteration. In [1] the "product-like" side of the construction, was used to force a maximal almost disjoint family of arbitrary cardinality, say cardinality λ which in particular can be of countable cofinality, while the "finite support" side of the construction was used to add a cofinal family of dominating reals, which has a prescribed size. Thus in the final generic extension, this cofinal family gives a prescribed size of the bounding number, say λ_0, and so a prescribed lower bound of $\mathfrak{a}$. Then an isomorphism of names argument provides that there are no mad families of intermediate cardinalities μ, i.e. cardinality μ such that $\lambda_0 \leqslant \mu < \lambda$. In order this isomorphism of names argument to work we need to assume CH, as well as $\aleph_2 \leqslant \lambda_0$.

We introduce a forcing notion, which adds a maximal cofinitary group of arbitrary size and which enjoys certain combinatorial properties, allowing for the poset to be iterated along a template (see [4, Definition 2.4]). Similarly to the mad families case, we use this poset along the "product-like" side of an appropriate template iteration to add a maximal cofinitary group of desired cardinality, say λ. The "finite support" side of this construction is used to add a cofinal family Φ of slaloms, each of which localizes the corresponding ground model reals. Using a combinatorial characteristic of $\mathrm{add}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$ which is due to Bartoszyński, we obtain that in the final generic extension both of those cardinal invariants have the size of the family Φ. However by a result of J. Brendle, O. Spinas and Y. Zhang (see [3]) the uniformity of the meager ideal $\mathrm{non}(\mathcal{M})$ is less than or equal to the minimal size of a maximal cofinitary group (denoted $\mathfrak{a}_g$), and so we obtain that $|\Phi|$ is a lower bound for $\mathfrak{a}_g$. An isomorphism of names argument, which is almost identical to the maximal almost disjoint families case, provides that in the final generic extension there are no maximal cofinitary groups of intermediate cardinalities, i.e. cardinalities μ such that $|\Phi| \leqslant \mu < \lambda$. Again for the isomorphism of names argument to work we have to assume CH, as well as $\aleph_2 \leqslant |\Phi|$. Thus we obtain:

Theorem 1. *([4, Theorem 5.9]) It is consistent with the usual axioms of set theory that the minimal size of a maximal cofinitary group is of countable cofinality.*

The above two constructions, dealing with mad families and cofinitary groups respectively, can be generalized in the following sense. We define two classes of forcing notions, which in a natural way generalize our poset for adding a maximal cofinitary group and Hechler poset for adding a dominating real, respectively. We refer to these posets as finite function posets with the strong embedding property (see [4, Definitions 3.16 and 3.17]) and good σ-Suslin forcing notions (see [4, Definitions 3.14 and 3.15]) respectively. We generalize the template iteration techniques of [1], so that arbitrary representatives of the above two classes can be iterated along a template (see [4, Definition 3.21 and Lemma 3.22]) and establish some basic combinatorial properties of this generalized iteration. We fix some notation: whenever $\mathcal{T}$ is a template, $\mathbb{Q}$ is a finite function poset with the strong embedding property and $\mathbb{S}$ is a good σ-Suslin forcing notion, we denote by $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ the iteration of $\mathbb{Q}$ and $\mathbb{S}$ along $\mathcal{T}$ ([4, Definition 3.21])). For example we show that:

Lemma 1. *([4, Lemma 3.27]) If $\mathbb{Q}$ is Knaster, then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is Knaster.*

Let $\mathfrak{a}_p$ and $\mathfrak{a}_e$ denote the minimal size of a maximal family of almost disjoint permutations on ω and the minimal size of a maximal almost disjoint family of functions in ${}^\omega\omega$ respectively. Let $\mathcal{T}_0$ be the template used in the proof of the consistency of $\mathfrak{a}$ being of countable cofinality (see [1]). Our main results states the following.

Theorem 2. *([4, Theorem 6.1]) Let $\bar{\mathfrak{a}} \in \{\mathfrak{a}, \mathfrak{a}_p, \mathfrak{a}_g, \mathfrak{a}_e\}$. There are a finite function poset with the strong embedding property $\mathbb{Q}_{\bar{\mathfrak{a}}}$ and a good σ-Suslin poset $\mathbb{S}_{\bar{\mathfrak{a}}}$ such that if V is a model of CH then $V^{\mathbb{P}(\mathcal{T}_0, \mathbb{Q}_{\bar{\mathfrak{a}}}, \mathbb{S}_{\bar{\mathfrak{a}}})} \vDash cof(\bar{\mathfrak{a}}) = \omega$.*

The most interesting case is indeed, the maximal cofinitary groups case. In fact for each of the other three cases, the forcing notion $\mathbb{Q}_{\bar{\mathfrak{a}}}$ is closely related to the forcing notion for adding a maximal cofinitary group discussed earlier.

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Short extenders forcings and collapses

Moti Gitik

A way of collapsing cardinals inside critical intervals is suggested. We study the PCF structure of models obtained as a combination of the gap 2 short extenders forcing with the supercompact Prikry forcing. Effects of combinations of the gap 2 short extenders forcing with Levy collapses are studied as well.

Results of the following type are obtained:

Assume GCH. Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ so that for every $n < \omega$, κ_n carries an extender E_n such that

(1) if $j_n : V \to M_n \simeq Ult(V, E_n)$, then $M_n \supseteq H(\kappa_n^{+n+2})$,
(2) the normal measure $E_n(\kappa_n)$ of E_n concentrates on the set $\{\nu < \kappa_n \mid \nu \text{ is a } \nu^{+n+2}\text{–supercompact cardinal}\}$.

Theorem 1. *Let $\langle \delta_n \mid n < \omega \rangle$ be a sequence of regular cardinals in $\prod_{n<\omega} \kappa_n$. Then there is a forcing extension with an increasing sequence $\langle \eta_n \mid n < \omega \rangle$ such that*

(1) *η_n is the immediate successor of a cardinal of cofinality δ_n, for every $n < \omega$,*
(2) $\bigcup_{n<\omega} \eta_n = \kappa$,
(3) $\mathrm{tcf}(\prod_{n<\omega} \eta_n / finite) = \kappa^{++}$.

The effects of adding a real to models of set theory

Mohammad Golshani

Given a model V of ZFC and a real R, let $V[R]$, if it exists, be the smallest model of ZFC with the same ordinals as V such that $V \subseteq V[R]$ and $R \in V[R]$.

In [2] we discussed the effects of adding a real to the power set function. Assuming the existence of a proper class of measurable cardinals, we showed that it is possible to force Easton's theorem by adding a single real. More precisely:

Theorem 1. *Let M be a model of $ZFC + GCH +$ there exists a proper class of measurable cardinals. In M let $F : REG \longrightarrow CARD$ be an Easton function, i.e a definable class function such that*

- $\kappa \leqslant \lambda \longrightarrow F(\kappa) \leqslant F(\lambda)$, *and*
- $cf(F(\kappa)) > \kappa$.

Then there exists a pair (W, V) of cardinal preserving extensions of M such that
(a) $W \models GCH$,

(b) $V = W[R]$ for some real R,
(c) $V \models \forall\kappa \in REG, 2^\kappa \geqslant F(\kappa)$.

The main result of our article answers an open question of Shelah and Woodin [5] by showing that it is possible to violate GCH at all infinite cardinals by adding a single real to a model of GCH:

Theorem 2. *Assume the consistency of an $H(\kappa^{+3})$-strong cardinal κ. Then there exists a pair (W, V) of models of ZFC such that:*
(a) W and V have the same cardinals,
(b) GCH holds in W,
(c) $V = W[R]$ for some real R,
(d) GCH fails at all infinite cardinals in V.

In [1] we showed that it is possible to code a real by two Cohen reals in a cofinality preserving way:

Theorem 3. *Suppose that R is a real in V. Then there are two reals a and b such that:*
(a) a and b are Cohen generic over V,
(b) all of the models $V, V[a], V[b]$ and $V[a,b]$ have the same cardinals,
(c) $R \in L[a,b]$.

Using this result and the previous results it is possible to get some interesting results.

In [3] and [4] we studied the interrelation between Cohen reals over different models of ZFC. A basic fact about Cohen reals is that adding λ-many Cohen reals cannot produce more than λ-many of Cohen reals [1]. More precisely, if $\langle s_\alpha : \alpha < \lambda\rangle$ are λ-many Cohen reals over V, then in $V[\langle s_\alpha : \alpha < \lambda\rangle]$ there are no λ^+-many Cohen reals over V. In these papers, we showed that this is not the case, if instead of dealing with one universe V we consider two universes. In particular we have the following:

Theorem 4. *Suppose that κ is a strong cardinal and $\lambda > \kappa$ is a regular cardinal. Then there is a cardinal preserving generic extension V^* of V such that V and V^* have the same reals, and such that adding $\kappa-$many Cohen reals over V^* adds $\lambda-$many Cohen reals over V.*

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[1] By "λ-many Cohen reals" we mean "a generic object $\langle s_\alpha : \alpha < \lambda\rangle$ for the poset $\mathbb{C}(\lambda)$ of finite partial functions from $\lambda \times \omega$ to 2".

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An Easton-like Theorem for Zermelo-Fraenkel Set Theory Without Choice

Peter Koepke
(joint work with Anne Fernengel)

In Zermelo-Fraenkel set theory without the Axiom of Choice the θ-function

$$\theta(\kappa) = \sup\{\nu \in \mathrm{Ord} \mid \text{there is a surjection } f : \mathcal{P}(\kappa) \to \nu\}$$

provides a "surjective" substitute for the exponential function 2^κ. Obviously

- $\theta(\kappa)$ is a cardinal $> \kappa^+$;
- $\kappa < \lambda$ implies $\theta(\kappa) \leqslant \theta(\lambda)$.

We show that these are the only restrictions that ZF imposes on $\theta(\kappa)$.

Theorem 1. *Let M be a ground model of* ZFC + GCH +Global Choice + *there are no inaccessible cardinals. In M, let F be a function defined on the class of infinite cardinals such that*

(i) *$F(\kappa)$ is a cardinal $> \kappa^+$;*
(ii) *$\kappa < \lambda$ implies $F(\kappa) \leqslant F(\lambda)$.*

Then there is an extension N of M which satisfies ZF, *preserves cardinals and cofinalities, and such that $\theta(\kappa) = F(\kappa)$ holds for all cardinals in N.*

This is a version of Easton's Theorem [1] for *all* infinite cardinals, irrespective of cofinalities. The model N is defined as an inner model of a class generic extension $M[G]$ of M. Some techniques are inspired by [2]. Detailed proofs and further investigations of the model and its variants, in particular concerning cardinal arithmetic properties and the amount of choice possible, will be presented in the PhD thesis of Anne Fernengel.

Let

$$A = \bigcup_{\kappa \in \mathrm{Card}} \{\kappa\} \times F(\kappa)$$

We say that $t = (t, <_t)$ is an *F-tree* if

- $t \subseteq A$ and t is the field of the binary relation $<_t$;
- $<_t$ is strict and transitive;
- $(\kappa, \nu) <_t (\lambda, \mu) \to \kappa < \lambda$;
- $(\lambda, \mu) \in t \wedge \kappa \in \mathrm{Card} \cap \lambda \to \exists! \nu\, (\kappa, \nu) <_t (\lambda, \mu)$; hence the predecessors of $(\lambda, \mu) \in t$ with $\lambda = \aleph_\alpha$ are linearly ordered by $<_t$ in ordertype α, i.e., t is a tree;
- there are finitely many maximal elements $(\lambda_0, \mu_0), \ldots, (\lambda_{n-1}, \mu_{n-1}) \in t$ such that

$$t = \{(\kappa, \nu) \mid \exists i < n\, (\kappa, \nu) \leqslant_t (\lambda_i, \mu_i)\}.$$

Let the class forcing $P = (P, \leqslant)$ consist of all conditions

$$p : (t, <_t) \to V$$

such that $(t, <_t)$ is an F-tree, and for all $(\lambda, \mu) \in t$:

- if $\lambda = \aleph_0$ then $p(\lambda, \mu) \in \mathrm{Fn}(\aleph_0, 2, \aleph_0)$
- if $\lambda = \kappa^+$ is the successor of $\kappa \in \mathrm{Card}$ then $p(\lambda, \mu) \in \mathrm{Fn}([\kappa, \lambda), 2, \lambda)$
- otherwise $p(\lambda, \mu) = \varnothing$

Here $\mathrm{Fn}(D, 2, \lambda)$ is the Cohen forcing
$\{h \mid h : \mathrm{dom}(h) \to 2, \mathrm{dom}(h) \subseteq D, \mathrm{card}(h) < \lambda\}$, partially ordered by $\supseteq$.

For $p, q \in P$ with $p : (t, <_t) \to V$ and $q : (s, <_s) \to V$ set $p \leqslant q$ iff

- $t \supseteq s$ and $<_t \supseteq <_s$
- $\forall (\lambda, \mu) \in s \; p(\lambda, \mu) \supseteq q(\lambda, \mu)$

Let G be M-generic for P. For $(\lambda, \mu) \in A$ let

$$c_{\lambda,\mu} = \{\xi \mid \exists p \in G\, (p : (t, <_t) \to V \wedge \exists (\kappa, \nu) \leqslant_t (\lambda, \mu)\, p(\kappa, \nu)(\xi) = 1)\}$$

be the "μ-th Cohen subset" of λ.

For limit $\delta < F(\lambda)$ define surjections

$$S_{\lambda,\delta} : \{c_{\lambda,\mu} \mid \mu < \delta\} \to \{0\} \cup (\mathrm{Lim} \cap \delta)$$

by $S_{\lambda,\delta}(c_{\lambda,\mu}) = \mu^*$ where μ^* is the largest element of $\{0\} \cup (\mathrm{Lim} \cap \delta)$ which is $\leqslant \mu$. Around each $S_{\lambda,\delta}$ define a "cloud" $\tilde{S}_{\lambda,\delta}$ of similar functions:

$\tilde{S}_{\lambda,\delta} = \{\sigma \circ S_{\lambda,\delta} \mid \sigma$ is a permutation of δ, which is the identity except for finitely many arguments $\}$.

The predicate $\mathcal{S}$ collects the equivalence classes:

$$\mathcal{S} = \bigcup \left\{ \tilde{S}_{\lambda,\delta} \mid \lambda \in \mathrm{Card} \text{ and } \delta \text{ is a limit ordinal } < F(\lambda) \right\}.$$

The model N is generated over M by the Cohen sets, the surjections $S_{\lambda,\delta}$ and the predicate $\mathcal{S}$.

$$N = (\mathrm{HOD}_{\mathcal{S}}(M \cup \{c_{\lambda,\mu} \mid (\lambda, \mu) \in A\} \cup \{S_{\lambda,\delta} \mid (\lambda, \delta) \in A \wedge \delta \in \mathrm{Lim}\}))^{M[G]}$$

The analysis of N is based on symmetry properties of P.

An *F-permutation* is a bijection $\pi : A \leftrightarrow A$ such that

$$\forall (\lambda, \mu) \in A \exists \mu' < F(\lambda)\, \pi(\lambda, \mu) = (\lambda, \mu'),$$

i.e., π preserves the levels of A. An F-permutation π canonically extends to a bijection of the class of all F-trees and to an automorphism of P. An F-permutation π is called *small* if for all assignments $\pi(\lambda, \mu) = (\lambda, \mu')$ there is $\gamma \in \{0\} \cup \mathrm{Lim}$ such that $\mu, \mu' \in [\gamma, \gamma + \omega)$.

Lemma 1. *If $p, q \leqslant r$ and $p \upharpoonright \mathrm{dom}(r)$, $q \upharpoonright \mathrm{dom}(r)$ are compatible then there exists a small F-permutation π such that*

- *$\pi \upharpoonright \mathrm{dom}(r) = \mathrm{id}$ and so $\pi(r) = r$*
- *p and $\pi(q)$ are compatible in P.*

A small F-permutation will only permute the Cohen sets $\dot{c}^H_{\lambda,\mu}$ by a finite difference in the second index μ. Since the functions $S_{\lambda,\delta}$ are invariant with respect to such small pertubations, one obtains:

Lemma 2. *Let φ be an $\in$-formula, $(\kappa_0, \mu_0), \ldots, (\kappa_{m-1}, \mu_{m-1}) \in A$, and $(\lambda_0, \delta_0), \ldots, (\lambda_{n-1}, \delta_{n-1}) \in A$ with limit ordinals δ_i. Let $\vec{x} \in M$. Let $p : (t, <_t) \to V$ be a condition such that $(\kappa_0, \mu_0), \ldots, (\kappa_{m-1}, \mu_{m-1}) \in t$. Let $(\bar{t}, <_t) \subseteq (t, <_t)$ be the subtree $\bar{t} = \{(\kappa, \nu) \mid \exists i < m\, (\kappa, \lambda) \leqslant_t (\kappa_i, \mu_i)\}$. Then, using canonical names,*

$$p \Vdash \varphi\left(\dot{c}_{\kappa_0,\mu_0}, \ldots, \dot{c}_{\kappa_{m-1},\mu_{m-1}}, \overrightarrow{\check{x}}, \dot{S}_{\lambda_0,\delta_0}, \ldots, \dot{S}_{\lambda_{n-1},\delta_{n-1}}, \dot{\mathcal{S}}\right)$$

iff

$$p \upharpoonright \bar{t} \Vdash \varphi\left(\dot{c}_{\kappa_0,\mu_0}, \ldots, \dot{c}_{\kappa_{m-1},\mu_{m-1}}, \overrightarrow{\check{x}}, \dot{S}_{\lambda_0,\delta_0}, \ldots, \dot{S}_{\lambda_{n-1},\delta_{n-1}}, \dot{\mathcal{S}}\right).$$

This implies the following approximation property:

Lemma 3. *Let $X \in N$, $X \subseteq$ Ord. Then there are indices $(\kappa_0, \mu_0), \ldots, (\kappa_{m-1}, \mu_{m-1}) \in A$ such that $X \in M\left[c_{\kappa_0,\mu_0}, \ldots, c_{\kappa_{m-1},\mu_{m-1}}\right]$.*

By the lemma, N is a cardinal preserving extension of M. For all cardinals $\lambda \in M$ and limit ordinals $\delta < F(\lambda)$, $S_{\lambda,\delta} \in N$ yields a surjection from $\mathcal{P}(\lambda) \cap N$ onto $(\{0\} \cup \mathrm{Lim}) \cap \delta$, thus $\theta(\lambda) \geqslant F(\lambda)$. Further permutation arguments show that $N \models \theta(\lambda) = F(\lambda)$, as required.

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Forcing Square with Finite Conditions

John Krueger

In previous work we introduced the idea of an adequate set of models and showed how to use adequate sets as side conditions in forcing with finite conditions. For countable sets of ordinals M and N, we define the *comparison point* of M and N, denoted by $\beta_{M,N}$. This ordinal has cofinality ω_1 and satisfies that any ordinal γ which is both either in M or a limit point of M, and either in N or a limit point of N, is below $\beta_{M,N}$. We then define a finite set of countable subsets of ω_2 to be *adequate* if for all M and N in A, either $M \cap \beta_{M,N} \in Sk(N)$, $N \cap \beta_{M,N} \in Sk(M)$, or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ (where Sk denotes the Skolem hull in an appropriate structure on $H(\omega_2)$).

We gave several examples of forcing with adequate sets, including forcing posets for adding a generic function on ω_2, adding a nonreflecting stationary subset of ω_2, adding a Kurepa tree on ω_1, and adding a club to a fat stationary subset of ω_2. The main result of the present talk is to define a forcing poset using adequate sets which adds a $\square_{\omega_1}$-sequence.

The idea of using models as side conditions in forcing goes back to Todorčević, where the method was applied to add generic objects of size ω_1 with finite approximations. In the original context of applications of PFA, the preservation of ω_2 was not necessary. To preserve ω_2, Todorčević introduced the requirement of a system of isomorphisms on the models in a condition.

In the present talk we introduce the idea of a coherent adequate set of models. A coherent adequate set is essentially an adequate set which also satisfies the existence of a system of isomorphisms in the sense of Todorčević. Combining these two ideas turns out to provide a powerful method for forcing with side conditions. An adequate set A is *coherent* if for all M and N in A:

(1) if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, then M and N are isomorphic;
(2) if $M \cap \beta_{M,N} \in Sk(N)$, then there exists N' in A such that $M \in Sk(N')$ and N and N' are isomorphic;
(3) if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ and $K \in A \cap Sk(M)$, then $\sigma_{M,N}(K) \in A$.

As an application of the idea of coherent adequate sets, we define a forcing poset which adds a square sequence on ω_2 with finite conditions. Let $\mathbb{P}$ be the forcing poset whose conditions are pairs (x, A) satisfying:

(1) x is a finite set of triples of the form $\langle \alpha, \gamma, \gamma' \rangle$, where $\alpha \in \omega_2 \cap cof(\omega_1)$ and $\gamma < \gamma' < \alpha$;
(2) if $\langle \alpha, \gamma, \gamma' \rangle$ and $\langle \alpha, \beta, \beta' \rangle$ are distinct in x, then $[\gamma, \gamma') \cap [\beta, \beta') = \emptyset$;
(3) A is a finite coherent adequate set;
(4) for all $M \in A$ and $\langle \alpha, \gamma, \beta \rangle \in x$, if $\alpha \in M$ then either γ and β are in M, or $\sup(M \cap \alpha) < \gamma$;
(5) if M and M' are in A and $M \cap \beta_{M,M'} = M' \cap \beta_{M,M'}$, then for any triple $\langle \alpha, \gamma, \beta \rangle \in Sk(M) \cap x$, $\sigma_{M,M'}(\langle \alpha, \gamma, \beta \rangle) \in x$.

Let $(y, B) \leqslant (x, A)$ if $x \subseteq y$ and $A \subseteq B$.

The main result of the talk is that the forcing poset $\mathbb{P}$ is strongly proper, ω_2-c.c., and adds a $\square_{\omega_1}$-sequence.

Borel complexity and automorphisms of C*-algebras

Martino Lupini

Suppose that X is a compact metrizable space, and Homeo(X) is the group of auto-homeomorphisms of X. This is a Polish group when endowed with the compact-open topology. If Γ is a countable discrete group, then an *action* of Γ on X is a group homomorphism from Γ to Homeo(X). Two actions ϕ, ϕ' are *conjugate* if there is an auto-homeomorphism α of X such that

$$\alpha \circ \phi(\gamma) \circ \alpha^{-1} = \phi'(\gamma)$$

for all $\gamma \in \Gamma$. Let us consider the particular case when Γ is the group of integers. In this case actions can be identified with single auto-homeomorphisms of X, and conjugacy of actions corresponds to conjugacy in Homeo(X). This motivates the problem of looking at the Borel complexity of the relation of conjugacy in Homeo(X).

Recall that if E and F are Borel equivalence relation on standard Borel spaces X and Y, then E is Borel reducible to F if there is a Borel function $f : X \to Y$ such that xEy iff $f(x)Ff(y)$ for all $x, y \in X$. An equivalence relation is called *classifiable* if it is Borel reducible to the orbit equivalence relation associated with a Borel action on a standard Borel space of the Polish group S_∞ of permutations of $\mathbb{N}$. A classifiable equivalence relation is *Borel complete* if it has maximal Borel complexity among classifiable equivalence relation.

The observation that, if X is zero-dimensional, then $\mathrm{Homeo}(X)$ is isomorphic to the Polish group of automorphisms of the countable Boolean algebra of clopen subsets of X allows one to conclude that the auto-homeomorphisms of X are classifiable up to conjugacy whenever X is zero-dimensional. It is moreover proved in [2] that if X is the Cantor space then the relation of conjugacy in $\mathrm{Homeo}(X)$ is Borel-complete. It is shown in [3] that the auto-homeomorphisms of the unit interval $[0,1]$ are classifiable up to conjugacy, while the relation of conjugacy in $\mathrm{Homeo}\left([0,1]^2\right)$ is not classifiable. This is one of the earliest applications of Hjorth's theory of turbulence.

Suppose that $G \curvearrowright X$ is a continuous action of a Polish group on a Polish space. If $1 \in V \subset G$ and $x \in U \subset X$ are open, then the local orbit $\mathcal{O}(x, U, V)$ is the smallest subset of U with the property that $x \in \mathcal{O}(x, U, V)$ and moreover if $y \in \mathcal{O}(x, U, V)$ and $\gamma \in V$ are such that $\gamma \cdot y \in U$, then $\gamma \cdot y \in \mathcal{O}(x, U, V)$. The action $G \curvearrowright X$ is called *preturbulent* if every orbit is dense and every local orbit is dense; if moreover every orbit is meager then the action is called turbulent. The main result of Hjorth's theory of turbulence asserts that if $G \curvearrowright X$ is a preturbulent Polish group action, $S_\infty \curvearrowright Y$ is an action of S_∞ on a standard Borel space, and $f : X \to Y$ is a Borel function mapping G-orbits into S_∞-orbits, then there is a G-invariant dense G_δ subset C of X such that $f[C]$ is contained in a single S_∞-orbit. As a consequence if $G \curvearrowright X$ is moreover turbulent then the corresponding orbit equivalence relation is not classifiable.

The problem of classifying homeomorphisms of compact metrizable spaces up to conjugacy can be seen as a particular case of classifying automorphisms of separable unital C*-algebras up to conjugacy. A *C*-algebra* is a complex algebra A with involution $x \mapsto x^*$ endowed with a norm making it a Banach space, and moreover satisfying the identities

$$\|xy\| \leqslant \|x\| \, \|y\|$$

and

$$\|x^* x\| = \|x\|^2$$

for $x, y \in A$. A C*-algebra is *unital* if it has a multiplicative identity, and *separable* if it is separable in the topology induced by the norm. In the following all C*-algebras will be assumed to be separable and unital. An *automorphism* of A is a bijection $\alpha : A \to A$ preserving all the operations and, hence, the norm; see [1, II.1.6.6]. The group $\mathrm{Aut}(A)$ of automorphisms of A is a Polish group with respect to the topology of pointwise convergence in norm.

If X is a compact metrizable space, then the space $C(X)$ of complex-valued continuous function on X is a C*-algebras with respect to the pointwise operations and the supremum norm. It is a classical result of Gelfand and Naimark that commutative C*-algebras are exactly those of this form; see [1, II.2.2.4]. It is easy to see that in this case the group $\mathrm{Aut}(C(X))$ of automorphisms of $C(X)$ is isomorphic as Polish group to $\mathrm{Homeo}(X)$. Therefore when A is an arbitrary C*-algebra we can think about $\mathrm{Aut}(A)$ as the group of auto-homeomorphisms of a *noncommutative space*.

The main theorem from [6] asserts that if $\mathcal{Z}$ is the Jiang-Su algebra, then the action of $\mathrm{Aut}(\mathcal{Z})$ on itself by conjugation is generically turbulent, i.e. turbulent on an invariant comeager set. Moreover if A is any $\mathcal{Z}$-stable C*-algebra, i.e. absorbing the Jiang-Su algebra tensorially, then the automorphisms of A are not classifiable up to conjugacy. This was inspired by one of the main results of [5], where the authors show that the automorphism group of the hyperfinite II_1 factor acts generically turbulently on itself by conjugation.

The Jiang-Su algebra has been constructed in [4] as direct limit of suitable building blocks called *dimension drop algebras*, to provide an example of an infinite dimensional simple nuclear C*-algebra with the same K-theory as the algebra of complex numbers. Since then the classification program of simple nuclear C*-algebras has focused on trying to classify $\mathcal{Z}$-stable C*-algebras. A currently open conjecture formulated by Toms and Winter in 2009 asserts that for simple nuclear C*-algebras $\mathcal{Z}$-stability is equivalent to several others regularity properties, such as finite nuclear dimension. (Nuclear dimension can be seen as the noncommutative analogue of the notion of dimension for compact metrizable spaces.)

By the previous discussion, if X is zero-dimensional compact metrizable space, then the automorphisms of $C(X)$ are classifiable up to conjugacy. It is an open problem to determine whether there is an example of an infinite-dimensional *simple* (i.e. with no nontrivial ideals) C*-algebra A such that conjugacy in $\mathrm{Aut}(A)$ is classifiable. In view of the main result from [6], such an algebra can not be $\mathcal{Z}$-stable. More generally it would be interesting to have some general results concerning the Borel complexity of the relation of conjugacy in auto-homeomorphisms groups of compact metrizable spaces, or automorphisms groups of (simple, nuclear) C*-algebras.

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Abstract of the talk "Full group of Minimal Homeomorphisms and descriptive set theory"

Julien Melleray
(joint work with Tomás Ibarlucia)

The talk focused on some descriptive-set-theoretic questions about the properties of full groups of minimal homeomorphisms of a Cantor space. Let us first recall that a *minimal homeomorphism* φ of a Cantor space X is a homeomorphism such that, for all $x \in X$, the φ-orbit of x is dense. These maps are an analogue, in the setting of topological dynamics, of the ergodic measure-preserving transformations studied in ergodic theory. In the measure-theoretic context, Dye introduced in a series of two papers [1], [2] the notion of a *full group*: the full group $[\Gamma]$ of a measure-preserving action of a countable group Γ on a standard probability space (X, μ) is the group of all measure-preserving transformations which map (almost) each Γ-orbit onto itself. This definition only depends on the equivalence relation induced by the action of Γ (where two points are equivalent iff they are in the same Γ-orbit; in the measure-theoretic context we neglect all sets of measure zero); conversely, if two countable groups Γ, Λ act ergodically on a standard probability space, and $[\Gamma]$, $[\Lambda]$ are isomorphic (as abstract groups), then Dye proved that the two associated equivalence relations are isomorphic. In that context, the right notion of isomorphism of equivalence relations is *orbit equivalence*: there exists a measure-preserving automorphism which maps (almost) every orbit for the first action onto an orbit for the second action.

This definition of full group and orbit equivalence can easily be translated to the context of topological dynamics: the full group of an action by homeomorphisms on a Cantor space is made up of all homeomorphisms which map orbits onto themselves, and two minimal actions of countable groups Γ, Λ on a Cantor space X are orbit equivalent if there is an homeomorphism of X which maps each Γ-orbit onto a Λ-orbit. Again the full group of an action only depends on the relation it induces, and Giordano–Putnam–Skau [4] proved the analogue of Dye's theorem in that context: if two minimal actions have isomorphic full groups, then they are orbit equivalent.

Full groups have been investigated in depth in ergodic theory; they come equipped with a natural Polish group topology: if Γ is a countable group acting by measure-preserving automorphisms on a standard probability space (X, μ), then one can define a complete, separable distance on the full group of this action by setting

$$d(S,T) = \mu(\{x \colon S(x) \neq T(x)\}) \ .$$

We have a good understanding of how full groups sit inside the group $\mathrm{Aut}(X, \mu)$ of all measure-preserving automorphisms, endowed with its usual (and unique) Polish group topology: the full group of an ergodic action of a countable group is always dense in $\mathrm{Aut}(X, \mu)$, and Wei [7] observed that this group is always a countable intersection of a countable union of closed subsets of $\mathrm{Aut}(X, \mu)$ (which essentially follows from the fact that the distance on the full group is lower-semicontinuous with respect to the topology induced by the Polish topology of $\mathrm{Aut}(X, \mu)$); in particular, full groups are always Borel subsets of the ambient group. Full groups have proved to be useful to study and understand countable measure-preserving equivalence relations, we refer to [6] for an in-depth discussion.

Going back to the context of topological dynamics: it would be desirable that the full group of a minimal action of a countable group on a Cantor space admit a compatible Polish group topology, since it would enable one to employ descriptive-set-theoretic techniques to understand these groups (and, via the Giordano–Putnam–Skau theorem mentioned above, properties of the associated equivalence relations). No such topology has been introduced in earlier work, which might lead one to suspect that this is an impossible task.

Theorem 1 ([5])**.** *Let Γ be a countable group acting minimally by homeomorphisms on a Cantor space X. Then there does not exist a second countable, Hausdorff group topology on $[\Gamma]$ for which the Baire category theorem holds.*

In the case $\Gamma = \mathbb{Z}$, i.e. the action is induced by a single minimal homeomorphism φ, the closure of $[\varphi]$ inside the homeomorphism group is well-understood: it follows from a result of Glasner–Weiss [3] that this closure coincides with the group of all homeomorphisms which preserve all the φ-invariant measures (as far as I know, describing the closure of a minimal action of a countable group is an open problem, even in the amenable case). It is natural to wonder whether an analogue of Wei's theorem holds in this setting.

Theorem 2 ([5])**.** *Let φ be a minimal homeomorphism of a Cantor space X. Then the full group $[\varphi]$ is a coanalytic, non Borel subset of the homeomorphism group Homeo(X).*

Note that above $\mathrm{Homeo}(X)$ is endowed with its unique Polish group topology (which coincides with the topology of uniform convergence for some/any compatible metric). We do not know whether the above result holds true for minimal actions of any countable group (though it seems natural to suspect so); this is because the proof uses, in a technical step, a theorem of Glasner–Weiss [3] that is only known to hold for $\mathbb{Z}$-actions (using work of Giordano–Matui–Putnam–Skau, one can extend this result to actions of $\mathbb{Z}^d$, but I know nothing beyond that).

We advocate the study of *closures* of minimal $\mathbb{Z}$-actions, which are complete invariants of orbit equivalence (when one restricts one's attention to $\mathbb{Z}$-actions) and have a built-in (and unique) compatible Polish group topology. We prove that these groups are always topologically simple (hence, so is the full group), but do not know whether they are always simple.

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Specialising all Aronszajn Trees and establishing the Ostaszewski club

Heike Mildenberger

(joint work with Saharon Shelah)

We prove:

Theorem 1. *It is consistent relative to* ZFC *that all Aronszajn trees are special and that the club principle holds.*

Thus we give a strong negative answer to the question whether the Ostaszewski club principle imply the existence of a Souslin tree. The question was asked by Juhász in 1976 [6].

Ostaszewski [7] introduced the club principle, also written ♣, for a topological construction:

Definition 1. *The club principle says: There is a sequence* $\langle C_\alpha : \alpha < \omega_1, \lim(\alpha)\rangle$ *such that for every uncountable* $X \subseteq \omega_1$ *the set*

$$\{\alpha < \omega_1 : C_\alpha \textit{ is a cofinal subset of } X \cap \alpha\}$$

is stationary. We call such a sequence $\langle C_\alpha : \alpha < \omega_1, \lim(\alpha)\rangle$ *a ♣-sequence.*

We recall the main notions about trees:

Definition 2. (1) $(\mathbf{T}, <_\mathbf{T})$ *is called an Aronszajn tree if*

(a) $|\mathbf{T}| = \aleph_1$,

(b) $(\mathbf{T}, <_\mathbf{T})$ *is a partial order such that for any* $t \in \mathbf{T}$, $\text{pred}(t) = \{s \in \mathbf{T} : s <_\mathbf{T} t\}$ *is well-ordered; there is a least element in* $<_\mathbf{T}$ *called the root of* $\mathbf{T}$,

(c) *for* $\alpha < \omega_1$, *the level* $L_\alpha = \{t \in \mathbf{T} : \text{pred}(t) \cong \alpha\}$ *of* $(\mathbf{T}, <_\mathbf{T})$ *is (at most) countable,*

(d) $(\mathbf{T}, <_\mathbf{T})$ *has no uncountable branch.*

(2) *An Aronsajn tree* $(\mathbf{T}, <_{\mathbf{T}})$ *is special if it has a specialistation function. A function* $f\colon \mathbf{T} \to \omega$ *is called a specialisation of* $(T, <_{\mathbf{T}})$ *or we say* f *specialises* $(\mathbf{T}, <_{\mathbf{T}})$ *iff* $\forall s, t \in \mathbf{T}(s <_{\mathbf{T}} t \to f(s) \neq f(t))$.

(3) *A Souslin tree is an Aronszajn tree in which all antichains are countable.*

(4) *The diamond principle,* $\diamondsuit$, *introduced by Jensen* [3] *says: There is a sequence* $\langle S_\alpha : \alpha < \omega_1\rangle$ *such that for every* $X \subseteq \omega_1$ *the set*

$$\{\alpha < \omega_1 : S_\alpha = X \cap \alpha\}$$

is stationary. We call such a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ *a* $\diamondsuit$*-sequence.*

If an Aronszajn tree is special then it is the union of countably many antichains, and hence the tree is not a Souslin tree. So our main theorem answers Juhász' question in a strong negative form. Shelah showed [8, Ch. IX] that "There is no Souslin tree" does not imply "All Aronszajn trees are special".

We define the tree creatures which will be used later to describe the branching of the countable trees that will serve as forcing conditions. For $n \in \omega\setminus\{0\}$, we write

$$\mathrm{spec}^{\mathbf{T}}_n = \{\eta : u \subseteq \mathbf{T} \text{ is finite}, \eta\colon u \to [0, n) \wedge (\eta(x) = \eta(y) \to \neg(x <_{\mathbf{T}} y))\}.$$

Definition 3. *A* creature *is a tuple* $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}))$ *with the following properties:*

(a) *The first component,* $i(\mathbf{c})$, *is called the kind of* $\mathbf{c}$ *and is just a natural number.* $\mathbf{c}$ *is an* i*-creature if* $i(\mathbf{c}) = i$.

(b) *The second component,* $\eta(\mathbf{c})$, *is called the base of* $\mathbf{c}$. *We require* ($\eta(\mathbf{c}) = \emptyset$ *and* $i(\mathbf{c}) = 0$) *or* ($i(\mathbf{c}) = i > 0$ *and* $0 \neq |\mathrm{dom}(\eta(\mathbf{c}))| < n_{1,i-1}$, *and* $\eta(\mathbf{c}) \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i-1}}$).

(c) $\mathrm{pos}(\mathbf{c})$ *is a non-empty subset of* $\{\eta \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i}} : \eta(\mathbf{c}) \subsetneqq \eta \wedge |\mathrm{dom}(\eta)| < n_{1,i}\}$.

Definition 4. *For an* i*-creature* $\mathbf{c}$ *we define* $\mathrm{nor}^0(\mathbf{c})$ *as the maximal natural number* m *such that* $m = 0$ *or*

(α) *if* $a \subseteq n_{2,i}$ *and* $|a| \leqslant m$ *and* $B_0, \dots, B_{m-1}$ *are branches of* $\mathbf{T}$, *then there is* $\nu \in \mathrm{pos}(\mathbf{c})$ *such that* $(\forall x \in (\bigcup_{\ell<m} B_\ell \cap \mathrm{dom}(\nu))\setminus\mathrm{dom}(\eta(\mathbf{c})))(\nu(x) \notin a)$,

(β) $\max\{\frac{|\mathrm{dom}(\nu)|}{n_{1,i}} : \nu \in \mathrm{pos}(\mathbf{c})\} \leqslant \frac{1}{m}$,

Definition 5. *Let* $\mathbf{T}$ *be an Aronszajn tree. We define a notion of forcing* $\mathbb{Q} = \mathbb{Q}_{\mathbf{T}}$ *with set of elements* $\mathbb{Q}$ *and a preorder* $\leqslant_{\mathbb{Q}}$.

(A) $p \in \mathbb{Q}$ *iff* $p = (i(p), T^p, <_p)$ *has the following properties:*

(a) *There is a set* $T^p \subseteq {}^{\omega>}\mathrm{spec}^{\mathbf{T}}$ *such that* $(T^p, <_p)$ *is a* $\mathbf{T}$*-tree with* ω *levels, the* ℓ*-th level of which is denoted by* $(T^p)^{[\ell]}$. *For* $t \in (T^p)^{[\ell]}$ *we also write and* $\mathrm{ht}_p(t) = \mathrm{ht}_{T^p}(t) = \ell$.

(b) T^p *has a root, the unique element of level 1, called* $\mathrm{rt}(p)$.

(c) *We have* $i(p) = i < \omega$ *such that the following holds: For any* $1 \leqslant \ell < \omega$ *and* $s \in (T^p)^{[\ell]}$ *there is an* $i + \ell - 1$ *creature*
$\mathbf{c}_{p,s}$ *such that*
$\eta(\mathbf{c}_{p,s}) = \mathrm{last}(s)$,

$\text{pos}(\mathbf{c}_{p,s}) = \{\text{last}(t) : t \in (T^p)^{[\ell+1]} : t \in \text{suc}_p(s)\}$.
We also write $i(s) = i$ for $i(\mathbf{c}_{p,s}) = i$.

(d) *For every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of T^p with $t_\ell = (\eta_0, \dots, \eta_\ell)$ we have $\lim_{\ell \to \omega} \text{nor}^0(\mathbf{c}_{p,t_\ell}) = \omega$.*

(B) *The order $\leqslant = \leqslant_{\mathbb{Q}}$ is given by letting $p \leqslant q$ (q is stronger than p, we follow the Jerusalem convention) iff $i(p) \leqslant i(q)$ and there is a projection $\text{pr}_{q,p}$ which satisfies*

(a) *$\text{pr}_{q,p}$ is a function from T^q to T^p preserving the absolute height: $\text{ht}_{T^q}(t) + i(q) = \text{ht}_{T^p}(\text{pr}_{q,p}(t)) + i(\text{pr}_{q,p}(\text{rt}(q)))$.*

(b) *If $t \in T^q$ then $\text{last}(t) \supseteq \text{last}(\text{pr}_{q,p}(t))$. This holds of course not only for the last element of the sequence t but for all elements, since T^q is downward closed.*

(c) *If t_1, t_2 are both in $\text{dom}(q)$ and if $t_1 \leqslant_q t_2$, then $\text{pr}_{q,p}(t_1) \leqslant_p \text{pr}_{q,p}(t_2)$.*

(d) *For every $t \in T^q$, $i(\mathbf{c}_{q,t}) = i(\mathbf{c}_{p,\text{pr}_{q,p}(t)})$.*

(e) *For any $\ell \in \omega$, $t \in (T^q)^{[\ell]} \Rightarrow \text{pr}_{q,p}(t) \in (T^p)^{[\ell+i(q)-i(p)]}$.*

(f) *For any $\ell \in \omega$: If $s \in (T^q)^{[\ell]}$ and $r \in (T^q)^{[\ell+1]}$ and $s \leqq_q r$, $\text{pr}_{q,p}(s) = t'$, $\text{pr}_{q,p}(r) = t$, then $\text{dom}(\text{last}(t)) \cap \text{dom}(\text{last}(s)) = \text{dom}(\text{last}(t'))$.*

The iterands $\mathbb{Q}_{\mathbf{T}}$ have a strong form of Axiom A.

Starting from a ground model with $\diamondsuit$ and $2^{\aleph_1} = \aleph_2$, we iterate such forcings with countable supports, using a book-keeping device, with iteration length ω_2. With the techniques from [4], that bear on [2] and [1], we show that in the resulting model the club principle holds. A name for the club sequence is based on the diamond in the ground model and is explicitly given. The preprint is posted in [5].

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Completely proper forcing and the Continuum Hypothesis

Justin Moore

In this talk I will answer a question of Shelah concerning which forcing axioms are consistent with the Continuum Hypothesis (CH). The forcing axiom for a class $\mathfrak{C}$ of partial orders is the assertion that if $\mathcal{D}$ is a collection of $\aleph_1$ cofinal subsets of a partial order, then there is an upward directed set G which intersects each element of $\mathcal{D}$. The well known Martin's Axiom for $\aleph_1$ dense sets ($MA_{\aleph_1}$) is the forcing axiom for the class of *c.c.c.* partial orders. Foreman, Magidor, and Shelah have isolated the largest class of partial orders $\mathfrak{C}$ for which a forcing axiom is consistent with ZFC. Shelah has established certain sufficient conditions on a class $\mathfrak{C}$ of partial orders in order for the forcing axiom for $\mathfrak{C}$ to be consistent with CH. His question concerns to what extent his result was sharp. The main result discussed in this talk, especially taken together with joint work with Aspero and Larson, shows that these conditions are close to optimal.

The solution to Shelah's problem has some features which are of independent interest. Suppose that T is a collection of countable closed subsets of ω_1 which is closed under taking closed initial segments and that T has the following property:

(1) Whenever s and t are two elements of T with the same supremum δ and $\lim(s) \cap \lim(t)$ is unbounded in δ, then $s = t$.

(Here $\lim(s)$ is the set of limit points of s.) Here, we regard T as a set-theoretic tree by declaring $s \leqslant t$ if s is an initial part of t. Observe that this condition implies that T contains at most one uncountable path: any uncountable path would have a union which is a closed unbounded subset of ω_1 and such subsets of ω_1 must have an uncountable intersection. In fact it can be shown that this condition implies that

$$\{(s,t) \in T^2 : (\mathrm{ht}_T(s) = \mathrm{ht}_T(t)) \wedge (s \neq t)\}$$

can be decomposed into countably many antichains. Jensen and Kunen, in unpublished work, have constructed examples of ω_1-Baire trees which are special off the diagonal under the assumption of $\diamondsuit$, but this article represents the first such construction from CH.

The main result of the talk asserts that if the Continuum Hypothesis is true, then there is tree T which satisfies (1) together with the following additional properties:

(1) T has no uncountable path;
(2) for each t in T, there is a closed an unbounded set of δ such that $t \cup \{\delta\}$ is in T;
(3) T is proper as a forcing notion and remains so in any outer model with the same set of real numbers in which T has no uncountable path. Moreover T is complete with respect to a simple $\aleph_1$-completeness system $\mathbb{D}$ in the sense of Shelah.

This provides the first example of a consequence of the Continuum Hypothesis which justifies condition (1) in the next theorem.

Theorem 1. *(Shelah; Eisworth) Suppose that $\langle P_\alpha; Q_\alpha : \alpha \in \theta\rangle$ is a countable support iteration of proper forcings which:*

(a) *are complete with respect to a simple 2-completeness system $\mathbb{D}$;*
(b) *satisfy either of the following conditions:*
   (i) *are weakly α-proper for every $\alpha \in \omega_1$;*
   (ii) *are proper in every proper forcing extension with the same set of real numbers.*

Then forcing with P_θ does not introduce new real numbers.

On the descriptive set-theoretical complexity of the embeddability relation between uncountable models

Luca Motto Ros

Given an infinite cardinal κ and an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ, we denote by $\mathrm{Mod}^\kappa_\varphi$ the collection of all $\mathcal{L}$-structures of size κ which satisfy φ. The complexity of the embeddability relation $\sqsubseteq$ on collections of the form $\mathrm{Mod}^\kappa_\varphi$ has been widely studied in the literature, see e.g. [NW65, Lav71, Bau76, She84, Mek90, KS92, DS03, Tho06]. In all these works, only the *combinatorial* complexity of (the quotient order of) $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ has been considered. For example, Baumgartner constructed in [Bau76] a *reduction* between the relation of inclusion modulo nonstationary sets on $\{X \subseteq \kappa \mid X \text{ is stationary}\}$ (for a regular $\kappa > \omega$) into the embeddability relation between linear orders of size κ, i.e. a map f such that for all stationary $X, Y \subseteq \kappa$

$$X\backslash Y \text{ is nonstationary} \iff f(X) \sqsubseteq f(Y)$$

(with $f(X)$, $f(Y)$ linear orders of size κ).

On the other hand, since each embeddability relation $\sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$ can be easily construed as an analytic quasi-order, when $\kappa = \omega$ it is also very natural to use the standard notion of Borel reducibility to analyze its *descriptive set-theoretical* complexity, that is to compare $\sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$ with other analytic quasi-orders on standard Borel spaces by means of *Borel* (or more generally: definable) *reductions*. This is clearly a stronger approach which gives information on both the combinatorial and the topological complexity of $\sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$. In this direction, building on previous work of Louveau and Rosendal [LR05] we obtained the following universality result. (Recall that a graph is called *combinatorial tree* if it is connected and acyclic.)

Theorem 1 (Friedman-Motto Ros [FMR11])**.** *Assume* $\mathsf{ZF}+\mathsf{DC}$*. Then the embeddability relation on countable combinatorial trees is* strongly invariantly universal *for analytic quasi-orders on standard Borel spaces, that is, for every $\boldsymbol{\Sigma}^1_1$ quasi-order R on a standard Borel space X there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that:*

(1) *all models of φ are combinatorial trees;*
(2) *there is an isomorphism f between the quotient order of R and the quotient order of $\sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$ such that both f and f^{-1} admit Borel liftings.*

Roughly speaking, this means that:

(a) the embeddability relation between countable combinatorial trees is extremely complicated, as it contains a copy of any $\mathbf{\Sigma}^1_1$ quasi-order on a standard Borel space;
(b) every $\mathbf{\Sigma}^1_1$ quasi-order on a standard Borel space can be identified in a faithful way with an embeddability relation of the form $\sqsubseteq \restriction \mathrm{Mod}^\omega_\varphi$, and thus the study of arbitrary analytic quasi-orders may be reduced (up to Borel isomorphism) to the study of embeddability relations between countable structures.

It is then natural to ask whether also the embeddability relation between structures of size $\kappa > \omega$ has similar universality properties. This of course requires that all the notions involved in Theorem 1 be adapted to the new context, and hence one is naturally led to employ ideas and methods from generalized descriptive set-theory, a very active area of research which studies the topological properties of the generalized Baire space ${}^\kappa\kappa$ (and of other related spaces) by combining classical descriptive set-theoretical methods with forcing techniques, results in the combinatorics of infinite cardinals, and so on.

In a series of papers, we showed that Theorem 1 can indeed be generalized to many uncountable cardinals κ under suitable set-theoretical assumptions. Here is a small sample of the results we obtained — we refer the reader to [MR13, AMR14, MMR14] for the exact definitions of the notions involved as well as for more results of this kind.

Theorem 2 (Andretta-Motto Ros [AMR14])**.** *Assume* $\mathsf{ZF}+\mathsf{DC}$*. Then the embeddability relation on combinatorial trees of size* $\aleph_1$ *is strongly invariantly universal for* $\mathbf{\Sigma}^1_2$ *quasi-orders on standard Borel spaces.*

Theorem 3 (Andretta-Motto Ros [AMR14])**.** *Assume* ZFC *and that* $x^\#$ *exists for every* $x \in {}^\omega\omega$*. Then the embeddability relation on combinatorial trees of size* $\aleph_2$ *is strongly invariantly universal for* $\mathbf{\Sigma}^1_3$ *quasi-orders on standard Borel spaces.*

More generally, let $r \colon \omega \to \omega$ be defined by

$$r(n) = \begin{cases} 2^{k+1}-1 & \text{if } n = 2k+1 \\ 2^{k+1} & \text{if } n = 2k+2. \end{cases}$$

Theorem 4 (Andretta-Motto Ros [AMR14])**.** *Assume* $\mathsf{ZFC}+\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$*. Then the embeddability relation on combinatorial trees of size* $\aleph_{r(n)}$ *is strongly invariantly universal for* $\mathbf{\Sigma}^1_n$ *quasi-orders on standard Borel spaces.*

Theorems 2–4 are meaningful only when the size of the continuum is larger than the cardinals involved, but they are notably independent of the actual value of $2^{\aleph_0}$.

Similar results concerning the embeddability relation between structures of "small" uncountable size may be obtained also in choiceless settings:

Theorem 5 (Andretta-Motto Ros [AMR14]). *Assume* AD. *For every $n \in \omega$ the embeddability relation on combinatorial trees of size $\boldsymbol{\delta}^1_{2n+1}$ is universal for $\boldsymbol{\Sigma}^1_{2n+2}$ quasi-orders on standard Borel spaces.*

Even stronger results may be obtained when considering larger uncountable cardinals. We call *generalized tree* a partial order $(T, \leqslant)$ such that the set of $\leqslant$-predecessors of each $t \in T$ is linearly ordered by $\leqslant$.

Theorem 6 (Mildenberger-Motto Ros [MMR14], see also [MR13] for the special case of a weakly compact κ). *Assume* ZFC *and let $\kappa > \omega$ be such that $\kappa^{<\kappa} = \kappa$. Then the embeddability relation on generalized trees of size κ is strongly invariantly universal for quasi-orders on ${}^\kappa\kappa$ which are κ-analytic, i.e. continuous images of closed subsets of ${}^\kappa\kappa$.*

Many interesting open problems remain to be solved in this area, including the following:

Questions.

(1) Is the condition $\kappa^{<\kappa} = \kappa$ necessary in Theorem 6?
(2) What is the descriptive set-theoretical complexity of a given embeddability relation $\sqsubseteq \restriction \mathrm{Mod}^\kappa_\varphi$ for κ a singular cardinal? Can it consistently be strongly invariantly universal for κ-analytic quasi-orders on ${}^\kappa\kappa$?
(3) Can we replace combinatorial and generalized trees with e.g. linear orders or groups in Theorems 2–6?

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Higher analogues of PFA and applications

Itay Neeman

We describe several analogues of PFA to contexts that involve meeting $\aleph_2$ maximal antichains, and some applications of these analogues. Among the applications is an analogue of MRP, leading to reflecting sequences of length ω_2, with stationarily many reflection points. This analogue implies the failure of $\square_\kappa$ for $\kappa \geqslant \omega_2$, and with a modification appears to also lead to a model where there is a well ordering of $H(\omega_3)$ of order type ω_3 definable over $H(\omega_3)$ from a parameter contained in ω_2.

Complicated colorings

Assaf Rinot

The theory of strong colorings was born with the 1933 construction of Sierpiński of a symmetric function $c : \mathbb{R}^2 \backslash \Delta \to 2$ that does not admit an uncountable monochromatic square. For the construction, one fixes a well-ordering $<_W$ of the reals, and contrast it with the usual ordering $<$, by letting $c(x,y) = 1$ iff $(x < y \;\&\; x <_W y) \vee (y < x \;\&\; y <_W x)$. The separability of $\mathbb{R}$ then implies that for every uncountable set A of reals, and every $i < 2$, there exist $x <_W y$ in A such that $c(x,y) = i$.

In the 1960's, Erdős and his school initiated a systematic study of this sort of colorings, and introduced the following piece of notation. We say that $\lambda \not\rightarrow [\mu]^2_\theta$ holds provided that there exists a symmetric coloring of pairs $d : [\lambda]^2 \to \theta$ with the property that for every subset A of λ of size μ, and every color $\gamma < \theta$, there exist $\alpha < \beta$ in A such that $d(\alpha,\beta) = \gamma$. So, Sierpiński's partition is a witness to $2^{\aleph_0} \not\rightarrow [\aleph_1]^2_2$. Erdős, Hajnal and Rado proved [1] that, assuming the Generalized Continuum Hypothesis (GCH), $\lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+}$ holds for every infinite cardinal λ. On its face, the existence of such a coloring $d : [\lambda^+]^2 \to \lambda^+$ that attains all possible colors on all squares of unbounded subsets of λ^+ appears to be the strongest conceivable failure of Ramsey's theorem at the level of successor cardinals. However, one can ask for more. To see this, let us revisit Sierpiński's example. Since c is symmetric, we get that if A is a rectangle of the form $I \times J$ of disjoint real intervals I and J, then $c \restriction A$ is a constant function. So, while c admits no monochromatic uncountable squares, it does admit monochromatic uncountable rectangles. In contrast, the Erdős-Hajnal-Rado coloring does attain all colors even on rectangles. Of course, this appears to come with a price: the Erdős-Hajnal-Rado construction requires the GCH. We refer the reader to [3],[4] for a resolution of this particular aspect, and turn now to a further finer concept:

Definition 1 (Shelah, [6]). $\mathrm{Pr}_1(\lambda,\mu,\theta,\chi)$ *asserts the existence of a coloring* $d : [\lambda]^2 \to \theta$ *such that for any family* $\mathcal{A} \subseteq [\lambda]^{<\chi}$ *of size* μ, *consisting of pairwise disjoint sets, and every color* $\gamma < \theta$, *there exist* $a, b \in \mathcal{A}$ *with* $\sup(a) < \min(b)$ *satisfying* $d[a \times b] = \{\gamma\}$.

So, the Sierpiński example is a coloring satisfying $\mathrm{Pr}_1(2^{\aleph_0}, \aleph_1, 2, 2)$, that fails to satisfy $\mathrm{Pr}_1(2^{\aleph_0}, \aleph_1, 2, 3)$. This justifies the above parameter χ. Another justification, and in fact, the origin of this concept, is in its effect on questions concerning chain conditions of product of topological spaces and related objects. For example, if $\mathrm{Pr}_1(\lambda, \lambda, 2, \omega)$ holds, then there exist two λ-cc Boolean algebras whose product is not λ-cc (see Galvin [2], Todorčević [13][14], and Shelah [8]).

Now, what about the parameter θ? Here, we mention that, for example, Shelah proved [10] that $\mathrm{Pr}_1(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$ holds for every singular cardinal λ, whereas the question of whether $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \mathrm{cf}(\lambda))$ (or just $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, 2)$) holds for every singular cardinal λ is the oldest open problem of this field.

In a breakthrough made by Todorčević [15], he proved that $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ holds outright in ZFC. Moreover, in the presence of a nonreflecting stationary set, Todorčević's technology generalizes to arbitrary regular cardinals $> \aleph_1$, yielding:

Theorem 1 (Todorčević, [15]; Shelah [7]). *If $\lambda > \aleph_1$ is a regular cardinal that admits a nonreflecting stationary set, then $\lambda \nrightarrow [\lambda]^2_\lambda$. That is, $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \chi)$ holds for $\chi = 2$.*

This raises the question whether under the same hypothesis, the above holds true also for higher χ's?

This particular question and its variations were studied systematically by Shelah in a sequence of papers [5],[6],[7],[8],[9],[11],[12], and in his monograph [10]. Roughly speaking, the difficulty in establishing $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \chi)$ for $\chi > 2$ is the need for some room that allows to derive several oscillation functions (and then to contrast them), or to enforce repetitions that allows to find for every family $\mathcal{A} \subseteq [\lambda]^{<\chi}$ as in Definition 1, an equipotent subfamily $\mathcal{A}' \subseteq \mathcal{A}$ which is, *less diverse* or *tamed*, in various senses. In [8], this was obtained through the arithmetic assumption "$2^\chi < \lambda$", and in [9], $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \chi)$ was established for $\chi = \aleph_0$ and a cardinal λ that admits a nonreflecting stationary set $S \subseteq \lambda$, through the requirement that for all $\alpha \in S$, $\mathrm{cf}(\alpha)$ is at least the double-successor of χ. Later on, in [11], it was proved that $\mathrm{Pr}_1(\chi^{++}, \chi^{++}, \chi^{++}, \chi)$ holds for every regular cardinal χ.

In this paper, we eliminate the arithmetic hypothesis from [8], eliminate the "double successor" cofinality gap requirement from [9], obtain the main results of [11][12] as a corollary, and indeed increase χ from 2 to ω in Theorem 1 above. It is proved:

Main result 1. If λ, χ are regular cardinal, $\lambda > \chi^+$, and $E^\lambda_{\geqslant\chi}$ admits a nonreflecting stationary set, then $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \chi)$ holds.

Let us say a few words about the proof. In [11], Shelah introduced a coloring principle $\mathrm{Pr}_6(\kappa, \kappa, \theta, \chi)$, studied its validity, and provided a lifting theorem:

Theorem 2 (Shelah, [11]). $\mathrm{Pr}_6(\mu^+, \mu^+, \mu^+, \mu)$ *holds for every regular cardinal μ.*

Theorem 3 (Shelah, [11]). *If $\mathrm{Pr}_6(\kappa, \kappa, \theta, \chi)$ holds, $\chi \leqslant \kappa < \mathrm{cf}(\lambda) = \lambda$, and there exists a nonreflecting stationary subset of $E^\lambda_{\geqslant\kappa}$, then $\mathrm{Pr}_1(\lambda, \lambda, \theta, \chi)$ holds.*

Unfortunately, if θ is considerably smaller than λ, it is unclear how to infer $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\chi)$ from $\mathrm{Pr}_1(\lambda,\lambda,\theta,\chi)$.[1] So, for instance, it is unclear how to deduce the main result of [9] from the above strategy. Moreover, by Ramsey's theorem, $\mathrm{Pr}_6(\kappa,\kappa,\theta,\chi)$ fails for $\kappa=\aleph_0$ and $\theta\geqslant 2$, so if λ admits a nonreflecting stationary set, but every stationary subset of $E^{\lambda}_{>\omega}$ reflects, then the above theorem does not come into play.

In this paper, we introduce a relative of Pr_6 that allows to overcome these two barriers:

Definition 2. *$P\ell_6(\kappa,\chi)$ asserts the existence of a coloring $c:{}^{<\omega}\kappa\rightarrow\omega$ satisfying the following. For every sequence $\langle(u_\alpha,v_\alpha,\rho_\alpha)\mid\alpha<\kappa\rangle$ and $\varphi:\kappa\rightarrow\kappa$ with*

1. *φ is regressive. That is, $\varphi(\alpha)<\alpha$ for co-boundedly many $\alpha<\kappa$;*
2. *u_α and v_α are nonempty elements of $[{}^{<\omega}\kappa]^{<\chi}$;*
3. *$\alpha\in\mathrm{Im}(\eta)$ for all $\eta\in u_\alpha$;*
4. *$\rho_\alpha{}^\frown\langle\alpha\rangle\sqsubseteq\rho$ for all $\rho\in v_\alpha$,*

there exist $\alpha<\beta<\kappa$ with $\varphi(\alpha)=\varphi(\beta)$ such that $c(\eta^\frown\rho)=\ell(\eta)$ for all $\eta\in u_\alpha$ and $\rho\in v_\beta$.

At a first glance, it may seem that $P\ell_6$ puts an impossible task on c: decomposing a concatenated sequence back into its original ingredients. Yet, it proved:

Theorem 4. *$P\ell_6(\mu^+,\mu)$ holds for every regular cardinal μ.*

Theorem 5. *If $P\ell_6(\kappa,\chi)$ holds, $\chi\leqslant\kappa<\mathrm{cf}(\lambda)=\lambda$, and there exists a nonreflecting stationary subset of $E^{\lambda}_{\geqslant\chi}$, then $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\chi)$ holds.*

This time, the main result of [9] does follow as a corollary. Indeed, Theorem 1 above is improved to the following.

Corollary 1. *If $\lambda>\aleph_1$ is a regular cardinal that admits a nonreflecting stationary set, then $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\aleph_0)$ holds.*

In conclusion, note that the preceding is optimal, since Galvin proved that Martin's axiom entails the failure of $\mathrm{Pr}_1(\aleph_1,\aleph_1,2,\aleph_0)$.

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Automatic continuity for isometry groups

Marcin Sabok

In this talk, I presented a general framework for automatic continuity results for groups of isometries of metric spaces [1]. Automatic continuity is a phenomenon that connects the algebraic and topological structures and typically says that any map which preserves an algebraic structure must automatically be continuous. It has been extensively studied for derivations on C*-algebras and in the context of groups and their homomorphisms, one of the first automatic continuity results has been proved by Dudley who showed that any homomorphism from a complete metric or a locally compact group into a free group is continuous.

A topological group G has the *automatic continuity property* if for every separable (or even Polish) topological group H, any group homomorphism from G to H is continuous. Recall that any measurable homomorphism from a Polish group to a separable group must be continuous and the existence of non-measurable homomorphisms on groups such as $(\mathbb{R}, +)$ can be derived from the axiom of choice. So, similarly as amenability, automatic continuity property for a given group can be interpreted in terms of nonexistence (on this group) of pathological phenomena that can follow from the axiom of choice.

The Urysohn space $\mathbb{U}$ is the separable complete metric space which is *homogeneous* (i.e. any finite partial isometry of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$) and such that any finite metric space embeds into $\mathbb{U}$ isometrically. The analogue of the Urysohn space of diameter 1 also exists and is called the *Urysohn sphere* (or the *bounded Urysohn space of diameter 1*) and denoted by $\mathbb{U}_1$. The group of isometries of $\mathbb{U}$ is universal among Polish groups, i.e. any Polish group is its closed subgroup.

The question whether the group of isometries of the Urysohn space has the automatic continuity property has been raised by Melleray. One of the main applications of the work presented during this talk, is the following.

Theorem 1. *The groups of isometries of the Urysohn space and the Urysohn sphere have the automatic continuity property.*

Theorem 1 has some immediate consequences on the topological structure of the above groups.

Corollary 1. *The group* $\mathrm{Iso}(\mathbb{U})$ *has unique Polish group topology.*

Recall that a group is *minimal* if it does not admit any strictly corser (Hausdorff) group topology. The second corollary follows from minimality of the group of isometries of the Urysohn sphere, proved by Uspenskij.

Corollary 2. *The group* $\mathrm{Iso}(\mathbb{U}_1)$ *has unique separable group topology.*

Theorem 1 follows from the following abstract result, which isolates metric (or model-theoretic) properties of a metric structure that imply that the group of automorphisms (with the pointwise convergence topology) of the structure has the automatic continuity property. The definitions of a metric structure and the notions appearing in the statement of the theorem were given during the talk.

Theorem 2. *Suppose M is a homogeneous complete metric structure that has locally finite automorphisms, the extension property and admits weakly isolated sequences. Then the group* $\mathrm{Aut}(M)$ *has the automatic continuity property.*

Theorem 2 can be also applied to give a unified treatment of previously known automatic continuity results for automorphism groups of some metric structures. It is worth mentioning that up to now, these results have been proved with different methods, varying from case to case. During the talk, I discussed how to apply Theorem 2 to show the automatic continuity property for the group $\mathrm{Aut}(\mu)$ (the group of measure-preserving automorphism of the unit interval) and the group $U(\ell_2)$ (unitary operators of the infinite-dimensional separable Hilbert space).

Corollary 3 (Ben Yaacov, Berenstein, Melleray)**.** *The group* $\mathrm{Aut}(\mu)$ *has the automatic continuity property.*

Corollary 4 (Tsankov)**.** *The group* $U(\ell_2)$ *has the automatic continuity property.*

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Core Model Induction and Hod Mice

Grigor Sargsyan

We will introduce a new covering principle call Covering with Derived Models and show how it and its generalization can be used to get strength out of failure of square at a measurable cardinal.

Does Π^1_1 determinacy imply $0^\#$?

Ralf Schindler
(joint work with Cheng Yong)

A well–known theorem of T. Martin and L. Harrington says that Π^1_1 determinacy is equivalent with the existence of $0^\#$. The proof of the forward direction ("$\Longrightarrow$") goes through the following principle.

Definition 1. *We let* Harrington's Principle*,* HP *for short, denote the following statement:*

$$\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \longrightarrow \alpha \text{ is an } L\text{-cardinal}).$$

We may then state the above–mentioned result as follows.

Theorem 1. (Martin–Harrington–Silver) *Assume* ZF*. The following statements are equivalent.*

(1) Π^1_1 *determinacy.*
(2) HP*.*
(3) $0^\#$ *exists.*

Each one of the statements (1), (2), and (3) is Σ^1_3, so that it is natural to ask, as did H.W. Woodin, if in Theorem 1 we may relax ZF to second order arithmetic.

Definition 2. *[(i)]*
(1) $Z_2 = ZFC^- +$ *Every set is countable.*[1]
(2) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists* + *Every set is of cardinality* $\leqslant \beth_1$.
(3) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* + *Every set is of cardinality* $\leqslant \beth_2$.

Z_2, Z_3, and Z_4 correspond to second order arithmetic (SOA), third order arithmetic, and fourth order arithmetic, respectively. It is not hard to verify that "(1) $\Longrightarrow$ (2)" and "(3) $\Longrightarrow$ (1)" in Theorem 1 are both provable in Z_2. Also, "(2) $\Longrightarrow$ (3)" is provable in Z_4.

Our paper produces the following results.

Theorem 2. *The following theories are equiconsistent.*

(1) Z_2 + HP
(2) ZFC

Theorem 3. *The following theories are equiconsistent.*

(1) Z_3 + HP
(2) ZFC + *there exists a remarkable cardinal.*

We also investigate strengthenings of Harrington's Principle, HP(φ), over higher order arithmetic.

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[1] ZFC^- denotes ZFC with the Power Set Axiom deleted.

Definition 3. *Let $\varphi(-)$ be a Σ_2–formula in the language of set theory such that, provably in* ZFC*: for all α, if $\varphi(\alpha)$, then α is an inaccessible cardinal and $L \models \varphi(\alpha)$. Let* HP$(\varphi)$ *denote the statement:*

$$\exists x \in 2^\omega \forall \alpha(\alpha \text{ is } x\text{-admissible} \longrightarrow L \models \varphi(\alpha)).$$

Theorem 4. *Let $\varphi(-)$ be as in Definition 3. The following theories are equiconsistent.*

(1) $Z_2 + HP(\varphi)$.
(2) $ZFC + \{\alpha|\varphi(\alpha)\}$ *is stationary.*

Theorem 5. *Let $\varphi(-)$ be as in Definition 3. The following theories are equiconsistent.*

(1) $Z_3 + \mathsf{HP}(\varphi)$.
(2)

$$ZFC + \text{ there exists a remarkable cardinal } +$$
$$\{\alpha|\varphi(\alpha) \wedge \{\beta < \alpha|\varphi(\beta)\} \text{ is stationary in } \alpha\} \text{ is stationary.}$$

As a corollary, Z_4 is the minimal system of higher order arithmetic to show that HP, HP(φ), and $0^\sharp$ exists are pairwise equivalent with each other. The question whether Π^1_1 determinacy implies the existence of $0^\#$ in Z_3 remains unanswered.

This is joint workwith Cheng Yong.

Very good scales and the failure of SCH

Dima Sinapova
(joint work with Spencer Unger)

The square property was isolated by Jensen in his fine structure analysis of L. It is an "incompacness" property, that holds in canonical inner models. Square at κ states that there is a coherent sequence of closed unbounded subsets singularizing points $\alpha < \kappa^+$. There are various weakenings of this principle by allowing multiple guesses for each club. More precisely:

Definition 1. *$\square_{\kappa,\lambda}$ holds if there is a sequence $\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+\rangle$, such that*

- *$1 \leqslant |\mathcal{C}_\alpha| \leqslant \lambda$, and*
- *if $C \in \mathcal{C}_\alpha$, then C is a club subset of α with order type $\leqslant \kappa$, and if β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$.*

Weak square at κ, $\square^*_\kappa$ is the principle, $\square_{\kappa,\kappa}$.

Analyzing how much of the square principles holds in a given model indicates how far this model is from canonical inner models. It is difficult to avoid the weaker square principles at successors of singulars. Doing so requires large cardinals. It is especially difficult to obtain these failures while also violating the Singular Cardinal Hypothesis (SCH). For example, violating SCH is usually done by singularizing a large cardinal κ. Whenever this is done in a way that preserves κ^+, $\square_{\kappa,\omega}$ holds in the outer model. These difficulties are even more pronounced in the case of smaller cardinals.

In 1980's Woodin asked if the failure of SCH at $\aleph_\omega$ implies that weak square holds at $\aleph_\omega$. In [3] we give a partial answer to that question:

Theorem 1. *Suppose that in V, κ is a supercompact cardinal. Then there is a generic extension in which:*

(1) $\kappa = \aleph_\omega$,
(2) *for all* $n < \omega$, $\square_{\aleph_\omega, \aleph_n}$ *fails*
(3) *SCH fails at* $\aleph_\omega$.

To show this, we analyze the PCF structure of the model constructed in [2]. We show that there is a product on which there is no very good scale. It is a combinatorial fact that for κ singular and $\lambda < \kappa$, $\square_{\kappa,\lambda}$ implies that every product carries a very good scale. On the other hand, very good scales at κ imply the failure of simultaneous reflection for ω-many stationary subsets of κ^+. While in the final model, simultaneous reflection does fail, there is enough remnants of it to influence the pcf structure. A key point in our construction is the following:

Theorem 2. *Let $V[G]$ be the final model, where G is generic for a Prikry type forcing with interleaved collapses, making $\kappa = \aleph_\omega$. Let $\langle \lambda_n \mid n < \omega \rangle$ be the Prikry sequence below κ. Then there is no very good scale at $\prod_n \lambda_n^{+n+1}$.*

An immediate corollary is that intermediate forms of square fail. Analyzing the pcf further, we show that there is a very good scale at $\prod_n \lambda_n^{+n+2}$. We conclude with the following open questions:

Question Is it consistent to have failure of weak square at $\aleph_\omega$ and not SCH at $\aleph_\omega$?

Question Does the failure of SCH imply the existence of a very good scale?

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A dual Ramsey and Galois connections

Slawomir Solecki

We give a new Ramsey Theorem, the Dual Ramsey Theorem for trees. It is a common generalization of the Dual Ramsey Theorem of Graham - Rothshild and Leeb's Ramsey Theorem for trees. Its formulation involves a special kind of Galois connections – embedding projection pairs. It can be viewed as a "non-commutative" Ramsey theorem.

Silver antichains

Otmar Spinas
(joint work with Marek Wyszkowski)

Let Si be **Silver forcing** consisting of all **Silver functions** $f : a \to 2$ such that $a \subseteq \omega$ and $|\omega \backslash a| = \aleph_0$, ordered by extension. Alternatively, such f can be identified with the **Silver tree** $p_f = \{\nu \in 2^{<\omega} : \forall\, n \in \mathrm{dom}(f) \cap |\nu|\ \nu(n) = f(n)\}$.

The **Silver forcing ideal** $\mathcal{I}(Si)$ is defined (similarly as for other tree forcings) by $X \in \mathcal{I}\,(Si)$ iff $X \subseteq 2^\omega \wedge \forall p \in Si\, \exists\, q \in Si\, (q \subseteq p \wedge [q] \cap X = \emptyset)$. Note that $\mathcal{I}(Si)$ is a σ-ideal, hence the ideal coefficients add $(\mathcal{I}(Si))$ and cov $(\mathcal{I}(Si))$ are cardinal invariants between $\aleph_1$ and $2^{\aleph_0}$. Answering a question by Goldstern, Laguzzi and Loewe we show

Theorem 1. add $(\mathcal{I}(Si)) \leqslant \mathfrak{b}$.

Their other question whether add $(\mathcal{I}(Si)) \leqslant$ cov $(\mathcal{M})$ remains open. Here $\mathfrak{b}$ is the bounding number and $\mathcal{M}$ the meager ideal. This result follows from our analysis of maximal antichains of Si (especially Theorem 2 below). Note that Si has countably infinite maximal antichains.Therefore we define $\mathfrak{a}(Si)$ as the minimal size of an uncountable maximal antichain of Si.

Conjecture $\mathfrak{a}(Si) = \mathfrak{c}$

We can prove this conjecture to some extent:

Theorem 2. *If $A \subseteq Si$ is a maximal antichain consisting of infinite Silver functions, then $|A| = \mathfrak{c}$.*

Theorem 3. *If $A \subseteq Si$ is a maximal antichain such that there exist an uncountable $A_0 \subseteq A$ and $x \in 2^\omega$ such that $\forall f \in A_0\ f \subseteq x$, then $|A| = \mathfrak{c}$.*

Corollary 1. *$\mathfrak{d} \leqslant \mathfrak{a}(Si)$.*
($\mathfrak{d}$ is the dominating number.)

Basis problem for analytic gaps

Stevo Todorčević
(joint work with Antonio Avilés)

In this joint work with Antonio Avilés we investigate the following general question for every integer $k \geqslant 2$: Given a sequence $\mathcal{C}_1, ..., \mathcal{C}_k$ of pairwise disjoint monotone[1] families of infinite subsets of $\mathbb{N}$, is there any combinatorial structure present in the class of its restrictions[2] $\mathcal{C}_1|_M, ..., \mathcal{C}_k|_M$ to infinite subsets M of $\mathbb{N}$?

To see the relevance of this question, consider a sequence (x_n) of objects (functions, points in a topological space, vectors of a normed space, etc) and let $\mathcal{C}_i$

[1]by monotone we mean that $M \subseteq N$ and $N \in \mathcal{C}_i$ imply $M \in \mathcal{C}_i$
[2]$\mathcal{C}|_M = \{N \in \mathcal{C} : N \subseteq M\}$

be the collection of all infinite subsets M of $\mathbb{N}$ for which the corresponding subsequence of $(x_n)_{n\in M}$ has some property P_i that is inherited when passing to a subsequence. We want to know whether by passing to a subsequence of (x_n) we could get some canonical behavior in each such example. The concrete examples are always *analytic* so this will be the restriction we will be making. As we will see, this turns out to be the right dividing line between cases where the structure can be found and the cases where the structure is absent.

To proceed further we need some definitions.

Definition 1. *A preideal on a countable set N is a family I of subsets of N such that if $x \in I$ and $y \subset x$ is infinite, then $y \in I$.*

Definition 2. *Let $\Gamma = \{\Gamma_i : i \in n\}$ be a family of n many preideals on the set N and let $\mathfrak{X}$ be a family of subsets of n.*

(1) *We say that Γ is separated if there exist subsets $a_0, \ldots, a_{n-1} \subset N$ such that $\bigcap_{i\in n} a_i = \varnothing$ and $x \subset^* a_i$ for all $x \in \Gamma_i$, $i \in n$.*
(2) *We say that Γ is an $\mathfrak{X}$-gap if it is not separated, but $\bigcap_{i\in A} x_i =^* \varnothing$ whenever $x_i \in \Gamma_i$, $A \in \mathfrak{X}$.*

We will consider only two choices of the family $\mathfrak{X}$, when $\mathfrak{X} = [n]^2$ is the family of all subsets of n of cardinality 2, a $[n]^2$-gap will be called an n-gap, while when $\mathfrak{X}$ consists only of the total set $n = 0, \ldots, n-1$, then an $\mathfrak{X}$-gap will be called an n_*-gap. The notion of n_*-gap is more general than that of a n-gap, since it does not require the preideals to be pairwise orthogonal. On the other hand, the use of n-gaps is more natural in some contexts, and for many of the problems that we discuss here, questions about n_*-gaps can be reduced to questions about n-gaps.

The general question that our theory deals with is the following: Given a gap Γ on N, can we find an infinite set $M \subset N$ such that the restriction of Γ to M becomes a gap which is canonical in some sense? The restriction of a preideal I to M is the preideal $I|_M = \{x \in I : x \subset M\}$, and the restriction of a gap Γ is $\Gamma|_M = \{\Gamma_i|_M : i \in n\}$. Notice that $\Gamma|_M$ may not be in general a gap, as the preideals may become separated when restricted to M.

The orthogonal of I is the family $I^\perp$ consisting of all $x \subset N$ such that $x \cap y =^* \varnothing$ for all $y \in I$. The orthogonal of the gap Γ is $\Gamma^\perp = (\bigcup_{i\in n} \Gamma_i)^\perp$. The gap Γ is called dense if $\Gamma^\perp$ is just the family of finite subsets of N.

Definition 3. *Given Γ and Δ two n_*-gaps on countable sets N and M, we say that $\Gamma \leqslant \Delta$ if there exists a one-to-one map $\phi : N \longrightarrow M$ such that for every $i \in n$,*

(1) *if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$.*
(2) *If $x \in \Gamma_i^\perp$ then $\phi(x) \in \Delta_i^\perp$*

To make this precise we need the following definitions.

Definition 4. *An analytic n_*-gap Γ is said to be a minimal analytic n_*-gap if for every other analytic n_*-gap Δ, if $\Delta \leqslant \Gamma$, then $\Gamma \leqslant \Delta$.*

Definition 5. *Two minimal analytic n_*-gaps Γ and Γ' are called equivalent if $\Gamma \leqslant \Gamma'$ (hence also $\Gamma' \leqslant \Gamma$).*

The new classification theory of analytic k-gaps that we develop here uses three layers of arguments belonging to different areas of mathematics. The first layer requires a suitable extension of the analytic gap theorems proved by the author some twenty years ago that gives a particularly canonical *tree-representation* to these analytic k-gaps.

Once the tree-representation is settled, the second layer is a new Ramsey principle for trees, not covered by the Ramsey theory of strong subtrees of Milliken used in our previous paper. To prove it we needed to use some deep reasoning from topological dynamics. After this Ramsey theorem is applied, we get our first basic result that can be stated as follows.

Theorem 1. *Fix a positive natural number k. For every analytic k_*-gap Γ there exists a minimal analytic k_*-gap Δ such that $\Delta \leqslant \Gamma$. Moreover, up to equivalence, there exist only finitely many minimal analytic k_*-gaps.*

The third layer is a fine combinatorial analysis of the finite list of canonical k_*-gaps in order to recognize the minimal gaps but also with an eye towards answering other natural questions. For example, we give an expression for and some lower and upper estimates on the cardinality $N(k)$ of the irredundant list of basic k-gaps. For example, we have

$$2^{J(k-1)-k-1} < N(k) < k^{J(k)-k},$$

where $J(k)$ is an explicitly given function whose asymptotic behavior is

$$J(k) \sim \frac{3}{8\sqrt{2\pi k}} \cdot 9^k.$$

The number $J(k)$ is in fact of independent interest as it is equal to the cardinality of the set of all *oscillation types* that are directly used in defining the basic k-gaps on the index-set $k^{<\mathbb{N}}$ rather than on $\mathbb{N}$. They are suggested by our Ramsey-theoretic analysis of analytic k-gaps that give us embeddings from $k^{<\mathbb{N}}$ into $\mathbb{N}$ transferring the basic k-gaps into restrictions of arbitrary analytic k-gaps.

Before concluding, we must mention that, although we have been guided by multidimensional problems, our theory already gives new and deeper information about classical 2-gaps. So far, as already mentioned above, only the first layer of the theory for 2-gaps had been considered by the author some twenty years ago. The lists of 5 minimal 2-gaps and of 163 minimal 3-gaps provides a book where to check any three-dimensional question on definable gaps than one may ask. But the effort to get these somehow exotic lists should not be viewed only as an objective in itself, but such a task has guided us in developing the combinatorial tools that unravel the structure of general analytic k-gaps. We expect that this structure theory will soon find applications.

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No maximal eventually different families in Solovay's model

Asger Törnquist

There are several long-standing problems about definability of various kinds of almost disjoint families. One such problem asks "how definable" a maximal eventually different (med) family of functions $\omega \to \omega$ can be (as subsets of Baire space). Until recently, it wasn't even known if med families could be *closed*. Recently, I showed that med families can't be analytic. However, this talk focused on an extension of these ideas, which allows us to prove:

Theorem 1 (T., 2014)**.** *There are no med families in Solovay's model.*

The key concept that underpins the proof is that of a *polarization* of an eventually different family. If $\mathcal{A}$ is an eventually different family, then a polarization of $\mathcal{A}$ is a family of sets $F_n^0, F_n^1 \subseteq \mathcal{A}$, such that

$$\mathcal{A}^2 \backslash \Delta \subseteq \bigcup_{n<\omega} F_n^0 \times F_n^1,$$

and whenever $f \in F_n^0$ and $g \in F_n^1$, then $dom(f \cap g) \subseteq n$.

Polarizations are easy to construct for analytic eventually different family using the tree representation of the analytic set, but in Solovay's model, we can prove their existence by using an apparent strengthening of Todorcevic's Open Colouring Axiom, called OCA_∞, which previously has appeared in Ilijas Farah's work.

The polarization of an eventually different family $\mathcal{A}$ gives us a decomposition of the family into countably many pieces F_n^0, each element of which is avoided completely above n by the elements of F_n^1. The proof now proceeds by showing that an appropriately defined "generic" function $\omega \to \omega$ (in the sense of Cohen or Baire) is eventually different from all elements of $\mathcal{A}$, and that the polarization gives us the existence of a "sufficiently generic" element already in the ground model, whence the family $\mathcal{A}$ cannot be maximal. This argument crucially uses that $\mathcal{A}$ contains a perfect set (if it is uncountable), and that the sets F_n^i have the Baire property. Since all this, as well as OCA_∞, hold in Solovay's model, we obtain Theorem 1 above.

We don't at this stage know if OCA itself is enough to provide polarizations, nor do we know if the use of the perfect set property ultimately can be avoided in the above proof. In particular, we don't know if an inaccessible is really needed to provide a model of ZF in which there are no med (or mad) families.

Mixing and triple recurrence in probability groups

Anush Tserunyan

We consider a class of groups equipped with an invariant probability measure that respects the group structure, so that this class contains all compact Hausdorff groups equipped with the normalized Haar measure and is closed under taking ultraproducts with the induced Loeb measure; call such groups *probability groups* (this class of groups was also considered in [4]). One naturally defines the notion of a *measure-preserving actions* of such groups on probability spaces. Furthermore, we call a measure-preserving action $a : G \curvearrowright (X, \nu)$ of a probability group (G, μ) on a probability space (X, ν) *mixing* if for any $f_1, f_2 \in L^2(X, \nu)$, we have:

$$\text{for } \mu\text{-a.e. } g \in G, \ \int_X f_1(g \cdot_a f_2) d\nu = \int_X P_a(f_1) P_a(f_2) d\nu,$$

where $P_a(f)$ is the orthogonal projection of a function $f \in L^2(X, \nu)$ on the space of a-invariant functions. This reads as follows: for μ-a.e. $g \in G$, f_1 and $g \cdot_a f_2$ are probabilistically independent. Call the probability group (G, μ) itself *mixing* if all of its measure-preserving actions on probability spaces are mixing. At first this definition may look unlikely to be satisfied, but we actually have the following:

Proposition 1. *A probability group (G, μ) is mixing if and only if its right translation action on itself is mixing.*

This shows that ultra quasirandom groups considered by Bergelson and Tao in [2] are examples of mixing probability groups.

Our main result is that mixing (i.e. double recurrence) for probability groups can be bootstrapped to a triple recurrence:

Theorem 1. *Let (G, μ) be a mixing probability group. Then for any $f_1, f_2, f_3 \in L^\infty(G, \mu)$, we have*

$$(\forall^\mu g \in G) \ \int_G f_1(x)(g \cdot_l f_2)(x)(g \cdot_c f_3)(x) dx = \int_G f_1(x) P_l(f_2)(x) P_c(f_3)(x) dx,$$

where $\cdot_l$ and $\cdot_c$ are, respectively, the left translation and the conjugation actions of G on itself.

This in particular applies to ultra quasirandom groups, generalizing the corresponding multiple recurrence result of Bergelson-Tao proved in [2] and providing a considerably shorter proof that only uses basic measure theory and Hilbert spaces. However, the latter proof uses ideas close to the original proof by Bergelson-Tao, so it shouldn't be considered as a new proof. In particular, both of the proofs use suitable van der Corput tricks (see for example Lemmas 4.9 and 9.24 in [3] and Theorem 2.3 in [1]), all versions of which turn out to be hinging on an interesting Ramsey theorem for a general type of filters on semigroups recently obtained by the speaker.

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Borel reducibility and cardinal arithmetic

Jindrich Zapletal

The purpose of the talk is to explain one possible line of communication between classification theory of analytic equivalence relations and combinatorial set theory. It should be stressed that there are many other possibilities, and the whole connection has been severely neglected so far.

I will produce Borel equivalence relations E, F classifiable by countable structures such that E is not Borel reducible to F but the natural (and the only known to me) proof of the nonreducibility uses a classical result of Magidor on the consistency of the failure of the Singular Cardinal Hypothesis at $\aleph_\omega$. Other combinatorial results concerning small uncountable cardinals can be injected into the Borel reducibility hierarchy in a similar way. The following definitions will be central:

Definition 1. *(Kanovei) Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of the space X. Say that τ is* pinned *if $P \times P \Vdash \tau_{left}\ E\ \tau_{right}$. E is* pinned *if for every pinned name τ there is a point $x \in X$ such that $P \Vdash \tau\ E\ \check{x}$.*

Definition 2. *(Zapletal) Let E be an analytic equivalence relation on a Polish space X. Let P, Q be posets and τ, σ be pinned P and Q-names respectively for elements of X. Say $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ if $P \times Q \Vdash \tau\ E\ \sigma$.*

A Shoenfield absoluteness argument shows that $\bar{E}$ is an equivalence relation on pinned names. It is interesting to count the number of equivalence classes of $\bar{E}$. This can be done in several ways, I will use the most natural one:

Definition 3. *(Zapletal) Let E be an analytic equivalence relation on a Polish space X. The* pinned cardinal *$\kappa(E)$ is the smallest cardinal κ such that every pinned name has an $\bar{E}$-equivalent on a poset of size $< \kappa$, if such cardinal exists. If such a cardinal does not exist, let $\kappa(E) = \infty$. If E is a pinned equivalence relation then put $\kappa(E) = \aleph_1$.*

The pinned cardinal behaves in a natural way vis-a-vis the Borel reducibility hierarchy and usual operations on equivalence relations.

Theorem 1. *Let E, F be analytic equivalence relations on Polish spaces.*

(1) *if E is Borel reducible to F then $\kappa(E) \leqslant \kappa(F)$;*

(2) *if E is Borel then $\kappa(E) < \beth_{\omega_1}$;*
(3) *if E is analytic then $\kappa(E)$ is less than the first measurable cardinal or it is equal to ∞;*
(4) *if there is a measurable cardinal then $\kappa(E) = \infty$ iff E_{ω_1} is weakly reducible to E;*
(5) $\kappa(E^+) \leqslant (2^{<\kappa(E)})^+$.

The evaluation of the cardinal $\kappa(E)$ may return surprising values. I will comment on one possibility.

Definition 4. *Let ϕ be an $L_{\omega_1\omega}$ sentence. Say that ϕ is* set-like *if ϕ proves among other things that there is an extensional binary relation $\in$ which is well-founded. Let E_ϕ be the equivalence relation of isomorphism between countable models of ϕ.*

If ϕ is set-like then the isomorphism relation E_ϕ is Borel since isomorphisms between extensional well-founded structures are unique by the Mostowski collapse theorem. It is also relatively easy to evaluate the cardinal $\kappa(E_\phi)$ for set-like sentences ϕ:

Theorem 2. *If ϕ is set-like then $\kappa(E_\phi)$ is the cardinal κ such that ϕ has models of all sizes below κ and no model of size κ.*

Theorem 3. *For every countable ordinal $\alpha > 0$ there is a set-like sentence ϕ such that (provably) $\kappa(E_\phi) = \aleph_\alpha$. There are set-like sentences ψ and θ such that (provably) $\kappa(E_\psi) = (\aleph_\omega^{\aleph_0})^+$ and $\kappa(E_\theta) = \max\{\aleph_{\omega+1}, \mathfrak{c}\}^+$.*

It is now clear that E_ψ is not Borel reducible to E_θ: if h were such a reduction then one would pass to a generic extension due to Magidor in which $\aleph_\omega^{\aleph_0} > \aleph_{\omega+1}, \mathfrak{c}$, use the Shoenfield absoluteness to prove that h still must be a reduction in this extension, and obtain a contradiction with Theorem 1.

In a different vein, consider a different equivalence relation. Let f be a function defined on $\mathcal{P}(\omega)^2$ by the following formula $f(a,b) = (1, a \cap b)$ if a is lexicographically less than b and $a \cap b$ is finite, $f(a,b) = (-1, a \cap b)$ if b is lexicographically less than a and $a \cap b$ is finite, and $f(a,b)$ =trash otherwise. Let $C = \{x \in (\mathcal{P}(\omega))^\omega : \text{rng}(x)$ contains no infinite subsets d, e such that $f \restriction d \times e$ is constant$\}$. Let E be the equivalence relation on $(\mathcal{P}(\omega))^\omega$ connecting x, y if either $x, y \notin C$ or $\text{rng}(x) = \text{rng}(y)$. This is an analytic equivalence relation. We have

Theorem 4. *Assume that Martin's axiom for $\aleph_2$ holds. Then $\kappa(E) = \aleph_2$ if Chang's Conjecture holds, and $\kappa(E) \geqslant \aleph_3$ if Chang's Conjecture fails.*

There are many equivalence relations for which the value of the pinned cardinal remains a mystery.

Question Let E be the measure equivalence (connecting probability measures if they have the same ideal of null sets). What is $\kappa(E)$?

Mathias forcing and combinatorial covering properties of filters

Lyubomyr Zdomskyy
(joint work with David Chodounský, Dušan Repovš)

Every free filter $\mathcal{F}$ on ω gives rise to a natural forcing notion $\mathbb{M}_{\mathcal{F}}$ introducing a generic subset $X \in [\omega]^\omega$ such that $X \subset^* F$ for all $F \in \mathcal{F}$ as follows: $\mathbb{M}_{\mathcal{F}}$ consists of pairs $\langle s, F\rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F\rangle$ is stronger than $\langle t, G\rangle$ if $F \subset G$, s is an end-extension of t, and $s \setminus t \subset G$. $\mathbb{M}_{\mathcal{F}}$ is usually called *Mathias forcing associated with* $\mathcal{F}$.

Posets of the form $\mathbb{M}_{\mathcal{F}}$ are important in the set theory of reals and have been used to establish various consistency results, see, e.g., [2, 5] and references therein. One of the most fundamental questions about $\mathbb{M}_{\mathcal{F}}$ is whether it adds a dominating real, i.e., whether in ω^ω of $V^{\mathbb{M}_{\mathcal{F}}}$ there exists x such that for every $a \in \omega^\omega$ in the ground model the inequality $a(n) \leqslant x(n)$ holds for all but finitely many n. Such filters $\mathcal{F}$ admit the following topological characterization.

Theorem 1. *Let $\mathcal{F}$ be a filter. Then $\mathbb{M}_{\mathcal{F}}$ does not add dominating reals if and only if $\mathcal{F}$ has the Menger covering property as a subspace of $\mathcal{P}(\omega)$.*

Recall from [6] that a topological space X has the *Menger covering property* (or simply is Menger), if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

Theorem 1 has a number of applications. For instance, since analytic Menger sets of reals are σ-compact [1], it implies the following fact.

Corollary 1. *Let $\mathcal{F}$ be an analytic filter on ω. Then $\mathbb{M}_{\mathcal{F}}$ does not add a dominating real if and only if $\mathcal{F}$ is σ-compact.*

Following [4] we say that a family $\mathcal{U}$ of subsets of a set X is

- *an ω-cover*, if $X \notin \mathcal{U}$ and for every finite subset K of X there exists $U \in \mathcal{U}$ such that $K \subset U$;
- *a γ-cover*, if for every $x \in X$ the family $\{U \in \mathcal{U} : x \notin U\}$ is finite.

The Hurewicz property is defined in the same way as the Menger one, the only difference being that the family $\{\bigcup \mathcal{V}_n : n \in \omega\}$ must be a γ-cover of X. We say that a poset $\mathbb{P}$ is *almost ω^ω-bounding* if for every unbounded $X \subset \omega^\omega$, $X \in V$, we have that $1 \Vdash_{\mathbb{P}}$ "X is unbounded". This definition is easily seen to be equivalent to the standard one.

Theorem 2. *Let $\mathcal{F}$ be a filter. Then $\mathbb{M}_{\mathcal{F}}$ is almost ω^ω-bounding if and only if $\mathcal{F}$ is has the Hurewicz property.*

Theorem 2 turns out to have applications to general Hurewicz spaces, not only to filters. In order to formulate them we need to recall some definition. A Tychonov space X is called a *γ-space* [4] if every open ω-cover of X contains a γ-subcover. γ-spaces are important in the theory of function spaces as they are exactly those X for which the space $C_p(X)$ of continuous functions from X to $\mathbb{R}$ with the topology inherited from $\mathbb{R}^X$, has the Fréchet-Urysohn property.

For $a \in [\omega]^\omega$ and $n \in \omega$, $a(n)$ denotes the n-th element in the increasing enumeration of a. For $a, b \in [\omega]^\omega$, $a \leqslant^* b$ means that $a(n) \leqslant b(n)$ for all but finitely many n. A $\mathfrak{b}$-scale is an unbounded set $S = \{s_\alpha : \alpha < \mathfrak{b}\}$ in $([\omega]^\omega, \leqslant^*)$ such that $s_\alpha \leqslant^* s_\beta$ for $\alpha < \beta$. It is easy to see that $\mathfrak{b}$-scales exist in ZFC. For each $\mathfrak{b}$-scale S, $S \cup [\omega]^{<\omega}$ is $\mathfrak{b}$-concentrated on $[\omega]^{<\omega}$ in the sense that $|S \backslash U| < \mathfrak{b}$ for any open $U \supset [\omega]^{<\omega}$. For brevity, the union of a $\mathfrak{b}$-scale with $[\omega]^{<\omega}$, viewed as a subset of the Cantor space $\mathcal{P}(\omega)$, will be called *a $\mathfrak{b}$-scale set*. As an application of Theorem 2 we will get the following result answering [8, Problem 4.2] in the affirmative.

Corollary 2. *It is consistent with ZFC that every $\mathfrak{b}$-scale set is a γ-space.*

The study of the relation between $\mathfrak{b}$-scale sets and γ-spaces already has some history. First of all, in the Laver model all γ-subspaces of 2^ω are countable because they have strong measure zero [4]. Answering one of the questions posed in [4], Galvin and Miller [3] constructed under $\mathfrak{p} = \mathfrak{c}$ a $\mathfrak{b}$-scale set which is a γ-set. Their $\mathfrak{b}$-scale was a tower, where $S = \{s_\alpha : \alpha < \kappa\} \subset [\omega]^\omega$ is called a *tower* if $s_\alpha \subset^* s_\beta$ for all $\beta < \alpha$. Later Orenshtein and Tsaban proved [7] that if $\mathfrak{p} = \mathfrak{b}$ then any $\mathfrak{b}$-scale set is a γ-space provided that the corresponding $\mathfrak{b}$-scale is a tower. On the other hand, under $\mathfrak{b} = \mathfrak{c}$ there exists a $\mathfrak{b}$-scale set which fails to be a γ-space, see [8]. Also, it is easy to see that such $\mathfrak{b}$-scale sets exist under $\mathfrak{p} < \mathfrak{b}$. Thus $\mathfrak{p} = \mathfrak{b} < \mathfrak{c}$ holds in any model of Corollary 2.

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Self-iterability of extender models with fullness preserving iteration strategies

Martin Zeman

(joint work with Nam Trang)

This is a joint work in progress with Nam Trang. We investigate under which circumstances an extender model contains iteration strategies for its initial segments. Iterability is the crucial property of extender models that makes it possible to compare them. This, in turn, by results of Woodin and others, makes initial segments of extender models internally ordinal definable. Results on self-iterability were previously obtained by Schindler and Steel, under more modest hypothesis than ours, and with somewhat weaker conclusions than ours.

Given a transitive set A, the structure $\mathsf{Lp}(A)$ is the model-theoretic union of all premice that project to A, are sound above A, and whose all countable elementary substructures are (ω_1+1)-iterable. A premouse $\mathcal{M}$ is tame if an only if there is no extender on the extender sequence of $\mathcal{M}$ of length, say β and with critical point κ such that some $\delta \in [\kappa, \beta)$ is a Woodin cardinal in the sense of the initial segment $\mathcal{M}||\beta$. A tame premouse is full if and only if for every $\mathcal{M}$-cardinal λ, the structure $\mathsf{Lp}(\mathcal{M}||\lambda)$ is the initial segment $\mathcal{M}||\lambda^{+\mathcal{M}}$.

An iteration strategy for $\mathcal{M}$ is a partial function Σ which assigns to every iteration tree $\mathcal{T}$ on $\mathcal{M}$ which is in $\mathsf{dom}(\Sigma)$ a cofinal well-founded branch through $\mathcal{T}$. An iteration tree $\mathcal{T}$ is according to Σ if and only if for every limit ordinal $\alpha < \mathsf{lh}(\mathcal{T})$ we have $\Sigma(\mathcal{T} \upharpoonright \alpha) = [0, \alpha)_{\mathcal{T}}$. An iteration strategy Σ for $\mathcal{M}$ is fullness preserving if and only if for every iteration tree $\mathcal{T}$ on $\mathcal{M}$ of limit length according to Σ and every limit $\alpha < \mathsf{lh}(\mathcal{T})$, the α-th model $\mathcal{M}_\alpha^{\mathcal{T}}$ on $\mathcal{T}$ is full, granting the branch $[0, \alpha)_{\mathcal{T}}$ does not involve any truncation.

Theorem 1. *Assume $\mathcal{N}$ is a tame premouse with a Woodin cardinal which has a fullness preserving iteration strategy. Let δ be the least Woodin cardinal in $\mathcal{N}$. Then for every $\gamma < \delta$, the initial segment $\mathcal{N}||\gamma$ has an iteration strategy in $\mathcal{N}$. More generally, if $\mathcal{N}$ either does not reach a measurable limit of Woodin cardinals or else has a largest Woodin cardinal then for every Woodin cardinal δ in $\mathcal{M}$, letting $\tilde{\delta}$ be the supremum of Woodin cardinals smaller than $\mathcal{N}$, for every $\mathcal{N}$-cardinal $\gamma \in (\tilde{\delta}, \delta)$ the initial segment $\mathcal{N}||\gamma$ is iterable when using only extender with indices in the interval $(\tilde{\delta}, \delta)$ and their images.*

As mentioned above, methods of Woodin and others yield the following corollary.

Corollary 1. *If $\mathcal{N}$ is a tame premouse with a fullness preserving iteration strategy then the initial segment of its extender sequence $E^{\mathcal{N}} \upharpoonright \delta$ is ordinal definable inside $\mathcal{N}$. More generally, if $\mathcal{N}$ either does not reach a measurable limit of Woodin cardinals or else $\mathcal{N}$ has a largest Woodin cardinal and δ' is the supremum of all Woodin cardinals in $\mathcal{N}$ then the initial segment of the extender sequence $E^{\mathcal{N}} \upharpoonright \delta'$ is ordinal definable inside $\mathcal{N}$.*

Reporter: Laura Fontanella

Participants

Prof. Dr. Jörg Brendle
Group of Logic, Statistics & Informatics
Graduate School of System Informatics
Kobe University
Rokko-dai 1-1, Nada
Kobe 657-8501
JAPAN

Dr. Clinton Conley
Department of Mathematics
Cornell University
584 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Dr. Scott Cramer
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. James W. Cummings
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
UNITED STATES

Prof. Dr. Natasha Dobrinen
Department of Mathematics
University of Denver
Aspen Hall, Room 717
2280 S. Vine St.
Denver, CO 80217-3364
UNITED STATES

Prof. Dr. Mirna Dzamonja
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
UNITED KINGDOM

Prof. Dr. Todd Eisworth
Department of Mathematics
Ohio University
321 Morton Hall
Athens, OH 45701-2979
UNITED STATES

Prof. Dr. Ilijas Farah
Dept. of Mathematics & Statistics
York University
4700 Keele Street
Toronto Ont. M3J 1P3
CANADA

Dr. Vera Fischer
Kurt Gödel Research Center
for Mathematical Logic
Universität Wien
Währinger Str. 25
1090 Wien
AUSTRIA

Dr. Laura Fontanella
Kurt Gödel Research Center
for Mathematical Logic
Universität Wien
Währinger Str. 25
1090 Wien
AUSTRIA

Prof. Dr. Sy-David Friedman
Kurt Gödel Research Center
for Mathematical Logic
Universität Wien
Währinger Str. 25
1090 Wien
AUSTRIA

Prof. Dr. Su Gao
Department of Mathematics
University of North Texas
P.O.Box 311430
Denton, TX 76203-1430
UNITED STATES

Dr. Stefan Geschke
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Prof. Dr. Moti Gitik
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Dr. Mohammad Golshani
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
Niavaran Square
P.O. Box 19395
Teheran
IRAN, ISLAMIC REPUBLIC OF

Prof. Dr. Radek Honzik
Faculty of Mathematics and Physics
Charles University
118 00 Praha 1
CZECH REPUBLIC

Prof. Dr. Ronald Björn Jensen
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin
GERMANY

Prof. Dr. Peter Koepke
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. John Krueger
Department of Mathematics
University of North Texas
P.O.Box 311430
Denton, TX 76203-1430
UNITED STATES

Prof. Dr. Jean Larson
Department of Mathematics
University of Florida
358 Little Hall
P.O.Box 118105
Gainesville, FL 32611-8105
UNITED STATES

Prof. Dr. Paul B. Larson
Department of Mathematics
Miami University
Oxford, OH 45056
UNITED STATES

Prof. Dr. Alain Louveau
Equipe d'Analyse; Boite 186
Université Pierre et Marie Curie
(Université Paris VI)
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Martino Lupini
Dept. of Mathematics & Statistics
York University
4700 Keele Street
Toronto Ont. M3J 1P3
CANADA

Prof. Dr. Menachem Magidor
Institute of Mathematics
The Hebrew University
Edmond J. Safra Campus
91904 Jerusalem
ISRAEL

Prof. Dr. Julien Melleray
Institut Camille Jordan
Université Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Dr. Carmi Merimovich
Department of Computer Science
Tel-Aviv Academic College
2 Rabenu Yehoram St.
Tel Aviv 68182
ISRAEL

Prof. Dr. Heike Mildenberger
Abteilung für Mathematische Logik
Universität Freiburg
Eckerstr. 1
79104 Freiburg
GERMANY

Prof. Dr. Benjamin Miller
Institut für Mathematische
Logik und Grundlagenforschung
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. William J. Mitchell
Department of Mathematics
University of Florida
358 Little Hall
P.O.Box 118105
Gainesville, FL 32611-8105
UNITED STATES

Prof. Justin Tatch Moore
Department of Mathematics
Cornell University
555 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Dr. Luca Motto Ros
Mathematisches Institut
Universität Freiburg
79104 Freiburg
GERMANY

Prof. Dr. Itay Neeman
Department of Mathematics
UCLA
405 Hilgard Ave.
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Dilip Raghavan
Department of Mathematics
National University of Singapore
10 Lower Kent Ridge Road
Singapore 119076
SINGAPORE

Dr. Assaf Rinot
Department of Mathematics
Bar-Ilan University
52 900 Ramat-Gan
ISRAEL

Dr. Marcin Sabok
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 21
ul. Sniadeckich 8
00-956 Warszawa
POLAND

Dr. Hiroshi Sakai
Group of Logic, Statistics & Informatics
Graduate School of System Informatics
Kobe University
Rokko-dai 1-1, Nada
Kobe 657-8501
JAPAN

Prof. Dr. Grigor Sargsyan
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Ralf-Dieter Schindler
Institut für Mathematische
Logik und Grundlagenforschung
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Dima Sinapova
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
Chicago, IL 60607-7045
UNITED STATES

Prof. Dr. Slawomir Solecki
Department of Mathematics
University of Illinois at Urbana
Champaign
273 Altgeld Hall
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Prof. Dr. Otmar Spinas
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Str. 4
24118 Kiel
GERMANY

Prof. Dr. John R. Steel
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Simon Thomas
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Stevo Todorcevic
Institute Mathématiques de Jussieu
Case 247
4, Place Jussieu
75252 Paris Cedex
FRANCE

Prof. Dr. Asger Törnquist
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
DENMARK

Dr. Todor Tsankov
Equipe de Logique Mathématique
Université Paris Diderot, Paris VII
UFR de Mathématiques
Case 7012
75205 Paris Cedex 13
FRANCE

Dr. Anush Tserunyan
Department of Mathematics
University of Illinois at
Urbana-Champaign
273 Altgeld Hall
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Dr. Robin Tucker-Drob
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd.
Piscataway NJ 08854-8019
UNITED STATES

Prof. Dr. Boban D. Velickovic
Equipe Logique Mathématique
Université Paris VII
2, Place Jussieu
75251 Paris Cedex 05
FRANCE

Dr. Matteo Viale
Dipartimento di Matematica
Universita degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Prof. Dr. W. Hugh Woodin
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Jindrich Zapletal
Department of Mathematics
University of Florida
358 Little Hall
Gainesville, FL 32611
UNITED STATES

Dr. Lyubomyr Zdomskyy
Kurt Gödel Research Center
for Mathematical Logic
Universität Wien
Währinger Str. 25
1090 Wien
AUSTRIA

Prof. Dr. Martin Zeman
Department of Mathematics
University of California, Irvine
Irvine, CA 92697-3875
UNITED STATES