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Mini-Workshop: Negative Curves on Algebraic Surfaces

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Abstract. Negative curves play a prominent role in the geometry of projective surfaces. They occur naturally as the irreducible components of exceptional loci of resolutions of surface singularities, at the same time, they are closely related to the geometry of the effective cone, and thus form an important building block of the Minimal Model Program. In the case of surfaces, classes of negative curves span extremal rays of the Mori cone. Any knowledge about them on a given surface reveals important information on linear series as well.

Mathematics Subject Classification (2010): Primary 14C17; Secondary 14G35.

Introduction by the Organisers

The miniworkshop *Negative curves on algebraic surfaces* gathered together a variety of mathematicians interested in this subject with a wide spread of backgrounds and professional experience. Participants came from several European countries (France, Germany, Great Britain, Hungary, Norway, Poland, Sweden) and from the United States. Their expertise ranged from advanced graduate students through post-docs to established senior researchers. This variety of experience and background greatly contributed to generating stimulating discussions during the workshop and the working group sessions, leading to what we believe will be the basis for several research collaborations in the near future.

The theme of the workshop

The primary goal of our meeting was to understand curves on algebraic surfaces, a classical area of mathematics, giving rise to a vast array of research with connections to subjects ranging from differential geometry and number theory to seemingly unrelated fields such as ergodic theory.

The workshop revolved around the following long standing conjecture which recently has attracted quite a lot of attention in the field of linear series as well (see e.g. [1], [3], [7], [8], [10], [12]).

Conjecture 1 (Bounded Negativity Conjecture (BNC))**.** *Let X be a smooth projective surface over the complex numbers. Then there exists a constant b_X such that*

$$(C^2) \geq -b_X$$

for all reduced effective curves C on X.

The conjecture is known to hold in a number of cases (as proven in [1]), nevertheless, it is wide open in general. It is of great interest because negative curves appear naturally in different branches of algebraic geometry: a natural source of examples are irreducible components of exceptional divisors on resolutions of surface singularities.

For a classical connection, elaborating on Nagata's famous conjecture, Segre [15], Gimigliano [9], Harbourne [11] and Hirschowitz [13] came to a striking geometric conjecture (part of what is called the SHGH-conjecture in the literature) stating that the only negative curves on the blow up of projective plane $\mathbb{P}^2$ in $s \geq 10$ points are the (-1) curves.

Considerable recent work leading to exciting partial results (see [5], [6], [2], [14], [4]), has been devoted to proving the SHGH conjecture; the efforts include large amounts of computer experiments.

In spite of these efforts, much of the area surrounding negative curves on surfaces remains to be explored. For instance, it is at present not even known if the self-intersection numbers of reduced curves on blow ups of $\mathbb{P}^2$ are bounded from below – as would be the case for blow ups of generic points in $\mathbb{P}^2$, with bound -1, if the SHGH conjecture is true.

The modus operandi of our workshop was to devote half of the available time to research in groups focusing on concrete subproblems. One of the working groups was in fact devoted to discussing exactly the above aspect of the conjecture. It turns out that one obtains interesting examples by looking at curves (on the blow ups of $\mathbb{P}^2$) coming from arrangements of lines in the projective plane.

It has been observed that there is an intriguing relation between highly negative curves on the one hand and counterexamples to a seemingly unrelated problem on containment relations between various symbolic and usual powers of ideals of planar points.

A second working group took up the question of boundedness for Shimura curves on ball quotients, in an attempt to reproduce the finiteness results of [1] for Shimura curves with negative self-intersection in quaternionic Hilbert modular surfaces. During the workshop we have learned that there is an unpublished result due to Margulis (communicated to us by Domingo Toledo and Misha Kapovich) that uses ergodic theoretic methods to show that there are only finitely many embedded totally geodesic subvarieties of at least half the dimension in certain locally symmetric varieties. This would cover at least the case of smooth Shimura curves on ball quotients.

There seems to be a reasonable chance that these ideas can be extended to the case of arbitrary Shimura curves, with the help of Ratner type theorems, but the status of such a statement is unclear. A major difference to the vector bundle methods applied in [1] in the Hilbert modular case is that the ergodic theoretic way does not give rise to effective BNC bounds.

The third working group considered the question of possible higher-dimensional generalizations of the Bounded Negativity Conjecture. Several results and examples were given; nevertheless, it appears that most of the natural generalizations are false.

The structure of the workshop

The aim of the workshop was twofold: to gather together experts working on the three different aspects mentioned above, and to stimulate collaboration by discussing open problems in the field.

For this reason every day consisted of two main activities:

- research talks, two to three in the morning; and
- working group discussions, in the afternoon.

A list of possible questions to work on during the workshop was distributed via email well ahead of the workshop. During the first day, final selections were made. Three main areas of interest emerged from the discussion: negative curves in arithmetic settings, bounded negativity for cycles on higher dimensional varieties, and local negativity and containment relations. Consequently three working groups were formed. The workshop was just the starting point to ignite collaborations on the chosen problems. The working groups continue their efforts. The outcome of these discussions will hopefully appear elsewhere.

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Mini-Workshop: Negative Curves on Algebraic Surfaces

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Abstracts

Arakelov (In)equalities

Stefan Müller-Stach
(joint work with Th. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, X. Roulleau, T. Szemberg)

Since this was the first talk of the conference, I intended to recall one of the main theorems of the seven authors paper [1], jointly with Th. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, X. Roulleau and T. Szemberg. It claims that bounded negativity holds for certain arithmetic curves.

Let $X = \Gamma\backslash\mathbb{H} \times \mathbb{H}$ be a compact (quaternionic) Hilbert modular surface, i.e., a quotient of two copies of the upper half plane by an arithmetic group. Furthermore, let $C \subset X$ be any integral curve on X.

Theorem 1: One has $K_X C + 2C^2 \geq 4\delta$, where $2\delta := K_X C + C^2 - 2(g-1)$ and g is the geometric genus of C. Equality holds if and only if C itself is a Shimura curve $\Gamma'\backslash\mathbb{H}$.

A proof using Higgs bundles can be found in [3]. As a corollary one gets $K_X C = 4(g-1)$ for Shimura curves, since

$$K_X C = 2K_X C + 2C^2 - 4\delta = 4(g-1).$$

We give a proof of the following Thm. 3.5 in [1]:

Theorem 2: Let C be any integral Shimura curve on X as above. Then $C^2 \geq -6c_2$ is bounded from below and for the geometric genus one has $g \leq 1 + c_2 + \sqrt{c_2^2 + c_2\delta}$. In particular, there exist only a finite number of such curves with $C^2 < 0$.

Proof: We use the following inequality of Miyaoka [2]:

$$P(\alpha) = \frac{\alpha^2}{2}(C^2 + 3K_X C - 6(g-1)) - 2\alpha(K_X C - 3(g-1)) + 3c_2 - c_1^2 \geq 0$$

for all α in $[0,1]$. In our case we get

$$P(\alpha) = \alpha^2(3\delta - C^2) + \alpha(C^2 - 2\delta) + c_2 \geq 0.$$

The minimum is attained at

$$\alpha_0 = \frac{2\delta - C^2}{2(3\delta - C^2)}.$$

We may assume that $C^2 < 0$ and hence $C^2 < 2\delta$. Then α_0 is non-zero and in $[0,1]$. Non-negativity of $P(\alpha_0)$ implies that

$$C^2 \geq 2\delta - 2c_2 - 2\sqrt{c_2^2 + c_2\delta}.$$

So for $\delta \geq 3c_2$, one gets

$$C^2 \geq 2\delta - \frac{2\delta}{3} - \frac{4\delta}{3} = 0.$$

We may thus assume that $C^2 < 0$ and $\delta < 3c_2$. Hence

$$C^2 \geq 2\delta - 6c_2 \geq -6c_2.$$

Miyaoka [2] has also shown that for $C \neq \mathbb{P}^1$ and $K_X C > 3(g-1)$, one has

$$(K_X C - 3(g-1))^2 - c_2(K_X C + \delta - 2(g-1)) \leq 0.$$

From this we get in a similar way

$$g \leq 1 + c_2 + \sqrt{c_2^2 + c_2\delta} \leq 1 + 3c_2$$

if $C^2 < 0$. Hence one has boundedness of the invariants $g = g(C)$ and $K_X C$ which implies the finiteness. QED.

Remarks: In the proof it is sufficient to have that $K_X C > 3(g-1)$. In this way, the proof also works for surfaces other than Hilbert modular surfaces, but only for curves with the property that $K_X C > 3(g-1)$. However, on ball quotients $X = \Gamma\backslash\mathbb{B}_2$ the method fails, as there one has $K_X C + 3C^2 = 6\delta$, and hence $K_X C = 3(g-1)$. On the first day of the conference Domingo Toledo informed us that one can probably use Ratner type theorems to prove bounded negativity also in the case of ball quotients. More generally, for embedded totally geodesic subvarieties in locally symmetric varieties of at least half the dimension, one should have finiteness. Apparently this result has already been known to Margulis, Gromov and others around 1990 or earlier. During the Mini-Workshop we tried to understand this idea. The method does not give effective lower bounds however.

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Twisted Teichmüller curves

Martin Möller
(joint work with Christian Weiss)

The Hilbert modular surfaces $X_D = \mathbb{H}^2/\mathrm{SL}_2(\mathfrak{o}_D)$ carry an infinite collection of well-studied Hirzebruch-Zagier cycles. These can be defined as the twists of the diagonal in $\mathbb{H}^2$ by a matrix $M \in \mathrm{GL}_2(K)$, i.e. as the image of $\{(M\,z, M^\sigma\,z),\ z \in \mathbb{H}\}$ in X_D, where σ is a generator of the Galois group of $K = \mathbb{Q}(\sqrt{D})$. In the context of the bounded negativity conjecture these curves have been candidates but turned out to not provide a counterexample: For given D only a finite number of them have negative self-intersection.

Hirzebruch-Zagier cycles are special instances of the following class of of curves. An immersed algebraic curve $C \to X_D$ is called *Kobayashi geodesic*, if the universal covering has the form $z \mapsto (z, \varphi(z)$ after maybe renumbering the components and with an appropriate choice of a base point. The first examples of Kobayashi geodesics arose from modular embeddings of triangle groups ([1]). An infinite series of these curves have been constructed in [3], arising from dynamically optimal billiard tables with a genus two unfolding. They are called *Teichmüller curves* and there are one or two such curves C for given $D > 5$. If $z \mapsto (z, \varphi(z))$ is the universal covering map of $C \to X_D$ and if $M \in \mathrm{GL}_2(K)$, then the image of $\{(M\,z, M^\sigma\,\varphi(z)),\ z \in \mathbb{H}\}$ is again a Kobayashi geodesic C_M in X_D, that is called *twisted Teichmüller curve* in [5].

All the classification questions solved for Hirzebruch-Zagier cycles (volumes, components, intersection numbers and geometry of the intersection points) since the 70s have their natural analogs for twisted Teichmüller curves. The first question has been answered by Weiss in [5]. We state the case with least number of technical conditions.

Theorem 1 ([5]). *For $D \equiv 5 \mod 8$ the volume of C_M equals the volume of C times $\deg(X_D(M) \to X_D)$, where $X_D(M)$ is the level covering of X_D uniformized by the group $\mathrm{SL}_2(\mathfrak{o}_D) \cap M\,\mathrm{SL}_2(\mathfrak{o}_D)\,M^{-1}$.*

The volumes of twisted diagonals is much smaller (for given M). In fact Weiss proves that in many cases the preimage of a Teichmüller curve in a level covering of a Hilbert modular surface is connected. On the other hand it is well-known that the preimage of the diagonal decomposes into many components.

Weiss also gives a bounds for the number of different twisted Teichmüller curves for given $\det(M)$ and computer provides evidence for a classification, but this problem as well as the other questions are presently open.

There are other constructions of Teichmüller curves, notably using Prym covers, that can be used to construct Kobayashi geodesics on Hilbert modular surfaces. To a Kobayashi geodesic C on can associate the ratio $\lambda_2 = (C \cdot \mathcal{L}_1)/(C \cdot \mathcal{L}_2)$ where $\mathcal{L}_i$ are the line bundles associated to the natural foliations of the Hilbert modular surface. This invariant is called *Lyapunov exponents* and it is invariant under twisting. It is thus a ways to detect truly different constructions of Kobayashi geodesics.

Theorem 2 ([2], [4], [5])**.** *There exist Kobayashi geodesics on Hilbert modular surface with Lyapunov exponents* 1*,* 1/2*,* 1/3*,* 1/5*,* 1/7*.*

Here, Hirzebruch-Zagier cycles correspond to $\lambda_2 = 1$ and the curves constructed in [3] correspond to $\lambda_2 = 1/3$. No other rational numbers are presently known to arise as Lyapunov exponents of a Kobayashi geodesic and the construction of Kobayashi curves with other Lyapunov exponents is an interesting open problem.

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Distinguished Line Bundles for Complex Surface Automorphisms with Positive Entropy

Paul Reschke

By work of Gromov [4], Yomdin [6], and Friedland [3], the topological entropy of a compact Kähler surface automorphism is exhibited completely in the cohomological actions induced by the automorphism. A consequence of this result is that the entropy of any such automorphism is either zero or the logarithm of a Salem number. (A Salem number is a real algebraic integer greater than one whose Galois conjugates are itself, its inverse, and possibly some even number of points on the unit circle.) We present a refined cohomological characterization of the condition that an automorphism f of a complex projective surface X has positive entropy: f has entropy $\log(\lambda) > 0$ if and only if there is a nef and big line bundle $L \in \mathrm{Pic(X)}$ such that $S(f^*)L = 0$, where $S(t)$ is the minimal polynomial for the Salem number λ. Moreover, we show that this condition is sharp: if f has entropy $\log(\lambda) > 0$, then there is ample line bundle $L \in \mathrm{Pic(X)}$ such that $S(f^*)L = 0$ if and only if no curve on X is periodic for f. (Here, $S(f^*)$ is the naturally defined $\mathbb{Z}$-module endomorphism obtained from the $\mathbb{Z}$-module endomorphism f^* and the polynomial $S(t) \in \mathbb{Z}[t]$.)

An idea of the Workshop is to use dynamics to prove the Bounded Negativity Conjecture for complex projective surfaces that admit automorphisms with positive entropy. When attention is restricted to dynamically minimal automorphisms–that is, automorphisms that give an infinite orbit to every curve with -1 as its self-intersection–work by Cantat [2] shows that automorphisms with positive entropy only arise on tori, K3 surfaces, Enriques surfaces, or certain blow-ups of $\mathbb{P}^2$. (More generally, any automorphism with positive entropy will only arise on a blow-up of one these types of surfaces.) The Bounded Negativitiy Conjecture follows from the adjunction formula in the first three cases; the configurations of blow-ups in most of the known examples of automorphisms with positive entropy on rational surfaces are such that the surfaces have effective anti-canonical bundles–so that the Bounded Negativity Conjecture again follows from the adjunction formula. However, Bedford and Kim [1] have shown that there are automorphisms with positive entropy on rational surfaces where no multiple of the anti-canonical bundle is effective. Proving the Bounded Negativity Conjecture for surfaces admitting dynamically minimal automorphisms with positive entropy would thus yield some new information; removing the constraint of dynamical minimality would of course yield a great deal of new information.

We note that the proof of the refined cohomological characterization of positive entropy described above has a connection to the Bounded Negativity Conjecture. Given an automorphism f of a complex projective surjective X whose entropy is $\log(\lambda) > 0$, there are unique nef (but not big) classes $e+$ and e_- in $\mathrm{NS(X)}_{\mathbb{R}}$ such that $f^*e_+ = \lambda e_+$, $f^*e_- = \lambda^{-1}e_-$, and $e_+.e_- = 1$. The starting point for the refined characterization is the observation by Kawaguchi [5] that a nef and big class of the form $u = ae_+ + be_-$ with $a > 0$ and $b > 0$ will have intersection 0 with a curve on X if and only if the curve is periodic for f. To prove the characterization, we take a line bundle $L \in \mathrm{Pic(X)}$ satisfying $L^2 > 0$ and $S(f^*)L = 0$ (where $S(t)$ is the minimal polynomial for λ) and show that the sequence

$$\{\lambda^{-n}((f^*)^n + ((f^{-1})^*)^n)[L]\}_{n\in\mathbb{N}}$$

(which approaches $([L].e_+)e_+ + ([L].e_-)e_-$) eventually becomes nef and big. This conclusion would be immediate if the Bounded Negativity Conjecture were known, but instead requires a more subtle argument (using the effectiveness of some multiple of L) for the time being.

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Real counterexample to the containment $I^{(3)} \subset I^2$ and line arrangements

Piotr Pokora

The first part of my talk is based upon the joint work with A. Czapliski, A. Główka - Habura, G. Malara, Magdalena Lampa - Baczyńska, Patrycja Łuszcz - Świdecka and Justyna Szpond [1], and the second part is based upon my recent unpublished researches.

The so-called *containment problem* is very interesting subject in the theory of linear series and also combinatorial algebraic geometry. Roughly speaking, we are interested in the relations between the symbolic and ordinary powers of homogeneous ideals of points, lines and other subvarieties of projective spaces. One of the most interesting and recent problem can be formulated in the following way.

Problem 1. Consider a set of points $Z = \{P_1, ..., P_s\} \in \mathbb{P}^2$ over an arbitrary field and denote by $I = I(P_1) \cap ... \cap I(P_s)$, where $I(P_i)$ is the ideal of a point P_i for $i \in \{1, ..., s\}$. Does the containment

$$I^{(3)} \subset I^2 \tag{1}$$

hold?

In the situation of points, $I^{(m)} = I(P_1)^m \cap ... \cap I(P_s)^m$ and $I^{(m)}$ corresponds to the set of homogeneous polynomials vanishing along Z with multiplicities at least m.

One year ago M. Dumnicki, T. Szemberg and H. Tutaj - Gasińska [2] gave the first counterexample to the containment (1) over the complex numbers. Today we also know that there exist some counterexamples to the above containment over fields of finite characteristic [3]. In [1] using the example of the Böröczky extremal configurations of lines over $\mathbb{R}$ (which deliver the maximal possible number of triple points) we showed that (1) doesn't hold.

This is worth to point out that this kind of line configurations is relevant for the so-called Harbourne constants.

Let $\mathcal{P} = \{P_1, \ldots, P_2\} \subset \mathbb{P}^2$ be a set of fixed points and denote by $f : X_{\mathcal{P}} \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at $\mathcal{P}$.

Then we define the following constants

$$H(\mathbb{P}) = \inf_{C \text{ reduced curve on } X_{\mathcal{P}}} \frac{C^2}{s}.$$

$$H(s) = \inf_{\mathcal{P} \in (\mathbb{P}^2)^{(s)}} H(\mathcal{P}).$$

$$H = \inf_{s} H(s).$$

The last constant is called the global Harbourne constant.

In the same spirit one defines a special variation of the above constants for configurations of lines and we denote them by $H_L(\mathcal{P}), H_L(s), H_L$ respectively. To clarity the infimum in $H_L(\mathcal{P})$ is taken over proper transforms of line arrangements.

The main result for H_L can be formulated in the following way.

Theorem 2. *Suppose that L is a configuration of d lines over the complex numbers. Then $H_L \geq -4$.*

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Observational connections between local negativity and containment conjecture counterexamples.

Brian Harbourne

Synergistic progress has been made recently on two seemingly independent problems, the first being to find new upper bounds on the extremal average negativity of plane curves and the second being to find failures of certain containments of symbolic powers of ideals of points in ordinary powers. Reasons for this synergism remain mysterious.

More specifically, the first problem is to compute

$$\inf \frac{(\deg(C))^2 - \sum_i (\mathrm{mult}_{p_i}(C))^2}{s}$$

where the infimum is taken over all reduced plane curves C. Here, for simplicity, take the points p_i to be the singular points of C and s to be the number of singular points. (For later use, let $\phi(C)$ denote $\frac{(\deg(C))^2 - \sum_i (\mathrm{mult}_{p_i}(C))^2}{s}$.) The second problem is to find ideals $I \subset k[\mathbf{P}^2]$ of finite point sets $Z \subset \mathbf{P}^2$ such that $I^{(2r-1)} \not\subseteq I^r$ for some value of $r \geq 2$, where k is the ground field. (Here, given points $p_1, \ldots, p_s$, we define the symbolic power $I^{(m)}$ by $I^{(m)} = \cap_i I(p_i)^m$.)

The connection between these two problems is purely observational: the singular points of reduced curves C with $\phi(C) < -2$ seem to give examples of points sets Z whose ideals I have $I^{(2r-1)} \not\subseteq I^r$ (usually with $r = 2$), and point sets Z whose ideals I satisfy $I^{(2r-1)} \not\subseteq I^r$ seem to be the singular points of reduced curves C with $\phi(C) < -2$. Examples of either phenomenon are quite rare, so this correlation could just be coincidence, and yet examples of the one have been used to find examples of the other which are unlikely to have been found otherwise.

In fact, over the complex numbers, as of the time of this talk, only three classes of examples of curves C with $\phi(C) < -2$ were known. The first example consists of taking C_n to be the $3n$ lines defined by $(x^n - y^n)(x^n - z^n)(y^n - z^n)$. Then the points are the $n^2 + 3$ points consisting of the n^2 points of intersection of $x^n - y^n = 0$ and $x^n - z^n = 0$ and the three coordinate vertices. In this case $\phi(C_n) > -3$, but $\lim_{n\to\infty} \phi(C_n) = -3$. The second class is similar, but defined over the reals [C. et al]. The third, due to P. Pokora, takes C to be a configuration of 21 lines known as the Klein configuration. In this case, C has 49 singular points and $\phi(C) = -3$.

The first example above initially arose as a curve C whose set Z of singular points have ideal I with $I^{(3)} \not\subseteq I^2$ [DST]. It was then noticed by T. Szemberg that $\phi(C)$ was about -3. Then Pokora gave the example where C is the curve consisting of the 21 lines of the Klein configuration, which has $\phi(C) = -3$. A version of the Klein configuration can be defined in $\mathbf{P}^2(\mathbf{F})$ where $\mathbf{F}$ is a finite field with $|\mathbf{F}| = 7$; the speaker checked that the singular points of C in the finite field case give an ideal I with $I^{(3)} \not\subseteq I^2$. Thus the connection has run both ways. Moreover, additional instances of the connection occur in positive characteristics [BCH, HS]. For example, the result of [DST] carries over without significant change [HS] to positive characteristics. All other previously known examples in positive characteristics of finite points sets in $\mathbf{P}^N$ having ideals I for which $I^{(Nr-N+1)} \not\subseteq I^r$ involve taking all but one of the points of $\mathbf{P}^N$ over a finite field $\mathbf{F}$ [HS]. The example coming from the Klein configuration in characteristic 7 is new in that one excludes not just 1 point but 8 points (specifically, one excludes the $\mathbf{F}$-points of the conic $x^2 + y^2 + z^2 = 0$).

The fact that examples of reduced plane curves C with $\phi(C) < -2$ seem to give rise to examples of ideals I with $I^{(2r-1)} \not\subseteq I^r$, and vice versa, raises the question of whether there is a deeper reason for this connection. The main goal of this talk is to raise the question of whether such a deeper reason exists, and if so, what it might be.

The examples in positive characteristic merit a brief comment. For the examples for which one takes all but one point of $\mathbf{P}^2(\mathbf{F})$ over a finite field $\mathbf{F}$, the curve C consists of all of the lines defined over $\mathbf{F}$ which do not contain the excluded point. In this case one always obtains $\phi(C) = \frac{s^4-(s^2+s)s^2}{s^2+s} \approx -s$ and, for certain values of r depending on the characteristic and on $|\mathbf{F}|$, one has $I^{(2r-1)} \not\subseteq I^r$. (Full disclosure requires mentioning that taking C to be the union of all of the lines in $\mathbf{P}^2(\mathbf{F})$ also gives $\phi(C) \approx -3$, but the ideal I of all of the points does not seem to lead to any examples of $I^{(2r-1)} \not\subseteq I^r$.)

Motivation for the first problem comes from the Bounded Negativity Problem; this is the question of whether it is true, given fixed points $p_1, \ldots, p_s \in \mathbf{P}^2$, that $\inf((\deg(C))^2 - \sum_i(\mathrm{mult}_{p_i}(C))^2)$ is finite, where the infimum is over all reduced curves $C \subset \mathbf{P}^2$. In exploring this problem, one looks for curves C for which $(\deg(C))^2 - \sum_i(\mathrm{mult}_{p_i}(C))^2$ is especially negative. In order to compare examples for different choices of s and different choices of points p_i, it is helpful to consider

$$\lambda = \inf \frac{(\deg(C))^2 - \sum_i (\mathrm{mult}_{p_i}(C))^2}{s}$$

where the infimum now is taken over all reduced curves C, all sets of s points p_i and all s.

By taking s points on any given plane curve C, we see that $\lambda \leq -1$. By taking the $s = \binom{d}{2}$ singular points of a union of d general lines as $d \to \infty$, we see that $\lambda \leq -2$. It was only in the last few weeks that examples $\phi(C) < -2$ have been found, so examples are still quite rare.

Examples of ideals I of points in $\mathbf{P}^N$ with $I^{(Nr-N+1)} \not\subseteq I^r$ are also still quite rare. Motivation for looking for them is the theorem of Ein-Lazarsfeld-Smith [ELS] that $I^{(Nr)} \subseteq I^r$ holds for all such ideals I and all r. Huneke then asked if it is always true that $I^{(3)} \subseteq I^2$ when $N = 2$. In fact, it is possible to show in many cases that $I^{(Nr-N+1)} \subseteq I^r$ holds for ideals I of points in $\mathbf{P}^N$, leading to the containment conjecture of the speaker that $I^{(Nr-N+1)} \subseteq I^r$ always holds [B. et al]. The examples above show that this conjecture is false, but understanding when and why failures of the containment occur is an open problem.

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Finiteness of extremal rays in the cone theorem

John Lesieutre

Suppose that X is a projective variety of dimension n with terminal singularities. The basic structure theorem for the Mori cone $\overline{\mathrm{NE}}(X)$ is the cone theorem:

Theorem 1 ([2]). *Suppose that (X, Δ) is a dlt pair with Δ effective. Then*

(1) *There exists a countable set of rational curves $C_j \subset X$, with $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim X$, such that*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(2) *If H is any ample divisor on X and $\epsilon > 0$,*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta+\epsilon H)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].$$

In the statement of (1), the number of extremal rays can indeed be infinite: there might be infinitely many rays accumulating on the hyperplane $K_X^{\perp}$. The standard example of this phenomenon is to take X the blow-up of $\mathbb{P}^2$ at nine or more very general points. This is a smooth rational surface containing infinitely many (-1)-curves, each generating an extremal class on $\overline{\mathrm{NE}}(X)_{K_X<0}$. Aside from this example and some simple variants on it (e.g. products, projective bundles), there are not many examples known in which the number of rays is not simply finite.

Early on in the study of the minimal model program, it was asked whether this might always be the case, so that the only examples of divisors with infinitely many rays are uniruled.

Conjecture 2 ([1], Problem 4-2-5). *Suppose that (X, Δ) is a klt pair with $\kappa(X, \Delta) \geq 0$. Then the set of curves C_j in Theorem 1 may be taken to be finite.*

There are several cases in which Conjecture 2 is easily verified.

Proposition 3. *The decomposition*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j]$$

holds with only a finite sum on the right hand side if (X, Δ) is a klt pair satisfying any of the following conditions:

(1) *either $K_X + \Delta$ or Δ is big;*
(2) $\dim X = 2$ *and $\kappa(X, \Delta) \geq 0$;*
(3) *X is smooth,* $\dim X = 3$, *$\Delta = 0$, and $\kappa(X, \Delta) \geq 0$.*

Uehara pointed out that this question has a negative answer in the generality of klt pairs: there exists a Calabi-Yau threefold X and a divisor Δ such that there are infinitely many $(K_X + \Delta)$-negative flopping curves [3][4]. Uehara suggests that the conjecture might nevertheless hold in the case $\Delta = 0$.

In dimension 3, the conjecture is closely related to the study of surfaces with infinitely many negative curves. If X is a terminal threefold of Kodaira dimension 0, then $K_X = \sum a_i E_i$, with E_i a set of rigid divisors. If there are infinitely many K_X-negative extremal rays, then infinitely many must be contained in some component E_0. Only finitely many of these give rise to divisorial contractions, and so there are infinitely many flipping contractions of curves contained in E_0. These curves are all contractible on E_0 itself, and so their strict transforms give infinitely many negative curves on the minimal resolution $\tilde{E}_0$.

Conjecture 4. *Suppose that Y is a terminal projective threefold. Then the number of minimal models X_i that can be reached via the K_Y-MMP is finite.*

This conjecture includes cases in which Y has infinitely many minimal models!

Proposition 5. *Conjecture 2 implies Conjecture 4.*

Proof. Let T be the decision tree for the K_Y-MMP: that is, let T be a tree with a node for each variety Z_i that can be encountered in the course of a run of the K_Y-MMP, and an edge between two nodes if the nodes are connected by a flip or divisorial contraction.

Conjecture 2 implies that each node of T has at most finitely many children, since at each stage of the MMP there are only finitely many choices of extremal contraction. Moreover, T does not contain any infinite paths, by termination of flips in dimension 3. By König's lemma, a finitely branching tree with no infinite paths must in fact be finite, and so there are only finitely many possible end results of the MMP. □

It seems to be difficult, however, to control the number of possible outcomes of the MMP without directly bounding the number of extremal rays at each step and proving Conjecture 2 directly. The following slightly restricted version of Conjecture 4 seems more amenable to other approaches, and indeed is closely related to questions about Zariski decomposition in dimension 3.

Conjecture 6. *Suppose that Y is a terminal projective threefold. Then the number of minimal models X_i that can be the outcome of the K_Y-MMP with scaling by an ample divisor H on Y is finite.*

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Quantitative aspects of Zariski chamber decompositions on surfaces

Thomas Bauer

(joint work with Michael Funke, Sebastian Neumann, David Schmitz)

Zariski chambers are natural pieces into which the big cone of an algebraic surface decomposes – they account for the variation of the stable base locus of big line bundles. From joint work of the author with Küronya and Szemberg [2] they are known to be locally polyhedral and locally finite in number. Beyond these geometric properties we are interested in Zariski chambers from two further perspectives:

- a *metric* point of view (How big are the chambers?), and
- a *combinatorial* point of view (How many chambers are there on the surface?).

For a smooth projective variety X and a big divisor D on X, denote by

$$\mathbf{B}(D) := \bigcap_{m>0} \mathrm{Bs}(mD)$$

the *stable base locus* of the linear series $|D|$. A guiding question is

How does $\mathbf{B}(D)$ *vary with* D*?*

In studying this question, it is preferable to work with an approximation introduced by Ein, Lazarsfeld, Mustata, and Popa [6], the *augmented base locus*

$$\mathbf{B}_+(D) := \bigcap_{A} \mathbf{B}(D - A)$$

(where the intersection is taken over all ample $\mathbb{Q}$-divisors A). It has the advantage of depending only on the numerical equivalence class of D, and it can be extended to real classes in the big cone $\mathrm{Big}(X)$. The big cone then naturally decomposes into subsets where $\mathbf{B}_+(D)$ is constant, and the basic question then is:

How do these subsets look like?

While this seems to be a hard problem in general, the picture is particularly pleasant when X is a surface: In that case, there is by [2] a locally finite decomposition of $\mathrm{Big}(X)$ into rational locally polyhedral subcones (called *Zariski chambers*) such that the following holds:

(i) In the interior of each of the subcones the base loci $\mathbf{B}$ and $\mathbf{B}_+$ are constant.
(ii) On each of the subcones the volume function is given by a single polynomial of degree two.
(iii) In each subcone the support of the negative part of the Zariski decomposition of the divisors in the subcone is constant.

The metric aspect: How big are the chambers? In work with D. Schmitz [4] we investigated the question whether it is possible to compare Zariski chambers with respect to their "size". A natural approach is suggested by work of Peyre [7] and Derenthal [5], who (in an arithmetic context) employ a concept of *nef cone volume* for Del Pezzo surfaces. The latter is defined as the volume of the cross section

$$\mathrm{Nef}(X) \cap \{\xi \in \mathrm{NS}_{\mathbb{R}}(X) \mid -K_X \cdot \xi = 1\}$$

or, alternatively, of the truncated cone

$$\mathrm{Nef}(X) \cap \{\xi \in \mathrm{NS}_{\mathbb{R}}(X) \mid -K_X \cdot \xi \le 1\}$$

(One checks that there is a well-defined Lebesgue measure on the Néron-Severi vector space $\mathrm{NS}_{\mathbb{R}}(X)$.) This volume concept naturally generalizes to arbitrary smooth surfaces (in fact, to arbitrary smooth varieties) and can be used to measure the sizes of Zariski chambers $\Sigma \subset \mathrm{Big}(X)$ through their *cone volume* $\mathrm{Vol}(\Sigma, A)$ with respect to an ample line bundle A. We provide in [4] a method to determine the cone volumes of Zariski chambers from the nef cone volumes of blow-downs:

Theorem. *Let X be a smooth projective surface of Picard number ρ, and let Σ be a Zariski chamber that is supported by a set S of s irreducible curves. Then:*

- *If S contains a curve E with $E^2 < -1$, then* $\mathrm{Vol}(\Sigma_P) = \infty$.

- *Otherwise S consists of s pairwise disjoint (-1)-curves and*

$$\mathrm{Vol}\big(\Sigma, -K_X\big) = \frac{(\rho - s)!}{\rho!} \cdot \mathrm{Vol}\big(\mathrm{Nef}(\pi_S(X)), -K_X\big) ,$$

where π_S is the blow-down of S.

This result allows one in particular to explicitly determine the volumes of all Zariski chambers on Del Pezzo surfaces.

The combinatorial aspect: How many chambers are there on the surface? Given a smooth projective surface X, consider the *chamber number*

$$z(X) := \#\{\text{Zariski chambers on } X\} \in \mathbb{N} \cup \{\infty\} .$$

In view of [2], the number $z(X)$ is the answer to the following questions that amount to asking how "complicated" X is from the point of view of linear series:

- How many different stable base loci can occur in big linear series on X?
- How many essentially different Zariski decompositions can big divisors on X have?
- How many "pieces" does the volume function $\mathrm{vol}\colon \mathrm{Big}(X) \to \mathbb{R}$ have?

If X contains only finitely many negative curves (e.g., when X is a Del Pezzo surface), then finding $z(X)$ is a finite problem, as $z(X)$ is determined by the number of negative definite principal submatrices of the intersection matrix of the negative curves on X. While this information may be sufficient from a purely theoretical point of view, it provides no immediate practical way to actually compute $z(X)$, because the naive approach of checking *all* principal submatrices has exponential complexity – for instance, on the blow-up of $\mathbb{P}^2$ in 8 general points there are 2^{240} such submatrices. In work with M. Funke and S. Neumann [1] we developed an algorithm that computes $z(X)$ via a backtracking strategy that drastically reduces the number of matrices to check. This can be applied in particular to Del Pezzo surfaces:

Theorem. *Let X_r be the blow-up of $\mathbb{P}^2$ in r general points with $1 \le r \le 8$.*

(i) *The number $z(X_r)$ of Zariski chambers on X_r is given by the following table. ($N(X_r)$ denotes the number of negative curves.)*

r	1	2	3	4	5	6	7	8
$N(X_r)$	2	3	6	10	16	27	56	240
$z(X_r)$	2	5	18	76	393	2 764	33 645	1 501 681

(ii) *The maximal number of curves that occur in the support of a Zariski chamber on X_r is r.*

Interestingly, it turns out that the computation of chamber numbers is much more challenging on surfaces like the Schur quartic [9]

$$x(x^3 - y^3) - z(z^3 - w^3) ,$$

which is famous for containing 64 lines (cf. [10] and [8]). When we tried to use the algorithm from [2] in order to determine how many chambers are supported by the 64 lines, we found that it takes an inordinate amount of time – to the point of becoming impractical in such situations. Only with an improved algorithm [3], which makes the computation of a large number of determinants much more efficient, was it possible to find out that the Schur quartic has precisely

$$8\,260\,383\,569$$

Zariski chambers that are supported by lines.

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Local positivity and Newton–Okounkov bodies

Alex Küronya
(joint work with Victor Lozovanu)

This is an account of joint unpublished work with Victor Lozovanu. Newton–Okounkov bodies serve to capture the behaviour of we all global sections of all multiples of a given big Cartier divisor at the same time. Building on earlier work of Okounkov and many others, in their current form Newton–Okounkov bodies were first studied by Kaveh–Khovanskii [3] and Lazarsfeld–Mustaţă [5]. For explicit examples see [5] and [4] for instance. Here we will explore the implications to local positivity of line bundles.

Let X be a smooth projective variety of dimension n over the complex number field, $Y_\bullet$ an admissible flag, D a big line bundle on X. The choice of the flag $Y_\bullet$ gives rise to a rank n valuation $\nu_{Y_\bullet}$ on the function field $\mathbb{C}(X)$ of X, which, evaluated on the global sections of multiples of D, yields a convex body $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$, the Newton–Okounkov body of D.

A result of Jow [2] claims that the set of all Newton–Okounkov bodies forms a universal numerical invariant, more specifically, if D and D' are two big Cartier divisors on X, and

$$\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$$

for all admissible flags $Y_\bullet$ on X, then D and D' are numerically equivalent. To take this philosophy a step further, it is feasible to expect that local positivity of D at a point $x \in X$ is determined by the set

$$\{\Delta_{Y_\bullet}(D) \mid (Y_\bullet)_n = \{x\}\} \ .$$

Our aim is to recover projective geometric information from Okounkov bodies associated to the divisor D. The more specific goal is characterize divisors having a given point in their restricted/augmented base loci. Combining our observations with results from [1], we arrive at descriptions of nef/ample divisors in terms of their Okounkov bodies. The main statements go as follows:

Theorem A. With notation as above, $x \in \mathbb{B}_(D)$ if and only if there exists a flag $Y_\bullet$ centered at the point $x \in X$ such that $0 \in \Delta_{Y_\bullet}(D)$.

As a consequence, D is nef if and only if for every $x \in X$ there exists a flag $Y_\bullet$ centered at x such that $0 \in \Delta_{Y_\bullet}(D)$.

Theorem B. With notation as above, $x \in \mathbb{B}_+(D)$ if and only if there exists a flag $Y_\bullet$ centered at the point x with Y_1 ample, and a positive real number $\epsilon > 0$ such that $\Delta_\epsilon \subseteq \Delta_{Y_\bullet}(D)$.

As a consequence D is ample precisely if this condition holds for all points $x \in X$.

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Global generation of toric vector bundles and cohomology vanishing

Sandra Di Rocco
(joint work with K. Jabbusch, G Smith)

The projectivization of a toric vector bundle is an algebraic variety characterized by a reach combinatorial structure coming from the torus action and at the same time having a geometrical structure less rigid compared to the one of toric varieties. They provide therefore an important testing ground in Algebraic Geometry.

Recently the existence of indecomposable toric vector bundles E such that $\mathbb{P}(E)$ is not a Mori dream space has been shown in [2], making this class of variety particularly interesting in connection with the Minimal Model Program.

Let X be a toric variety and let T be the dense torus acting on it.

Definition. A *toric vector bundle* is a locally free sheaf E whose total space $V(E)$ has a T action with respect to which the projection map $\pi : E \to X$ is T-equivariant. Moreover it is required that T acts linearly on the fibres of E.

If E is a toric vector bundle then $\mathbb{P}(E)$ also inherits a T-action for which the projection map $\mathbb{P}(E) \to X$ is T-equivariant.

Notice, moreover, that $\mathbb{P}(E)$ is a toric variety if and only if $E = L_1 \oplus \ldots \oplus L_r$ for line bundles L_i on X.

Toric vector bundles have been extensively studied, see [6, 7, 8]. They are characterized by a sequence of decreasing filtered vector spaces, indexed by the edges of the fan defining the toric variety X.

Klaychko used the theory of toric vector bundles on $\mathbb{P}^2$ as main ingredient in his prove of the Horn conjecture.

Our motivation comes from two central conjectures in Algebraic Geometry.

The first one is the belief (more than an explicit conjecture) that smooth toric varieties should be projectively normal. Recall that a projective algebraic variety $X \hookrightarrow \mathbb{P}^N$ is *projectively normal* if and only if the multiplication maps

$$Sym^k H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(k))$$

are surjective for all $k \geq 2$. M. Green in [3, 4] proved an interesting connection between projective normality and higher syzygies with vanishing properties of vector bundles. Consider the vector bundle given by the kernel of the evaluation map: $M_L = \ker(H^0(X, L) \otimes \mathcal{O}_X \to L)$. Then $H^1(X, \wedge^k M_L \otimes L^j) = 0$ for $k \leq 2$ and $j \geq 1$ would prove that the embedding is projectively normal and the ideal is generated by quadrics. Vanishing for higher k extend to criteria for the so called Green's property N_p.

When X is toric the vector bundle M_L is a toric vector bundle. In [5] it was established that M_L is moreover globally generated.

Because globally generated line bundles on toric varieties have vanishing higher cohomology it is natural to ask whether this property extends to higher rank. An affirmative answer would prove the above long standing conjecture.

Our second motivation is Fujita conjecture. This conjecture states that if L is an ample line bundle on a complex manifold X then $K_X \otimes L^m$ is globally generated for $m \geq \dim(X) + 1$ and it is very ample for $m \geq \dim(X) + 2$. The globally generation statement has been shown to be true up till dimension 4 (Reider proved it in dimension 2, Ein and Lazarsfeld in dimension 3 and Kawamata in dimension 4.) In its full generality the conjecture is known to be true only for toric varieties, as shown by Mustaţă. For the tautological line bundle, ξ_E, on a projectivized vector bundle $\mathbb{P}(E)$ the conjecture is directly related to the global generation or very ampleness of the toric vector bundle $Sym^k(E) \otimes K_X \otimes \det(E)$ for $t \geq \dim(X)$.

When X is a toric variety it is again a question about the global generation of a toric factor bundle.

For line bundles on toric varieties there is a tight connection between generation of global sections and numerical positivity. This relation is essentially due to the convex geometrical interpretation of the global sections of L. Recall that any line bundle has an associated lattice polytope, P_L, cut out by hyperplanes with normal vectors dual to the rays of Σ and whose lattice points correspond to a basis for the vector space $H^0(X, L)$. Viceversa any such polytope defines a line bundle on X. Let $\sigma(t)$ be the collection of cones of dimension t in the defining fan Σ and let m_σ be the associated character in the dual lattice, then the following statements are equivalent:

(1) L is globally generated
(2) L is globally generated at each torus fixed point $x(\sigma)$, for all $\sigma \in \Sigma(\dim(X))$.
(3) $P_L = \mathrm{Conv}\{m_\sigma\}_{\sigma\in\Sigma(\dim(X))}$.
(4) $L \cdot C_\rho =$ distance$(m_\sigma, m_{\sigma'}) \geq 0$ for each invariant curve associated to $\rho = \sigma \cap \sigma' \in \Sigma(\dim(X) - 1)$.
(5) $L \cdot C \geq 0$ for every curve C.

To what extent these properties carry out to higher rank bundles? An example of a nef but not globally generated toric vector bundle was found in [5], where explicit criteria for nefness and ampleness are established. Our aim is to answer the following two questions:
Q1: Which numerical positivity implies global generation? Are ample toric vector bundles globally generated?
Q1: Do globally generated vector bundles enjoy higher cohomology vanishing?

In order to gain a better understanding we propose a more convex geometrical interpretation of global sections of vector bundles which allows to have an effective criterion for global generation.

Let rank$(E) = r$. To every $\sigma \in \Sigma(\dim(X))$ we associate a *compatible basis* $e(\sigma) = (e_1, \ldots, e_r)$ of $\mathbb{C}^r$ and to each e_i a (possibly empty) polytope P_{e_i}, defined using the filtration of the vector bundle E. We denote such collection with $\mathcal{P}_E = \{P_{e_i}, e_i \in e(\sigma)\}$ and call it the *parliament of polytopes* associated to E. Let L_i be the line bundle on X defined by the polytope P_{e_i}. We show that

$$H^0(X, E) = \bigoplus_{m\in\mathcal{P}_E} \mathbb{C}\chi^m$$

where χ^m is the global section defined by the character m.

Moreover we prove the following criterion.

Proposition [1] The following assertions are equivalent

(1) E is globally generated.
(2) E is globally generated at all fixed points $x(\sigma)$.
(3) For every $\sigma \in \Sigma(\dim(X))$ $P_{e_i} \neq \emptyset$ for every $e_i \in e(\sigma)$ and the corresponding decomposable vector bundle $L_1 \oplus \ldots \oplus L_r$ is globally generated at $x(\sigma)$.

Thanks to this criterion we were able to easily construct toric vector bundles and check their global generation, providing answers to the two questions above.
A1[1] There are ample but not not globally generated toric vector bundles on smooth toric varieties.
A2 [1] There are globally generated toric vector bundles on smooth toric varieties with non vanishing higher cohomology.

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Local Positivity and Cayley Polytopes

Anders Lundman

For a smooth projective variety X and a line bundle $\mathscr{L}$ on X there are various notions for measuring the local positivity of $\mathscr{L}$ at a point $x \in X$. We consider the *osculating space* $\mathbb{T}^k_x(X,\mathscr{L})$ of order k for various $k \in \mathbb{N} := \{0,1,\dots\}$. Recall that $\mathbb{T}^k_x(X,\mathscr{L})$ is defined as $\mathbb{P}(\mathrm{im}(j^k_x))$ where $\mathrm{im}(j^k_x)$ is the image of the jet map

$$j^k_x : \mathrm{H}^0(X,\mathscr{L}) \to \mathrm{H}^0(X,\mathscr{L} \otimes (\mathscr{O}_X/\mathfrak{m}_x^{k+1})).$$

Observe that when $k=1$ the osculating space $\mathbb{T}^1_x(X,\mathscr{L})$ is simply the projective tangent space at x. In this setting we say that $\mathscr{L}$ is *k-jet spanned* if j^k_x is onto. It is natural to ask to what extent fixing the dimension of the osculating space at every point determines the pair $(X,\mathscr{L})$. One theorem in this direction is the following characterization of the k-th Veronese embedding.

Theorem([7]) *Let* $N = \begin{pmatrix} n+k \\ k \end{pmatrix} - 1$, *then a closed embedding of a projective smooth* n*-fold* $X \hookrightarrow \mathbb{P}^N$, *over any algebraically closed field, is the* k*-th Veronese embedding of* $\mathbb{P}^n$ *if and only if* $\mathbb{T}^k_x(X,\mathscr{L}) \cong \mathbb{P}^N$ *for all points* $x \in X$.

Similarly there are characterizations of balanced rational normal surface scrolls [1] and abelian varieties [6] in terms of their osculating spaces. Here we are interested in the case when $(X,\mathscr{L})$ is a smooth polarized toric variety. As might be expected there are simple combinatorial characterisations of the dimension of $\mathbb{T}^k_x(X, \mathscr{L})$ in terms of the polytope $P_{\mathscr{L}}$ associated to $(X, \mathscr{L})$ (see [5] and [11]). Moreover, in [11], David Perkinson has characterized all polarized smooth toric surfaces and threefolds $(X, \mathscr{L})$ such that for every point $x \in X$ and for a fixed $k \in \mathbb{N}$, $\mathscr{L}$ is k-jet spanned, but not $(k+1)$-jet spanned at x. If one, in his classification, considers only embeddings given by a complete linear series $|\mathscr{L}|$, then $(X, \mathscr{L})$ has the structure of a projective bundle with fibers isomorphic to $(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(k))$. Explicitly in the case of a surface, $(X, \mathscr{L})$ is either a Veronese embedding or a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In the case of threefolds $(X, \mathscr{L})$ is either a Veronese embedding or a $\mathbb{P}^1$-bundle over a smooth toric surface.

The characterisation in [11] is based on convex geometry. The resulting classification is in fact stated in terms of the polytopes associated to $(X, \mathscr{L})$. That the complete embeddings appearing in the classification are projective bundles stems from the fact that the associated polytopes have a Cayley structure of type $[P_0 * P_1]^k$. Recall that if $P_0, \dots, P_r$ are polytopes in $\mathbb{R}^s$ then $[P_0 * \cdots * P_r]^k :=$ $\mathrm{Conv}((P_0 \times \vec{0}, P_1 \times k\hat{e}_1, \dots, P_r \times k\hat{e}_r) \subset \mathbb{R}^{s+r}$ where $e_1, \dots, e_r$ is a basis of $\mathbb{Z}^r$. Our main results is a generalization of Perkinsons classification to arbitrary dimension.

Theorem([10]) *Let $(X, \mathscr{L})$ be a smooth polarized toric variety and let $P_{\mathscr{L}}$ be the polytope associated to the complete linear series $|\mathscr{L}|$. The line bundle $\mathscr{L}$ is k-jet spanned but not $(k+1)$-jet spanned at every point $x \in X$ if and only if $P \cong [P_0 * P_1]^k$ for some lower dimensional polytopes P_0 and P_1 and every edge of P contain at least $k+1$ lattice points.*

In algebro geometric language if $(X, \mathscr{L})$ is associated to a Cayley polytope $P = [P_0 * P_1]^k$, then there exist an explicit birational morphism $\pi : X' \to X$, where X' is a projective fiber bundle with fiber $F \cong \mathbb{P}^1$ and $\pi^*\mathscr{L}|_F \cong \mathscr{O}_{\mathbb{P}^1}(k)$ for all fibers F. Here $X' = \mathbb{P}(L_0 \oplus L_1)$, where the L_i are line bundles on the toric variety associated to (the inner-normal fan of) the Minkowski sum $P_0 + P_1$ (see [4] for details).

An other way of measuring the local positivity of a nef line bundle $\mathscr{L}$ on a smooth projective variety X is via so called Seshadri constants. For any point $x \in X$ Jean-Pierre Demailly [3] defined the Seshadri constant at x as the real number:

$$\epsilon(X, \mathscr{L}; x) := \inf_{C \subseteq X} \frac{\mathscr{L} \cdot C}{m_x(C)}.$$

Here the infimum is taken over all irreducible curves C passing through x and $m_x(C)$ is the multiplicity of C at x. Unfortunately Seshadri constants are in general very hard to compute and as a consequence there are few classification results in the general setting. However when X is toric one might expect that Seshadri constants could be captured by convex geometric properties of the polytope associated to $(X, \mathscr{L})$. This is indeed the case and in [8] Atsushi Ito showed that on toric varieites local positivity expressed in terms of Seshadri constants is related to the existens of a Cayley structure. More to the point Ito showed that if $(X, \mathscr{L})$

is a polarized toric variety, then $P_{\mathscr{L}} \cong [P_0 * P_1]^1$ if and only if $\epsilon(X, \mathscr{L}, x) = 1$ at a very general point. To incorporate local positivity expressed in terms of Seshadri Constants we extend our main result to the following form.

Theorem([10]) *Let $(X, \mathscr{L})$ be a smooth polarized toric variety, let $P_{\mathscr{L}}$ be the corresponding smooth polytope and let $k \in \mathbb{N}$. Then the following are equivalent:*

(1) *$s(\mathscr{L}, x) = k$ at every point $x \in X$.*
(2) *$s(\mathscr{L}, x) = k$ at the fixed points and at the general point.*
(3) *$\epsilon(X, \mathscr{L}; x) = k$ at every point $x \in X$.*
(4) *$\epsilon(X, \mathscr{L}; x) = k$ at the fixed points and at the general point.*
(5) *$P_{\mathscr{L}} \cong [P_0 * P_1]^k$ for some lower dimensional polytopes P_0 and P_1 and every edge of P has length at least k.*

Here $s(\mathscr{L}, x)$ is the largest natural number k such that $\mathscr{L}$ is k-jet spanned at $x \in X$

The above result has two facets. On the one hand it gives a characterisation of a large class of generalised Cayley polytopes and thereby generalise the characterisations of Perkinsson and Ito (in the smooth setting). It would be intriguing to find a similar algebro geometric characterisation of all general Cayley polytopes, at least in the smooth setting. On the other hand our results provide an equivalence between Seshadri constants and the numbers $s(\mathscr{L}, x)$ for smooth toric varieties. The exact nature of the relationship between $s(\mathscr{L}, x)$ and $\epsilon(X, \mathscr{L}; x)$ is in general an open and interesting question, very much related to Demailly's original motivation for introducing Seshadri constants [2], [9].

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Mori dream hypersurfaces in products of projective spaces

John Christian Ottem

In my talk I presented a few results about the birational structure of hypersurfaces in products of projective spaces. These hypersurfaces are in many respects simple as algebraic varieties, but it turns out that they can have surprisingly complicated behaviour from the viewpoint of birational geometry. For example, a hypersurface of tridegree $(2, 2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, has infinite birational automorphism group and the effective cone is rational non-polyhedral.

A natural question is when such hypersurfaces are so-called Mori dream spaces. These varieties were introduced by Hu and Keel in [2] as a class of varieties with good birational geometry properties. By definition, a variety is Mori dream if its Cox ring is finitely generated. This ring is essentially defined as

$$\mathcal{R}(X) = \bigoplus_{D \in Pic(X)} H^0(X, D).$$

Choosing a presentation for the Cox ring gives an embedding of X into a simplicial toric variety Y such that each small modification of X is induced from a modification of the ambient toric variety Y (see [2]). From this one shows that the Minimal Model Program can be carried out for any divisor and has a combinatorial structure as in the case of toric varieties.

Being a Mori dream space is a relatively strong condition and there are classical examples of varieties that are not. Perhaps the most famous of these is Nagata's counterexample to Hilbert's 14th problem, in which he proves that the blow-up of $\mathbb{P}^2$ along the base-locus of a general cubic pencil has infinitely many (-1)-curves. This blow-up is clearly not a Mori dream space since each of the (-1)-curves would require a generator of the Cox ring.

When X is a hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$, the Picard number is 2, so this phenomenon can not occur. However, there are other obstructions to having finitely generated Cox ring. Here the main interesting case occurs when $m = 1$. In the case $m, n \geq 2$, it is straightforward to show that the Cox ring of X is quotient of that of $\mathbb{P}^m \times \mathbb{P}^n$ by the defining polynomial, and so X is clearly a Mori dream space. For hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^n$ we have the following:

Theorem. *Let X be a very general hypersurface of bidegree (d, e) in $\mathbb{P}^1 \times \mathbb{P}^n$ and let $H_i = \mathrm{pr}_i^* \mathcal{O}(1)$. Then X is a Mori dream space if and only if it belongs to the following cases:*

- *$d \leq n$ in which case the Cox ring has the following presentation*

$$\mathcal{R}(X) = k[x_0, x_1, y_0, \ldots, y_n, z_1, \ldots, z_d]/I$$

  where I is generated by $d + 1$ forms of bidegree dH_2.
- *$e = 1$, in which case X is a projective bundle over $\mathbb{P}^1$.*

In all other cases, we have

$$\overline{\mathrm{Eff}(X)} = \overline{\mathrm{Mov}(X)} = \mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0}(neH_2 - dH_1),$$

but $\mathrm{Eff}(X)$ is not closed. Hence X is not a Mori dream space.

In each case one can describe the birational structure of X fairly explicitly using the defining polynomial of X. To show that some of the hypersurfaces are not Mori dream spaces, a degeneration argument is used.

As a pleasant by-product we get simple analogues of many classical pathologies in birational geometry:

(i) A surface with $\mathrm{Nef}(X), \mathrm{Eff}(X), \mathrm{Mov}(X)$ all rational polyhedral, but $\mathrm{Eff}(X)$ not closed. This is true for surfaces of large bidegrees in $\mathbb{P}^1 \times \mathbb{P}^2$.

(ii) A rational surface with infinitely many (-1)-curves. The blow-up of $\mathbb{P}^2$ along the base-locus of two degree e curves embeds as a $(1, e)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$. For example, Nagata's example is a $(1, 3)$-hypersurface.

(iii) A non-ample line bundle with positive intersection numbers with any curve. If X is a hypersurface of bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, then the line bundle $L = \mathcal{O}(2H_2 - H_1)$ satisfies $L^2 = 0$ and $L \cdot C > 0$ for every curve $C \subset X$. Such examples where first constructed by Mumford using projective bundles over curves of genus ≥ 2.

(iv) A Calabi-Yau threefold with infinitely many $(-1, -1)$-curves. If X has tridegree $(2, 2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, then one can consider the two projections $p_{13}, p_{23} : X \to \mathbb{P}^1 \times \mathbb{P}^2$. These projections are generically $2 : 1$, and so X possesses two pseudoautomorphisms σ_1, σ_2 given by interchanging the two sheets of the double cover. It can be shown that σ_1, σ_2 generate an infinite subgroup of $Bir(X)$.

(v) A rationally connected variety which is not birational to a log Fano. Finally, using these results, I constructed a counterexample to a question of Cascini and Gongyo [1], which asks whether every rationally connected variety is birational to a log Fano variety. It turns out that many Fano fibrations over $\mathbb{P}^1$ are not birational to a log Fano, since they are so-called *birationally superrigid*, meaning that they have essentially only one Mori fiber structure.

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Extremal rays on hyperkähler manifolds and relations to Brill-Noether theory

Andreas Leopold Knutsen

(joint work with C. Ciliberto, and with M. Lelli-Chiesa and G. Mongardi)

A hyperkähler manifold is a simply-connected compact Kähler manifold X carrying an everywhere non-degenerate holomorphic 2-form, unique up to scale. In particular, it has even dimension. By the Beauville–Bogomolov decomposition theorem, hyperkähler manifolds form one of the three basic building blocks —

besides Calabi–Yau varieties with zero irregularity and abelian varieties — for smooth varieties with trivial first Chern class.

Apart from two examples of O'Grady in dimensions 6 and 10, the only examples known of such manifolds are Hilbert schemes $S^{[k]}$ of k points on $K3$ surfaces S and generalized Kummer varieties $K^{[k-1]}$ (which are the subsets of the Hilbert scheme of k points on abelian surfaces whose sum under the group operation is zero), together with their deformations. It is known by results of Huybrechts [7] and Boucksom [3] that *rational curves* determine the nef and ample cones, just like for $K3$s. In other words, extremal rays on hyperkähler manifolds are generated by limits of classes of rational curves. In the case of the examples above, rational curves in $S^{[k]}$ or $K^{[k-1]}$ correspond to curves on the surfaces with some partial normalization carrying a g^1_k. In the talk I gave an idea of the study of linear series on *normalizations* of curves on $K3$ surfaces [4] and abelian surfaces [8] and the role of vector bundle methods, as e.g. in [9]. In particular, I showed how these methods give restrictions on the existence of indecomposable rational curves in $S^{[k]}$ and $K^{[k]}$ compatible with recent results in [1, 2]. More precisely, assume that C is a *nodal* curve on a $K3$ surface of arithmetic genus $p = p_a(C)$ such that a *partial normalization* at δ of its nodes carries a g^1_k (and no normalization at fewer nodes does). Let R be the rational curve in $S^{[k]}$ determined by this g^1_k. Then the class of R in $N_1(S^{[k]}, \mathbb{Z})$ is given by

$$R \equiv C - (p - \delta + k - 1)\mathfrak{r}_k, \tag{1}$$

cf. [4], where the notation is explained as follows: We have $N_1(S^{[k]}, \mathbb{Z}) \cong N_1(S, \mathbb{Z}) \oplus \mathbb{Z}[\mathfrak{r}_k]$, where $\mathfrak{r}_k$ is the class of a fiber of the Hilbert-Chow morphism $\mu : S^{[k]} \to \mathrm{Sym}^k(S)$ over a generic point of the diagonal. The embedding of $N_1(S, \mathbb{Z})$ in $N_1(S^{[k]}, \mathbb{Z})$ is given by sending the class of a curve C to the class of the curve $\{p_1 \cup \cdots \cup p_{k-1} \cup x\}_{x \in C}$, with p_i some fixed points outside C. By abuse of notation we denote the curve class in $N_1(S^{[k]}, \mathbb{Z})$ still by C. By [4, 5], if the class of R is not decomposable into two effective nontrivial classes, then

$$\rho(p, \alpha, k\alpha + \delta) \geq 0, \quad \text{where } \alpha := \left\lfloor \frac{p - \delta}{2(k-1)} \right\rfloor. \tag{2}$$

This special Brill-Noether inequality implies the inequality

$$q(R) = 2(p-1) - \frac{(p - \delta + k - 1)^2}{2(k-1)} \geq -\frac{k+3}{2},$$

where q is the *Beauville-Bogomolov ($\mathbb{Q}$-valued) quadratic form* on homology, a result predicted by a conjecture of Hassett and Tschinkel [6] (and subsequently proved to hold for any effective 1-cycle R on any polarized variety deformation equivalent to $S^{[k]}$ with S a $K3$ surface by Bayer and Macrì in [1]).

On the other hand, by [4], the inequality (2) is also a *sufficient* condition for the existence of a δ-nodal curve in a *primitive* linear system $|C|$ on a general polarized $K3$ surface $(S, \mathcal{O}_S(C))$ with a normalization carrying a g^1_k, and such that no normalization at fewer nodes carries a g^1_k. This proves the existence of rational curves with classes as in (1). In many cases, as explained in [4], these generate extremal rays.

Similar results hold on generalized Kummer varieties, cf. [8].

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Higher rank interpolation and Bridgeland stability

Jack Huizenga

(joint work with Izzet Coskun)

Fix a zero-dimensional scheme $Z \subset \mathbb{P}^2$. It is a classical problem to determine the minimal integer $n_0 = n_0(Z)$ such that Z imposes independent conditions on curves of degree n_0. Equivalently, this amounts to proving a cohomology vanishing statement $H^1(I_Z(n_0)) = 0$. We then have that for all $n \geq n_0$, Z imposes independent conditions on curves of degree n.

The *higher-rank interpolation problem* asks for a description of the set of rational numbers $\mu \in \mathbb{Q}$ which are the slopes of vector bundles E such that $E \otimes I_Z$ has no cohomology. If E is a vector bundle such that $E \otimes I_Z$ has no cohomology, we say that E *satisfies interpolation for* Z. A first result says that the sets of rational numbers which can occur as solutions to the higher-rank interpolation problem are all infinite rays.

Proposition ([1])**.** Fix a zero-dimensional scheme $Z \subset \mathbb{P}^2$. There is a unique real number $\mu_0(Z) \in \mathbb{R}$ with the following properties.

(1) If $\mu > \mu_0(Z)$, then there is a vector bundle of slope μ with interpolation for Z.
(2) If $\mu < \mu_0(Z)$, then no vector bundle of slope μ has interpolation for Z.

Thus the higher-rank interpolation essentially amounts to computing the invariant $\mu_0(Z)$ for various schemes Z. Our results carry out this program for certain types of schemes.

Theorem ([1, 2, 3])**.** The invariant $\mu_0(Z)$ is explicitly computable if

(1) Z is a general collection of points,
(2) Z is a complete intersection scheme, or
(3) Z is a monomial scheme.

Motivation for the higher-rank interpolation problem comes from the birational geometry of Hilbert schemes of points and, more generally, moduli spaces of semistable sheaves. As a consequence of the previous theorem, we explicitly determine the cone of effective divisors on any Hilbert scheme of points in $\mathbb{P}^2$.

We will explain how Bridgeland stability provides a key insight into determining the solution to the higher-rank interpolation problem.

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Self-intersection numbers of curves on Hilbert Modular Surfaces

Sonia Samol

In 1976 Hirzebruch and Zagier ([4]) calculated the intersection numbers of Hirzebruch-Zagier curves T_N on Hilbert Modular surfaces. The aim of this text is to present the formula they calculated for the self-intersection numbers of these curves, following the book of van der Geer ([3]).

Let $p \equiv 1 \bmod 4$ be a prime, $K = \mathbb{Q}(\sqrt{p})$, $\mathcal{O}$ the ring of integers of K and $\mathfrak{a}$ an ideal in $\mathcal{O}$ with Norm($\mathfrak{a}$)= A.
With $\mathrm{SL}_2(\mathcal{O}, \mathfrak{a}) = \left\{ T \in \begin{pmatrix} \mathcal{O} & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O} \end{pmatrix}, \det T = 1 \right\}$ the quotient $X^{\mathfrak{a}} = \mathbb{H}^2 / SL_2(\mathcal{O}, \mathfrak{a})$ is a non-compact complex surface with finitely many singularities which can be compactified by adding the cusps to $X^{\mathfrak{a}}$ and resolving the singularites created. Then one gets the Hirzebruch compactification $\bar{X^{\mathfrak{a}}} = X^{\mathfrak{a}} \cup \bigcup_k S_k$ where the S_k are rational curves.

Definition: A skew-hermitian matrix

$$B = \begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{D} \end{pmatrix}$$

is called $\mathfrak{a}-$integral if a and b are integrals and $\lambda \in \mathfrak{a}^{-1}$, where λ' is the conjugate of λ.
If there is no integer $n > 1$ with $(\frac{a}{n}, \frac{b}{n}, \frac{\lambda}{n}) \in \mathbb{Z}^2 \times \mathfrak{a}^{-1}$, then B is called primitive.

Definition: For a primitive, $\mathfrak{a}-$integral, skew-hermitian matrix B the curve F_B is defined as the image of the set

$$\left\{(z_1,z_2)\in\mathbb{H}^2\cup\mathbb{P}^1(K):(z_2,1)B\begin{pmatrix}z_1\\1\end{pmatrix}=0\right\}$$

in $X^{\mathfrak{a}}$. With Norm($\mathfrak{a}$)$=A$ the curve F_N is defined as

$$F_N:=\bigcup_{\substack{B\text{ as above}\\ \det(B)=\frac{N}{A}}} F_B.$$

Franke ([2]) showed that for a prime discriminant p and $p^2\nmid N$ the curve F_N consists of only one component.
The Hirzebruch-Zagier curve T_N is the defined as

$$T_N=\bigcup_{\substack{t\geq 1\\ t^2|N}} F_{\frac{N}{t^2}},$$

so for N squarefree one gets $T_N=F_N$ irreducible. Furthermore, F_N is not empty if $\chi_p(NA)\neq 1$, where $\chi_p(n)=\left(\frac{n}{p}\right)$ is the Legendre symbol, and compact if N is not the norm of an ideal in the genus of $\mathfrak{a}$.
For the self-intersection number of the curves T_N we get the following formula.

Theorem (Hirzebruch-Zagier [4]**):**

$$T_N^2=\frac{1}{2}\sum_{n|N} n\left(H_p\left(\frac{N^2}{n^2}\right)+I_p\left(\frac{N^2}{n^2}\right)\right)\left(\chi_p(n)+\chi_p\left(\frac{NA}{n}\right)\right),$$

with

$$H_p(n)=\sum_{\substack{x\in\mathbb{Z}\\ x^2\leq 4n\\ x^2\equiv 4n \bmod p}} H\left(\frac{4n-x^2}{p}\right),$$

$$H(n)=\begin{cases}-\frac{1}{12} & \text{if } n=0\\ \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right) & \text{else}\end{cases},$$

$$h'(\Delta)=\begin{cases}\frac{1}{3} & \text{if } \Delta=-3\\ \frac{1}{2} & \text{if } \Delta=-4\\ h(\Delta) & \text{if } \Delta\equiv 0 \text{ or } 1 \bmod 4, \Delta\leq -4\end{cases},$$

where $h(\Delta)$ is the class number of positive definite primitive binary integral quadratic forms with discriminant Δ and

$$I_p(n)=\frac{1}{\sqrt{p}}\sum_{\substack{\lambda\in\mathcal{O}\\ \lambda>0,\lambda'>0\\ \lambda\lambda'=n}}\min(\lambda,\lambda').$$

The aim is now to find a bound $b(X^{\mathfrak{a}})$ such that $T_N^2 \geq b(X^{\mathfrak{a}})$ for all squarefree N. In [1] $T_N^2 \geq -6c_2(\bar{X^{\mathfrak{a}}})$ was proven. We have $c_2(\bar{X^{\mathfrak{a}}})$ =vol$(X^{\mathfrak{a}})$+$l(X^{\mathfrak{a}})$, where $l(X^{\mathfrak{a}})$ comes from the cusps, and vol$(X^{\mathfrak{a}}) = [SL_2(\mathcal{O}) : SL_2(\mathcal{O}, \mathfrak{a})]2\zeta_K(-1)$ with $\zeta_K(-1)$ the Dedekind zeta-function. For K a real quadratic field with discriminant p

$$\zeta_K(-1) = \frac{1}{60}\sum_{x\in\mathbb{Z}} \sigma_1\left(\frac{p-x^2}{4}\right),$$

where $\sigma_1(x) = 0$ if $x \notin \mathbb{Z}_{\geq 1}$ and $\sigma_1(x) = \sum_{d|x} d$ if $x \in \mathbb{Z}_{\geq 1}$.

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Convergence of normalised log-eigenvalues of Hermitian metrics on big line bundles

Catriona Maclean

(joint work with Huayi Chen)

This talk is a report on the contents of the preprint [4].

Suppose that X is a projective smooth variety over a field K which is either the complex numbers or a non-archimedean field, and let L be a big line bundle on X. We assume that this line bundle is equipped with two continuous Hermitian metrics, ϕ and ψ and that X is equipped with a volume form μ. Integrating against μ we then obtain associated Hermitian metrics on each of the spaces $H^0(nL) = V_n$, which we will denote by ϕ_n and ψ_n. We will be concerned with the following question : how does the relative geometry of ϕ_n and ψ_n behave as $n \to \infty$?

More precisely, there is for each n a basis for V_n, $(e_1, \ldots, e_{d_n})$, which is orthonormal for ϕ_n and orthogonal for ψ_n : if we denote the number $\psi_n(e_i, e_i)$ by λ_i for all i then it is fairly easy to see that

$$\log(\max\lambda_i) \sim_{n\to\infty} n\max(\log(\psi/\phi))$$

which suggests that the normalisation $\mu_i = \frac{\log(\lambda_i)}{n}$ may have more interesting convergence properties.

More precisely, we consider the following question : let now Z_n be a random variable which is equal to μ_i with probabilty $\frac{1}{d_n}$. Does the sequence of random variables Z_n converge in probability ? Note that this question is well-defined for any multiplicative linear series $V_\bullet$, where $V_n \subset H^0(nL)$ for all $n \geq 0$, and it is in this form that we have studied this question.

In the case where V_n is the complete linear series $H^0(nL)$ and the base field is the complex numbers, Boucksom and Berman have proved in [2] that the sequence $\mathbb{E}(Z_n)$ converges (and have established an important formula for the limit in terms of equilibrium Monge-Ampere metrics.) Moreover, when mooreover L is ample and ϕ and ψ are Kähler metrics, Berndtsson [1] has proved the convergence (and has, again, given an important formula for the limit distribution.) We prove the following theorem :

Theorem
With X, L, ϕ, ψ, V_n and Z_n as above, then provided V_n is of Lazarsfeld-Mustata ample type, the sequence of random variables Z_n converges in probability.

One of the main innovations in the proof is the use of Newton-Okounkov bodies in this context. The Newton-Okounkov Δ_{V_n} body is a compact convex body in $\mathbb{R}^d$ associated to a multiplicative linear series $V_\bullet \subset H^0(nL)$: see [5] and [6] for more details. It is sometimes possible to rewrite functions of the linear series V_n as a sum over the points of $\Delta(V_n) \cap (\mathbb{Z}/n)^d$, which is a convenient context for proving convergence theorems (see [3], [7] for more details.)

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Intersection of special cycles on orthogonal Shimura varieties

Fritz Hörmann

This was mainly a historical talk about Siegel-Weil theory which started with the question of determining representation numbers $R(T)$ of a positive-definite integral quadratic form Q. Siegel [6, 7] found the correct generalization of this question to indefinite forms: The representation numbers have to be replaced by volumes of certain submanifolds $Z(T)$ of arithmetic quotients of the Grassmannian associated with the given indefinite quadratic form. As in the former case, those are indexed by symmetric integral $g \times g$ matrices T. In certain cases, these locally symmetric varieties are, in fact, algebraic (Shimura varieties of orthogonal type) and the volumes of the $Z(T)$ can be interpreted as their degree in the sense of algebraic geometry. This was interesting for this workshop because, in dimension 2, these Shimura varieties give rise to a lot of examples of surfaces (Hilbert modular or Shimura surfaces) with an interesting class of distinguished curves on them. The main classical theorems state that the representation numbers $R(T)$, resp. the volumes of the $Z(T)$, are encoded as Fourier coefficients of modular forms (theta functions) and are, at least in an average over the genus, Eisenstein series, whose Fourier coefficients can be given in a rather explicit way in terms of Euler products. The theory of Siegel was considerably advanced by Weil [8]: He realized that the above results are basically a reformulation of the fact that the Tamagawa number of any orthogonal group is 2. Kudla and Millson in the 80's and 90's refined this theory [4] and considered the cohomology classes of the $Z(T)$ in the Betti cohomology. The result is that, again, the generating series

$$\Theta_B(\tau) = \sum_T [Z(T)]_B \cup e^{g-\mathrm{rk}(T)} \exp(2\pi i \mathrm{tr}(T\tau))$$

is a Siegel modular form of genus g (with values in a Betti cohomology group of the locally symmetric variety). Here e is a certain (Euler) cohomology class. Furthermore, the formula

$$\Theta_B(\tau_1) \cup \Theta_B(\tau_2) = \Theta_B(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}) \tag{1}$$

for the cup product of two generating series was proven. Together with the previous result, this gives an explicit formula for the cup product of two classes $[Z(T_1)]_B$ and $[Z(T_2)]_B$. In the Shimura variety case, one might consider the generating series $\Theta_{CH}(\tau)$ with values in the corresponding Chow group of (a compactification of) this variety instead. Kudla conjectures (cf. [5]):

1. $\Theta_{CH}(\tau)$ is a modular form itself.
2. Formula (1) holds for the intersection product.

Conjecture 1 was proven for $g = 1$ by Borcherds [1] and recently for $g = 2$ by Bruinier [2]. Conjecture 2 is open. However, in the end of the talk, I sketched a proof of Conjecture 2 for surfaces. In particular, combined with the Siegel-Weil theory explained above, this gives a nice proof and interpretation of formulas for the intersection number of two Hirzebruch-Zagier divisors on Hilbert modular or Shimura surfaces, respectively. I also reported briefly about my own work [3] about Kudla's more recent program on the Arakelov analogue of this theory.

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The effect of fattening in dimension 3

Tomasz Szemberg
(joint work with Thomas Bauer)

Let Z be an arbitrary subscheme in $\mathbb{P}^n$ and let I_Z be its homogeneous ideal. A basic question in polynomial interpolation is to determine the *initial degree* of I_Z defined as

$$\alpha(I_Z) = \min\{\, d : \ (I_Z)_d \neq 0 \,\}.$$

This is a difficult task in general. For example the Nagata Conjecture predicts that if Z is a *fat points scheme* $Z = mP_1 + \ldots + mP_s$ of $s \geq 10$ general points in $\mathbb{P}^2$, then

$$\alpha(I_Z) > m \cdot \sqrt{s}.$$

This conjecture remains widely open for over 55 years. See [3] for recent generalizations to general configurations of linear subspaces in $\mathbb{P}^n$.

Fat points schemes have been studied extensively for a long time. Recently, Bocci and Chiantini in [2] proposed a new approach. Assume that Z is a reduced subscheme of $\mathbb{P}^n$ supported on a finite number of points. Their program is to study the *initial sequence*

$$\alpha(I_Z),\ \alpha(I_Z^{(2)}),\ \alpha(I_Z^{(3)}), \ldots$$

and conclude out of its growing pattern geometrical information about Z, here $J^{(m)}$ denotes the m–th *symbolic power* of an ideal J.

In a joint paper with Thomas Bauer [1] we state a conjecture relating subschemes Z with the initial sequence relatively minimal growth to star configurations, see [4] for a very enjoyable introduction to star configurations.

Conjecture 1. *Let Z be a finite set of points in projective space $\mathbb{P}^n$ and let $I = I_Z$ be the radical ideal defining Z. If*

$$d := \alpha(I^{(n)}) = \alpha(I) + n - 1\,, \tag{1}$$

then either

$\alpha(I) = 1$, i.e., Z is contained in a single hyperplane H in $\mathbb{P}^n$

or

Z consists of all intersection points (i.e., points where n hyperplanes meet) of a general configuration of d hyperplanes in $\mathbb{P}^n$, i.e., Z is a star configuration. *For any polynomial in $I^{(n)}$ of degree d, the corresponding hypersurface decomposes into d such hyperplanes.*

This Conjecture has been proved for $n = 2$ by Bocci and Chiantini in [2, Theorem 1.1]. The content of my talk was to present its proof in dimension $n = 3$.

Theorem (Bauer, Szemberg)**.** The above Conjecture holds in dimension 3.

There are several interesting possible generalizations of Conjecture 1 to star configurations of higher dimensional linear subspaces. We refer to [1] for details.

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