Mathematisches Forschungsinstitut Oberwolfach

Report No. 17/2014

DOI: 10.4171/OWR/2014/17

Real Algebraic Geometry With A View Toward Systems Control and Free Positivity

Organised by
Didier Henrion, Toulouse
Salma Kuhlmann, Konstanz
Victor Vinnikov, Beer-Sheva

6 April – 12 April 2014

Abstract. New interactions between real algebraic geometry, convex optimization and free non-commutative geometry have recently emerged, and have been the subject of numerous international meetings. The aim of the workshop was to bring together experts, as well as young researchers, to investigate current key questions at the interface of these fields, and to explore emerging interdisciplinary applications.

Mathematics Subject Classification (2010): 12D15, 13P10, 13P15, 14M12, 14P05, 14P10, 14Q05, 14Q10, 47LXX, 49M29, 49N35, 52A20, 52A27, 52A41, 90C22, 90C26, 93B25, 93B40, 93B51.

Introduction by the Organisers

Hilbert, in the 1880s and 1890s, was the first to study the connection between positive polynomials and sums of squares. He considered this topic so important that he included a central open question in his list of mathematical problems. Artin–Schreier's solution of Hilbert's 17th problem in the 1920s marked the beginning of real algebra. The subject continued to develop, with the fundamental Positivstellensatz (a description of positive polynomials on a basic closed semialgebraic set) discovered by Krivine and Stengle in the 1960s and the 1970s, and it was reinvigorated in the 1990s through new connections with functional analysis and moment problems, resulting in new denominator free Positivstellensätze of Schmüdgen, Putinar and Jacobi–Prestel.

The discovery of efficient algorithms, combined with the growing power of electronic computation in general, gave results on positive polynomials, sums of squares, and moment problems a central role in polynomial optimization. Conversely, problems in optimization have been shaping very recent research directions in real algebra. The implementation of semidefinite programming has generated difficult questions in convex real algebraic geometry. Relations to control theory and engineering initiated questions on positivity in a non-commutative setting. Analogues from the traditional commutative theory served successfully as excellent guiding principles in the non-commutative case, leading to startling recent results.

The aim of the workshop was to bring together researchers working in

(**A**) real algebraic geometry (positive polynomials, sums of squares, and moment problems),
(**B**) linear matrix inequalities (LMIs), systems and control, optimization, and
(**C**) non-commutative and free positivity.

While some of the participants were well aware of the connections between the different areas, others were more or less of newcomers to the interdisciplinary scene. There was also a large number of young researchers, both graduate students and juniour faculty members.

To create a synergy between these different groups the organizers have asked 11 participants to give talks of 50 minutes that would present a topic to a mixed audience of non-specialists and specialists. These survey-expository talks were scheduled during the morning sessions, with regular research talks of 40 minutes scheduled in the afternoons (except for Friday, when there were no survey-expository talks). The survey-expository talks were roughly divided according to the three main areas mentioned above,

(**A**) M. Schweighofer, M. Marshall, Y. Yomdin, B. Reznick, and H. Woerdeman,
(**B**) J.-B. Lasserre, P. Brändén, and M. Safey El Din,
(**C**) K. Schmüdgen, J. W. Helton, and J. Ball,

though of course some of them crossed the boundaries.

Here is a summary of some of the main topics discussed at the workshop.

Positive polynomials and sums of squares

M. Schweighofer described representations of positive and strictly positive polynomials as weighted sums of squares, including the matrix valued case and with an emphasis on degree bounds (both of which are of particular importance for applications). A variety of far reaching new results on degree bounds were the main topic of the talk of G. Blekherman. B. Reznick discussed *explicit constructions* of positive polynomials that are not sums of squares, from Hilbert's original papers through the first explicit examples of Motzkin and Robinson till the most recent results. T. Netzer showed an interesting new application of real algebra techniques to a class of graph-theoretic problems. F. Vallentin presented some very explicit results on invariant sums of squares. The talk of J. Ball was primarily devoted to the free non-commutative setting, but he discussed the commutative case as a motivation, with certain weighted hermitian sums of squares decompositions that originated with the work of J. Agler in the early 1990s and played since a central role in operator theory and function theory on the unit polydisc in $\mathbb{C}^d$. These "Agler decompositions" were a central theme in the talks of H. Woerdeman (were they were related to both the author's earlier work on the trigonometric moment problem and to determinantal representations of polynomials), and of M. Dritschel.

Moment problems

M. Marshall gave a state-of-the-art survey, including a new approach based on localization, for the (full) moment problem on a basic closed semialgebraic set; he discussed both the *existence* and the *uniqueness* of the representing measure. M. Infusino described the moment problem in an infinite dimensional setting; the results were hitherto unknown to most participants, and led to a lively discussion. Y. Yomdin presented a broad subject of moment vanishing problems and related topics that were virtually unknown in the real algebra community. H. Woerdeman described his work on the truncated two-dimensional trigonometric moment problem for a class of absolutely continuous measures.

LMIs and hyperbolic polynomials

LMI is the name traditionally used in systems control to refer to spectrahedra which are feasibility sets of semidefinite programming problems. Spectrahedral cones are always hyperbolic cones associated to some hyperbolic polynomial, and one of the main conjectures in the area (the *generalized Lax conjecture*) is that these two classes of cones actually coincide; this is equivalent to the existence of certain determinantal representation for hyperbolic polynomials. P. Brändén gave a survey of hyperbolic polynomials and spectrahedral cones, icluding the recent application to the proof of the Kadisonn–Singer conjecture in operator algebras (by Marcus, Spielman, and Srivastava). C. Hanselka presented a new proof (using quadratic forms rather than algebraic geometry) of the fact that any homogeneous hyperbolic polynomial in three variables admits a determinantal representation. E. Shamovich introduced a natural generalization of the notion of hyperbolicity for higher codimensional subvariaties of the projective space. M. Kummer gave a startling new result on determinantal representations, obtained using real algebra and the full power of the Positivstellensatz, that goes some of the way towards establishing the generalized Lax conjecture.

Systems control and optimization

In his talk, J.-B. Lasserre surveyed the use of algebraic certificates of positivity and the resulting hierarchies of semidefinite programming problems in non-convex optimization. A. A. Ahmadi described the use of linear matrix inequalities (LMI) to approximate the joint spectral radius, a quantity ruling the decay or growth rate of the norm of the product of given matrices. J. W. Helton argued that the structure of dimension-free LMI problems, ubiquitous in linear robust control problems, can be inferred from properties of non-commutative polynomials. J. Ball complemented this point of view, and he explained how results from non-commutative algebra and operator theory can be exploited in linear robust control. LMI problems coming from sum-of-squares decomposition of multivariate polynomials are now common in systems theory, especially in signal processing, and H. Woerdermann explored such connections in the bivariate trigonometric case for filtering problems.

Non-commutative and free positivity

K. Schmüdgen gave a general overview of the setting, methods, and results of non-commutative real algebraic geometry. J. Cimpric presented new non-commutative Postivsellensätze for matrix polynomials. The talk of A. Thom discussed the case of group rings with applications and relations to a variety of problems in group theory and operator algebras. J. W. Helton gave an overview of the free real algebraic geometry, with a special emphasis on the motivation coming from LMIs appearing in systems control. Free polynomial optimization was discussed by S. Burgdorf. The talk of J. Ball described function theory, operator theory, and system theory, related to the free non-commutative versions of such classical domains as the unit polydisc or the unit ball in $\mathbb{C}^d$, with a central role being played by certain non-commutative weighted hermitian sums of squares decompositions.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting J. William Helton and Bruce Reznick in the "Simons Visiting Professors" program at the MFO.

Workshop: Real Algebraic Geometry With A View Toward Systems Control and Free Positivity

Table of Contents

Abstracts

On LP and SDP Certificates of Positivity

Jean-Bernard Lasserre

In many problems in control, optimal and robust control, one has to solve global optimization problems of the form: $\mathbf{P}: \; f^* = \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$, or, equivalently, $f^* = \max\{\lambda : f - \lambda \geq 0 \text{ on } \mathbf{K}\}$, where f is a polynomial (or even a semi-algebraic function) and $\mathbf{K}$ is a basic semi-algebraic set. One may even need solve the "robust" version $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K};\; h(\mathbf{x}, \mathbf{u}) \geq 0, \forall \mathbf{u} \in \mathbf{U}\}$ where $\mathbf{U}$ is a set of parameters. For instance, some static output feedback problems can be cast as polynomial optimization problems whose feasible set $\mathbf{K}$ is defined by a polynomial matrix inequality (PMI). And robust stability regions of linear systems can be modeled as parametrized polynomial matrix inequalities (PMIs) where parameters $\mathbf{u}$ account for uncertainties and (decision) variables $\mathbf{x}$ are the controller coefficients.

Therefore, to solve such problems one needs *tractable characterizations* of polynomials (and even semi-algebraic functions) which are nonnegative on a set, a topic of independent interest and of primary importance because it also has implications in many other areas.

We will review two kinds of *tractable* characterizations of polynomials which are nonnegative on a basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$.

• The first type of characterization is when knowledge on $\mathbf{K}$ is through its defining polynomials, i.e., $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0,\; j = 1, \ldots, m\}$, in which case some powerful certificates of positivity can be stated in terms of:

- a weighted linear combination of the (g_j) whose weights are some sums of squares (SOS), or
- positive linear combinations of powers of the defining polynomials (g_j) (with positive scalars as weights).

For instance, depending on which type of positivity certificate is chosen, this allows to define a hierarchy of (semidefinite) SDP-relaxations or LP-relaxations of problem $\mathbf{P}$. In both hierarchies each optimization problem is convex (an SDP or an LP) and each hierarchy provides an associated monotone sequence of *lower bounds* converging to f^*. In fact, finite convergence of the SOS hierarchy is generic and one may also extract global minimizers.

• The second type of characterization is when knowledge on $\mathbf{K}$ is through *moments* of a measure whose support is $\mathbf{K}$. In this case, checking whether a polynomial is nonnegative on $\mathbf{K}$ reduces to solving a sequence of *generalized eigenvalue* problems associated with a countable (nested) family of real symmetric matrices of increasing size. When applied to $\mathbf{P}$ this results in a monotone sequence of *upper bounds* converging to the global minimum f^*, which complements the previous sequence of upper bounds.

These two (dual) characterizations provide convex *inner* (resp. *outer*) approximations (by spectrahedra) of the convex cone of polynomials nonnegative on $\mathbf{K}$.

Positive polynomials, sums of squares, degree bounds and semidefinite representations

Markus Schweighofer
(joint work with Christoph Hanselka)

All students of mathematics should know that every polynomial in one variable nonnegative on the real line is a sum of two squares of polynomials:

Theorem 1. *Suppose $f \in \mathbb{R}[X]$ and $f \geq 0$ on $\mathbb{R}$. Then there exist $p, q \in \mathbb{R}[X]$ such that $f = p^2 + q^2$.*

Proof. By the *fundamental theorem of algebra*, f is a product of linear polynomials in $\mathbb{C}[X]$ corresponding to the multiset of complex roots of f (i.e., the roots counted with multiplicity). Since f is nonnegative, the real factors appear with an even multiplicity. Since f is real, the non-real factors appear in complex-conjugated pairs. Any division of the multiset into two complex-conjugated parts, now leads to a complex polynomial $p + \mathring{\imath} q$ ($p, q \in \mathbb{R}[X]$) such that

$$f = (p - \mathring{\imath} q)(p + \mathring{\imath} q) = p^2 + q^2$$

where $\mathring{\imath} \in \mathbb{C}$ denotes the imaginary unit. □

Note that this theorem can be reformulated in the following more systematic style (since a complex polynomial taking real values on the line is automatically real):

For all $f \in \mathbb{C}[X]$ with $f \geq 0$ on $\mathbb{R}$, there exists $p \in \mathbb{C}[X]$ such that $f = p^* p$.

Here $\mathring{\imath}^* = -\mathring{\imath}$ and $X^* = X$: We denote by $*$ the complex conjugation and extend it on an involution on polynomial rings by considering the variables to be formally self-adjoint. We also have obvious *degree bounds* in the above: If $d \in \mathbb{N}$ such that $\deg f \leq 2d$, then $\deg p \leq d$ follows immediately.

The following non-trivial generalization of Theorem 1 to *matrix polynomials positive semidefinite on the real line* was folklore at least since the 1960s (see for example [1]). Here $*$ acts as before but in addition transposes the matrices.

Theorem 2. *Suppose $F \in \mathbb{C}[X]^{s \times s}$ and $F \succeq 0$ on $\mathbb{R}$. Then there exists $P \in \mathbb{C}[X]$ such that $F = P^* P$.*

Taking the trace on both sides of the equation $F = P^* P$ yields $\sum_{i=1}^{s} F_{ii} = \sum_{i,j=1}^{s} P_{ij}^* P_{ij}$. Using this, it is easy exercise to show that we get the same kind of automatic degree bounds as before. The most elementary proof of Theorem 1 has been given (for the case $F \in \mathbb{R}[X]^{s \times s}$) by Choi, Lam and Reznick [2, Section 7]. The rough idea of their proof is by completing the square successively with respect to the different variables one by each. To compensate for the impossibility of division in the polynomial ring, during this process multipliers have to be introduced which can be neutralized using the fundamental theorem of algebra. In [2], this neutralization involves very tricky computations. In the first part of the talk, we present a new and very "clean" way to do this neutralization using basic linear algebra instead of computations. This yields arguably the easiest known proof of Theorem 2.

With a little more work, this new argument also allows to show that the determinant of the factors in the factorization can be described with the maximal possible freedom (compare to the proof of Theorem 1):

Theorem 3 (Hanselka & S., Ball & Rodman). *Suppose* $F \in \mathbb{C}[X]^{s\times s}$ *and* $g \in \mathbb{C}[X]$ *such that* $F \succeq 0$ *on* $\mathbb{R}$ *and* $\det F = g^* g$. *Then there exists* $P \in \mathbb{C}^{s\times s}$ *such that* $F = P^* P$ *and* $\det P = g$.

Theorem 3 was already known in the case where g and g^* have no common zero [1, Theorem 3]. Our question whether it is already known in the above stated general form, reached Joe Ball and Leiba Rodman who negated it and at the same gave an alternative unpublished proof which is however based on a considerable amount of the theory of matrix polynomials [3]. Our investigations were initially motivated by the fact that the algorithm described in [4] to compute the decomposition in Theorem 2 seems to use (at least weaker versions of) Theorem 3 [5] even though no version of this theorem is stated let alone proved in [4] (note also that the authors of [4] claim to give a system-theoretic proof of Theorem 2 [4, page 5660, last paragraph] which does not seem to be the case since they use the equation $Q^*(\lambda_i^*)v_i = 0$ in [4, page 5665] without any proof but this equation is almost equivalent to the existence of the decomposition).

The second part of the talk was a survey on modern versions of Theorems 1 and 2 which we state in the following synthesized way:

Theorem 4 (Schmüdgen 1991, Putinar 1993, Hol & Scherer 2005). *Let* $m, n \in \mathbb{N}$, $g_1, \dots, g_m \in A := \mathbb{R}[X_1, \dots, X_n]$ *and set* $g_0 := 1 \in A$. *Consider the basic closed semialgebraic set*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\},$$

and the matricial quadratic modules

$$T^{(s)} := \left\{ \sum_{i=0}^{m} g_i \sum_j P_{ij}^* P_{ij} \mid P_{ij} \in A^{s\times s} \right\} \qquad (s \in \mathbb{N}),$$

and the ordinary quadratic module $T := T^{(1)}$. *The following are equivalent:*

(a) *There exist* $t \in \mathbb{N}$ *and* $h_1, \dots, h_t \in A$ *such that* $\prod_{i\in I} h_i \in T$ *for all* $I \subseteq \{1, \dots, t\}$ *and* $\{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \dots, h_t(x) \geq 0\}$ *(and therefore also its subset* S*) is compact.*
(b) *There exists* $h \in T$ *such that* $\{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ *is compact.*
(c) *There exists* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^n X_i^2 \in T$.
(d) *For all* $p \in A$ *there is an* $N \in \mathbb{N}$ *such that* $N + p \in T$.
(e) S *is compact and for every* $f \in A$ *with* $f > 0$ *on* S, *we have* $f \in T$.

(f) *S is compact and for all $s \in \mathbb{N}$ and all $F \in \mathbb{R}[X]^{s\times}$ with $F \succ 0$ on S, we have $f \in T^{(s)}$.*

The backward implications are obvious whereas the forward implications are not: We don't know if there is an easy direct proof of (a)⇒(b). The hardest implications are (a)⇒(c) (or even (b)⇒(c)): This is the essence of Schmüdgen's celebrated 1991 theorem [6] whose first algebraic proof was found by Wörmann [7]. All proofs of Schmüdgen's Theorem use Krivine's (classical) Positivstellensatz from real algebraic geometry [8] (reproved by Stengle [9] and Prestel [10]). The implication (c)⇒(d) is just a few lines of tricky identities. Implication (d)⇒(e) was just a by-product in the article [11] by Putinar but got famous due to its numerous applications. The easiest known proof today stems from Marshall [14]. Finally, (d)⇒(f) is a theorem due to Hol & Scherer [12]. See [13, 14, 15].

The advantage of modern versions of Theorems 1 and 2 such as Theorem 4 is that they work in severable variables instead of only one and that they allow to consider positivity on arbitrary basic closed semialgebraic sets.

The big drawback of the modern versions is that there no obvious or "clean" degree bounds. The degree bounds instead depend on the geometry [13, 17, 18, 19, 20]. Indeed, the validity of these theorems even strongly relies on the possibility of huge degree cancellations. Related to this, strict positivity is in general needed although the certificate is only for nonnegativity.

An ingenious idea of Helton and Nie however surmounts partially these difficulties in cases where S is stricly convex and the polynomial to represent is of degree one [20, 21, 22, 23]. In Theorem 4, instead of applying (e) to the degree one polynomial, they apply (f) to the Hessians of certain polynomials defining the set S locally and write the degree one polynomial as a double integral over an expression involving this Hessian. This leads to strong theorems about semidefinite representability of large classes of convex semialgebraic sets. Indeed, it is an open question if all convex semialgebraic sets are projections of spectrahedra (i.e., solution sets of linear matrix inequalities). Even if one is interested in sums of squares representations of polynomials rather than matrix polynomials, it can be of great help to study the case of matrix polynomials.

References

[1] V.A. Jakubovič: Factorization of symmetric matrix polynomials, Dokl. Akad. Nauk SSSR 194, 1970, 532–535

[2] M.D. Choi, T.Y. Lam, B. Reznick: Real zeros of positive semidefinite forms I, Math. Z. 171 (1980), no. 1, 1–26

[3] I. Gohberg, P. Lancaster, L. Rodman: Matrix polynomials, Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1982

[4] E. Aylward, S. Itani, P.A. Parrilo: Explicit SOS decompositions of univariate polynomial matrices and the Kalman-Yakubovich-Popov Lemma, Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, 2007

[5] B. Mangold: Quadratsummen von Matrixpolynomen in einer Variablen, Bachelor's thesis, November 2013, Universität Konstanz

[6] K. Schmüdgen: The K-moment problem for compact semi-algebraic sets, Math. Ann. 289 (1991), no. 2, 203–206
[7] R. Berr, T. Wörmann: Positive polynomials on compact sets, Manuscripta Math. 104 (2001), no. 2, 135–143
[8] J.-L. Krivine: Anneaux préordonnés, J. Analyse Math. 12 1964 307–326
[9] G. Stengle: A Nullstellensatz and a Positivstellensatz in semialgebraic geometry, Math. Ann. 207 (1974), 87–97
[10] A. Prestel: Lectures on formally real fields, Rio de Janeiro: IMPA (1975), reprinted in: Lecture Notes in Mathematics 1093, Springer-Verlag, Berlin, 1984
[11] M. Putinar: Positive polynomials on compact semi-algebraic sets, Indiana Univ. Math. J. 42 (1993), no. 3, 969–984
[12] C.W.J. Hol, C.W. Scherer: Matrix sum-of-squares relaxations for robust semi-definite programs, Math. Program. 107 (2006), no. 1-2, Ser. B, 189–211
[13] A. Prestel, C.N. Delzell: Positive polynomials, From Hilbert's 17th problem to real algebra, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001
[14] M. Marshall: Positive polynomials and sums of squares, Mathematical Surveys and Monographs, 146, American Mathematical Society, Providence, RI, 2008
[15] M. Laurent; Sums of squares, moment matrices and optimization over polynomials, Emerging applications of algebraic geometry, 157270, IMA Vol. Math. Appl., 149, Springer, New York, 2009, updated versions available at `http://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf`
[16] A. Prestel: Bounds for representations of polynomials positive on compact semi-algebraic sets, Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999), 253260, Fields Inst. Commun., 32, Amer. Math. Soc., Providence, RI, 2002
[17] M. Schweighofer: On the complexity of Schmüdgen's Positivstellensatz, J. Complexity 20 (2004), no. 4, 529543
[18] J. Nie, M. Schweighofer: On the complexity of Putinar's Positivstellensatz, J. Complexity 23 (2007), no. 1, 135150
[19] R. Hess: Die Sätze von Putinar und Schmüdgen für Matrixpolynome mit Gradschranken, Diplomarbeit, January 2013, Universität Konstanz
[20] J.W. Helton, J. Nie: Semidefinite representation of convex sets, Math. Program. 122 (2010), no. 1, Ser. A, 21–64
[21] J.W. Helton, J. Nie: Sufficient and necessary conditions for semidefinite representability of convex hulls and sets, SIAM J. Optim. 20 (2009), no. 2, 759–791
[22] R. Sinn: Spectrahedra and a relaxation of convex semialgebraic sets, Diplomarbeit, April 2010, Universität Konstanz
[23] T.-L. Kriel: Semidefinite representability of closed semi-algebraic sets, Bachelor's thesis, September 2012, Universität Konstanz

Noncommutative Positivity and Positivstellensätze

Konrad Schmüdgen

This talk is about various versions of Positivstellensätze for general noncommutative $*$-algebras. Let A be a unital $*$-algebra with involution $a \to a^*$. There are various methods to define positive elements of A. An important and powerful method is based on $*$-representations. A $*$-representation of A is a homomorphism π of A into the algebra of linear operators on a unitary space $(V_\pi, \langle\cdot,\cdot\rangle)$ such that

$$\langle \pi(a)\varphi, \psi\rangle = \langle \varphi, \pi(a^*)\psi\rangle, \quad \varphi, \psi \in V_\pi,\ a \in A.$$

If R is a family of $*$-representations of A, then the set

$$A(R)_+ = \{a = a^* \in A : \langle \pi(a)\varphi, \varphi\rangle \geq 0 \quad \text{for } \varphi \in V_\pi, \pi \in R\}$$

is a quadratic module. Noncommutative Positivstellensätze express elements of such quadratic modules $A(R)_+$ in terms of (weighted) sums of squares. Here elements of the form x^*x with $x \in A$ are called (hermitean) squares in A. A number of different types of such Positivstellensätze are reviewed in the talk. Among them are Positivstellensätze for algebras of free type without denominators (as proved by W. Helton, S. McCullough and their coworkers) and Positivstellensätze for Weyl algebras and enveloping algebras (as proved by the author). Matrices over algebras lead to unexpected new phenomena and new types of Positivstellensätze. This was elaborated in joint work with Y. Savchuk [1].

In the second part of the talk a generalization of Marshall's Positivstellensatz [2] to noncommutative $*$-algebras is presented [3, Theorem 4]. This result leads to a unified approach to a number of strict Positivstellensätze for Weyl algebras, enveloping algebras and quantum algebras obtained by the author. The main technical tool for these results are appropriate bounded $*$-algebras of fractions.

Finally, a number of open problems are stated (mainly taken from [1, Section 11]. One important open problem concerns the Weyl algebra: Is each hermitean element of the Weyl algebra which is represented by a positive symmetric operator in the Schrödinger representation a sum of squares with denominator?

References

[1] Savchuk, Y. and Schmüdgen, K., Positivstellensätze for algebras of matrices, Linear Algebra Appl. **436**(2012), 758–788.

[2] Marshall, M., Extending the Archimedean Posotivstellensatz to the non-comapct case, Can. Math. Bull. **44**(2001), 223–230.

[3] Schmüdgen, K., Algebras of fractions and strict Positivstellensätze for $*$-algebras, J. Reine Angew. Math. **647**(2010),57-86.

Computation of the Joint Spectral Radius with Optimization Techniques

Amir Ali Ahmadi

1. The JSR

Given a finite set of real $n \times n$ matrices $\mathcal{A} := \{A_1, \ldots, A_m\}$, their *joint spectral radius* $\rho(\mathcal{A})$ is defined as

$$\rho(\mathcal{A}) = \lim_{k\to\infty} \max_{\sigma \in \{1,\ldots,m\}^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}, \tag{1}$$

where the quantity $\rho(\mathcal{A})$ is independent of the norm used in (1). The joint spectral radius (JSR) is a natural generalization of the spectral radius of a single matrix and it characterizes the maximal growth rate that can be obtained by taking products of arbitrary length, of all possible permutations of $A_1, \ldots, A_m$. This concept was introduced by Rota and Strang [7] in the early 60s and has since emerged in many areas of application such as stability and control of switched linear systems, computation of the capacity of codes, continuity of wavelet functions, convergence of consensus algorithms, and many others; see [5] and references therein. In particular, the switched linear dynamical system

$$x_{k+1} = A_i x_k, \quad i \in \{1, \ldots, m\}, \tag{2}$$

is *asymptotically stable under arbitrary switching* (i.e., attracts all initial conditions $x_0 \in \mathbb{R}^n$ to the origin for all possible switching sequences $\{A_i\}$) if and only if $\rho(\mathcal{A}) < 1$.

In almost all application areas mentioned above, the question of interest is to compute the JSR given the input matrices or simply decide whether it is less than one. In recent years, several algorithms have emerged to compute (or approximate) the JSR using optimization methods. These techniques often involve a search for a Lyapunov function of some algebraic structure that certifies stability of the dynamical system in (2).

2. Our recent work on the subject

Semidefinite programming for upper bounding the JSR (with Jungers, Parrilo, Roozbehani [3]). In this work, we describe a "general recipe" for designing polynomially sized semidefinite programs that give upper bounds on the JSR. Our paper establishes a correspondence between sets of LMIs and certain families of finite automata whose states denote unknown Lyapunov functions as SDP decision variables and whose transitions are labeled with matrices $A_1, \ldots, A_m$. Our main theorem states that if the finite automaton satisfies a certain language-theoretic property, then a feasible solution to the SDP certifies an upper bound on the JSR. These upper bounds come with worst-case approximation guarantees. For example, we describe an SDP with m decision matrices of size $n \times n$ and m^2 LMI constraints that produces an upper bound $\hat{\rho}$ on ρ with the guaranteed accuracy of

$$\frac{1}{\sqrt[4]{n}} \hat{\rho}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}(\mathcal{A}).$$

Dynamic programming for JSR of rank-one matrices (with Parrilo [4]). We show that in the special case where the matrices $A_1, \ldots, A_m$ have rank one, the JSR can be computed exactly in $O(m^3 + m^2 n)$. The algorithm is based on dynamic programming and comes from a connection we make between this problem and the *maximum cycle mean problem* in graph theory [6].

Existence of "bad" examples for optimization-based techniques (with Jungers [2]). We show that for any positive integer d, there are families of switched linear systems—in fixed dimension and defined by two matrices only—that are stable under arbitrary switching but do not admit (i) a polynomial Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a piecewise quadratic Lyapunov function with $\leq d$ pieces. This implies that there cannot be an upper bound on the size of the linear and semidefinite programs that search for such stability certificates. This is in contrast with (non-switched) linear systems that, if stable, always admit a *quadratic* Lyapunov function.

Sos-convex Lyapunov functions (with Jungers [1]**).** We show that if the JSR is less than one, there always exists an *sos-convex* Lyapunov function that proves it. Sos-convex Lyapunov functions are polynomial Lyapunov functions that have an algebraic certificate of convexity; they can be found efficiently by semidefinite programming. We show via an explicit example, however, that the minimum degree of an sos-convex Lyapunov function can be arbitrarily higher than a (non-convex) polynomial Lyapunov function. On the other hand, we show that ensuring convexity of a Lyapunov function is crucial for certifying stability of *nonlinear* switched systems.

3. Open questions

We state two fundamental open questions regarding the computation of the JSR.

Problem 1 (decidability of stability). Is there an algorithm that can take as input m matrices of dimension n with rational entries, halt in finite time, and output the correct yes-no answer as to whether their JSR is strictly less than one?

Problem 2 (the rational finiteness conjecture). Consider any product $A_{\sigma_1} \dots A_{\sigma_k}$ of some length k with $\sigma_i \in \{1, \dots, m\}$. It is well-known and easy to establish that

$$(\rho(A_{\sigma_1} \dots A_{\sigma_k}))^{1/k} \leq \rho(A_1, \dots, A_m).$$

Note that the ρ on the left is simply the spectral radius while the ρ on the right is the JSR. If $A_1, \dots, A_m$ have rational entries, is it true that equality is always achieved at a finite k?

A positive answer to Problem 2 implies a positive answer to problem 1.

References

[1] A. A. Ahmadi and R. Jungers. SOS-convex Lyapunov functions with applications to nonlinear switched systems. In *Proceedings of the IEEE Conference on Decision and Control*, 2013.

[2] A. A. Ahmadi and R. Jungers. On complexity of Lyapunov functions for switched linear systems. In *Proceedings of the 19th World Congress of the International Federation of Automatic Control*, 2014.

[3] A. A. Ahmadi, R. Jungers, P. A. Parrilo, and M. Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. *SIAM Journal on Optimization and Control*, 2014. To appear.

[4] A. A. Ahmadi and P. A. Parrilo. Joint spectral radius of rank one matrices and the maximum cycle mean problem. In *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012.

[5] R. Jungers. *The joint spectral radius: theory and applications*, volume 385 of *Lecture Notes in Control and Information Sciences*. Springer, 2009.

[6] R. Karp. A characterization of the minimum cycle mean in a digraph. *Discrete mathematics*, 23(3):309–311, 1978.
[7] G. C. Rota and W. G. Strang. A note on the joint spectral radius. *Indag. Math.*, 22:379–381, 1960.

Positive matrix polynomials

Jaka Cimprič

Let $\mathcal{M}_n(\mathbb{R}[x])$ be the algebra of all real matrix polynomials of size n in several variables and write $\mathcal{S}_n(\mathbb{R}[x]) = \{H \in \mathcal{M}_n(\mathbb{R}[x]) \mid H^T = H\}$. In my talk I discussed the following problems:

(1) (One-sided Real Nullstellensatz) Given $G_1, \ldots, G_m \in \mathcal{M}_n(\mathbb{R}[x])$, characterize all $F \in \mathcal{M}_n(\mathbb{R}[x])$ such that $F(a)v = 0$ for all $a \in \mathbb{R}^d$ and all $v \in \mathbb{R}^n$ satisfying $G_1(a)v = \ldots = G_m(a)v = 0$.

(2) (One-sided Positivstellensatz) Given $G_1, \ldots, G_m \in \mathcal{S}_n(\mathbb{R}[x])$, characterize all $F \in \mathcal{S}_n(\mathbb{R}[x])$ such that $v^T F(a)v > 0$ for all $a \in \mathbb{R}^d$ and all $v \in \mathbb{R}^n$ for which $v^T G_i(a)v \geq 0$ for all i.

The first problem has been solved in [1] by the following result:

Theorem. Given $G_1, \ldots, G_m, F \in \mathcal{M}_n(\mathbb{R}[x])$, the following are equivalent:

(1) $F(a)v = 0$ for all $a \in \mathbb{R}^d$ and $v \in \mathbb{R}^n$ for which $G_1(a)v = \ldots = G_m(a)v = 0$.

(2) F belongs to the smallest real left ideal of $\mathcal{M}_n(\mathbb{R}[x])$ which contains $G_1, \ldots, G_m$. (Here a left ideal J of $\mathcal{M}_n(\mathbb{R}[x])$ is real if for every $H_1, \ldots, H_k \in \mathcal{M}_n(\mathbb{R}[x])$ such that $H_1^T H_1 + \ldots + H_k^T H_k \in J + J^T$ we have that $H_1, \ldots, H_k \in J$.)

For $m = 1$, the second problem is related to the famous Finsler's Lemma. I discussed the following conjecture and its variants:

Conjecture. For every $F, G \in \mathcal{S}_n(\mathbb{R}[x])$, the following are equivalent:

(1) $v^T F(a)v > 0$ for all $a \in \mathbb{R}^d$ and $v \in \mathbb{R}^n$ for which $v^T G(a)v \geq 0$.

(2) There exist sos polynomials s, t and matrix polynomials $A_1, \ldots, A_k$ such that $(1 + s)F = I_n + \sum_{i=1}^k A_i^T A_i + tG$.

This conjecture is false in general (even for $G = g \cdot I_n$ where $g \in \mathbb{R}[x]$ and I_n is the identity matrix) but it is true in several interesting special cases. A more general variant of this conjecture (where the terms $A_i^T A_i$ have weights) is true for $G = g \cdot I_n$ but even that variant fails in general.

The details will appear in [2].

References

[1] J. Cimprič, A Real Nullstellensatz for Free Modules, J. Algebra **396**, 143–150.
[2] J. Cimprič, Finsler's Lemma for Matrix Polynomials, in preparation.

Freeness of Line Arrangements

Jean Vallès

A hyperplane arrangement is a finite collection of distinct hyperplanes $\mathcal{A} = (H_1, \ldots, H_m)$ of $\mathbf{k}^n$ when it is affine of $\mathbb{P}^n_{\mathbf{k}}$ when it is projective. One could say that the story of hyperplane arrangements $\mathcal{A}$, in particular of line arrangements, begins with the following two problems:

(1) counting the number of connected components of $\mathbb{R}^n \setminus \mathcal{A}$,
(2) prove Sylvester's affirmation: "any finite set of points of the real plane, having the property that any two secant line is a three secant line, is aligned". Such sets of points are called Sylvester-Gallai-Configurations (SGC for short).

I first recall the beautiful formula of Zaslavski (cf. [4]) that gives an answer to problem (1) in terms of combinatorics of $\mathcal{A}$; more precisely the number of chambers and also the number of bounded chambers are given by the evaluation of the Poincaré polynomial in 1 and -1. Then I present the Hesse and Dual Hesse arrangements (cf. [1]) showing that there exists SGC over the field $\mathbb{C}$.

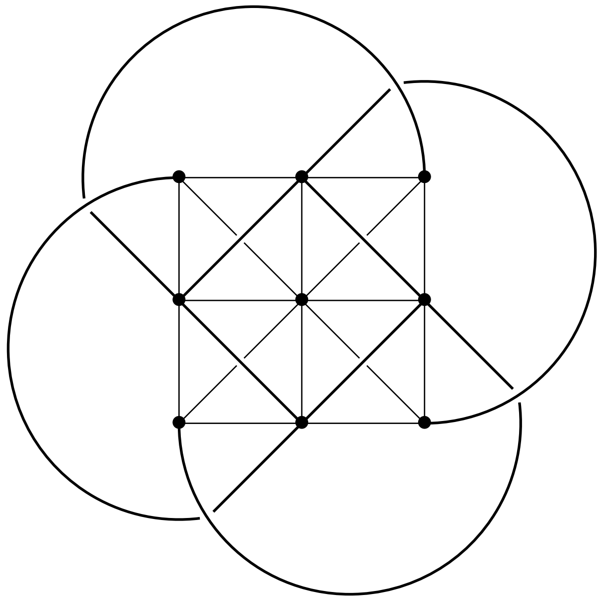

Figure 1. Hesse arrangement of 12 lines

The vector bundle $\mathcal{T}_{\mathbb{P}^n}(-\log \mathcal{A})$ of vector fields tangent to the line arrangement $\mathcal{A}$ was introduced by Saito (cf. [2]). When this vector bundle splits as a sum of line bundles of degrees $(-a_n \leq \ldots \leq -a_1)$ the corresponding arrangement is called **free with exponents** $(a_1 \leq \ldots \leq a_n)$ (cf. [1]). For instance, on the projective plane, the Hesse and Dual Hesse arrangements are free with exponents $(4, 7)$ and $(4, 4)$.

One main issue in the theory of arrangements is to what extent the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log \mathcal{A})$ depends on the combinatorial type of $\mathcal{A}$, defined as the isomorphism type of the lattice $L_{\mathcal{A}}$ of intersections of hyperplanes in $\mathcal{A}$. This lattice is partially ordered by reverse inclusion, and is equipped with a rank function given by codimension (cf. [1]). An important conjecture of Terao (reported in [1]) asserts that if $\mathcal{A}$ and $\mathcal{A}'$ have the same combinatorial type, and $\mathcal{T}_{\mathbb{P}^n}(-\log \mathcal{A})$ splits as a direct sum of line bundles (i.e. $\mathcal{A}$ is free), the same should happen to $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}'})$.

In *Logarithmic bundles and Line arrangements, an approach via the standard construction*, written in collaboration with Daniele Faenzi (cf. arXiv:1209.4934), we prove this conjecture on the complex projective plane up to 12 lines.

Our point of view is to study the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log \mathcal{A})$ relating it to the finite collection Z of points in the dual space $\check{\mathbb{P}}^n$ associated with $\mathcal{A}$ (we write $\mathcal{A} = \mathcal{A}_Z$ when $Z = \{z_1, \dots, z_m\}$ satisfies $H_i = H_{z_i}$ for all i, where $H_z \subset \mathbb{P}^n$ denotes the hyperplane corresponding to a point $z \in \check{\mathbb{P}}^n$). Our first result is that $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z})$ is obtained via the so-called standard construction from the ideal sheaf $I_Z(1)$. More precisely, denoting by $\mathbb{F}$ the incidence variety $\mathbb{F} = \{(x, y) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid x \in H_y\}$ and by p and q the projections onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$, we prove that:

$$\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z}) \simeq p_*(q^*(I_Z(1))).$$

Next we show that a line arrangement $\mathcal{A}_Z$ with a point of multiplicity k is free with exponents $(k, k + r)$ if and only if $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z})) = k(k + r)$. Here, by definition, $\mathcal{A}_Z$ free with exponents $(k, k + r)$ means that $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z}) \simeq \mathcal{O}_{\mathbb{P}^2}(-k) \oplus \mathcal{O}_{\mathbb{P}^2}(-r-k)$, and we write Chern classes on $\mathbb{P}^n$ as integers, with obvious meaning. For real arrangements, using a theorem of Ungar (cf. [3]), we push this criterion to points of multiplicity $k - 1$, under the assumption that $k \leq 3r + 5$.

References

[1] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[2] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[3] Peter Ungar. $2N$ noncollinear points determine at least $2N$ directions. *J. Combin. Theory Ser. A*, 33(3):343–347, 1982.

[4] Thomas Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 1(issue 1, 154):vii+102, 1975.

Hyperbolic polynomials and spectrahedral cones

Petter Brändén

The first part of this talk is a survey talk on hyperbolicity cones and spectrahedral cones, with focus on recent developments on the generalized Lax conjecture. In the second part we show how the recent proof by Marcus, Srivastava and Spielman [9] of the Kadison–Singer problem may be generalized to the setting of hyperbolic polynomials.

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros. If $A_1, \dots, A_n$ are symmetric real matrices, let

$A(\mathbf{x}) = x_1A_1 + x_2A_2 + \cdots + x_nA_n$. If $A(\mathbf{e}) \succ 0$, then the polynomial $\det(A(\mathbf{x}))$ is hyperbolic with respect to $\mathbf{e} = (e_1, \ldots, e_n)^T$. The *hyperbolicity cone* of a hyperbolic polynomial h may be defined as the closure of the connected component of

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

which contains $\mathbf{e}$. Hyperbolicity cones are closed basic semialgebraic convex cones, see [12]. The hyperbolicity cone of $\det(A(\mathbf{x}))$ is a *spectrahedral cone*:

$$\Lambda_+ = \{\mathbf{x} \in \mathbb{R}^n : x_1A_1 + x_2A_2 + \cdots + x_nA_n \succeq 0\}.$$

The *generalized Lax conjecture* predicts that the converse is true:

Conjecture 0.1 (Generalized Lax conjecture)**.** *Hyperbolicity cones are spectrahedral.*

The celebrated Helton–Vinnikov theorem [5] implies that Conjecture 0.1 holds for three variables. Indeed, the Helton–Vinnikov theorem states that each hyperbolic polynomial in three variables of degree d is a definite determinantal polynomial defined by matrices of size d. This solved a conjecture of Peter Lax from 1959. Algebraic conjectures that imply of Conjecture 0.1 were formulated in [5]. These were disproved in [1]. Further recent progress on Conjecture 0.1 are reported in [2, 7, 8, 10, 11, 13].

Recently Marcus, Spielman and Srivastava [9] proved the notorious Kadison–Singer problem [6] by proving a stronger version of a conjecture of Weaver [14]. Below we show how these results may be generalized [3] to concern hyperbolic polynomials and hyperbolicity cones.

Suppose h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$. The *eigenvalues* of $\mathbf{x} \in \mathbb{R}^n$ are $\lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x})$, where

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{i=1}^{d} (t - \lambda_i(\mathbf{x})).$$

The *trace*, *rank* and *spectral radius* of $\mathbf{x} \in \mathbb{R}^n$ are defined as for matrices:

$$\mathrm{tr}(\mathbf{x}) = \sum_{i=1}^{d} \lambda_i(\mathbf{x}), \quad \mathrm{rank}(\mathbf{x}) = \#\{i : \lambda_i(\mathbf{x}) \neq 0\} \quad \text{and} \quad \rho(\mathbf{x}) = \max_{1 \leq i \leq d} |\lambda_i(\mathbf{x})|.$$

If the lineality space of Λ_+ is trivial, then ρ is a norm on $\mathbb{R}^n$.

Theorem 0.2. *Let* $\epsilon > 0$*. Suppose* h *is hyperbolic with respect to* $\mathbf{e}$*. Let* $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \Lambda_+$ *be such that*

$\mathrm{rank}(\mathbf{u}_i) \leq 1$ *for all* $1 \leq i \leq m$,
$\mathrm{tr}(\mathbf{u}_i) \leq \epsilon$ *for all* $1 \leq i \leq m$, *and*
$\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m = \mathbf{e}$.

Then there is a partition $A \cup B = \{1, \ldots, m\}$ *such that*

$$\max\left\{\rho(\mathbf{u}_A - \mathbf{e}/2), \rho(\mathbf{u}_B - \mathbf{e}/2)\right\} \leq \sqrt{2}\epsilon + \epsilon^2,$$

where $\mathbf{u}_C = \sum_{j\in C} \mathbf{u}_j$.

Theorem 0.2 follows from the following theorem.

Theorem 0.3. *Let* $\epsilon > 0$. *Suppose* h *is hyperbolic with respect to* $\mathbf{e}$. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_m$ *be random vectors in* Λ_+ *(with finite supports) such that*

$\mathrm{rank}(\mathbf{v}_i) \le 1$ *for all* $1 \le i \le m$,
$\mathbb{E}\mathrm{tr}(\mathbf{v}_i) \le \epsilon$ *for all* $1 \le i \le m$, *and*
$\mathbb{E}(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m) = \mathbf{e}$.

Then

$$\mathbb{P}\left[\lambda_{\max}\left(\sum_{i=1}^m \mathbf{v}_i\right) \le (1+\sqrt{\epsilon})^2\right] > 0.$$

The main work in proving Theorem 0.3 goes into bounding the zeros of *mixed characteristic polynomials*:

$$\chi_{\mathbf{u}_1,\dots,\mathbf{u}_m}(t) = (1 - D_{\mathbf{u}_1})\cdots(1 - D_{\mathbf{u}_m})h(\mathbf{x})\Big|_{\mathbf{x}=t\mathbf{e}},$$

where $\mathbf{u}_1, \dots, \mathbf{u}_m \in \Lambda_+$ and

$$D_{\mathbf{u}} = \sum_{j=1}^n u_j \frac{\partial}{\partial x_j}, \qquad \text{where } \mathbf{u} = (u_1, \dots, u_n)^T.$$

Problem 0.4. *Maximize the largest zero of* $\chi_{\mathbf{u}_1,\dots,\mathbf{u}_m}(t)$ *under the constraints* $\mathbf{u}_1, \dots, \mathbf{u}_m \in \Lambda_+$, $\mathrm{tr}(\mathbf{u}_i) \le \epsilon$ *for all* $1 \le i \le m$, *and* $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m = \mathbf{e}$.

It has been conjectured the maximum in Problem 0.4 is attained for

$$\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_{k-1} = (\epsilon/d)\mathbf{e}, \mathbf{u}_k = (\rho/d)\mathbf{e}, \mathbf{u}_{k+1} = \cdots = \mathbf{u}_m = 0,$$

where $\rho \le \epsilon$ and k are determined by the constraints. An equivalent [3] conjecture is the following:

Conjecture 0.5. *The maximal zero in Problem 0.4 is achieved for vectors which are all in the interior of* Λ_+, *or identically zero.*

References

[1] P. Brändén, *Obstructions to determinantal representability,* Adv. Math., **226** (2011), 1202–1212.

[2] P. Brändén, *Hyperbolicity cones of elementary symmetric polynomials are spectrahedral*, Optimization letters, to appear.

[3] P. Brändén, Unpublished.

[4] L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech. **8** (1959), 957–965.

[5] J. Helton, V. Vinnikov, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. **60** (2007), 654–674.

[6] R. V. Kadison, I. M. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400.

[7] M. Kummer, *A note on the hyperbolicity cone of the specialized Vámos polynomial*, `arXiv:1306:4483`.

[8] M. Kummer, *Determinantal representations and the Bézout matrix*, `arXiv:1308:5560`.

[9] A. Marcus, D. A. Spielman, N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem*, `arXiv:1306.3969`.

[10] T. Netzer, R. Sanyal, *Smooth hyperbolicity cones are spectrahedral shadows*, Math. Prog. B, special issue 'Lifts of Convex Sets in Optimization', to appear.
[11] T. Netzer, A. Thom, *Hyperbolic polynomials and generalized Clifford algebras*, Discrete and Computational Geometry, to appear.
[12] J. Renegar, *Hyperbolic programs, and their derivative relaxations*, Found. Comput. Math., **6** (2006), 59–79.
[13] J. Saunderson, P. A. Parrilo, *Polynomial-sized semidefinite representations of derivative relaxations of spectrahedral cones*, `arXiv:1208.1443`.
[14] N. Weaver, *The Kadison-Singer problem in discrepancy theory*, Discrete Math. **278** (2004), 227–239.

Free Real Algebraic Geometry and Linear Matrix Inequalities

Bill Helton

Thanks to Klep and McCullough for many contributions to the talk. The subject is polynomials in free noncommutative variables $x = (x_1, \ldots, x_g)$. We evaluate them on matrix variables $X = (X_1, \ldots, X_g)$. To define properties such as

$$p \text{ is } PosSemiDef \quad \Rightarrow q \text{ is } \quad PosSemiDef$$

Free Real Algebraic Geometry

Here is a viewpoint to where things stand in Free RAG.

Free Postivstellensatz	Status	Free Behavior versus Classical
$q \succ 0$ on $p \succeq 0$ compact	Good	Similar
$q \succeq 0$ on a free convex set	Great	Better
$q \succeq 0$ on $p \succeq 0$	Poor	?
Free Nullstellensatz for finitely generated left ideals		
Hilbert (complex)	Good	Better
Real Nullstellensatz	Good	Better

For algebras which are not free see Schmüdgen's talk.

A tracial free RAG is emerging, due to Klep and Schweighofer.

Free Convexity

Free basic semi algebraic sets are convex if and only if they are solution sets to some LMI.

There are theorems classifying "analytic" free changes of variables from one free convex set to a ball (they are extremely rare). More general change of variables theory is progessing but a very open area.

Alas, the talk ran out of time.

Real Algebraic Geometry, Positive Polynomials, and Moments

Murray Marshall

A connection between positive linear functionals and measures is provided by the extended version of Haviland's theorem given in [11]:

Theorem 1. *Suppose A is an $\mathbb{R}$-algebra, X is a Hausdorff space and $\hat{}: A \to C(X)$ is an $\mathbb{R}$-algebra homomorphism. Let $\operatorname{Pos}(X) := \{a \in A \mid \hat{a} \geq 0$ on $X\}$. Suppose there exists $p \in \operatorname{Pos}(X)$ such that for each integer $k \geq 1$ the set*

$$X_k := \{\alpha \in X \mid \hat{p}(\alpha) \leq k\}$$

is compact. Then, for any linear function $L : A \to \mathbb{R}$ satisfying $L(\operatorname{Pos}(X)) \subseteq [0, \infty)$, there exists a positive Borel measure μ on X such that $\forall\ a \in A,\ \ L(a) = \int \hat{a}\, d\mu$.

See [5], [6] for the original version of Haviland's theorem. The representation theorem of T. Jacobi [7] also plays a central role in the theory:

Theorem 2. *Suppose A is an $\mathbb{R}$-algebra, M is an archimedean quadratic module of A, and*

$$X_M := \{\alpha : A \to \mathbb{R} \mid \alpha \text{ is an } \mathbb{R}\text{-algebra homomorphism, } \alpha(a) \geq 0\ \forall a \in M\}.$$

For $a \in A$ define $\hat{a} : X_M \to \mathbb{R}$ by $\hat{a}(\alpha) := \alpha(a)$. Give X_M the weakest topology such that $\hat{a}$ is continuous for each $a \in A$. Then, for $a \in A$, the following are equivalent:

(1) $\hat{a} \geq 0$ on X_M.

(2) $a + \epsilon \in M$ for all real $\epsilon > 0$.

See [1], [8] and [16] for early versions of Jacobi's theorem. See [12] for a simple proof. Applications of Haviland's theorem and Jacobi's theorem are given in [4] and [11]. The results in [4] extend results presented earlier, in the group algebra case, in [2]. Some of the results in [11] extend results presented earlier, in the preordering case, in [9] and [10]. Recent results in [13] explain how the results in [11] can be used:

– to reformulate the multivariate moment problem in terms of extension of PSD linear functionals on $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \dots, x_n]$ to PSD linear functionals on the localization of $\mathbb{R}[\underline{x}]$ at p, for suitably chosen p, e.g., $p = 1 + x_1^2 + \cdots + x_n^2$ or $p = (1 + x_1)^2 \cdots (1 + x_n^2)$;

– to prove new results concerning existence and uniqueness of the measure μ and density of $\mathbb{C}[\underline{x}]$ in the Lebesgue Space $\mathcal{L}^s(\mu)$ for various $s \in [1, \infty)$;

– to give new proofs of old results of B. Fuglede [3], A.E. Nussbaum [14], L.C. Petersen [15], M. Putinar and F.-H. Vasilescu [17] and [18], and K. Schmüdgen [19], results which were proved previously using the theory of strongly commuting self-adjoint operators on Hilbert space.

Another application of results in [11], which was overlooked at the time when [13] was being prepared, and which extends a result of A.E. Nussbaum [14], is the following one:

Theorem 3. *Suppose $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD. Suppose for $j = 1, \dots, n-1$*

$$(1) \quad \exists \text{ a sequence } \{q_{jk}\}_{k=1}^{\infty} \text{ in } \mathbb{C}[\underline{x}] \text{ such that } \lim_{k\to\infty} L(|1-(1+x_j^2)q_{jk}\overline{q_{jk}}|^2) = 0.$$

Then there exists a positive Borel measure μ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition (1) holds also for $j = n$ then the measure μ is determinate.

References

[1] E. Becker, N. Schwartz, Zum Darstellungssatz von Kadison-Dubois. *Arch. Math.* **39** 421–428 (1983).

[2] C. Berg, J.P.R. Christensen, P. Ressel, *Positive definite functions on abelian semigroups*, Math. Ann. 223, 253–272, (1976).

[3] B. Fuglede, The multidimensional moment problem. *Expositiones Mathematicae* **1**, 47–65 (1983).

[4] M. Ghasemi, S. Kuhlmann, M. Marshall, Application of Jacobi's representation theorem to locally multiplicatively convex topological R-algebras, *J. Funct. Anal.* **266**, 1041–1049 (2014).

[5] E.K. Haviland, On the momentum problem for distribution functions in more than one dimension. *Amer. J. Math.* **57**, 562–572 (1935).

[6] E.K. Haviland, On the momentum problem for distribution functions in more than one dimension II. *Amer. J. Math.* **58**, 164–168 (1936).

[7] T. Jacobi, A representation theorem for certain partially ordered commutative rings. *Math. Z.* **237**, 259–273 (2001).

[8] J.L. Krivine, Anneaux préordonnés. *J. Analyse Math.* **12**, 307–326 (1964).

[9] S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multidimensional moment problem. *Trans. Amer. Math. Soc.* **354**, 4285–4301 (2002).

[10] M. Marshall, Extending the Archimedean Positivstellensatz to the non-compact case, *Canad. Math. Bull.* **44**, 223–230 (2001).

[11] M. Marshall, Approximating positive polynomials using sums of squares, *Canad. Math. Bull.* **46**, 400–418 (2003).

[12] M. Marshall, Positive polynomials and sums of squares, *Mathematical Surveys and Monographs*, **146**. Amer. Math. Soc., Providence, RI, 2008.

[13] M. Marshall, Application of localization to the multivariate moment problem, *Math. Scand.,* to appear.

[14] A.E. Nussbaum, Quasi-analytic vectors, *Ark. Math.* **6** (10), 179–191 (1965).

[15] L.C. Petersen, On the relation between the multidimensional moment problem and the one dimensional moment problem, *Math. Scand.* **51**, 361–366 (1982).

[16] M. Putinar, Positive polynomials on compact sets, *Indiana Univ. Math. J.* **42**, 969–984 (1993).

[17] M. Putinar, F.-H. Vasilescu, Solving moment problems by dimension extension, *Annals of Math.* **149**, 1087–1107 (1999).

[18] M. Putinar, F.-H. Vasilescu, A uniqueness criterion in the multivariate moment problem, *Math. Scand.* **92**, 295–300 (2003).

[19] K. Schmüdgen, On determinacy notions for the two dimensional moment problem, *Ark. Math.* **29**, 277–284 (1991).

Positivestellensätze in Graph Theory

Tim Netzer
(joint work with Andreas Thom)

It is an important topic in extremal graph theory to describe universal inequalities between homomorphism- or subgraph-densities. For this purpose one can use sums of squares techniques in certain graph algebras. We will introduce the topic and then prove several Positivstellensätze. They all show that up to an arbitrarily small error, any valid inequality can be proven via sums of squares.

Real Symmetric Determinantal Representations of Ternary Hyperbolic Polynomials

Christoph Hanselka

In 1958 [5] conjectured, that every hyperbolic polynomial in three variables has a definite symmetric determinantal representation: A polynomial $h \in \mathbb{R}[X, Y, Z]$ homogeneous of degree d is called *hyperbolic* with respect to a direction $e \in \mathbb{R}^3$, if $h(e) > 0$ and every real line parallel to e intersects the zero set of h in only real points. In other words that means for every $a \in \mathbb{R}^3$ the univariate polynomial $h(Te + a) \in \mathbb{R}[T]$ has only real roots. h is said to admit a *definite symmetric determinantal representation*, if there exist real symmetric matrices $A, B, C \in \mathrm{Sym}_d(\mathbb{R})$ such that

$$h = \det(XA + YB + ZC)$$

and $xA + yB + zC$ is positive definite for some point $(x, y, z) \in \mathbb{R}^3$.

The result has been proven by Helton and Vinnikov in the 2000's in [4], based on earlier works in [1, 2] using the theory of theta functions on the jacobian of the curve defined by the polynomial. The connection to Lax' conjecture has been noted in [6].

We present a new and in most parts quite elementary proof by characterizing the characteristic polynomials of symmetric matrices over the univariate polynomial ring $\mathbb{R}[X]$ via a hyperbolicity like condition. Viewing the coordinate ring of the affine curve belonging to such a polynomial as an $\mathbb{R}[X]$-algebra, we get the trace form of that extension, which is an everywhere positive semidefinite quadratic form by the realness assumption on its roots. Restricting this form to a suitabe fractional ideal will let us find an orthonormal basis with respect to this form. From there it is easy to get a symmetric matrix with the given characteristic polynomial. A similar approach has been used by Bender in [3], where he considers characteristic polynomials of matrices over any integral domain and specifically over the integers.

References

[1] Joseph A. Ball and Victor Vinnikov. Zero-pole interpolation for meromorphic matrix functions on an algebraic curve and transfer functions of 2D systems. *Acta Appl. Math.*, 45(3):239–316, 1996.

[2] Joseph A. Ball and Victor Vinnikov. Zero-pole interpolation for matrix meromorphic functions on a compact Riemann surface and a matrix Fay trisecant identity. *Amer. J. Math.*, 121(4):841–888, 1999.

[3] Edward A. Bender. Classes of matrices over an integral domain. *Illinois J. Math.*, 11:697–702, 1967.

[4] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5):654–674, 2007.

[5] P. D. Lax. Differential equations, difference equations and matrix theory. *Comm. Pure Appl. Math.*, 11:175–194, 1958.

[6] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. The Lax conjecture is true. *Proc. Amer. Math. Soc.*, 133(9):2495–2499 (electronic), 2005.

Determinantal Representations of Space Curves

Eli Shamovich

(joint work with Victor Vinnikov)

The notion of a real hypersurface hyperbolic with respect to a real point off the hypersurface has been described by L. Hörmander and L. Gårding in their study of linear partial differential equations (see in particular [1]). It is clear that if a hypersurface admits a definite determinantal representation, then it is hyperbolic. It was shown by J. W. Helton and V. Vinnikov in [2] that every hyperbolic plane curve admits a definite determinantal representation. However, for hypersurfaces in $\mathbb{P}^d$, where $d \geq 3$ this theorem fails.

In this talk we will generalize the notion of hyperbolicity to subvarieties $X \subset \mathbb{P}^d$ of arbitrary codimension ℓ. The reference point will be an $\ell - 1$-dimensional real subspace of $\mathbb{P}^d$. We will also define the notion of Livsic-type determinantal representations. Both of this notions are intimately tied to the associated hypersurface to X in $\mathrm{Gr}(\ell - 1, d)$, the Grassmannian of $\ell - 1$-dimensional planes in $\mathbb{P}^d$. In particular we will focus on a special subclass of Livsic-type determinantal representations that we call very reasonable. For those representations the associated hypersurface admits a determinantal representation as well.

Next we will consider the case of curves and show that every curve admits a very reasonable Livsic-type determinantal representation. Furthermore, if the curve is real and hyperbolic with respect to some $\ell - 1$-dimensional real subspace, then the curve admits a Livsic-type determinantal representation that is definite.

References

[1] L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech. **8** (1959), 957–965.

[2] J.W. Helton and V. Vinnikov, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. **60** (2007), 654–674.

Noncommutative functions and balls, and robust control

Joseph A. Ball
(joint work with Gilbert Groenewald, Sanne ter Horst)

We consider first the classical case of holomorphic functions of a single variable. We define the classical Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ to consist of all holomorphic functions from the unit disk $\mathbb{D}$ into the space of contractive operators $\overline{\mathcal{B}}\mathcal{L}(\mathcal{U},\mathcal{Y})$ between two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. The following result gives two other equivalent formulations for the class $\mathcal{S}(\mathcal{U},\mathcal{Y})$.

Theorem 1. Given a function $S\colon \mathbb{D} \to \mathcal{L}(\mathcal{U},\mathcal{Y})$, the following are equivalent:

1. (1) S is in the Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$.
2. (2) The kernel $K_S(z,w)) := \frac{I-S(z)S(w)^*}{1-z\overline{w}}$ is a positive kernel on $\mathbb{D}$ ($K_S(z,w) = H(z)H(w)^*$ for some $H\colon \mathbb{D} \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ for some Hilbert space $\mathcal{X}$).
3. (3) There exists a system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with $\|\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\| \le 1$ so that $S(z)$ has the transfer-function realization $S(z) = D + zC(I - zA)^{-1}B$.

For holomorphic functions mapping the polydisk $\mathbb{D}^d$ into contraction operators from $\mathcal{U}$ to $\mathcal{Y}$, such a result is not possible due to the failure of the von Neumann inequality if $d > 2$ (see [2] for a full discussion). To remedy the situation we define the *Schur-Agler class* $\mathcal{SA}_d(\mathcal{U},\mathcal{Y})$ on the polydisk $\mathbb{D}^d$ (the d-fold Cartesian product of the unit disk $\mathbb{D}$ with itself) to consist of all $S\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ such that $\|S(T)\| \le 1$ whenever $T = (T_1, \dots, T_d)$ is a d-tuple of strict contraction operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{H}$; here we use the standard multivariable notation: if $n = (n_1, \dots, n_d) \in \mathbb{Z}^d_+$, then we set $z^n = z_1^{n_1} \cdots z_d^{n_d}$ and $T^n = T_1^{n_1} \cdots T_d^{n_d}$ and we define $S(T) = \sum_{n \in \mathbb{Z}^d_+} S_n \otimes T^n \in \mathcal{L}(\mathcal{U} \otimes \mathcal{H}, \mathcal{Y} \otimes \mathcal{H})$ assuming that the series converges in some suitable topology. The following result is due to Agler [1].

Theorem 2. Given a function $S\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U},\mathcal{Y})$, the following are equivalent:

1. (1) S is in the Schur-Agler class $\mathcal{SA}_d(\mathcal{U},\mathcal{Y})$.
2. (2) There is a Hilbert space space $\mathcal{X}$ and a spanning orthogonal family of projections $\{P_k\colon k = 1, \dots, d\}$ on $\mathcal{X}$ (so $P_kP_j = \delta_k, jP_k$ and $\sum_{k=1}^d P_k = I_{\mathcal{X}}$) and a function $H\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ so that the defect kernel $I - S(z)S(w)^*$ can be decomposed as $I - S(z)S(w)^* = H(z)(I - L(z)L(w)^*)H(w)^*$ where we set $L(z) = \sum_{k=1}^d z_kP_k$.
3. (3) There is a Hilbert space space $\mathcal{X}$ and a spanning orthogonal family of projections $\{P_k\colon k = 1, \dots, d\}$ on $\mathcal{X}$ as in statement (2) above together with a system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with $\|\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\| < 1$ so that $S(z)$ has the multidimensional transfer-function realization $S(z) = D + C(I - L(z)A)^{-1}L(z)B$ where as in (2) $L(z) = z_1P_1 + \cdots + z_dP_d$.

We remark that this result has been extended to the setting where the polydisk $\mathbb{D}^d$ is replaced by a domain $\mathcal{D}$ in $\mathbb{C}^d$ defined by $\mathcal{D}_Q = \{z \in \mathbb{C}^d\colon \|Q(z)\| < 1\}$ for some prescribed matrix polynomial in d variables (see [5, 6]); in this case the statements remain the same but one replaces $L(z)$ by $Q(z)$.

We discuss next a noncommutative version of Theorem 2. Rather than a holomorphic function $S(z)$ in the commuting complex variables $z = (z_1, \dots, z_d)$, we now let $z = (z_1, \dots, z_d)$ be a d-tuple of freely noncommuting indeterminates; thus monomials such as $z_1 z_2$ and $z_2 z_1$ are to be considered as distinct. We let $\mathbb{M}_d$ be the monoid of all words $\alpha = i_N \cdots i_1$ where each letter i_j comes from the alphabet consisting of the letters $1, \dots, d$; we also include the empty word $\emptyset$ in $\mathbb{M}_d$ which serves as the unit element for the multiplication defined by concatenation of words. For $\alpha = i_N \cdots i_1$ we associate the noncommutative monomial $z^\alpha = z_{i_N} \cdots z_{i_1}$. Given a d-tule $T = (T_1, \dots, T_d)$ of operators on some Hilbert space $\mathcal{H}$, an adaptation of the noncommutative functional calculus leads to the definition $T^\alpha = T_{i_N} \cdots T_{i_1}$ (where now the products are operator composition rather than just concatenation of symbols). Given a noncommutative formal power series $S(z) = \sum_{\alpha \in \mathbb{M}_d} S_\alpha z^\alpha$ with coefficients S_α in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ (the space of bounded linear operators from the Hilbert space $\mathcal{U}$ to the Hilbert space $\mathcal{Y}$), written as $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$, and given a d-tuple of operators $T = (T_1, \dots, T_d)$ as above, we define $S(T) \in \mathcal{L}(\mathcal{U} \otimes \mathcal{H}, \mathcal{Y} \otimes \mathcal{H})$ by $S(T) = \sum_{\alpha \in \mathbb{M}_d} S_\alpha \otimes T^\alpha \in \mathcal{L}(\mathcal{U} \otimes \mathcal{H}, \mathcal{Y} \otimes \mathcal{H})$ whenever the infinite series converges in some appropriate sense. For $L(z) = E_1 z_1 + \cdots + E_d z_d$ a linear pencil with coefficients $E_j \in \mathbb{C}^{M \times N}$ finite matrices, we say that the formal power series $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ is in the noncommutative Schur-Agler class $\mathcal{SA}_L(\mathcal{U}, \mathcal{Y})$ if $S(T)$ is defined for any $T = (T_1, \dots, T_d)$ for which $\|L(T)\| < 1$ and then also $\|S(T)\| \leq 1$. Here we define $L(T) = \sum_{k=1}^d E_k \otimes T_k \in \mathcal{L}(\mathbb{C}^N \otimes \mathcal{H}, \mathbb{C}^M \otimes \mathcal{H})$ whenever $T = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{H})^d$. A special case of the following result (where the pencil $L(z)$ is required to be of a special admissible form) was obtained in [8]; the more general form stated here is a consequence of still more general results obtained in [3].

Theorem 3. *Given a formal power series $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ and given a matrix pencil $L(z) = E_1 z_1 + \cdots + E_d z_d$, the following are equivalent.*

(1) *$S(z)$ is in the noncommutative Schur-Agler class $\mathcal{SA}_L(\mathcal{U}, \mathcal{Y})$.*
(2) *$S(z)$ has a* noncommutative Agler decomposition*: there exists a Hilbert space $\mathcal{X}$ and a formal power series $H(z) \in \mathcal{L}(\mathcal{X} \otimes \mathbb{C}^M, \mathcal{Y})$ so that*

$$I - S(z)S(w)^* = H(z)\,(I - (I_\mathcal{X} \otimes L(z))(I_\mathcal{X} \otimes L(w))^*)\,H(w)^*.$$

(3) *There is a Hilbert space $\mathcal{X}$ and a system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \begin{bmatrix} \mathcal{X} \otimes \mathbb{C}^M \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \otimes \mathbb{C}^N \\ \mathcal{Y} \end{bmatrix}$ with $\|\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\| \leq 1$ such that $S(z)$ is given by*

$$S(z) = D + C(I - (I_\mathcal{X} \otimes L(z))A)^{-1}(I_\mathcal{X} \otimes L(z))B.$$

Remark 3.1: We remark that the proof of (1) $\Rightarrow$ (2) in Theorem 3 from [8, 3] is adapted from the proof of (1) $\Rightarrow$ (2) in Theorem 2 for the commutative case in [1]: there is an infinite-dimensional cone-separation argument, working with formal power series coefficients in [8] and working with the results of noncommutative operator-tuple evaluation of formal power series (or "noncommutative functions" in the sense of Kaliuzhnyi-Verbovetskyi–Vinnikov [15]) in [3]. Similarly, the proof of (2) ⇒ (3) in Theorem 3 is yet another instance of the "lurking isometry" argument used originally in [1].

Remark 3.2: An observation of Alpay and Kaliuzhnyi-Verbovetskyi [4] is that, in the definition of the noncommutative Schur-Agler class $\mathcal{SA}_L(\mathcal{U}, \mathcal{Y})$, it suffices to check the contractivity of S when acting on finite matrices of arbitrary sizes, i.e., on $T \in \cup_{n=1}^{\infty} \mathcal{B}_L((\mathbb{C}^{n \times n})^d)$. Here in general we let $\mathcal{B}_L(\mathcal{L}(\mathcal{H})^d)$ denote the set of d-tuples $T = (T_1, \dots, T_d)$ of operators on $\mathcal{H}$ such that $\|L(T)\| < 1$. As a consequence of this observation, we may consider an element S of the formal noncommutative Schur-Agler class $\mathcal{SA}_L(\mathcal{U}, \mathcal{Y})$ as a function on $\cup_{n=1}^{\infty} \mathcal{B}_L(\mathbb{C}^{n \times n})^d$ with values in $\cup_{n=1} \overline{\mathcal{B}}\mathcal{L}(\mathcal{U}, \mathcal{Y}) \otimes \mathbb{C}^{n \times n}$ (the closed unit ball of the space of operators $\mathcal{L}(\mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$ which is *graded* in the sense that S maps $\mathcal{B}_L(\mathbb{C}^{n \times n})^d$ into $\overline{\mathcal{B}}\mathcal{L}(\mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$ for each $n = 1, 2, \dots$. It is easily verified from the tensor structure of this functional calculus that the following additional axioms are satisfied:

(I) $S(\Gamma T \Gamma^{-1}) = \Gamma S(T) \Gamma^{-1}$ for any $T = (T_1, \dots, T_d) \in \mathcal{B}_L((\mathbb{C}^{n \times n})^d)$ (where $\Gamma T \Gamma^{-1} = (\Gamma T_1 \Gamma^{-1}, \dots, \Gamma T_d \Gamma^{-1})$ and it is assumed that Γ is such that $\Gamma T \Gamma^{-1}$ is again in the noncommutative ball $\mathcal{B}_L((\mathbb{C}^{n \times n})^d)$).

(II) $S(T^{(1)} \oplus T^{(2)}) = S(T^{(1)}) \oplus S(T^{(2)})$ for all $T^{(1)} \in \mathcal{B}_L((\mathbb{C}^{n_1 \times n_1})^d)$ and $T^{(2)} \in \mathcal{B}_L((\mathbb{C}^{n_2 \times n_2})^d)$.

A result of Kaliuzhnyi-Verbovetskyi–Vinnikov [15] having roots in the much earlier work of Taylor [18] says that, with mild additional assumptions (local boundedness), conversely any graded function S mapping $\cup_{n=1}^{\infty} \mathcal{B}_L(\mathbb{C}^{n \times n})^d$ into $\cup_{n=1} \overline{\mathcal{B}}(\mathcal{L}(\mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$ satisfying (I) and (II) (called a "noncommutative function") arises from a formal power series $S(z) = \sum_\alpha S_\alpha z^\alpha$ as above with explicit formulas for the "Taylor-Taylor" coefficients S_α. In view of all these observations, we see that condition (1) in Theorem 3 can be adjusted to read:

(1′) S is a noncommutative function mapping the noncommutative ball $\cup_{n=1}^{\infty} \mathcal{B}_L(\mathbb{C}^{n \times n})^d$ to the noncommutative ball $\cup_{n=1}^{\infty} \overline{\mathcal{B}}(\mathcal{L}(\mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$.

We note that Agler-McCarthy [3] use the noncommutative-function formulation from the start in their proof of Theorem 3; they also are able to allow a general noncommutative polynomial matrix $Q(z)$ in place of the noncommutative matrix pencil $L(z)$. We also note that Helton-Klep-McCullough [13, 14] have studied such noncommutative ball maps which are proper: the boundary of the noncommutative ball on the domain side maps to the boundary of the noncommutative ball on the range side.

Finally, it is of interest to point out that the intersection of the Schur-Agler class $\mathcal{SA}_L(\mathcal{U}, \mathcal{Y})$ with linear functions can be identified with *completely contractive* maps between certain operator spaces (see [17]). In the case when the operator spaces are operator algebras, one can read off the Arveson-Stinespring dilation theorem (see [17, Corollary 7.7]) from the linear case of the realization theorem (part (3) of Theorem 3): details on these matters will appear in [10].

There is a converse question: given a noncommutative matrix pencil $L(z)$ and a formal power series $S(z)$ defined via a system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as in statement (3) of Theorem 3, when is it the case that S is in the noncommutative Schur-Agler class

$\mathcal{SA}_L(\mathcal{U},\mathcal{Y})$? In the context of this question, it is more convenient to formulate the question with the *strict* noncommutative Schur-Agler class $\mathcal{SA}_L^o(\mathcal{U},\mathcal{Y})$ rather than the Schur-Agler class as discussed up to now: here we say that $S \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ is in the *strict* Schur-Agler class $\mathcal{SAL}^o(\mathcal{U},\mathcal{Y})$ if $\|S(T)\| \le r$ for some $r<1$ for all $T \in \mathcal{L}(\mathcal{H})^d$ with $\|L(T)\| < 1$. The following result appears in [9], at least for the case where the pencil $L(z)$ is assumed to have a special "admissible" form which we do not go into here. Results of this type go under the name *strict Bounded Real Lemma* in the engineering literature (see e.g. [11] and [19]).

Theorem 4 *With L an "admissible" matrix pencil and $S(z) \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ defined via a compatible system matrix $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ as in part (3) of Theorem 3, then $S \in \mathcal{SA}_L^o(\mathcal{U},\mathcal{Y})$ if and only if there exists an intertwining pair (H,H') ($H>0$, $H'>0$ and $(I_{\mathcal{X}} \otimes L(z))H = H'(I_{\mathcal{X}} \otimes L(z))$) so that the strict Linear Matrix Inequality*

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & I \end{bmatrix} < 0,$$

holds, or equivalently, there exist an invertible intertwining pair (Γ,Γ') so that

$$\left\| \begin{bmatrix} \Gamma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Gamma'^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| < 1.$$

It is interesting to note that the analogue of Theorem 4 for the commutative case (i.e., given $S(z)$ defined on the polydisk via a system matrix $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ as in Theorem 2, characterize in terms of A,B,C,D when $S(z)$ is in the commutative Schur-Agler class $\mathcal{SA}_d(\mathcal{U},\mathcal{Y})$) fails; this is one instance of the sharp contrasts between the commutative multivariable setting and the free noncommutative multivariable setting.

It can be argued that Theorem 4 appeared earlier, at least implicitly, in the robust control literature of the 1980s and 1990s as we now explain. For $L(z) = E_1 z_1 + \cdots + z_d E_d$ with coefficients $E_j \in \mathbb{C}^{M\times N}$ and a matrix $A \in \mathbb{C}^{N\times M}$, we define the *structured singular value* $\mu_L(A)$ by

$$\mu_L(A) = \frac{1}{\inf\{\|L(z)\| \colon I - L(z)A \text{ singular, } z=(z_1,\dots,z_d)\in\mathbb{C}^d\}}.$$

Motivation comes from the analysis of Linear-Fractional-Transformation (LFT) models for structured uncertainty in Robust Control Theory (see [11, 19]). The key property of $\mu_L(A)$ is the characterization of when $\mu_L(A)<1$:

$$\mu_L(A) < 1 \Leftrightarrow I - L(z)A \text{ invertible for all } z \in \mathbb{C}^d \text{ with } \|L(z)\| \le 1.$$

There was intensive effort to compute $\mu_L(A)$ in the 1980s and 1990s (see [19] and the references there). An early observation was that there is a computable upper bound given by

$$\widehat{\mu}_L(A) = \inf\{\|DAD'^{-1}\| \colon D, D' \text{ invertible with } D'L(z) = L(z)D \text{ for all } z \in \mathbb{C}^d\}.$$

It is not hard to see that $\widehat{\mu}_L(A) < 1$ if and only if there exist a strictly positive intertwining pair (H,H') (so $H>0$, $H'>0$ with $H'L(z) = L(z)H$ for all $z \in \mathbb{C}^d$)

which solves the strict LMI

$$A^*HA - H' < 0.$$

It is always the case that $\mu_L(A) \le \widehat{\mu}_L(A)$ but there are a number of results in the literature explaining how the gap in between can be arbitrarily bad in various precise senses. There are also results on the high level of computational complexity involved in computing μ_L exactly. On the other hand at some point it occurred to people to consider the relaxed problem where one lets the variable $z = (z_1, \dots, z_d)$ be operators $T = (T_1, \dots, T_d)$ on a fixed separable Hilbert space $\mathcal{H}$ (e.g., $\mathcal{H} = \ell^2$ for concreteness). This leads one to consider instead the relaxed version of $\mu_L(A)$ defined by

$$\widetilde{\mu}_L(A) = \frac{1}{\inf\{\|L(T)\| \colon I - L(T)(A \otimes I_{\mathcal{H}}) \text{ singular}, T = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{H})^d\}}.$$

Then it is not difficult to establish that

$$\mu_L(A) \le \widetilde{\mu}_L(A) \le \widehat{\mu}_L(A).$$

Moreover, one can find structured uncertainty interpretations for the quantity $\widetilde{\mu}_L(A)$ (in terms of robustness of stability and performance with respect to time-varying parameter uncertainty rather than just stationary parameter uncertainty, or, in a more sophisticated version, with respect to specification of admissibility sets for the set of input-output pairs of the true plant around the set of input-output pairs for the nominal plant—see [11] for details) which are just as compelling as the original motivation for $\mu_L(A)$. The following result, announced without proof in [12], due essentially to Paganini [16] which in turn drew on earlier work of Shamma and Magretski-Treil (see [11] for details), then gives a structured-uncertainty interpretation for the upper bound $\widehat{\mu}(A)$; again this result is obtained explicitly in [16, 11] only for linear pencils L of a certain special form which we do not define precisely here.

Theorem 5 *With notation as above, $\widetilde{\mu}(A) = \widehat{\mu}(A)$. Thus, $I - L(T)(A \otimes I_{\mathcal{H}})$ is invertible for all $T = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{H})^d$ with $\|L(T)\| \le 1$ if and only if there is a strictly positive structured solution (H, H') of the strict LMI $A^*HA - H' < 0$.*

What's more, Theorems 3, 4, 5 are closely related in that one can easily get from one to another by simple known manipulations. In particular, an attempt was made to derive Theorem 5 from Theorem 4 in [9] but a technical gap remained; based on the analysis in [16] it is now possible to complete this proof (details will appear in [10]). Conversely, it is possible to go from Theorem 5 to Theorem 4 by simple manipulations (especially the Main Loop Theorem [19, page 284]). It is also interesting to note that the proof of Theorem 5 from [16, 11] also uses a cone-separation argument, but in a more elementary finite-dimensional setting when compared to the argument in the proof of Theorem 3 in [8, 3].

References

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, in: Topics in Operator Theory: Ernst D. Hellinger Memorial Volume (L. de Branges, I. Gohberg, and J. Rovnyak, eds.) pp. 47–66, *Oper. Th. Adv. and Appl.* **48**, Birkhäuser, Basel, 1990.

[2] J. Agler and J.E. McCarthy, *Pick Interpolation and Hilbert Functions Spaces*, Graduate Studies in Mathematics **44**, Amer. Math. Soc., 2002.

[3] J. Agler and J.E. McCarthy, *Global holomorphic functions in several non-commuting variables*, arXiv:1305.1636v1, 2013.

[4] D. Alpay and D. Kaliuzhnyi-Verbovetskyi, *Matrix-J-unitary non-commutative rational formal power series*, in: The State Space Method: Generalizations and Applications (D. Alpay and I. Gohberg, ed.), pp. 49–113, *Oper. Th. and Appl.* **161**, Birkhäuser, Basel, 2005.

[5] C.-G. Ambrozie and D. Timotin, *A von Neumann type inequality for certain domains in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **131** (2003) no. 3, 859–869.

[6] J.A. Ball and V. Bolotnikov, *Realization and interpolation for the Schur-Agler-class functions on domains with matrix polynomial defining function in* $\mathbb{C}^n$, J. Funct. Anal. **213** (2004), 45–87.

[7] J.A. Ball. G. Groenewald, and T. Malikorn, *Structured noncommutative multidimensional linear systems*, SIAM J. Control and Optimization **44** (2006) no. 4, 174–1528.

[8] J.A. Ball. G. Groenewald, and T. Malikorn, *Conservative structured noncommutative multidimensional linear systems*, in: The State Space Method: Generalizations and Applications (D. Alpay and I. Gohberg, ed.), pp. 179–223, *Oper. Th. and Appl.* **161**, Birkhäuser, Basel, 2005.

[9] J.A. Ball, G. Groenewald, and T. Malikorn, *Bounded real lemma for structured noncommutative multidimensional linear systems and robust control*, Multidimensional Sys. Signal Process. **17** (2006), 119–150.

[10] J.A. Ball, G. Groenewald, and S. ter Horst, *Bounded real lemma and structured singular value versus diagonal scaling: the free noncommutative setting*, in preparation.

[11] G.E. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach*, Texts in Applied Mathematics **36**, Springer, 2000.

[12] W.-M. Lu, K. Zhou, and J.C. Doyle, *Stabilization of uncertain linear systems: an LFT approach*, IEEE Trans. Automat. Control **41** no. 1 (1996), 50–56.

[13] J.W. Helton, I. Klep, and S. McCullough, *Noncommutative ball maps*, J. Funct. Anal. **257** (2009), 47–87.

[14] J.W. Helton, I. Klep, and S. McCullough, *Analytic mappings between noncommutative pencil balls*, J. Math. Anal. Appl. **376** (2011), 407–428.

[15] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of Noncommutative Function Theory*, arXiv:1212.6345v1, 2012.

[16] F. Paganini, *Sets and Constraints in the Analysis of Uncertain Systems*, PhD thesis, California Institute of Technology, 1996.

[17] V.L. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, 2002.

[18] J.L. Taylor, *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252.

[19] K. Zhou (with J.C. Doyle and K. Glover), *Robust Optimal Control*, Prentice Hall, 1996.

Vanishing of Polynomial Moments and of Iterated Integrals

Yosef Yomdin

We consider momentlike expressions of the form

$$m_k = \int_\Omega P^k(x) q(x) dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and P and q are polynomials in $x \in \mathbb{R}^n$. We study here the "Moment vanishing problem" for m_k, i.e. our goal is to give necessary and sufficient conditions on Ω, P, q for vanishing of $m_k,\ k = 0, 1, \ldots,$.

Recently, moment vanishing problems of various specific forms appeared in Qualitative Theory of ODEs, in Inverse problems (in particular, in Algebraic signal sampling), in Representation Theory, and in study of Algebras of Differential Operators, as related to the Jacobian Conjecture. In particular, let us consider the Abel differential equation

$$y' = p(x)y^2 + q(x)y^3. \tag{1}$$

A solution y(x) of (1) is called "closed" on [a,b] if y(a)=y(b). Study of closed solutions of (1) is directly related to the classical Hilbert 16-th (= Smale 13-th) and the Poincare Center-Focus problems. The last problem consists (in case of equation (1)) in providing necessary and sufficient conditions on p, q, a, b for (1) to have all its solutions closed (center).

It turns out that a rather accurate "first order" approximation of such "center conditions" is provided by the vanishing of $m_k = \int_a^b P^k(x)q(x)dx$, where $P = \int p$. Higher order approximations of the center conditions are provided by the vanishing of the higher Melnikov functions, which are linear combinations of iterated integrals of the form $\int p \int q \int q \ldots \int p$.

In recent 20 years an important algebraic-analytic structure has been connected to the Moment vanishing: Composition Algebra of polynomials. The study of this structure significantly improved our understanding of moment vanishing, as well as of closed solutions of (1). Very recently, a complete and effective solution of the one-dimensional polynomial Moment vanishing problem has been finally produced by F. Pakovich in [8]. On this base, a serious progress has been achieved in [2, 3] in study of center conditions for polynomial Abel equation (1).

In this talk some of these new results and some open questions related to moment vanishing have been presented. In particular, we've started with a very short review of the classical results of J. Wermer ([10]) describing vanishing conditions for moments on a closed curve in $\mathbb{C}^n$. We've also mentioned a remarkable development of Wermer's theory by G. Henkin and P. Dolbeault ([5]), and by R. A. Walker ([9]). In somewhat more details recent results of Pakovich ([8]) have been presented. Very recent results of M. Briskin on vanishing of iterated integrals on Pakovich spaces ([1]) have been shortly mentioned. And only in answering the auditory's questions the Mathieu conjectire ([7]), its solution for tori by Duistermaat and Van der Kallen ([6]), and some recent developments by W. Zhao towards the Jacobian Conjecture (compare [4] and references therein) have been mentioned.

References

[1] M. Briskin, *Vanishing of Second Melnikov function on Pakovich subspaces*, in preparation.

[2] M. Briskin, F. Pakovich, Y. Yomdin, *Algebraic Geometry of the Center-Focus problem for Abel Differential Equation*, arXiv:1211.1296.

[3] M. Briskin, F. Pakovich, Y. Yomdin, *Parametric Center-Focus Problem for Abel Equation*, arXiv:1312.1609.

[4] J.-P. Francoise, F. Pakovich, Y. Yomdin, W. Zhao, *Moment Vanishing Problem and Positivity: Some Examples*, Bull. Sci. Math. **135** (2011), no. 1, 1032.

[5] P. Dolbeault, G. Henkin, *Surfaces de Riemann de bord donne dans* $\mathbb{CP}^n$, Asp. Math., **26** (1994), 163–187.
[6] J. J. Duistermaat, W. van der Kallen, *Constant terms in powers of a Laurent polynomial,* Indag. Math. (N.S.) **9** (1998), no. 2, 221-231.
[7] O. Mathieu, *Some conjectures about invariant theory and their applications,* Algébre non commutative, groups quantiques et invariants (Reims 1995), 263-279, Semin. Congr., **2**, Soc. Math. France, Paris, 1997.
[8] F. Pakovich, *Generalized "second Ritt theorem" and explicit solution of the polynomial moment problem*, Compositio Math. **149** (2013), 705-728.
[9] R. A. Walker, *Boundaries of Holomorphic 1-Chains Within Holomorphic Line*, J. Geom. Anal. **20** (2010), 226–241.
[10] J. Wermer, *The hull of a curve in* $\mathbb{C}^n$, Ann. Math. **68** (1958), 550–561.

Exact algorithms in effective real algebraic geometry: from complexity results to practical computations

Mohab Safey El Din

Many important results in combinatorial and computational geometry (see *e.g.* [12, 22]), in theoretical computer science (see *e.g.* results on non-negative matrix factorization [1] or game theory [16]) rely on effective real algebraic geometry. Polynomial system solving over the reals has also many applications in engineering sciences, *e.g.* in robotics [2], and control theory [11] among other areas.

Typical computational challenges in real algebraic geometry are: deciding the emptiness of semi-algebraic sets, performing geometric operations such as projection (quantifier elimination), answering connectivity queries (roadmaps), computing the real dimension or computing the Euler-Poincaré characteristic, Betti numbers, etc.

Huge efforts have been invested during the last 25 years to derive algorithms that improve the doubly exponential complexity in n of Cylindrical Algebraic Decomposition [13]. This has led to algorithms for deciding the emptiness of semi-algebraic sets (in time $(s\,D)^{O(n)}$) [7], performing one-block quantifier elimination [6], computing the real dimension [19], answering connectivity queries (in time $(s\,D)^{O(n^2)}$) [12, 8]; see [9] for a self-contained overview.

Critical point methods are at the heart of these results. They consist in extracting important properties of semi-algebraic sets from the *critical points* of a well-chosen map. These are points at which the differential (of the map) is not surjective; local extrema of the map are reached at its critical points. These methods were used in combination with the introduction of infinitesimals that deform the input. This allows us to obtain cheap reductions to smooth and bounded semi-algebraic sets but affects the cost of arithmetic operations and hence practical performance.

It has been a long-standing problem to obtain efficient implementations for real-world problems based on critical point methods. Indeed, it requires to improve the exponents in the complexity bounds by introducing new algebraic and geometric techniques to avoid the use of infinitesimals. One successful research direction is to identify properties of critical points or polar varieties and to exploit them computationally using algorithms of elimination theory.

This trend started with [3] and has been developed for a decade (*e.g.* [21, 20, 4, 17, 14] and references therein) to understand the properties of these sets of points. We refer to [5] for an exposition of properties of polar varieties. Once these properties are understood, they can be exploited to design geometric procedures for solving.

In this talk, we present an overview of critical point methods. We highlight recent advances that lead to practically fast algorithms for deciding the existence of real solutions of polynomial systems.

References

[1] S. Arora, R. Ge, R. Kannan, and A. Moitra. Computing a nonnegative matrix factorization–provably. In *Proceedings of the 44th symposium on Theory of Computing*, pages 145–162. ACM, 2012.

[2] C. Bajaj. *Some applications of constructive real algebraic geometry.* Springer, 1994.

[3] B. Bank, M. Giusti, J. Heintz, and G.-M. Mbakop. Polar varieties and efficient real equation solving: the hypersurface case. *Journal of Complexity*, 13(1):5–27, 1997.

[4] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. Generalized polar varieties: geometry and algorithms. *Journal of Complexity*, 21(4):377–412, 2005.

[5] B. Bank, M. Giusti, J. Heintz, M. Safey El Din, and E. Schost. On the geometry of polar varieties. *Applicable Algebra in Engineering, Communication and Computing*, 2010.

[6] S. Basu, R. Pollack, and M.-F. Roy. On the combinatorial and algebraic complexity of quantifier elimination. *J. ACM*, 43(6):1002–1045, Nov. 1996.

[7] S. Basu, R. Pollack, and M.-F. Roy. A new algorithm to find a point in every cell defined by a family of polynomials. In *Quantifier elimination and cylindrical algebraic decomposition.* Springer-Verlag, 1998.

[8] S. Basu, R. Pollack, and M.-F. Roy. Computing roadmaps of semi-algebraic sets on a variety. *Journal of the AMS*, 3(1):55–82, 1999.

[9] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics.* Springer-Verlag, second edition, 2006.

[10] S. Basu, M.-F. Roy, M. Safey El Din, and E. Schost. A baby-step giant-step roadmap algorithm for general real algebraic sets. *submitted to Foundations of Computational Mathematics*, 2012.

[11] V. D. Blondel. Simultaneous stabilization of linear systems and interpolation with rational functions. In *Open problems in mathematical systems and control theory*, pages 53–59. Springer, 1999.

[12] J. Canny. *The complexity of robot motion planning.* PhD thesis, MIT, 1987.

[13] G. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decompostion. In *Automata Theory and Formal Languages*, pages 134–183. Springer, 1975.

[14] J.-C. Faugère, M. Safey El Din, and P.-J. Spaenlehauer. Critical points and Gröbner bases: the unmixed case. In J. van der Hoeven and M. van Hoeij, editors, *ISSAC*, pages 162–169. ACM, 2012.

[15] M. Giusti, G. Lecerf, and B. Salvy. A Gröbner-free alternative for polynomial system solving. *Journal of Complexity*, 17(1):154–211, 2001.

[16] K. A. Hansen, M. Koucky, N. Lauritzen, P. B. Miltersen, and E. P. Tsigaridas. Exact algorithms for solving stochastic games. In *Proc. 43rd annual ACM Symposium on Theory of Computing*, pages 205–214. ACM, 2011.

[17] G. Jeronimo, D. Perrucci, and J. Sabia. On sign conditions over real multivariate polynomials. *Discrete & Computational Geometry*, 44(1):195–222, 2010.

[18] P. Koiran. Randomized and deterministic algorithms for the dimension of algebraic varieties. In *Proc. 38th Annual Symposium on Foundations of Computer Science.*, pages 36–45. IEEE, 1997.

[19] P. Koiran. The Real Dimension Problem is $NP_{\mathbb{R}}$-complete. *Journal of Complexity*, 15(2):227–238, 1999.

[20] M. Safey El Din. Testing sign conditions on a multivariate polynomial and applications. *Mathematics in Computer Science*, 1(1):177–207, December 2007.

[21] M. Safey El Din and E. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In *ISSAC'03*, pages 224–231. ACM, 2003.

[22] J. Solymosi and T. Tao. An incidence theorem in higher dimensions. *Discrete & Computational Geometry*, pages 1–26, 2012.

Some old and new results and examples on psd forms which are not sos

Bruce Reznick

(joint work with Greg Blekherman)

I am pleased to report on my activities at Oberwolfach Workshop was "1415: Real Algebraic Geometry With A View Toward Systems Control and Free Positivity". I gave a morning survey talk "Some old and new results and examples on psd forms which are not sos" at 09:15 on Thursday 10 April. This talk was "blackboard" and not "beamer", so I cannot link to a .pdf. All the material in the "old" part can be found in [8, 9, 10]; the "new" material is in papers yet to be written with Greg Blekherman – see [1].

There are two main questions in the new material: one relates to maximizing the number of zeros which an irreducible psd ternary form of a given degree may have, the other involves the construction of extremal hyperplanes which separate certain well-known psd-not-sos ternary sextics from the cone of sos ternary sextics.

Let $P_{3,2k}$ denote the cone of psd ternary forms of degree $2k$ and let $\Sigma_{3,2k} \subseteq P_{3,2k}$ denote the cone of sos ternary forms: $p \in P_{3,2k}$ if and only if $p(x, y, z) \geq 0$ for $(x, y, z) \in \mathbb{R}^3$ and $p \in \Sigma_{3,2k}$ if and only if there exist ternary forms h_j of degree k so that $p = \sum_{j=1}^{m} h_j^2$. We review two essential "old" examples of forms in $P_{3,6} \setminus \Sigma_{3,6}$ from the 1960s. The first is Robinson's simplification [11] of Hilbert's original construction [4]. Let $\mathcal{H} := \{(\pm 1, \pm 1, 1), (\pm 1, 0, 1), (0, \pm 1, 1)\}$ be eight points in $\mathbb{R}^3$, viewed projectively. There is a pencil of ternary cubics vanishing at $\mathcal{H}$, generated by $F(x, y, z) = x(x^2 - z^2)$ and $G(x, y, z) = y(y^2 - z^2)$. As an illustration of Cayley-Bacharach, F and G have a 9th common zero at $(0, 0, 1)$. Let $K(x, y, z) = (x^2 - z^2)(y^2 - z^2)(z^2 - x^2 - y^2)$. Then K is singular at $\mathcal{H}$ and for some $\lambda > 0$, $F^2 + G^2 + \lambda K \in P_{3,6}$ (i.e., is psd.) Let $R = F^2 + G^2 + 1 \cdot K$; R turns out to be symmetric as well as psd:

$$R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2.$$

Since $R(0, 0, 1) = 1 > 0$ and any cubic which vanishes on $\mathcal{H}$ is zero there, it follows that R is not sos. (In fact, R has two additional zeros at infinity: $(1, \pm 1, 0)$.)

The second example is Motzkin's original example [7], which is proved in detail in, e.g., [9, p.257]. The same argument shows that for any $c \in (0, 3]$,

$$M_c(x, y, z) := x^4y^2 + x^2y^4 + z^6 - cx^2y^2z^2 \in P_{3,6} \setminus \Sigma_{3,6}.$$

For the first question, in 1980, Choi, Lam and I [2] proved that there is an integer $\alpha(k)$ with the property that if $p(x, y, z)$ is a real psd ternary form of degree $2k$ which has more than $\alpha(k)$ distinct zeros (viewed projectively), then there exists an indefinite ternary form h so that $p = h^2q$. We showed that $\alpha(2) = 1$, $\alpha(2) = 4$, $\alpha(3) = 10$. More generally, $k^2 \le \alpha(k) \le \frac{3}{2}k(k-1) + 1$ for $k \ge 4$, with the upper bound coming from Petrovskii's work on ovals. We (Greg and I) are now able to show that $\alpha(4) \ge 17$. This uses a variation of Robinson's argument; starting with a 4×4 grid, rather than a 3×3 one, and creating three zeros at infinity. (See [1] for the explicit zeros and coefficients of several examples.)

In 1893, Hilbert [5] proved that for a psd form p of degree $2k$, there exists a psd multiplier p_1 of degree $\le 2k-4$ so that pp_1 is a sum of three squares of forms. The 17-zero form has the property that there is a unique quadratic multiplier making it sos, but the product is a sum of four squares, not three, so an octic may genuinely need a quartic multiplier. We also construct an octic with 16 zeros, two of which vanish to 4th order in one direction. This strongly suggests that $\alpha(4) = 18$ is possible. (The ovals bound is 19). In the 1990s, my PhD student William Harris gave a family of even symmetric decics [3] with 30 zeros, so $\alpha(5) \ge 30$. One particular instance of this family, W, has no quadratic multiplier making it sos, let alone a sum of three squares:

$$W(x, y, z) = 16\sum x^{10} - 36\sum x^8y^2 + 20\sum x^6y^4 + 57\sum x^6y^2z^2 - 38\sum x^4y^4z^2.$$

(The sums above should be taken so as to make W symmetric.)

We conjecture that $\alpha(k) \ge k^2 + 1$ for all $k \ge 3$, and using these examples, and results from [2] can prove it for all but five values of k, the largest being for forms of degree 46.

One application arises in $Q_{3,2k}$, the dual cone to $P_{3,2k}$ under the standard "Fischer" inner product. It is proved in [8] that if $\mathcal{Z}(f) = \{(a_i, b_i, c_i)\}$ and the $|\mathcal{Z}(f)|$ linear forms $\{(a_ix + b_iy + c_iz)^{2k}\}$ are linearly independent, then any form

$$\sum_{i=1}^{|\mathcal{Z}(f)|} \lambda_i(a_ix + b_iy + c_iz)^{2k}, \quad (\lambda_i > 0)$$

has no other expression as a sum of $2k$-th powers of linear forms. The *a priori* lower bound on "maximal width" is $\frac{(k+1)(k+2)}{2}$. These examples of ternary forms with many zeros satisfy the linear independence condition and produce wider examples: e.g., 30 vs. the "minimal" upper bound of 21 when $k = 5$.

We turn briefly to separating hyperplanes. Suppose $q \in Q_{n,2k}$ and $[q, h^2] = 0$ for a specific form f of degree k. A consideration of $[q, (h + th')^2]$ for real t shows that $[q, hh'] = 0$ for all h'. The family $M_c \notin \Sigma_{3,6}$ for $c > 0$, hence for each such c, there exists $q_c \in Q_{3,6}$ so that $[q_c, M_c] < 0$. However, it is now easy to show that there is *no* single q so that $[q, M_c] < 0$ for *all* such $c > 0$. Indeed, if

$[q, x^4y^2 + x^2y^4 + z^6 - cx^2y^2z^2] < 0$ for all $c > 0$, then $[q, x^4y^2 + x^2y^4 + z^6] = [q, x^4y^2] + [q, x^2y^4] + [q, z^6] \le 0$, which implies that $[q, x^4y^2] = [q, x^2y^4] = [q, z^6] = 0$, and so $[q, (x^2y)^2] = 0$. It follows from the previous remark that $[q, x^2y^2z^2] = 0$, a contradiction.

On the other hand, using the work of Blekherman as a suggestive guide, it is not hard to construct specific separating hyperplanes based on evaluation at the 9 points of an Cayley-Bachrach set for ternary cubics. For example, if

$$H(p) := 3\sum_{\pm,\pm'} p(\pm 1, \pm' 1, 1) + \sum_{\pm} p(\pm 3, 0, 1) + \sum_{\pm} p(0, \pm 3, 1) - \tfrac{196}{31} p(0, 0, 1),$$

then it is not hard to show that $H(M_3) < 0$ but $H(q^2) \ge 0$ for any ternary cubic q. There is considerable flexibility in this (extremal) example; "3" can be replaced by any real $t > 2$. This example is simpler than the one in [6, pp.16-17].

For the Robinson form R, we can show that for $\beta > 0$,

$$H_\beta(p) := \sum_{\pm,\pm'} p(\pm 1, \pm' 1, 1), + \beta\left(\sum_{\pm} p(\pm 1, 0, 1) + \sum_{\pm} p(0, \pm 1, 1)\right) - \tfrac{4\beta}{4+\beta} p(0, 0, 1)$$

is such an extremal hyperplane separating R from $\Sigma_{3,6}$. Other separating hyperplanes for R were discussed in [8, pp. 142-146].

We hope that proofs and generalizations of all these assertions will appear in a future publication.

References

[1] Blekherman, G. and B. Reznick, *PSD ternary forms with lots of zeros*, Research Report for MFO-1415.

[2] Choi, M. D., T. Y. Lam and B. Reznick, *Real zeros of positive semidefinite forms, I*, Math. Z., **171** (1980), 1–25, (MR81d.10012).

[3] Harris, W. *Real even symmetric ternary forms*, J. Algebra **222** (1999), 204245, MR1728161 (2001a:11070).

[4] Hilbert, D., *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), 342–350; see Ges. Abh. 2, 154–161, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.

[5] Hilbert, D., *Über ternäre definite Formen*, Acta Math. **17** (1893) 169–197; see Ges. Abh. 2, 345–366, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1965.

[6] Iliman, S. and T. De Wolff, *Separating inequalities for nonnegative polynomials that are not sums of squares*, arXiv:1201.4061v2.

[7] Motzkin, T, S., *The arithmetic-geometric inequality*, pp. 205–224 in *Inequalities* (O. Shisha, ed.) Proc. of Sympos. at Wright-Patterson AFB, August 19–27, 1965 , Academic Press, New York, 1967; also in Theodore S. Motzkin: Selected Papers, Birkhäuser, Boston, (D. Cantor, B. Gordon and B. Rothschild, eds.), (MR36 #6569).

[8] Reznick, B., *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. **96** (1992), no. 463, (MR93h:11043).

[9] Reznick, B., *Some concrete aspects of Hilbert's 17th Problem*, Contemp. Math., **253** (2000), 251–272, (MR2001i:11042).

[10] Reznick, B., *On Hilbert's construction of positive polynomials*, arXiv:0707.2156.

[11] Robinson, R. M., *Some definite polynomials which are not sums of squares of real polynomials*, Izdat. "Nauka" Sibirsk. Otdel. Novosibirsk, (1973) pp. 264–282, (Selected questions

of algebra and logic (a collection dedicated to the memory of A. I. Mal'cev), abstract in Notices AMS, **16** (1969), p. 554, (MR49#2647)

The autoregressive filter problem for two variables, and related problems

HUGO J. WOERDEMAN

The two variable *autoregressive filter problem* concerns the following. Given $c_k = \overline{c}_{-k} \in \mathbb{C}$, $k = (k_1, k_2) \in \{0, \ldots, n\} \times \{0, \ldots m\} =: \Lambda$. Find a polynomial $p(z, w) = \sum_{r=0}^{n} \sum_{s=0}^{m} p_{rs} z^r w^s$ so that $p \neq 0$ on $\overline{\mathbb{D}}^2$ (i.e., p is *stable*) and

$$\widehat{\frac{1}{|p|^2}}(k) := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-ik_1\theta} e^{-ik_2\phi}}{|p(e^{i\theta}, e^{i\phi})|^2} d\theta d\phi = c_k, \ k = (k_1, k_2) \in \Lambda.$$

Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and $\overline{\mathbb{D}}$ is its closure.

The solution to the one-variable analog can be traced back to the works of Carathéodory, Toeplitz, Szegő, Yule and Walker around the 1920s. While in this classical one variable case positive definiteness of the finite Toeplitz matrix $(c_{k-l})_{k,l=0}^n$ is necessary and sufficient for the existence of a solution, the two variable case is significantly more involved and was solved in 2004 in [4]. As a corollary a two variable Fejér-Riesz factorization is derived in [4]: a two variable trigonometric polynomial $q(z, w)$ of degree (n, m), which is strictly positive on the bitorus, can be factored as $|p|^2$ with p as above if and only if

$$\operatorname{rank}[(c_{k-l})_{k \in \{1,\ldots,n\} \times \{0,\ldots,m\}, l \in \{0,\ldots,n\} \times \{1,\ldots,m\}}] = nm,$$

where $c_k = \widehat{\frac{1}{q}}(k)$ is the kth Forurier coefficient of $\frac{1}{q}$; see also [2, Theorem 3.4.1], where the matrix valued version appears.

A major step in the proof of the autoregressive filter result is the derivation of a two-variable analog of the Christoffel-Darboux formula: for a stable degree (n, m) polynomial $p(z, w)$ with its reverse defined by $\overleftarrow{p}(z, w) = \sum_{r=0}^{n} \sum_{s=0}^{m} \overline{p}_{rs} z^{n-r} w^{m-s}$, we have that

$$p(z, w)\overline{p(z_1, w_1)} - \overleftarrow{p}(z, w)\overline{\overleftarrow{p}(z_1, w_1)} =$$
$$(1 - w\overline{w_1}) \sum_i g_i(z, w)\overline{g_i(z_1, w_1)} + (1 - z\overline{z_1}) \sum_i h_i(z, w)\overline{h_i(z_1, w_1)},$$

where $g_1, \ldots, g_m$ are polynomials of degree at most $(n, m-1)$ and $h_1, \ldots, h_n$ are polynomials of degree at most $(n-1, m)$. The polynomials g_i and h_i may be found via Cholesky factorizations of the inverse of the doubly Toeplitz matrix $(c_{k-l})_{k,l \in \Lambda}$. A three or more variable analog of the Christoffel-Darboux formula has not been established yet. It is clear, as we will see below, that such a generalization will not be straightforward.

As observed in [3], it is interesting to note that the two variable Christoffel-Darboux formula provides a way to prove von Neumann's inequality for two variable rational inner functions. The classical von Neumann inequality states that for an analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ and a Hilbert space strict contraction T we have that $\|f(T)\| \leq 1$. If we now take a pair $T = (T_1, T_2)$ of commuting strict contractions on some Hilbert space and use the Christoffel-Darboux formula above, then

$$p(T_1,T_2)p(T_1,T_2)^* - \overleftarrow{p}(T_1,T_2)\overleftarrow{p}(T_1,T_2)^* =$$

$$\sum_i g_i(T_1,T_2)(I - T_2T_2^*)g_i(T_1,T_2)^* + \sum_i h_i(T_1,T_2)(I - T_1T_1^*)h_i(T_1,T_2)^*$$

is positive semidefinite. Thus

$$\|\frac{\overleftarrow{p}}{p}(T_1,T_2)\| \leq 1 = \|\frac{\overleftarrow{p}}{p}\|_\infty = \sup_{(z,w)\in\mathbb{D}^2} |\frac{\overleftarrow{p}(z,w)}{p(z,w)}|.$$

In other words, $\frac{\overleftarrow{p}}{p}$ satisfies the von Neumann inequality. The two variable von Neumann inequality goes back to [1], where the proof uses the theory of unitary dilations. An overview of the results above may be found in the monograph [2].

In three or more variables, many open questions remain. In [5] it was shown that for the stable polynomial

$$p(z_1,z_2,z_3) = 1 + \frac{11}{60}\, z_1z_2z_3\left(z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2 - 2z_1z_2z_3^2 - 2z_1z_2^2z_3 - 2z_1^2z_2z_3\right),$$

triples of commuting strict contractions (T_1,T_2,T_3) exist so that $\|\frac{\overleftarrow{p}}{p}(T_1,T_2,T_3)\| > 1$. Partial results seem to indicate that the theory of determinantal representations may be crucial to obtain a full understanding here. Indeed, denoting $Z_n = \oplus_{i=1}^d z_i I_{n_i}$, one may consider the polynomial $p(z_1,\dots,z_d) = \det(I - KZ_n)$, where K is a contraction. It was shown in [5] that the rational inner function $f(z) = \frac{z^n\bar{p}(1/z)}{p(z)} = \det[(Z_n - K^*)(I - KZ_n)^{-1}]$ satisfies

$$\|f(T_1,\dots,T_d)\| \leq 1 \tag{0.1}$$

for d-tuples of commuting strict contractions $(T_1,\dots,T_d)$. To what extent determinantal representations provide a complete characterization of rational inner functions thst satisfy the von Neumann inequality, is a topic of ongoing research.

References

[1] T. Ando. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)* 24 1963 88–90.

[2] M. Bakonyi and H. J. Woerdeman. *Matrix completions, moments, and sums of Hermitian squares.* Princeton University Press, Princeton, NJ, 2011.

[3] B. J. Cole and J. Wermer, Ando's theorem and sums of squares. *Indiana Univ. Math. J.* 48 (1999), no. 3, 767–791.

[4] J. S. Geronimo and H. J. Woerdeman, Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables. *Ann. of Math.* (2) 160 (2004), no. 3, 839–906.

[5] A. Grinshpan, D. S. Kaliuzhnyi-Verbovetskyi, and H. J. Woerdeman. Norm-constrained determinantal representations of multivariable polynomials. *Complex Anal. Oper. Theory* 7 (2013), 635–654.

Can moment relaxation methods solve (efficiently) polynomial optimization problems?

Bernard Mourrain
(joint work with Abril Bucero)

We study the problem of computing the infimum of a real polynomial function f on a closed basic semialgebraic set S and the points where this infimum is reached, if they exist:

$$\begin{array}{ll} \inf_{\mathbf{x}\in\mathbb{R}^n} & f(\mathbf{x}) \\ s.t. & g_1^0(\mathbf{x}) = \cdots = g_{n_1}^0(\mathbf{x}) = 0 \\ & g_1^+(\mathbf{x}) \geq 0, ..., g_{n_2}^+(\mathbf{x}) \geq 0 \end{array}$$

where $f, g_i^0, g_j^+ \in \mathbb{R}[\mathbf{x}]$, $\mathbf{g} = (\mathbf{g}^0; \mathbf{g}^+)$. The corresponding semi-algebraic set is

$$\mathcal{S}(\mathbf{g}) \quad = \quad \{\mathbf{x} \in \mathbb{R}^n \mid g_1^0(\mathbf{x}) = \cdots = g_{n_1}^0(\mathbf{x}) = 0, g_1^+(\mathbf{x}) \geq 0, ..., g_{n_2}^+(\mathbf{x}) \geq 0\}.$$

Relaxation methods based on Semi-Definite Programming are known to compute a lower approximation of the optimum. We show that exact relaxations which reach the minimum in a finite number of steps can be constructed to compute the minimum and points where the minimum is reached or to detect that there is no minimizer.

More precisely, when the infimum is reached, a Semi-Definite Program hierarchy constructed from the Karush-Kuhn-Tucker ideal is always exact and the vanishing ideal of the KKT minimizer points is generated by the kernel of the associated moment matrix in that degree, even if this ideal is not zero-dimensional. This relaxation allows to detect when there is no KKT minimizer.

In the case of optimization with no constraint, using the gradient of f yields an exact relaxation. In the case of constraints, the usual approach consists in introducing Lagrange multipliers. The corresponding gradient constraints yield an exact relaxation, which defines the Karush-Kuhn-Tucker minimizers. We analyze the variety defined by the KKT equations and its relation with the Fritz John variety, containing all the minimizer points of f on $\mathcal{S}(\mathbf{g})$. We show that these minimizers can be computed by an exact relaxation. We prove that the exactness of the relaxation depends only on the real points which satisfy these constraints. This exploits representations of positive polynomials as elements of the preordering modulo the KKT ideal, which only involves polynomials in the initial set of variables.

The approach provides a uniform treatment of different optimization problems considered previously. Applications to global optimization, optimization on semialgebraic sets defined by regular sets of constraints, optimization on finite semialgebraic sets and real radical computation are presented.

For the effective solution of the optimization problem, we introduce border basis reduction techniques, which combined with a reconstruction method from moment sequences yields an algorithm to compute the minimum of f and the minimizers when they are in finite number.

References

[1] Marta Abril Bucero and Bernard Mourrain. Exact relaxation for polynomial optimization on semi-algebraic sets. http://hal.inria.fr/hal-00846977
[2] Marta Abril Bucero and Bernard Mourrain. Border Basis relaxation for polynomial optimization. http://hal.inria.fr/hal-00981546

Singular numbers of products of operators (including matrices)

Ken Dykema
(joint work with Hari Bercovici, Benoît Collins, Wing-Suet Li, Dan Timotin)

1. The additive Horn problem and Schubert calculus

In 1962, A. Horn [7] conjectured an answer to the question, (asked by Weyl in 1912): which triples (α, β, γ) arise as eigenvalue sequences $\big(\lambda(A), \lambda(B), \lambda(C)\big)$ (each listed according to multiplicity and in nonincreasing order) for $n \times n$ Hermitian matrices A, B, C satisfying $A + B + C = 0$?

Horn defined sets $H^{(n)}$ of triples (I, J, K) of subsets of $\{1, \ldots, n\}$, with $|I| = |J| = |K|$, by a recursive algorithm. We will call such (I, J, K) *Horn triples*. Hown's conjecture is now a theorem, due to work of many authors, prinicipally Klyachko [9], Knutson and Tao [11] and S. Johnson [8]. (Also others were involved; see Fulton's review article [6] for a fuller description.)

Theorem ([8], [9], [11]). A triple (α, β, γ) of nonincreasing real sequences

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \beta = (\beta_1, \ldots, \beta_n), \qquad \gamma = (\gamma_1, \ldots, \gamma_n)$$

arises as eigenvalue sequences $(\lambda(A), \lambda(B), \lambda(C))$ of $n \times n$ Hermitian matrices A, B and C satisfying $A + B + C = 0$ if and only if

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j + \sum_{k=1}^{n} \gamma_k = 0$$

and for all $(I, J, K) \in H^{(n)}$, we have

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k \leq 0. \tag{1.1}$$

The inequality (1.1) is called the (additive) *Horn inequality* associated to the triple (I, J, K).

It is now recognized that Horn's set of triples $H^{(n)}$ is the set of all (I, J, K) satisfying the *Littlewood–Richardson rule*: $c^{(n)}_{IJK} > 0$. The LR-coefficient $c^{(n)}_{IJK}$ is obtained by a counting algorithm. These are commonly denoted $c^{\nu}_{\lambda,\mu}$ for partitions of integers λ, μ and ν, defined from I, J and K.

Schubert calculus can be used to show that the Horn inequality for a triple (I, J, K) always holds. A *flag* in $M_n(\mathbb{C})$ is a family $E = (E_1, \ldots, E_n)$ of projections with $E_\ell \le E_{\ell+1}$ and $\operatorname{rank}(E_\ell) = \ell$. Writing

$$I = \{i(1) < i(2) < \cdots < i(r)\}$$

the *Schubert variety* $\mathcal{S}(E, I)$ is the set of all projections P such that $\operatorname{rank}(P) = r$ and

$$\operatorname{rank}(P \wedge E_{i(\ell)}) \ge \ell \qquad (1 \le \ell \le r).$$

To prove that the Horn inequality (1.1) always holds, is suffices to show that for arbitrary flags E, F and G in $M_n(\mathbb{C})$, we have $\mathcal{S}(E, I) \cap \mathcal{S}(F, J) \cap \mathcal{S}(G, K) \neq \emptyset$. Finding $P \in \mathcal{S}(E, I) \cap \mathcal{S}(F, J) \cap \mathcal{S}(G, K)$ is called *solving the Schubert intersection problem.*

The cohomology ring of the Grassmannian can be used to show

$$c_{IJK}^{(n)} > 0 \qquad \Longrightarrow \qquad \mathcal{S}(E, I) \cap \mathcal{S}(F, J) \cap \mathcal{S}(G, K) \neq \emptyset$$

and this proves the "easier direction" of Horn's conjecture (that the Horn inequalities hold whenever $A + B + C = 0$).

Belkale [1] showed that the set of Horn inequalities for those (I, J, K) with $c_{IJK}^{(n)} = 1$ determines the convex body defined by all the Horn inequalities (so those with $c_{IJK}^{(n)} > 1$ are redundant).

2. A multiplicative Horn problem

A multiplicative version of Horn's problem is: given $n \times n$ matrices A and B whose singular numbers are known, what can the singular numbers of AB be? The answer, for invertible matrices A and B, is the following result of Klyachko. (Another multiplicative Horn problem, about eigenvalues of unitary matrices, was solved by Belkale [2].)

Theorem ([10]). For decreasing, strictly positive sequences of length n, α, β and γ, there exist matrices $A, B, C \in M_n(\mathbb{C})$ whose singular numbers are α, β, γ, respectively, and such that $ABC = 1$ if and only if

$$\prod_{i=1}^{n} \alpha_i \prod_{j=1}^{n} \beta_j \prod_{k=1}^{n} \gamma_k = 1$$

and, for all (I, J, K) with $c_{IJK}^{(n)} = 1$,

$$\prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j \prod_{k \in K} \gamma_k \le 1$$

In particular, α, β, γ are singular number sequences $s(A)$, $s(B)$, $s(C)$ of some matrices with $ABC = 1$ if and only if $\log \alpha$, $\log \beta$, $\log \gamma$ are eigenvalue sequences $\lambda(A')$, $\lambda(B')$, $\lambda(C')$ of some Hermitian matrices with $A' + B' + C' = 0$.

We generalize the above theorem to the case of non-invertible matrices. The goal is, for arbitrary sequences α, β with $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$, to find

$$S_{\alpha,\beta} := \{s(AB) \mid A, B \in M_n(\mathbb{C}),\ s(A) = \alpha,\ s(B) = \beta\}.$$

Theorem ([3]). $S_{\alpha,\beta}$ is the set of $\nu = (\nu_j)_1^n$, $\nu_1 \geq \cdots \geq \nu_n \geq 0$ such that for all (I, J, K) with $c_{IJK}^{(n)} = 1$, we have

$$\prod_{i\in I} \alpha_i \prod_{j\in J} \beta_j \leq \prod_{k\in \overline{K}} \nu_k \quad \text{and} \quad \prod_{i\in I^c} \alpha_i \prod_{j\in J^c} \beta_j \geq \prod_{k\in \overline{K}^c} \nu_k, \tag{2.1}$$

where $\overline{K} = \{n + 1 - k \mid k \in K\}$.

The proof depends on an interpolation result of Bercovici, Li and Timotin, from [5].

3. Horn problems in finite von Neumann algebras

Our goal is to investigate certain infinite dimensional analogues of the Horn problems, namely, in finite von Neumann algebra. To summarize without going into detail, a self-adjoint element b in a finite von Neumann algebra, instead of an eigenvalue seqncence, has a non-increasing, right continuous *eigenvalue function* $\lambda_b : [0, 1] \to \mathbb{R}$ and for an arbitrary element b in a finite von Neumann algebra, instead of the sequence of singular numbers, there is the *singular value function* $s_b = \lambda_{|b|}$.

Theorem ([4]). Real-valued, nonincreasing, right continuous functions f, g, h on $[0, 1]$ arise as the eigenvalue functions of self–adjoint elements a, b, c in a finite von Neumann algebra satisfying $a + b + c = 0$ if and only if for every n, their discretizations $f^{(n)}, g^{(n)}, h^{(n)}$, obtained by averaging over the intervals $[\frac{j}{n}, \frac{j+1}{n}]$, satisfy the Horn inequalities (1.1) for all $(I, J, K) \in H^{(n)}$ with $c_{IJK}^{(n)} = 1$.

Theorem ([3]). Nonnegative, nonincreasing right-continuous functions u, v, w on $[0, 1]$ arise as singular value functions of elements a, b, c in a finite von Neumann algebra satisfying $ab = c$ if and only if for every n, the discretizations of their logarithms $(\log u)^{(n)}$, $(\log v)^{(n)}$, $(\log w)^{(n)}$, satisfy the log versions of the inequalities (2.1), namely

$$\sum_{i\in I} (\log u)_i^{(n)} + \sum_{j\in J} (\log v)_j^{(n)} \leq \sum_{k\in \overline{K}} (\log w)_k^{(n)}$$
$$\sum_{i\in I^c} (\log u)_i^{(n)} + \sum_{j\in J^c} (\log v)_j^{(n)} \geq \sum_{k\in \overline{K}^c} (\log w)_k^{(n)},$$

where $-\infty$ is allowed as a value, for all $(I, J, K) \in H^{(n)}$ with $c_{IJK}^{(n)} = 1$.

Both of these results can be seen as modest evidence for a positive solution to Connes' embedding problem. Indeed, they say that as far as eigenfunctions of sums and singular values of products goes, the possibilities in arbitrary finite von

Neumann algebras are the same as in von Neumann algebras that are embeddable in R^ω.

The proofs of the two theorems above depend crucially on the following constructive solution of the Schubert intersection problem, which may also be interesting for strictly finite dimensional phenomena.

Theorem ([4]). Given (I, J, K) with $c^{(n)}_{IJK} = 1$, there is a lattice polynomial Q (involving operations $\wedge$ and $\vee$) in $3(n-1)$ variables, such that for any flags E, F and G in $M_n(\mathbb{C})$, letting

$$P = Q(E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}, G_1, \ldots, G_{n-1}),$$

we have $P \in \mathcal{S}(E, I) \cap \mathcal{S}(F, J) \cap \mathcal{S}(G, K)$.

References

[1] P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compos. Math. **129** (2001), 67–86.

[2] P. Belkale, *Quantum generalization of the Horn conjecture*, J. Amer. Math. Soc. **21** (2008), 365–408.

[3] H. Bercovici, B. Collins, K. Dykema, W. Li, *Characterization of singular numbers of products of operators in matrix algebras and finite von Neumann algebras*, preprint, arXiv:1306.6434.

[4] H. Bercovici, B. Collins, K. Dykema, W. Li, D. Timotin, *Intersections of Schubert varieties and eigenvalue inequalitites in an arbitrary finite factor*, J. Funct. Anal. (2010), **258** 1579–1627.

[5] H. Bercovici, W. Li, D. Timotin, *The Horn conjecture for sums of compact selfadjoint operators*, Amer. J. Math. **131** (2009), 1543–1567.

[6] W. Fulton *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), 209–249.

[7] A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math **12** (1962), 225–241.

[8] S. Johnson, *The Schubert calculus and eigenvalue inequalities for sums of Hermitian matrices*, Ph.D. thesis, University of California, Santa Barbara (1979).

[9] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), 419–445.

[10] A. Klyachko, *Random walks on symmetric spaces and inequalities for matrix spectra*, Linear Algebra Appl. **319** (2000), 37–59.

[11] A. Knutson, T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055–1090.

The moment problem on infinite dimensional basic semi-algebraic sets

Maria Infusino

(joint work with Tobias Kuna and Aldo Rota)

The infinite dimensional moment problem naturally arises from applied fields dealing with the analysis of complex systems like many-body systems in statistical mechanics, spatial ecology, stochastic geometry, etc. Since such a system consists of a huge number of identical components, the essence of its investigation is to evaluate selected characteristics (usually correlation functions), which encode the most relevant properties of the system. These characteristics are indeed the only ones that give a reasonable picture of the qualitative behaviour of the system.

It is therefore fundamental to investigate whether a given candidate correlation function actually represents the correlation function of some random distribution. This problem is well-known as *realizability problem* and can be interpreted as a moment problem posed in an infinite dimensional setting.

The classical multivariate moment problem asks whether a given multisequence of real numbers $(m_\alpha)_{\alpha\in\mathbb{N}_0^d}$ is the moment sequence of some non-negative finite measure with fixed support $K \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$. However, at an early stage, this problem has also been generalized to the case of infinitely many variables (see [1] for more details on this topic). Here, each m_n in the starting sequence $(m_n)_{n\in\mathbb{N}_0}$ is an element of the tensor product of n copies of a certain infinite dimensional space (e.g. for each n, m_n is a generalized function of n variables in $\mathbb{R}^d$) and the support of the measure is assumed to be a non-linear subset of this space (examples of supports are the set of all L^2 functions, the cone of all non-negative generalized functions, the set of all signed measures). In this general setting, the realizability problem is nothing but a moment problem on a space of functions.

This interpretation has been first introduced in [7], where the authors give necessary and sufficient conditions for a pair of symmetric functions $\rho_1(x)$ and $\rho_2(x, y)$, $x, y \in X$, to be the first two correlation functions of a point process, namely a probability on the space of all locally finite configurations of points in a topological space X. Basically, they provide a solution for a particular instance of the realizability problem on point configuration spaces, treating it as an infinite dimensional truncated moment problem. (Recall that the moment problem is called *truncated* if the starting sequence is finite and *full* when it is an infinite one). The results presented in this talk are based on this approach, which is thought to shed some light on the problem to identify relevant explicit necessary and sufficient realizability conditions out of the rather inexplicit ones already known in literature.

We have recently observed in [5] that the various spaces of configurations, as well as several other function spaces on which the realizability problem has been posed in the applications, are actually defined by uncountable many polynomial constraints on the space of generalized functions on $\mathbb{R}^d$. This led us to analyze the full infinite dimensional moment problem on basic semi-algebraic sets of generalized functions on $\mathbb{R}^d$. It is well-known that the semi-algebraic structure of the support of the representing measure plays a fundamental role in the study of the classical finite dimensional moment problem (see e.g. [2, 14, 13, 12, 8, 10]). Combining the techniques recently developed in [9] to treat the moment problem on closed basic semi-algebraic sets of $\mathbb{R}^d$ with the classical results about the moment problem on nuclear spaces [1, 15], we derived necessary and sufficient conditions for an infinite sequence of generalized functions to be the moment sequence of a finite measure concentrated on a basic semi-algebraic set. In this way, we determined realizability conditions that can be more easily verified than the Haviland type conditions introduced by Lenard in [11]. Our conditions are indeed given by the non-negativity of the generalized Riesz functional on the quadratic module generated by the polynomials defining the semi-algebraic set. The new ideas employed in the proof of these conditions can be also applied to the finite dimensional moment problem and allow us to extend the result in [9] to basic semi-algebraic sets of $\mathbb{R}^d$ defined by an uncountable family of polynomials.

As already mentioned, the semi-algebraic structure is a common feature in many instances of the realizability problem. For example, our result applies to the set of all Radon measures, the set of all the measures having bounded Radon-Nikodym density w.r.t. the Lebesgue measure, the set of all sub-probabilities, the set of all multiple and simple configurations. In particular, the realizability problem for point configurations is very important in applications in statistical mechanics. In this context, it is more convenient and also more natural to have a characterization of the support of the realizing measure in terms of correlation functions rather than moment functions as in [5]. A vital problem for the applicability of our result is to find the smallest possible number of realizability conditions, namely to determine a representation of the considered support as a semi-algebraic set defined by as few polynomials as possible, [6].

One question that is still open is to generalize the results already obtained for the full realizability problem on basic semi-algebraic sets of generalized functions to the truncated case. In fact, it would be extremely interesting to obtain realizability conditions of positive semidefinite type as in [5] rather than the ones of the Haviland type given in [7], which are still the only known conditions for the truncated case. A result of this kind would be a great step forward in the theory of both finite and infinite dimensional moment problem, and it would have a strong impact on the applications.

The results obtained in [5] also suggest to investigate the question of approximating polynomials non-negative on a basic semi-algebraic set of generalized functions via elements of the associated quadratic module. This is a crucial problem in real algebraic geometry and it has been extensively studied in relation to the finite dimensional moment problem. In the case when the basic semi-algebraic set K is compact in $\mathbb{R}^d$, using the famous Putinar Positivstellensatz [13], it is possible to prove that the cone of non-negative polynomials on K coincides with the closure of the associated quadratic module w.r.t. the finest locally convex topology on $\mathbb{R}[x_1, \ldots, x_d]$. In the case of K non-compact, there have been negative results which suggested to consider the closure of the quadratic module w.r.t. other kind of topologies. This idea was recently developed in [4] and generalized in [3]. It would be interesting to investigate these techniques in the infinite dimensional case, since a result in this direction could be employed to get approximations for the solutions of polynomial optimization problems and thus bounds for some material properties of great physical interest.

References

[1] Y. M. Berezansky and Y. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Naukova Dumka, Kiev (1988). In Russian. English translation: Kluwer Academic Publishers, Dordrecht (1995).

[2] C. Berg and P. H. Maserick, *Polynomially positive definite sequences*, Mathematische Annalen, **259** (1982), 487–495.

[3] M. Ghasemi, S. Kuhlmann, and M. Marshall, *Application of Jacobi's representation theorem to locally multiplicatively convex topological* $\mathbb{R}$*-algebras*, J. Funct. Anal., **266**, no. 2 (2014), 1041–1049.
[4] M. Ghasemi, S. Kuhlmann, and E. Samei, *The moment problem for continuous positive semidefinite linear functionals*, Arch. Math. (Basel), **100**, no. 1 (2013), 43–53.
[5] M. Infusino, T. Kuna, and A. Rota, *The full infinite dimensional moment problem on semi-algebraic sets of generalized functions*, ArXiv e-prints, arXiv:1311.2175, submitted.
[6] M. Infusino and T. Kuna, *The full moment problem on subsets of probabilities and configurations*, in preparation.
[7] T. Kuna, J. L. Lebowitz, and E. R. Speer, *Necessary and sufficient conditions for realizability of point processes*, The Annals of Applied Probability, **21**, no. 4 (2011), 1253–1281.
[8] J. B. Lasserre, *Moments, positive polynomials and their applications*, Volume 1 of Imperial College Press Optimization Series, Imperial College Press, London, (2010).
[9] J. B. Lasserre, *The* **K***-moment problem for continuous linear functionals*, Trans. Amer. Math. Soc., **365**, no. 5 (2013), 2489–2504.
[10] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, Emerging Application of Algebraic Geometry, **149** (2008), 157–270.
[11] A. Lenard, *States of classical statistical mechanical systems of infinitely many particles I*, Arch. Rational Mech. Anal., **59** (1975), 219–239.
[12] V. Powers and C. Scheiderer, *The moment problem for non-compact semialgebraic sets*, Adv. Geom., **1**, no.1 (2001), 71–88.
[13] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J., **42**, no.3 (1993),969–984.
[14] K. Schmüdgen, *The* $K-$*moment problem for compact semi-algebraic sets*, Math. Ann., **289** (1991),203–206.
[15] S. N. Šifrin, *Infinite-dimensional symmetric analogues of the Stieltjes problem of moments*, Ukrain. Mat. Ž., **26** (1974) 696–701, 718.

Non-commutative Real Algebraic Geometry and Applications to Group Rings

Andreas Thom
(joint work with Tim Netzer and Jesse Peterson)

Let Γ be a discrete countable group. Let H be a complex Hilbert space and $U(H)$ the group of unitary operators on H. A *unitary representation* is a homomorphism $\pi\colon \Gamma \to U(H)$. There are always many unitary representations, the most obvious being the left regular representation $\lambda\colon \Gamma \to U(\ell^2\Gamma)$, where $\ell^2\Gamma$ is a Hilbert space with an orthonormal basis $\{\delta_g \mid g \in \Gamma\}$ and $\lambda(g)\delta_h := \delta_{gh}$. We denote by $\mathbb{C}[\Gamma]$ the complex group ring $\mathbb{C}[\Gamma] := \left\{\sum_{g\in\Gamma} a_g g \mid \text{finite sum}\right\}$.

This is a ring with the obvious addition and with multiplication extending the group multiplication. Setting $\left(\sum_g a_g g\right)^* = \sum_g \bar{a}_g g^{-1}$, we endow $\mathbb{C}[\Gamma]$ with an involution. We define the cone of hermitian sums of squares as follows:

$$\Sigma^2(\mathbb{C}[\Gamma]) := \left\{\sum_{i=1}^{n} \xi_n^* \xi_i \mid n \in \mathbb{N}, \xi_i \in \mathbb{C}[\Gamma]\right\}.$$

Lemma 0.1. *The element* $1 \in \mathbb{C}[\Gamma]$ *is an algebraic interiour point in the convex cone* $\Sigma^2(\mathbb{C}[\Gamma])$.

Corollary 0.2. *The following conditions are equivalent:*

(1) For all $\varepsilon > 0$, $a + \varepsilon \in \Sigma^2(\mathbb{C}[\Gamma])$.
(2) $\pi(a) \geq 0$ in all unitary representations.

Proof. Use a standard separation argument and the GNS construction. □

We denote by $\mathbb{F}_n$ the free group on n generators. A more refined characterization of positivity is available for the free group.

Theorem 0.3. *The following conditions are equivalent:*

(1) We have $a \in \Sigma^2(\mathbb{C}[\mathbb{F}_n])$.
(2) $\pi(a) \geq 0$ in all unitary representations.
(3) $\pi(a) \geq 0$ in all finite-dimensional unitary representations.

The corresponding result also holds for $\mathbb{Z}^2$ by a deep result of Claus Scheiderer; see [7]. It fails for $\mathbb{Z}^3$.

Conjecture 0.4. *The same holds for fundamental groups of surfaces.*

It is known that $2 \Leftrightarrow 3$ for surface groups by a deep result of Lubotzky-Shalom; see [1]. The equivalence $2 \Leftrightarrow 3$ for $\mathbb{F}_2 \times \mathbb{F}_2$ is equivalent to a positive solution to Connes' Embedding Problem; see [3, 4].

Let A, B be unital $*$-algebras. We say that a linear map $\varphi\colon A \to B$ is completely positive if for all $n \in \mathbb{N}$, the map $1_{M_n} \otimes \varphi\colon M_n(A) \to M_n(B)$ maps sums of hermitian squares to sums of hermitian squares. We denote the hermitian part of A by A_h and call A real-reduced if $a_1^* a_1 + \cdots + a_n^* a_n = 0$ implies $a_1 = \cdots = a_n = 0$.

Theorem 0.5 (see [2])**.** *Let A be a real reduced unital $*$-algebra. If $a \in A$ is hermitian and $a \notin \Sigma^2(A)$, then there exists a completely positive $\mathbb{R}$-linear functional*

$$\varphi\colon A_h \to \mathbf{R},$$

for some real closed field $\mathbf{R} \supset \mathbb{R}$, such that $\varphi(1) = 1$ and $\varphi(a) < 0$.

This result allows to give a new proof of Theorem 0.3.

Proof of Theorem 0.3: Take $a \in \mathbb{C}[\mathbb{F}_n]_h$ with $a \notin \Sigma^2(\mathbb{C}[\mathbb{F}_n])$. Find $\varphi\colon \mathbb{C}[\mathbb{F}_n]_h \to \mathbf{R}$, for some real closed field $\mathbf{R} \supset \mathbb{R}$, such that $\varphi(1) = 1$ and $\varphi(a) < 0$. Perform the GNS-construction (π, H, ξ) over $\mathbf{C}$ and cut down to a finite-dimensional subspace to find contractions

$$A_1, \ldots, A_n \in M_d(\mathbf{C}),$$

such that $\langle a(A_1, \ldots, A_n)\xi, \xi\rangle = \varphi(a) < 0$. Lift the $A_i's$ to unitaries over $\mathbf{C}$:

$$U_i := \begin{pmatrix} A_i & \sqrt{1 - A_i A_i^*} \\ \sqrt{1 - A_i^* A_i} & -A_i^* \end{pmatrix}, \quad 1 \leq i \leq n.$$

Tarski's Transfer Principle gives unitary matrices over $\mathbb{C}$. □

Another consequence of the same technique:

Theorem 0.6 (Helton-Klep-McCullough)**.** *Let $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ be the free algebra on self-adjoint generators. Let $L(x)$ be a monic linear pencil with coefficients in symmetric $k \times k$-matrices. The following conditions are equivalent:*

(1) $p \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ lies in the quadratic module generated by $L(x)$, i.e.,

$$p = \sum_i s_i^* s_i + \sum_j v_j^* L(x) v_j.$$

(2) $p(A_1, \ldots, A_n) \geq 0$ whenever $A_1, \ldots, A_n \in M_d(\mathbb{C})_h$ satisfy

$$L(A_1, \ldots, A_n) \geq 0$$

as a $kd \times kd$-matrix.

Proof. Same argument as before. □

There are explicit bounds on d. Indeed, for fixed $p \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ with k summands of degree m, it is enough check $M_d(\mathbb{C})_h$ with $d = mk$. Using this approach it is harder to get bounds on the degrees of s_i's and v_j's in the representation

$$p = \sum_i s_i^* s_i + \sum_j v_j^* L(x) v_j.$$

Any approach like this to $\mathbb{F}_2 \times \mathbb{F}_2$ should first be able to deal with the group $\mathbb{Z}^2$.

Question 0.7. *Can one do such an argument to understand the case $\mathbb{Z}^2$ better?*

We now turn to a discussion of Kazhdan's Property (T). Assume that Γ is generated by $g_1, \ldots, g_n$. We set $\Delta := 2n - \sum_{i=1}^n g_i + g_i^{-1}$ – the Laplace operator with respect to the generating set. It is well-known that the group Γ has Kazhdan's Property (T) if and only there exists $\varepsilon > 0$, such that

$$\mathrm{sp}(\pi(\Delta)) \subset \{0\} \cup [\varepsilon, 4n]$$

for all unitary representations of Γ. Equivalently, we get:

Corollary 0.8. *The group Γ has Kazhdan's Property (T) if and only there exists $\varepsilon > 0$, such that*

$$\pi(\Delta^2 - \varepsilon\Delta) \geq 0$$

for all unitary representations of Γ.

We set

$$\omega_{\mathbb{C}}(\Gamma) := \left\{ \sum_g a_g g \in \mathbb{C}[\Gamma] \mid \sum_g a_g = 0 \right\}.$$

This is an two-sided ideal in $\mathbb{C}[\Gamma]$, called the augmentation ideal. Clearly, $\Delta \in \omega_{\mathbb{C}}(\Gamma)$. It was shown in [2] that $\Sigma^2\omega_{\mathbb{C}}(\Gamma) = \Sigma^2(\mathbb{C}[\Gamma]) \cap \omega_{\mathbb{C}}(\Gamma)$.

Theorem 0.9 (see [2])**.** *If $H_1(\Gamma) = 0$, then Δ is an interiour point of the convex cone*

$$\Sigma^2(\omega_{\mathbb{C}}(\Gamma)) \subset \omega_{\mathbb{C}}(\Gamma)_h.$$

Proof. Use the Completely Positive Separation Theorem. □

Remark 0.10. Ozawa showed with an elementary argument that the element Δ is an interiour point of $\Sigma^2\omega_{\mathbb{R}}(\Gamma) \subset \omega_{\mathbb{R}}(\Gamma)_h$ for every Γ.

Theorem 0.11 (Ozawa, see [5]). *The group* Γ *has Kazhdan's property (T) if and only if there exists* $\varepsilon > 0$*, such that*

$$\Delta^2 - \varepsilon\Delta \in \Sigma^2(\omega_{\mathbb{C}}(\Gamma)).$$

Let $\Gamma := \langle g_1, \ldots, g_n \mid R\rangle$, where $R \subset \mathbb{F}_n$ is a set of defining relations. The group Γ is called *finitely presented* if one can choose R finite.

Theorem 0.12 (Shalom, 2000). *Every group* Γ *with Kazhdan's Property (T) is a quotient of a finitely presented group with Kazhdan's Property (T).*

Ozawa's new proof [5]*:* Any computation that shows that $\Delta^2 - \varepsilon\Delta \in \Sigma^2(\omega_{\mathbb{C}}(\Gamma))$ can only use finitely many relations $r_1, \ldots, r_m \in R$. But then, $\Lambda := \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m\rangle$ also has Kazhdan's Property (T), and $\Lambda \twoheadrightarrow \Gamma$. □

We end by mentioning some results about traces on group rings. For any $*$-algebra A, we denote by $[A, A]$ the linear span of all commutators in A, i.e.,

$$[A, A] := \operatorname{span}\{ab - ba \mid a, b \in A\}.$$

A trace on A is a linear functional $\tau\colon A \to \mathbb{C}$ that satisfies $\tau(ab) = \tau(ba)$, i.e., $\tau([A, A]) = 0$.

Theorem 0.13 (Klep-Schweighofer). *The following conditions are equivalent if and only if the Connes Embedding Problem has a positive solution.*

(1) For all $\varepsilon > 0$*,* $a + \varepsilon \in \Sigma^2(\mathbb{C}[\mathbb{F}_2]) + [\mathbb{C}[\mathbb{F}_2], \mathbb{C}[\mathbb{F}_2]]$*.*
(2) For all $n \in \mathbb{N}$ *and pairs* $u, v \in U(n)$*, we have* $\operatorname{tr}(a(u, v)) \geq 0$*.*

Surprisingly, for many groups the following analogue can be proved:

Theorem 0.14 (see [6]). *If* Γ *is a group like* $SL_3(\mathbb{Z})$ *or* $SL_2(\mathbb{Z}[1/2])$*, then the following conditions are equivalent.*

(1) For all $\varepsilon > 0$*,* $a + \varepsilon \in \Sigma^2(\mathbb{C}[\Gamma]) + [\mathbb{C}[\Gamma], \mathbb{C}[\Gamma]]$*.*
(2) For all f.d. representation $\pi\colon \Gamma \to U(n)$*, we have* $\operatorname{tr}(\pi(a)) \geq 0$*.*

Proof. We give complete classification of positive traces on $\mathbb{C}[\Gamma]$, using a detailed study of the von Neumann algebra generated in the GNS-representation, and ideas coming from Ergodic Theory and the proof of Margulis' Normal Subgroup Theorem. □

The corresponding problem for $SL_2(\mathbb{Z})$ is again much harder and equivalent to Connes' Embedding Problem, see [4].

Corollary 0.15. *Let* $\Gamma := SL_2(\mathbb{Q})$*. Let* $a = \sum_g a_g g \in \mathbb{C}[\Gamma]$*. The following conditions are equivalent:*

(1) We have $a \in \Sigma^2(\mathbb{C}[\Gamma]) + [\mathbb{C}[\Gamma], \mathbb{C}[\Gamma]]$*.*
(2) The inequalities $a_e > 0$ *and* $\sum_g a_g > 0$ *hold.*

Proof. One can show that $a \mapsto a_e$ and $a \mapsto \sum_g a_g$ are the only two (extremal) positive traces on $\mathbb{C}[\Gamma]$. □

References

[1] A. Lubotzky and Y. Shalom, *Finite representations in the unitary dual and Ramanujan groups*, Discrete geometric analysis: proceedings of the first JAMS Symposium on Discrete Geometric Analysis, December 12-20, 2002, Sendai, Japan.

[2] T. Netzer and A. Thom, *Real Closed Separation Theorems and Applications to Group Algebras*, Pacific J. Math.**263** (2013), No. 2, 435–452.

[3] N. Ozawa, *About the QWEP Conjecture*, Int. J. Math., **15** (2004), 501–530.

[4] N. Ozawa, *About the Connes embedding conjecture: algebraic approaches.* Jpn. J. Math. **8** (2013), no. 1, 147–183.

[5] N. Ozawa, *Noncommutative real algebraic geometry of Kazhdan's property (T).* preprint, (see arXiv:1312.5431).

[6] J. Peterson and A. Thom, *Character Rigidity for Special Linear Groups*, to appear in J. Reine Angew. Math., (see arXiv:1303.4007).

[7] C. Scheiderer, *Sums of squares on real algebraic surfaces*, manuscripta math. **119** (2006), 395–410.

Degree Bounds in Rational Sums of Squares Representations

Grigoriy Blekherman

(joint work with Greg Smith and Maurizio Velasco)

1. Main Theorem

Let $X \subset \mathbb{RP}^n$ be an irreducible non-degenerate curve with dense real points with vanishing ideal I and coordinate ring $R = \mathbb{R}[x_0, \dots, x_n]/I$ with the standard grading.

Let $P_{X,2s} \subset R_{2s}$ be the cone of forms of degree $2s$ that are nonnegative on X and $\Sigma_{X,2s} \subset R_{2s}$ be the cone of sums of squares of forms on degree d on X. We are interested in the following question:

Question 1.1. *Given an even degree $2s$, find an integer $k \in \mathbb{N}$ depending on s and invariants of X only, such that for every form $p \in P_{2s}$ there exists $q \in \Sigma_{2k}$ such that $pq \in \Sigma_{2s+2k}$.*

We note that such a bound exists by the Positivstellensatz, however we are interested in explicit bounds, which should be better than bounds known for general semialgebraic sets [2].

We also observe that such q does not, in general, provide a certificate of nonnegativity of p. This is because X may contain isolated points. However, this is the obstruction for an irreducible curve: such a sum of squares multiplier q is certificate of nonnegativity of p, whenever every real connected component of X is 1-dimensional.

Let $\mathrm{reg}_{\mathrm{H}}$ and $\mathrm{reg}_{\mathrm{CM}}$ denote the Hilbert and Castelnuovo-Mumford regularity of X respectively. Our main contribution is the following theorem:

Theorem 1.2. *Let $X \subset \mathbb{P}^n$ be a reduced irreducible non-degenerate curve with dense real points. Let d and g be the degree and the arithmetic genus of X. If*

$$k = \max\left(\operatorname{reg}_{\mathrm{H}}(X), \operatorname{reg}_{\mathrm{CM}}(X) - s - 2, \left\lfloor \frac{2g-1}{d} \right\rfloor + 1\right),$$

then for every form $p \in P_{X,2s}$ nonnegative on X there exists $q \in \Sigma_{X,2k}$ such that

$$pq \in \Sigma_{X,2s+2k}.$$

It is known that for curves we have $\operatorname{reg}_{\mathrm{CM}}(X) \leq d - n + 2$ and $\operatorname{reg}_{\mathrm{H}}(X) \leq \operatorname{reg}_{\mathrm{CM}}(X) - 1$ [3]. Thus we immediately obtain the following corollary:

Corollary 1.3. *Let $X \subset \mathbb{P}^n$ be a reduced irreducible non-degenerate curve with dense real points of degree d and arithmetic genus g. Let*

$$k = \max\left(d - n + 1, \left\lfloor \frac{2g-1}{d} \right\rfloor + 1\right).$$

Then for any form p nonnegative on X there exists a sum of squares q of degree $2k$ such that pq is a sum of squares on X.

For smooth curves, where arithmetic genus is equal to geometric genus, we can use Castelnuovo bound for geometric genus (see [1]) to show that $\left\lfloor \frac{2g-1}{d} \right\rfloor + 1 \leq d - n + 1$, so we obtain the following:

Corollary 1.4. *Let $X \subset \mathbb{P}^n$ be a smooth irreducible non-degenerate curve with dense real points of degree d and arithmetic genus g. Let*

$$k = d - n + 1.$$

Then for any form p nonnegative on X there exists a sum of squares q of degree $2k$ such that pq is a sum of squares on X.

For a planar curve X we have $g = \frac{1}{2}(d-1)(d-2)$, and therefore $\left\lfloor \frac{2g-1}{d} \right\rfloor + 1 = \left\lfloor \frac{(d-1)(d-2)-1}{d} \right\rfloor + 1 = \left\lfloor \frac{d^2-3d+1}{d} \right\rfloor = d-2$; also $\operatorname{reg}_{\mathrm{CM}}(X) = d$, while $\operatorname{reg}_{\mathrm{H}}(X) = d-2$. Thus we have the following uniforms bound for planar curves:

Corollary 1.5. *Let $X \subset \mathbb{P}^2$ be a irreducible planar curve with dense real points of degree d. Let*

$$k = d - 2.$$

Then for any form p nonnegative on X there exists a sum of squares q of degree $2k$ such that pq is a sum of squares on X.

We are currently working on understanding the tightness of the above bounds.

References

[1] Harris, J., A bound on the geometric genus of projective varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 8 (1981), 35–68.

[2] Lombardi, H., Perruci, D., Roy, M.-F., An elementary recursive bound for effective Positivstellensatz and Hilbert 17th problem, arXiv:1404.2338 (2014).

[3] Nagel, Uwe, On Castelnuovo's regularity and Hilbert functions, Comp. Math. 76 (1990), 265–275.

Determinantal Representations of Hyperbolic Polynomials

Mario Kummer

A homogeneous polynomial $h \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \dots, x_n]$ is said to be *hyperbolic with respect to* $e \in \mathbb{R}^n$, if h does not vanish in e and if for every $v \in \mathbb{R}^n$, the univariate polynomial $h(te+v) \in \mathbb{R}[t]$ has only real roots. The *hyperbolicity cone* $\mathrm{C}_h(e)$ of h at e is the set of all $v \in \mathbb{R}^n$ such that no zero of $h(te+v)$ is strictly positive. Hyperbolicity cones are semi-algebraic convex cones, as shown for example in [3].

The interest in hyperbolic polynomials was originally motivated by the theory of partial differential equations (see for example [2, 6]). But lately, interest arose in the area of optimization, especially semidefinite optimization (see for example [4, 5, 7]). In particular the connection to polynomials with a *definite determinantal representation* has attracted much attention: We say that a homogeneous polynomial $h \in \mathbb{R}[\underline{x}]$ has a definite determinantal representation, if there are symmetric matrices $A_1, \dots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ such that

$$h = \det(x_1 \cdot A_1 + \ldots + x_n \cdot A_n)$$

and if $A(e) = e_1 A_1 + \ldots + e_n A_n$ is positive definite for some $e \in \mathbb{R}^n$. It is easy to see that such polynomials are hyperbolic with respect to e. An important result of Helton and Vinnikov [5] says that conversely every hyperbolic polynomial in three variables has a definite determinantal representation. This holds no longer true for more than three variables. Actually Brändén [1] found a hyperbolic polynomial h in four variables such that no power h^N admits a definite determinantal representation (a good review about these topics can be found in [8]). But still one can ask whether some multiple of any hyperbolic polynomial admits a determinantal representation. This is exactly what we will prove in the case where the hyperbolic polynomial has no real singularities:

Theorem. Let $h \in \mathbb{R}[\underline{x}]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. Assume that h has no real singularities, i.e. $\nabla h(v) \neq 0$ for all $0 \neq v \in \mathbb{R}^n$. Then there is a hyperbolic polynomial $q \in \mathbb{R}[\underline{x}]$, such that $q \cdot h$ has a definite determinantal representation.

This result can be seen as a first step towards the Generalized Lax Conjecture:

Conjecture. Let $h \in \mathbb{R}[\underline{x}]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. Then there is a hyperbolic polynomial $q \in \mathbb{R}[\underline{x}]$, such that $\mathrm{C}_h(e) \subseteq \mathrm{C}_q(e)$ and such that $q \cdot h$ has a definite determinantal representation.

References

[1] P. Brändén. Obstructions to determinantal representability. *Adv. Math.*, **226(2)**, (2011) 1202–1212.

[2] L. Gårding. Linear hyperbolic differential equations with constant coefficients. *Acta Math.*, **85**, (1951) 2–62.

[3] L. Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, **8**, (1959) 957–965.

[4] O. Güler. Hyperbolic polynomials and interior point methods for convex programming. *Math. Oper. Res.*, **22(2)**, (1997) 350–377.

[5] J. W. Helton, V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, **60(5)**, (2007) 654–674.

[6] P. D. Lax. Differential equations, difference equations and matrix theory. *Comm. Pure Appl. Math.*, **11**, (1958) 175–194.
[7] J. Renegar. Hyperbolic programs, and their derivative relaxations. *Found. Comput. Math.*, **6(1)**, (2006) 59–79.
[8] V. Vinnikov. LMI Representations of Convex Semialgebraic Sets and Determinantal Representations of Algebraic Hypersurfaces: Past, Present, and Future. *Mathematical Methods in Systems, Optimization, and Control*, (2012) 325–349.

Spectrahedral cones generated by rank 1 matrices

Roland Hildebrand

Let $\mathcal{S}_n$ be the real vector space of $n \times n$ real symmetric matrices and $S_+(n) \subset \mathcal{S}_n$ the cone of positive semi-definite matrices. A subset C of a real affine space A is called a *spectrahedron* if for some n there exists an injective affine map $f : A \to \mathcal{S}_n$ such that $C = f^{-1}[S_+(n)]$. The map f identifies A with an affine subspace of $\mathcal{S}_n$, and hence spectrahedra can be seen as intersections of the cone $S_+(n)$ with affine subspaces of $\mathcal{S}_n$. A spectrahedron may have several in a natural sense non-equivalent such representations, and therefore one must distinguish the spectrahedron C as a subset of affine space from the pair (C, f), comprised of C and its representation f. If A is equipped with the structure of a vector space and the map f is linear, then C is called a *spectrahedral cone*.

Spectrahedra appear as the feasible sets of semi-definite programs and are thus of importance for convex optimization [5]. The facial structure of spectrahedra and spectrahedral cones has been studied in [4]. By their construction, spectrahedra are convex basic semi-algebraic sets. More precisely, they are *algebraic interiors* [2], i.e., they possess a representation as the closure of a connected component of the Positivstellen set $\{x \in A \,|\, p(x) > 0\}$ for some determinantal polynomial p.

Here we consider spectrahedral cones satisfying the following property.

Property 0.1. *The cone $K \subset \mathbb{R}^k$ has a spectrahedral representation by an injective linear map $f : \mathbb{R}^k \to \mathcal{S}_n$ such that every matrix in the image $f[K]$ can be represented as a sum of rank 1 matrices in $f[K]$.*

We shall call such spectrahedral cones *rank 1 generated* (ROG). One of our main results is that for a ROG cone, all representations possessing Property 0.1 are equivalent.

Definition 0.2. Let $L \subset \mathcal{S}_n$, $L' \subset \mathcal{S}_{n'}$ be linear subspaces of matrix spaces, and suppose that $n \leq n'$. We call L, L' *isomorphic* if there exists an injective linear map $f : \mathbb{R}^n \to \mathbb{R}^{n'}$ with coefficient matrix $A \in \mathbb{R}^{n' \times n}$ such that the induced map $\tilde{f} : \mathcal{S}_n \to \mathcal{S}_{n'}$ given by $\tilde{f} : X \mapsto AXA^T$ takes L onto L'.

Theorem 0.3. *Let $K \subset \mathbb{R}^k$ be a k-dimensional ROG cone and let the linear maps $f : \mathbb{R}^k \to \mathcal{S}_n$, $f' : \mathbb{R}^k \to \mathcal{S}_{n'}$ define spectrahedral representations of K, satisfying the condition in Property 0.1. Then the images of f, f' are isomorphic, and the isomorphism can be chosen as an extension of $f' \circ f^{-1}$.*

For ROG cones, there is hence essentially no difference between the cone as a subset of $\mathbb{R}^k$ and its representation as a linear section of a positive semi-definite matrix cone $S_+(n)$. In particular, a given point x of a ROG cone K is represented by a matrix of the same rank independently of the representation, and hence the rank is an intrinsic property of the point.

In what follows, we shall consider spectrahedra and spectrahedral cones directly as affine or linear sections of a positive semi-definite matrix cone. We may also assume that the spectrahedron contains a positive definite matrix by possibly reducing the size of the matrices. The condition of being ROG can equivalently be stated in terms of bounded spectrahedra. Namely, the conic hull K of a bounded spectrahedron C not containing the zero matrix is ROG if and only if C is the convex hull of the rank 1 matrices in C. Therefore, if C is the compact section of a ROG spectrahedral cone, then minimizing a linear function over the nonconvex set of rank 1 matrices in C is equivalent to minimizing this linear function over the bounded spectrahedron C.

Thus the ROG property is in close relation to the exactness of semi-definite relaxations of nonconvex problems in the case when the relaxation is obtained by dropping a rank constraint. Many nonconvex optimization problems which are arising in computational practice fall into this framework, i.e., they can be cast as semi-definite programs with an additional rank constraint. It is this rank constraint which makes the problem nonconvex and difficult to solve. At the same time, dropping the rank constraint provides a convenient way of relaxing the problem into an easily solvable semi-definite program.

Examples of ROG cones are

- The cone of all Hankel matrices in $S_+(n)$.
- The 15-dimensional moment cone of the ternary quartics, represented by 6×6 matrices.
- The positive semi-definite matrix cone $S_+(n)$.
- Spectrahedral cones of codimension 1, i.e., sets $\{X \in S_+(n) \mid \langle X, Q\rangle = 0\}$, where Q is an indefinite quadratic form.

New ROG cones can be obtained from given ones by a number of procedures.

Faces of ROG cones are again ROG.

Direct products of ROG cones are ROG. Conversely, if a ROG cone K is a direct product of m lower-dimensional cones, then the factors are also ROG. The representation of K as a linear section of $S_+(n)$ induces a decomposition $\mathbb{R}^n = \bigoplus_{i=1}^m H_i$ into a direct sum of subspaces. This justifies the definition of *simple* ROG cones as those which are not nontrivial direct products.

It can be shown that a simple ROG cone of degree n has dimension at least $2n-1$. An example of simple cones of minimal dimension are the cones of positive semi-definite Hankel matrices, but there exist also other examples.

Full extensions of ROG cones are ROG cones, defined as follows.

Definition 0.4. Let k, n be positive integers, $k < n$, and let $K' = L' \cap S_+(n-k)$, $K = L \cap S_+(n)$ be spectrahedral cones, where $L' \subset \mathcal{S}_{n-k}$, $L \subset \mathcal{S}_n$ are linear subspaces. We call K a *full extension* of K' if there exists a direct sum decomposition $\mathbb{R}^n = H \oplus E$ into subspaces of dimensions $n-k, k$, respectively, and a corresponding direct sum decomposition of $\mathcal{S}_n$ into subspaces $L_E = span\{xy^T + yx^T \,|\, x \in \mathbb{R}^n,\ y \in E\}$, $L_H = span\{xx^T \,|\, x \in H\}$, with the following properties. The inclusion $L_E \subset L$ holds, and the subspaces $L' \subset \mathcal{S}_{n-k}$, $L_H \cap L \subset \mathcal{S}_n$ are isomorphic.

Every ROG cone of codimension 2 must be a full extension of $S_+(1) \times S_+(2)$.

Intertwinings are constructed from pairs of lower-dimensional ROG cones.

Definition 0.5. Let K_1, K_2 be ROG cones, let $\Phi_1 \subset K_1$, $\Phi_2 \subset K_2$ be faces which are isomorphic to a positive semi-definite matrix cone $S_+(k)$, and let $f : span\Phi_1 \to span\Phi_2$ be an isomorphism between Φ_1, Φ_2. Define the linear subspace $\Lambda = \{(X_1, X_2) \,|\, X_1 \in span\Phi_1,\ X_2 \in span\Phi_2,\ f(X_1) + X_2 = 0\}$ of the direct product $spanK_1 \times spanK_2$. Then the projection K of the direct product $K_1 \times K_2$ on the quotient space $(spanK_1 \times spanK_2)/\Lambda$ is called an *intertwining* of K_1, K_2.

It can be shown that intertwinings are also ROG. The families of ROG cones which have been obtained from chordal graphs in [1],[3] can be constructed as intertwinings and direct products of positive semi-definite matrix cones only. However, the class of ROG cones which can be obtained as intertwinings of matrix cones $S_+(k)$ is much richer and contains continuous families parameterized by an arbitrary number of real parameters.

We have also the following result on the structure of the set of extreme rays of a ROG cone.

Theorem 0.6. *Let K be a ROG cone of degree n. Then the number of its isolated extreme rays does not exceed n. Let the rank 1 matrices $x_1x_1^T, \dots, x_kx_k^T \in S_+(n)$ generate the isolated extreme rays of K. Then the vectors $x_1, \dots, x_k \in \mathbb{R}^n$ are linearly independent, and there exists a subspace $H \subset \mathbb{R}^n$ of dimension $n-k$, transversal to the linear span of $x_1, \dots, x_k$, with the following properties. Let F be the face of $S_+(n)$ given by the convex hull of the set $\{zz^T \,|\, z \in H\}$, and define the cone $K_F = F \cap K$. Then the cone K_F is a ROG cone without isolated extreme rays, and K is isomorphic to the direct product $K_F \times \mathbb{R}_+^k$.*

The discrete and the continuous part of the set of extreme rays of a ROG cone K thus generate separate factors of K. The factor generated by the discrete part is isomorphic to the nonnegative orthant.

Question: For a ROG cone $K \subset S_+(n)$, the set $\{x \in \mathbb{R}^n \,|\, xx^T \in K\}$ defines a projective algebraic variety. Which irreducible components can occur?

References

[1] Jim Agler, J. William Helton, Scott McCullough, and Leiba Rodman. Positive semidefinite matrices with a given sparsity pattern. *Linear Algebra Appl.*, 107:101–149, 1988.

[2] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Applied Math.*, 60(5):654–674, 2007.

[3] Vern I. Paulsen, Stephen C. Power, and Roger R. Smith. Schur products and matrix completions. *J. Funct. Anal.*, 85:151–178, 1987.

[4] Motakuri Ramana and A.J. Goldman. Some geometric results in semidefinite programming. *J. Global Optim.*, 7:33–50, 1995.
[5] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of semidefinite programming. Theory, Algorithms, and Applications*, volume 27 of *International Series in Operations Research and Management Science*. Kluwer Academic Publishers, Boston, 2000.

Realizations via Preorderings with Application to the Schur Class

Michael A. Dritschel

The classical realization theorem gives a variety of characterizations of those functions which are in the Schur class on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$; that is, those functions in the closed unit ball of $H^\infty(\mathbb{D})$.

Jim Agler found a method for extending this result to the polydisk, though for dimension d greater than 2, one must consider a (possibly) restricted algebra of functions along with a different norm. The unit ball for such an algebra is now commonly known as the Schur-Agler class; the term Schur class usually being reserved for the unit ball of $H^\infty(X)$ when X is a domain in $\mathbb{C}^d$. Among other things, the realization theorem states that a complex function φ on the polydisk is in the Schur-Agler class if and only if it has a so-called Agler decomposition, expressing $1 - \varphi\varphi^*$ as an element of a cone generated by products of certain positive kernels and kernels of the form $1-\psi\psi^*$, where ψ is a coordinate function. Other equivalent conditions include the existence of a transfer function representation and a von Neumann type inequality for suitably restricted tuples of commuting contractions. The equivalence of all of these conditions makes no a priori assumptions about φ, and it is this which enables the use of the realization theorem in such applications as Pick interpolation. These results have been vastly generalized, in the spirit of Agler's work.

The Agler decomposition has its analogues in real algebraic geometry. For example, if we have a domain in $\mathbb{R}^n$ which is the non-negativity set of a finite collection of polynomials including the polynomial which is identically equal to 1 (a so-called basic semi-algebraic set), and these polynomials also include $1-\psi_i^2$ where the ψ_is are constant multiples of the coordinate functions, then Putinar's theorem states that a strictly positive polynomial is in the quadratic module generated by these polynomials; that is, it is in the cone generated by finite sums of squares of polynomials times the individual polynomials defining the semi-algebraic set. If the condition regarding the coordinate functions is dropped, the statement of Putinar's theorem is in general false, even if the semi-algebraic set is assumed to be compact. However the situation can be salvaged in the compact setting by replacing the quadratic module by a preordering; that is, by considering the cone generated by finite sums of squares of polynomials times the various products of the polynomials defining the semi-algebraic set. This is the content of Schmüdgen's theorem. Further refinements are possible. For example, if only two polynomials define the compact semi-algebraic set then one can get by with the quadratic module in Schmüdgen's theorem (see the book of Prestel and Delzell), which because of Andô's theorem is analogous to what happens in the complex case with Agler's realization theorem.

Back in the complex function setting, recent work of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman shows that on the polydisk for dimension greater than 2, one can recover the entire Schur class by using the appropriate variant of a preordering. The caveat is that they find it necessary to assume that the function they are considering is already known to be in the Schur class, and so there is no direct application to Pick interpolation in the Schur class. Also, they are missing the crucial transfer function representation needed for interpolation results, though in a separate paper, they prove in a later paper that a form of the von Neumann inequality is available. A particularly interesting aspect of this last work is that the tuples of operators the authors are considering have a unitary dilation, obtained by showing that they induce a completely contractive representation of $H^\infty(\mathbb{D}^d)$ and then applying the standard machinery.

This talk has several goals. The first is to place the work of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman in the context of test functions on a set X, in this way allowing for a much broader class of function algebras. For example, there are analogues of the algebra $H^\infty(\mathbb{D}^d)$ and the multivariable analogue of the disk algebra, $A(\mathbb{D}^d)$, in this setting, which we denote by $H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ and $A(\mathcal{K}_{\Lambda,\mathcal{H}})$, respectively. The set X can be topologized and closed in an appropriate norm, which allows us to make sense of the analogue of the disk algebra in this context; that is, elements of our analogue of H^∞ which extend continuously to the closure of X.

Secondly, we give a full version of the realization theorem include the transfer function representation and analogues of the von Neumann inequality, at least when we have a so-called ample preordering. None of the realization theorems requires the precondition that the function under consideration is already in $H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$. Thus in principle in the ample case, such applications as Agler-Pick interpolation are possible.

Even if the preordering is not ample, we can show that elements of our generalized Schur-Agler class have a transfer functions representation, but not that everything with a transfer function representation is in our algebra. However, the transfer functions into $\mathcal{L}(\mathcal{H})$ do form the unit ball of an algebra having a matrix norm structure, and so form an operator algebra. We then show in this setting that certain types of representations (the Brehmer representations over the analogue of the disk algebra and the weakly continuous Brehmer representations over the analogue of H^∞) are completely contractive, implying the existence of a dilation of the representation to a so-called boundary representation (without the assumption of irreducibility). This in turn coincides with representations which are contractive on the auxiliary test functions, meaning that such representations are also completely contractive. Since when the preordering is ample the unit ball of $H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ coincides with those functions having transfer function representations, the same then goes for this algebra.

Of particular interest is the case of the polydisk, since then the ample preordering gives $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. Since the auxiliary test functions are not given constructively, determining if a representation is contractive on these is difficult. However they are matrix valued functions, and so we are able to conclude that for $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$, any weakly continuous representation which is 2^{d-1}-contractive is completely contractive, and that weak continuity can be dispensed with for the corresponding analogue of the disk algebra.

Finally, we indicate that in the setting of ample preorderings, Andô's theorem allows us to instead consider so-called nearly ample preorderings instead. With this we are able to recover the full power of the results of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman, and at the same time improve the result mentioned in the previous paragraph by proving that when $d \geq 2$, 2^{d-2}-contractive weakly continuous representations of $H^\infty(\mathcal{K}_{\Lambda,\mathcal{H}})$ are completely contractive, with a similar statement to the polydisk case for $A(\mathcal{K}_{\Lambda,\mathcal{H}})$.

Free polynomial optimization

Sabine Burgdorf

Free polynomial optimization can be interpreted in two ways: one can consider this as optimization of polynomials in free variables or as polynomial optimization free of any dimension constraints. We want to emphasize the similarity of asymptotic convergence of a Lasserre hierarchy for polynomial optimization with respect to free positivity (i.e. positive semidefiniteness) and to trace-positivity.

To fix the setup let $\mathbb{R}\langle \underline{X} \rangle$ denote the $\mathbb{R}$-algebra generated freely by n noncommuting variables $X_1, \ldots, X_n$. Its elements are called nc polynomials. We want to evaluate these polynomials in tuples of symmetric matrices or self-adjoint operators, hence we endow $\mathbb{R}\langle \underline{X} \rangle$ with an involution * modeling the transpose operation on matrices/operators, i.e. * is the (unique) involution which fixes $\mathbb{R}$ and $\{X_1, \ldots, X_n\}$ pointwise.

1. Free positivity

Let $p, g_1, \ldots, g_r \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric. The optimization problem we are interested in is the following

$$(1.1) \qquad p^{min} = \inf_{(H,\varphi,\underline{A})} \langle \varphi, p(\underline{A})\varphi \rangle \quad \text{s.t. } g_i(\underline{A}) \succeq 0 \text{ for } i = 1, \ldots, r$$

where H is a Hilbert space, φ a unit vector in H and $\underline{A}$ a tuple of bounded self-adjoint operators on H. Basically we want to find the smallest eigenvalue p can attain when evaluated in self-adjoint operators satisfying some additional constraints. For applications we refer e.g. to the talk of Bill Helton.

The classical Lasserre relaxation using quadratic modules to model positivity can be transferred to this setup. For this we define the free quadratic module with respect to $g = (g_1, \ldots, g_r)$ as

$$QM(g) := \{f \in \mathbb{R}\langle \underline{X} \rangle \mid f = \sum h_i^* h_i + \sum h_{ij}^* g_i h_{ij}; h_i, h_{ij} \in \mathbb{R}\langle \underline{X} \rangle\}$$

and its truncated counterpart

$$QM(g)_k := \{f \in QM(g) \mid \deg h_i^* h_i, \deg h_{ij}^* g_i h_{ij} \leq 2k\}.$$

Since by construction for $f \in QM(g)$ and every tuple $\underline{A}$ of self-adjoint operators the evaluated polynomial $f(\underline{A})$ is positive semidefinite, the following relaxation gives a lower bound on p^{min} for any $k \geq \deg p$:

$$p_k = \sup \lambda \ \text{ s.t. } p - \lambda \in QM(g)_k. \tag{1.2}$$

One key feature of this relaxation is that it is computable with a semidefinite program (SDP). Another feature is that when $N - \sum X_i^2 \in QM(g)$ for some $N \in \mathbb{N}$ the hierarchy converges asymptotically to p^{min} – although one might have $p_k < p^{min}$ for all $k \geq \deg p$. This follows directly from a Positivstellensatz of Helton and McCullough [2, Theorem 1.2]

Theorem 1.1 (Helton, McCullough)**.** *Let $p, g_i \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric and such that $QM(g)$ contains $N - \sum X_i^2$ for some $N \in \mathbb{N}$. If p is positive semidefinite for all tuples $\underline{A}$ of symmetric operators with $g_i(\underline{A}) \succeq 0$ then for all $\varepsilon > 0$ we have $f + \varepsilon \in QM(g)$.*

Pironio, Navascués and Acín gave a constructive proof for this showing additionally that in the limit the infimum is actually a minimum [4, Theorem 1].

Theorem 1.2 (Pironio, Navascués, Acín)**.** *Let $p, g_i \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric and such that $QM(g)$ contains $N - \sum X_i^2$ for some $N \in \mathbb{N}$. Then $p_k \to p^{min}$ as $k \to \infty$. Furthermore, there exists a Hilbert space H, a normalized vector $\varphi \in H$ and a tuple $\underline{A} \in B(H)^n$ of self-adjoint operators such that $g_i(\underline{A}) \succeq 0$ and $p^{min} = \langle \varphi, p(\underline{A})\varphi \rangle$.*

We want to emphasize that the Hilbert space H might be *infinite dimensional.* If the set $K = \{\underline{A} \in B(H)^n \mid A_j \text{ symmetric}, g_i(\underline{A}) \succeq 0\}$ is convex there exists by the addendum of [2, Theorem 1.2] a finite dimensional Hilbert space H where the optimum is attained. But in general a guarantee to get a finite dimensional optimizer is not known, and even in the convex case it is not clear whether the corresponding SDP always gives a finite dimensional optimizing solution as output. For a further discussion of open problems we refer to Chapter 21 in [1].

2. Trace positivity

The condition of being positive semidefinite under all evaluations is quite strict. A natural attempt to get a broader set of positive polynomials is to consider – instead of the smallest eigenvalue – the trace a polynomial can attain under evaluations in self-adjoint operators. When one defines an appropriate Lasserre hierarchy as for free positivity it shows the same behavior in terms of asymptotic convergence. To be more precise let again $p, g_1, \dots, g_r \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric. The optimization problem we are interested in is now the following

$$p^{min} = \inf_{((N,\tau),\underline{A})} \tau(p(\underline{A})) \quad \text{s.t. } g_i(\underline{A}) \succeq 0 \text{ for } i = 1, \dots, r \tag{2.1}$$

where (N, τ) is a finite von Neumann algebra with trace τ and $\underline{A}$ a tuple of self-adjoint operators in N. The Lasserre relaxation works here as well. Since the trace of a commutator $pq - qp$ is always 0 under all evaluations in self-adjoint operators, it is natural to extend the free quadratic module $QM(g)$ by commutators of polynomials, i.e. we define the tracial quadratic module as

$$TQM(g) := QM(g) + [\mathbb{R}\langle \underline{X} \rangle, \mathbb{R}\langle \underline{X} \rangle]$$

and its truncated counterpart

$$TQM(g)_k := QM(g)_k + [\mathbb{R}\langle \underline{X} \rangle, \mathbb{R}\langle \underline{X} \rangle].$$

As in the free case the relaxation

$$p_k = \sup \lambda \ \text{ s.t. } p - \lambda \in TQM(g)_k. \tag{2.2}$$

gives a lower bound on p^{min} for any $k \geq \deg p$ and is in fact also computable with an SDP. By the following Positivstellensatz of Klep and Schweighofer [3, Theorem 3.12] we get asymptotic convergence of the hierarchy for the hypercube.

Theorem 2.1 (Klep, Schweighofer)**.** *Let $p \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric and $g_i = 1 - X_i^2$ for $i = 1, \dots, r$. If $\tau(p(\underline{A})) \geq 0$ for all tuples $\underline{A}$ of self-adjoint contractions in finite von Neumann algebras (N, τ) then for all $\varepsilon > 0$ we have $f + \varepsilon \in TQM(g)$.*

Additionally, the proof of Pironio et al. in the free case can be transferred to a constructive proof at least if $TQM(g)$ contains $1 - \sum X_i^2$.

Theorem 2.2. *Let $p, g_i \in \mathbb{R}\langle \underline{X} \rangle$ be symmetric and such that $TQM(g)$ contains $1 - \sum X_i^2$. Then $p_k \to p^{min}$ as $k \to \infty$. Furthermore, there exists a finite von Neumann algebra (N, τ) and a tuple $\underline{A} \in N^n$ of self-adjoint operators such that $g_i(\underline{A}) \succeq 0$ and $p^{min} = \tau(p(\underline{A}))$.*

The statement can likely be extended to the case where $TQM(g)$ contains $N - \sum X_i^2$ (for some $N \in \mathbb{N}$) instead of $1 - \sum X_i^2$ using an argument involving ultrapowers. Again, the underlying Hilbert space which is associated to the von Neumann algebra of the optimizing solution can be *infinite dimensional*. In summary, we get the same asymptotic behavior as in the free case, which indicates that optimizing the trace over finite von Neumann algebras is the appropriate setup instead of optimizing the trace only on matrices of finite size. Only if Connes' embedding conjecture is true, both optimization problems (finite and infinite dimensional) would always lead asymptotically to the same result.

Nevertheless, for applications one is in general more interested in finite dimensional solutions (as in the free case). For example, trace-positivity appears naturally in the dual formulation of the completely psd cone, which is a generalization of the cone of completely positive matrices and consists of symmetric matrices which have a Gram representation given by tuples of positive semidefinite matrices. This cone plays a crucial role in the investigation of quantum graph parameters. Unfortunately, there is almost no knowledge about when one can find a finite dimensional optimal solution apart from an analog flatness criterion as in the classical case. Therefore further investigation is needed.

References

[1] M.F. Anjos, J.B. Lasserre, *Handbook on Semidefinite, Conic and Polynomial Optimization* International Series in Operations Research & Management Science, **166** (2012), Springer.

[2] J.W. Helton, S. McCullough, *A Positivstellensatz for non-commutative polynomials*, Tansactions AMS **356** 9 (2004) 3721–3737.

[3] I. Klep, M. Schweighofer, *Connes' embedding problem and sums of hermitian squares* Advances in Mathematics **217** 4 (2008) 1816–1837.

[4] S. Pironio, M. Navascués, A. Acín, *Convergent relaxations of polynomial optimization problems with non-commuting variables* SIAM J. Optim. **20** 5 (2010) 2157–2180.

Invariant SOS

Frank Vallentin

Testing that a given polynomial is a sum of squares (SOS) is a fundamental computational problem in real algebraic geometry. Using the Gram matrix method this can be reduced to the feasibility problem of semidefinite optimization. Gatermann and Parrilo [1] developed a general theory to simplify the matrices occurring in the Gram matrix method when the polynomial at hand is invariant under the action of a finite matrix group.

In this talk—where I do not claim any originality (I simply think that it is beautiful mathematics on the boundary of pure and applicable mathematics which deserves to be more widely known)—I will consider this problem for a polynomial invariant under a finite group generated by reflections. In this case the computation can be done rather concretely on the basis of the Chevalley-Shephard-Todd theory of finite reflection groups (see for example the book by Humphreys [2]).

Here is the theorem: Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be a finite reflection group. It is acting on the polynomial ring $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x]$ by

$$(gp)(x) = p(g^{-1}x).$$

The invariant ring is

$$\mathbb{C}[x]^G = \{p \in \mathbb{C}[x] : gp = p \text{ for all } g \in G\}.$$

It is generated by basic generators $\sigma_1, \dots, \sigma_n$:

$$\mathbb{C}[x]^G = \mathbb{C}[\sigma_1, \dots, \sigma_n],$$

and it is even a free algebra. The coinvariant algebra is

$$\mathbb{C}[x]_G = \mathbb{C}[x]/(\sigma_1, \dots, \sigma_n)$$

which is a graded algebra of dimension $|G|$. In particular,

$$\mathbb{C}[x] = \mathbb{C}[x]^G \otimes \mathbb{C}[x]_G$$

holds. The action of G on the invariant algebra $\mathbb{C}[x]_G$ is equivalent to the regular representation of G. Let $\widehat{G}$ be the set of irreducible unitary representations of G up to equivalence. Then there are homogeneous polynomials

$$\varphi_{ij}^{\pi}, \quad \text{with} \quad \pi \in \widehat{G},\ 1 \leq i,j \leq d_\pi,$$

where d_π is the degree of π, which form a basis of the coinvariant algebra such that

$$g\varphi_{ij}^{\pi} = (\pi(g)_j)^{\mathsf{T}} \begin{pmatrix} \varphi_{i1}^{\pi} \\ \vdots \\ \varphi_{id_\pi}^{\pi} \end{pmatrix},\ i = 1,\dots,d_\pi$$

where $\pi(g)_j$ is the j-th column of the unitary matrix $\pi(g) \in \mathrm{U}(d_\pi)$. Now the cone of G-invariant SOS polynomials equals

$$\left\{ p \in \mathbb{R}[x] : p = \sum_{\pi \in \widehat{G}} \langle P^\pi, Q^\pi \rangle, P^\pi \text{ is Hermitian SOS matrix polynomial in } \sigma_i \right\}.$$

Here $\langle A, B\rangle = \mathrm{Tr}(B^*A)$ denotes the trace inner product, the matrix P^π is a Hermitian SOS matrix polynomial in the variables $\sigma_1,\dots,\sigma_n$, i.e. there is is a matrix L^π with entries in $\mathbb{C}[x]^G$ such that $P^\pi = (L^\pi)^* L^\pi$ holds and $Q^\pi \in (\mathbb{C}[x]^G)^{d_\pi \times d_\pi}$ is defined componentwise by

$$[Q^\pi]_{kl} = \sum_{i=1}^{d_\pi} \varphi_{ki}^{\pi} \overline{\varphi_{li}^{\pi}}.$$

The computational value of this theorem is that one only has to determine basic invariants $\sigma_1,\dots,\sigma_n$ and a basis φ_{ij}^{π} of the coinvariant algebra. These computations are *independent* of the degree of the polynomial p.

References

[1] K. Gatermann, P.A. Parrilo, *Symmetry groups, semidefinite programs, and sums of squares*, J. Pure Appl. Algebra **192** (2004), 95–128.

[2] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1992.

Theta ranks and matroid minors

Raman Sanyal

(joint work with Francesco Grande)

Let $V \subset \mathbb{R}^d$ be a finite configuration of points. Every linear function (polynomial of degree ≤ 1) which is non-negative on V has a representation as a sum-of-squares on V. That is, there are polynomials $h_1,\dots,h_m \in \mathbb{R}[\mathbf{x}]$ such that

$$\ell(p) \ = \ h_1^2(p) + h_2^2(p) + \cdots + h_m^2(p)$$

for all $p \in V$. The *Theta rank* $\mathrm{TH}(V)$ of V is the smallest k such that such a representation exists for all non-negative $\ell(\mathbf{x})$ and $\deg h_i \leq k$ for $i = 1,\dots,m$. It is easily seen that $\mathrm{TH}(V) \leq |V| - 1$ as $\sqrt{\ell(\mathbf{x})}$ can be interpolated by a polynomial of degree at most $|V|-1$ on V. This is a rough upper bound as can be verified at the 0/1-cube $V=\{0,1\}^d$ which has Theta rank $\mathrm{TH}(V)=1$. Since we are dealing with linear polynomials, basic convexity assures us that it suffices to only consider *facet-defining* linear functions, that is, those non-negative $\ell(\mathbf{x})$ such that $V\cap\{\ell(\mathbf{x})=0\}$ spans a hyperplane. Here, we also make the general assumption that V is not contained in a hyperplane. The *levelness* of V is defined as

$$\mathrm{lev}(V) \;:=\; \max\{|\ell(V)| : \ell \text{ facet-defining}\}$$

Geometrically, this is the minimal number l such that for every facet-defining hyperplane H, l parallel copies of H suffice to cover all of V. The Theta rank was introduced by Gouveia, Parillo, and Thomas [2] and it was shown that

$$\mathrm{TH}(V) \;\leq\; \mathrm{lev}(V)-1$$

and $\mathrm{TH}(V)=1$ in fact characterizes 2-level configurations. It is in general a (computationally) difficult problem to determine the theta rank of point configurations and we do not have an understanding of point configurations with given or bounded Theta rank.

In this talk I will consider special point configurations that arise from combinatorial objects whose structure allows for insights into geometric/combinatorial properties of the Theta rank. More precisely, we consider *matroid basis configurations.* A *matroid* $M=(E,\mathcal{B})$ is a pair consisting of a finite ground set E and a collection of basis $\mathcal{B}\subseteq 2^E$ subject to the basis exchange condition: For any $B_1,B_2\in\mathcal{B}$ and $x\in B_1\setminus B_2$ there is a $y\in B_2\setminus B_1$ such that $(B_1\setminus x)\cup y\in\mathcal{B}$. Matroids make prominent appearances in algebraic geometry [1], combinatorial optimization [5], and geometric combinatorics [4], to name just a few. There is a multitude of characterizations of matroids and we refer the reader to Oxley's book [4] for more. The associated point configuration is then $V_M=\{\mathbf{1}_B\in\{0,1\}^E : B\in\mathcal{B}\}$ where $\mathbf{1}_B$ is the characteristic vector of $B\subseteq E$. If $M(G)$ is the matroid associated to a connected graph G, then $\mathcal{B}$ is the collection of spanning trees of G and the convex hull of $V_{M(G)}$ is the spanning tree polytope.

Similar to graphs, there is a rich theory of *minors* of matroids, i.e., matroids obtained by deletion $M\backslash e$ or contraction M/e of elements $e\in E$. Minors are used to describe forbidden substructures for 'minor-closed' properties. Combining geometric and combinatorial reasoning, we show the following.

Theorem. The class of matroids with Theta rank $\leq k$ as well as the class of matroids with levelness $\leq k$ are minor-closed.

For graphic matroids, the Graph Minor Theorem states that the list of forbidden minors is finite. For matroids finiteness does not necessarily hold. For the first non-trivial instance, we can nevertheless give a complete and finite list of forbidden minors.

Theorem. The four rank-3 matroids on 6 elements $M(K_4), \mathcal{W}^3, P_6, Q_6$ are the forbidden minors for the class of Theta rank = 1 matroids.

Restricted to graphs, this recover the ubiquitous *serial-parallel* networks. As an application, we give the following two equivalent characterizations of this class of matroids:

The Theta rank 1 matroids are precisely those matroids M for which the variety V_M is cut out by quadrics. This is a necessary but in general not a sufficient condition for Theta rank 1; see [2].

The Theta rank 1 matroids are precisely those matroids M for which the matroid base polytope conv(V_M) has *minimal PSD-rank* in the sense of [3]. Again, in general, this implication is strict.

References

[1] Gel′fand, I.M., Goresky, R.M., MacPherson, R.D., Serganova, V.V., *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63**, no. 3, 301–316, 1987.
[2] Gouveia, J., Parrilo, P.A., Thomas, R.R., *Theta bodies for polynomial ideals*, SIAM J. Optim. **20**, no. 4, 2097–2118, 2010.
[3] Gouveia, J., Robinson, R.Z., Thomas, R.R., *Polytopes of Minimum Positive Semidefinite Rank*, Discrete Comput. Geom. **50**, no. 3, 679–699, 2013.
[4] Oxley, J., *Matroid Theory*, Second Edition, Oxford University Press, New York, 2011.
[5] Schrijver, A., *Combinatorial optimization, Polyhedra and efficiency*, Vol. B, Matroids, trees, stable sets, Chapters 39–69, Algorithms and Combinatorics **24** B, Springer-Verlag, Berlin, 2003.

Reporter: Rainer Sinn

Participants

Marta Abril Bucero
INRIA Méditerranée
2004 route des Lucioles
06560 Sophia-Antipolis Cedex
FRANCE

Dr. Amir Ali Ahmadi
Dept. of Business Analytics & Math. Sciences
IBM Thomas J Watson Research Center
1 Rogers Street
Cambridge MA 02142
UNITED STATES

Prof. Dr. Joseph A. Ball
Department of Mathematics
Virginia Polytechnic Institute & State University
Blacksburgh, VA 24061
UNITED STATES

Prof. Dr. Greg Blekherman
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
UNITED STATES

Prof. Dr. Petter Bränden
Department of Mathematics
Royal Institute of Technology
100 44 Stockholm
SWEDEN

Sabine Burgdorf
CWI
Postbus 94079
1090 GB Amsterdam
NETHERLANDS

Prof. Dr. Laurent Busé
INRIA Sophia Antipolis
EPI Galaad
2004 route des Lucioles
06902 Sophia Antipolis Cedex
FRANCE

Prof. Dr. Jaka Cimpric
FMF - Matematika
University of Ljubljana
Jadranska 19
1111 Ljubljana
SLOVENIA

Prof. Dr. Etienne de Klerk
Tilburg University
Faculty of Economic Sciences
Warandelaan 2
P.O. Box 90153
5000 LE Tilburg
NETHERLANDS

Prof. Dr. Michael A. Dritschel
School of Mathematics and Statistics
The University of Newcastle
Newcastle upon Tyne NE1 7RU
UNITED KINGDOM

Prof. Dr. Ken Dykema
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Dr. Charu Goel
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz
GERMANY

Prof. Dr. Danielle Gondard
Institut de Mathématiques
Université Pierre et Marie Curie
Boite 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Sebastian Gruler
Fachbereich Mathematik u. Statistik
Universität Konstanz
Postfach 5560
78457 Konstanz
GERMANY

Christoph Hanselka
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz
GERMANY

Prof. Dr. J. William Helton
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. Didier Henrion
LAAS - CNRS
7, Avenue du Colonel Roche
31077 Toulouse Cedex 4
FRANCE

Prof. Dr. Roland Hildebrand
Laboratoire Jean Kuntzmann
Université Joseph Fourier
Tour IRMA
B.P.53
38041 Grenoble Cedex
FRANCE

Dr. Maria Infusino
Department of Mathematics
University of Reading
Whiteknights
P.O.Box 220
Reading RG6 6AX
UNITED KINGDOM

Kai Kellner
Goethe-Universität Frankfurt
FB 12 - Institut für Mathematik
Robert-Mayer-Str. 10
Postfach 111932
60325 Frankfurt am Main
GERMANY

Milan Korda
Ecole Polytechnique Fédérale de
Lausanne (EPFL)
STI IGM LA3,
ME C2 396 (Bâtiment ME)
Station 9
1015 Lausanne
SWITZERLAND

Prof. Dr. Salma Kuhlmann
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz
GERMANY

Mario Kummer
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz
GERMANY

Dr. Jean Bernard Lasserre
LAAS - CNRS
7, Avenue du Colonel Roche
31077 Toulouse Cedex 4
FRANCE

Prof. Dr. Monique Laurent
Centrum Wiskunde & Informatica
(CWI)
Postbus 94079
1090 GB Amsterdam
NETHERLANDS

Dr. Maria Lopez-Quijorna
Fachbereich Mathematik u. Statistik
Universität Konstanz
Postfach 5560
78434 Konstanz
GERMANY

Prof. Dr. Murray Marshall
Department of Mathematics
University of Saskatchewan
106 Wiggins Road
Saskatoon Sask. S7N 5E6
CANADA

Prof. Dr. Bernard Mourrain
INRIA Sophia Antipolis
2004 route des Lucioles
06902 Sophia Antipolis Cedex
FRANCE

Simone Naldi
LAAS-CNRS
7, Avenue du Colonel Roche
31077 Toulouse Cedex 4
FRANCE

Dr. Tim Netzer
Mathematisches Institut
Universität Leipzig
04081 Leipzig
GERMANY

Dr. Dmitrii V. Pasechnik
Department of Computer Science
Oxford University
Wolfson Building
Parks Road
Oxford OX1 3QD
UNITED KINGDOM

Dr. Juan Francisco Pons Llopis
Lab. de Mathématiques et leurs
Applications
Université de Pau et des Pays de l'Adour
Avenue de l'Université
P.O. Box 1155
64013 Pau Cédex
FRANCE

Prof. Dr. Janez Povh
Fakulteta za informacijske studije v
Novem mestu
Sevno 13, Ulica talcev 3
8000 Novo Mesto
SLOVENIA

Prof. Dr. Victoria Powers
Department of Mathematics &
Computer Science
Emory University
400, Dowman Dr.
Atlanta, GA 30322
UNITED STATES

Prof. Dr. Alexander Prestel
Fachbereich Mathematik u. Statistik
Universität Konstanz
Postfach 5560
78457 Konstanz
GERMANY

Prof. Dr. Bruce Reznick
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana, IL 61801-2975
UNITED STATES

Prof. Dr. Mohab Safey El Din
Université Pierre et Marie Curie
UPMC - LIP6, project POLSYS,
Boite courrier 169
4 Place Jussieu, UFR 920
75252 Paris Cedex 05
FRANCE

Prof. Dr. Raman Sanyal
Institut für Mathematik
Freie Universität Berlin
Arnimallee 2
14195 Berlin
GERMANY

Prof. Dr. Claus Scheiderer
Fakultät für Mathematik
Universität Konstanz
78457 Konstanz
GERMANY

Prof. Dr. Konrad Schmüdgen
Fakultät für Mathematik/Informatik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig
GERMANY

Prof. Dr. Markus Schweighofer
FB Mathematik u. Statistik
Universität Konstanz
78457 Konstanz
GERMANY

Eli Shamovich
Department of Mathematics
Ben Gurion University of the Negev
PO BOX 653
Beer Sheva 84105
ISRAEL

Rainer Sinn
FB Mathematik u. Statistik
Universität Konstanz
Postfach 5560
78457 Konstanz
GERMANY

Prof. Dr. Andreas B. Thom
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04103 Leipzig
GERMANY

Prof. Dr. Philippe Trébuchet
Université Pierre et Marie Curie
UPMC - LIP6, Equipe APR
Boite courrier 169
4 Place Jussieu, UFR 920
75252 Paris Cedex 05
FRANCE

Prof. Dr. Elias Tsigaridas
Université Pierre et Marie Curie
UPMC - LIP6, project POLSYS,
Boite courrier 169
4 Place Jussieu, UFR 920
75252 Paris Cedex 05
FRANCE

Prof. Dr. Frank Vallentin
Mathematisches Institut
Universität zu Köln
50931 Köln
GERMANY

Prof. Dr. Jean Vallès
Lab. de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
BP 1155
64013 Pau Cedex
FRANCE

Prof. Dr. Victor Vinnikov
Department of Mathematics
Ben-Gurion University of the Negev
P.O.Box 653
Beer-Sheva 84105
ISRAEL

Tillmann Weisser
Fachbereich Mathematik u. Statistik
Universität Konstanz
Postfach 5560
78457 Konstanz
GERMANY

Prof. Dr. Hugo J. Woerdeman
Department of Mathematics
Drexel University
Korman Center 206
3141 Chestnut St.
Philadelphia, PA 19104
UNITED STATES

Prof. Dr. Yosef N. Yomdin
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76 100
ISRAEL