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Discrete Geometry

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Abstract. Several significant new developments have been reported in many branches of discrete geometry at the workshop. The area has strong connections to other fields of mathematics for instance topology, algebraic geometry, combinatorics, and harmonic analysis. Discrete geometry is very active with hundreds of open questions and many solutions. There was a large number of young participants eager to work on these questions, and the future of discrete geometry is very safe.

Mathematics Subject Classification (2010): 52-xx.

Introduction by the Organisers

Discrete Geometry is a classic yet modern and rapidly developing field of mathematics. It deals with the structure and complexity of discrete geometric objects like finite point sets in the plane and intersection patterns of convex sets in high dimensional spaces. Classical problems such as Kepler's conjecture or Hilbert's third problem on decomposing polyhedra, as well as works by Minkowski, Steinitz, Hadwiger, and Erdős form the heritage of this area. In the last decade several outstanding problems have been solved. For instance, Erdős distinct distance problem was solved by Guth and Katz using methods of algebraic geometry. The solution gave impetus to the use of algebra in discrete geometry, and several lectures at the workshop demonstrated this new phenomenon. Another example is the use of algebraic (often equivariant) topology in discrete geometry and important developments in this area and in algorithmic aspects of topology have been reported on the meeting. Also, topological generalization of the first selection lemma by

Gromov has lead to further significant results in this direction. Moreover, semialgebraic relations and their use in discrete geometry form a new area of research leading, for instance, to a powerful and surprising extension of Szemerédi's famous regularity lemma in discrete geometry. Connections to symplectic geometry and Mahler's conjecture and billiard trajectories have been reported.

Discrete geometry is an interdisciplinary area and has many relations to other fields of mathematics like algebra, topology, combinatorics, computational geometry, probability and discrepancy theory. It is also in the front line of applications like geographic information systems, mathematical programming, coding theory, solid modeling, computational structural biology, and crystallography.

The workshop had 52 participants. There was a series of eight survey talks giving an overview of some important developments in discrete geometry and related areas:

- Karim Adiprasito: Some observations on the geometry and combinatorics of simplicial polytopes
- Roman Karasev: Bang's problem and symplectic invariants
- Nabil Mustafa: The use of geometric separators for combinatorial optimization problems
- János Pach: Three cornerstones of extremal graph theory
- Marcus Schaefer: $\exists\mathbb{R}$, or the real logic of drawing graphs
- Eric Sedgwick: Embeddability in $\mathbb{R}^3$ is decidable
- Joshua Zahl: Space curve arrangements with many incidences
- Günter M. Ziegler: Tight and non-tight topological Tverberg type theorems

In addition, there were 23 shorter talks and a problem session on Tuesday evening (chaired by Günter Rote). The collection of open problems resulting from this session can be found in the report. The program left enough time for research and discussions in the friendly and stimulating atmosphere of the Oberwolfach Institute. In particular, there were several informal sessions, organized and attended by smaller groups of the participants, on specific topics of common interest. On Wednesday we had a very pleasant excursion up North in the valley.

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Workshop: Discrete Geometry

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Abstracts

Some observations on the geometry and combinatorics of simplicial polytopes

Karim Adiprasito

Define the g-vector $g(\Delta) = (g_0(\Delta), \cdots, g_d(\Delta))$ of a simplicial $(d-1)$-complex using the relation $g_i(\Delta) = \sum_{k=0}^{i} (-1)^{i-k} \binom{d+1-k}{i-k} f_{k-1}(\Delta)$, where f_i denotes the number of i-dimensional elements (called faces) of a simplicial or polytopal complex.

The study of simplicial polytopes was revolutionized by the work of McMullen Stanley and Billera–Lee, who completely characterized the face numbers of simplicial polytopes by characterizing the numbers g_i, $i \le \frac{d}{2}$ as the ranks of the graded components of some commutative polynomial ring generated in degree 1. Nevertheless, this connection between combinatorics and algebra is mostly numeric, and remains to be understood better.

In my talk, I expanded on this connection by providing a inequality between the Betti numbers of induced subcomplexes of a simplicial polytope and the g-numbers. As a corollary, we obtain a quantitative generalization of the celebrated generalized lower bound theorem. In particular, I also present a proof of a conjecture of Gil Kalai concerning the relation of g-numbers and the shape of a polytope, previously established with J. Samper using different methods.

Elementary approach to closed billiard trajectories in asymmetric normed spacess

Arseniy Akopyan

(joint work with Alexey Balitskiy, Roman Karasev, and Anastasia Sharipova)

We consider billiards in convex bodies and estimate the minimal length of a closed billiard trajectory. This kind of estimates is rather useful in different practical applications, see further references on this subject in [3].

In [1] Shiri Artstein-Avidan and Yaron Ostrover presented a unified symplectic approach to handle billiards in a convex body $K \in V$ (here V is a real vector space), whose trajectory length (and therefore the reflection rule) is given by a norm with unit ball T° (polar to a body $T \in V^*$ containing the origin); throughout this paper we use *possibly non-standard* notation for this norm $\|\cdot\|_T$ with T lying in the dual space. We denote by $\xi_T(K)$ the minimal $\|\cdot\|_T$-length of a closed billiard trajectory in K.

We emphasize that in this work the norm need not be symmetric, that is need not satisfy $\|q\| = \|-q\|$. The other possible term is "Minkowski billiard", but Minkowski norms are sometimes assumed to be symmetric and we want to distinguish from this particular case.

We use elementary and efficient approach, developed by K. Bezdek and D. Bezdek in [3] for the Euclidean norm. It turns out that this approach remains valid without change for possibly asymmetric norm, it allows to give elementary proofs of most results of [1], worry less about the non-smoothness issues.

Consider an n-dimensional real vector space V, a smooth convex body $K \subset V$, and define

$$\mathcal{P}_m(K) = \{(q_1, \dots, q_m) : \{q_1, \dots, q_m\} \text{ does not fit into } \alpha K + t \text{ with } \alpha \in (0,1),\ t \in V\}.$$

Theorem 1. *For smooth convex bodies $K \in V$ and $T \in V^*$, the length of the shortest closed billiard trajectory in K with norm $\|\cdot\|_T$ equals*

$$\xi_T(K) = \min_{m \geq 1} \min_{P \in \mathcal{P}_m(K)} \ell_T(P).$$

Moreover, the minimum is attained at $m \leq n+1$.

Here we mention several corollaries from this Theorem.

The monotonicity of the shortest billiard trajectory (Folklore)**.**

$$\xi_T(K) \leq \xi_T(L) \text{ when } K \subseteq L$$

The Brunn–Minkowski-type inequality (S. Artstein-Avidan and Y. Ostrover, 2011)**.** *For any two convex smooth bodies $K_1, K_2 \subseteq \mathbb{R}^n$, one has:*

$$\xi_T(K_1 + K_2) \geq \xi_T(K_1) + \xi_T(K_2).$$

Moreover, equality holds if and only if their exists a closed curve which, up to homothety, is a length-minimizing billiard trajectory in both K_1 and K_2.

We will mention the theorem which motivate the authors for developing their technique.

Theorem 2 (D. Bezdek and K. Bezdek, 2009)**.** *Let K be a convex body in $\mathbb{R}^n$. Then any of the shortest billiard trajectories in K is of period at most $n+1$.*

Here is the corollary of this theorem.

Corollary (A. Akopyan, B. Balitskiy, R. Karasev, A. Sharipova, 2013)**.** *Any shortest billiard trajectory in the body of constant width 1 on the Euclidean plane has period 2.*

Open Question. *Is the same true for bodies of constant width in higher dimensions?*

This technique also gives an elementary proof for the following theorem which related with the Mahler conjecture [2].

Theorem 3 (S. Artstein-Avidan, R. Karasev, Y. Ostrover, 2013)**.** *If K and T are centrally symmetric and polar to each other ($T = K^\circ$) then K is 2-periodic with respect to T and $\xi_T(K) = 4$. Moreover, a 2-bouncing billiard trajectory passes through every point on ∂K.*

In other words, the shortest billiard trajectory in the unit ball in the Minkowski space has length 4.

For non-symmetric case we can prove the following.

Theorem 4 (A. Akopyan, B. Balitskiy, R. Karasev, A. Sharipova, 2013)**.** *If $K \subset \mathbb{R}^n$ is a convex body containing the origin in its interior then*

$$\xi_{K^\circ}(K) \geq 2 + 2/n,$$

and the bound is tight.

This approach can help to prove the existence of billiard trajectories in non-smooth convex bodies.

Observation (A. Akopyan, B. Balitskiy, R. Karasev, A. Sharipova, 2013)**.** *Suppose K is convex body in $\mathbb{R}^d$ with property that for any point $q \in \partial K$ either ∂K is smooth at q, or the cone of unit tangent vectors to K at q has diameter less than $\pi/2$. Then K has a closed billiard trajectory.*

Here we assume the "standard" definition of billiard, that is all bounces happens in smooth points of the boundary.

Open Question. *Is it possible to extend Bezdeks' approach to the Hyperbolic or Spherical geometry. Or Riemannian and Finsler manifolds?*

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Small subset sums

Gergely Ambrus

(joint work with Imre Bárány, Victor Grinberg)

Consider a finite dimensional, real normed space: that is, $\mathbb{R}^d$ endowed with a symmetric norm $\|.\|$, whose unit ball is B, a symmetric, convex body in $\mathbb{R}^d$. Let V be a set of n vectors of norm at most one, whose centroid is at the origin: $V = \{v_1, \ldots, v_n\} \subset B$, with $\sum_1^n v_i = 0$. We are interested in finding a subset V_k of V with a fixed cardinality $k \in [n]$, so that the sum of the vectors in V_k has small norm. Steinitz's lemma [3, 2] guarantees that there exists an ordering of the vectors, along with each partial sum has norm at most d – hence, we can choose subsets of arbitrary cardinality for which the upper estimate d holds. The constant d is optimal in the Steinitz lemma. However, for the present question, we are able to prove a stronger, general estimate:

Theorem 1. *Let $\|.\|$ be a symmetric norm in $\mathbb{R}^d$, and let V be a set of n vectors of norm at most 1, which sum to 0. Then for every $k \in [n]$, there exists a subset*

$V_k \subset V$ consisting of k vectors, so that

$$\left\| \sum_{v \in V_k} v \right\| \le \begin{cases} d/2 & \text{if } d \text{ is even;} \\ (d+1)/2 & \text{if } d \text{ is odd.} \end{cases}$$

The proof uses the method linear dependencies, like many of the related results [1]. The estimate is sharp in the general case. An example showing that is the following: let $v_1, \dots, v_d$ be the vectors of the standard orthonormal basis, and take $v_{d+1} = (-1, \dots, -1)$. The vector set V consists of $\{v_1, \dots, v_{d+1}\}$. Define $\|.\|$ to be the norm whose unit ball is $\operatorname{conv}\{\pm v_1, \dots, \pm v_{d+1}\}$, and set k to be $d/2$ when d is even, and $(d+1)/2$ for d odd. It is straightforward to deduce that the sum of any k vectors has norm at least k.

One expects that for more specific norms, a stronger estimate may be given. We indeed manage to verify this in two instances; the estimate given below is asymptotically sharp.

Theorem 2. *Let $\|.\|$ be the ℓ_2 (Euclidean) or the ℓ_∞ (maximum) norm in $\mathbb{R}^d$. There exists an absolute constant C, so that for any finite set V of vectors of norm at most 1, summing to 0, and for every k which is not greater than the cardinality of V, there exists a subset $V_k \subset V$ consisting of k vectors, whose sum has norm at most $C\sqrt{d}$.*

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On the Grünbaum mass partition problem

Pavle V. M. Blagojević

(joint work with Florian Frick, Albert Haase, Günter M. Ziegler)

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be an integrable density function on $\mathbb{R}^d$. A *mass distribution* on $\mathbb{R}^d$ determined by f is a finite Borel measure $\mu(X) \coloneqq \int_X f d\mu$.

An affine hyperplane $H = \{x \in \mathbb{R}^d : \langle x, v\rangle = a\}$, given by the vector $v \in \mathbb{R}^d$ and constant $a \in \mathbb{R}$, determines two open halfspaces

$$H^0 = \{x \in \mathbb{R}^d : \langle x, v\rangle > a\}, \quad H^1 = \{x \in \mathbb{R}^d : \langle x, v\rangle < a\}.$$

Let $\mathcal{H}$ be an arrangement of k hyperplanes in $\mathbb{R}^d$, and $g = (i_1, \dots, i_k) \in (\mathbb{Z}/2)^k = \{0,1\}^k$. The *orthant* determined by $\mathcal{H}$ and g is the intersection of halfspaces

$$\mathcal{O}_g^{\mathcal{H}} = H_1^{i_1} \cap \dots \cap H_k^{i_k}.$$

An arrangement of hyperplanes $\mathcal{H} = \{H_1, \dots, H_k\}$ *equiparts* the collection of mass distributions $\mathcal{M} = \{\mu_1, \dots, \mu_j\}$ in $\mathbb{R}^d$ if for every $g \in (\mathbb{Z}/2)^k$ and every $\ell \in \{1, \dots, j\}$

$$\mu_\ell(\mathcal{O}_g^{\mathcal{H}}) = \tfrac{1}{2^k}\mu_\ell(\mathbb{R}^d).$$

As a generalization of the Ham-Sandwich theorem, Grünbaum [3, Sec. 4.(v)] suggested the following general mass partition problem.

Problem. Determine the function $\Delta : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\begin{aligned}\Delta(j,k) = {} & \text{minimal dimension } d \text{ such that for every collection of } j \text{ mass} \\ & \text{distributions } \mathcal{M} = \{\mu_1, \dots, \mu_j\} \text{ in } \mathbb{R}^d \text{ there exists a hyperplane} \\ & \text{arrangement } \mathcal{H} = \{H_1, \dots, H_k\} \text{ in } \mathbb{R}^d \text{ that equiparts } \mathcal{M}.\end{aligned}$$

In this language the Ham-Sandwich theorem states that $\Delta(d,1) = d$. A lower bound for the function Δ, based on measures concentrated along a moment curve in $\mathbb{R}^d$, was given by Avis [1]:

$$\tfrac{2^k-1}{k} j \le \Delta(j,k).$$

The best upper bound to date, based on a Fadell–Husseini index calculation, is due to Mani-Levitska et al. [5, Thm. 39]:

$$\Delta(2^e + r, k) \le 2^{e+k-1} + r.$$

In this talk we present two results from [2]. The first result is a degree based proof of a slight generalization of a result of Hadwiger [4].

Theorem 1. $\Delta(2,2) = 3$.

The second result is a correct and elementary proof for a claim announced by Živaljević [6, Thm. 2.1].

Theorem 2. $\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2$.

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Geometric Permutations of Non-Overlapping Unit Balls Revisited

Otfried Cheong
(joint work with Jae-Soon Ha, Xavier Goaoc, Jungwoo Yang)

Given four congruent balls A, B, C, D in $\mathbb{R}^d$ that have disjoint interior and admit a line that intersects them in the order $ABCD$, we show that the distance between the centers of consecutive balls is smaller than the distance between the centers of A and D. This allows us to give a new short proof of the previously proven result [1] that n interior-disjoint congruent balls admit at most three geometric permutations, two if $n \geq 7$. We also make a conjecture that would imply that $n \geq 4$ such balls admit at most two geometric permutations, and show that if the conjecture is false, then there is a counter-example of a highly degenerate nature.

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A point in a (nd)-polytope is the barycenter of n points in its d-faces

Michael Gene Dobbins

In this talk I show that for any positive integers n, d and any target point in a (nd)-dimensional convex polytope P, it is always possible to find n points in the d-dimensional faces of P such that the center of mass of these points is the given target point. Equivalently, the n-fold Minkowski sum of the d-skeleton of P is a copy of P scaled by n. This verifies a conjecture by Takeshi Tokuyama, and may be viewed as loosely analogous to Carathéodory's Theorem. The proof uses equivariant topology.

On a (p, q)-property for hypergraphs

Vladimir L. Dol'nikov
(joint work with Ilya I. Bogdanov)

In this talk we will tell about the chromatic and piercing number of an r-graphs with a (p, q)-property.

The notion of a (p, q)-property was initially introduced by Hugo Hadwiger and Hans Debrunner [1, 2] for families of convex subsets in $\mathbb{R}^d$ in a connection with an investigation of Helly and Helly–Gallai numbers of these families.

Definition 1. Let p and q be integers such that $p \geq q \geq 2$. We say that a family of sets $\mathcal{F}$ has a (p, q)-property and write $\mathcal{F} \in \Pi_{p,q}$ provided $\mathcal{F}$ has at least p members, and among every p members some q have a common point.

By definition, put $\tau(\mathcal{F}) = \inf_{A \bigcap X \neq \varnothing,\ A \in \mathcal{F}} |X|$. The number $\tau(\mathcal{F})$ is called the *piercing number* of a family $\mathcal{F}$ and

$$M(p,\ q,\ \mathcal{F}) = \sup_{\mathcal{F}_0 \subset \mathcal{F}, \mathcal{F}_0 \in \Pi_{p,q}} \tau(\mathcal{F}_0),$$

where $\mathcal{F}$ is a certain family of convex sets in $\mathbb{R}^d$.

By Helly's Theorem, $M(p,\ p,\ \mathcal{F}) = 1$, where $p \geq d+1$. If $q \leq d$, $d \geq 2$, then $M(p,\ q,\ \mathcal{F}) = \infty$ for the family of all convex sets $\mathcal{F}$ in $\mathbb{R}^d$.

Hadwiger and Debrunner proved [1] that $M(p,\ q,\ \mathcal{F}) = p - q + 1$, if $d + 1 \leq q \leq p < \frac{d}{d-1}(q-1)$. N.Alon and D.Kleitman [3] have solved Hadwiger-Debrunner conjecture and proved that if $d + 1 \leq q \leq p$, then $M(p,\ q,\ \mathcal{F}) < \infty$ for every $p,\ q,\ d \in \mathbb{N}$ and for the family of all convex sets $\mathcal{F}$ in $\mathbb{R}^d$.

Hadwiger and Debrunner considered [2] also other families of convex sets.

At first, with the abstract point of view this problems were considered in paper [4].

Let us recall necessary definitions.

Definition 2. A *hypergraph* is a pair $G = (V, E)$, where V is a set, and $E \subseteq 2^V$. The elements of V are called *vertices* of G, and the elements of E are called its *(hyper)edges*. A hypergraph $G = (V, E)$ is an *r-graph*) if all its edges have cardinality r. Thus, 2-graphs are just usual simple graphs.

Let $G = (V, E)$ be a hypergraph. A set of its vertices U is called *independent*, if it contains no edges. Evidently, a set $X = V \setminus U$, where U is a independent set, is a transversal of the family of edges E. By definition, put $\tau(G) = \tau(E)$.

A *coloring* (proper) of a hypergraph is a partition of the set V into several disjoint independent parts: $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_t$. The subsets $V_1, \ldots, V_t$ are called *colors*. The *chromatic number* $\chi(G)$ of a hypergraph G is the minimal number of colors in its coloring.

Let $G = (V, E)$ be a hypergraph, and p, q be integer numbers with $p \geq q \geq 1$. We say that G satisfies a *(p, q)-property* if $|V| \geq p$, and every subset $V' \subseteq V$ with $|V'| \geq p$ contains an independent subset with q elements.

If $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, then we have such hypergraph $G = (V, E)$. The set of vertices V is the family $\mathcal{F}$ and a subfamily $e \subset \mathcal{F}$ forms the edge if and only if $1 \leq |e| \leq d + 1$ and $\cap_{v \in e} v = \varnothing$.

Then a nonempty set of vertices (a subfamily) $U \subset V = \mathcal{F}$ is a independent set iff the subfamily U has a nonempty intersection. The chromatic number $\chi(G)$ of this hypergraph is equal to the piercing number $\tau(\mathcal{F})$ of the family $\mathcal{F}$.

Theorem 3. *Suppose r, p, q, $p \geq q \geq r$ are positive integers.*

(1) *If $p > \frac{r}{r-1}(q-1)$, then for every N there exists an r-graph G satisfying the (p, q)-property such that $\chi(G) > N$.*

(2) *If $p \leq \frac{r}{r-1}(q-1)$, then for every an r-graph G satisfying the (p, q)-property we have*

$$\chi(G) \leq \left\lceil \frac{p-q}{r-1} \right\rceil + 2.$$

(3) *If* $p \le \frac{r}{r-1}(q-1)$*, then there exist* r*-graphs satisfying the* (p,q)*-property such that*

$$\left\lceil \frac{p-q}{r-1} \right\rceil + 1 \le \chi(G).$$

Remark 1. If $r = 2$ and the conditions of the theorem are fulfilled, then $\chi(G) \le p-q+1$. And it is not hard to show that there exist a simple graph with a (p,q)-property, $p \le 2q-2$, such that $\chi(G) = p-q+1$.

Essentially, this result for $r = 2$ (points 2,3) was proved Hadwiger and Debrunner [1, 2] although they proved it for graphs of intersections of rectangles.

For arbitrary graphs this result follows from the paper of P. Erdös and T. Gallai [5], where they proved that in this case $\tau(G) \le p-q$.

Point 1 follows from a result of P. Erdös (1959).

First a nontrivial case is an estimate $\chi(G)$ for an r-graph with a $(2r, 2r-1)$-property. By the theorem 1 we have $2 \le \chi(G) \le 3$.

Is it true that such an r-graph is bichromatic (bipartite)? We can prove that such r-graph is bichromatic if $r = 2,3,4,5$.

For $r = 2,3$ this assertion follows from the paper P. Erdös and T. Gallai [5] . For $r = 4,5$ this assertion follows from this theorem.

Theorem 4. *Suppose* G *is a hypergraph such that* $|e_1 \cap e_2| \le 1$ *for every* $e_1, e_2 \in E$ *and* $|e| \ge 4$ *for every* $e \in E$*. Then* $\chi(G) = 2$.

Remark 2. The conditions $|e| \ge 4$ is important. If G is the finite projective plane of the order 2 and the edges are the lines, then $\chi(G) = 3$.

Definition 5. By definition, put $MH(p,\ q,\ ,r) = \sup_G \tau(G)$, where G is an r-graph with the (p,q)-property, $p \ge q \ge r$.

We can prove following theorems.

Theorem 6. *If* $p > \frac{r}{r-1}(q-1)$*, then* $MH(p,\ q,\ r) = \infty$*. If* $p \le \frac{r}{r-1}(q-1)$*, then* $MH(p,\ q,\ r) \le r(p-q)$*.*

Zs. Tuza [6] found best (asymptotically sharp) estimate for $p(q)$, when $MH(p,\ q,\ ,r) \le p-q, p \le p(q)$.

First a nontrivial estimate is an estimate $MH(p,\ q,\ r)$ for $p = 2r$ and $q = 2r-1$.

Theorem 4. If $r = 2,3$, then $MH(2r,\ 2r-1,\ r) = 1$, and $MH(2r,\ 2r-1,\ r) = r$ for $r > 3$ and there exist a finite projective plane of order $r-1$.

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Toward the Hanani–Tutte Theorem for Clustered Graphs

Radoslav Fulek

The weak variant of Hanani–Tutte theorem says that a graph is planar, if it can be drawn in the plane so that every pair of edges cross an even number of times. Moreover, we can turn such a drawing into an embedding without changing the order in which edges leave the vertices. We prove a generalization of the weak Hanani–Tutte theorem that also easily implies the monotone variant of the weak Hanani–Tutte theorem by Pach and Tóth [1]. Thus, our result can be thought of as a common generalization of these two neat results. In other words, we prove the weak Hanani-Tutte theorem for strip clustered graphs, whose clusters are linearly ordered vertical strips in the plane and edges join only vertices in the same cluster or in neighboring clusters with respect to this order. In order to prove our main result we first obtain a forbidden substructure characterization of embedded strip clustered planar graphs.

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Expansion for Simplicial Complexes

Anna Gundert

Roughly speaking, a graph is an expander if it is sparse and at the same time well-connected. Such graphs have found various applications, in theoretical computer science as well as in pure mathematics. Expander graphs have, e.g., been used to construct certain classes of error correcting codes, in a proof of the PCP Theorem, a deep result in computational complexity theory, and in the theory of metric embeddings. See, e.g., the surveys [10] and [14] for these and other applications.

In recent years, the combinatorial study of simplicial complexes - considering them as a higher-dimensional generalization of graphs - has attracted increasing attention and the profitability of the concept of expansion for graphs has inspired the search for a corresponding higher-dimensional notion, see, e.g., [9, 15, 22, 24]

The expansion of a graph G can be measured by the *Cheeger constant* [1]

$$h(G) := \min_{\substack{A\subseteq V\\ 0<|A|<|V|}} \frac{|V||E(A,V\setminus A)|}{|A||V\setminus A|}.$$

[1]Often the Cheeger constant is defined by $\phi(G) = \min_{A\subseteq V, 0<|A|\leq|V|/2} \frac{|E(A,V\setminus A)|}{|A|}$. Since $\phi(G)\leq h(G)\leq 2\phi(G)$, the two concepts are closely related. Both are also called *(edge) expansion ratio.*

Here $E(A, V \setminus A)$ is the set of edges with one endpoint in A and the other in $V \setminus A$. A straightforward higher-dimensional analogue is the following *Cheeger constant* of a k-dimensional simplicial complex X with *complete* $(k-1)$-skeleton, studied in [8, 22]:

$$h(X) := \min_{\substack{V = \sqcup_{i=0}^{k} A_i \\ A_i \neq \varnothing}} \frac{|V||F(A_0, A_1, \ldots, A_k)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_k|}.$$

Here $F(A_0, A_1, \ldots, A_k)$ is the set of k-dimensional faces of X with exactly one vertex in each set A_i.

For graphs, this combinatorial notion of expansion is connected to the spectra of certain matrices associated with the graph: the adjacency matrix and the Laplacian. This connection between combinatorial and spectral expansion properties of a graph is established, e.g., by the discrete Cheeger inequality [1, 2, 4, 25]. For a graph G with second smallest eigenvalue $\lambda(G)$ of the Laplacian $L(G)$ and maximum degree $d_{\max}$, it states that

$$\lambda(G) \leq h(G) \leq \sqrt{8 d_{\max} \lambda(G)}.$$

A different approach to generalizing expansion is hence to consider higher-dimensional analogues of graph Laplacians. Higher-dimensional Laplacians were first introduced by Eckmann [5] in the 1940s and have since then been used in various contexts, see [12] for an example. For a recent exposition and systematic development of the basic properties of combinatorial Laplacians, see also [11].

The Cheeger inequality for graphs has proven to be a useful tool. Computing the Cheeger constant is difficult, from the standpoint of complexity theory [20, 3] but often also for explicit examples. The lower bound – even though easy to prove – hence gives a helpful, polynomially computable, lower bound on the Cheeger constant. Parzanchevski, Rosenthal and Tessler [22] recently showed the following analogue of this lower bound of the Cheeger inequality for k-dimensional simplicial complexes with complete $(k-1)$-skeleton. Denote by $\lambda(X)$ the smallest non-trivial eigenvalue of the higher-dimensional Laplacian. More precisely, $\lambda(X)$ is the smallest eigenvalue of the upper Laplacian $L_{k-1}^{\mathrm{up}}(X)$ on $(B^{k-1}(X;\mathbb{R}))^{\perp}$, see, e.g., [11] for the relevant definitions. For a k-dimensional simplicial complex X with complete $(k-1)$-skeleton the result in [22] states that

$$\lambda(X) \leq h(X).$$

An extension to arbitrary complexes (k-complexes with non-complete $(k-1)$-skeleton) can be found in [8]. See [22] for a discussion of possible upper bounds.

A different higher-dimensional analogue of edge expansion, going by the name of *combinatorial or cohomological expansion*, is based on concepts from algebraic topology, more specifically on cohomological notions. It emerged in various contexts as a useful notion. Linial, Meshulam and Wallach [13, 21] used the combinatorial expansion of the complete complex, containing all possible simplices on a fixed set of vertices, to study the cohomological properties of random complexes. Gromov suggested this notion when examining more geometrical notions of expansion: Any expander graph possesses the following geometric overlap property.

When mapped to the real line $\mathbb{R}$, it exhibits a point in $\mathbb{R}$ that is covered by the images of a lot of edges. The higher-dimensional analogue of this situation is captured by the *overlap number* of a simplicial complex. Gromov [7] showed that any combinatorially expanding complex has a large overlap number. See also [19] for a more combinatorial treatment of Gromov's proof.

For a k-dimensional complex X with complete $(k-1)$-skeleton, this notion can be described[2] by

$$\min_{f \in C^{k-1}(X, \mathbb{Z}_2)} \frac{|V| \cdot |\delta_X f|}{|\delta_{K_n^k} f|}.$$

Here δ_X denotes the $\mathbb{Z}_2$-coboundary operator of a complex X, $|\cdot|$ denotes the Hamming norm, and K_n^k is the complete k-dimensional complex on n vertices (the k-skeleton of the $(n-1)$-simplex). As this seems to be an important and useful concept, one might wish for an analogue of the Cheeger inequality also for this notion of expansion. It was, however, shown that an analogue of the lower bound can not exist, see [9, 24], where infinite families of examples are presented whose combinatorial expansion tends to zero while the spectral expansion is bounded away from zero. [24] also contains a similar counterexample for an analogue of the upper bound.

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[2] Usually, one considers

$$\phi(X) := \min_{f \in C^{k-1}(X, \mathbb{Z}_2)} \frac{|\delta_X f|}{|[f]|},$$

where $|[f]| = \min\{|f + \delta g| : g \in C^{k-2}(X; \mathbb{Z}_2)\}$, which generalizes the parameter $\phi(G)$ for graphs mentioned in footnote 1. For k-complexes with complete $(k-1)$-skeleton, the two notions are closely related because of expansion properties of the complete complex, see [8] for more details.

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Bang's problem and symplectic invariants

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(joint work with Arseniy Akopyan and Fedor Petrov)

1. Introduction

We start from recalling the classical problem attributed to Alfred Tarski and Thøger Bang and the known results on this problem, in particular those of Keith Ball, that give motivation to the whole discussion in this text.

The earliest version of this problem appeared when Tarski studied [21, 22] certain degree of equivalence $\tau(x)$ of a unit square Q and a rectangle P of size $x \times \frac{1}{x}$, defined as the smallest number of parts one has to cut the rectangle into to assemble the square from the parts. To solve a particular case of this problem and show that $\tau(n) = n$ for natural numbers n, Henryk Moese [16] inscribed a disk K into Q and noticed that this disk cannot be covered by less than n parts P_i of P. The solution used the trick of projecting the sphere in $\mathbb{R}^3$ onto K and counting the areas of the preimages of P_i on the sphere. By the way, this gave the solution of what was called later "the Bang problem" for the round disk K and the Euclidean norm.

Bang had [9] a different (non-volumetric) solution of the more general problem: If a convex body $K \in \mathbb{R}^n$ is covered by planks $P_1, \dots, P_m$ (a plank is a set bounded by a pair of parallel hyperplanes) then the sum of Euclidean widths of the planks is at least the Euclidean width of K. After that, Bang conjectured [9] that whenever a convex body K is covered by planks $P_1, \dots, P_m$, the sum of relative widths of the planks is at least 1. Here the relative width is the width of P_i in the norm with the unit ball $K - K$ (the symmetrization of K), and this version would certainly imply the original result of Bang.

The best to date result on Bang's conjecture belongs to Ball [6], who established it for all centrally symmetric convex bodies K. For non-symmetric bodies the problem remains open.

There is essentially one general approach to Bang's problem known so far, designed by Bang himself. For any plank P_i we take an orthogonal (in a fixed Euclidean metric) segment I_i such that $|I_i|$ is slightly more than the width $w(P_i)$. The first easy step is to show that the Minkowski sum $I_1 + \dots + I_m$ can be translated to fit into any given convex body of minimal width 1; and the main lemma of Bang asserts that at least one point of this Minkowski sum $I_1 + \dots + I_m$ is not covered by P_i. The proof is given by optimizing a cleverly chosen quadratic function of m variables. This lemma immediately proves the Euclidean case of Bang's problem and is also used in Ball's proof of the general symmetric case. The same approach was used by Vladimir Kadets [15] to show that any convex covering of the unit Euclidean ball in $\mathbb{R}^n$ has the sum of inradii at least 1.

In contrast to the general case, the approach of Moese to the cases of dimension 2 and 3 is volumetric. The crucial observation is that for a plank P_i the area of its intersection with the round two-sphere $S^2 \subset \mathbb{R}^3$ is proportional to the plank width. It is also known that the volumetric approach fails in larger dimensions.

In this paper we are going to propose another "quantitative" approach to the Bang conjecture based on certain invariants of symplectic manifolds and Hamiltonian systems, introduced by Hofer and Zehnder (see their nice book [14]), with first nontrivial examples given previously by M. Gromov [13]. This approach has already proved to be useful in a series of works [4, 5, 3], and allows either to solve a problem in convex geometry by symplectic methods or provides a good intuition to pose the "right questions" in convex geometry.

Even for the known results by Ball [6, 8] such a new approach would be useful, because Ball's proofs are very long and technical. However, at the moment we are only able to handle these results in a particular case of "almost parallel planks", the general case being dependent on some conjectured property of symplectic capacities.

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2. Symplectic invariants

Let us try to relate the Bang conjecture to some notions of symplectic geometry. Denote by V the ambient vector space of the convex body K, and let V^* be its dual. Consider some norm $\|\cdot\|$ on V. Let $\|\cdot\|_*$ be the dual norm on V^*, with the unit ball B°. In the Bang conjecture the natural choice of the norm $\|\cdot\|$ is the norm with the unit ball $K-K$ (the Minkowski sum of K with its centrally symmetric image), but we do not restrict ourselves and allow arbitrary norms here.

We always assume that the norms $\|\cdot\|$ and $\|\cdot\|_*$ are sufficiently smooth. For the Bang conjecture this is not a problem, since the conjecture allows going to the limit. We also assume that K has sufficiently smooth boundary when this is needed in the argument.

Our idea is to start from a covering of K by planks $P_1,\dots,P_m$ with widths $w_1,\dots,w_m$ (measured in the norm $\|\cdot\|$) and show that a certain symplectic invariant of a certain subset of $V\times V^*$ is bounded in terms of $\sum_i w_i$.

First of all, the space $V\times V^*$ is, in more general terms, the cotangent space of the manifold V. The cotangent space always inherits the canonical symplectic structure, which in this particular case is given by the formula

$$\omega((q_1,p_1),(q_2,p_2)) = \langle q_1,p_2\rangle - \langle q_2,p_1\rangle.$$

Now we are going to consider the set $K\times B^\circ \subset V\times V^*$, which is a sort of a unit disk bundle over K. Now we want to bound the *displacement energy* of $K\times B^\circ$. The definition of the displacement energy operates with a time dependent Hamiltonian $H(q,p,t)$ on $V\times V^*\times[T_1,T_2]$, whose *total oscillation* is defined to be

$$\|H\| = \int_{T_1}^{T_2} \sup_{q,p} H(q,p,t) - \inf_{q,p} H(q,p,t)\, dt,$$

see [14, Ch. 5] or [19] for the detailed definitions. Sometimes the segment $[T_1,T_2]$ is normalized to be $[0,1]$, but actually this and the following definitions do not depend on this normalization. It is always possible to scale the time segment by α and multiply the Hamiltonian by $1/\alpha$ without changing the result of the Hamiltonian flow.

The *displacement energy* of $K\times B^\circ$, denoted by $e(K\times B^\circ)$, is, roughly speaking, the minimal $\|H\|$ such that the corresponding time dependent Hamiltonian flow φ_t takes $K\times B^\circ$ off itself, that is

$$(K\times B^\circ)\cap\varphi_{T_2}(K\times B^\circ)=\varnothing.$$

Again, more details can be found in [14, Ch. 5].

What we want to establish in order to attack the Tarski–Bang problem is the following:

Conjecture 1. *If K can be covered with a finite set of planks with the sum of relative widths equal to w then $e(K\times B^\circ)\le 2w$.*

Bang's conjecture would follow if we also estimate the value $e(K\times B^\circ)$ from below, in the particular case when $B=K-K$. This can indeed be done in some important cases, but the truth is that this does not work fully, see Section 3 for more details.

We cannot prove this conjecture so far, but we sketch an argument for a particular case of it:

Theorem 2. *The previous conjecture holds if the $\|\cdot\|_*$-unit normals $n_i \in V^*$ of the planks can be chosen so that for every nonnegative coefficients c_i, at least one of which is* 1 *the inequality holds:*

$$\left\| \sum_i c_i n_i \right\|_* \geq 1. \tag{1}$$

Let us call the assumption *almost parallel planks.* In the Euclidean case this is guaranteed by a simpler assumption that $(n_i, n_j) \geq 0$ for any pair of indices.

Proof. Let a plank P_i have width w_i. Consider the Hamiltonian $H_i(q,p,t)$ defined for $t \in [2i-2, 2i]$ so that H_i is independent of t and y, equals zero for q on one side of P, equals w_i for q on the other side of P_i, and changes linearly in q inside P_i. The effect of the corresponding (discontinuous!) flow is as follows: $(q,p) \in V \times V^*$ remains fixed if q is outside P_i and gets shifted by $(0, 2n_i)$, where $n_i = d_q H_i$ is the unit normal to P_i, for q inside P_i. From the definition it follows that the part $(K \cap P_i) \times B^\circ$ gets shifted outside $K \times B^\circ$, spending the total oscillation $2w_i$.

The idea is to shift this way everything outside $K \times B^\circ$ in a sequence of such steps for all planks P_i. The total oscillation of such a sequence of Hamiltonians is therefore twice the sum of widths. To make this idea work we need some care. First, the function $H_i(q,p,t)$ is not smooth in q and therefore the Hamiltonian flow is discontinuous. This could be remedied by a certain smoothening changing the value of dH in a small neighborhood of $\partial P_i \times V^*$; but after that we have to keep in mind that some parts near boundaries of planks are "incompletely shifted". A more serious problem, is that we have made several shifts and it usually happens that something, previously shifted outside $K \times B^\circ$, returns inside $K \times B^\circ$ on a subsequent shift. It is easy to check that nothing returns back under the assumption (1); and in this case the proof passes. ☐

Remark 3. Here we observe a strange phenomenon. The classical method of Bang works better when the planks are far from parallel, see [7] for an impressive example. But the displacement energy approach presented above likes the opposite situation, when the planks are almost parallel.

Remark 4. Yaron Ostrover has noted in the private communication that the proof of the above theorem does not use the convexity of K. This may be useful, though lower bounds for the symplectic invariants of $K \times B^\circ$ seem less accessible for non-convex K.

It is well known that the Hofer–Zehnder symplectic capacity $c_{HZ}(U)$ (see the definition and discussion in [14]) gives a lower bound for the displacement energy of U, where U is an open bounded set in $V \times V^*$. So the following version of the conjecture would also be sufficient for many purposes:

Conjecture 3. *If K can be covered with a finite set of planks with the sum of relative widths equal to w then $c_{HZ}(K \times B^\circ) \le 2w$.*

The ultimate version of this conjecture would be the subadditivity property of the Hofer–Zehnder (or similar) capacity:

Conjecture 4. *If a convex body $X \subset \mathbb{R}^{2n}$ in the standard symplectic space $\mathbb{R}^{2n}$ is covered by a finite set of convex bodies $\{X_i\}$ then, for some symplectic capacity,*

$$\sum_i c(X_i) \ge c(X).$$

It is easy to give many examples when this property is violated when some of X_i are non-convex, see the full version of this text. So here we expect a deep fact that essentially ties the convex geometry and symplectic geometry. We do not discuss the definitions of different capacities here, but in the next section we interpret them in a rather elementary way.

Why do we believe this subadditivity might be true? One evidence is the result of Keith Ball [8]: When the unit ball in $\mathbb{C}^n$ is covered by unitary cylinders Z_i of radii r_i then $\sum_i r_i^2 \ge 1$. Here $\mathbb{C}^n$ is endowed with some Hermitian metric and a unitary cylinder of radius r is an r-neighborhood of a complex hyperplane. A unitary cylinder is a particular case of a symplectic cylinder with any capacity equal to πr_i^2, so this is indeed a particular case of Conjecture 4.

It is also checked by hand that the subadditivity holds if we partition the standard ball $B^{2n} \subset \mathbb{R}^{2n}$ in two convex parts.

3. Billiards and capacity

Assuming Conjecture 3, in order to solve the Bang problem it remains to prove that $c_{HZ}(K \times B^\circ) \ge 2$, where B° is the polar to $K - K$. Fortunately, the paper [4] provides a nice elementary description of this capacity (all the bodies are assumed to be sufficiently smooth):

Theorem 5 (Artstein-Avidan, Ostrover, 2011)**.** *The Hofer–Zehnder capacity of $K \times B^\circ$ is equal to the length of the shortest closed billiard trajectory in K, where the length is measured in the norm $\|\cdot\|$ and the reflection rule reflects the momentum coordinate from one point on ∂B° to the other point on ∂B° by combining it with a multiple of the normal to ∂K at the hit point.*

Remark 5. In [4] closed geodesics of ∂K were also considered as a particular case of a billiard trajectory, with length measured with $\|\cdot\|$ norm. But in [2] it was shown that such closed trajectories can never be shorter than the ordinary bouncing trajectories.

It is possible to give the lower bounds for the lengths of closed billiard trajectories in a convex body K, measuring the lengths by a norm with unit ball B. This was done in [5, 2] by relatively elementary methods. Here we cite the results:

Theorem 6 (Artstein-Avidan, Karasev, Ostrover, 2013)**.** *Let $\|\cdot\|$ be a smooth norm and let its dual $\|\cdot\|_*$ be also smooth. Then any closed billiard trajectory in the unit ball B, being measured with $\|\cdot\|$, has length at least 4.*

Evidently, the segment $[q, -q] \in B$, where $\|q\| = 1$, passed forth and back is a closed billiard trajectory of length 4 in the norm associated with B. Theorem 6 asserts that this is the shortest one, and together with Conjecture 3 (or some other similar conjecture) would imply Ball's theorem from [6] about the Bang problem for centrally symmetric case. Clearly, this result together with Theorem 2 already gives a symplectic proof for the particular case of Ball's theorem, when the "almost parallel planks" assumption in (1) is imposed.

For possibly non-symmetric convex bodies a similar result was established in [2]. In this theorem we allow a norm to violate the reflexivity property $\|q\| = \|-q\|$, that is we consider a *Finsler norm*:

Theorem 7 (Akopyan, Balitskiy, Karasev, Sharipova, 2014). *Let $\|\cdot\|$ be a smooth non-symmetric norm in $\mathbb{R}^n$ and let its dual $\|\cdot\|_*$ be also smooth. Then any closed billiard trajectory in the unit ball K, measured with $\|\cdot\|$, has length at least $2+2/n$.*

See also [2] for the discussion of its relation to the non-symmetric case of Mahler's problem.

Now we prove one more estimate related to the non-symmetric case of Bang's problem. It resembles the previous one but does not seem to be equivalent.

Theorem 8. *Let K be a smooth strictly convex body in $\mathbb{R}^n$. Consider the norm with the unit ball $B = K - K$, then any closed billiard trajectory in K with this norm has length at least $1 + \frac{1}{n}$.*

Remark 6. This estimate is obviously tight for $n = 1, 2$, and is actually tight for $n \geq 3$, as it was checked by Yoav Nir [17, Ch. 4]. In fact, a closed polygonal line with vertices at the centers of facets of a simplex K is a closed billiard trajectory in K with respect to the norm with unit ball $K - K$.

Remark 7. Assuming something like Conjecture 3 this theorem would imply a weaker result than the Bang conjecture, that is the sum of relative widths of planks would be proved to be at least $\frac{n+1}{2n}$. This is not what was conjectured by Bang, but would be a good step toward the Bang conjecture. Again, for "almost parallel planks", like in Theorem 2, this weaker Bang conjecture with sum $\frac{n+1}{2n}$ already follows from Theorems 2 and 8.

In order to proceed we need a simple lemma:

Lemma 9. *Let K be a convex body in $\mathbb{R}^n$ and $\|\cdot\|$ be the norm with unit ball $K - K$. If $C \in \mathbb{R}^n$ is a connected graph with total $\|\cdot\|$-length h, then C can be covered by a translate of the homothet hK.*

Proof of the Lemma. We may assume that C has straight line segments as edges. For an edge $[a, b]$ the inequality

$$\|a - b\| \leq \delta$$

just means that $[a, b]$ can be covered with a translate of δK. So we cover all edges of C (that is the whole C) by translates $\delta_1 K + t_1, \dots, \delta_m K + t_m$ with

$$\delta_1 + \delta_2 + \dots + \delta_m \leq h.$$

Then we observe that if two sets $\delta_i K + t_i$ and $\delta_j K + t_j$ intersect then they can be covered by a single set $(\delta_i + \delta_j)K + t'$. Using the connectedness of C we can repeat this step several times to cover the whole C with a translate of $(\delta_1 + \dots + \delta_m)K$. □

Proof of Theorem 8. By [2, Theorem 2.1] the shortest closed billiard trajectory in K has at most $n+1$ bounce points $\{q_i\}_{i=1}^m$ and cannot be covered by a smaller positive homothet of K. By Lemma 9 we have:

$$\sum_{i=2}^{m} \|q_i - q_{i-1}\| \geq 1.$$

If L is the length of the closed polygonal line $q_1, q_2, \dots, q_m, q_1$ then the above inequality is a lower bound for L minus the length of the segment $[q_m, q_1]$. The same argument applies to any other segment, and since some of them has length at least $\frac{L}{n+1}$ (remember that $m \leq n+1$) then

$$\left(1 - \frac{1}{n+1}\right) L \geq 1,$$

that is $L \geq \frac{n+1}{n}$. □

As a more elementary example of this activity, we want to mention another result, implicit in [10]:

Theorem 10 (D. Bezdek, K. Bezdek., 2009)**.** *For a convex body K of constant width* 1 *in the plane with the Euclidean norm, any shortest closed billiard trajectory has length* 2 *and must be a diameter of K passed twice.*

As was noted by Alexey Balitskiy (private communication), the proof of [10, Theorem 1.2] proves this assertion as well. Example 11 shows that it cannot be generalized to arbitrary norm without additional assumptions. Also, the generalization of Theorem 10 for the Euclidean norm in dimensions more than 2 is open.

Returning to the Tarski–Bang problem we conclude that Conjecture 3 (or 1) would imply the following claim: If for a smooth strictly convex body K there are no closed billiard trajectories in K with $\|\cdot\|$-length less than 2 then the Bang conjecture holds for K. It is easy to verify the assumption when K is the Euclidean ball of unit diameter and $\|\cdot\|$ is the Euclidean norm. Theorem 6 gives an affirmative answer for the case of symmetric K (Ball's theorem), but the following example is unpleasant in view of the general Bang conjecture:

Example 11. *If K is the triangle in the plane, then the small triangle formed by its midpoints of sides is a closed billiard trajectory and has relative length* 3/2*. The triangle is not smooth, but it can be smoothened without increasing the number* 3/2 *too much. Thus the billiard approach is* not sufficient *to establish the Bang conjecture already in this simple case.*

Moreover, for the Euclidean norm, the triangle of unit width has a billiard trajectory along the midpoints of length $\sqrt{3}$. So the billiard approach fails even for the known case of the Bang theorem, which estimates the sum of Euclidean widths of the covering planks by the minimal width *of K.*

4. Two directions of planks

In this section we prove a particular case of Bang's problem with elementary methods. It is independent of the other parts of this paper, but we thought it makes sense to confirm another particular case of the conjecture. One may check that it does not follow from the result under the "almost parallel" assumption.

Theorem 12. *Let a convex body $K \subset \mathbb{R}^n$ be covered by a family of planks $P_1, \dots, P_m$, whose normals have only two distinct directions. Then the sum of widths of the planks in the norm with the unit ball $K - K$ is at least 1, that is the Bang conjecture holds in this case.*

Proof. If all the planks are parallel to each other then the assertion is evidently true. Assume there are two distinct normals $n_1, n_2 \in V^*$ (we put $V = \mathbb{R}^n$). Obviously, the projection

$$\pi : V \to \mathbb{R}^2, \quad \pi(x) = (n_1(x), n_2(x))$$

reduces the problem to the following planar case: The projection (denote it by K again) is inscribed in the unit square $abcd$ (let a be the left bottom and b be left top), that is K contains the points on every side of $abcd$. Let those points be p, q, r, s ($p \in ad$, $q \in ab$, $r \in bc$, $s \in cd$), we allow some of the to coincide. Assume K to be covered by a set of horizontal and vertical planks with sum of widths (now widths can be considered Euclidean) less than 1. Also choose such a covering with the minimal number of planks.

If there are only two planks then the result is well known, see [18] or [11, Lemma 10.1.1]. So we assume that there are k vertical planks and at least k horizontal planks (we interchange the axes if needed).

Consider the points of K not covered with the vertical planks, they split into $k+1$ convex sets $M_1 \cup M_2 \cup \dots \cup M_{k+1}$ ordered from left to right, we allow some of them to be empty. These sets have to be covered with horizontal planks and this reduces to cover their projection to the $0y$ axis with a set of segments. Definitely, one needs at most $k+1$ segments to cover those projections, and we now that k segments are really needed. Now consider the cases:

(1) The set $M_1 \ni q$ is nonempty and its projection to $0y$ has no intersection with the projections of other M_i's. Then one horizontal plank is needed to cover M_1 separately from the other parts. But it makes sense to replace this plank with a vertical one, indeed, the set M_1 contains the triangle qc_1d_1 homothetic to qcd, whose vertical and horizontal widths coincide. Therefore the vertical width of M_1 is at least its horizontal width. So we replace the horizontal plank of M_1 with a vertical one and merge this vertical plank with the first vertical plank in the list. After that the sum of widths does not increase and the number of planks does decrease.

(2) The case when the projection of $M_{k+1} \ni s$ to $0y$ does not intersect the other projections of M_i's is considered similarly.

(3) The set M_1 is empty and M_{k+1} is also empty. Then the projections of M_i's to $0y$ can be covered with $k-1$ segments, but we have assumed that the number of segments is at least k.
(4) $M_1 = \varnothing$, M_{k+1} is not empty and its projection to $0y$ intersects some of the projections of other M_i's. Again, in this case at most $k-1$ horizontal planks are sufficient.
(5) Similar to the previous case, when we interchange M_1 and M_{k+1}.
(6) Both the projections of $M_1 \ni q$ and $M_{k+1} \ni s$ to $0y$ are nonempty and both of them intersect other M_i's. Again, we know that we really need at least k horizontal planks to cover M_i's. This may only happen when the projections of M_1 and M_{k+1} do intersect and the projections of M_i's with $2 \le i \le k$ are disjoint from them and are disjoint from each other. Therefore a horizontal plank P_h cover both M_1 and M_{k+1}. Hence $P_h \ni q, s$. Other sets $M_2, \dots, M_k$ then have to be disjoint from P_h and the total number of needed horizontal planks is precisely k.

Interchanging the vertical and horizontal direction we find a vertical stripe $P_v \ni p, r$. The segment qs intersects the boundary of P_v at q_1, s_1, lying in their respective M_i, M_{i+1}. But then M_i and M_{i+1} intersect P_h, which is a contradiction.

□

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Steinhaus' circle lattice point problem, revisited

Hiroshi Maehara

In 1957, H. Steinhaus posed the following problem in elementary mathematics [2,3,4]. Is there a circle in the plane that contains in its interior exactly n lattice points, for any given n? (A lattice point means a point whose coordinate are all integers.) And Steinhaus himself proved that for every natural number n, there exists a circle of area n which contains in its interior exactly n lattice points (see Honsberger [1] p. 118). We prove the following.

Let X be a compact region of area n in the plane.

(1) If X is a strictly convex region (i.e., a convex region whose boundary curve contains no line segment) or a region bounded by an irreducible algebraic curve, then X can be translated to a position where it covers exactly n lattice points.
(2) If X is a (possibly concave) polygon or a general convex region, then X can be rotated and translated so that it covers exactly n lattice points.

Problem: Is there a planar region of area n such that no congruent copy of it can contain exactly n lattice points?

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Upper and Lower Bounds of Long Dual-Paths in Line Arrangements

Tillmann Miltzow
(joint work with Udo Hoffmann, Linda Kleist)

Given a line arrangement $\mathcal{A}$ up to lower order terms we show that there exists a path of length $n^2/3$ in the dual graph of $\mathcal{A}$ formed by its faces. This is tight. We also consider bichromatic line arrangements and define alternating paths as paths that alternatingly cross red and blue lines. We describe an example of a line arrangement with $3n$ blue and $2n$ red lines with no alternating path longer than $14n$. Further we show that a random coloring of any line arrangement has an alternating path of length $\Omega(n^2/\log n)$.

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About an Erdős-Grünbaum conjecture concerning piercing of non bounded convex sets

Luis Montejano
(joint work with Amanda Montejano, Edgardo Roldán-Pensado, Pablo Soberón)

In this talk, we will discuss the number of compact sets needed in an infinite family of convex sets with a local intersection structure to imply a bound on its piercing number, answering a conjecture of Erdős and Grünbaum. Namely, if in an infinite family of convex sets in $\mathbb{R}^d$ sets we know that out of every p there are q which are intersecting, we determine if having some compact sets implies a bound on the number of points needed to intersect the whole family. We also study variations of this problem.

The Use of Geometric Separators for Combinatorial Optimization Problems

Nabil H. Mustafa
(joint work with Rajiv Raman, Saurabh Ray)

Separators are by now a widely-used tool for designing efficient algorithms. The planar graph separator theorem of Lipton and Tarjan (1977) has found many uses in the design of exact and approximation algorithms for optimization problems. In this talk I will survey some recent work on the construction of new separators for geometric objects, as well as their use in algorithmic design for combinatorial optimization problems. Specifically, this series of recent work was initiated by the interesting result of Adamaszek and Wiese (2013), who proved a new separator theorem extending the separators of Fox and Pach (2009), which they then used to design a QPTAS for the geometric independent-set problem. Their separator theorem was made optimal and generalized independently by several authors (Har-Peled (2014), Mustafa, Raman and Ray (2014)). Further application of these separators in the design of QPTAS for the geometric set-cover problem was subsequently given by Mustafa, Raman and Ray (2014).

Three cornerstones of extremal graph theory

János Pach

By *Ramsey's theorem*, any system of n segments in the plane has roughly $\log n$ members that are either pairwise disjoint or pairwise intersecting. Analogously, by Ramsey's theorem on hypergraphs, any set of n points $p(1), \ldots, p(n)$ in the plane has a subset of roughly $\log \log n$ elements with the property that the orientation of $p(i)p(j)p(k)$ is the same for all triples from this subset with $i < j < k$. The elements of such a subset form the vertex set of a convex polygon. However, in both cases we know that there exist much larger "homogeneous" subsystems satisfying the above conditions.

By the *Kővári-Sós-Turán (and Erdős) theorem*, the incidence graph of n points and n lines in the plane has at most $n^{3/2}$ edges. The celebrated Szemerédi-Trotter theorem states that a much stronger result is true: the maximum number of incidences between n points and n lines in the plane is $O(n^{4/3})$, and the order of magnitude of this bound is tight.

The *Szemerédi regularity lemma* and its far reaching extensions belong to the most applicable tools of modern combinatorics. It follows that, for every $\varepsilon > 0$, the vertex set of any sufficiently large graph can be partitioned into a bounded number of almost equal parts such that the bipartite graphs induced by all but an at most ε fraction of all pairs of parts behave like random graphs (with an error of at most ε). However, for intersection graphs of segments, a much stronger partition theorem was proved by Pach and Solymosi: the bipartite graphs induced by all but an at most ε fraction of all pairs of parts are either complete or empty!

What is behind this favorable behavior? One of the common features of the above problems is that the underlying graphs and triple-systems are *semi-algebraic*, that is, they can be defined by a small number of polynomial equations and inequalities in terms of the coordinates of the segments, points, and lines. It turns out that (1) *Ramsey's theorem,* (2) the *Kővári-Sós-Turán theorem*, and (3) *Szemerédi's regularity lemma*, as well as a number of other related results in extremal combinatorics, can be substantially strengthened for semi-algebraic graphs and hypergraphs.

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Iterated bisections of simplices

Igor Pak
(joint work with Karim Adiprasito)

We survey known results in Numerical Analysis on the (Rivara) iterated bisection of simplices. The main problems on periodicity and degeneration are completely resolved in 2 dimensions, but until now remained open in higher dimensions. We report on the progress in 3 dimensions. Joint work with Karim Adiprasito.

Indecomposable coverings with unit discs

Dömötör Pálvölgyi
(joint work with János Pach)

Let $\mathcal{C}$ be a family of sets in $\mathbb{R}^d$, and let $P \subseteq \mathbb{R}^d$. We say that $\mathcal{C}$ is an *m-fold covering of P* if every point of P belongs to at least m members of $\mathcal{C}$. A 1-fold covering is simply called a *covering.*

Definition. A subset $C \subseteq \mathbb{R}^d$ is said to be *cover-decomposable* if there exists a positive integer $m = m(C)$ such that every m-fold covering of $\mathbb{R}^d$ with *translates* of C can be decomposed into two coverings.

The problem of characterizing cover-decomposable sets was proposed by Pach [4] in 1980. In [4], the following conjecture was made.

Conjecture (Pach)**.** Every planar convex set C is cover-decomposable.

Winkler [9] even suggested that for the unit disc $m(C) = 4$.
We disprove this conjecture by showing it does not hold even for the unit disk.[1]

Theorem 1. *The unit disk is not cover-decomposable.*

Our construction generalizes to most planar convex sets C.

Theorem 2. *Let C be an open plane convex set with a smooth boundary, which has two parallel supporting lines with positive curvature at the points of tangencies. Then C is not cover-decomposable.*

[1] In a 30 years old, unpublished manuscript [3], the opposite was claimed.

It easily follows from Theorem 1 that the unit ball in $\mathbb{R}^d$ is not cover-decomposable in any dimension $d \geq 3$. This weaker statement was first proved in [6].

On the positive side, it was shown in [5] that every centrally symmetric convex *polygon* (i.e., open polygonal region) is cover-decomposable. It took almost 25 years to generalize this statement to all convex polygons [8], [7]. One may first believe that there exists an absolute constant m' such that every m'-fold covering of the plane with translates of any convex polygon Q splits into two coverings. Since the unit disk can be approximated by convex n-gons with n tending to infinity, this would imply by compactness that every m'-fold covering of the plane with unit disks also splits into two coverings, contradicting Theorem 1. Therefore, we have the following.

Corollary. *Given a convex polygon Q, let $m(Q)$ denote the smallest positive integer m such that every m-fold covering of the plane with translates of Q can be decomposed into two coverings. Then* $\sup m(Q) = \infty$, *where the sup is taken over all convex polygons Q.*

The answer to the following question may still be positive.

Problem. Does there exist for any $n > 3$ an integer $m(n)$ such that every convex n-gon Q satisfies $m(Q) \leq m(n)$?

For any triangle T, there is an affine transformation of the plane that takes it to an equilateral triangle T_0, we have $m(T) = m(T_0)$ and, hence, $m(3)$ is finite. We have been unable to answer this question even for $n = 4$.

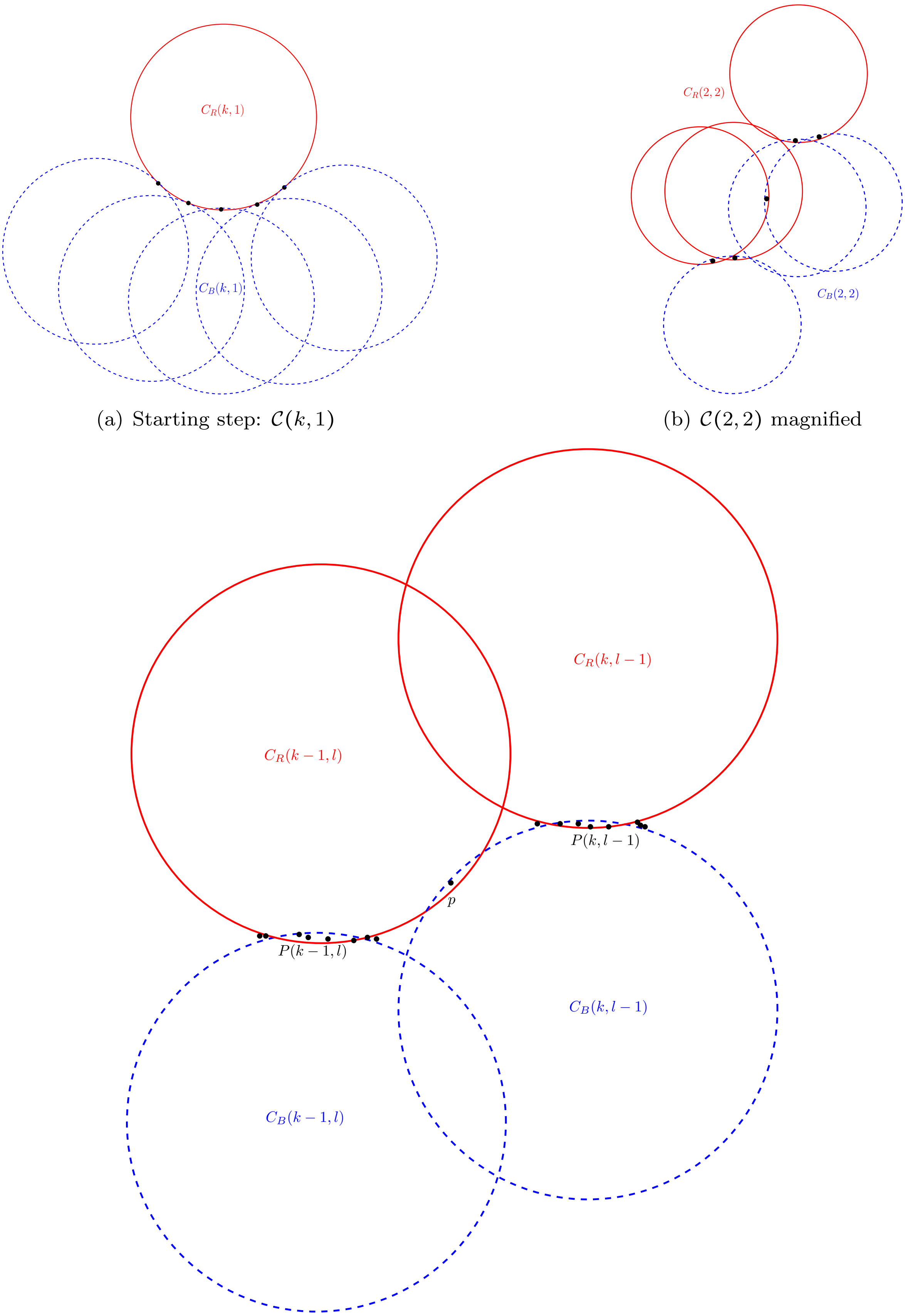
The fact that every open triangle is cover-decomposable was generalized by Keszegh and Pálvölgyi [1] in the following way: There is an absolute constant m' such that every m'-fold covering of the plane with homothetic copies of a triangle can be decomposed into two coverings.[2] Using the idea of the proof of our Theorem 1, Kovács [2] showed that the last statement cannot be extended to convex polygons with more than 3 sides. More precisely, for every such polygon Q and for every positive integer m, there is an m-fold covering of the plane with homothetic copies of Q that cannot be split into two coverings.

We end this report by giving an inductive proof by picture for the following asymmetric finite "dual" form of the above questions.

Lemma 3. *For any positive integers k, l and for any $\varepsilon > 0$, there is a finite point set P and a finite family of open unit disks $\mathcal{C} = \mathcal{C}_R \cup \mathcal{C}_B$ with the following properties.*

(1) *Any disk $C \in \mathcal{C}_R$ (resp. $\mathcal{C}_B$) contains precisely k (resp. l) points of P;*
(2) *For any coloring of P with red and blue, there is a disk in $\mathcal{C}_R$ such that all of its points are red or a disk in $\mathcal{C}_B$ such that all of its point are blue.*
(3) *Any two points of P are at a distance less than ε from each other.*
(4) *For any two disks $C, C' \in \mathcal{C}_R$ or $C, C' \in \mathcal{C}_B$, we have $d(C, C') < \varepsilon$.*
(5) *For any two disks $C \in \mathcal{C}_R$ and $C' \in \mathcal{C}_B$, we have $2 - \varepsilon < d(C, C') < 2$.*

[2] A *homothetic copy* of a set is a translate of a dilated copy of it, where the coefficient of the dilation is positive. Consequently, any set and any homothetic copy or, in short, *homothet* of it are in "parallel position."



(a) Starting step: $\mathcal{C}(k,1)$

(b) $\mathcal{C}(2,2)$ magnified

(c) Induction step

FIGURE 1. The construction.

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Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d+1$ times

Attila Pór
(joint work with I. Bárány and J. Matoušek)

A curve γ in $\mathbb{R}^d$ is a continuous mapping of a closed interval. We call γ $(\le k)$-crossing if it intersects every hyperplane at most k times. There are no $(\le (d-1))$-crossing curves in $\mathbb{R}^d$ since any d points are on a hyperplane. The $(\le d)$-crossing curves are often called convex curves like the moment curve $\{(t, t^2, \dots, t^d) \mid t \in [0,1]\}$.

Theorem 1. *For all $d \ge 2$ there exists $M(d)$ such that every $(\le d+1)$- crossing curve in $\mathbb{R}^d$ can be subdivided into at most $M(d)$ convex curves.*

This result implies a tight lower bound for order type homogenous subsequences of points based on previous work of Eliáš, Roldán, Safernová and Matoušek.
For a sequence $P = (p_1, \dots, p_n)$ we say it is order type homogeneous if every $(d+1)$-tuple has the same sign (orientation). A sequence $P = (p_1, \dots, p_n)$ in $\mathbb{R}^d$ is strong order type homogeneous if for every $1 \le k \le d$ the projection of P onto the k-dimensional space spanned by the first k coordinates is order type homogeneous. By Ramsey theory there exists a least integer $N = \mathrm{OT}_d(n)$ such that every sequence of points $P = (p_1, \dots, p_N)$ in general position has an order type homgeneous subsequence of length at least n.

Theorem 2. *(Suk)*
$OT_d(n) \le twr_d(O(n))$
where $twr_d(a)$ is the tower function of hight d.

Similarly by Ramsey theory there exists a least integer $N = \mathrm{OT}^*_d(n)$ such that every sequence of points $P = (p_1, \dots, p_N)$ in general position has a strong order type homgeneous subsequence of length at least n.

Theorem 3. *(Eliáš, Roldán, Safernová and Matoušek)*
$OT^*_d(n) \geq twr_d(n-d)$

Our result implies that

Theorem 4. $OT_d(n) \geq OT^*_d(\Omega(n))$

and establishes a tight lower bound for $\mathrm{OT}_d(n)$.

Polynomials vanishing on Cartesian products: The Elekes-Szabó Theorem revisited

Orit E. Raz
(joint work with Micha Sharir, Frank De Zeeuw)

Let $F \in \mathbb{C}[x,y,z]$ be a polynomial of degree d, and let $A, B, C \subset \mathbb{C}$ with $|A| = |B| = |C| = n$. In a recent work, we show that either F vanishes on at most $O(n^{11/6})$ points of the Cartesian product $A \times B \times C$, or F locally has a special group-related form. This improves a theorem of Elekes and Szabó [1], and generalizes the result of Raz et al. in [2]. We prove the same statement over $\mathbb{R}$, and extend it to the case where A, B, C have different sizes. This result provides a unified tool for improving bounds in various Erdős-type problems in geometry. In the talk I will present the main ideas of the proof, and will mention several applications. This is a joint work with Micha Sharir and Frank de Zeeuw

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Points with distinct circumradii and anti-Ramsey

Edgardo Roldán-Pensado
(joint work with Leonardo Martínez, Miguel Raggi)

In 1975, inspired by the observations from Esther Szekeres and his results with George Szekeres, Paul Erdős posed the following problem:

"Is it true that for every k there is an n_k such that if there are given n_k points in the plane in general position (i.e. no three on a line no four on a circle) one can always find k of them so that all the $\binom{k}{3}$ triples determine circles of distinct radii?"

This problem is similar to the Erdős-Szekeres Theorems. As is the case with these theorems, the existence of n_k can be established using Ramsey Theory if the existence of n_6 can be verified. However, establishing the existence of n_6 is not completely trivial and the bound obtained from this method is an exponential tower.

Three years later, in 1978, Erdős published a paper where he claimed a positive answer to the question with $n_k \le 2\binom{k-1}{2}\binom{k-1}{3} + k$. However, he inadvertently left out a non-trivial case for which his method does not work. It seems that Erdős remained unaware of this and even restated the result in 1985.

We address this issue and give a polynomial bound for n_k.

Theorem 1 (Martínez, R.). *There is a constant C such that for any Ck^9 points in the plane in general position (i.e. no four on a line or circle) there are k of them so that all their triples determine circles with distinct radii.*

The proof of this Theorem is based on Erdős' argument but is slightly more involved. It uses Bézout's Theorem from algebraic geometry.

In the proof we need bounds for the first values of n_k. Using almost completely combinatorial methods we obtain the bounds $n_4 \le 9$ and $n_5 \le 37$. The only geometrical fact we use is that through a pair of points there are at most two circles of a given radius.

Then we tried to obtain a more combinatorial result that could be applied in more general settings. Let $H = (V, E)$ be the complete k-uniform hypergraph on N vertices. A $(k-1)$-sunflower is a set $S \subset E$ of edges such that $\#(\bigcap S) = k - 1$.

Theorem 2 (Martínez, Raggi, R.). *Assume that the edges of H are coloured so that no monochromatic $(k-1)$-sunflower has more than λ edges. If N is large enough then there exists $V' \subset V$ with n vertices such that hypergraph induced by V' is heterochromatic.*

From this we obtain the following results.

Corollary. *Let $X \subset \mathbb{R}^d$ be set of N points with no $d+2$ on a hyperplane or sphere. If N is large enough, there is a set $Y \subset X$ with n points such that no two different simplexes with vertices in Y have the same circumradius.*

Corollary. *Let $X \subset \mathbb{R}^d$ be a set of N points with no $d+1$ on a hyperplane. If N is large enough, there is a set $Y \subset X$ with n points such that no two different simplexes with vertices in Y have the same volume.*

Corollary. *Let $X \subset \mathbb{R}^d$ be a set of N points with no $d+1$ on a hyperplane. If N is large enough, there is a set $Y \subset X$ with n points such that no two different simplexes with vertices in Y are similar.*

On Kinetic Delaunay Triangulations

Natan Rubin

Let P be a collection of n points in the plane, each moving along some straight line at unit speed. We obtain an almost tight upper bound of $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, on the maximum number of discrete changes that the Delaunay triangulation $\mathsf{DT}(P)$

of P experiences during this motion. Our analysis is cast in a purely topological setting, where we only assume that (i) any four points can be co-circular at most three times, and (ii) no triple of points can be collinear more than twice; these assumptions hold for unit speed motions.

We discuss this work in connection with other classical results on the combinatorial complexity of geometric structures.

Many triangulated odd-spheres

Francisco Santos
(joint work with Eran Nevo, Stedman Wilson)

For $d \geq 3$ fixed and n large, Kalai [5] constructed $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ combinatorially distinct n-vertex triangulations of the d-sphere (the *squeezed* spheres) and concluded from Stanley's upper bound theorem for simplicial spheres an upper bound of $2^{O(n^{\lceil d/2 \rceil} \log n)}$ for the number of such triangulations. That is, if we define

$$s_d(n) := \text{number of } n\text{-vertex simplicial spheres of dimension } d,$$

Kalai proved

$$\Omega(n^{\lfloor d/2 \rfloor}) \leq \log s_d(n) \leq O(n^{\lceil d/2 \rceil} \log n).$$

For even d the difference between the upper and lower bound is only a $\log n$ (in the exponent, though), but in odd dimension $d = 2k - 1$ the gap is much bigger, from $2^{\Omega(n^{k-1})}$ to $2^{O(n^k \log n)}$. Most strikingly, for $d = 3$ the gap is from $2^{\Omega(n)}$ to $2^{O(n^2 \log n)}$. Pfeifle and Ziegler [8] reduced this gap by constructing $2^{\Omega(n^{5/4})}$ combinatorially different n-vertex triangulations of the 3-sphere. We here improve both constructions, to obtain:

Theorem 1. $\log s_d(n) \geq \Omega(n^{\lceil d/2 \rceil})$.

Although Kalai does not mention this, his spheres are *geodesic*, that is, they can be realized geodesically in the standard sphere. The number $g_d(n)$ of geodesic spheres can be bounded by noticing that a geodesic sphere is determined by the order type of the set of its vertices (considered as a $(d+1)$-dimensional vector configuration) together with its $\lceil d/2 \rceil$-skeleton. This was shown by Dey [2] for straight-line triangulations of polytopes in $\mathbb{R}^{d+1}$, but the proof carries over to the geodesic sphere case. Since the number of different order types is "small" [3, 1], the number of geodesic spheres is bounded by the possible $\lceil d/2 \rceil$-skeleta. In even dimension this does not improve on the upper bound obtained from the UBT, but in odd dimension it gets rid off the $\log n$ factor. That is, putting together Kalai's construction and Dey's result one gets:

$$\Omega(n^{k-1}) \leq \log g_{2k-1}(n) \leq O(n^k).$$

The spheres we construct in Theorem 1 are not (as far as we know) geodesic, but a variation of our construction still gives more geodesic spheres than constructed by Kalai:

Theorem 2.

$$\log g_{2k-1}(n) \geq \Omega(n^{k-1+1/k}).$$

For example, for the first open case of $d = 3$ we construct $2^{\Omega(n^2)}$ spheres in total and $2^{\Omega(n^{3/2})}$ geodesic ones, while Kalai's construction in this case gives only $2^{\Omega(n)}$ and the best previous construction (by Pfeifle and Ziegler [8]) gave $2^{\Omega(n^{5/4})}$ spheres, which were not (guaranteed to be) geodesic.

Our Theorem 1 follows from constructing a polyhedral 3-sphere with $\Omega(n^2)$ combinatorial *bipyramids* (or their natural generalization to higher dimension) among its facets. A bipyramid is the unique simplicial 3-polytope with 5 vertices.

The idea of the construction (in dimension three for simplicity) is as follows. Consider a certain simplicial 3-ball K with n vertices and $\Theta(n^2)$ tetrahedra. Then:

- Find particular $\Theta(n)$ simplicial 3-balls contained in K, with disjoint interiors and with $\Theta(n)$ tetrahedra each.
- On the boundary of each such 3-ball find particular $\Theta(n)$ pairs of adjacent triangles (each pair forms a square), such that these squares have disjoint interiors.
- Replace the interior of each such 3-ball with the cone from a new vertex over each boundary square (forming a bipyramid) and over each remaining boundary triangle (forming a tetrahedron).
- Show that the particular 3-balls and squares chosen have the property that the above construction results in a polyhedral 3-ball. Adding a cone over the boundary results in a polyhedral 3-sphere.

In higher dimension the idea is the same, replacing "squares" and "bipyramids" to "k-polytopes with $k+2$-vertices". Such polytopes are free-sums of two lower dimensional simplices, and have two minimal non-faces and two triangulations. With this, we prove the following:

Lemma 3. *There are $(2k-1)$-spheres with n vertices and with $\Omega(n^k)$ facets that are not simplices. Each such facet has $k+2$ vertices and their minimal non-faces (two in each facet) are all distinct.*

Each of the facets in the above lemma can then be triangulated in two ways to obtain a triangulation of the $2k-1$-sphere. The fact that all minimal non-faces are distinct implies that all these $2^{\Omega(n^k)}$ triangulations are valid simplicial complexes, which gives Theorem 1.

We have two specific constructions providing Lemma 3, one based in the join of k paths and one based in the boundary complex of the cyclic $2k$-polytope. The latter gives a better constant inside the $\Theta(\cdot)$ notation ($4n^2/25$, versus $2n^2/25$ bipyramids in the former, for the case $k = 2$) but the former is somehow simpler to describe. Details can be found in the full version of this report [7].

In order to get Theorem 2 we use the same strategy except the 3-ball K needs to be "straight" (to be precise, a geometric triangulation of a polytope) and the balls we retriangulate need to be star-convex. The first condition is satisfied in our original construction already, but to get the second condition we have to reduce the number of non-simplicial facets as follows, which implies Theorem 4:

Lemma 4. *There are geodesic $2k-1$-spheres with n vertices and with $\Omega(n^{k-1+1/k})$ facets that are not simplices.*

Let us now discuss some by-products of our construction:

- Erickson conjectured that there are no 4-polytopes or 3-spheres on n vertices with $\Omega(n^2)$ non-simplicial facets. Lemma 3 refutes this for 3-spheres, but we leave the question open for 4-polytopes. For them we can only prove the following, which is the first construction of 4-polytopes with more than $O(n^{1+\epsilon})$ non-simplicial facets (a construction of cubical 4-polytopes with $\Theta(n\log(n))$ non-simplicial facets is due to Joswig and Ziegler [4]):

  Theorem 5. *There are 4-dimensional polytopes with n vertices and with $\Theta(n^{3/2})$ facets that are bipyramids.*

- Triangulating these bipyramids appropriately we also get:

  Theorem 6. *There are 4-dimensional simplicial polytopes with n vertices and with $\Theta(n^{3/2})$ edges of degree three.*

  The dual to the polytopes in this theorem have n facets and $\Theta(n^{3/2})$ triangles, which answers the following question of Ziegler: Can a simple 4-polytope with n facets have more than $O(n)$ non-quadrilateral 2-faces?

- The sphere of Theorem 5 can be triangulated in $2^{\Theta(n^{3/2})}$ ways. These triangulations cannot be all polytopal, simply because they are too many. But we can prove $2^{\Omega(n\log n)}$ of them to be polytopal, which gives a new construction of as many 4-polytopes as the Goodman-Pollack bound allows (a previous construction was given by Shemer [9]).

  A variation of this idea shows that there is a point configuration of n points in $\mathbb{R}^3$ having $2^{\Theta(n^2)}$ triangulations, $2^{\Theta(n\log n)}$ of them regular.

All our constructions are done in the PL-category. (All simplicial 3-spheres are PL [6], but non-PL simplicial spheres exist in any dimension ≥ 5.)

To finish with an open question, we would ask whether $\lim_{n\to\infty} \log g_d(n)/\log s_d(n)$ is, for fixed d, zero or positive. Observe that individual non-geodesic spheres are easy to construct; for example, a 3-sphere containing a trefoil knot on five edges or less cannot be geodesic.

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$\exists\mathbb{R}$, or the Real Logic of Drawing Graphs

Marcus Schaefer

We reported on $\exists\mathbb{R}$, a complexity class introduced recently to capture the complexity of problems involving real numbers and geometry.

Let us consider the rectilinear crossing number problem: does a given graph G have a straight-line drawing with at most k crossings? This problem is easily seen to be **NP**-hard, but does it lie in **NP**.[1] In a way, an answer to that question has been known since 1991, when Bienstock [3] showed that the rectilinear crossing number problem is not only **NP**-hard, but encodes what appears to be a much larger theory: the existential theory of the real numbers.

Definition 1. The existential theory of the real numbers, ETR, is the set of true sentences of the form

$$(\exists x_1, \dots, x_n)[\ \varphi(x_1, \dots, x_n)],$$

where φ is a quantifier-free $(\vee, \wedge, \neg)$-Boolean formula over the signature $(0, 1, +, *, <, \le, =)$ interpreted over the universe of real numbers.

ETR is a very expressive language, for example, it in turn effectively encodes the rectilinear crossing number problem, so that it can be said that from a computational complexity point of view the two problems have the same decision complexity. So when asking whether the rectilinear crossing number problem is in **NP**, we are really asking whether a rather powerful and expressive theory can be decided in **NP**. While that is not impossible (and some people believe it to be true), tools for such a proof would probably have to come from real algebraic geometry and logic, not graph theory. One small case in point: **SSQRT**, the *sum of square roots* problem—deciding which of two sums of square roots is larger—at this point has only been located in the counting hierarchy [2], which is significantly beyond **NP**. However, compared to the power of ETR, it appears to be a tiny special case.

So we think it worthwhile to consider the problems which have the same computational complexity as ETR as its own separate complexity class, we call it $\exists\mathbb{R}$.

[1] A question asked explicitly, for example, in [5].

Definition 2. We say a decision problem $L \subseteq \{0,1\}^*$ is $\exists\mathbb{R}$-*hard*, if there is a polynomial-time reduction from any problem in ETRto L; L is $\exists\mathbb{R}$-*complete* if it is $\exists\mathbb{R}$-hard, and reduces to ETR.

In terms of classical complexity theory, $\exists\mathbb{R}$ is located between **NP** (it is easy to see that it encodes Boolean satisfiability) and **PSPACE** (a far from obvious result due to Canny [6].

1. Complete Problems

The definition of ETR is closely related to (multivariate) polynomials, so it is not surprising that it is easy to find problems about polynomials which are $\exists\mathbb{R}$-complete. Polynomials can be described in various ways, ranging from very succinct descriptions, such as straight-line programs, to explicit representations, as sums of monomials. We assume that polynomials are explicitly represented, e.g. $f(x,y) = 3xy + y + 6x + 2$, rather than $f(x,y) = (3x+1)(y+2)$.

The following problems are $\exists\mathbb{R}$-complete:

- 4-**Feasibility:** given a multivariate polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of total degree at most 4, does it have a root? (A detailed proof can be found in Schaefer, Štefankovič [17]; in the real computation model of Blum-Shub-Smale, a similar result is true, see [4, Section 5.4].
- **Hilbert's Homogenous Nullstellensatz:** does a family $f_i : \mathbb{R}^n \to \mathbb{R}$ of homogenous multivariate have a non-trivial root (zero being the trivial root). See Schaefer [16].
- **Brouwer Fixed Point:** Does a family of polynomials $f_i : B_n(0,1) \to B_n(0,1)$ have a fixed point in $B(0,1/2)$? See Schaefer, Štefankovič [17].

And one can imagine natural variations of these problems; however, our main interest lies in applications to graph realization problems. In that area, the original $\exists\mathbb{R}$-complete problem is due to Mnëv [12] and his universality theorem, which has been reproved in various ways and strengthened several times (e.g. Shor [18], and Richter-Gebert [14]). Call a pseudoline arrangement *stretchable* if there is an equivalent straight-line arrangement.

Theorem 3 (Mnëv [12])**.** *Stretchability is* $\exists\mathbb{R}$-*complete.*

For a nice recent exposition of Mnëv's result, from the computational complexity point of view, see Matoušek [10]. There have been many other $\exists\mathbb{R}$-hardness results over the years. Some recent and relevant results include:

- **Intersection graphs:** of segments (Kratochvíl, Matoušek [8]), unit disks or disks (McDiarmid, Müller [11, 7]), ellipses or convex sets (Schaefer [15]).
- **Non-rigidity of a linkage:** Due to Schaefer [16], using Abbott's [1] reduction from Hilbert's Homogenous Nullstellensatz.
- **Straight-line realizability:** of a complete graph with a prescribed set of pairs of edges that have to cross. Due to Kyncl [9].

Most hardness results start with a known $\exists\mathbb{R}$-complete problem such as stretchability, or segment intersection graphs; these are to $\exists\mathbb{R}$-hardness as the clique or independent set problem are to **NP**-hardness. Occasionally, however, one has to go back all the way to ETR: a small modification of Mnëv's proof shows that the collinearity problem (given a set of points, and for each triple of points the information whether they are collinear or not) is $\exists\mathbb{R}$-complete, answering a question by Scott Aaronson.

Theorem 4. *Deciding collinearity logic is* $\exists\mathbb{R}$*-complete.*

We have not been able to show this more easily via a reduction from segment intersection graphs, or stretchability, say.

There are many problems in discrete geometry and graph drawing which make good candidates for $\exists\mathbb{R}$-completeness. To mention a few: Lombardi planarity, RAC graphs, bounding the number of guards in an art gallery, packing puzzles, matchstick graphs, Gabriel graphs, and visibility graphs (only recently shown **NP**-hard).

There are also some problems which, although requiring some real computation, don't appear to be as hard as $\exists\mathbb{R}$. Minimum Weight Triangulation (only recently shown **NP**-hard by Mulzer and Rote [13]) and the Euclidean Traveling Salesperson problem lie in the class $\mathbf{NP}^{\mathbf{SSQRT}}$, which is **NP**, with a **SSQRT** oracle, that is, we can ask questions of the type $\sqrt{3}+\sqrt{7}+\sqrt{10}<\sqrt{4}+\sqrt{12}$. We suspect that $\mathbf{NP}^{\mathbf{SSQRT}}$ is a separate complexity class, smaller than $\exists\mathbb{R}$, but there have been no relativized separations (we do not even know whether $\mathbf{NP} \subsetneq \exists\mathbb{R} \subsetneq \mathbf{PSPACE}$ is possible in a relativized world). A starting point would be to find complete problems for $\mathbf{NP}^{\mathbf{SSQRT}}$, and Minimum Weight Triangulation and the Euclidean Traveling Salesperson seem to be good starting points.

2. Universality Phenomena

We should mention that Mnëv [12] proved a much stronger result than $\exists\mathbb{R}$-completeness: he proved that stretchability is universal for semi-algebraic sets, that is every semi-algebraic set is homotopy equivalent to a realization space of a pseudo-line arrangement. Universality theorems show that a problem is algebraically hard, whereas $\exists\mathbb{R}$-completeness shows that the problem is computationally hard. In Mnëv's proof, the two coincide, but that is not always the case, since homotopy equivalence may be witnessed by highly inefficient reductions which do not yield hardness results (and such examples are known). So it would be interesting to have a new notion of *efficient homotopy or stable equivalence* which captures both the algebraic and the computational aspect.

The typical reductions encountered in $\exists\mathbb{R}$-hardness proofs are *geometric* in the sense that an algebraic solution to one problem can be obtained from the other by simple projection of coordinates. E.g. Bienstock [3] reduces Stretchability to a rectilinear crossing number problem $\mathrm{rcr}(G)\le k$. From the coordinates of a drawing realizing $\mathrm{rcr}(G)\le k$, he can read off the equations of a straight-line arrangement realizing the given pseudo-line arrangement.

This immediately implies that what could be called *weak universality* phenomena: we know that ETR may require exponential precision in a realization:

$x_1 = 1/2$, $x_2 = x_1^2, \ldots, x_n = x_{n-1}^2$; by Mnëv's universality theorem, so does stretchability, and then by Bienstock's reduction, so does the rectilinear crossing number problem. Problems which "encode" equality, e.g. linkage rigidity, show a similar behavior, in that they encode all algebraic numbers.

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Embeddability in $\mathbb{R}^3$ is decidable

Eric Sedgwick
(joint work with Jiří Matoušek, Martin Tancer, Uli Wagner)

The Problem. Given a 2-dimensional complex K, a collection of triangular faces along with identifications of their edges and/or vertices, does the complex K embed in $\mathbb{R}^3$?

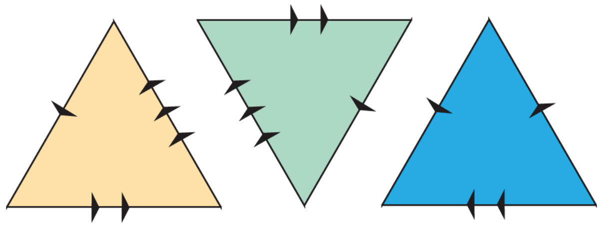

Figure 1. Does this 2-complex embed in $\mathbb{R}^3$?

This question belongs to two lines of inquiry. First, this problem is an important special case, $\text{EMBED}_{2\to 3}$, of the more general question $\text{EMBED}_{k\to d}$: given a k-dimensional simplicial complex K, does K embed in $\mathbb{R}^d$. Of course, $\text{EMBED}_{1\to 2}$ is graph planarity, which is decidable in linear time. $\text{EMBED}_{k\to d}$, for $d \geq 4$, was shown to be decidable in polynomial time when $k < (2d-2)/3$ in a series of papers by Čadek, Krčál, Matoušek, Sergaerert, Vokřínek and Wagner [2, 3, 4, 9]; and was shown by Matuošek, Tancer and Wagner to be NP-hard when $k < (2d-2)/3$ and even undecidable when $k \geq d-1 \geq 4$ [11].

Our main result is that $\text{EMBED}_{2\to 3}$ is decidable [12]. That is, there is an algorithm that, when given a 2-dimensional complex K, decides whether K embeds in $\mathbb{R}^3$. It is not hard to see that a 2-dimensional complex K embeds in $\mathbb{R}^3$ if and only one of a finite number of thickenings of K is a 3-manifold X that embeds in $\mathbb{R}^3$ (or equivalently, embeds in S^3).

We show $\text{EMBED}_{2\to 3}$ is decidable by demonstrating that this last question is decidable; namely there is an algorithm that, when given a triangulated 3-manifold X, decides whether X embeds into the 3-sphere S^3. This approach makes available decision procedures and tools from the second line of inquiry, 3-manifold topology. Well known decision algorithms include: Haken's algorithm to recognize the unknot [6], Rubinstein [14] and Thompson's [15] algorithm to recognize S^3, and Jaco and Sedgwick's algorithm to determine whether a knot manifold embeds in S^3 [8].

Embedding a 3-manifold. One complicating factor is that a given 3-manifold X may embed in S^3 in many, even an infinite number, of ways. A result of Fox [5] partially simplifies matters by guaranteeing that if X embeds in S^3, then X admits an embedding into S^3 so that its complement is a thickened (and likely knotted and linked) graph, see the following figure. Given X, the goal is then to determine whether S^3 can be obtained by attaching some thickened graph to the boundary of X. The Rubinstein/Thompson algorithm can then be used determine whether the obtained manifold is S^3 .

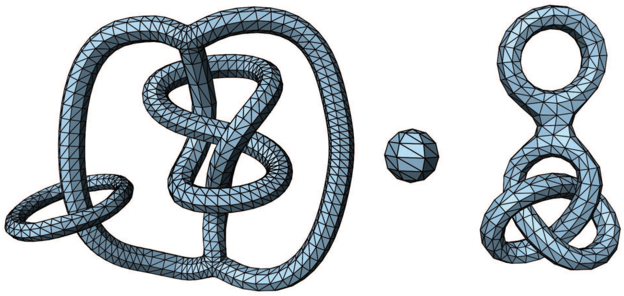

FIGURE 2. A Fox embedding: the 3-manifold X is the complement of a graph in S^3 (equivalently, $\mathbb{R}^3$).

Spherical boundary components (thickened isolated vertices, the ball in the figure) are easy to handle. There is but one, up to homeomorphism, 3-manifold that can be obtained by attaching a ball to a spherical boundary component X. Unfortunately, for each non-spherical boundary component there are an infinite number of ways of attaching a thickened graph and in general we expect to obtain an infinite number of manifolds in this manner. In essence, the figure could be misleading. The edges of the boundary triangulation are pictured to be short and straight. If this were the case, the number of tetrahedra required to fill in the thickened graph would be quite reasonable. But, a priori, it is quite possible that boundary edges wrap around the boundary in an unbounded fashion. Since the number of tetrahedra required to fill in the missing thickened graph depends on the degree of this wrapping, it suffices to show that the degree of wrapping is bounded.

Our main technical result is the *short meridian theorem*. Roughly speaking, it states that, after applying known simplifications to the topology and triangulation of X, there is an embedding for which the degree of wrapping is bounded by a function of the number of tetrahedra in X. In turn, this bounds the number of thickened graph attachments that needs to be considered and thus obtain the desired result.

The proof uses and extends *normal surface theory*, the principal tool for the aforementioned 3-manifold algorithms, to account for the presence of annuli. We also utilize Li's result on *thin position for graphs in* S^3 [10], Bachman, Derby-Talbot, and Sedgwick's result on *almost normal surfaces with boundary* [1], Jaco and Rubinstein's *0-efficient triangulations* [7], and our recent result demonstrating how to *untangle* systems of curves in a surface[13].

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Mass partitions using hyperplanes with fixed directions

Pablo Soberón

(joint work with Roman Karasev and Edgardo Roldán-Pensado)

During the talk we discussed properties of partitions induced by nested hyperplane cuts with fixed directions. Moreover, we add the restriction that each hyperplane has also prescribed the region it's going to be cutting. We showed that using these partitions one can generalise Alon's classic necklace splitting theorem [1] and its high-dimensional versions [2]. Our main result is that for any t measures in $\mathbb{R}^d$, there is a partition as described above using $t(k-1)$ cuts so that the resulting parts may be distributed among k sets $A_1, A_2, \ldots, A_k$ that have the same size in each measure. We also described how these partitions imply a result regarding fixed-direction paths in the plane. For any t measures in the plane, there is a path using only vertical and horizontal segments and at most $t-1$ turns so that it splits each measure by half.

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The number of double-normal pairs in a set of n points

Konrad Swanepoel

(joint work with János Pach)

A *double-normal pair* of a finite set S of points from $\mathbb{R}^d$ is a pair of points $\{\boldsymbol{p}, \boldsymbol{q}\}$ from S such that S lies in the closed strip bounded by the hyperplanes through $\boldsymbol{p}$ and $\boldsymbol{q}$ perpendicular to $\boldsymbol{pq}$. A double-normal pair $\boldsymbol{pq}$ is *strict* if $S \smallsetminus \{\boldsymbol{p}, \boldsymbol{q}\}$ lies in the open strip. We answer some questions of Martini and Soltan [1] by proving the following.

(1) A set of $n \geq 3$ points in the plane has at most $3\lfloor n/2 \rfloor$ double-normal pairs. This bound is sharp for each such n.

(2) A set of $n \geq 8$ points on the 2-sphere has at most $17n/4 - 6$ double-normal pairs (as a subset of $\mathbb{R}^3$). This bound is sharp for infinitely many values of n. There exist n-element point sets on the 2-sphere with at least $17n/4 - O(\sqrt{n})$ double-normal pairs.

(3) A set of $n \geq 4$ points on the 2-sphere has at most $2n-2$ strict double-normal pairs. This bound is sharp.

(4) For $d \geq 3$, the maximum number of double-normal pairs [strict double-normal pairs] in a set of n points in $\mathbb{R}^d$ is asymptotically $\frac{1}{2}(1 - \frac{1}{k(d)})n^2 + o(n^2)$ [resp. $\frac{1}{2}(1 - \frac{1}{k'(d)})n^2 + o(n^2)$], where $k(d)$ and $k'(d)$ satisfy $\lceil d/2 \rceil \leq k'(d) \leq k(d) \leq d - 1$ and asymptotically $k(d) \geq k'(d) \geq d - O(\log d)$. In particular, the maximum number of double-normal pairs (or strict double-normal pairs) in $\mathbb{R}^3$ is $\frac{1}{4}n^2 + o(n^2)$.

This work will appear in [2] and [3].

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Van Kampen–Flores type non-embeddability result for $2k$-dimensional manifolds

Martin Tancer

(joint work with Xavier Goaoc, Isaac Mabillard, Pavel Paták, Zuzana Safernová, Uli Wagner)

The classical question of topological graph theory is to determine the (orientable or non-orientable) *genus* of the complete graph with n vertices, that is the minimum genus of an (orientable or non-orientable) 2-manifold into which that graph embeds. This question was answered by Ringel and Youngs (see [3] for a detailed discussion); the orientable genus of the complete graph on n equals $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ whereas the non-orientable genus equals $\left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$ for $n \neq 7$ (and equals 3 for $n = 7$).

This question naturally generalizes to higher dimension, where one may wonder what is the minimum topological complexity of a manifold into which $\Delta_n^{(k)}$, the k-dimensional skeleton of the n-dimensional simplex, embeds. This line of enquiry started in the 1930's when Van Kampen [4] and Flores [1] showed that $\Delta_{2k+2}^{(k)}$ does not embed into $\mathbb{R}^{2k}$ (the case $k = 1$ corresponding to the non-planarity of the complete graph on five vertices). Perhaps surprisingly, little else seems known and the following conjecture of Kühnel [2, Conjecture B] remains essentially untouched:

Conjecture 1. *Let k, n, b be integers. If $\Delta_n^{(k)}$ embeds in a compact, $(k-1)$-connected $2k$-manifold with kth Betti number b then $\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b$.*

(In Conjecture 1 we work with homology and Betti numbers with $\mathbb{Z}_2$ coefficients.)

We prove a result of similar spirit, with weaker bound, but also with weaker assumptions. One of the relaxations of the assumptions is to replace embeddings with so called almost embeddings. This relaxation actually helps with setting up the right proof method.

We define *almost-embedding* of a simplicial complex K, with geometric realization $|K|$, into a topological space X as a continuous map $f : |K| \to X$ such that any disjoint simplices $\sigma, \tau \in K$ have disjoint images $f(\sigma), f(\tau)$.

Our main result is the following.

Theorem 2. *Let k, n, b be integers. If $\Delta_n^{(k)}$ almost embeds into a compact $2k$-manifold with kth Betti number b, then $n \leq 2b\binom{2k+2}{k} + 2k + 5$.*

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Saturated simple and k-simple topological graphs

Géza Tóth
(joint work with Jan Kynčl, János Pach, Radoš Radoičić)

A *simple topological graph* G is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing. G is called *saturated* if no further edge can be added without violating this condition. We construct saturated simple topological graphs with n vertices and $O(n)$ edges. For every $k > 1$, we give similar constructions for k-*simple topological graphs*, that is, for graphs drawn in the plane so that any two edges have at most k points in common. We also show that in any k-simple topological graph, any two independent vertices can be connected by a curve that crosses each of the original edges at most $2k$ times. Another construction shows that the bound $2k$ cannot be improved. Several other related problems are also considered.

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On permutations induced by projection between lines with individual centers

Pavel Valtr

(joint work with Martin Balko, Alfredo García, Ferran Hurtado, Javier Tejel)

Let α, β be two horizontal lines and let $S = \{s_1, \dots, s_n\}$ be a set of n points between these two lines in the plane. We project n points $a_1, \dots, a_n \in \alpha$ to points $b_1, \dots, b_n \in \beta$ in such a way that $a_i s_i b_i$ are collinear for each $i = 1, \dots, n$. Suppose the points $a_1, \dots, a_n$ lie in this left-to-right order on α. We may obtain different permutations of $b_1, \dots, b_n$ on β for different instances of $A = \{a_1, \dots, a_n\}$. For fixed $S = \{s_1, \dots, s_n\}$, let $p(S)$ be the number of distinct permutations of $B = \{b_1, \dots, b_n\}$ obtained in the above way. Obviously, $1 \le p(S) \le n!$. Both bounds can be attained. In the talk it was shown that $p(S) \ge 2^{\Omega(n)}$, provided S has $\Omega(n)$ distinct y-coordinates. Also, it was outlined how to prove that $\mathbb{E}p(S) \le c^{n \log \log n}$ holds for a certain constant c and for a certain randomly chosen set $S = \{s_1, \dots, s_n\}$.

Space Curve Arrangements with Many Incidences

Joshua Zahl

(joint work with Larry Guth)

In 2010, Guth and Katz proved that if a collection of N lines in $\mathbb{R}^3$ contained more than $N^{3/2}$ 2–rich points, then many of these lines must lie on planes or reguli. I will discuss some generalizations of this result to space curves in $\mathbb{R}^3$. This is joint work with Larry Guth.

Tight and non-tight topological Tverberg type theorems

Günter M. Ziegler
(joint work with Pavle V. M. Blagojević, Benjamin Matschke, and Florian Frick)

A short history of (tight) Tverberg theorems. The history of "Tverberg type" multiple intersection theorems (after the classical convexity results of Helly and Radon, and the non-embeddability results of van Kampen and Flores, etc.) starts with Birch's 1959 paper "On 3N points in a plane" [5], which contained the following three achievements.

Theorem 1: *Any $3N$ points in the plane can be partitioned into N triangles that have a point in common.*

Theorem 1*: *Any $3N-2$ points in the plane can be partitioned into N subsets whose convex hulls have a point in common.*

Conjecture: *Any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into r subsets whose convex hulls have a point in common.*

We note that Birch's Theorem 1*, as well as his conjecture, which was proved in full by Helge Tverberg in 1964, fifty years ago (see [17]), and thus is now known as Tverberg's theorem [13], are *tight*: This is not only evident from concrete configurations, but also from a general position argument: If $(r-1)(d+1)$ points in $\mathbb{R}^d$ in general position are partitioned into r subsets, then not even their *affine* hulls intersect, as one sees from a codimension count. (See Kalai [9] for far-reaching conjectured extensions of this.)

The tightness of the results also means that for a generic point configuration the number of intersection points, known as *Tverberg points*, is finite. With this finiteness it is also very natural to ask for the minimal number of Tverberg r-partitions, which according to Sierksma's conjecture should be $(r-1)!^d$; only a much weaker result is proven [15]; see [10].

In a modern version (from a point of view pioneered in [1]), Tverberg's theorem says that *for $d \geq 1$, $r \geq 2$, $N := (r-1)(d+1)$, and any affine map $f : \Delta_N \to \mathbb{R}^d$, the N-dimensional simplex Δ_N contains r points $x_1, \dots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N but whose images coincide: $f(x_1) = \cdots = f(x_r)$.*

In this version, the "topological version" for continuous maps f is natural. This is a breakthrough result by Bárány, Shlosman & Szűcs [4] from 1981, extended from primes r to prime powers by Özaydin [12] in 1987 — which, however, remains a conjecture for the case when r is not a prime power and $d \geq 2$:

The topological Tverberg theorem/conjecture [4] [12]: *Let $d \geq 1$ and $r \geq 2$, and $N := (r-1)(d+1)$. For any continuous map $f : \Delta_N \to \mathbb{R}^d$ the N-dimensional simplex Δ_N contains r points $x_1, \dots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N whose images coincide: $f(x_1) = \cdots = f(x_r)$.*

Again this result is "tight": It fails for *all* general-position maps f if N is replaced by a smaller number.

Colored versions of Tverberg type theorems. In an influential 1989 Computational Geometry paper, Bárány, Füredi, and Lovász observed: *"we need a colored version of Tverberg's theorem."* For their purposes they needed only a very small special case: *Let A, B, C be sets of t points in the plane, then one can find $r = 3$ disjoint triples consisting of one point of each of the three sets such that the convex hulls of the triples have a point in common.* (Here the points in A, B, and C are interpreted as having three different colors.) Bárány et al. proved this for $t = 7$, asserted they also had a proof for $t = 4$, but also noted that they had no counterexample even for $t = 3$. So, in particular, their result was not tight.

The call for a colored version of Tverberg's theorem was seen as a challenge, and attacked immediately. The first answer, by Imre Bárány and David Larman 1990, treated the case of $3r$ points in the plane, with three different colors. In particular, they suggested the following.

The Bárány–Larman colored Tverberg conjecture [3]**:** *Let $d \geq 1$, $r \geq 2$, and $N \geq N(r, d)$ sufficiently large. Assume that $f : \Delta_N \longrightarrow \mathbb{R}^d$ is affine (or at least continuous), where the $N + 1$ vertices of Δ_N carry $d + 1$ different colors, and every color class has size at least r. Then Δ_N has r disjoint* rainbow faces *whose images under f intersect.*

Here a *rainbow face* refers to a d-dimensional face of the simplex Δ_N whose $d + 1$ vertices carry the $d + 1$ different colors. In the case $d = 2$ thus we have at least $3r$ points in the plane, which carry the three different colors. In this situation there should be r rainbow triangles that have a point in common. For $d = 2$ Bárány and Larman proved this, and thus obtained a "sharp" colored version of Birch's Theorem 1. However, for $d > 2$ they did not obtain a finiteness result for $N(r, d)$.

In a "Note added in proof," Bárány and Larman announced that Živaljević and Vrećica had proven the finiteness of $N(r, d)$ — but indeed, they haven't. In their celebrated 1992 paper [16] (see [10]) they established the following result.

Živaljević and Vrećica's colored Tverberg theorem [16]**:** *Let $d \geq 1$ and r a prime. Assume that $f : \Delta_N \longrightarrow \mathbb{R}^d$ is continuous, where the $N + 1$ vertices of Δ_N carry $d + 1$ different colors, and every color class has size at least $2r - 1$. Then Δ_N has r disjoint* rainbow faces *whose images under f intersect.*

Via Bertrand's postulate, the condition that r is prime may be dropped if we make the color classes bigger, e.g. of size at least $4r - 1$. However, even if in the Bárány–Larman conjecture $N(r, d)$ is taken to be *very* large, then this still does not imply that *all* color classes get large, say larger than $2r - 1$. Thus Živaljević and Vrećica's colored Tverberg theorem does establish the colored Tverberg result suggested by Bárány–Füredi–Lovász, but it does *not* even yield finiteness of $N(r, d)$ in the Bárány–Larman problem.

Furthermore, neither the Bárány–Larman result for $d = 2$ nor the Živaljević–Vrećica version is tight: A tight version should generalize Birch's Theorem 1*, and thus not use more than $3r - 2$ points in the plane!

A first sharp version, with a proof of $N(r, d) = r(d + 1)$ in the case that $r + 1$ is prime, was obtained only very recently, as a (not quite direct) consequence of the "tight colored Tverberg theorem" below (announced in 2009, to be published 2014, see [7]). For this, a substantial change in the concepts of *colored* and *rainbow* was needed: We allow for more than $d+1$ colors, and a rainbow face does not need to use all the different colors, and instead of requiring that all color classes have size at least r, they now are required to have less than r elements (so, indeed, no color is used in all the blocks of a partition into r subsets). Here it is:

Tight colored Tverberg theorem [7]**:** *Let $d \geq 1$, r prime, $N \geq (r-1)(d+1)$, and $f : \Delta_N \longrightarrow \mathbb{R}^d$ continuous, where the $N+1$ vertices of Δ_N are colored and each color class C_i has size $|C_i| \leq r-1$ (so there are at least $d+2$ different colors). Then Δ_N has r disjoint* rainbow faces *whose images under f intersect.*

This is also the first colored Tverberg theorem that has (the r prime case of) the topological Tverberg theorem as a special case. Our original proof for this in [7] used equivariant obstruction theory. (A part of the proof was rephrased in terms of degrees by Vrećica and Živaljević [14], with an error in the value given for the degree; it was also elaborated on by Matoušek, Tancer, and Wagner [11].) A substantially different proof via Fadell–Husseini index, with further applications, such as the finiteness of $N(r,d)$ for prime r, was provided in [8].

Colored versions via constraints. If one is content with a non-tight result, then indeed one can get a colored topological Tverberg theorem "nearly for free" from the original topological Tverberg theorem, by allowing for extra points but adding constraints, as follows. (See [6] for details.)

Lemma (A Tverberg unavoidable subcomplex)**.** Let $d \geq 1$, let r be a prime power, and $N \geq (r-1)(d+1)$. If $f : \Delta_N \to \mathbb{R}^d$ is continuous, then for any set C of at most $2r-1$ vertices of Δ_N, every Tverberg r-partition for f has a block that has at most one vertex in C.

Indeed, in this setting Tverberg r-partitions exist, and by the pigeonhole principle not all r blocks of a Tverberg r-partition can have at least two vertices in C.

Weak colored Tverberg theorem [6]**:** *Let $d \geq 1$, let r be a prime power, and $N \geq 2(r-1)(d+1)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of Δ_N are colored by $d+1$ colors, where each color class C_i has cardinality at most $2r-1$, then there are r pairwise disjoint rainbow faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \varnothing$.*

Proof. For $0 \leq i \leq d$, let $g_i : \Delta_N \to \mathbb{R}$ be the Euclidean distance from the subcomplex of Δ_N formed by the faces that have at most one vertex in the color class C_i. Now consider the continuous map $F := (f, g_0, \dots, g_d) : \Delta_N \longrightarrow \mathbb{R}^{2d+1}$. According to the topological Tverberg theorem, this map F has a Tverberg r-partition into r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$, such that $f(x_1) = \dots = f(x_r)$ for points $x_k \in \sigma_k$ in disjoint faces $\sigma_k \subset \Delta_N$, and such that $g_i(x_1) = \dots = g_i(x_r)$ for $0 \leq i \leq d$. However, by the lemma for each color i one of the faces σ_k has only one vertex in C_i, that is, $g_i(x_k) = 0$, and this implies that *all* faces σ_k have only one vertex in C_i. □

This "weak colored Tverberg theorem" is stronger than the one by Živaljević and Vrećica [16].

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Open Problems in Discrete Geometry

Collected by Tillmann Miltzow

Problem 1 (Tillmann Miltzow)**.** Given n red, n blue and n green lines in the plane in general position. These lines form a line arrangement. Does there always exist an *alternating path* of length $\Theta(n^2)$ in the dual of the line arrangement?

A path in the dual is said to be *alternating* if it does not cross over two edges of the same color consecutively.

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Problem 2 (Hiroshi Maehara)**.** Let ABC be a spherical triangle with fixed arc-lengths a, b, c on a sphere of (variable) radius $r > (a+b+c)/(2\pi)$. By the spherical cosine law, we have

$$\cos \sphericalangle A = \frac{\cos(a/r) - \cos(b/r)\cos(c/r)}{\sin(b/r)\sin(c/r)}.$$

Find an elementary proof of the fact that $\cos \sphericalangle A$ is monotone increasing function of r.

Remark. It follows from the Alexandrov-Toponogov comparison theorem that $\sphericalangle A$ is a monotone decreasing function of r, and hence $\cos \sphericalangle A$ is monotone increasing function of r, see e.g. [1, Theorem 3.91, p. 189]. (Arseniy Akopyan informed me on this theorem.)

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Problem 3 (Michael Gene Dobbins)**.** What is the minimum volume among all convex sets in $\mathbb{R}^d$ that have width at least 1 in all directions? This is one variation of a question posed by Kakeya in 1917 [1], who additionally required that a unit length segment can be continuously rotated inside the set. Pál showed that the answer is given by an equilateral triangle of area $1/\sqrt{3}$ [2]. Further insight may be gained from the proof of a recent generalization of Pál's result [3].

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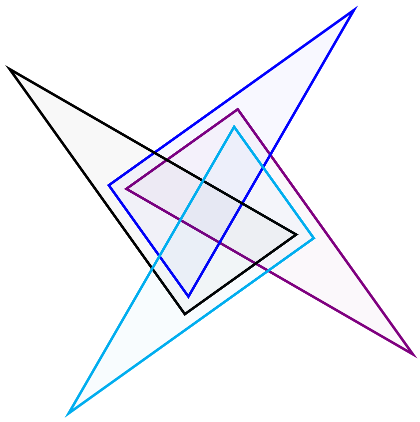
Problem 4 (Michael Gene Dobbins)**.** We say an arrangement of convex sets is in convex position when the sets are indexed by $1, \ldots, n$ and for every triple $\{K_i, K_j, K_k\}$, $i < j < k$: each set intersects the boundary of the convex hull of the triple in a single connected component, these components are pairwise disjoint, and the sets appear counter-clockwise around the boundary in order according to their indices $(\cdots i, j, k \cdots)$. We define a metric on pairs of arrangements by taking the maximum of the Hausdorff distance among pairs of sets with the same index. By k-gon we mean a convex polygon with *at most* k vertices.

Is the space of n k-gons in convex position modulo affine transformations contractible? Difficulties may arise when the sets intersect, as seen in the figure below.



Four triangles in convex position.

This question arose from research with Andreas Holmsen and Alfredo Hubard generalizing the Erdős-Szekeres Theorem to arrangements of convex sets [1]. In a forthcoming paper, we show in particular that this space is contractible if the sets are not required to be k-gons.

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Problem 5 (Dömötör Pálvölgyi)**.** The Voronoi game is a game played on a compact metric space with a probability measure by two players. The players alternate in placing a facility on a single point in the space. The game lasts for a fixed number of rounds. At the end of the game, the space is divided between the two players: each player receives the area which is closer to his or her facilities, or in other words, the sum of the areas of the corresponding regions in the Voronoi diagram.

A very interesting special case is the one-round game, where the first player claims t vertices, then the second player claims one. Denote by $VR_{t:1}(G, 1)$ the fraction that the first player gets after an optimal play on G and the minimum over all G for such a game by $VR_{t:1} = \inf_G VR_{t:1}(G, 1)$.

We know very little about $VR_{t:1}$, it is even possible that $VR_{t:1} = 0$ for every t or maybe already $VR_{2:1} > 0$. Recently it was shown by David Speyer that $VR_{2:1} \leq \frac{10}{21}$ and then by Sam Zbarsky that $VR_{t:1} \leq \frac{t-1}{t+1}$. Is this bound sharp?

See http://mathoverflow.net/questions/148466 for discussions and related results.

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Problem 6 (Pablo Soberón)**.** What is the minimum number $n = n(t, d, k)$ such that for any t probability measures in $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into n convex pieces so that its parts can be distributed into k sets $A_1, A_2, \ldots, A_k$ satisfying

$$\mu_j(A_i) = \frac{1}{k}$$

for all i, j.

Problem 7 (Boris Bukh)**.** Let $P \subset \mathbb{R}^2$ be a set of n points, and let $f \colon S \to \{-1, +1\}$ be a two-coloring. The discrepancy of a line l is defined as

$$D(l) = \left| \sum_{\substack{p \in P \\ p \text{ left of } l}} f(p) - \sum_{\substack{p \in P \\ p \text{ right of } l}} f(p) \right|.$$

Is it true that for every $\alpha > 0$ there exists $\beta < \alpha$ with the following property: whenever P and f are such that $D(l) \leq \alpha n$ for every line l, there exists a point $q \in \mathbb{R}^2$ such that $D(l) \leq \beta n$ for all lines l passing through q?

Problem 8 (Luis Montejano)**.** Let X be a collection of n points in general position in $\mathbb{R}^3$.

- If $|X| = 12$, then X does not admit a transversal line to the convex hull of all 6-sets of X.
- If $|X| = 9$, then X admits a transversal line to the convex hull of all 6-sets of X.
- The cyclic polytope with 11 points does not admit a transversal line to the convex hull of all its 6-sets, and there is a collection X of 11 points in general position that admits a transversal line to the convex hull of all 6-sets of X.

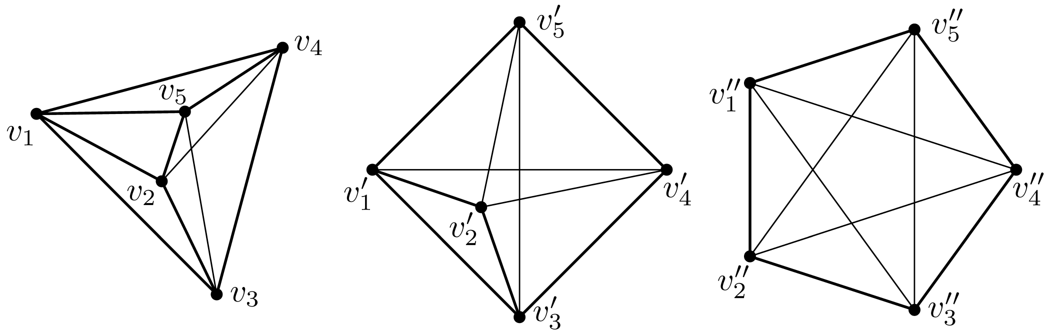
Problem: Give a collection of 10 points in general position in $\mathbb{R}^3$ without a transversal line to the convex hull of all its 6-sets.

Problem 9 (Roman Karasev)**.** For an acute triangle T, denote by $\xi(T)$ the perimeter of its orthic triangle, i.e., the triangle formed by the bases of T's altitudes. Is it true that whenever T is covered by a set $S_1, \ldots, S_N$ of other acute triangles, then

$$\xi(T) \leq \sum_{i=1}^{N} \xi(S_i)?$$

Problem 10 (Emo Welzl, with Alexander Pilz)**.** Crossing-Maximal Order Types

Given two finite equal-size point sets P and Q in general position in the plane, a bijection $P \to Q$, $p \mapsto p'$, is called *crossing-preserving* if whenever the segment pq crosses the segment rs (for four distinct points p, q, r and s in P) then $p'q'$ crosses $r's'$. If such a mapping exists, we say that Q *crossing-dominates* P, in symbols $P \leq_x Q$. And if $P \leq_x Q$ and, moreover, there is no such mapping from Q to P we say that Q *strictly crossing-dominates* P, in symbols $P <_x Q$.



Points sets P, P', and P'' on five points. The mappings $p \mapsto p'$ and $p' \mapsto p''$ are crossing-preserving, therefore $P \leq_x P' \leq_x P''$; in fact,$P <_x P' <_x P''$.

A point set P is *crossing-maximal* if there is no point set Q with $P <_x Q$. Point sets in convex position are crossing-maximal, but other examples exist – we would like to know how many.

Our question is how the ratio of the number of order-types of crossing-maximal point sets on n points versus the number of all order-types of point sets with n points behaves, as n grows. We know that for each n up to 5, there is a unique maximal order-type (convex position), 3 out of the 16 order-types for 6 points (18%) constitute a maximal type, 17/135 (13%) is the ratio for 7, 489/3′315 for 8 (15%) points, and 28′103/155′517 (18%) for 9. (Note here that point sets of distinct order-type can be equivalent in the $\leq_x$ relation.)

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