Mathematisches Forschungsinstitut Oberwolfach

Dirichlet Form Theory and its Applications

Organised by
Sergio Albeverio, Bonn
Zhen-Qing Chen, Seattle
Masatoshi Fukushima, Osaka
Michael Röckner, Bielefeld

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ABSTRACT. Theory of Dirichlet forms is one of the main achievements in modern probability theory. It provides a powerful connection between probabilistic and analytic potential theory. It is also an effective machinery for studying various stochastic models, especially those with non-smooth data, on fractal-like spaces or spaces of infinite dimensions. The Dirichlet form theory has numerous interactions with other areas of mathematics and sciences.

This workshop brought together top experts in Dirichlet form theory and related fields as well as promising young researchers, with the common theme of developing new foundational methods and their applications to specific areas of probability. It provided a unique opportunity for the interaction between the established scholars and young researchers.

Introduction by the Organisers

The workshop *Dirichlet Form Theory and Its Applications*, organized by Sergio Albeverio (University of Bonn, Germany), Zhen-Qing Chen (University of Washington, USA), Masatoshi Fukushima (Osaka University, Japan) and Michael Röckner (University of Bielefeld, Germany) was well attended with 52 participants with broad geographic representation from Canada, China, France, Germany, Italy, Korea, Romanian, Spain, Japan, UK, and USA. Women and young researchers had a strong presence among the invited participants. This workshop was a nice blend of researchers with various backgrounds, but sharing a common interests in the subject of this workshop. The workshop had 27 invited talks with 7 short communications, leaving plenty of time for discussions. To accommodate the travel

delay caused by the Germany train driver strike on the day of arrival for the workshop, we shifted and started the program Monday afternoon of October 20, 2014. It worked out well. Most young researchers at the workshop were invited to present their work.

To create some focus for the very broad topic of the workshop, we had chosen a few areas of concentration, including (i) Theory of Dirichlet forms; (ii) Stochastic analysis and potential theory on infinite dimensional spaces; (iii) Analysis on fractals and percolation clusters; (iv) Jump type processes and non-local operators. Many talks and discussions were directly related to these themes, and covered diverse topics and areas including the following:

- Fukushima decomposition for quasi-regular semi-Dirichlet forms (Zhi-Ming Ma)
- Heat kernel estimates: for strongly local regular Dirichlet forms (Alexander Grigoryan), for killed symmetric Lévy processes in half spaces (Panki Kim), for Liouville Brownian motion in $\mathbb{R}^2$ (Naotaka Kajino), and for Brownian motion on a space of varying dimension (Shuwen Lou).
- Invariance principle for random walks in ergodic random media and for ergodic random conductance models, and for random walks on trees (Jean-Dominique Deuschel, Takashi Kumagai and Anita Winter).
- Geometric and functional inequalities for Dirichlet forms, and curvature-dimension conditions with applications including optimal mass transport (Michel Ledoux and Karl-Theodor Sturm)
- Trace Dirichlet forms for random walks on trees associated to p-adic numbers (Jun Kigami) and for reflected Brownian motion in Euclidean domains (Lucian Beznea).
- Dirichlet forms on the cone of random measures with applications in ecology (Yuri Kondratiev and Diana Putan)
- Intrinsic ultracontractivity for skew product diffusion processes (Matsuyo Tomisaki)
- Criticality and subcriticality for (generalized) Schrödinger forms as related to properties of the spectrum of time changed processes (Masayoshi Takeda)
- BV functions and Skorohod equations on Wiener spaces (Masanori Hino)
- Sticky reflected distorted Brownian motion for the wetting model (Robert Vosshall)
- Long time asymptotics for the paths of symmetric Markov processes (Yuichi Shiozawa)
- Instability properties of sequential limits of Dirichlet forms (Toshihiro Uemura)
- Local Dirichlet structure on the Poisson space associated with an energy form and a lent particle method (Laurent Denis)
- Stochastic averaging via Dirichlet forms (Max von Renesse)

- Differential calculus for Dirichlet forms on noncommutative spaces (Fabio Cipriani)

Several talks were related to or motivated by mathematical physics: scaling limit of interface models (Torben Fattler), Dirichlet forms related to the stochastic quantization (Hiroshi Kawabi), magnetic energy forms and Feynman-Kac-Ito formulae, including non-local setups (Michael Hinz).

There were three talks devoted to stochastic partial differential equations and infinite dimensional stochastic functional equations (Wilhelm Stannat, Rongchan Zhu and Xiangchan Zhu).

Potential theory for stochastic processes was also covered during this workshop. The topic includes minimal thinness of symmetric Lévy processes (Renming Song), ergodicity of infinite dimensional Feller-Markov processes (Fuzhou Gong), regularity of harmonic functions for non-local operators (Moritz Kassmann), Malliavin smoothness for SDEs with singular coefficients (Tusheng Zhang). Andreas Eberle presented a talk on a random walk Metropolis algorithm in high dimension, while Cheng Ouyang talked about a geometric derivation of Lévy's arcsine law for occupation time on hypersurfaces of a Riemannian manifold.

The diversity of the topics and participants stimulated many in-depths discussions. In particular, it became clear how vividly the approach of the Dirichlet form theory interacts with other areas of analysis and probability, as well as geometry and mathematical physics. Also the interplay of theory and applications was greatly enhanced. The workshop helped not only strengthening the exiting connections but also making new friends and fostered new collaborations among the participants who came from diverse fields of mathematics and geographic locations.

Workshop: Dirichlet Form Theory and its Applications

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Abstracts

Some Recent Results on Quasi-regular Semi-Dirichlet Forms
ZHI-MING MA

(joint work with Wei Sun, Li-Fei Wang)

In this talk we present some results on quasi-regular semi-Dirichlet forms (cf. [4]). In particular, we present our recent results on Fukushima type decomposition for semi-Dirichlet forms and discuss some related topics (cf.[5]) .

Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form which is not necessarily local. We impose the following assumption.

Assumption 1.3 There exist $\{V_n\} \in \Theta$ and locally bounded function $\{C_n\}$ on $\mathbb{R}$ such that for each $n \in \mathbb{N}$, if $u, v \in D(\mathcal{E})_{V_n,b}$ then $uv \in D(\mathcal{E})$ and

$$\mathcal{E}(uv, uv) \leq C_n(\|u\|_\infty + \|v\|_\infty)(\mathcal{E}_1(u, u) + \mathcal{E}_1(v, v)).$$

We obtain the following result.

Theorem 1.4 Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 1.3. Then for $u \in D(\mathcal{E})_{loc}$ the following two assertions are equivalent to each other.

(i) u admits a Fukushima type decomposition. That is, there exist $M^{[u]} \in \mathcal{M}_{loc}^{I(\zeta)}$ and $N^{[u]} \in \mathcal{L}_c$ such that

(1) $\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E.$

(ii) u satisfies Condition (S) specified below.

$$(S): \quad \mu_u(dx) := \int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx) \text{ is a smooth measure.}$$

Moreover, if u satisfies Condition (S), then the decomposition (1) is unique up to the equivalence of local AFs, and the continuous part of $M^{[u]}$ belongs to $\dot{\mathcal{M}}_{loc}$.

In the above Theorem, $N^{[u]} \in \mathcal{L}_c$ means that $N^{[u]}$ is a local AF and there exists $\{E_n\} \in \Theta$ such that for all n, $t \mapsto N_{t \wedge \tau_{E_n}}$ is continuous and is of zero quadratic variation. $M^{[u]} \in \mathcal{M}_{loc}^{I(\zeta)}$ means that $M^{[u]}$ is a locally square integrable MAF on the set $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$, with ζ being the lifetime of X and ζ_i the totally inaccessible part of ζ; and $N^{[u]}$ is a local AF which is continuous and has zero quadratic variation on $I(\zeta)$.

It is worth to point out that Assumption 1.3 is weaker than the condition of local control in [3] and the condition ($\mathcal{E}$.5) in [6]. We are very grateful to Professor Oshima for sending us his new book [6]. The condition ($\mathcal{E}$.5) in [6] stimulated us to formulate Assumption 1.3.

Remarks
1. Theorem 1.4 extends the corresponding result of [3].
2. Theorem 1.4 is an extension of [6, Theorem 5.1.5].

3. Theorem 1.4 extends the corresponding results of [1, Theorem 5.5.1] and [2, Theorem 4.2] from the symmetric case to the semi-Dirichlet form case.

4. In Theorem 1.4 if we use $\mathcal{M}_{loc}^{[\![0,\zeta[\![}$ instead of $\mathcal{M}_{loc}^{I(\zeta)}$, then the uniqueness of the decomposition may fail to be true.

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Generalized capacity, Harnack inequality, and heat kernels on metric measure spaces

Alexander Grigor'yan

(joint work with Jiaxin Hu, Ka-Sing Lau)

Let (M, d) be a locally compact separable metric space, whose metric balls $B(x, r)$ are precompact. Let μ be a Radon measure on M with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form in $L^2(M, \mu)$. Denote by $\{P_t\}_{t \geq 0}$ the associated heat semigroup and by $p_t(x, y)$ the integral kernel of the operator P_t should it exists. Our aim is to obtain convenient equivalent conditions for sub-Gaussian estimates of the heat kernel $p_t(x, y)$.

Set $V(x, r) = \mu(B(x, r))$ and assume that for all $x \in M$ and $0 < r \leq R$,

$$(1) \qquad C^{-1} \left(\frac{R}{r}\right)^{\alpha'} \leq \frac{V(x, R)}{V(x, r)} \leq C \left(\frac{R}{r}\right)^{\alpha},$$

for some constants $\alpha', \alpha > 0$.

Fix a function $\Psi : (0, \infty) \to (0, \infty)$ and assume that it is a continuous increasing bijection satisfying for all $0 < r \leq R$

$$(2) \qquad C^{-1} \left(\frac{R}{r}\right)^{\beta} \leq \frac{\Psi(R)}{\Psi(r)} \leq C \left(\frac{R}{r}\right)^{\beta'}$$

for some constants $1 < \beta \leq \beta'$. We refer to $\Psi(r)$ as a gauge function.

We say that (UE) holds if the heat kernel $p_t(x, y)$ exists and satisfies the following *upper estimate*

$$(3) \qquad p_t(x, y) \leq \frac{C}{V(x, \Psi^{-1}(t))} \exp\left(-\frac{1}{2} t \Phi\left(c \frac{d(x, y)}{t}\right)\right)$$

for all $t > 0$ and μ-almost all $x, y \in M$, where

$$\Phi(s) := \sup_{r>0} \left\{ \frac{s}{r} - \frac{1}{\Psi(r)} \right\}.$$

For example, if $\Psi(r) = r^\beta$ for $\beta > 1$, then supremum here is attained at $r = (s/\beta)^{-\frac{1}{\beta-1}}$ which yields

$$\Phi(s) = C s^{\beta/(\beta-1)}.$$

In this case (UE) coincides with sub-Gaussian upper bound

$$p_t(x,y) \leq \frac{C}{V\left(x, t^{1/\beta}\right)} \exp\left(-c \left(\frac{d^\beta(x,y)}{t} \right)^{\frac{1}{\beta-1}} \right).$$

We say (NLE) holds if the heat kernel $p_t(x,y)$ exists and satisfies the following *near-diagonal lower estimate*

$$(4) \qquad\qquad p_t(x,y) \geq \frac{c}{V\left(x, \Psi^{-1}(t)\right)},$$

for all $t > 0$ and μ-almost all $x, y \in M$ such that $d(x,y) \leq \varepsilon \Psi^{-1}(t)$. Under the latter condition, the term $t\Phi\left(c\frac{d(x,y)}{t}\right)$ in (3) is bounded by a constant, so that the upper bound (UE) is consistent with (NLE).

We say that (PI) (=Poincaré inequality) holds if there is $\sigma \in (0,1)$ and $C > 0$ such that, for any ball $B = B(x,r)$ and for any function $f \in \mathcal{F}$,

$$\int_{\sigma B} (f - m)^2 \, d\mu \leq C \Psi(r) \int_B d\Gamma \langle f, f \rangle,$$

where Γ is the energy measure of $(\mathcal{E}, \mathcal{F})$ and $m = \frac{1}{\mu(\sigma B)} \int_{\sigma B} f d\mu$.

Definition. Let Ω be an open subset of M and $A \subset \Omega$ be a Borel set. For any measurable function u on Ω, define the *generalized capacity* $\mathrm{cap}_u(A, \Omega)$ by

$$\mathrm{cap}_u(A, \Omega) = \inf \left\{ \int_\Omega u^2 d\Gamma \langle \varphi, \varphi \rangle : \varphi \in \mathrm{cutoff}(A, \Omega) \right\}.$$

Generalized capacity condition. We say that the *generalized capacity condition* $(\mathrm{Gcap}_\leq)$ holds if, for any $u \in \mathcal{F}$ and for any two concentric balls $B_1 := B(x, R)$ and $B_2 := B(x, R+r)$,

$$\mathrm{cap}_u(B_1, B_2) \leq C_1 \int_{B_2 \setminus B_1} d\Gamma \langle u, u \rangle + \frac{C_2}{\Psi(r)} \int_{B_2 \setminus B_1} u^2 d\mu.$$

By definition of cap_u, we can restate $(\mathrm{Gcap}_\leq)$ as follows: for any $u \in \mathcal{F}$ there is some $\varphi \in \mathrm{cutoff}(B_1, B_2)$ such that

$$(5) \qquad \int_{B_2 \setminus B_1} u^2 d\Gamma \langle \varphi, \varphi \rangle \leq C_1 \int_{B_2 \setminus B_1} d\Gamma \langle u, u \rangle + \frac{C_2}{\Psi(r)} \int_{B_2 \setminus B_1} u^2 d\mu.$$

Condition (5) is very close to the condition (CSA) (*cutoff Sobolev inequality in annulus*) that was introduced recently by Andres and Barlow. Namely, condition

(CSA) means the following: there exists $\phi \in$ cutoff (B_1, B_2) such that (5) holds for any $u \in \mathcal{F}$. Condition $\left(\mathrm{Gcap}_{\leq}\right)$ is obviously weaker than (CSA) as we allow φ in (5) to depend on u.

Theorem 1. *Assuming* (1), *we have*

$$
\begin{aligned}
(UE) + (NLE) \quad &\Leftrightarrow \quad (PI) + \left(\mathrm{Gcap}_{\leq}\right) \\
&\Leftrightarrow \quad (PI) + (CSA).
\end{aligned}
$$

Define the Faber-Krahn inequality (FK) with the gauge function $\Psi(r)$ as follows. For an open set $\Omega \subset M$, denote by $\lambda_{\min}(\Omega)$ the bottom of the spectrum of the Dirichlet Laplacian in Ω. The condition (FK) means that, for any ball B of radius r and any $\Omega \subset B$,

$$
\lambda_{\min}(\Omega) \geq \frac{c}{\Psi(r)} \left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\nu}.
$$

For a conservative Dirichlet form $(\mathcal{E}, \mathcal{F})$, Andres and Barlow '12 have proved the following equivalence:

$$
(6) \qquad\qquad (UE) \Leftrightarrow (FK) + (CSA),
$$

where (CSA) is the cutoff Sobolev inequality in annulus, mentioned above. The following theorem is our second main result that provides a slight improvement of (6).

Theorem 2. *Assuming that $V(x, r)$ satisfies the volume doubling and that $(\mathcal{E}, \mathcal{F})$ is conservative, we have the equivalence*

$$
(UE) \Leftrightarrow (FK) + (\mathrm{Gcap}_{\leq}).
$$

Criticality and Subcriticality of Generalized Schrödinger Forms
Masayoshi Takeda

In [5], we define the subcriticality, criticality and superciriticality for a Schrödinger form and characterize these properties in terms of the bottom of the spectrum of a time changed process. In the process, we prove the existence of a harmonic function (or ground state) of the Schrödinger form and show that it has a bounded, positive, continuous version with invariance for its Schrödinger semigroup. As an application of this fact, we derive some condition for the maximum principle and Liouville property of Schrödinger operators.

Let X be a locally compact separable metric space and m a positive Radon measure on X with full topological support. Let $\mathbf{M} = (\mathbf{P_x}, \mathbf{X_t}, \zeta)$ be an m-symmetric Hunt process with life time $\zeta = \inf\{t > 0 : X_t = \infty\}$. We assume that $\mathbf{M}$ is irreducible, strong Feller and, in addition, that $\mathbf{M}$ generates a *regular* Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X; m)$. Let $\mu = \mu^+ - \mu^-$ be a signed Radon measure

such that the positive (resp. negative) part μ^+ (resp. μ^-) belongs to the local Kato class (resp. the Kato class). We consider a Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$:

$$\mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) + \int_X u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^\mu)(= \mathcal{D}(\mathcal{E}) \cap L^2(X; \mu^+)).$$

We denote by A_t^μ the positive continuous additive functional in the Revuz correspondence to μ, and define the Feynman-Kac semigroup $\{p_t^\mu\}_{t \geq 0}$ by

$$p_t^\mu f(x) = E_x \left(e^{-A_t^\mu} f(X_t) \right).$$

We denote by $\mathcal{D}_{loc}(\mathcal{E})$ the set of functions locally in $\mathcal{D}(\mathcal{E})$ and introduce a function space by

$$\mathcal{H}^+(\mu) = \{h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C(X) : h > 0, \ p_t^\mu h \leq h\}.$$

Suppose that $\mathcal{H}^+(\mu)$ is not empty and take $h \in \mathcal{H}^+(\mu)$. We then define the bilinear form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ on $L^2(X; h^2 m)$ through h-transform of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$:

(1)
$$\begin{cases} \mathcal{E}^{\mu,h}(u, u) = \mathcal{E}^\mu(hu, hu) \\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(X; h^2 m) : hu \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{cases}$$

We see that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ turns out to be a regular Dirichlet form on $L^2(X; h^2 m)$. Consequently, if $\mathcal{H}^+(\mu)$ is not empty, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is non-negative, $\mathcal{E}^\mu(u, u) \geq 0$ for all $u \in \mathcal{D}(\mathcal{E}^\mu)$. The semigroup $\{p_t^{\mu,h}\}_{t \geq 0}$ generated by $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is expressed as

$$p_t^{\mu,h} f(x) = \frac{1}{h(x)} p_t^\mu(hf)(x).$$

We define the subcriticality, criticality and superciriticality for the Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ as follows: $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *subcritical* (resp. *critical*) if $\mathcal{H}^+(\mu)$ is not empty and $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient (resp. recurrent) for some $h \in \mathcal{H}^+(\mu)$. Besides, $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *supercritical* if $\mathcal{H}^+(\mu)$ is empty. We show that these definitions are well-defined ([5, Lemma 3.2, Lemma 3.3]).

We denote by $\mathbf{M}^{\mu^+} = (P_x^{\mu^+}, X_t)$ the subprocess of $\mathbf{M}$ by the multiplicative functional $\exp(-A_t^{\mu^+})$ and by $(\mathcal{E}^{\mu^+}, \mathcal{D}(\mathcal{E}^{\mu^+}))$ the Dirichlet form generated by $\mathbf{M}^{\mu^+}$. Suppose the negative part μ^- is non-trivial and Green-tight with respect to $\mathbf{M}^{\mu^+}$. We then define $\lambda(\mu)$ by

(2) $$\lambda(\mu) := \inf \left\{ \mathcal{E}(u, u) + \int_X u^2 d\mu^+ : u \in \mathcal{D}(\mathcal{E}), \int_X u^2 d\mu^- = 1 \right\}.$$

We show in [4, Theorem 2.1] that the minimizer of (2) exists in the extended Dirichlet space $\mathcal{D}_e(\mathcal{E}^{\mu^+})$. Moreover, we prove in [5] that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, critical and supercritical, if and only if $\lambda(\mu) > 1$, $\lambda(\mu) = 1$ and $\lambda(\mu) < 1$ respectively. In fact, if $\lambda(\mu) > 1$, then the gauge function $g^\mu(x) = E_x^{\mu^+}(\exp(A_\zeta^{\mu^-}))$ is bounded and in $\mathcal{H}^+(\mu)$. If $\lambda(\mu) = 1$, the minimizer h has a bounded, positive, continuous version in $\mathcal{H}^+(\mu)$ with p_t^μ-invariance, $p_t^\mu h = h$ ([5, Lemma 4.15, Lemma 4.16, Corollary 4.17]).

Let

(3) $\qquad \mathcal{H}^{ba}(\mu) = \{h \in \mathcal{B}(X) : h \text{ is bounded above, } p_t^\mu h \geq h\}$

and define the maximum principle:

(**MP**) If $h \in \mathcal{H}^{ba}(\mu)$, then $h(x) \leq 0$ for all $x \in X$.

We then have the next theorem:

Theorem 1. Assume that

(**A**) $E_x\left(\exp(-A_\infty^{\mu^+}); \zeta = \infty\right) = 0.$

Then (**MP**) holds if and only if $\lambda(\mu) > 1$.

"If part" follows from the fact that $\lambda(\mu) > 1$ is equivalent to the gaugeability of μ,
$\sup_{x \in X} E^{\mu^+}(\exp(A_\zeta^{\mu^-}) < \infty$. If $\lambda(\mu) \leq 1$, then the minimizer h mentioned above belongs to $\mathcal{H}^{ba}(\mu)$ and (**MP**) does not holds. In [1], they define a maximum principle for a uniformly elliptic operator of second order, $L = M + c = a_{i,j}\partial_i\partial_j + b_i\partial_i + c$, on a general bounded domain D of $\mathbb{R}^d$. Let u_0 be a solution to the equation $Mu = -1$ vanishing on ∂D in a suitable sense: define $\mathcal{S}$ by the set of sequences $\{x_n\}_{n=1}^\infty \subset D$ such that x_n converges to a point of the boundary ∂D and $u_0(x_n)$ converges to 0. They say that L satisfies the *refined maximum principle*, if $Lh \geq 0$ on D, h is bounded above, and $\limsup_{n \to \infty} h(x_n) \leq 0$ for any $\{x_n\}_{n=1}^\infty \in \mathcal{S}$, then $h \leq 0$ on D. Moreover, they prove that L satisfies the refined maximum principle if and only if the principal eigenvalue of $-L$ is positive.

Note that u_0 equals $E_x(\tau_D)$, where P_x is the diffusion process with generator M and τ_D is the first exit time from D. We see that if D is bounded (more generally, Green-bounded, i.e., $\sup_{x \in D} E_x(\tau_D) < \infty$), then $\mathcal{S}$ is identical to the set of sequences $\{x_n\}$ such that $x_n \to \infty$ $(x_n \to \partial D)$ and $E_{x_n}(\exp(-\tau_D)) \to 1$ as $n \to \infty$. Considering this fact, we define

(4) $\qquad \mathcal{S} = \left\{\{x_n\} \subset X : x_n \to \infty \text{ and } E_{x_n}(e^{-\zeta}) \to 0 \text{ as } n \to \infty\right\}.$

We see that $\mathcal{S}$ equals

$\qquad \widetilde{\mathcal{S}} = \{\{x_n\} \subset X : x_n \to \infty \text{ and for any } \epsilon > 0, P_{x_n}(\zeta > \epsilon) \to 0 \text{ as } n \to \infty\}.$

Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local and set

$$\widetilde{\mathcal{H}}^{ba}(\mu) = \left\{ h : \begin{array}{l} h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C(X) \text{ is bounded above, } \mathcal{E}^\mu(h, \varphi) \leq 0 \text{ for} \\ \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(X), \limsup_{n \to \infty} h(x_n) \leq 0 \text{ for } \forall\{x_n\} \in \mathcal{S}. \end{array} \right\}$$

Following [1], we here define the *refined maximum principle*:

(**RMP**) If $h \in \widetilde{\mathcal{H}}^{ba}(\mu)$, then $h(x) \leq 0$ for all $x \in X$.

We can show that $\widetilde{\mathcal{H}}^{ba}(\mu) \subset \mathcal{H}^{ba}(\mu)$ and have the next theorem as a corollary:

Theorem 2. Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local. Under Assumption (**A**) in Theorem 1

(5) $$\lambda(\mu) > 1 \Longrightarrow (\mathbf{RMP}).$$

We would like to emphasis that in the case above if D is bounded and L is symmetric, the principal eigenvalue λ_0 of $-L$ is positive if and only if $\lambda(\mu) > 1$, $d\mu = c(x)dx$. However, $\lambda(\mu) > 1$ does not imply $\lambda_0 > 0$ for a general domain D, while $\lambda_0 > 0$ generally implies $\lambda(\mu) > 1$.

Let us introduce

$$\mathcal{H}^b(\mu) = \{h \in \mathcal{B}_b(X) : \ p_t^\mu h = h\}$$

and define the Liouville property by

($\mathbf{L}$) If $h \in \mathcal{H}^b(\mu)$, then $h(x) = 0$ for all $x \in X$.

We see, as a corollary of Theorem 1, that

(6) $$\lambda(\mu) > 1 \Longrightarrow (\mathbf{L}).$$

Let

$$\widetilde{\mathcal{H}}^b(\mu) = \{h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C_b(X) : \mathcal{E}^\mu(h, \varphi) = 0, \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(X)\}.$$

We then see that $\widetilde{\mathcal{H}}^b(\mu) \subset \mathcal{H}^b(\mu)$ if $\mathbf{M}$ is conservative. Thus if we denote by ($\widetilde{\mathbf{L}}$) the property that $\widetilde{\mathcal{H}}^b(\mu) = \{0\}$, we see from Theorem 1 that if $\mathbf{M}$ is conservative and $E_x(\exp(-A_\infty^{\mu^+})) = 0$, then

(7) $$\lambda(\mu) > 1 \Longrightarrow (\widetilde{\mathbf{L}}).$$

Let us apply (7) to a transient Brownian motion $\mathbf{M} = (P_x, B_t)$. We know that under ($\mathbf{A}$), i.e., $E_x\left(\exp(-A_\infty^{\mu^+})\right) = 0$,

(8) $$\inf\left\{\frac{1}{2}\mathbb{D}(v, v) + \int_{\mathbb{R}^d} v^2 d\mu^+ : u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} v^2 d\mu^- = 1\right\} > 1 \Longrightarrow (\widetilde{\mathbf{L}}).$$

In [2], Grigor'yan and Hansen prove this fact in the case that $\mu^- = 0$. The equation (8) tells us that if μ^- is small in the sense that the right hand side of (8) holds, ($\widetilde{\mathbf{L}}$) still follows. It is shown in [3] that if $\sup_{x \in \mathbb{R}^d} E_x\left(\exp(-A_\infty^{\mu^-})\right) < \infty$, ($\widetilde{\mathbf{L}}$) is equivalent to ($\mathbf{A}$).

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Markov processes on the Lipschitz boundary for the Neumann and Robin problems

Lucian Beznea
(joint work with Speranţa Vlădoiu)

In a series of works (cf. [Arendt, W., ter Elst, A.F.M., *J. Operator Theory*, 2012], [Arendt, W., ter Elst, A.F.M., *J. Differential Equations*, 2011], and [Arendt, W., Mazzeo, R., *Ulmer Seminare Heft*, 2007]) Arendt, ter Elst, and Mazzeo constructed C_0-semigroups of operators and the corresponding infinitesimal generators on the L^2-space of the boundary of a bounded open Euclidean set $\mathcal{O}$ with a Lipschitz boundary Γ, associated to the Neumann and Robin problems.

Our aim is to associate Markov processes on the Lipschitz boundary to the Neumann and Robin problems. It turns out that the transition functions of the obtained processes on the boundary, regarded as families of $L^2(\Gamma)$-operators, are precisely the C_0-semigroups mentioned above.

The first step of our approach is to construct sub-Markovian C_0-semigroups of contractions on the space $L^p(\Gamma)$, $p > 1$, induced by the boundary conditions. Then we show that these semigroups are actually given by the transition functions of some standard Markov processes with state space Γ. Note that the L^p-method we use was initially tailored for constructing martingale solutions for singular stochastic differential equations on Hilbert spaces. In our frame the necessary hypotheses are fulfilled, due to several embedding results for function spaces on the boundary.

Using a result from [Bass, R.F., Hsu, P.E., *Annals of Probab.*, 1991] we show that for the Lipschitz boundary the boundary process of the Neumann problem is the time changed (induced by a continuous additive functional carried by Γ) of the reflected Brownian motion. We emphasize the connections with the Dirichlet forms approach; cf. [Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet Forms and Symmetric Markov Processes.* 2nd edition, De Gruyter, 2011]

The Robin problem we study is given by the operator $\frac{\partial}{\partial n} + \beta$, where β is a function on the boundary. We consider this operator as a perturbation with β of the normal derivative operator $\frac{\partial}{\partial n}$. Actually, the connection with the operator perturbation theory is deeper. In the case when β is positive the boundary process of the Robin problem is obtained from the boundary process of the Neumann problem by killing it with the multiplicative functional induced by β. If β is a negative function then we apply a Kato type perturbation method; cf. [Getoor, R.K., *Potential Anal.*, 1999] and [Beznea, L., Boboc, N., *Potential Anal. 30*, 2009]. Namely, under an L^p-Kato type condition imposed to the function β, with respect to the reflected Brownian motion on the closure of $\mathcal{O}$, we perturb with β the C_0-semigroup on $L^p(\Gamma)$ associated to the Neumann problem and we obtain the L^p-semigroup on the boundary, generated by the Robin boundary operator $\frac{\partial}{\partial n} + \beta$. This semigroup is not necessary sub-Markovian, however, after a standard modification, we may arrive at the transition function of the requested boundary process. A key argument in the proof is that the L^p-Kato condition for β on the

closure of $\mathcal{O}$ is transformed, using the Revuz formula, in a similar Kato condition for β on $L^p(\Gamma)$. As in the case of the Neumann problem, the obtained boundary process is also produced by a time change of the diffusion on $\overline{\mathcal{O}}$ of the Robin problem.

Dirichlet form theory and its applications

LAURENT DENIS

(joint work with Nicolas Bouleau)

Let $(X, \mathcal{X}, \nu, , \gamma)$ be a local symmetric Dirichlet structure which admits a carré du champ operator. This means that $(X, \mathcal{X}, \nu)$ is a measured space, ν is σ-finite and the bilinear form $e[f, g] = \frac{1}{2} \int \gamma[f, g] \, d\nu$ is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ γ (cf. Fukushima-Oshima-Takeda [7] in the locally compact case and Bouleau-Hirsch [6] in a general setting). $(X, \mathcal{X}, \nu, , \gamma)$ is called the bottom space.

Consider a Poisson random measure N on $[0, +\infty[\times X$ with intensity measure $dt \times \nu$.

A Dirichlet structure may be constructed canonically on the probability space of this Poisson measure that we denote $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1, \mathbb{D}, \Gamma)$. We call this space the upper space.

$\mathbb{D}$ is a set of functions in the domain of Γ, in other words a set of random variables which are functionals of the random distribution of points. The main result is the following formula:

For all $F \in \mathbb{D}$

$$\Gamma[F] = \int_0^{+\infty} \int_X \epsilon^-(\gamma[\epsilon^+ F]) \, dN$$

in which ϵ^+ and ϵ^- are the creation and annihilation operators.

Moreover, in all the examples we consider, it satisfies the (EID) property i.e. for any d and all U with values in $\mathbb{R}^d$ whose components are in the domain of the form, the image by U of the measure with density with respect to Λ the determinant of the carré du champ matrix is absolutely continuous with respect to the Lebesgue measure i.e.

$$U_*[(\det \gamma[U, U^t]) \cdot \Lambda] \ll \lambda^d.$$

Let us explain the meaning and the use of this formula on an example.

Consider a centered Lévy process without gaussian part Y such that its Lévy measure, $\sigma = k(x)dx$, is infinite with density dominating near 0 a positive continuous function, and such that $1 + \Delta Y_s \neq 0$ a.s.

We equip the space $\mathbb{R} \setminus \{0\}$ with the Dirichlet form on $L^2(\sigma)$ with carré du champ operator

$$\gamma[u](x) = x^2 u'^2(x) \mathbf{1}_{\{|x| < \epsilon\}},$$

$\epsilon > 0$ being fixed.

We want to study the existence of density for the pair $(Y_t, \mathcal{E}xp(Y)_t)$ where $\mathcal{E}xp(Y)$ is the Doléans exponential of Y.

$$\mathcal{E}xp(Y)_t = e^{Y_t} \prod_{s \le t} (1 + \Delta Y_s) e^{-\Delta Y_s}.$$

$1^0/$ We add a particle (α, y) i.e. a jump to Y at time $\alpha \le t$ with size y:

$$\varepsilon^+_{(\alpha,y)}(\mathcal{E}xp(Y)_t) = e^{Y_t + y} \prod_{s \le t} (1 + \Delta Y_s) e^{-\Delta Y_s} (1 + y) e^{-y} = \mathcal{E}xp(Y)_t (1 + y).$$

$2^0/$ We compute $\gamma[\varepsilon^+ \mathcal{E}xp(Y)_t](y) = (\mathcal{E}xp(Y)_t)^2 y^2$.
$3^0/$ We take back the particle:

$$\varepsilon^- \gamma[\varepsilon^+ \mathcal{E}xp(Y)_t] = \left(\mathcal{E}xp(Y)_t (1 + y)^{-1} \right)^2 y^2$$

we integrate w.r.t. N and that gives the upper carré du champ operator (lent particle formula):

$$\begin{aligned} \Gamma[\mathcal{E}xp(Y)_t] &= \int_{[0,t] \times \mathbb{R}} \left(\mathcal{E}xp(Y)_t (1 + y)^{-1} \right)^2 y^2 N(d\alpha, dy) \\ &= \sum_{\alpha \le t} \left(\mathcal{E}xp(Y)_t (1 + \Delta Y_\alpha)^{-1} \right)^2 \Delta Y_\alpha^2. \end{aligned}$$

By a similar computation the matrix $\underline{\underline{\Gamma}}$ of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is given by

$$\underline{\underline{\Gamma}} = \sum_{\alpha \le t} \begin{pmatrix} 1 & \mathcal{E}xp(Y)_t (1 + \Delta Y_\alpha)^{-1} \\ \mathcal{E}xp(Y)_t (1 + \Delta Y_\alpha)^{-1} & \left(\mathcal{E}xp(Y)_t (1 + \Delta Y_\alpha)^{-1} \right)^2 \end{pmatrix} \Delta Y_\alpha^2.$$

Hence under hypotheses implying (EID), the density of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is yielded by the condition

$$\dim \mathcal{L} \left(\begin{pmatrix} 1 \\ \mathcal{E}xp(Y)_t (1 + \Delta Y_\alpha)^{-1} \end{pmatrix} \quad \alpha \in JT \right) = 2$$

where JT denotes the jump times of Y between 0 and t.

Making this in details we obtain

Let Y be a real Lévy process with infinite Lévy measure with density dominating near 0 a positive continuous function, then the pair $(Y_t, \mathcal{E}xp(Y)_t)$ possesses a density on $\mathbb{R}^2$.

Then, the same technic permits to establish existence of density and even regularity (by iteration of the lent particle method) of the solution of SDEs driven by a random Poisson measure of the following form:

$$(1) \qquad X_t = x + \int_0^t \int_X c(s, X_{s-}, u) \tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s-}) dZ_s$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times X \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ and Z is a semimartingale independent of N.

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Heat flows, geometric and functional inequalities

Michel Ledoux

(joint work with D. Bakry)

Heat flow and semigroup interpolations have grown over the years as a major tool for proving geometric and functional inequalities. The talk will survey some of these developments, with main illustrations on logarithmic Sobolev inequalities, heat kernel bounds, isoperimetric-type comparison theorems, Brascamp-Lieb inequalities and noise stability. Transportation cost inequalities from optimal mass transport are also part of the picture as consequences of new Harnack-type inequalities. The geometric analysis involves Ricci curvature lower bounds via, as a cornerstone, equivalent gradient bounds on the diffusion semigroups. Inspired by the Dirichlet form theory, many of these achievements may be developed in the general context of Markov Triples.

Selected topics developed in the lecture:

- Logarithmic Sobolev form of the Li-Yau parabolic inequality
- (Gaussian) isoperimetric-type inequalities
- Optimal transport and Harnack inequalities
- Brascamp-Lieb inequalities and noise stability

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Invariance Principle for symmetric Diffusions in a degenerate and unbounded stationary and ergodic Random Medium

JEAN-DOMINIQUE DEUSCHEL

(joint work with Alberto Chiarini)

We are interested in the study of reversible diffusions in a random environment. Namely, we are given an infinitesimal generator L^ω in divergence form

$$(1) \qquad L^\omega u(x) = \nabla \cdot (a^\omega(x) \nabla u(x)), \quad x \in \mathbb{R}^d$$

where $a^\omega(x)$ is a symmetric d-dimensional matrix depending on a parameter ω which describes a random realization of the environment.

The environment is modeled as a probability space $(\Omega, \mathcal{G}, \mu)$ on which a measurable group of transformations $\{\tau_x\}_{x \in \mathbb{R}^d}$ is defined. The random field $\{a^\omega(x)\}_{x \in \mathbb{R}^d}$ will be constructed taking a random variable $a : \Omega \to \mathbb{R}^{d \times d}$ and defining $a^\omega(x) := a(\tau_x \omega)$. We assume that the random environment $(\Omega, \mathcal{G}, \mu)$, $\{\tau_x\}_{x \in \mathbb{R}^d}$ is stationary and ergodic.

It is well known that when $x \to a^\omega(x)$ is bounded and uniformly elliptic, uniformly in ω, then a quenched invariance principle holds for the diffusion process X_t^ω associated with L^ω. This means that, for μ-almost all $\omega \in \Omega$, the scaled process $X_t^{\epsilon,\omega} := \epsilon X_{t/\epsilon^2}^\omega$ converges in distribution to a Brownian motion with a non-trivial covariance structure as ϵ goes to zero; this is known as diffusive limit. See for example the classic result of Papanicolau and Varadhan [10] where the coefficients are assumed to be differentiable, and [9] for measurable coefficients and more general operators.

Recently, a lot of effort has been put into extending this result beyond the uniform elliptic case. For example [5] consider a non-symmetric situation with uniformly elliptic symmetric part and unbounded antisymmetric part and the recent paper [3] proves an invariance principle for divergence form operators $Lu = e^V \nabla \cdot (e^{-V} \nabla)$ where V is periodic and measurable. They only assume that $e^V + e^{-V}$ is locally integrable. For what concerns ergodic and stationary environment recent results have been achieved in the case of random walk in random environment in [1], [2]. In this work moments of order greater than one are needed to get an invariance principle in the diffusive limit; [1] and the techniques therein are the main inspiration for our paper.

The aim of our work is to prove a quenched invariance principle for an operator L^ω of the form (1) with a random field $a^\omega(x)$ which is ergodic, stationary and possibly unbounded and degenerate. We assume that a is symmetric and that there exist Λ, λ, $\mathcal{G}$-measurable, positive and finite, such that:

(a.1) for all $\omega \in \Omega$ and $\xi \in \mathbb{R}^d$

$$\lambda(\omega)|\xi|^2 \le \langle a(\omega)\xi, \xi \rangle \le \Lambda(\omega)|\xi|^2;$$

(a.2) there exist $p, q \in [1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\mathbb{E}_\mu[\lambda^{-q}] < \infty, \quad \mathbb{E}_\mu[\Lambda^p] < \infty,$$

(a.3) as functions of x, $\lambda^{-1}(\tau_x\omega), \Lambda(\tau_x\omega) \in L_{loc}^\infty(\mathbb{R}^d)$ for μ-almost all $\omega \in \Omega$.

Since $a^\omega(x)$ is meant to model a random field, it is not natural to assume its differentiability in $x \in \mathbb{R}^d$. Accordingly, the operator defined in (1) does not make any sense, and the techniques coming from Stochastic differential equations and Itô calculus are not very helpful neither in constructing the diffusion process, nor in performing the relevant computation.

The theory of Dirichlet forms is the right tool to approach the problem of constructing a diffusion. Instead of the operator L^ω we shall consider the bilinear form obtained by L^ω, formally integrating by parts, namely

$$(2) \qquad \mathcal{E}^\omega(u,v) := \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}^\omega(x)\partial_i u(x)\partial_j v(x)dx$$

for a proper class of functions $u, v \in \mathcal{F}^\omega \subset L^2(\mathbb{R}^d, dx)$, more precisely $\mathcal{F}^\omega$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, dx)$ with respect to $\mathcal{E}^\omega + (\cdot, \cdot)_{L^2}$. It is a classical result of Fukushima [6] that it is possible to associate to (2) a diffusion process X^ω as soon as $(\lambda^\omega)^{-1}$ and Λ^ω are locally integrable. As a drawback, the process cannot in general start from every $x \in \mathbb{R}^d$, and the set of exceptional points may depend on the realization of the environment. Assumption $(a.3)$ is designed to address this issue. We use assumption $(a.2)$ and ergodicity of the environment to grant that the process X^ω starting from any $x \in \mathbb{R}^d$ does not explode for almost all realization of the environment.

Once the diffusion process X^ω is constructed, the standard approach to diffusive limit theorems consists in showing the weak compactness of the rescaled process and in the identification of the limit. In the case of bounded and uniformly elliptic coefficients the compactness is readily obtained by the Aronson-Nash estimates for the heat kernel. In order to identify the limit, we use the standard technique used in [5], [7] and [9]; namely, we decompose the process X_t^ϵ into a martingale part, called the *harmonic coordinates* and a fluctuation part, called the *correctors*. The martingale part is supposed to capture the long time asymptotic of X_t^ϵ, and will characterize the diffusive limit.

The challenging part is to show that the correctors are uniformly small for almost all realization of the environment, this is attained generalizing Moser's arguments [8] to get a maximal inequality for positive subsolutions of uniformly elliptic, divergence form equations. In this sense the relation $1/p + 1/q < 2/d$ is designed to let the Moser's iteration scheme working. This integrability assumption firstly appeared in [4] in order to extend the results of De Giorgi and Nash to degenerate elliptic equations. A similar condition was also recently exploited in [11] to obtain estimates of Nash - Aronson type for solutions to degenerate parabolic equations. They look to a parabolic generator of the form $\mathcal{L}u = \partial_t u - e^{-V}\nabla \cdot (e^V \nabla u)$, with the assumption that $\sup_{r \geq 1} |r|^{-d} \int_{|x| \leq r} e^{pV} + e^{-qV} dx < \infty$.

We want to stress out that condition $(a.3)$ is not needed to prove the sublinearity of the corrector, nor his existence, we used it only to have a more regular density of the semigroup associated to X^ω and avoid some technicalities due to exceptional sets in the framework of Dirichlet form theory.

Once the correctors are showed to be sublinear, the standard invariance principle for martingales gives the desired result.

Theorem 1 Assume $(a.1)$, $(a.2)$ and $(a.3)$ are satisfied. Let $\mathbf{M}^\omega := (X_t^\omega, \mathbb{P}_x^\omega)$, $x \in \mathbb{R}^d$, be the minimal diffusion process associated to $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d, dx)$. Then the following hold

(i) For μ-almost all $\omega \in \Omega$ the limits

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_0^\omega[X_t^\omega(i) X_t^\omega(j)] = \mathbf{d}_{ij} \quad i, j = 1, ..., d$$

exist and are deterministic constants.

(ii) Denote by W_t a standard Brownian motion. For μ-almost all $\omega \in \Omega$, the family of processes $X_t^{\omega, \epsilon} := \epsilon X_{t/\epsilon^2}^\omega$, $\epsilon > 0$, converges in distribution as $\epsilon \to 0$ to $\mathbf{D}^{1/2} W_t$, where $\mathbf{D} = [\mathbf{d}_{ij}]$ is a positive definite matrix.

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Dirichlet forms on the cone of Radon measures

YURI KONDRATIEV

(joint work with Eugene Lytvynov, Anatoly Vershik)

In the representation theory, so-called quasi-regular representations of a group G in a space $L^2(\Omega, \mu)$ play an important role. Here Ω is a homogeneous space for the group G and μ is a (probability) measure which is quasi-invariant with respect to

the action of G on Ω. However, in the case studied in this paper as well as in similar cases, the measure is not quasi-invariant with respect to the action of the group, so that one cannot define a quasi-regular unitary representation of the group. Hence, the problem of construction of a representation of the Lie algebra, of a Laplace operator and other operators from the universal enveloping algebra is highly non-trivial. Moreover, we will deal with the situation in which the representation of the Lie algebra cannot be realized but, nevertheless, the Laplace operator may be defined correctly.

An important example of a quasi-regular representation is the following case. Let X be a smooth, noncompact Riemannian manifold, and let $G = Diff_0(X)$, the group of C^∞ diffeomorphisms of X which are equal to the identity outside a compact set. Let Ω be the space $\Gamma(X)$ of locally finite subsets (configurations) in X, and let μ be the Poisson measure on $\Gamma(X)$. Then the Poisson measure is quasi-invariant with respect to the action of $Diff_0(X)$ on $\Gamma(X)$, and the corresponding unitary representation of $Diff_0(X)$ in the L^2-space of the Poisson measure was studied in [7], see also [3]. Developing analysis associated with this representation, one naturally arrives at a differential structure on the configuration space $\Gamma(X)$, and derives a Laplace operator on $\Gamma(X)$, see [1]. In fact, one gets a certain lifting of the differential structure of the manifold X to the configuration space $\Gamma(X)$. Hereby the Riemannian volume on X is lifted to the Poisson measure on $\Gamma(X)$, and the Laplace–Beltrami operator on X, generated by the Dirichlet integral with respect to the Riemannian volume, is lifted to the generator of the Dirichlet form of the Poisson measure. The associated diffusion can be described as a Markov process on $\Gamma(X)$ in which movement of each point of configuration is a Brownian motion in X, independent of the other points of the configuration. Let us now recall the definition of a semidirect product of groups. Assume that G and Θ are groups, and denote by $Aut(\Theta)$ the group of all automorphisms of Θ. Assume that $\varphi : G \to Aut(\Theta)$ is a group homomorphism. Then the semidirect product of G and Θ is the Cartesian product $G \times \Theta$ equipped with the group operation

$$(g_1, \theta_1)(g_2, \theta_2) = (g_1 g_2, \theta_1(\alpha(g_1)\theta_2)).$$

Let $C_0(X \to R_+)$ denote the multiplicative group of continuous functions on X with values in $R_+ = (0, \infty)$ which are equal to one outside a compact set. (Analogously, we could have considered $C_0(X)$, the additive group of real-valued continuous functions on X with compact support.) The group of diffeomorphisms, $Diff_0(X)$, naturally acts on X, hence on $C_0(X \to R_+)$. We will consider the group

$$G$$

the semidirect product of $Diff_0(X)$ and $C_0(X \to R_+)$. This group and similar semidirect products and their representations play a fundamental role in mathematical physics and quantum field theory. Even more important than G are the semidirect products in which the space $C_0(X \to R_+)$ is replaced by a current space, i.e., a space of functions on X with values in a Lie group. Note that, in our case, this Lie group, R_+, is commutative. The case of commutative group,

studied in this paper, is also important and it is interesting to find relations with the theory of random fields and with infinite dimensional dynamical systems.

A wide class of representations of a group like G is obtained by considering a probability measure on a space of locally finite configurations These studies were initiated in [7], and almost at the same time in [3], but in less generality. See also [2, 6, 8].

The group G naturally acts on the space $M(X)$ of Radon measures on X. So a natural question is to identify a class of laws of random measures (equivalently, probability measures on $M(X)$) which are quasi-invariant with respect to the action of the group G and which allow for corresponding analysis, like Laplace operator, diffusion in $M(X)$, etc. We will search for such random measures within the class of laws of measure-valued Lévy processes whose intensity measure is infinite. Each measure μ from this class is concentrated on the set $K(X)$ of discrete Radon measures of the form $\sum_{i=1}^{\infty} s_i \delta_{x_i}$, where δ_{x_i} is the Dirac measure with mass at x_i and $s_i > 0$. Furthermore, μ-almost surely, the configuration $\{x_i\}_{i=1}^{\infty}$ is dense in X, in particular, the set $\{x_i\}_{i=1}^{\infty}$ is not locally finite.

A noteworthy example of a measure from this class is the gamma measure. In the case where X is compact, it was proven in [4] that the gamma measure is the unique law of a measure-valued Lévy process which is quasi-invariant with respect to the action of the group $C_0(X \to R_+)$ and which admits an equivalent σ-finite measure which is projective invariant (i.e., invariant up to a constant factor) with respect to the action of $C_0(X \to R_+)$. The latter (σ-finite) measure was studied in [5] and was called there the infinite dimensional Lebesgue measure. We also note that, in papers [4, 2, 8, 6], the gamma measure was used in the representation theory of the group $SL(2, F)$, where F is an algebra of functions on a manifold.

In this paper, we first single out a class of laws of measure-valued Lévy processes μ which are quasi-invariant with respect to the action of the group $C_0(X \to R_+)$. However, since the intensity measure of μ is infinite, the measure μ is not quasi-invariant with respect to the action of the diffeomorphsim group $Diff_0(X)$, and, consequently, it is not quasi-invariant with respect to the action of the group G. Thus, we do not have a quasi-regular representation of G in $L^2(K(X), \mu)$.

Nevertheless, the action of the group G on $K(X)$ allows us to introduce the notion of a directional derivative on $K(X)$, a tangent space, and a gradient. Furthermore, we introduce the notion of partial quasi-invariance of a measure with respect to the action of a group. We show that the measure μ is partially quasi-invariant with respect to G, and this essentially allows us to construct an associated Laplace operator.

We note that, for each measure μ under consideration, we obtain a quasi-regular representation of the group $C_0(X \to R_+)$ on $L^2(K(X), \mu)$ and a corresponding integration by parts formula. Furthermore, there exists a filtration $(F_n)_{n=1}^{\infty}$ on $K(X)$ such that the σ-algebras F_n generate the σ-algebra on which the measure μ is defined, and hence the union of the spaces $L^2(K(X), F_n, \mu)$ is dense in $L^2(K(X), \mu)$. The action of the group $Diff_0(X)$ on $K(X)$ transforms each

σ-algebra F_n into itself, and the restriction of μ to F_n is quasi-regular with respect to the action of $Diff_0(X)$. This implies a quasi-regular representation of $Diff_0(X)$ in $L^2(K(X), F_n, \mu)$, and we also obtain an integration by parts formula on this space. It should be stressed that the σ-algebras F_n are not invariant with respect to the action of the group $C_0(X \to R_+)$. Despite the absence of a proper integration by parts formula related to the Lie algebra g of the Lie group G, using the above facts, we arrive at a proper Laplace operator related to the Lie algebra g, and this Laplace operator is self-adjoint in $L^2(K(X), \mu)$.

We next prove that the Laplace operator is essentially self-adjoint on a set of test functions. Assuming that the dimension of the manifold X is ≥ 2, we then explicitly construct a diffusion process on $K(X)$ whose generator is the Laplace operator.

Finally, we notice that a different natural choice of a tangent space leads to a different, well defined Laplace operator in $L^2(K(X), \mu)$. Using the theory of Dirichlet forms, we can prove that the corresponding diffusion process in $K(X)$ exists. However, its explicit construction is still an open problem, even at a heuristic level.

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Dirichlet forms on p-adic numbers and random walks on the associated tree

JUN KIGAMI

Stochastic Processes on p-adic numbers $\mathbb{Q}_p$ have been studied by many authors. For example, there are papers by Evans [4], Aldous-Evans [3], Albeverio-Karwowski [1, 2]. In this talk, I will show some duality between random walks on trees and

jump processes on the p-adic numbers. For simplicity, I will focus on 3-regular tree and 2-adic numbers.

Let T be 3-regular tree. For each edge xy, we associated a conductance $C_{xy} > 0$ which satisfies $C_{xy} = C_{yx}$. Then define a energy $(\mathcal{E}, \mathcal{F})$ associated with $\{C_{xy}\}$ by

$$\mathcal{E}(u,v) = \sum_{(x,y):\text{edge of } T} C_{xy}(u(x) - u(y))(v(x) - v(y))$$

$$\mathcal{F} = \{u | u : T \to \mathbb{R}, \mathcal{E}(u,u) < +\infty\}.$$

We always associate a random walk on T whose transition probability from x to y is given by $P(x,y) = \frac{C_{xy}}{\sum_{z \in T} C_{xz}}$.

Fix a reference point $\phi \in T$. An infinite sequence of edges $x_0 x_1 x_2 \ldots$ is called an infinite ray form ϕ if $x_0 = \phi$, $x_{i-1} x_i$ is an edge for any i and it contains no loop. Let $\Sigma = \{\text{an Infinite ray from } \phi\}$. Note that Σ is the Cantor set. For an edge xy of T, we define

$$T_y^x = \{\text{a vertex belonging to the opposite side of } x \text{ with respect to } y\} \cup \{y\},$$

$$\Sigma_y^x = \{\phi x_1 x_2 \ldots | \text{an infinite ray from } \phi, \, x_m \in T_y^x \text{ for sufficiently large } m,\}$$

$$\mathcal{E}_{T_y^x}(u,u) = \sum_{(p,q):\text{edge of } T_y^x} C_{pq}(u(p) - u(q))^2,$$

$$R_y^x = \sup \left\{ \frac{u(y)^2}{\mathcal{E}_{T_y^x}(u,u)} \Big| u : \text{finite support}, u \not\equiv 0 \right\}$$

R_y^x is thought of as an effective resistance between y and Σ_y^x as one point. In the following we assume that $R_y^x < +\infty$ for any x and y. This ensures that the associated random walk is transient and hence all the machinery of Martin boundary theory works. Now we present our first theorem.

Theorem For any $x_0 x_1 x_2 \ldots \in \Sigma$,

$$(0) \quad \prod_{m=0}^{\infty} \frac{R_{x_m}^{x_{m+1}}}{R_{x_m}^{x_{m+1}} + r_{x_m x_{m+1}}} = 0 \quad \text{if and only if} \quad \prod_{m=0}^{\infty} \frac{R_{x_m}^{x_{m-1}}}{R_{x_{m+1}}^{x_m} + r_{x_m x_{m+1}}} = 0,$$

where $R_{x_0}^{x^{-1}}$ is defined as the effective resistance between $x_0 = \phi$ and Σ.

One of the main aims of this talk is to explain what this theorem means. To do so, we need to introduce the random walk. Let $\{X_n\}_{n \geq 1}$ be the random walk on T induced by the transition prob. $\{P(x,y)\}_{xy: \text{ vertex of } T}$. Then the classical theory of Martin boundary by Doob, Hunt and Cartier gives the following theorem.

Theorem There exists a Σ-valued random variable $X(\omega)$ such that a.s. ω,

$$\lim_{n \to \infty} X_n(\omega) = X(\omega)$$

Due to the above theorem, we may define the hitting probability ν_ϕ by

$$\nu_\phi(A) = P_\phi(X(\omega) \in A).$$

In [5], we obtain a formula to describe relation between the hitting probability and effective resistances. Making use of it, we have the following fact.

Proposition

$$(1) \qquad \nu_\phi(\{\omega\}) = \prod_{m=0}^{\infty} \frac{R_{x_m}^{x_{m-1}}}{R_{x_{m+1}}^{x_m} + r_{x_m x_{m+1}}}.$$

The above equality (1) gives the meaning of the right-hand side of (0).

To explain the meaning of the other side of (0), we need to introduce 2-adic numbers $\mathbb{Q}_2$, which is defined by

$$\mathbb{Q}_2 = \Big\{ \sum_{n=-\infty}^{+\infty} a_n 2^n \Big| a_n \in \{0,1\}, \exists N \in \mathbb{Z}, a_n = 0 \text{ if } n < N \Big\}$$

Let $\phi_* = \phi_0 \phi_1 \phi_2 \ldots \in \Sigma$ be a infinite ray from ϕ. Then $\mathbb{Q}_2$ can be identified with $\Sigma^+ = \Sigma \backslash \{\phi_*\}$. For any $x \in T$, there exists unique infinite ray $x_0 x_1 \ldots$ from x and $m \in \mathbb{Z}$ such that $x_{n-m} = \phi_n$ for sufficiently large n. Define $\pi : T \to T$ by $\pi(x) = x_1$, $|x| = m$ and $T_m = \{x | x \in T, |x| = m\}$. Then we can identify Σ^+ and $\mathbb{Q}_2$ in the following way:

$$\sum_{n=-\infty}^{\infty} a_n 2^n \underset{\text{bijective}}{\longleftrightarrow} \rho = (\rho_m)_{m \in \mathbb{Z}} : \rho_m \in T_m, \rho_{m-1} = \pi(\rho_m).$$

Also for any $\rho, \tau \in \Sigma^+$, there exists $m \in \mathbb{Z}$ such that $\rho_i = \tau_i$ for any $i \le m$ and $\rho_{m+1} \ne \tau_{m+1}$. Using this number m, we define the confluence $\rho \wedge \tau$ of ρ and τ as $\rho \wedge \tau = \rho_m = \tau_m$. Also we denote $m = N(\rho, \tau)$. Furthermore, set $\Sigma_x^+ = \Sigma_x^{\pi(x)}$.

Next we are going to construct a Dirichlet form on Σ^+ from a pair $\Gamma = (\lambda, \mu)$ where $\lambda : T \to [0, \infty), \forall \rho \in \Sigma^+, \lim_{m \to \infty} \lambda(\rho_m) = +\infty$ and μ is a radon measures on Σ^+ satisfying $\mu(\Sigma_x^+) > 0$ for any $x \in T$. Assume $\sum_{m=-\infty}^{-1} |\lambda(\phi_{m+1}) - \lambda(\phi_m)| < +\infty$. Then

$$J_\Gamma(\rho, \tau) = \frac{1}{2} \sum_{m=-\infty}^{N(\rho, \tau)-1} \frac{\lambda(\rho_{m+1}) - \lambda(\rho_m)}{\mu(\Sigma_{\rho_{m+1}}^+)}$$

converges absolutely. Moreover if we assume that $J_\Gamma(\rho, \tau) \ge 0$ for any $\rho, \tau \in \Sigma^+$, we obtain the next theorem.

Theorem Define

$$\mathcal{Q}_\Gamma(u, v) = \mathcal{Q}_\Gamma^c(u, v) + \lambda_I(u, v)_\mu \quad \text{and} \quad \mathcal{D}_\Gamma = \{u | u \in L^2(\Sigma^+, \mu), \mathcal{Q}_\Gamma^c(u, u) < +\infty\},$$

where

$$\mathcal{Q}_\Gamma^c(u, v) = \int_{\Sigma^+ \times \Sigma^+} J_\Gamma(\rho, \tau)(u(\rho) - u(\tau))(v(\rho) - v(\tau)) \mu(d\rho)\mu(d\tau).$$

and $\lambda_I = \inf_{x \in T} \lambda(x) = \lim_{m \to -\infty} \lambda(\phi_m)$. Then $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is a regular Dirichlet form on $L^2(\Sigma^+, \mu)$.

To relate transient random walks and the Dirichlet forms on $\mathbb{Q}_2$, we return to the classical theory of Martin boundaries of random walks T. Let $K_\phi(x, y)$ be the Martin kernel associated with the random walk and define

$$\big(h(u)\big)(x) = \int_\Sigma K_\phi(x, \rho) u(\rho) \nu_\phi(d\rho).$$

Then $h(u)$ is the harmonic function with the boundary value $u : \Sigma \to \mathbb{R}$. The following theorem has been obtained in [5].

Theorem Define

$$\mathcal{F}_\Sigma = \{u | u \in L^2(\Sigma^+, \nu_\phi), h(u) \in \mathcal{F}\} \quad \text{and} \quad \mathcal{E}_\Sigma(u, v) = \mathcal{E}(h(u), h(v))$$

for $u, v \in \mathcal{F}_\Sigma$. Then $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a regular Dirichlet form on $L^2(\Sigma, \nu_\phi)$.

To obtain a Dirichlet form on Σ^+, we need to define a blow up version of the hitting measure ν_ϕ as follows. Let $R_x = R_x^{\pi(x)}$. Define a Radon measure ν_* on Σ^+ inductively by $\nu_*(\Sigma_\phi^+) = 1$ and

$$\nu_*(\Sigma_x^+) = \frac{R_{\pi(x)}}{r_{x\pi(x)} + R_x} \nu_*(\Sigma_{\pi(x)}^+).$$

and define $\lambda_* : T \to [0, \infty)$ by $\lambda_*(x) = (\nu_*(\Sigma_x^+) R_x)^{-1}$. Using ν_*, we construct a Dirichlet form $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ as

$$\mathcal{F}_{\Sigma^+} = \{u | u \in \mathcal{F}_\Sigma, u(\phi_*) = 0, u \in L^2(\Sigma^+, \nu_*)\} \quad \text{and} \quad \mathcal{E}_{\Sigma^+} = \mathcal{E}_\Sigma|_{\mathcal{F}_{\Sigma^+} \times \mathcal{F}_{\Sigma^+}}.$$

Theorem Let $\Gamma_* = (\lambda_*, \nu_*)$. Then $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+}) = (\mathcal{Q}_{\Gamma_*}, \mathcal{D}_{\Gamma_*})$.

By definitions,

$$\lambda_I = \lim_{m \to \infty} \lambda_*(\phi_m) = (R_\phi^{\phi_1})^{-1} \prod_{m=0}^\infty \frac{R_{\phi_m}^{\phi_{m+1}}}{R_{\phi_m}^{\phi_{m+1}} + r_{\phi_m \phi_{m+1}}}$$

Note that the most right expression of the above equality is the left side of (0). Hence by (0), we have

$$\nu_\phi(\{\phi_*\}) = 0 \Leftrightarrow \lambda_I = 0 \Leftrightarrow (\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+}) \text{ is conservative.}$$

This is what the equivalence (0) means.

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Global Dirichlet Heat Kernel Estimates for Symmetric Lévy Processes in Half-space

Panki Kim

(joint work with Zhen-Qing Chen)

In this paper, we derive explicit sharp two-sided estimates for the Dirichlet heat kernels of a large class of symmetric (but not necessarily rotationally symmetric) Lévy processes on half spaces for all $t > 0$. These Lévy processes may or may not have Gaussian component. When Lévy density is comparable to a decreasing function with damping exponent β, our estimate is explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent β of Lévy density.

Classical Dirichlet heat kernel is the fundamental solution to the heat equation on an open set with zero boundary values. Except for a few special cases, an explicit form of Dirichlet heat kernel is impossible to obtain. Thus the best thing we can hope for is to establish sharp two-sided estimates of Dirichlet heat kernels. See [16] for upper bound estimates on the Dirichlet heat kernel for diffusion and [20] for the lower bound estimate of the Dirichlet heat kernel for diffusion in bounded $C^{1,1}$ domains.

The generator of a discontinuous Lévy process is an integro-differential operator and so it is a non-local operator. Dirichlet heat kernels (if they exist) of the generators of discontinuous Lévy processes on an open set D are the transition densities of such Lévy processes killed upon leaving D. Due to this relation, obtaining sharp estimates on Dirichlet is a fundamental problem both in probability theory and in analysis.

Before [7], sharp two-sided estimates for the Dirichlet heat kernel of any non-local operator on open sets are unknown. Jointly with Renming Song, in [7] for the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ with zero exterior condition, we succeeded in establishing sharp two-sided estimates in any $C^{1,1}$ open set D and over any finite time interval (see [1] for an extension to non-smooth open sets). When D is bounded, one can easily deduce large time heat kernel estimates from short time estimates by a spectral analysis. The approach developed in [7] provides a road map for establishing sharp two-sided heat kernel estimates of other discontinuous processes in open subsets of $\mathbb{R}^d$ (see [3, 8, 9, 11, 12, 13, 18]). In [15, 10, 11], sharp two-sided estimates for the Dirichlet heat kernels $p_D(t, x, y)$ of $\Delta^{\alpha/2}$ and of $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ are obtained for all $t > 0$ in two classes of unbounded open sets: half-space-like $C^{1,1}$ open sets as well in exterior open sets. Since the estimates in [15, 10, 11] hold for all $t > 0$, they are called global Dirichlet heat kernel estimates. An important question in this direction is that on how general discontinuous Lévy processes one can prove sharp two-sided global Dirichlet heat kernel estimates in open subsets of $\mathbb{R}^d$.

We conjectured in [10, (1.9)] that, when D is half space-like $C^{1,1}$ open set, the following two sided estimates holds for a large class of rotationally symmetric Lévy process X whose Lévy exponent of X is $\Phi(|\xi|)$: there are constants $c_1, c_2, c_3 \geq 1$

such that for every $(t, x, y) \in (0, \infty) \times D \times D$,

$$\frac{1}{c_1} \left(\frac{1}{\sqrt{t \, \Phi(1/\delta_D(x))}} \wedge 1 \right) \left(\frac{1}{\sqrt{t \, \Phi(1/\delta_D(y))}} \wedge 1 \right) p(t, c_2(y - x)) \leq p_D(t, x, y)$$

$$(1) \qquad \leq c_1 \left(\frac{1}{\sqrt{t \, \Phi(1/\delta_D(x))}} \wedge 1 \right) \left(\frac{1}{\sqrt{t \, \Phi(1/\delta_D(y))}} \wedge 1 \right) p(t, c_3(y - x))$$

where $p(t, x)$ is the transition density of X. Very recently, this conjecture was proved for unimodal rotationally symmetric pure jump Lévy process with global upper and lower scaling condition in [3].

In this paper, we mainly focus on (1) when D is a half space and we prove that (1) holds for a large class of (not necessarily rotationally) symmetric Lévy processes. Our symmetric Lévy processes may or may not have Gaussian component. Once the global heat kernel estimates are obtained on upper half space $\mathbb{H}$, one can then use the "push inward" method introduced in [15] to extend the results to half-space-like $C^{1,1}$ open sets. See [3].

Note that, for all symmetric Lévy process in $\mathbb{R}$, except compound Poisson, the survival probability of its subprocess in half line $(0, \infty)$ is comparable to $(t^{-1/2} \Phi(1/x)^{-1/2} \wedge 1)$ (see [19, Theorem 4.6] and [2, Theorem 2.6]). This fact, which is used several times in this paper, is essential in our approach.

In general, the explicit estimates of the transition density $p(t, y)$ in $\mathbb{R}^d$ depend heavily on the corresponding Lévy measure and Gaussian component (see [5, 14]). On the other hand, for a large class of symmetric Lévy processes, the scale-invariant parabolic Harnack inequality holds with the explicit scaling in terms of Lévy exponent (see [14, Theorem 4.12] and [5, Theorem 4.11]). Motivated by this, we first develop a rather general version of the upper bound under the assumption that parabolic Harnack inequality holds. Moreover, we show that this assumption on $p(t, x)$ and the upper bound of $p_D(t, x)$ imply a very useful lower bound of $p_D(t, x)$.

Jointly with Takashi Kumagai, we have established the two-sided heat kernel estimates for large class of symmetric Markov processes (see [14, 5, 6]). If we assume the jumping kernels of our Lévy process satisfy the assumptions of [5, 6, 14], using the two-sided heat kernel estimates in [5, 6, 14] and our lower bounds, we prove that the conjecture (1) holds for such symmetric Lévy processes and for $D = \mathbb{H}$. Furthermore, our estimate is explicit in terms of the distance to the boundary, the Lévy exponent and the damping exponent β of Lévy density.

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Continuity and estimates of the transition density of the Liouville Brownian motion

Naotaka Kajino

(joint work with Sebastian Andres (Universität Bonn))

1. Introduction: Liouville Brownian motion

The purpose of this talk was to present the results on the transition density of the Liouville Brownian motion in a recent preprint [1].

The *Liouville Brownian motion* has been constructed by Garban, Rhodes and Vargas in [5], and in a weaker form also by Berestycki in [2], as the canonical diffusion process on the "Riemannian manifold"

$$(\mathbb{R}^2, g_{\gamma,X}), \qquad g_{\gamma,X} := e^{\gamma X(\cdot)} g_{\text{Euc}},$$

where g_{Euc} denotes the standard Riemannian metric on $\mathbb{R}^2$, $\gamma \in (0, 2)$, and $X(\cdot)$ is a (massive) Gaussian free field on $\mathbb{R}^2$. A significant mathematical difficulty here is that, because of the logarithmic singularity of its covariance kernel, $X(\cdot)$ can be rigorously constructed only as a *distribution*-valued random variable and not as a random function on $\mathbb{R}^2$, so that its exponential $e^{\gamma X(\cdot)}$ is a highly ill-defined object.

The original motivation for this line of study comes from the search of the correct theory of quantum gravity in theoretical physics, where it is believed that the two-dimensional version of quantum gravity, called *Liouville quantum gravity*, should be described as two-dimensional surfaces equipped with "Riemannian metric" $g_{\gamma,X}$ as above; see, e.g., [11, 12] and references therein in this connection.

The ultimate aim of the study of Liouville quantum gravity is to understand what the geometry under the "Riemannian metric" $g_{\gamma,X}$ should look like. Note here that it is not clear what "the geometry" should even mean; in fact, it is still not known how to make rigorous sense of the "Riemannian metric" $g_{\gamma,X}$ or at least its associated distance structure. To achieve a rigorous construction of this associated distance structure is part of the motivations of the recent studies of random planar maps by Le Gall and Miermont. In their studies, a random distance function on the two-dimensional sphere has been obtained as the Gromov-Hausdorff limit of certain random discrete objects and is strongly believed to correspond to the above "Riemannian metric" $g_{\gamma,X}$ with $\gamma = \sqrt{8/3}$ for various reasons; see [9] and references therein for details.

The construction of the Liouville Brownian motion is another attempt to extract any geometric information of $(\mathbb{R}^2, g_{\gamma,X})$ from other rigorous mathematical objects. While the "metric" $g_{\gamma,X}$ and the associated distance structure are very difficult to make sense of, the "volume measure" $e^{\gamma X(z)}dz$ can be rigorously constructed for $\gamma \in (0, 2)$ as a random Radon measure $M := M_\gamma$ on $\mathbb{R}^2$ with full topological support, thanks to Kahane's theory of *Gaussian multiplicative chaos* in [7] (see also [11]). Then by the conformal invariance of the two-dimensional Brownian motion $B = \{B_t\}_{t \in [0,\infty)}$, it is not difficult to figure out that the canonical diffusion process $\mathcal{B} = \{\mathcal{B}_t\}_{t \geq 0}$ on $(\mathbb{R}^2, g_{\gamma,X})$ should be given as the time change of B by the positive continuous additive functional $F = \{F_t\}_{t \geq 0}$ of B with Revuz measure M:

$$\mathcal{B}_t = B_{F_t^{-1}},$$

where $F = \{F_t\}_{t \geq 0}$ should admit a formal expression "$F_t = \int_0^t e^{\gamma X(B_s)}ds$" but needs to be made rigorous sense of. The first construction of $F = \{F_t\}_{t \geq 0}$ has been done in [5], where $\mathcal{B} = \{\mathcal{B}_t\}_{t \geq 0}$ is called the *Liouville Brownian motion* (LBM). (A summary of the actual construction of $F = \{F_t\}_{t \geq 0}$ can be found in [1, Section 2, Appendices A and B].) Then some (weak) properties of the LBM have been obtained in [4, 10].

In this talk, the author talked about the results in [1] on the continuity and (rough but still best known) estimates of the transition density of the LBM.

2. Main results

Recall that $\gamma \in (0, 2)$ is fixed and that $M = M_\gamma$ is a random Radon measure on $\mathbb{R}^2$ with full topological support defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. All the statements in this section are valid for $\mathbb{P}$-a.e. realization of the measure M and the associated LBM $\mathcal{B} = \{\mathcal{B}_t\}_{t \geq 0}$. Note that the LBM $\mathcal{B} = \{\mathcal{B}_t\}_{t \geq 0}$ is M-symmetric by the general result [3, Theorem 6.2.1-(i)] from time change theory.

For each non-empty open subset U of $\mathbb{R}^2$, let $\mathcal{B}^U = \{\mathcal{B}_t^U\}_{t \geq 0}$ denote the LBM killed upon the exit time $\tau_U := \{t \in [0, \infty) \mid \mathcal{B}_t \in \mathbb{R}^2 \setminus U\}$ from U.

Theorem 1. For any non-empty open set $U \subset \mathbb{R}^2$ the following hold:
(1) There exists a (unique) continuous function $p^U = p_t^U(x, y) : (0, \infty) \times U \times U \to [0, \infty)$ such that $P_x[\mathcal{B}_t^U \in dy] = p_t^U(x, y)M(dy)$ for any $(t, x) \in (0, \infty) \times U$.
(2) $\mathcal{B}^U$ is strong Feller, i.e., for any bounded Borel measurable function $u : U \to \mathbb{R}$,

$$P_t^U u := \int_U p_t^U(\cdot, y)u(y)M(dy) = E_{(\cdot)}[u(\mathcal{B}_t)\mathbf{1}_{\{t < \tau_U\}}]$$

is a continuous function on U.
(3) If U is connected, then $p_t^U(x, y) \in (0, \infty)$ for any $(t, x, y) \in (0, \infty) \times U \times U$.

Theorem 2. Set $p_t(\cdot, \cdot) := p_t^{\mathbb{R}^2}(\cdot, \cdot)$. Then for any bounded $U \subset \mathbb{R}^2$ and any $\beta \in (\frac{1}{2}(\gamma + 2)^2, \infty)$, there exist random constants $C_k = C_k(X, \gamma, U, \beta) \in (0, \infty)$, $k = 1, 2$, such that for any $(t, x, y) \in (0, \frac{1}{2}] \times \mathbb{R}^2 \times U$,

$$p_t(x, y) = p_t(y, x) \leq C_1 t^{-1} \log(t^{-1}) \exp\left(-C_2 \left(\frac{|x - y|^\beta \wedge 1}{t}\right)^{\frac{1}{\beta - 1}}\right).$$

The main results in [6] play crucial roles in the proofs of Theorems 1 and 2.

Remark 3. (1) Results similar to Theorems 1 and 2 have been proved independently and simultaneously in [8], where the on-diagonal estimate is $t^{-(1+\delta)}$ for any $\delta \in (0, \infty)$ instead of $t^{-1} \log(t^{-1})$ and the range of admissible β is of the form $(\beta(\gamma), \infty)$ with $\beta(\gamma) > \frac{1}{2}(\gamma + 2)^2$ and $\lim_{\gamma \uparrow 2} \beta(\gamma) = \infty$.
(2) Unfortunately, Theorem 2 alone does not even exclude the possibility that β could be taken arbitrarily close to 2, which in the case of the two-dimensional torus has been in fact disproved in a recent result [8, Theorem 5.1] showing an off-diagonal lower bound of $p_t(x, y)$ and thereby that $\beta \in (1, \infty)$ satisfying the upper bound of $p_t(x, y)$ as in Theorem 2 for small t must be at least $2 + \gamma^2/4$.

Theorem 4. For M-a.e. $x \in \mathbb{R}^2$, for any open subset U of $\mathbb{R}^2$ with $x \in U$ there exist $C_3 = C_3(X, \gamma, |x|) \in (0, \infty)$ and $t_0(x) = t_0(X, \gamma, x, U) \in (0, \frac{1}{2}]$ such that

$$p_t^U(x, x) \geq C_3 t^{-1}\left(\log(t^{-1})\right)^{-34} \qquad \text{for any } t \in (0, t_0(x)].$$

Corollary 5. For M-a.e. $x \in \mathbb{R}^2$, for any open subset U of $\mathbb{R}^2$ with $x \in U$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t^U(x, x)}{-\log t} = 2.$$

For each non-empty bounded open subset U of $\mathbb{R}^2$, let $\{-\lambda_n^U\}_{n\in\mathbb{N}} \subset (-\infty, 0]$ be the eigenvalues of the infinitesimal generator of $\mathcal{B}^U$, enumerated in the decreasing order and repeated according to multiplicity, and define

$$Z_U(t) := \int_U p_t^U(x,x)M(dx) = \sum_{n=1}^{\infty} e^{-\lambda_n^U t}, \qquad t \in (0, \infty).$$

Corollary 6. For any non-empty bounded open subset U of $\mathbb{R}^2$,

$$\lim_{t\downarrow 0} \frac{2\log Z_U(t)}{-\log t} = 2.$$

Remark 7. It is not clear to the authors of [1] whether the counterpart of Corollary 6 for the eigenvalue counting function $N_U(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^U \leq \lambda\}$ also holds.

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Geometric Deviation of the Lévy's Occupation Time Arcsine Law

Cheng Ouyang

(joint work with Elton P. Hsu)

We study Lévy's arcsine law near a hypersurface on a Riemannian manifold. More specifically, let $W = \{W_t, t \geq 0\}$ be a standard one-dimensional Brownian motion starting from the origin. Lévy's occupation arcsine law (Lévy [1, 2]) states that

the time spend by W in the positive half line $\mathbb{R}_+ = [0, \infty)$ up to time 1 obeys the arcsine law:

$$\mathbb{P}\left[\int_0^1 I_{\mathbb{R}_+}(W_s)\,ds \le u\right] = \frac{2}{\pi}\arcsin\sqrt{u}.$$

For someone with a background in stochastic analysis on manifolds, it is natural to ask what becomes of this law for a Brownian motion on a Riemannian manifold near a hypersurface. To formulate the problem, let M be a Riemannian manifold and N a smooth hypersurface in M. Locally N divides M into two disjoint parts M_+ and M_-. Take a point $o \in M$ as the starting point of a Riemannian Brownian motion W. How does one describe the total time T the Brownian motion W spends in, say, M_+ up to time t? We expect that the arcsine law still holds in the limit as $t \to 0$. The main result is that the deviation from the arcsine law is expressed in terms of the mean curvature of the hypersurface. More precisely, we have both almost surely and in L^p

$$T = t\int_0^1 \mathbb{I}_{\mathbb{R}_+}(B_u)\,du + t\sqrt{t}H\int_0^1 u\,dL_u + o(t^{\frac{3}{2}}), \quad \text{as } t \downarrow 0.$$

Here $L = \{L_u, 0 \le u \le 1\}$ denotes the local time of B (at 0) and H the mean curvature of the hypersurface N at o.

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Construction of sticky reflected distorted Brownian motion and application to the wetting model

ROBERT VOSSHALL

(joint work with Torben Fattler and Martin Grothaus)

In [Fun05, Sect. 15.1] J.-D. Deuschel and T. Funaki investigated the scalar field $\phi_t := \big(\phi_t(x)\big)_{x \in \Lambda}$, $t \ge 0$, for the dynamical wetting model on a finite volume $\Lambda \subset \mathbb{Z}^d$, $d \in \mathbb{N}$, described by the stochastic differential equations

$$(1) \qquad d\phi_t(x) = -\mathbf{1}_{(0,\infty)}\big(\phi_t(x)\big) \sum_{\substack{y \in \Lambda \\ |x-y|=1}} V'\big(\phi_t(x) - \phi_t(y)\big)\,dt$$

$$+\mathbf{1}_{(0,\infty)}\big(\phi_t(x)\big)\sqrt{2}dB_t(x) + d\ell_t^0(x), \quad x \in \Lambda,$$

subject to the conditions:

$$\phi_t(x) \geq 0, \quad \ell_t^0(x) \text{ is non-decreasing with respect to } t, \quad \ell_0^0(x) = 0,$$

$$\int_0^\infty \phi_t(x) \, d\ell_t^0(x) = 0,$$

$$\beta \ell_t^0(x) = \int_0^t \mathbf{1}_{\{0\}}\big(\phi_s(x)\big) \, ds \quad \text{for fixed } \beta > 0,$$

where $\ell_t^0(x)$ denotes the *central local time* of $\phi_t(x)$ at 0, $(B_t(x))_{x \in \Lambda}$ is a standard Brownian motion and the pair interaction potential V is symmetric, strictly convex and C^2. ϕ_t, $t \geq 0$, is an element of the state space $E = [0,\infty)^n$ for some $n \in \mathbb{N}$.

The dynamical wetting model is a fundamental mathematical model for the physical description of interfaces from a mesoscopic point of view with reflection on a hard wall and pinning effect. This pinning effect is also known as sticky boundary behavior and describes the competing effects of reflection from the wall and a very local force which attracts the interface to the wall. Thus, the system of SDEs (1) is called sticky reflected distorted Brownian motion or distorted Brownian motion with delayed reflection.

Motivated by the dynamical wetting model, we discuss the construction and analysis of sticky reflected distorted Brownian motion for general drift functions via Dirichlet form techniques. Let $I = \{1,\ldots,n\}$, $I_+(x) := \{i \in I | \ x_i > 0\}$, $E_+(B) := \{x \in E| \ I_+(x) = B\}$ for $B \subset I$ and $dm_{n,\beta} := \prod_{i=1}^n (dx_i + \beta d\delta_0^i)$, where dx_i denotes the Lebesgue measure and $d\delta_0^i$ denotes the Dirac measure in 0 with respect to the i-th component. Assume that $\varrho : E \to [0,\infty)$ is given such that

(i) $\varrho > 0 \ m_{n,\beta}$-a.e.
(ii) $\varrho \in C^1(E)$
(iii) $\sqrt{\varrho_{|E_+(B)}}$ is in the Sobolev space $H^{1,2}(E_+(B))$ for every $\emptyset \neq B \subset I$.

Set $\mu = \varrho m_{n,\beta}$. Under these assumptions our main result states that there exists a μ-symmetric conservative diffusion process $\mathbf{M} = \big(\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbf{P}_x)_{x \in E}\big)$ which is properly associated to the recurrent, strongly local, regular, symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ given by the closure of

$$\mathcal{E}(f,g) = \int_E \sum_{i=1}^n \mathbf{1}_{\{x_i \neq 0\}} \partial_i f \partial_i g \, d\mu \quad \text{for } f,g \in C_c^1(E)$$

on $L^2(E;\mu)$ and which is a weak solution to

$$d\mathbf{X}_t^i = \mathbf{1}_{(0,\infty)}(\mathbf{X}_t^i) \, \partial_i \ln \varrho(\mathbf{X}_t) dt + \mathbf{1}_{(0,\infty)}(\mathbf{X}_t^i)\sqrt{2}dB_t^i + \frac{1}{\beta}\mathbf{1}_{\{0\}}(\mathbf{X}_t^i)dt, \quad i = 1,\ldots,n,$$

for quasi-every starting point $x \in E$, where $B_t = (B_t^1,\ldots,B_t^n)$ is a standard Brownian motion. This result is obtained by a Fukushima decomposition of the process. Moreover, if additionally $\varrho > 0$ on E, it holds for $F \in L^1(E;\mu)$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F(\mathbf{X}_s)ds = \frac{\int_E F d\mu}{\mu(E)}$$

$\mathbf{P}_x$-a.s. for quasi-every $x \in E$.

Under the stronger assumption that the density ϱ is of the form $\varrho = \exp(-H)$ with

$$H(x_1, \cdots, x_n) = \frac{1}{2} \sum_{\substack{i,j \in \{0, \ldots, n+1\} \\ |i-j|=1}} V(x_i - x_j),$$

where $x_0 := x_{n+1} := 0$ and $V : \mathbb{R} \to [-b, \infty)$, $b \in [0, \infty)$, is in $C^2(\mathbb{R})$, symmetric and strictly convex, we give the main ideas for the proof of the existence of a diffusion associated to $(\mathcal{E}, D(\mathcal{E}))$ which possesses even the strong Feller property, i.e., its transition semigroup $(p_t)_{t \geq 0}$ fulfills $p_t(\mathcal{B}_b(E)) \subset C_b(E)$. Thus, in this case the preceding results can be improved from "quasi everywhere" to "everywhere" statements. The connections of the sticky reflected distorted Brownian motion and the associated Dirichlet form to random time changes and Girsanov transformations are presented. In particular, we explain how the results of [CK09] by Z.-Q. Chen and K. Kuwae are applied to our setting, using additionally an approximation argument, in order to show that the strong Feller property of the n-dimensional sticky Brownian motion is preserved under Girsanov transformation.

All results of this talk are contained in [FGV14] and [GV14].

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Brownian Motion on Spaces of Varying Dimension

Shuwen Lou

(joint work with Zhen-Qing Chen)

In my joint work with Zhen-Qing Chen, we study Brownian motion on state spaces with varying dimension, the simplest case of which is an infinite 1-dimensional pole installed on $\mathbb{R}^2$. However, it is known that a singleton in the plane is polar (i.e. of zero capacity) with respect to Brownian motion, by which we mean 2-dimensional Brownian motion does not hit a singleton in finite time. Consequently, Brownian motion cannot be constructed in the usual sense on such a state space because once a Brownian motion particle is on the plane, it will never have the chance to climb up the pole. Therefore, we will need to define Brownian motion with varying dimension as a "darning" process. Roughly speaking, we "collapse" (or short) a small closed disk $B_\epsilon = B(0, \epsilon)$ on $\mathbb{R}^2$ into a singleton a^* which therefore

has positive capacity, namely Brownian motion on the plane does hit a closed disk in finite time with probability one, and then we install a pole at a^*.

Another equivalent way to construct our example of Brownian motion of varying dimension (BMVD in abbreviation) is that, after removing the $\epsilon-$disc from $\mathbb{R}^2$, we glue an open "tube" wuth radius ϵ along ∂B_ϵ, and then we run $2-$dimensional Brownian motion on this space. When we speed up the circular movement of Brownian motion along the tube, the circular position of the trajectories becomes "invisible". It can be proved that the BMVD constructed in the first paragraph is the limiting process of this by ignoring the circular position of the paths.

To be precise, in my thesis, the state space of BMVD is defined in the following way. Suppose for $\epsilon > 0$, B_ϵ is the closed disc on $\mathbb{R}^2$ centered at $(0,0)$ with radius ϵ. Let $D_0 := \mathbb{R}^2 \backslash B_\epsilon$. By identifying B_ϵ with a singleton denoted by a^*, we can introduce a topological space $E := D_0 \cup \{a^*\} \cup \mathbb{R}_+$, with a neighborhood of a^* defined as $\{a^*\} \cup (D_1 \cap \mathbb{R}_+) \cup (D_2 \cap D_0)$ for some neighborhood D_1 of 0 in $\mathbb{R}_+$ and D_2 of B_ϵ in $\mathbb{R}^2$. Fix $p > 0$. Let m_p be the measure on E whose restriction on $\mathbb{R}_+$ and D_0 is the Lebesgue measure times p and 1, respectively, where the constant p measures the "speed" of Brownian motion with varying dimension restricted on $\mathbb{R}_+$. We set $m_p(\{a^*\}) = 0$.

Fix $p > 0$. Brownian motion with varying dimension (BMVD in abbreviation) as an m_p-symmetric diffusion process on E is defined as follows. (see, for example, [2])

(1) Its part process in $\mathbb{R}_+$ or D_0 has the same law as standard Brownian motion in $\mathbb{R}_+$ or D_0.
(2) It admits no killings on a^*.

It is shown in [1] by the theory of darning processes that there exists an unique strong Markov process on (E, m_p) satisfying the above two conditions. It can be inferred from the above definition of BMVD that such a process spends zero amount of time under Lebesgue measure (i.e. zero sojourn time) at a^*.

The primary motivation of our research is to study the properties of BMVD on the state space E defined in the last two paragraphs, in particular, by deriving the global sharp two-sided estimates of the transition density functions. Although it can be shown that the BMVD we study is also associated to the Laplacian operator on its state space, one cannot expect its transition density function to be in the same Gaussian type as above. Indeed it turns out its heat kernel embodies both 1-dimensional Gaussian type and 2-dimensional Gaussian type. The reason is that the volume-doubling property near the darning point a^* is sabotaged by the varying dimension of the state space. Therefore the behavior of its heat kernel not only depends on the region of the points, but also presents significant change when t grows from nearing zero to infinity.

Theorem 1 Let $T > 0$ be fixed. Then when $t \in (0, T]$, there exist positive constants C_i, $1 \le i \le 22$, such that the transition density $p(t, x, y)$ of BMVD satisfies the following estimates.

(1) For $x, y \in \mathbb{R}_+$,

$$\frac{C_1}{\sqrt{t}} e^{-\frac{C_2|x-y|^2}{t}} \le p(t, x, y) \le \frac{C_3}{\sqrt{t}} e^{-\frac{C_4|x-y|^2}{t}}.$$

(2) For $x \in \mathbb{R}_+$, $y \in D_0 \cup \{a^*\}$,

$$\frac{C_5}{\sqrt{t}} e^{-\frac{C_6\rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_7}{\sqrt{t}} e^{-\frac{C_8\rho(x,y)^2}{t}}, \quad \text{when } |y|_\rho \le 1;$$

whereas

$$\frac{C_9}{t} e^{-\frac{C_{10}\rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_{11}}{t} e^{-\frac{C_{12}\rho(x,y)^2}{t}}, \quad \text{when } |y|_\rho > 1,$$

(3) For $x, y \in D_0 \cup \{a^*\}$, when $|x|_\rho < 1$, $|y|_\rho < 1$,

$$\frac{C_{13}}{\sqrt{t}} e^{-\frac{C_{14}\rho(x,y)^2}{t}} + \frac{C_{13}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{15}|x-y|_e^2}{t}} \le p(t, x, y)$$

$$\le \frac{C_{16}}{\sqrt{t}} e^{-\frac{C_{17}\rho(x,y)^2}{t}} + \frac{C_{16}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{18}|x-y|_e^2}{t}};$$

otherwise

$$\frac{C_{19}}{t} e^{-\frac{C_{20}\rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_{21}}{t} e^{-\frac{C_{22}\rho(x,y)^2}{t}},$$

It is easy to see that the above three cases cover all the possible locations of the points $x, y \in E$, up to switching x and y. The large time heat kernel estimates for BMVD are given by the next theorem, which is very different from the small time estimates.

Theorem 2 Let $T > 0$ be fixed. Then when $t \in [T, \infty)$, there exist positive constants C_i, $23 \le i \le 40$, such that the transition density $p(t, x, y)$ of BMVD satisfies the following estimates:
(1) For $x, y \in D_0$,

$$\frac{C_{23}}{t} e^{-\frac{C_{24}\rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_{25}}{t} e^{-\frac{C_{26}\rho(x,y)^2}{t}}.$$

(2) For $x \in \mathbb{R}_+$, $y \in D_0 \cup \{a^*\}$,
when $|y|_\rho \le 1$,

$$\frac{C_{27}}{t}\left[1 + \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \cdot \log t\right] e^{-\frac{C_{28}\rho(x,y)^2}{t}} \le p(t, x, y)$$

$$\le \frac{C_{29}}{t}\left[1 + \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \cdot \log t\right] e^{-\frac{C_{30}\rho(x,y)^2}{t}};$$

when $|y|_\rho > 1$,

$$\frac{C_{31}}{t}\left[1 + \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 + \log\left(1 + \frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{32}\rho(x,y)^2}{t}} \leq p(t,x,y)$$

$$\leq \frac{C_{33}}{t}\left[1 + \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 + \log\left(1 + \frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{34}\rho(x,y)^2}{t}},$$

(3) For $x,y \in \mathbb{R}_+$,

$$\frac{C_{35}}{\sqrt{t}}\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|}{\sqrt{t}}\right)e^{-\frac{C_{36}|x-y|^2}{t}} + C_{35}\left[\frac{1}{t} + \frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{C_{37}(|x|^2+|y|^2)}{t}}$$

$$\leq p(t,x,y)$$

$$\leq \frac{C_{38}}{\sqrt{t}}\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|}{\sqrt{t}}\right)e^{-\frac{C_{39}|x-y|^2}{t}}$$

$$+ C_{38}\left[\frac{1}{t} + \frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{C_{40}(|x|^2+|y|^2)}{t}}$$

Remark 1 In the above two theorems, T can be chosen as an arbitrary positive constant. Changing the T value will only result in possible different values of C_i's. The forms of the heat kernel bounds will remain the same.

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Scaling limit of interface models

TORBEN FATTLER

(joint work with Martin Grothaus)

In [2] a solution of a stochastic partial differential equation with sticky boundary behavior using Dirichlet form techniques was obtained in finite dimension. The authors construct a Brownian motion with drift in $E := [0,\infty)^n$, $n \in \mathbb{N}$, where the behavior of the dynamics at the boundary ∂E is governed by the competing effects of reflection from and pinning at the boundary (sticky boundary behavior). The problem is posed in an $L^2(E;\mu)$-setting, where the reference measure is given by $\mu = \varrho m$. Here ϱ denotes a positive density function, integrable with respect to the measure m and fulfilling the so-called Hamza condition. The measure m is of that kind, that $m(\partial E) \neq 0$. Such a reference measure is needed in order to realize the so-called Wentzell boundary condition. In literature, this type of boundary condition is typical for modeling sticky boundary behavior. In providing a Skorokhod decomposition of the constructed stochastic process the authors could identify this process as weak solution to the underlying stochastic differential equation for quasi

every starting point with respect to the associated Dirichlet form. Note that in [1] the non-existence of a strong solution is verified. That the boundary behavior of the constructed process indeed is sticky, is obtained by proving ergodicity of the constructed process. Therefore, the authors are able to show that the occupation time on specified parts of the boundary is positive. In order to obtain the Skorokhod decomposition it is needed that ϱ is a continuously differentiable density on E such that for all $B \subset I$, ϱ is almost everywhere positive on $E_+(B)$ with respect to the Lebesgue measure and for all $\emptyset \neq B \subset I$, $\sqrt{\varrho|_{E_+(B)}}$ is in the Sobolev space of weakly differentiable functions on $E_+(B)$, square integrable together with its derivative. Here $E_+(B) := \{x \in E \mid x_i > 0 \text{ for all } i \in B \text{ and } x_i = 0 \text{ for all } i \in I \setminus B\}$ for $B \subset I := \{1, \ldots, n\}$ with $E_+(B) \subset \partial E$ for $I \neq B \subset I$. ϱ continuously differentiable on E implies that the drift part $\big(\partial_j \ln(\varrho)\big)_{j \in I}$ is continuous on $\{\varrho > 0\}$. Lipschitz continuity and boundedness of the logarithmic derivative of ϱ is not needed.

In this talk we present ideas to tackle the infinite dimensional setting. I.e., we outline the construction of an infinite dimensional sticky Ornstein–Uhlenbeck process. It is planed to discuss the following issues:

(i) *Scaling limit of finite dimensional distorted sticky Brownian motions*
Here we study an appropriate scaling limit of the conservative, local, strongly regular Dirichlet forms considered in [2] in the critical regime on a suitably nice domain of functions. We intend to apply an approach developed in [4] in order to obtain a limiting infinite dimensional gradient Dirichlet form having as invariant measure the law of the stochastic process $(\beta_t)_{t \geq 0}$, where $\beta_t := |b_t|$, $t \geq 0$, and $(b_t)_{t \geq 0}$ is a Brownian bridge between 0 and 0.

(ii) *Construction and identification of the limiting process*
Our aim is to follow an approach developed in [3] to obtain a Skorokhod decomposition of the at first in the abstract sense existing limiting process in order to identify the process as solution of a stochastic partial differential equation with sticky boundary condition. In particular, it is planed to apply an integration by parts formula for the limiting invariant measure associated with the limiting infinite dimensional gradient Dirichlet form obtained in (i) to tackle this item.

(iii) *Approximation of infinite dimensional sticky Ornstein–Uhlenbeck processes*
In this part of the project we plan to investigate stronger types of convergence. Our goal is to show Kuwae–Shioya convergence of Dirichlet forms. This would imply convergence in finite dimensional distributions of the associated stochastic processes. Moreover, we plan to use the Lyons-Zheng decomposition to show tightness of the stochastic processes associated to the Dirichlet forms studied in [2] in the critical regime.

From the point of view of applications the infinite dimensional sticky Ornstein–Uhlenbeck process can be seen as a fluctuation limit of a dynamical wetting model (also known as Ginzburg-Landau dynamics). In dimension 2 a dynamical wetting

model describes the motion of an interface resulting from wetting of a solid surface by a fluid.

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Stochastic nerve axon equations

WILHELM STANNAT

When analyzing neural activity in the brain, stochasticity on the molecular level, in particular channel noise but also synaptic noise, has to be taken into account. We therefore develop a rigorous mathematical theory for the analysis and numerical approximation of spatially extended conductance based neuronal models with noise, realized as stochastic evolution systems on L^2-spaces w.r.t. the Lebesgue measure. Besides generating the action potential traveling along the nerve axon, these systems exhibit a rich phenomenology, like propagation failure, backpropagation, spontaneous pulse solutions and annihilation of pulses, that cannot be modelled using their deterministic counterparts only.

As a prototypical example consider the following system of nonlinear equations, derived by Hodgkin and Huxley in the 1950ies describing the dynamics of a single neuron in the squid's giant axon in terms of the membrane potential v:

$$(1) \qquad \tau\partial_t v = \lambda^2 \partial_{xx} v - g_K n^4(v - E_K) - g_{Na} m^3 h(v - E_{Na}) - g_L(v - E_L)$$

and three stochastic ordinary differential equations for the gating variables $\bar{p} = (n, m, h)$ – describing the probability of certain ion channels being open – given by

$$(2) \qquad \partial_t p = \alpha_p(v)(1 - p) - \beta_p(v)p, \qquad p \in \{m, n, h\}.$$

Here τ (resp. λ) denote specific time (resp. space) constants, g_{Na}, g_K, g_L denote conductances, E_{Na}, E_K, E_L resting potentials and

$$\alpha_p(v) = a_p^1 \frac{v + A_p}{1 - e^{-a_p^2(v + A_p)}} \quad \text{and} \quad \beta_p(v) = b_p^1 e^{-b_p^2(v + B_p)}$$

for some constants a_p^i, $b_p^i > 0$, A_p, $B_p \in \mathbb{R}$ (see [1] or Section 5 of [4] for data matching the standard Hodgkin-Huxley neuron).

To take into account fluctuations in the ion channel concentrations we add additive noise to the membrane potential and multiplicative noise to the gating variables

in such a way that the value of the gating variables remain between 0 and 1. The resulting system of a stochastic reaction diffusion equation coupled to stochastic ordinary evolution equations is then a special example of the following (density controlled) stochastic evolution equation

$$(3) \quad \begin{aligned} dv(t) &= \left(\nu\Delta v(t) + f(v(t), \bar{p}(t))\right) dt + \sigma dW(t) \\ dp_i(t) &= g_i(v(t), p_i(t)) \, dt + \sigma_i(v(t), \bar{p}(t)) dW_i(t), \quad i = 1, \ldots, d \end{aligned}$$

on the space $H \times H_d$, where $H = L^2(0,1)$ and $H_d := \prod_{i=1}^d H$. Here Δ denotes the Laplacian with Neumann boundary conditions. Let $V := H^{1,2}(0,1)$ be the domain of its associated Dirichlet form.

The following crucial assumptions are imposed on the coefficients:

(A) $f \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$, $g_i \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ with

$$|f(v, \bar{p})|, |\nabla f(v, \bar{p})| \leq L\left(1 + |v|^{r-1}\right)\left(1 + \rho(\bar{p})\right), \qquad \partial_v f(v, \bar{p}) \leq L\left(1 + \rho(\bar{p})\right)$$
$$|g_i(v, p_i)|, |\nabla g_i(v, p_i)| \leq L\left(1 + \rho_i(v)\right)\left(1 + |p_i|\right), \qquad \partial_{p_i} g_i(v, p_i) \leq L$$

    for constants $L > 0$, $2 \leq r \leq 4$, some locally bounded functions $\rho : \mathbb{R}^d \to \mathbb{R}^+$ and $\rho_i : \mathbb{R} \to \mathbb{R}^+$ exponentially bounded.

(B) There exist $K \geq 0$ and $\kappa_K < 0$ such that $\partial_v f(v, \bar{p}) \leq -\kappa_K$ for all $|v| > K$, $\bar{p} \in [0,1]^d$ and

$$g_i(v, p_i) \geq 0 \text{ (resp. } \leq 0) \text{ if } p_i \leq 0 \text{ (resp. } p_i \geq 1)$$

(C) $\sigma \in L_2(H, V)$, in particular it admits an integral kernel of the form

$$(\sigma w)(x) = \int_0^1 \sigma(x, y) w(y) \, dy, \quad \sigma \in W^{1,2}\left((0,1)^2\right)$$

    Also, $\sigma_i : H \times H_d \to L_2(H)$ with integral kernels

$$\left(\sigma_i(w_1, \bar{p}) w_2\right)(x) = 1_{\{0 \leq \bar{p} \leq 1\}} \int_0^1 \sigma_i\left(w_1(x), \bar{p}(x), x, y\right) w_2(y) \, dy$$

    with $\sigma_i(w_1, \bar{p}) \in L^2\left((0,1)^2\right)$ being Lipschitz continuous in the first two variables and of at most linear growth.

Formulated in words, we assume that all functions are locally Lipschitz continuous, but we do not prescribe any a priori control on the constants. Furthermore, f as a function of v and g_i as a function of p_i, with all other variables fixed, satisfy a one-sided Lipschitz condition. In order to deal with the growth of the Lipschitz constants in terms of ρ and ρ_i, our analysis is based on L^∞-estimates for both variables v and $\bar{p}$. To this end assumption **(B)** implies that for initial condition $\bar{p}_0$ satisfying $0 \leq \bar{p}_0 \leq 1$ the solution of (3) stays within 0 and 1. Assumption **(C)** on the noise terms is standard from the mathematical point of view. Being additive in the membrane potential and multiplicative in the gating variables has been justified in applications.

The solution to (3) will be considered to be a stochastic process in $\mathcal{V} \times \mathcal{P}$, where

$$\mathcal{V} := \Big\{ v \in C\big([0,T]; H\big) \ \mathcal{F}_t\text{-adapted} : \|v(t)\|_{L^\infty(0,1)} \leq R_t \ P\text{-a.s. for some}$$

$$\text{adapt. stoch. process with } E\Big(\exp(\frac{\alpha}{2} R_T^2)\Big) < \infty \text{ for some } \alpha > 0 \Big\}$$

$$\mathcal{P} := \Big\{ \bar{p} \in C\big([0,T]; H_d\big) \ \mathcal{F}_t\text{-adapted} : 0 \leq \bar{p}(t) \leq 1 \ P\text{-a.s.} \Big\}.$$

We then have the following result:

Theorem Let $p \geq \max\{2(r-1), 4\}$, $v_0 \in L^p(\Omega, \mathcal{F}_0, P; H)$ be independent of W with Gaussian moments in $E := C([0,1])$ and $\bar{p}_0 \in L^p(\Omega, \mathcal{F}_0, P; H_d)$ with $0 \leq \bar{p}_0 \leq 1$ P-a.s. Then there exists a unique variational solution $(v, \bar{p})$ to (3) with $v \in \mathcal{V}$ and $\bar{p} \in \mathcal{P}$.

The proof of this theorem is based on a fixed point iteration and solving each equation without coupling (see [4] for details). It is not covered by any of the existing results on existence and uniquenes of variational solutions to stochastic evolution equations.

We also derive explicit error estimates for finite difference approximations to (3), in particular a pathwise convergence rate of order $\sqrt{h}$, where h denotes the spatial grid size, and a strong convergence rate of order h in special cases. Our results complement previous work on stochastic FitzHugh-Nagumo systems (see [3]).

Having established existence and uniqueness of variational solutions for general conductance based neuronal models with noise our aim is to develop a mathematical rigorous multiscale analysis of the dynamics, in particular w.r.t. the action potential. This has been achieved so far in the much simpler scalar-valued case for the stochastic bistable reaction-diffusion equation

$$(4) \qquad\qquad \partial_t v = \nu \partial_{xx}^2 v + f(v) + \sigma \partial_t W(t, x)$$

where $f \in C^1(\mathbb{R})$ now satisfies the assumptions $f(0) = f(a) = f(1) = 0$ for some $a \in (0,1)$, $f'(0) < 0$, $f'(a) > 0$, $f'(1) < 0$ and $f < 0$ on $(0,a)$ (resp. $f > 0$ on $(a,1)$). It is well-known that under these assumptions on the reaction term (4) admits in the deterministic case a traveling-wave solution $v(t,x) = v^{TW}(x - ct)$ connecting 0 and 1, where $c \in \left(-2\sqrt{\nu f'(a)}, 2\sqrt{\nu f'(a)}\right)$ is called the wave-speed. v^{TW} can be seen as a persistent pulse, i.e. the analogue of the action potential obtained in the limit of persistent excitation.

Numerical simulations indicate that the solution v of (4) can be decomposed into the traveling wave $v^{TW}(x + C(t))$ shifted by some stochastic process $C(t)$, depending on the driving noise term, such that the difference $u(t,x) := v(t,x) - v^{TW}(x + C(t))$ remains bounded uniformly in time with large probability. References [5, 6] provide a rigorous statements and proofs of this decomposition for stochastic Nagumo equations and more general stochastic bistable reaction-diffusion equations. In particular, a dynamical description of the above mentioned spatial

shift $C(t)$ in terms of a random ordinary differential equation is provided. Similar results for stochastic neural field equations have been obtained in [2].

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Functions of locally bounded variation on Wiener spaces

Masanori Hino

Functions of bounded variation (BV functions) on abstract Wiener spaces were first studied in [1, 2] for their applications in stochastic analysis, but recently BV functions on infinite dimensional spaces have attracted attention for a variety of reasons. In this report, we newly introduce the space $\dot{BV}_{\mathrm{loc}}$ of functions of locally bounded variation (local BV functions) on the abstract Wiener space (E, H, μ). Needless to say, there could be several ways of localizing the concept of bounded variation. Here we adopt the ideas in the theory of (quasi-)regular Dirichlet forms, which are suitable for application to stochastic analysis. Indeed, we show that a Dirichlet form of type $\mathcal{E}^{\rho}(f, g) = \frac{1}{2} \int_E \langle \nabla f, \nabla g \rangle_H \, \rho \, d\mu$ associated with a nonnegative function ρ in $\dot{BV}_{\mathrm{loc}}$ with some extra assumptions provides a diffusion process that has the Skorokhod representation. This result is regarded as a natural generalization of [2, Theorem 4.2], where ρ is assumed to be a BV function instead. We also consider the classical Wiener space on $\mathbb{R}^d$ as E and provide a sufficient condition for an open set O of $\mathbb{R}^d$ so that the indicator function of the set of all paths staying in the closure $\overline{O}$ is a local BV function. Accordingly, we can construct the (modified) reflecting Ornstein–Uhlenbeck process with the Skorokhod representation on the set of paths staying in $\overline{O}$ under a rather weak condition on O. This is a study related to another paper [3], in which a sufficient condition was given for the above-mentioned indicator function to be a BV function on either pinned path spaces or one-sided pinned spaces.

The full paper is available at `http://arxiv.org/abs/1403.3151`

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Averaging principle for diffusion processes via Dirichlet forms

Max von Renesse

(joint work with Florent Barret)

We study diffusion processes driven by a Brownian motion with regular drift in a finite dimensional setting. The drift has two components on different time scales, a fast conservative component and a slow dissipative component. Such systems were considered before by Freidlin-Wentzell, Freidlin-Weber and others. Using the theory of Dirichlet forms and Mosco-convergence we obtain simpler proofs, interpretations and new results of the averaging principle for such processes when we speed up the conservative component. As a result, one obtains an effective process with values in the space of connected level sets of the conserved quantities. The use of Dirichlet forms provides a simple and nice way to characterize this process and its properties.

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Three-dimensional Navier-Stokes equations driven by space-time white noise

Xiangchan Zhu

(joint work with Rongchan Zhu)

In this paper, we consider 3D Navier-Stokes equation driven by space-time white noise: Recall that the Navier-Stokes equations describe the time evolution of an incompressible fluid and are given by

$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + \xi$$
$$u(0) = u_0, \quad div u = 0$$

(1)

where $u(x,t) \in \mathbb{R}^3$ denotes the value of the velocity field at time t and position x, $p(x,t)$ denotes the pressure, and $\xi(x,t)$ is an external force field acting on the fluid. We will consider the case when $x \in \mathbb{T}^3$, the three-dimensional torus. Our mathematical model for the driving force ξ is a Gaussian field which is white in time and space.

This paper aims at giving a meaning of the equation (1) when ξ is space-time white noise and obtain local (in time) solution. Such a noise might not be relevant for the study of turbulence. However, in other cases, when a flow is subjected to an external forcing with very small time and space correlation length, a space-time white noise can be considered. The main difficulty in this case is that ξ is so singular that the non-linear term is not well-defined.

Thanks to the regularity structure theory introduced by Martin Hairer in [2] and the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski in [1] we can solve this problem and obtain existence and uniqueness of the local solutions to the three dimensional Navier-Stokes equations driven by space-time white noise. In the theory of regularity structures, the right objects, e.g. regularity structure that could possibly take the place of Taylor polynomials can be constructed. The regularity can also be endowed with a model, which is a concrete way of associating every distribution to the abstract regularity structure. Multiplication, differentiation, the living space of the solutions, and the convolution with singular kernel can be defined on this regularity structure and then the equation has been lifted on the regularity structure. On this regularity structure, the fixed point argument can be applied to obtain local existence and uniqueness of the solutions. Furthermore, we can go back to the real world with the help of another central tool of the theory the reconstruction operator $\mathcal{R}$. If ξ is a smooth process, $\mathcal{R}u$ coincides with the classic solution of the equation.

In this paper we first apply Martin Hairer's regularity structure theory to solve three dimensional Navier-Stokes equations driven by space-time white noise. First as in the two dimensional case we write the nonlinear term $u \cdot \nabla u = \frac{1}{2} div(u \otimes u)$ and construct the associated regularity structure . We construct different admissible models to denote different realizations of the equations corresponding to different noises. Then for any suitable models, we obtain local existence and uniqueness of

solutions by fixed point argument. Finally, we renormalized models of approximation such that the solutions to the equations associated with these renormalized models converge to the solution of the 3D Navier-Stokes equation driven by space-time white noise in probability, locally in time.

The theory of paracontrolled distribution combines the idea of Gubinelli's controlled rough path, which is defined by the following: Let $\Delta_j f$ be the jth Littlewood-Paley block of a distribution f, define

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{j \geq -1} \sum_{i < j-1} \Delta_i f \Delta_j g, \quad \pi_0(f,g) = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

Formally $fg = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g)$. Observing that if f is regular $\pi_<(f,g)$ behaves like g and is the only term in the Bony's paraproduct not raising the regularities, the authors in [1] consider paracontrolled ansatz of the type

$$u = \pi_<(u',g) + u^\sharp,$$

where $\pi_<(u',g)$ represents the "bad-term" in the solution, g is some distribution we can handle and $u^\sharp$ is regular enough to define the multiplication required. Then to make sense of the product of uf we only need to define gf.

In the second part of this paper we apply paracontrolled distribution method to three dimensional Navier-Stokes equations driven by space-time white noise. First we split the equation into four equations and consider the approximation equations. By using paracontrolled ansatz we obtain uniform estimates for the approximation equations and moreover we also get the local Lipschitz continuity of solutions with respect to initial values and some extra terms independent of the solutions. Then we do suitable renormalisation for these terms and prove their convergence in suitable spaces.

Moreover we study approximations to 3D Navier-Stokes (NS) equation driven by space-time white noise by paracontrolled distribution. In order to make the approximating equation converge to 3D NS equation driven by space-time white noise, we should subtract some drift terms in approximating equations. These drift terms, which come from renormalizations in the solution theory, converge to the solution multiplied by some constant depending on approximations.

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<div align="center">

Recent Results on Curvature-Dimension Conditions for Diffusion Operators and for Metric Measure Spaces

KARL-THEODOR STURM
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The Γ-calculus of Bakry-Émery-Ledoux has been developed over the last decades and has found lot of applications in many fields of mathematics. Among the recent progresses in this context are

- the definiton of the Ricci and N-Ricci tensors for diffusion operators;
- the self-improving property of the Bochner inequality in full strength and generality;
- the Ricci bounds for diffusion operators under conformal transformations and time changes.

According to fundamental work of Ambrosio-Gigli-Savaré and recent contributions of Erbar-Kuwada-Sturm there is a one-to-one correspondence between the energetic curvature-dimension condition of Bahry-Émery (formulated in terms of the Γ_2-operator) and the entropic curvature-dimension condition of Lott-Sturm-Villani (formulated in terms of optimal transport). This in particular allows to deduce new transformation formulas for Ricci bounds on metric measure spaces under conformal transformations and time changes. Among the applications are the "convexification" of non-convex domains and the definition of the Neumann Laplacian in non-convex domains as gradient flow of the Boltzmann entropy.

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Intrinsicultracontractivity and semismall perturbation for skew product diffusion operators

Matsuyo Tomisaki

Let $R = [R_t, P_r^R]$ be a one-dimensional diffusion process on $I = (l_1, l_2)$ ($0 \le l_1 < l_2 \le \infty$)with scale function s^R, speed measure m^R and no killing measure (ODDP with $(s^R, m^R, 0)$ for brief). Let $\Theta = [\Theta_t, P_\theta^\Theta]$ be the spherical Brownian motion on $S^{d-1} \subset \mathbb{R}^d$ with generator $(1/2)\Delta^\Theta$, Δ^Θ being the spherical Laplacian on S^{d-1}. Let ν be a Radon measure on I and assume that the support of ν coincides with I. We set $\mathbf{f}(t) = \int_I l^R(t, r) \, d\nu(r)$, where $l^R(t, r)$ is the local time of R. We also set $M = I \times S^{d-1}$ and denote by m^X the product measure $m^R \otimes m^\Theta$. The skew product of the ODDP R and the spherical Brownian motion Θ with respect to the PCAF $\mathbf{f}(t)$ is given by $X = [X_t = (R_t, \Theta_{\mathbf{f}(t)}), P_{(r,\theta)}^X = P_r^R \otimes P_\theta^\Theta, \ (r, \theta) \in M]$. We study [SSP] (i.e., 1 is a semismall perturbation of the generator of X) and [IU] (i.e., the associated heat kernel is intrinsically ultracontractive).

Let $(\mathcal{E}^R, \mathcal{F}^R)$ be the symmetric bilinear form given by

$$\mathcal{E}^R(u, v) = \int_I \frac{du}{ds^R} \frac{dv}{ds^R} \, ds^R,$$

$\mathcal{F}^R = \{u \in L^2(I, m^R) : u \text{ is absolutely continuous w.r.t. } ds^R \text{ and } \mathcal{E}^R(u, u) < \infty\}$.

We set $\mathcal{C}^R = \{u \circ s^R : u \in C_0^\infty(J)\}$, where $J = s^R(I)$ and $C_0^\infty(A)$ is the set of all infinitely differentiable functions on A with compact support. Then $(\mathcal{E}^R, \mathcal{F}^R)$ is a regular, strongly local, irreducible Dirichlet form on $L^2(I, m^R)$ possessing $\mathcal{C}^R$ as

its core and corresponding to the ODDP R (see [1], [3]). We denote by $\left(\mathcal{E}^\Theta, \mathcal{F}^\Theta\right)$ the Dirichlet form on $L^2(S^{d-1}, m^\Theta)$ corresponding to Θ, where m^Θ stands for the standard spherical element of S^{d-1}, so that $|S^{d-1}| = \int_{S^{d-1}} dm^\Theta$ is the total area of S^{d-1}. We set

$$\mathcal{E}^X(f,g) = \int_{S^{d-1}} \mathcal{E}^R(f(\cdot,\theta), g(\cdot,\theta)) \, dm^\Theta(\theta) + \int_I \mathcal{E}^\Theta(f(r,\cdot), g(r,\cdot)) \, d\nu(r),$$

for $f, g \in \mathcal{C}^X$, where $\mathcal{C}^X = \{f(s^R(r), \theta) : f \in C_0^\infty(J \times S^{d-1})\}$ and $J = s^R(I)$. Then by means of Theorems 1.1 and 7.2 of [2] and Theorems 3.1 and 5.1 of [5], we obtain the following result. We note that R is not necessarily conservative.

Proposition 1   *The form $(\mathcal{E}^X, \mathcal{C}^X)$ is closable on $L^2(M, m^X)$. The closure $(\mathcal{E}^X, \mathcal{F}^X)$ is a regular Dirichlet form, it is corresponding to the skew product X, and the semigroup $\{p_t^X, \ t > 0\}$ is irreducible.*

Note that there exists the transition probability density $p^X(t, x, y)$ with respect to m^X so that $p_t^X f(x) = \int_M p^X(t, x, y) f(y) \, dm^X(y)$ for $t > 0$, $x \in M$, and $f \in C_b(M)$. $p^X(t, x, y)$ is continuous in (t, x, y). The α-Green function $G^X(\alpha, x, y)$ corresponding to the skew product X is given by $G^X(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p^X(t, x, y) \, dt$ for $\alpha > 0$ and $x, y \in M$.

Now we state our results. Let us fix a point $c_o \in I$ arbitrarily and set $J_{\mu,\lambda}(r) = \int_{(c_o, r]} d\mu(\xi) \int_{(c_o, \xi]} d\lambda(\eta)$ for $r \in I$, where μ and λ are Borel measures on I. The first result is concerned with [SSP], that is, 1 is a semismall perturbation of $L^X + \alpha$ on M, where L^X is the selfadjoint operator in $L^2(M, m^X)$ associated with the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$.

Theorem 2   *Let $\alpha > 0$. Assume $\min\{J_{s^R, m^R}(l_i), \ J_{m^R, s^R}(l_i)\} < \infty$ for $i = 1, 2$, that is, both end points l_i $(i = 1, 2)$ are not natural for the ODDP R. Let $j = 1$ or 2. Assume one of the following conditions $(S1)_j$ and $(S2)_j$.*

$(S1)_j$ $|s^R(l_j)| < \infty$, $\left|\int_{(l_j, c_o)} ds^R(\xi) \int_{(\xi, c_o)} d\nu(\eta)\right| = \infty$, *and* $\left|\int_{(l_j, c_o)} d\nu(\xi)\right| = \infty$.

$(S2)_j$ $|s^R(l_j)| = \infty$ *and* $\left|\int_{(l_j, c_o)} s^R(\xi) \, d\nu(\xi)\right| = \infty$. *Then for any $\varepsilon > 0$ there is a*

compact set $K \subset M$ such that

$$\lim_{|x| \to l_j} G^X(\alpha, x, y)^{-1} \int_{M \setminus K} G^X(\alpha, x, z) G^X(\alpha, z, y) dm^X(z) < \varepsilon,$$

for $y \in M \setminus K$.

We next introduce the following conditions.

$(E1)$ For each $n \geq 1$, $\beta \geq 0$ and $\gamma_n = \frac{1}{2}n(n + d - 2)$, let $h^{(n)}$ be a positive β-harmonic function corresponding to an ODDP with $(s^R, m^R, \gamma_n \nu)$, and put $s^{(n)}(\xi) = \int_{(c_o, \xi]} h^{(n)}(\eta)^{-2} ds^R(\eta)$ and $m^{(n)}(\xi) = \int_{(c_o, \xi]} h^{(n)}(\eta)^2 dm^R(\eta)$. Then $\min\{J_{s^{(n)}, m^{(n)}}(l_i), J_{m^{(n)}, s^{(n)}}(l_i)\} < \infty$ for $i = 1, 2$, that is, both end points l_i $(i = 1, 2)$ are not natural for an ODDP with $(s^{(n)}, m^{(n)}, 0)$.

(E2)   There exists a positive constant B_0 satisfying $\int_I u\,d\nu \geq B_0 \int_I u\,dm^{\mathrm{R}}$ for $u \in C_0(I)$. Here $C_0(I)$ is the set of all continuous functions on I with compact support.

Under one of the conditions (Ej), $j = 1, 2$, we obtain the following.

Proposition 3   *Assume* $\min\{J_{s^{\mathrm{R}}, m^{\mathrm{R}}}(l_i),\ J_{m^{\mathrm{R}}, s^{\mathrm{R}}}(l_i)\} < \infty$ *for* $i = 1, 2$, *that is, both end points* l_i $(i = 1, 2)$ *are not natural for the ODDP* R*. Further assume one of* (Ej), $j = 1, 2$. *Then there is a complete orthonormal base of eigenfunctions* $\{\phi_j^{\mathrm{X}}\}_{j=0}^{\infty}$ *with eigenvalues* $0 \leq \lambda_0^{\mathrm{X}} < \lambda_1^{\mathrm{X}} \leq \lambda_2^{\mathrm{X}} \leq \cdots$ *repeated according to multiplicity, so that*

$$p^{\mathrm{X}}(t, x, y) = \sum_{j=0}^{\infty} e^{-\lambda_j^{\mathrm{X}} t} \phi_j^{\mathrm{X}}(x) \phi_j^{\mathrm{X}}(y), \qquad t > 0,\ x, y \in M,$$

where the series converges uniformly on any compact set of $(0, \infty) \times M \times M$. *Further* $\lambda_0^{\mathrm{X}} = \lambda_0^{\mathrm{R}}$, $\phi_0^{\mathrm{X}}(x) = \phi_0^{\mathrm{R}}(|x|)/\sqrt{|S^{d-1}|}$, *and there exists the limit*

$$\lim_{|y| \to l_i} p^{\mathrm{X}}(t, x, y)/\phi_0^{\mathrm{X}}(y) = q(t, x, (l_i, \theta)), \quad t > 0,\ x \in M,$$

for $y = (r, \theta) \in M$. *The convergence is uniform in* $(t, x, \theta) \in K$ *for any compact set* $K \subset (0, \infty) \times M \times S^{d-1}$.

Finally we consider a sufficient condition for $p^{\mathrm{X}}(t, x, y)$ satisfying the following condition [IU] (i.e., intrinsic ultracontractivity).

[IU]   For any $t > 0$, there exists a constant $C_t > 0$ such that

$$p^{\mathrm{X}}(t, x, y) \leq C_t\ \phi_0^{\mathrm{X}}(x) \phi_0^{\mathrm{X}}(y), \quad x, y \in M.$$

We may assume that $m^{\mathrm{R}}(l_1) < 0 < m^{\mathrm{R}}(l_2)$ and $s^{\mathrm{R}}(l_1) < 0 < s^{\mathrm{R}}(l_2)$ without loss of generality. We fix points $c_i \in I$, $i = 1, 2$, such that $m^{\mathrm{R}}(c_1) < 0 < m^{\mathrm{R}}(c_2)$ and $s^{\mathrm{R}}(c_1) < 0 < s^{\mathrm{R}}(c_2)$. Let $j = 1$ or 2, and consider following conditions:

$(A1)_j$ $\left| \int_{(l_j, c_j)} m^{\mathrm{R}}(\xi)\,ds^{\mathrm{R}}(\xi) \right| < \infty$ and $\left| \int_{(l_j, c_j)} \frac{\mu_j(\xi)}{m^{\mathrm{R}}(\xi)}\,dm^{\mathrm{R}}(\xi) \right| < \infty$, where

$\mu_j(\xi) = \sup \left\{ |m^{\mathrm{R}}(\eta)\{s^{\mathrm{R}}(\eta) - s^{\mathrm{R}}(l_j)\}| : l_j \wedge \xi < \eta < l_j \vee \xi \right\}$.

$(A2)_j$ $\left| \int_{(l_j, c_j)} m^{\mathrm{R}}(\xi)\,ds^{\mathrm{R}}(\xi) \right| = \infty$ and $\left| \int_{(l_j, c_j)} \frac{\nu_j(\xi)}{s^{\mathrm{R}}(\xi)}\,ds^{\mathrm{R}}(\xi) \right| < \infty$, where

$\nu_j(\xi) = \sup \left\{ |s^{\mathrm{R}}(\eta)\{m^{\mathrm{R}}(\eta) - m^{\mathrm{R}}(l_j)\}| : l_j \wedge \xi < \eta < l_j \vee \xi \right\}$.

Theorem 4   *Assume* $\min\{J_{s^{\mathrm{R}}, m^{\mathrm{R}}}(l_i),\ J_{m^{\mathrm{R}}, s^{\mathrm{R}}}(l_i)\} < \infty$ *for* $i = 1, 2$, *that is, both end points* l_i $(i = 1, 2)$ *are not natural for the ODDP* R*. Assume* $(A1)_1$ *or* $(A2)_1$ *[resp.* $(A1)_2$ *or* $(A2)_2$*] if* $\max\{J_{s^{\mathrm{R}}, m^{\mathrm{R}}}(l_1), J_{m^{\mathrm{R}}, s^{\mathrm{R}}}(l_1)\} = \infty$ *[resp.* $\max\{J_{s^{\mathrm{R}}, m^{\mathrm{R}}}(l_2), J_{m^{\mathrm{R}}, s^{\mathrm{R}}}(l_2)\} = \infty$*], that is,* l_1 *[resp.* l_2*] is exit or entrance for the ODDP* R*. Further assume* $(E2)$. *Then [IU] holds for* $p^{\mathrm{X}}(t, x, y)$.

By means of some results due to Murata[4], we obtain the following. **Corollary**

5   *Under the same assumption as Theorem 4, the following [SP] and [SSP] hold.*
[SP]   1 *is a small perturbation of* $L^{\mathrm{X}} + \alpha$ *on* M, *i.e., for any* $\varepsilon > 0$ *there exists a*

compact subset K of M such that

$$\int_{M \setminus K} G^{\mathrm{X}}(\alpha, x, z) G^{\mathrm{X}}(\alpha, z, y)\, dm^{\mathrm{X}}(z) \leq \varepsilon G^{\mathrm{X}}(\alpha, x, y), \qquad x, y \in M \setminus K.$$

[**SSP**]   1 is a semismall perturbation of $L^{\mathrm{X}} + \alpha$ on M, i.e., for any $\varepsilon > 0$ there exists a compact subset K of M such that

$$\int_{M \setminus K} G^{\mathrm{X}}(\alpha, x^0, z) G^{\mathrm{X}}(\alpha, z, y)\, dm^{\mathrm{X}}(z) \leq \varepsilon G^{\mathrm{X}}(\alpha, x^0, y), \qquad y \in M \setminus K,$$

where x^0 is a reference point fixed in M. [SSP] of Corollary 5 is different from that of Theorem 2. The assumption of Corollary 5 does not imply that of Theorem 2, and vice versa.

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Ergodicity and Exponential Ergodicity of Feller-Markov Processes on Infinite Dimensional Polish Spaces

Fuzhou Gong

(joint work with Huiqian Li, Yong Liu, Yuan Liu, Dejun Luo)

There exists a long literature of studying the ergodicity and asymptotic stability for various semigroups from dynamic systems and Markov chains. Abundant theories and applications have been established for compact or locally compact state spaces (for example see [10]). Actually, in the field of stochastic partial differential equations, the uniqueness of ergodic measures can be derived from the *strong Feller* property besides *topological irreducibility*, which has been a routine to deal with the equations with non-degenerate additive noise. The solutions of these kinds of stochastic partial differential equations just determine the *strong Feller* semigroups on some Banach spaces, there exist the transition densities with respect to the ergodic measures, and the semigroups consisting of compact operators under very week integrable conditions with respect to the ergodic measures. Hence, all most of results for long time behavior of Feller-Markov processes on locally compact spaces can extend to these cases. However, it seems very hard to extend all of them to infinite dimensional or general Polish settings in non-strong Feller cases. Recent years, people have developed many new approaches to more

complicated models. For instance, the *asymptotic strong Feller* property, as a celebrating breakthrough, was presented by Hairer and Mattingly in [7], which can be applied to deal with the uniqueness of ergodicity for 2D Navier-Stokes equations with degenerate stochastic forcing. Some notable contributions to this subject came from Lasota and Szarek in [9] along with their sequential works for *equicontinuous* semigroups (for example see [12],[13],[8]). Indeed, the equicontinuity is adaptable to many known stochastic partial differential equations containing the 2D Navier-Stokes equations with degenerate stochastic forcing. However, on one hand, it seems far from being necessary in the theoretical sense. On the other hand, there also exist non-equicontinuous semigroups, or it is too complicated to prove their equicontinuity. For example, for the Ginzburg-Landau $\nabla\varphi$ interface model introduced by Funaki and Spohn in [4], it seems hopeless to prove the equicontinuity. In this talk, we will give the sharp criteria or equivalent characterizations about the ergodicity and asymptotic stability for *Feller* semigroups on Polish spaces with full generality. To this end we will introduce some new notions, especially the *eventual continuity* of Feller semigroups, which seems very close to be necessary for the ergodic behavior in some sense and also allows the sensitive dependence on initial data in some extent. Furthermore, we will revisit the unique ergodicity and prove the asymptotic stability of stochastic 2D Navier-Stokes equations with degenerate stochastic forcing according to our criteria. This is the joint work with Yuan Liu.

If the Feller-Markov process is asymptotic stable, how to estimate the convergence rate of it to ergodic measure? More importantly, how to estimate to the exponential convergence rate for the exponential ergodic Feller-Markov process? If the state spaces of Feller-Markov processes are Hilbert spaces or Banach spaces consisted by functions,then the above problems are concerned with the below problem: how to use the information of coefficients in partial differential operators to get the information of spectrum of the operators? In particular, spectral gap of the operators concern with the exponential ergodicity of the corresponding Feller-Markov processes. There exists a long literature of studying this problem from theory of diffusion processes and partial differential equations, and there are a lot of interesting problems need to answer. Among them there is a *fundamental gap conjecture* observed by Michiel van den Berg in [3] and was independently suggested by Ashbaugh and Benguria in [2] and S.T.Yau in [14], which gave an optimal lower bound of $\lambda_1 - \lambda_0$, the distance between the first two Dirichlet eigenvalues of a Schrödinger operator $-\Delta + V$ on a bounded uniformly convex domain Ω with a weakly convex potential V, which concerns with the exponential ergodicity of its ground state transformation semigroup. By introducing the notion of *modulus of convexity* for functions, and studying the relationship between the modulus of convexity for V and the modulus of log-concavity for the first eigenfunction (i.e. ground state) of Schrödinger operator $-\Delta + V$ through that of the one dimensional corresponding problems, Andrews and Clutterbuck in [1] recently solved the fundamental gap conjecture. More interestingly, they proved a fundamental

gap comparison theorem, that compare the fundamental gap of the Schrödinger operator $-\Delta + V$ with that of the one dimensional corresponding operator.

Note that, for the spectral gap of Schrödinger Operators and Diffusion Operators there are some sharp results for exponential integrability conditions of potential functions and diffusion coefficients such as Theorem 4.5 in [11] and Corollary 7.2 in [5] in the literature. However, there was no nice estimates on the spectral gap or ground state. Roughly speaking, we have to make some control on the "derivative" of potential functions or diffusion coefficients, otherwise a high-frequency vibration on them will impact heavily on the spectral gap or ground state, but make no difference to the integrability.

In this talk we extend the fundamental gap comparison theorem of Andrews and Clutterbuck to the infinite dimensional setting. More precisely, we proved that the fundamental gap of the Schrödinger operator $-\mathcal{L}_* + V$ ($\mathcal{L}_*$ is the Ornstein–Uhlenbeck operator) on the abstract Wiener space is greater than that of the one dimensional operator $-\frac{d^2}{ds^2} + s\frac{d}{ds} + \tilde{V}(s)$, provided that $\tilde{V}$ is a modulus of convexity for V in Malliavin calculus sense. Similar result is established for the diffusion operator $-\mathcal{L}_* + \nabla F \cdot \nabla$.

There are also examples to show that, the sharp exponential integrability for diffusion coefficients ∇F such as Corollary 7.2 in [5] can not be used but the above result can.

This is jointed work with Yong Liu, Yuan Liu and Dejun Luo, and the paper about the above results on spectral gap comparison has been published in [6].

Furthermore, we give the probabilistic proofs of fundamental gap conjecture and spectral gap comparison theorem of Andrews and Clutterbuck in finite dimensional case via the coupling by reflection of the diffusion processes. This is the jointed work with Huaiqian Li and Dejun Luo.

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Escape rate of symmetric jump-diffusion processes

Yuichi Shiozawa

In this report, we are concerned with the "upper escape rate" of symmetric Hunt processes generated by regular Dirichlet forms. In order to explain this notion, we start with the Brownian motion on $\mathbb{R}^d$. Let $(\{B_t\}_{t\geq 0}, P)$ be the Brownian motion on $\mathbb{R}^d$ starting from the origin. Then Khintchine's law of the iterated logarithm says that

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1, \quad P\text{-a.s.}$$

Hence if we define

$$R_\varepsilon(t) = \sqrt{(2+\varepsilon)t\log\log t}$$

for $\varepsilon > 0$, then

$$P\left(\text{there exists } T > 0 \text{ such that } |B_t| \leq R_\varepsilon(t) \text{ for all } t \geq T\right) = 1.$$

The function R_ε is called an *upper rate function* or *upper radius* of the Brownian motion on $\mathbb{R}^d$. This function expresses the forefront of the Brownian particle for all sufficiently large time. Note that R_ε is *not* an upper rate function for any $\varepsilon \leq 0$.

A similar result as above is also obtained for symmetric stable processes on $\mathbb{R}^d$. For $0 < \alpha < 2$, let $(\{X_t\}_{t\geq 0}, P)$ be a symmetric α-stable process on $\mathbb{R}^d$ starting from the origin. If we define

$$R_{\alpha,\varepsilon}(t) = t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}}$$

for $\varepsilon > 0$, then Khintchine [8] proved that for any $\varepsilon > 0$,

$$(1) \qquad P\left(\text{there exists } T > 0 \text{ such that } |X_t| \leq R_{\alpha,\varepsilon}(t) \text{ for all } t \geq T\right) = 1.$$

Similarly to the Brownian case, $R_{\alpha,\varepsilon}$ is *not* an upper rate function for any $\varepsilon \leq 0$.

Grigor'yan [3], Grigor'yan-Hsu [4] and Hsu-Qin [5] studied about the upper escape rate of Brownian motions on complete Riemannian manifolds M. Their results are summarized as follows: let

$$\psi(R) := \int_6^\infty \frac{r}{\log m(B_x(r)) + \log\log r}\,\mathrm{d}r,$$

where m is the Riemannian volume and $B_x(r)$ is an open ball in M with radius $r > 0$ and center $x \in M$. If $\lim_{R\to\infty}\psi(R) = \infty$, then the Brownian motion on M is conservative. Furthermore, the inverse function $\psi^{-1}(t)$ is an upper rate function

for the Brownian motion on M. Recently, Ouyang [9] extended this result to symmetric diffusion processes generated by strongly local regular Dirichlet forms. In particular, his result shows how the growth rate of the diffusion coefficient affects the range of sample paths for large time. Huang [6] and Huang-S [7] also extended the result to Markov chains on weighted graphs, which are jump processes with finite jump ranges.

We extend the results as we mentioned above to symmetric jump-diffusion processes. We can then observe how the growth/degeneracy rate of the coefficient affects the range of sample paths for large time. In this report, we do not state our theorem; our purpose here is to give typical examples related to symmetric stable processes.

Let $c(x, y)$ be a positive measurable function on $\mathbb{R}^d \times \mathbb{R}^d$. For $\alpha \in (0, 2)$, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the bilinear form on $L^2(\mathbb{R}^d)$ given by

$$\mathcal{E}(u, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) \frac{c(x, y)}{|x - y|^{d+\alpha}} \, \mathrm{d}x \mathrm{d}y$$

$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^d) \mid \mathcal{E}(u, u) < \infty \right\}.$$

We denote by $C_0^\infty(\mathbb{R}^d)$ the totality of continuous functions on $\mathbb{R}^d$ with compact support and assume that the bilinear form $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(\mathbb{R}^d)$ defined as the closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ and $\mathbf{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d})$ an associated symmetric Hunt process on $\mathbb{R}^d$.

If $c(x, y)$ is bounded from above and below by positive constants, then $\mathbf{M}$ is called the symmetric stable-like process introduced by Z.-Q. Chen-Kumagai [2]. We then have a result similar to (1) by using exit time estimates from balls obtained by Barlow-Grigor'yan-Kumagai [1]. Using our result, we can obtain upper rate functions for $\mathbf{M}$ even if $c(x, y)$ is unbounded or degenerates.

(i) Assume that for $\delta \in [0, 1)$ and $q \in [0, \alpha)$,

$$c(x, y) \asymp \begin{cases} (1 + |x|)^2 (\log(2 + |x|))^\delta + (1 + |y|)^2 (\log(2 + |y|))^\delta, & |x - y| < 1, \\ (1 + |x|)^q + (1 + |y|)^q, & |x - y| \geq 1. \end{cases}$$

(i-a) If $\alpha - q > 1 - \delta$, then for some $c > 0$,

$$\limsup_{t \to \infty} \frac{|X_t - X_0|}{\exp(ct^{\frac{1}{1-\delta}})} \leq 1 \quad P_x\text{-a.s., a.e. } x \in \mathbb{R}^d.$$

(i-b) If $0 < \alpha - q \leq 1 - \delta$, then for any $\varepsilon > 0$, there exists $c > 0$ such that

$$\limsup_{t \to \infty} \frac{|X_t - X_0|}{\exp\left(ct^{\frac{1}{\alpha-q}} (\log t)^{\frac{1+\varepsilon}{\alpha-q}}\right)} \leq 1 \quad P_x\text{-a.s., a.e. } x \in \mathbb{R}^d.$$

The condition in (i-a) means that the coefficient of small jumps dominates that of big jumps. We see from [9] that the upper rate function here is the same with that of the symmetric diffusion process generated by the closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$, where

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^d} (1 + |x|)^2 \log(2 + |x|)^\delta |\nabla u(x)|^2 \, \mathrm{d}x, \ u \in C_0^\infty(\mathbb{R}^d).$$

The result in (i-b) may describe how big jumps affect the upper rate function. We also note that for $\delta = 1$, $\mathbf{M}$ is conservative ([10, 11]), but the upper rate function is not known.

(ii) Assume that

$$c(x, y) \asymp \left(1 \wedge \frac{1}{|x|^p} \right) \left(1 \wedge \frac{1}{|y|^p} \right)$$

for some $p > 0$. Then for any $\varepsilon > 0$, there exists $c > 0$ such that

$$\limsup_{t \to \infty} \frac{|X_t - X_0|}{ct^{\frac{2+p}{2(\alpha+p)}} (\log t)^{\frac{(1+\varepsilon)(2+p)}{2(\alpha+p)}}} \leq 1 \quad P_x\text{-a.s.,} \quad \text{a.e. } x \in \mathbb{R}^d.$$

Here we note that $(2 + p)/2(\alpha + p) < 1/\alpha$ for any $p > 0$. This means that for large time, the sample path range for $\mathbf{M}$ is narrower than that for the symmetric α-stable process.

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Minimal thinness with respect to symmetric Lévy processes

Renming Song

(joint work with Panki Kim and Zoran Vondracek)

Minimal thinness is a notion that describes the smallness of a set at a boundary point. Minimal thinness in the half-space was introduced by Lelong-Ferrand in [23], while minimal thinness in general open sets was developed by Naïm in [25];

for a more recent exposition see [3, Chapter 9]. A probabilistic interpretation in terms of Brownian motion was given by Doob, see e.g. [11].

A Wiener-type criterion for minimal thinness of a subset of the half-space (using a Green energy) has already appeared in [23]. A refined version of such a criterion (using the ordinary capacity) was proved in [12]. A general version of the Wiener-type criterion for minimal thinness in NTA domains was established by Aikawa in [1] using a powerful concept of quasi-additivity of capacity. In case of a smooth domain, Aikawa's version of the criterion implies several results obtained earlier, cf. [4, 10, 24, 27]. A good exposition of this theory can be found in [2, Part II, 7].

All of these results have been proved in the context of classical potential theory related to the Laplacian, or probabilistically, to Brownian motion. Even though the concept of minimal thinness for Hunt processes admitting a dual process (and satisfying an additional hypothesis) was studied by Föllmer [13], concrete criteria for minimal thinness with respect to certain integro-differential operators (i.e., certain Lévy processes) in the half space have been obtained only recently in [17]. To be more precise, in [17] the underlying process X belongs to a class of subordinate Brownian motions, where the Laplace exponents of the corresponding subordinators are complete Bernstein functions satisfying a certain condition at infinity. The first result of [17] was a necessary condition for minimal thinness of a Borel subset E of the half-space $\mathbb{H} \subset \mathbb{R}^d$, $d \geq 2$: If E is minimally thin in $\mathbb{H}$ (with respect to the process X) at the point $z = 0$, then

$$\int_{E \cap B(0,1)} |x|^{-d} \, dx < \infty \, .$$

Here $B(z, r)$ denotes the open ball centered at $z \in \mathbb{R}^d$ with radius $r > 0$. In the classical case this was proved in [4] for $d = 2$ and in [10] for $d \geq 3$. The method applied in [17] was based on a result of Sjögren, [26, Theorem 2]. The second result of [17] was a criterion for minimal thinness in $\mathbb{H}$ of a set under the graph of a Lipschitz function which in the classical case is due to Burdzy, see [8] and [14].

In this paper we generalize the results from [17] in several directions. We will always assume that $d \geq 2$. We work with a broader class of purely discontinuous symmetric transient Lévy processes and prove a version of Aikawa's Wiener-type criterion for minimal thinness of a subset of a (not necessarily bounded) κ-fat open set at any finite (minimal Martin) boundary point. By specializing to $C^{1,1}$ open sets, we get an integral criterion for minimal thinness in the spirit of [4, 10]. Moreover, in case the processes satisfy an additional assumption governing the global behavior, we obtain criteria for minimal thinness of a subset of half-space-like open sets at infinity. In the classical case of the Laplacian, such results are direct consequences of the corresponding finite boundary point results by use of the inversion with respect to a sphere and the Kelvin transform. In the case we study, this is much more delicate, since the method of Kelvin transform is not available.

The main tools of this paper are the recent results on the potential theory of symmetric Lévy processes contained in [5, 6, 7, 9, 15, 16, 17, 18, 19, 20, 21, 22].

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Contraction rates for Random Walk Metropolis in high dimensions

ANDREAS EBERLE

Let

$$\mu(dx) \;=\; \frac{1}{Z}\, e^{-H(x)}\, dx$$

be a probability measure on $\mathbb{R}^d$. We assume that H is smooth with bounded derivatives up to order 4. The corresponding Random Walk Metropolis (RWM) algorithm with Gaussian proposals is the time-homogeneuous Markov chain on $\mathbb{R}^d$ with transition step

$$x \;\mapsto\; Y \;:=\; x + \sqrt{h}Q\, I_{\{U < a(x,\, x+\sqrt{h}Q)\}}.$$

Here the step size $h > 0$ is a constant that has to be scaled appropriately w.r.t. the dimension, Q is a standard normal random vector in $\mathbb{R}^d$, and U is independent of Q and uniformly distributed on $(0,1)$. The acceptance probability is given by

$$a(x,y) \;=\; \exp(-(H(y) - H(x))^+) \qquad \text{or} \qquad a(x,y) = (1 + \exp(H(y) - H(x)))^{-1}.$$

In both cases, the detailed balance condition w.r.t. μ is satisfied.

In specific situations (e.g. when μ is a product measure or a perturbation of a Gaussian measure) it is known that there is a diffusion limit for the Markov chain given by an SDE or an SPDE when the step size is scaled of order $h = O(d^{-1})$ (Roberts et al., Mattingly/Pillai/Stuart). Hence well-known results for diffusions suggest that the RWM Markov chain might converge to equilibrium with a rate of order $O(h^{-1}) = O(d)$ for a corresponding scaling of the step sizes. On the other hand, there might be effects of the discretization that lead to a slower order of convergence or the need to choose smaller step sizes. Traditional approaches based on local contractions in total variation distance fail to produce bounds that do not degenerate exponentially in the dimension. Approaches based on isoperimetric inequalities have led to partial results. We propose a new approach that is based on contracitivity measured in a Kantorovich (L^1 Wasserstein) metric W_f w.r.t. an underlying distance of the form $d(x,y) = f(|x-y|)$ on $\mathbb{R}^d$. Here $f : [0,\infty) \to [0,\infty)$ is a very carefully chosen, strictly increasing concave function with $f(0) = 0$ and $f'(0+) = 1$ that is adapted to the problem. It has a discontinuity at 0 and is smooth on $(0,\infty)$. We prove that for a given ball in $\mathbb{R}^d$ there exists a corresponding explicit function f and an explicit constant $c > 0$ such that the transition kernel of RWM is contractive w.r.t. W_f with rate $1-ch$ provided h^{-1} is of order $O(d^2 \log d)$. The result is robust in the sense that the function f and the constant c depend only on bounds for the derivative of H but not on the dimension and on the particular choice of H. Since both total variation and the L^1 Wasserstein distance can be controlled by W_f, this implies bounds of the same order for convergence to equilibrium w.r.t. these more standard distances.

The following are the three key ingredients in the proof of contractivity:

(1) The proof is based on an explicit coupling of the transition kernels $p(x, dy)$ and $p(\tilde{x}, d\tilde{y})$ for $x, \tilde{x} \in \mathbb{R}^d$. The coupling is obtained by combining essentially optimal couplings for the proposal and for the acceptance/rejection step (but in general it is *not* an optimal coupling for the combination of both!) It is important that there exists an explicit coupling for the proposal step that is optimal w.r.t. W_f for *any* concave increasing function f.

(2) The second main ingredient in the proof is a general contraction theorem for Markov chains on metric spaces. This theorem states that given an upper bound for the expectation of the increase $\Delta R = |y - \tilde{y}| - |x - \tilde{x}|$ of the coupling distance during one step and a lower bound for the second moment of $(\Delta R)^-$ that depend only on $|x - \tilde{x}|$, one can find an appropriate function f and a strictly positive constant $c > 0$ such that

$$W_f(p(x, \cdot), p(\tilde{x}, \cdot)) \leq (1 - c) \cdot W_f(\delta_x, \delta_{\tilde{x}}).$$

whenever $|x - \tilde{x}| \leq r_1$ for some finite constant r_1. Both f and c depend only on the given bounds and on r_1 but not on the dimension.

(3) The third key ingredient is the application of the general contraction theorem to Random Walk Metropolis chains. This requires very careful estimates for ΔR. We distinguish several regimes where ΔR is controlled in different ways. It is important to derive very precise estimates as suboptimal estimates easily lead to a strong dimensional dependence of the bounds.

The results are contained in the forthcoming paper [1]. Some related previous results are contained in [2] and [3].

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Dirichlet forms on Noncommutative Spaces

FABIO CIPRIANI

(joint work with Uwe Franz, Daniele Guido, Anna Kula, Tommaso Isola, Jean-Luc Sauvageot)

The purpose of the talk is to illustrate how the notion of Dirichlet form can be generalized to develop a Potential Theory in spaces where the classical tools of analysis, like measure theory, topology and differential calculus, are unfitted or insufficient and their exclusive use would force us to treat these spaces as *singular*.

Properties of sets like

- spaces of irreducible unitary representations of *nonabelian* topological groups
- spaces of Clifford spinors of Riemannian manifolds
- space of leaves of Riemannian foliations
- space of orbits of *ergodic* dynamical systems
- spaces of random variables in Free Probability
- momentum space of electrons in quasi-crystals
- spaces of observables of spin systems in Quantum Statistical Mechanics
- *Quantum* Groups

are naturally encoded in algebras which, generalizing the familiar ones of measurable, continuous or smooth functions, are in general *noncommutative*.

0.1. **Noncommutative topology.** The topology of locally compact, Hausdorff spaces X is encoded in algebras $C_0(X)$ of continuous functions vanishing at infinity. These form the commutative subclass of the C*-algebras, defined as the involutive Banach algebras whose norm and involution are related by

$$\|a^*a\| = \|a\|^2.$$

Finite dimensional C*-algebras are direct sums of full matrix algebras $M_n(\mathbb{C})$ and any C*-algebra A can be realized as a closed, involutive subalgebra of the algebra $\mathcal{B}(h)$ of all bounded linear operators acting on a Hilbert space. An important rôle, for application to K-theory, is played by the C*-algebra $K(h) \subseteq \mathcal{B}(h)$ of all compact linear operators.

0.2. **Noncommutative measure theory.** C*-algebras A are vector spaces ordered by the *positive* cone $A_+ := \{a^*a \in A : a \in A\}$ and a preeminent rôle in the development of their theory is played by the class of positive linear functionals $\tau \in A_+^*$: the pair (A, τ) is seen as a noncommutative measure space because in the commutative case τ is represented by a positive measure m on X. Moreover, the celebrated Gelfand-Naimark-Segal construction allows to build a Hilbert space $L^2(A, \tau)$ in which A acts continuously in a parallel way in which continuous functions act on m-square integrable ones.

0.3. **Dirichlet forms on noncommutative measured spaces** (A, τ)**.** The theory of Dirichlet forms and Markovian symmetric semigroups on C*-algebras was initiated by L. Gross [22], [23] for the needs of applications to Quantum Field Theory and fully formulated by S. Albeverio and R. Høegh-Krohn in [1], [2], in case the positive functional τ is a trace: $\tau(ab) = \tau(ba)$ for $a, b \in A$. In this framework contribution were in particular given by J.-L. Sauvageot [25] and J.M. Lindsay-E.B. Davies [19]. The extension to normal positive functionals τ on von Neumann algebras has been performed in [21] and [4] while in [5] the theory has been extended to Kubo-Martin-Schwinger KMS-functionals on C*-algebras.

0.4. **Differential calculus in Dirichlet spaces.** A canonical differential calculus in Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ on trace C*-algebras (A, τ) commutative or not, has been developed by the author and J.-L. Sauvageot in [9]: any regular Dirichlet space supports a unique closable derivation $\partial : \mathcal{F} \cap A \to \mathcal{H}$ taking values in a Hilbert A-bimodule $\mathcal{H}$ in such a way that the Dirichlet form can be represented as $\mathcal{E}[a] = \|\partial a\|_{\mathcal{H}}^2$ and the derivation is a differential square root of the self-adjoint generator L of the associated Markovian semigroup e^{-tL}: $L = \partial^* \circ \partial$.

When applied to the Dirichlet integral of a Euclidean space or Riemannian manifold M, the above construction reproduce canonically the gradient operator ∇_M acting on the Hilbert module $L^2(TM)$ of square integrable vector fields. The familiar Beurling-Deny decomposition of a Dirichlet form can be

0.5. **Applications to Riemannian Geometry.** Applications in geometrical settings has been given to

i) the construction of the transverse heat semigroup of Riemannian foliations [25]

ii) a characterization of Riemannian manifolds M with positive curvature operator $\widehat{R} \geq 0$ as those for which the heat semigroup e^{-tD^2} generated by the Dirac Laplacian D^2 on the Clifford algebra of M is Markovian [11].

0.6. **Applications to fractals.** Applications to Analysis and Topology of p.c.f. fractals has been given to

i) the construction of elliptic pseudo-differential operators and topological invariants as Fredholm modules on p.c.f. fractals [12]

ii) the construction of differential 1-forms and their line integrals on the Sierpinki gasket and the development of a Hodge-de Rham theory [14]

iii) the construction of a Connes' Spectral Triple on the Sirpinski gasket for a noncommutative approach to its metric geometry with the emergence of a *energy spectral dimension* [15].

0.7. **Applications to weak amenability of von Neumann algebras.** A recent applications of Dirichlet forms with respect to non tracial states has been given to a characterization of von Neumann algebras $\mathcal{M}$ with Haagerup approximation property in terms of the discreteness of the spectrum of some Dirichlet form on $\mathcal{M}$ [3].

0.8. **Applications to Free Probability.** Applications of Dirichlet forms to problems in Free Probability has been given to prove factoriality of von Neumann algebras of random variables in Free Probability by the use of the Voiculescu's Dirichlet form [17], [18], [24], [26].

0.9. **Applications to Quantum Groups.** A Dirichlet form approach has been developed in [8] to the construction of Levy's Processes on Compact Quantum Groups with applications to the construction of associated Connes' Spectral Triples and weak amenability of the related von Neumann algebra.

0.10. **Multipliers of Dirichlet spaces.** Discussing the problem of the development of a geometry of Dirichlet spaces even in cases where energy and volume are distributed singularly, so that the class of Lipschiz functions (defined in terms of the Radon-Nikodym derivative of energy measures with respect to the volume one) is reduced to constants, we will report results of [13] were we introduced

- the multipliers algebra $\mathcal{M}(\mathcal{F})$ of a Dirichlet space $(\mathcal{E}, \mathcal{F})$.

By the use of the Deny Embedding Theorem and its extension to the noncommutative setting, we will show that

- the algebra of finite energy multipliers $\mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ is a quadratic form core
- the multipliers seminorm is related to Fukushima's capacitary inequalities.

The exposition will be concluded showing why $\mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ represents a natural, nontrivial weak Sobolev space intrinsically associated to the Dirichlet space.

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Magnetic energies and Feynman-Kac formulas based on regular Dirichlet forms

MICHAEL HINZ

We are interested in generalizations of the energy form

$$\mathcal{E}^{a,v}(f) = \frac{1}{2}\int_{\mathbb{R}^3}|(\nabla+ia)f|^2 dx + \int_{\mathbb{R}^3}|f|^2 v dx,$$

$f \in C_c^1(\mathbb{R}^3)$, and the Hamiltonian $H^{a,v}f = -\frac{1}{2}(\nabla+ia)^2 f + vf$, $f \in C_c^2(\mathbb{R}^3)$, for a particle in $\mathbb{R}^3$ subject to a stationary magnetic field $-\operatorname{curl} a$ and a stationary electric field $-\nabla v$. The semigroup $(P_t^{a,v})_{t>0}$ on $L_2(\mathbb{R}^3)$ associated with $H^{a,v}$ may be expressed using the Feynman-Kac-Itô formula

$$P_t^{a,v}f(x) = \mathbb{E}_x\left[e^{i\int_0^t a(B_s)\circ dB_s - \int_0^t v(B_s)ds}f(B_t)\right],$$

where $B = (B_t)_{t\geq 0}$ denotes Brownian motion in $\mathbb{R}^3$ and the stochastic integral is of Stratonovich type. Analytical results for magnetic energies and Hamiltonians on Euclidean spaces, domains, manifolds and graphs may for instance be found in [2, 6, 10, 16, 17], for the Feynman-Kac-Itô formula see [2, 9, 10, 17]. In [13] we studied magnetic energies for local Dirichlet forms on fractals, and the present results, taken from our paper [11], provide an additional probabilistic point of view. They also apply to nonlocal setups, [14].

Our starting point is a regular symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L_2(X,\mu)$ (a priori real), where X is a locally compact separable metric space and μ is a nonnegative Radon measure on X with full support. For convenience we also assume that $(\mathcal{E},\mathcal{F})$ is conservative. The generator is denoted by $(L,\operatorname{dom} L)$. See [4, 8] for background. We write $\mathcal{C} := \mathcal{F} \cap C_c(X)$, where $C_c(X)$ is the space of continuous compactly supported functions on X.

By $\mathcal{H}$ we denote the (a priori real) Hilbert space of generalized L_2-differential 1-forms on X associated with $(\mathcal{E}, \mathcal{F})$ as introduced by Cipriani and Sauvageot in [5]. It is obtained by endowing $\mathcal{C} \otimes \mathcal{C}$ with the Hilbert seminorm $\|\cdot\|_{\mathcal{H}}$ determined by

$$\|f \otimes g\|_{\mathcal{H}}^2 := \mathcal{E}(fg^2, f) - \frac{1}{2}\mathcal{E}(f^2, g^2),$$

factoring out zero seminorm elements and completing. By defining suitable actions of $\mathcal{C}$ on $\mathcal{H}$ and setting $\partial f := f \otimes \mathbf{1}$ one can obtain a derivation operator $\partial : \mathcal{C} \to \mathcal{H}$ that satisfies $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f)$, $f \in \mathcal{C}$. See e.g. [5, 11, 12]. By duality we also view $\mathcal{H}$ as *space of L_2-vector fields*. The operator ∂ extends to a closed operator from $L_2(X, \mu)$ into $\mathcal{H}$ with domain $\mathcal{F}$. Its adjoint is denoted by $-\partial^*$.

Let $P_t(x, dy)$, $t > 0$ denote the Markov transition kernels associated with $(\mathcal{E}, \mathcal{F})$. The space $\mathcal{H}$ is isometrically isomorphic to the Hilbert space $\mathbb{H}$, obtained by factoring and completing the space of all Borel functions a on $X \times X$ for which the Hilbert seminorm

$$\|a\|_{\mathbb{H}}^2 := \lim_{t \to 0} \frac{1}{2} \int_X \int_X |a(x, y)|^2 P_t(x, dy) \mu(dx)$$

is finite. The space $\mathbb{H}$ is an inverse limit of the spaces $L_2(X \times X, P_t(x, dy)\mu(dx))$ (with the projection being given by the identity), and any element $a \in \mathcal{H}$ may be viewed as an element of $\mathbb{H}$.

From now on we will also use the natural complexifications of $L_2(X, \mu)$, $(\mathcal{E}, \mathcal{F})$, $\mathcal{H}$ and $\mathbb{H}$, denoted by the same symbols, respectively. Given a real vector field $a \in \mathcal{H}$ we define a quadratic form (possibly extended real valued) by

$$(1) \qquad \mathcal{E}^a(f) := \lim_{t \to 0} \frac{1}{2} \int_X \int_X |f(x) - e^{ia(x,y)} f(y)|^2 P_t(x, dy)\mu(dx), \quad f \in \mathcal{C},$$

and refer to $\mathcal{E}^{a,v}(f)$ as the *energy of f with respect to the magnetic potential a*. By [11, Lemma 4.2] we have $\mathcal{E}^{a,v}(f) < +\infty$ for any $f \in \mathcal{C}$, and polarization yields a conjugate symmetric bilinear form on $\mathcal{C}$. For a suitable function v on X we may also consider the form $\mathcal{E}^{a,v}(f, g) := \mathcal{E}^a(f, g) + \int_X f \overline{g} v d\mu$, here v represents an additional *electric potential*.

Let $Y = (Y_t)_{t \geq 0}$ be the μ-symmetric Hunt process uniquely associated with $(\mathcal{E}, \mathcal{F})$ on X in the sense of [8]. Consider the Hilbert space $(\mathring{\mathcal{M}}, \mathbf{e})$ of martingale additive functionals $M = (M_t)_{t \geq 0}$ of Y of finite energy $\mathbf{e}(M) = \lim_{t \to 0} \frac{1}{2t} \mathbb{E}_\mu(M_t^2)$, see [4, 8]. By Nakao's theorem ([15, Theorem 5.1], [12, Theorem 9.1]) the spaces $\mathcal{H}$ and $\mathring{\mathcal{M}}$ are isometrically isomorphic under the linear and continuos extension of the map $\Theta(f \otimes g) := g \bullet M^f$, $f, g \in \mathcal{C}$, where the right hand side denotes the (Itô type) stochastic integral of g with respect to the martingale part M^f in the Fukushima decomposition $M_t^f + N_t^f$ of the additive functional given by $f(Y_t) - f(Y_0)$, cf. [8]. Let $M \mapsto \Lambda(M) = (\Lambda(M)_t)_{t \geq 0}$ denote Nakao's divergence functional, defined by $\Lambda(M)_t := N_t^w - \int_0^t w(Y_s)ds$, $M \in \mathring{\mathcal{M}}$, where w is the unique element of $\mathcal{F}$ such that $\mu_{\langle M^h, M \rangle}(X) = \mathcal{E}_1(w, h)$, $h \in \mathcal{F}$. Here $\mu_{\langle M^h, M \rangle}$ denotes the signed Revuz measure of the sharp bracket $\langle M^h, M \rangle$, see [3, 15]. For real valued

$a \in \mathcal{H}$ we set

$$\int_{Y([0,t])} a := \Theta(a)_t + \Lambda(\Theta(a))_t, \quad t \geq 0,$$

to define the *stochastic line integral (of Stratonovich type) of a along $Y([0,t])$*. For diffusions on manifolds this formula had been a theorem of Nakao's, [15, Theorem 5.2]. For suitable a the time antisymmetry of $\int_{Y([0,t])} a$ can be shown using results from [3], the local case was already settled in [7]. We now define operators $P_t^{a,v}$ by

$$P_t^{a,v} f(x) := \mathbb{E}_x[e^{i \int_{Y([0,t])} a - \int_0^t v(Y_s)ds} f(Y_t)].$$

Again under certain conditions on a and v it follows that $(P_t^{a,v})_{t \geq 0}$ is a strongly continuous symmetric semigroup of bounded linear operators on $L_2(X, \mu)$, [11, Theorem 8.1]. Hence there is a closed conjugate symmetric bilinear form $(\mathcal{Q}^{a,v}, \mathcal{F}^{a,v})$ on $L_2(X, \mu)$ uniquely associated with this semigroup. Under additional assumptions the forms $\mathcal{E}^{a,v}$ and $\mathcal{Q}^{a,v}$ agree on a suitable core, and $(\mathcal{Q}^{a,v}, \mathcal{F}^{a,v})$ is a closed extension of $\mathcal{E}^{a,v}$ restricted to this core. Let $C_0(X)$ denote the space of continuous functions on X vanishing at infinity. We quote [11, Theorem 9.2].

Theorem 3. *Assume the semigroup $(P_t)_{t \geq 0}$ associated with $(\mathcal{E}, \mathcal{F})$ is Feller with $C_0(X)$-generator $(L, \mathrm{dom}_{C_0(X)} L)$ and that $\mathcal{C}_L := \mathrm{dom}_{C_0(X)} \cap \mathcal{C}$ is dense in $L_2(X, \mu)$. Assume further that $(\mathcal{E}, \mathcal{F})$ admits a carré du champ in the sense of [1] and that the jump measure J of $(\mathcal{E}, \mathcal{F})$ is of form $J(dxdy) = n(x, dy)\mu(dx)$ with a kernel $n(x, dy)$. Let $a \in \mathcal{H}$ be a real vector field of form $a = \partial w + \eta$, where $w \in \mathrm{dom}\, L$ and $\eta \in \ker \partial^*$ and assume that the pure jump part a_j of a is antisymmetric. Let v be a real Borel function with uniformly bounded negative part. Then we have*

$$\mathcal{Q}^{a,v}(f,g) = \mathcal{E}^{a,v}(f,g), \quad f, g \in \mathcal{C}_L,$$

and the (lower bounded) self-adjoint operator $H^{a,v}$ associated with $(\mathcal{Q}^{a,v}, \mathcal{F}^{a,v})$ satisfies

$$H^{a,v} f(x) = (\partial_c + ia_c)^*(\partial_c + ia_c)f(x) + \int_X (f(x) - e^{ia_j(x,y)}f(y))n(x,dy) + v(x)f(x),$$

$f \in \mathcal{C}_L$, where ∂_c denotes the (strongly) local part of the derivation ∂ and a_c and a_j are the (strongly) local and the pure jump part of a, respectively.

For the case of an isotropic α-stable Lévy process Y on $\mathbb{R}^n$ with $0 < \alpha < 2$ and for suitable a and v we observe

$$\int_{Y([0,t])} a =$$

$$\lim_{\varepsilon \to 0} \left\{ \sum_{0 < s \leq t} a(Y_s, Y_{s-})\mathbf{1}_{\{|a(Y_s,Y_{s-})|>\varepsilon\}} + \int_0^t \int_{\{y \in X: |a(y,Y_s)| \leq \varepsilon\}} \frac{a(y, Y_s)}{|Y_s - y|^{n+\alpha}} dyds \right\},$$

the limit taken in L_2. By immaterial modifications of the above the form

$$\mathcal{E}^{a,v}(f) = \frac{1}{2} \int_X \int_X \frac{|f(x) - e^{ia(x,y)}f(y)|^2}{|x - y|^{n+\alpha}} dxdy + \int_X |f|^2 v dx, \quad f \in C_c^2(\mathbb{R}^n),$$

is seen to be closable, a closed extension is given by $(\mathcal{Q}^{a,v}, \mathcal{F}^{a,v})$, and for the associated operator $H^{a,v}$ we have

$$H^{a,v}f(x) = \int_X \frac{f(x) - e^{ia(x,y)}f(y)}{|x-y|^{n+\alpha}}dy + v(x)f(x), \quad f \in C_c^2(\mathbb{R}^n).$$

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Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions

Rongchan Zhu

(joint work with Michael Röckner, Xiangchan Zhu)

In this talk, we present a general framework for solving stochastic functional differential equations in infinite dimensions in the sense of martingale solutions, which can be applied to a large class of SPDE with finite delays, e.g. d-dimensional

stochastic fractional Navier-Stokes equation with delays, d-dimensional stochastic reaction-diffusion equation with delays, d-dimensional stochastic porous media equation with delays. Moreover, under local monotone conditions for the nonlinear term we obtain the existence and uniqueness of strong solutions to SPDE with delays.

Recently stochastic partial differential equations (SPDE) with delays have been paid a lot of attention. There is a large amount of literature on the mathematical theory and on applications of stochastic functional (or delay) differential equations. When one wants to model some evolution phenomena arising in physics, biology, engineering, etc., some hereditary characteristics such as after-effect, time-lag or time-delay can appear in the variables. Typical examples arise in the mathematical modelling of materials with termal memory, biochemical reactions, population models, etc. So, one is naturally lead to use functional differential equations which take into account the history of the system. However, in most cases, randomness affects the model , so that the system should be modelled by a stochastic functional equation.

There are a lot of studies about the existence and uniqueness of probabilistically strong solutions for various classes of nonlinear SPDEs with time delays in a variational framework: Let

$$V \subset H_0 \cong H_0^* \subset V^*$$

be a Gelfand triple and

(1)
$$dx(t) = [A_1(t, x(t)) + A_2(t, x_t)]dt + B(t, x_t)dW(t), t \in [0, \infty),$$
$$x(s) = \psi(s), s \in [-h, 0],$$

where $x_t(s) := x(t + s)$ and $A_1 : \mathbb{R}^+ \times V \to V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(V)$-measurable, $A_2 : \mathbb{R}^+ \times C([-h, 0]; H_0) \to V^*$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(C([-h, 0]; H_0))$-measurable, $B : \mathbb{R}^+ \times C([-h, 0]; H_0) \to L_2(U; H_0)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(C([-h, 0]; H_0))$-measurable. In the existing literature the authors suppose that A_1, i.e. the nonlinear term without delay, satisfies monotonicity condition, whereas A_2, i.e. the nonlinear term with delay, satisfies Lipschitz condition with respect to a suitable norm. If we consider more general equations with nonlinear terms not satisfying monotonicity condition such as Navier-Stokes equations and more interesting delay terms like $A_2(t, x) = \int_{-h}^0 f(x(t + r))dr \cdot \nabla x(t - r_1(t)) + \nabla b(x(t - r_2(t)))$, we cannot apply the above result. Here $r_1, r_2 : \mathbb{R} \to [0, h]$ and f is a bounded Lipschitz continuous function on $\mathbb{R}$.

To obtain the existence and uniqueness of solutions for the equations containing more general nonlinear terms and more interesting delays, we intended to use monotonicity trick and assumed that the nonlinear terms satisfy local monotonicity condition as Wei Liu and the first named author did in [1]. However, when we apply this method, the local monotonicity condition should be: There exists a locally bounded measurable function $\rho : V \to [0, +\infty)$ and $\rho_1 : C([-h, 0]; H_0) \to [0, +\infty)$

such that

$$_{V^*}\langle A_1(t, \xi(t)) - A_1(t, \eta(t)) + A_2(t, \xi_t) - A_2(t, \eta_t), \xi(t) - \eta(t)\rangle_V$$

$$+ \|B(t, \xi_t) - B(t, \eta_t)\|_2^2 \le [\rho(\eta(t)) + \rho_1(\eta_t)]\|\xi(t) - \eta(t)\|_{H_0}^2,$$

for $\xi, \eta \in C([-h, \infty), H_0) \cap L^p_{\mathrm{loc}}([-h, \infty); V)$ with $p \ge 1$. Here the middle norm is H_0-norm not the norm for the paths like $C([-h, 0], H_0)$-norm, which is not natural and cannot even cover the Lipschitz case mentioned above.

Instead, in this paper we take a different approach. First, we provide a general framework to prove the existence of solutions under very weak assumptions in the sense of D. W. Stroock and S.R.S. Varadhan's martingale problem , which can be applied to a large class of SPDEs with delays, such as stochastic fractional Navier-Stokes equations with delay in any dimensions and stochastic reaction-diffusion equation with delay and stochastic porous media equations with delay in any dimension. We also emphasize that for our existence results, we only assume continuity, coercivity and growth conditions written in terms of integrals over time, which enables us to cover a large number of equations with interesting delays . Second, under local monotonicity conditions for the nonlinear terms we obtain pathwise uniqueness for SPDEs with delays, which implies existence and uniqueness of (probabilistically) strong solution by the Yamada-Watanabe Theorem. Here we also emphasize that the local monotonicity condition we assume can cover the examples we mentioned above, which of course is weaker than the Lipschitz condition and (2) since instead of the middle norm H_0 we use the norm for the paths and the local term in the local monotonicity condition may depend on the paths of two solutions.

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Uniqueness of Dirichlet forms related to stochastic quantization under exponential interaction in two-dimensional finite volume

HIROSHI KAWABI

(joint work with Sergio Albeverio, Stefan Michalache and Michael Röckner)

The uniqueness problem for infinite dimensional diffusion operators naturally appears in several areas of mathematical physics including Euclidean quantum field theory and statistical mechanics. However it is still understood very insufficiently in the sense that there are several important types of infinite dimensional diffusion operators for which it is not known whether uniqueness holds or not. The most prominent example in which strong uniqueness (essential self-adjointness) is not known is the stochastic quantization of $P(\phi)_2$-quantum fields in infinite volume. Even in finite volume, this problem had been open for many years since Jona-Lasinio–Mitter [6] gave a mathematical approach to stochastic quantization of $P(\phi)_2$-quantum fields, and strong uniqueness of the corresponding generator

was only proved by Liskevich–Röckner [7] and Da Prato–Tubaro [5]. (We mention that the stochastic quantization of $P(\phi)_2$-quantum fields in infinite volume was achieved by Albeverio–Röckner [4] and L^p-uniqness of the generators of the stochastic quantization of both $P(\phi)_1$- and $\exp(\phi)_1$-quantum fields in infinite volume was proved by Albeverio–Kawabi–Röckner [3].)

In this report, we are concerned with strong uniqueness for the diffusion operators defined through Dirichlet forms given by space-time quantum fields with interactions of exponential type, called $\exp(\phi)_2$-measures (Høegh-Krohn's model of quantum fields [1]), in finite volume.

Let $\mathbb{T}^2 = [0, 2\pi]^2$ be the two dimensional torus and $H^\alpha(\mathbb{T}^2)$, $\alpha \in \mathbb{R}$ denotes the Sobolev space of order α with periodic boundary condition. Let μ_0 be the mean-zero Gaussian measure on $E := H^{-\delta}(\mathbb{T}^2)$, $\delta > 0$ with covariance operator $(1 - \Delta)^{-1}$. It is called the *space-time free field* (in finite volume). Let $:z^n:$, $n \in \mathbb{N} \cup \{0\}$ be the Wick power with respect to μ_0 and let $a \in \mathbb{R}$ be a *charge parameter*. We define the Wick exponential $:\exp(az):$ by

$$:\exp(az):\, := \sum_{n=0}^\infty \frac{a^n}{n!} :z^n:, \quad z \in E,$$

where the right-hand side converges in $L^2(\mu_0; E) := L^2(E, \mu_0; E)$ under the condition $-\sqrt{\pi/e} < a < \sqrt{\pi/e}$. We then define the $\exp(\phi)_2$-measure μ on E by

$$\mu(dz) := Z^{-1} \exp\big(-_E\langle :\exp(az):, \mathbf{1}_{\mathbb{T}^2}\rangle_{E^*}\big)\mu_0(dz),$$

where $\mathbf{1}_{\mathbb{T}^2}$ denotes the indicator function of $\mathbb{T}^2$ and $Z > 0$ is the normalization constant. The function $\mathbf{1}_{\mathbb{T}^2}$ is regarded as an element of $E^* = H^\delta(\mathbb{T}^2)$.

Let $\{e_k\}_{k \in \mathbb{Z}^2}$ be the basis in $H := L^2(\mathbb{T}^2)$ formed by the eigenfunctions of the periodic Laplacian Δ. We set $K := \mathrm{span}\{e_k : k \in \mathbb{Z}^2\}$ and $\mathcal{H} := H^\alpha(\mathbb{T}^2)$, $\alpha > 0$. Note that $K \subset E^* = H^\delta(\mathbb{T}^2)$ is a dense linear subspace of E. We define by $\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(K)$ the space of all smooth cylinder functions on E having the form

$$F(z) = f(\varphi_1^*(z), \ldots, \varphi_n^*(z)), \quad z \in E,$$

with $n \in \mathbb{N}$, $f = f(\xi_1, \ldots, \xi_n) \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\{\varphi_1, \ldots, \varphi_n\} \subset K$. Here φ_j^* denotes the unique continuous extension of the functional $(\varphi_j, \cdot)_{\mathcal{H}}$ to E, that is, φ_j^* is given by $\varphi_j^*(z) = \langle z, (1 - \Delta)^\alpha \varphi_j\rangle$ for $z \in E$. Since we have $\mathrm{supp}(\mu) = E$, two different functions in $\mathcal{F}C_b^\infty$ represent two different μ-classes. Note that $\mathcal{F}C_b^\infty$ is dense in $L^p(\mu)$ for all $p \geq 1$. We define the $\mathcal{H}$-Fréchet derivative $D_{\mathcal{H}}F : E \to \mathcal{H}$ by

$$D_{\mathcal{H}}F(z) := \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big)\varphi_j, \quad z \in E, \ F \in \mathcal{F}C_b^\infty.$$

Now we consider the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \frac{1}{2} \int_E \big(D_{\mathcal{H}}F(z), D_{\mathcal{H}}G(z)\big)_{\mathcal{H}}\mu(dz), \ F, G \in \mathcal{F}C_b^\infty.$$

Then we obtain

$$\mathcal{E}(F, G) = -\int_E \mathcal{L}_0 F(z) G(z) \mu(dz), \quad F, G \in \mathcal{F}C_b^\infty,$$

where

$$\begin{aligned}
\mathcal{L}_0 F(z) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big)(\varphi_i, \varphi_j)_{\mathcal{H}} \\
&\quad - \frac{1}{2} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big) \\
&\quad\quad \times \big\{ \langle z, (1-\Delta)\varphi_j \rangle + a \langle {:}\exp(az){:}, \varphi_j \rangle \big\}, \quad z \in E,
\end{aligned}$$

and it holds $\mathcal{L}_0 F \in L^p(\mu)$ for all $1 \le p < 1 + \frac{\pi}{a^2 e}$.

This means that the operator $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty)$. In particular, it implies that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable in $L^2(\mu)$. We denote by $\mathcal{D}(E)$ the completion of $\mathcal{F}C_b^\infty$ with respect to the $\mathcal{E}_1^{1/2}$-norm. Then $(\mathcal{E}, \mathcal{D}(E))$ is a Dirichlet form and the operator $\mathcal{L}_0$ has a self-adjoint extension $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$, called the Friedrichs extension, corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(E))$. The semigroup $\{e^{t\mathcal{L}_\mu}\}_{t \ge 0}$ generated by $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$ in $L^2(\mu)$ is Markovian, and since $\{e^{t\mathcal{L}_\mu}\}_{t \ge 0}$ is symmetric on $L^2(\mu)$, $\{e^{t\mathcal{L}_\mu}\}_{t \ge 0}$ can be extended as a family of C_0-semigroup of contractions in $L^p(\mu)$ for all $p \ge 1$.

It is a fundamental question whether the Friedrichs extension is the only closed extension generating a C_0-semigroup on $L^p(\mu), p \ge 1$, which for $p = 2$ is equivalent to the fundamental problem of essential self-adjointness of $\mathcal{L}_0$. Even if $p = 2$, in general there are many lower semi bounded self-adjoint extensions $\widetilde{\mathcal{L}}$ of $\mathcal{L}_0$ in $L^2(\mu)$ which therefore generate different symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathcal{L}}}\}_{t \ge 0}$ in $L^2(\mu)$. If, however, we have $L^p(\mu)$-uniqueness of $\mathcal{L}_0$ for some $p \ge 2$, there is hence only one semigroup which is strongly continuous and with generator extending $\mathcal{L}_0$ in $L^p(\mu)$. Consequently, in this case, only one such L^p-, hence only one such L^2-dynamics exists, associated with the $\exp(\phi)_2$-measure μ.

Our main theorem in this report is the following:

Theorem ([2]). *Under $\alpha, \delta > 1/2$, $2\alpha + \delta > 1$ and $-\sqrt{\pi/e} < a < \sqrt{\pi/e}$, we have the following:*

(1) *The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ is $L^p(\mu)$-unique for all $1 \le p < \frac{1}{2}\big(1 + \frac{\pi}{a^2 e}\big)$, i.e., there exists exactly one C_0-semigroup in $L^p(\mu)$ such that its generator extends $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$. In particular, for any $-\sqrt{\pi/e} < a < \sqrt{\pi/e}$, the Dirichlet form $(\mathcal{E}, \mathcal{D}(E))$ is the unique extension of $(\mathcal{E}_0, \mathcal{F}C_b^\infty)$ such that $\mathcal{F}C_b^\infty$ is contained in the domain of the associated generator.*

(2) *There exists a diffusion process $\mathbb{M} := (\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, (X_t)_{t \ge 0}, \{\mathbb{P}_z\}_{z \in E})$ such that the semigroup $\{P_t\}_{t \ge 0}$ generated by the unique extension of $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ satisfies the following identity for any bounded Borel measurable function $F : E \to \mathbb{R}$,*

and $t > 0$:

$$P_t F(z) = \int_\Theta F(X_t(\omega)) \mathbb{P}_z(d\omega), \quad \mu\text{-a.s. } z \in E.$$

Moreover, the diffusion process $\mathbb{M}$ is the unique μ-symmetric Hunt process with the state space E solving the stochastic quantization equation

$$dX_t = -\frac{1}{2}(1-\Delta)^{1-\alpha} X_t dt - \frac{a}{2}(1-\Delta)^{-\alpha} : \exp(aX_t) : dt + dB_t^{(\alpha)}, \quad X_0 = z$$

weakly for $\mathcal{E}_\mu$-q.e. (resp. $\mathcal{E}_{\mu_0}$-q.e.) $z \in E$, where $(B_t^{(\alpha)})_{t \geq 0}$ is under $\mathbb{P}_z$ an $(\mathcal{F}_t)_{t \geq 0}$-adapted $H^\alpha(\mathbb{T}^2)$-cylindrical Brownian motion starting at zero.

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Equilibrium States over the Cone of Discrete Measures

Diana Putan

(joint work with Yuri Kondratiev, Tanja Pasurek)

Gibbs measures are one of the main objects studied in mathematical statistical mechanics. They represent equilibrium states of systems of interacting particle systems. Moreover, they appear as the invariant measures for some diffusion or birth and death processes (e.g. [1], [3], [9]). Therefore, it is only natural that one is concerned with the study of their properties.

Let us focus on a particular setting. Consider systems of particles lying on the cone of discrete measures

$$\mathbb{K}(\mathbb{R}^d) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \middle| s_i \in \mathbb{R}_+^*, \ x_i \in \mathbb{R}^d \right\}.$$

This framework can be used to model complex systems with a non-trivial internal structure of their elements (e.g. ecological systems in the presence of biological

diversity). This situation appears to be somehow new in the literature. Such systems were considered recently in [5], [7] and [6]. In these papers, the role of equilibrium states is attributed to Gamma processes on the corresponding location spaces. We are able to extend the framework to what we will call generalized Lévy processes. To each particle $x \in \mathbb{R}^d$, we attach a positive characteristic (mark) s_x such that (s_x, x) is distributed according to some *generalized Lévy intensity measure* $\tau(ds, dx)$ on $(0, \infty) \times \mathbb{R}^d$. In this sense, we obtain an extension of some results concerning existence of Gibbs measures from [5] and [7], where the case $\tau(ds, dx) = \lambda(ds)m(dx)$ was considered, for $m(dx)$ the Lebesgue measure on $\mathbb{R}^d$ and $\lambda(ds) = e^{-s}/s\, ds$ the Gamma measure on $\mathbb{R}_+^* = (0, \infty)$.

The interaction of the system will be described via a bounded pair potential ϕ, in terms of the relative energy

$$H_\Delta(\eta\xi) := \int_\Delta \int_\Delta \phi(x, y)\eta(dx)\eta(dy) + 2\int_{\Delta^c} \int_\Delta \phi(x, y)\eta(dx)\xi(dy),$$

for η, ξ belonging to the cone of discrete measures $\mathbb{K}(\mathbb{R}^d)$ and for a finite volume $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$.

For each $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\beta > 0$, the *local Gibbs measures* with boundary conditions $\xi \in \mathbb{K}(\mathbb{R}^d)$ are given by

$$\mu_\Delta(d\eta|\xi) := \frac{1}{Z_\Delta(\xi)} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{L}_{\Delta,\tau}(d\eta),$$

where the corresponding *partition function* is given by

$$Z_\Delta^\beta(\xi) := \int_{\mathbb{K}(\Delta)} \exp\{-\beta H_\Delta(\eta_\Delta|\xi)\} \mathcal{L}_{\Delta,\tau}(d\eta_\Delta).$$

A probability measure μ on $\mathbb{K}(\mathbb{R}^d)$ is called a *Gibbs measure* (or *state*) with pair potential ϕ if it satisfies the *Dobrushin-Lanford-Ruelle* (*DLR*) equilibrium equation

$$\int_{\mathbb{K}(\mathbb{R}^d)} \pi_\Delta(B|\eta)\mu(d\eta) = \mu(B)$$

for all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$.

Two essential cases are considered. First, for a spatially bounded Lévy intensity measure $\tau(ds, dx)$, i.e. for which

$$\int s^i \tau(ds, Q_k) \leq M < \infty, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d,$$

we are able to prove the existence (by adapting arguments from [4]) and also uniqueness due to small interaction or first spatial moment of τ (based on the generalized Dobrushin-Pechersky criterion cf. [8] and [2]) of the Gibbs states.

Secondly, in a special case of unbounded Lévy intensity measure $\tau(ds, dx)$, where

$$\int s^i \tau(ds, Q_k) \leq C_i e^{a_i|k|}, \text{ for } i = 1, 2 \text{ and any } k \in \mathbb{Z}^d,$$

an existence result for the equilibrium states can be established.

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Intrinsic scaling properties for nonlocal operators

Moritz Kassmann

The talk and the corresponding publication [1] are about a technique that allows to treat integro-differential operators with an arbitrary order of differentiability less than 2. Note that, in general, this order is represented by a function and not by a number. Of special interest are differentiability orders that are below any positive fixed number.

Assume $\mathcal{L} \colon C_b^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is of the form

$$(1) \qquad \mathcal{L}u(x) = \int_{\mathbb{R}^d \setminus \{0\}} \big(u(x+h) - u(x) - \langle \nabla u(x), h \rangle \chi_{B_1}(h)\big) K(x,h) \, dh$$

$$(2) \qquad = \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} \big(u(x+h) - 2u(x) + u(x-h)\big) K(x,h) \, dh \,,$$

with $K(x,h) = K(x,-h)$ for all x and h. Let us assume $0 \leq \alpha < 2$. In the special case $K(x,h) \asymp |h|^{-d-\alpha}$ one obtains $\mathcal{L}u = -(-\Delta)^{\alpha/2}u$. In general, we assume that $K \colon \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$ is a measurable function satisfying the following

conditions:

(K_1)
$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^2) K(x,h) \, dh \leq K_0 \,,$$

(K_2)
$$K(x,h) = K(x,-h) \qquad (x \in \mathbb{R}^d,\ h \in \mathbb{R}^d) \,,$$

(K_3)
$$\kappa^{-1} \frac{\ell(|h|)}{|h|^d} \leq K(x,h) \leq \kappa \frac{\ell(|h|)}{|h|^d} \qquad (0 < |h| \leq 1)$$

for some numbers $K_0 > 0$, $\kappa > 1$ and some function $\ell \colon (0,1) \to (0,\infty)$ which is locally bounded and varies regularly at zero with index $-\alpha \in (-2,0]$. Possible examples could be $\ell(s) = 1$, $\ell(s) = s^{-3/2}$ and $\ell(s) = s^{-\beta} \ln(\frac{2}{s})^2$ for some $\beta \in (0,2)$.

Let us explain the above set-up in the simple case of constant-coefficient operators, i.e. translation invariant operators. In this case there is a connection between the characteristic exponent and the symbol of the operator $\mathcal{L}$. Note that $\widehat{\mathcal{L}f}(\xi) = -\psi(-\xi)\hat{f}(\xi)$ for any $f \in \mathcal{S}(\mathbb{R}^d)$, where $-\psi(-\xi)$ is the symbol (multiplier) of the operator $\mathcal{L}$. Let us define an auxiliary function. Note that (K_1) and (K_3) imply that $\int_0^1 s\,\ell(s)\,ds \leq c$ holds for some constant $c > 0$. Let $L \colon (0,1) \to (0,\infty)$ be defined by $L(r) = \int_r^1 \frac{\ell(s)}{s}\,ds$. Note that the function L is always decreasing. The following proposition explains, for the translation invariant case, how the choice of ℓ is connected with the differentiability order of $\mathcal{L}$.

Proposition: *Let $\mathcal{L} \colon \mathcal{S} \to \mathcal{S}$ be given by* (1). *Assume $K(x,h) := k(h)$ satisfies* (K_1)-(K_3). *There is a constant $c > 0$ such that*

$$c^{-1} L(|\xi|^{-1}) \leq \psi(\xi) \leq c L(|\xi|^{-1}) \quad \text{for } \xi \in \mathbb{R}^d,\ |\xi| \geq 5 \,.$$

For the presentation of our main result, we do not assume any regularity of $K(x,h)$ with respect to x. Note that we do assume that the corresponding martingale is well-posed. Our main result concerning regularity is the following result:

Theorem: *There exist constants $c > 0$ and $\gamma \in (0,1)$ so that for all $r \in (0,\frac{1}{2})$ and $x_0 \in \mathbb{R}^d$*

$$(3) \qquad |u(x) - u(y)| \leq c\|u\|_\infty \frac{L(|x-y|)^{-\gamma}}{L(r)^{-\gamma}}, \quad x,y \in B_{r/4}(x_0)$$

for all bounded functions $u \colon \mathbb{R}^d \to \mathbb{R}$ that are harmonic in $B_r(x_0)$ with respect to $\mathcal{L}$.

It is important to note that the result trivially holds if the function $L \colon (0,1) \to (0,\infty)$ satisfies $\lim_{r \to 0+} L(r) < +\infty$. This is equivalent to the condition $\int_{B_1} \frac{\ell(|h|)}{|h|^d}\,dh < +\infty$, which, in the case $K(x,h) = k(h)$, means that the Lévy measure is finite. One feature of this article is that our result holds true up to and across the phase boundary determined by whether the kernel $K(x,\cdot)$ is integrable (finite Lévy measure) or not.

The main ingredient in the proof of the above theorem is a new version of a result on hitting probabilities which we provide now. We define a measure μ by

$$(4) \qquad \mu(dx) = \frac{\ell(|x|)}{L(|x|)|x|^d} \, \chi_{B_1}(x) \, dx \, .$$

Moreover, for $a > 1$, we define a function $\varphi_a : (0,1) \to (0,1)$ by $\varphi_a(r) = L^{-1}(\frac{1}{a}L(r))$. The main auxiliary result for us is the following:

Proposition: *There exists a constant $c > 0$ such that for all $a > 1$, $r \in (0, \frac{1}{2})$ and measurable sets $A \subset B_{\varphi_a(r)} \setminus B_r$ with $\mu(A) \geq \frac{1}{2}\mu(B_{\varphi_a(r)} \setminus B_r)$*

$$\mathbb{P}_x(T_A < \tau_{B_{\varphi_a(r)}}) \geq \mathbb{P}_x(X_{\tau_{B_r}} \in A) \geq c\frac{\ln a}{a}$$

holds true for all $x \in B_{r/2}$.

The novelty here is the definition and use of the measure μ. It allows us to deal with the classical cases as well as with critical cases, e.g. given by $K(x,h) \asymp |h|^{-d}\chi_{B_1}(h)$.

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On instabilities of global path properties of symmetric Dirichlet forms under the Mosco convergence

TOSHIHIRO UEMURA

(joint work with Kohei Suzuki)

The Mosco convergence is originally used to study the existence of solutions of variational inequalities and it is also known that the convergence is equivalent to the strong convergences of the corresponding semigroups and resolvents(see [5, 6]). The strong convergence of semigroups implies the convergence of finite-dimensional distributions of the associated Markov processes when the closed forms in question are regular Dirichlet forms. This is a reason why the Mosco convergence has been used to show the weak convergence of stochastic processes in the theory of Markov processes. In the talk, we shall give sufficient conditions for the Mosco convergence of the following three types of the Dirichlet forms: symmetric diffusion processes satisfying the locally uniformly elliptic conditions, symmetric translation invariant Dirichlet forms and symmetric pure-jump type Dirichlet forms. We also show some instabilities of global path properties under the Mosco convergence such as recurrence or transience, and conservativeness or explosion by giving several examples.

To this end, we first recall the definition of the Mosco convergence: For a closed form $(\mathcal{E}, \mathcal{F})$ on a Hilbert space H (not necessarily densely defined), we let $\mathcal{E}(u, u) = \infty$ for every $u \in H \setminus \mathcal{F}$. The Mosco convergence of closed forms is defined as follows (see [6]): A sequence of closed forms $\mathcal{E}_n$ on a Hilbert space H is said to be convergent to $\mathcal{E}$ in the sense of Mosco if (i) for every u and every sequence

$\{u_n\}$ converging to u *weakly* in H, $\liminf_{n\to\infty} \mathcal{E}_n(u_n, u_n) \geq \mathcal{E}(u, u)$; (ii) for every u there exists a sequence $\{u_n\}$ converging to u in H so that $\limsup_{n\to\infty} \mathcal{E}_n(u_n, u_n) \leq \mathcal{E}(u, u)$.

Case (I): Let $A_n(x) = (a_{ij}^n(x))$ and $A(x) = (a_{ij}(x))$ be $d \times d$-symmetric matrix valued functions on $\mathbb{R}^d$ satisfying that

(I-i) $\forall K \subset \mathbb{R}^d$ (compact), $0 < \exists \lambda \leq 1$ s.t. $\forall \xi \in \mathbb{R}^d$,

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}^n(x)\xi_i\xi_j \leq \lambda|\xi|^2 \ dx\text{-a.e. } x \in K,$$

(I-ii) $\forall K \subset \mathbb{R}^d$ (compact), $\int_K \|A_n(x) - A(x)\|dx \to 0 \ (n \to \infty)$, where $\|A(x)\| = \sqrt{\sum_{i,j=1}^d a_{ij}(x)^2}$.

Then consider the following Dirichlet forms:

$$\mathcal{E}^n(u, v) = \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}^n(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx, \quad u, v \in \mathcal{F}^n := \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1^n}}$$

and

$$\mathcal{E}(u, v) = \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx, \quad u, v \in \mathcal{F} := \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}.$$

We obtain

Theorem 1. The Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco if the conditions **(I-i)** and **(I-ii)** are satisfied.

Example 1. (i) First consider $\{A^n(x)\}$ are the diagonal matrix-valued functions defined as follows: for $x \in \mathbb{R}^d$, $a_{ii}^n(x) = (2 + |x|)^2\big(\log(2 + |x|)\big)^{1+1/n}$ and $a_{ii}(x) = (2 + |x|)^2 \log(2 + |x|)$. Then the Dirichlet forms $\mathcal{E}^n$ converges to $\mathcal{E}$ in the sense of Mocso and the sequence $\{\mathcal{E}^n\}$ are explosive while the limit form $\mathcal{E}$ is conservative. (ii) If we now set for $x \in \mathbb{R}^d$, $a_{ii}^n(x) = (2 + |x|)^{2-1/n}\big(\log(2 + |x|)\big)^2$ and $a_{ii}(x) = (2 + |x|)^2\big(\log(2 + |x|)\big)^2$, then $\mathcal{E}^n$ converges to $\mathcal{E}$ in the sense of Mosco. We find also that $\{\mathcal{E}^n\}$ are conservative Dirichlet forms, but $\mathcal{E}$ is explosive(see [4]).

Case (II): Let $\{\varphi_n\}$ be a sequence of the characteristic exponents defined by symmetric convolution semigroups $\{\nu_t^n, t > 0\}_{n\in\mathbb{N}}$: for $x \in \mathbb{R}^d$, $e^{-t\varphi_n(t)} := \widehat{\nu_t}^n(x)\big(= \int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\nu_t^n(dy)\big)$. Let φ be also the characteristic exponent defined by a symmetric convolution semigroup $\{\nu_t, t > 0\}$. The Dirichlet forms corresponding $\{\nu_t^n, t > 0\}$ are defined by $\mathcal{E}^n(u, v) = \int_{\mathbb{R}^d} \widehat{u}(\xi)\overline{\widehat{v}(x)}\varphi_n(\xi)d\xi$ for $u, v \in \mathcal{F}^n = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}^n(u, u) < \infty\}$.

Theorem 2. The Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ converges to the Dirichlet form corresponding to φ in the sense of Mosco if and only if φ_n converges to φ *locally* in $L^1(\mathbb{R}^d)$.

Example 2. Assume $d = 1$. Let α and α_n be measurable functions on $\mathbb{R}$ satisfying $0 < \underline{\alpha} \leq \alpha_n(x)$, $\alpha(x) \leq \overline{\alpha} < 2$ a.e. and define Lévy measures on $\mathbb{R}$

as follows: $n_n(dx) = |x|^{-1-\alpha_n(x)}dx$ and $n(dx) = |x|^{-1-\alpha(x)}dx$. The by the Lévy-Khinchine formula, we find that the characteristic exponents corresponding to the Lévy measures n_n and n are $\varphi_n(x) = \int_{\mathbb{R}}\big(1 - \cos(x\xi)\big)n_n(d\xi)$ and $\varphi(x) = \int_{\mathbb{R}}\big(1 - \cos(x\xi)\big)n(d\xi)$, respectively. Then the corresponding Dirichlet forms are given by

$$\mathcal{E}^n(u,u) = \int_{\mathbb{R}}\big|\widehat{u}(\xi)\big|^2\varphi_n(\xi)d\xi = \iint_{x\neq y}\big(u(x+h)-u(x)\big)^2n_n(dh)dx$$

and

$$\mathcal{E}(u,u) = \int_{\mathbb{R}}\big|\widehat{u}(\xi)\big|^2\varphi(\xi)d\xi = \iint_{x\neq y}\big(u(x+h)-u(x)\big)^2n(dh)dx,$$

respectively. Then we can give a recurrent/transient criterion:

Proposition 1. When $\alpha(x) = 1 - \big(\log(|x|^2 + e^2)\big)^{-\varepsilon}$ for $x \in \mathbb{R}$, then the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent if and only if $\varepsilon \geq 1$.

Using this proposition, the following results hold:

(i) If $\alpha_n(x) = 1 + 1/n - \big(\log(|x| + e^2)\big)^{-1/2}$ and $\alpha(x) = 1 - \big(\log(|x| + e^2)\big)^{-1/2}$, the forms $\mathcal{E}^n$ are recurrent for any n (see [9]) but $\mathcal{E}$ is transient. Moreover $\mathcal{E}^n$ converges to $\mathcal{E}$ in the sense of Mosco since φ_n converges to φ *locally* in L^1.

(ii) If $\alpha_n(x) = 1 - \big(\log(|x| + e^2)\big)^{-(1-1/n)}$ and $\alpha(x) = 1 - \big(\log(|x| + e^2)\big)^{-1}$, we easily see that the Dirichlet forms $\mathcal{E}^n$ correpsonding to φ_n converges to the one $\mathcal{E}$ corresponding to φ in the sense of Mocso. Moreover the forms $\mathcal{E}^n$ are transient, while the form $\mathcal{E}$ is recurrent.

Similarly, we could give a sufficient conditions for the Mosco convergence of symmetric jump-type Dirichlet forms. Moreover we are also able to show some examples for which transient Dirichlet forms converges to a recurrent Dirichlet forms in the sense of Mosco and vice versa. The point is that the 'locally convergence' of jump kernels also implies the Mosco convergence (see [8]).

Remark. Consider the case where $d = 1$ and $I = (0,1)$. Take a measurable periodic function $a(x)$ on $\mathbb{R}$ with the period 1: $a(x) = a(x+1)$, $x \in \mathbb{R}$. We assume that there exists a $\lambda \geq 1$ so that $\lambda^{-1} \leq a(x) \leq \lambda < \infty$. If we put $a_n(x) := a(nx)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, then we easily see that the sequence of the functions $\{a_n(x)\}$ converges to the average $\overline{a} = \int_0^1 a(x)dx$ *weakly* in L^1 (in actually, it converges weakly$*$ in L^∞). Moreover it is known that the corresponding Dirichlet forms $\mathcal{E}^n(u,u) = \int_0^1 a_n(x)u'(x)^2dx$ converges to $\mathcal{E}(u,u) = \underline{a}\int_0^1 u'(x)^2dx$ in the sense of Γ with $\mathcal{F}^n = \mathcal{F} = W^{1,2}(0,1)$, where $\underline{a} = \big(\int_0^1 (1/a(x))dx\big)^{-1}$ is the *harmonic mean* of a.

Thus the L^1-convergence of a sequence of bounded functions $\{a_n(x)\}$ satisfying the uniformly elliptic condition to a function $a(x)$ implies the Mosco convergence, hence the Γ-converngence, of the corresponding Dirichlet forms. It clearly implies that $\{a_n(x)\}$ converges to $a(x)$ *weakly* in L^1. So the Mosco (or Γ) convergence of the corresponding Dirichlet forms does not follow in general if we only assume the *weakly L^1-convergence* on the functions $\{a_n(x)\}$. In a sense, this means that our

conditions **(I-i)** and **(I-ii)** can not be relaxed in order to obtain the Mosco (or Γ) convergence. This is a simple example from homogenization problems (see *e.g.* [1, 2, 3]).

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Invariance principle for variable speed random walks on trees

ANITA WINTER

(joint work with Siva Athreya, Wolfgang Löhr)

A *rooted metric measure tree* (T, r, ρ, ν) consists of a complete, separable and locally compact 0-hyperbolic metric space (T, r), a distinguished point $\rho \in T$, and a boundedly finite measure ν on $(T, \mathcal{B}(T))$ of full support. If (T, r) is in addition path-connected, we refer to (T, r, ρ, ν) as *rooted measure $\mathbb{R}$-tree*.

In this talk we construct the class of skip-free strong Markov processes on natural scale with values in a given rooted metric measure tree. For that purpose, fix a *rooted metric measure tree* (T, r, ρ, ν). As usual, denote by $\mathcal{A}$ the space of absolutely continuous functions $f : T \to \mathbb{R}$ and by $\mathcal{C}_\infty$ the space of continuous functions $f : T \to \mathbb{R}$ which vanish at infinity. We will make use of the fact that there is a unique measure $\lambda^{(T,r,\rho)}$ defined by the requirement that $\lambda^{(T,r,\rho)}((\rho, x]) = r(\rho, x)$ for all $x \in T$. This so-called *length measure* was introduced in [4] on $\mathbb{R}$-trees, where it can be understood as the analogue of the Lebesgue measure, and extended to possibly disconnected trees in [3]. Moreover, a notion of a *gradient* is given on trees, i.e., for all $f \in \mathcal{A}$ there exists a unique (up to $\lambda^{(T,r,\rho)}$-zero sets) locally integrable function ∇f such that for all $x, y \in T$,

$$f(y) - f(x) = \int_{(\rho, y]} \mathrm{d}\lambda^{(T,r,\rho)} \nabla f - \int_{(\rho, x]} \mathrm{d}\lambda^{(T,r,\rho)} \nabla f.$$

We consider then the bilinear form

$$\mathcal{E}(f,g) := \int_T \mathrm{d}\lambda^{(T,r,\rho)} \nabla f \nabla g$$

with

$$\mathcal{D}(\mathcal{E}) := \big\{ f \in L^2(\nu) \cap \mathcal{A} \cap \mathcal{C}_\infty : \nabla f \in L^2(\lambda^{(T,r,\rho)}) \big\}.$$

It is shown in [2] that the form is closed if the *global lower mass bound property* holds, i.e., $\inf_{x \in T} \nu(B(x,r)) > 0$ for all $r > 0$, and that it is closable in any case. It is also shown that the Dirichlet form $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$ is *regular* and that Dirac measures are of finite energy integral. This extends the results obtained earlier in [1] for $\mathbb{R}$-trees. Hence by standard Dirichlet form theory there is a unique strong Markov process $(X^x \ x \in T)$ associated with $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$, which we refer to as the *speed-ν motion on (T,r) starting in $x \in T$*.

Classical particular examples are standard Brownian motion which corresponds to $\mathbb{R}$ equipped with the Euclidian distance and the Lebesgue-measure, or the so-called speed-ν random walk on a discrete tree (T,r) which is a continuous time Markov chain which jumps from a vertex $x \in T$ to a neighboring vertex $x' \in T$ (i.e., such that $[x,x'] = \{x,x'\}$) with rate

$$\gamma_{x,x'} = \tfrac{1}{2\nu(\{x\})r(x,x')}.$$

The main result states the following *invariance principle*:

Theorem. Let (T,r,ρ,ν), (T_1,r_1,ρ_1,ν_1), (T_2,r_2,ρ_2,ν_2), ... be rooted metric measure trees, and let X be the speed-ν motion on (T,r) starting in ρ, and for all $n \in \mathbb{N}$, let X^n be the speed-ν_n motion on (T_n,r_n) starting in ρ_n. Assume that the following conditions hold:

(A0) For all $R > 0$,

$$\limsup_{n \to \infty} \sup \big\{ r_n(x,z) : x \in B_n(\rho_n, R), z \in T_n, [x,z] = \{x,z\} \big\} < \infty.$$

(A1) The sequence $(T_n, r_n, \rho_n, \nu_n)_{n \in \mathbb{N}}$ converges to (T,r,ρ,ν) pointed Gromov-vaguely.

(A2) The uniform local lower mass-bound property holds, i.e., for all $r, R > 0$,

$$\inf_{n \in \mathbb{N}} \inf_{x \in T_n \cap B_n(\rho_n, R)} \nu_n\big(B(x,r)\big) > 0.$$

Then X^n converges in the one-point compactification in path-space to a process Y, such that Y stopped at infinity has the same distribution as the speed-ν motion X. In particular, if X is conservative (i.e. does not hit infinity), then X^n converges weakly in path-space to X.

If $\sup_{n \in \mathbb{N}} \mathrm{diam}(T_n, r_n) < \infty$, where diam is the diameter, and we assume (A1) but not (A2), then X^n converges f.d.d. to X.

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Malliavin smoothness for stochastic differential equations with singular drifts

Tusheng Zhang

(joint work with O Menoukeu-Pamen,T. Meyer-Brandis, T. Nilssen, F. Proske)

In this paper we are mainly interested to study the following stochastic differential equation (SDE) given by

$$(1) \qquad dX_t = b(t, X_t)dt + dB_t, \ \ 0 \le t \le T, \ \ \ X_0 = x \in \mathbb{R}^d,$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a $d-$dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \pi)$. We denote by $\mathcal{F}_t$ the augmented filtration generated by B_t.

If b in (1) is of linear growth and (globally) fulfills a Lipschitz condition it is well known that there exists a unique global strong solution to the SDE (1). More precisely, there exists a continuous $\mathcal{F}_t-$adapted process X_t solving (1) such that

$$E\left[\int_0^T X_t^2 dt\right] < \infty.$$

Important applications, however, of SDE's of the type (1) to physics or stochastic control theory show that Lipschitz continuity imposed on the drift coefficient b is a rather severe restriction. For example, in statistical mechanics, where one is interested in solutions of (1) as functionals of the driving noise (i.e. strong solutions) to model interacting infinite particle systems, the drift b is typically discontinuous or singular. See e.g. [21] and the references therein.

Strong solutions of SDE's with non-Lipschitz coefficients have been investigated by many authors in the past decades. To begin with we mention the work of Zvonkin [52], where the author obtains unique strong solutions of (1) in the one-dimensional case, when b is merely bounded and measurable. The latter result can be regarded as a milestone in the theory of SDE's. Subsequently, this result was generalized by Veretennikov [49] to the multidimensional case. The tools used by these authors to derive strong solutions are based on estimates of solutions of parabolic partial differential equations and a pathwise uniqueness argument.

Other important and more recent results in this direction based on a pathwise uniqueness argument (in connection with other techniques due to Portenko [36] or the Skorohod embedding) can be e.g. found in Krylov, Röckner [21], Gyöngy, Krylov [16] or Gyöngy, Martínez [17]. We also refer to [11], where the authors employ a modified version of Gronwall's Lemma. In this context we shall also point out the paper of Davie [7], who even establishes uniqueness of strong solutions of

(1) for almost all Brownian paths in the case of bounded and measurable drift coefficients.

In this paper we further develop the new approach devised in [31] to construct strong solutions of SDE's with irregular drift coefficients which additionally yields the important insight that these solutions are Malliavin differentiable. See also [29] and [38]. More precisely, we derive the results in [31] without assuming a certain symmetry condition [31, Definition 3] on the drift b in (1), which severely restricts the class of SDE's to be studied. In particular, one of our main results is the extension of [31, Theorem 4] on the Malliavin differentiability of solutions of (1) for merely bounded Borel functions b from the one-dimensional to the multi-dimensional case.

Our approach is mainly based on Malliavin calculus. To be more precise, our technique relies on a compactness criterion based on Malliavin calculus and an approximation argument for certain generalized processes in the Hida distribution space which we directly verify to be strong solutions of (1). We remark that our construction method is different from the above mentioned authors' ones. The technique proposed in this paper is not based on a pathwise uniqueness argument (or the Yamada-Watanabe theorem). In fact we tackle the construction problem from the "opposite" direction and prove that strong existence in connection with uniqueness in law of solutions of SDE's enforces strong uniqueness.

The additional information that strong solutions of SDE's with merely measurable drift coefficients are Malliavin differentiable has important and interesting implications. Further, one major strength of our approach is that it exhibits great flexibility to be applied and generalized to the analysis of various important aspects of solutions of a broader range of stochastic equations with irregular coefficients besides finite dimensional SDE's. In the last section of this paper we illustrate this by first showing how our techniques imply Sobolev differentiability of the strong solution X_t of SDE (1) in the initial condition x. Together with the Malliavin differentiability of X_t this is then used to derive a useful stochastic representation of spatial derivatives of solutions to the Kolmogorov equation - known as Bismut-Elworthy-Li formula - which does not involve derivatives of the initial condition of the Kolmogorov equation. Secondly, we present the applicability of our techniques and results to the problem of well-posedness of stochastic transport equations with singular coefficients (see also [32]).

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Reporter: Shuwen Lou

Participants

Prof. Dr. Sergio Albeverio
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Patricia Alonso Ruiz
Institut für Stochastik
Universität Ulm
89069 Ulm
GERMANY

Prof. Dr. Lucian Beznea
Institute of Mathematics
University of Bucharest
P.O. Box 1-764
Bucharest 014700
ROMANIA

Prof. Dr. Nicolas Bouleau
51, Rue Gerard
75013 Paris
FRANCE

Prof. Dr. Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle, WA 98195
UNITED STATES

Iulian Cimpean
Institute of Mathematics
University of Bucharest
P.O. Box 1-764
Bucharest 014700
ROMANIA

Prof. Dr. Fabio Cipriani
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci, 32
20133 Milano
ITALY

Prof. Dr. Laurent Denis
Département de Mathématiques
Université du Maine, LMM
Avenue O. Messiaen
72085 Le Mans Cedex
FRANCE

Prof. Dr. Jean Dominique Deuschel
Institut für Mathematik
Sekr. MA 7-4
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Andreas Eberle
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. David Elworthy
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Dr. Torben Fattler
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67618 Kaiserslautern
GERMANY

Prof. Dr. Uta Freiberg
Fachbereich Mathematik
Institut f. Stochastik & Anwendungen
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Masatoshi Fukushima
Department of Mathematics
Faculty of Engineering
Kansai University
Suita
Osaka 564-8680
JAPAN

Prof. Dr. Fuzhou Gong
Academy of Mathematics & Systems
Science
CAS
55, Zhongguancun East Rd.
Beijing 100190
CHINA

Prof. Dr. Masha Gordina
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Alexander Grigoryan
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Martin Grothaus
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67618 Kaiserslautern
GERMANY

Prof. Dr. Masanori Hino
Graduate School of Engineering Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-8531
JAPAN

Michael Hinz
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Niels Jacob
Department of Mathematics
Swansea University
Singleton Park
Swansea SA2 8PP
UNITED KINGDOM

Dr. Naotaka Kajino
Department of Mathematics
Kobe University
Rokkodai-cho 1-1, Nada-ku
Kobe 657-8501
JAPAN

Prof. Dr. Moritz Kassmann
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Hiroshi Kawabi
Department of Mathematics
Faculty of Science
Okayama University
3-1-1 Tsushima-naka
Okayama 700-8530
JAPAN

Prof. Dr. Jun Kigami
Graduate School of Informatics
Kyoto University
Kyoto 606-8501
JAPAN

Prof. Dr. Panki Kim
Department of Mathematical Sciences
Seoul National University
Gwanak-ro, Gwanak-gu
Seoul 151-747
KOREA, REPUBLIC OF

Prof. Dr. Yuri Kondratiev
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld
GERMANY

Prof. Dr. Takashi Kumagai
Research Institute for Math. Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Kazuhiro Kuwae
Department of Mathematics &
Engineering
Graduate School of Science &
Technology
Kumamoto University
Kumamoto 860-8555
JAPAN

Prof. Dr. Michel Ledoux
Institut de Mathématiques
Université de Toulouse
31062 Toulouse Cedex 9
FRANCE

Dr. Shuwen Lou
Department of Mathematics
University of Illinois at Chicago
Chicago IL 60607-7045
UNITED STATES

Prof. Dr. Zhi-Ming Ma
Academy of Mathematics & Systems
Science
55, Zhong-Guan-Cun East Rd.
Beijing 100190
CHINA

Prof. Dr. Cheng Ouyang
Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60607-7045
UNITED STATES

Diana Putan
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Michael Röckner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. René Schilling
Institut für Mathematische Stochastik
Technische Universität Dresden
Helmholtzstr. 10
01069 Dresden
GERMANY

Prof. Dr. Yuichi Shiozawa
Graduate School of Natural Science and
Technology
Okayama University
3-1-1. Tsushima-naka
Okayama 700-8530
JAPAN

Prof. Dr. Renming Song
Department of Mathematics
University of Illinois at
Urbana-Champaign
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Prof. Dr. Wilhelm Stannat
Fachbereich Mathematik
Technische Universität Berlin
10623 Berlin
GERMANY

Prof. Dr. Karl-Theodor Sturm
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Wei Sun
Department of Mathematics & Statistics
Concordia University
1455 De Maisonneuve Blvd. West
Montreal, QC H3G 1M8
CANADA

Prof. Dr. Masayoshi Takeda
Mathematical Institute
Tohoku University
Aramaki, Aoba-ku
Sendai 980-8578
JAPAN

Prof. Dr. Matsuyo Tomisaki
Department of Mathematics
Nara Women's University
Kita-Uoya, Nishimachi
Nara 630-8506
JAPAN

Prof. Dr. Gerald Trutnau
Department of Mathematical Sciences
Seoul National University
599 Gwanak-Ro, Gwanak-Gu
Seoul 151-747
KOREA, REPUBLIC OF

Prof. Dr. Toshihiro Uemura
Department of Mathematics
Faculty of Engineering Science
Kansai University
Suita-shi
Osaka 564-8680
JAPAN

Prof. Dr. Max von Renesse
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig
GERMANY

Robert Vosshall
Fachbereich Mathematik
T. U. Kaiserslautern
Postfach 3049
67618 Kaiserslautern
GERMANY

Prof. Dr. Jian Wang
School of Mathematics & Computer
Science
Fujian Normal University
Fuzhou 350007
CHINA

Dr. Anita Winter
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Prof. Dr. Rongchan Zhu
School of Mathematics & Statistics
Beijing Institute of Technology
5 South Zhongguancun Street
Beijing 100081
CHINA

Prof. Dr. Tusheng S. Zhang
School of Mathematics
The University of Manchester
Manchester M13 9PL
UNITED KINGDOM

Dr. Xiangchan Zhu
School of Mathematical Sciences
Beijing Jiaotong University
Shangyuancun 3
Beijing 100044
CHINA