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# Mini-Workshop: Eigenvalue Problems in Surface Superconductivity

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Abstract. The aim of the meeting is to discuss several classes of Schrödinger equations appearing within the Ginzburg-Landau theory of superconductivity. The related problems are discussed from several perspectives including semiclassical analysis, PDE in non-smooth domains, geometric spectral theory and operator theory, which should provide a new insight into various phenomena appearing in superconducting systems.

*Mathematics Subject Classification (2010):* 35P15, 35P20, 65N25, 35C20, 35J25.

## Introduction by the Organisers

The idea of the meeting was to discuss in a concentrated way several particular classes of differential operators appearing in the theory of surface superconductivity. We mention explicitly the two most important representatives, which served as an initial motivation. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $n$ be the outward pointing unit normal vector at its boundary $\partial\Omega$ which is assumed to be sufficiently regular. If $A$ is a magnetic vector potential and $\lambda$ is a coupling constant, one is interested in the associated magnetic Neumann eigenvalue problem

$$(i\nabla + \lambda A)^2 u = Eu \text{ in } \Omega, \quad n \cdot (i\nabla + \lambda A)u = 0 \text{ at } \partial\Omega. \tag{1}$$

Another important example is the (zero-field) Robin eigenvalue problem

$$-\Delta u = Eu \text{ in } \Omega, \quad n \cdot \nabla u = \lambda u \text{ at } \partial\Omega. \tag{2}$$

The both problems may be obtained through the linearization of the respective Ginzburg-Landau functionals, and the lowest eigenvalues $E = E(\lambda)$ are known be related to the critical temperature at which the normal state becomes unstable, while the respective eigenfunction $u$ describes the associated density of the particles (Cooper pairs). In view of this correspondence, one is interested in the dependence of the eigenvalue and the eigenfunction on the geometry of $\Omega$ and the coupling constant $\lambda$. A considerable amount of literature is dedicated to the study of the problem (1) in the limit $\lambda \to +\infty$, which is based on various advanced tools from the semiclassical and pseudodifferential analysis. It appears that the respective eigenfunctions concentrate near the boundary, whose geometric properties determine the eigenvalue asymptotics. During the last years, a particular attention is given to domains whose boundaries have singularities like corners or edges, and in that case the study of new classes of model domains, such as sectors or cones, becomes of importance.

During recent contacts at various meetings we found out that very similar qualitative effects are valid for the Robin problem (2) in the strong coupling limit. Actually, the study of the Robin eigenvalue asymptotics appeared first in the study of reaction-diffusion equations, and its relevance to the theory of superconductivity was given a limited attention only. On the other hand, recently, it has attracted an increasing attention from the point of view of the shape optimization, numerical computation or non-smoothness effects. One may also mention the recent applications to the classical topics such as the extension theorems for Sobolev spaces or Hardy inequalities.

In view of the above observations, we invited a group of experts from different scientific communities in order to exchange new ideas and methods concerning the analysis of differential operators of the above types. The meeting was concentrated on several specific questions: such as the qualitative spectral theory of linear differential operators in bounded and unbounded domains, optimization of eigenvalues with respect to geometry, boundary value problems in non-smooth domains, semiclassical methods. Some non-linear models were discussed as well.

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**Mini-Workshop: Eigenvalue Problems in Surface Superconductivity**

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# Abstracts

## Boundary behavior of the Ginzburg-Landau order parameter in the surface superconductivity regime

Michele Correggi
(joint work with Nicolas Rougerie)

The Ginzburg-Landau (GL) theory of superconductivity, originally proposed in [GL], provides a phenomenological, macroscopic, description of the response of a superconductor to an applied magnetic field. The state of a superconductor is described in suitable units by an order parameter $\Psi : \mathbb{R}^2 \to \mathbb{C}$ and an induced magnetic vector potential $\frac{1}{\varepsilon^2}\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ generating an induced magnetic field $h = \frac{1}{\varepsilon^2} \operatorname{curl} \mathbf{A}$. The ground state of the theory is found by minimizing the energy

$$(1) \qquad \mathcal{G}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left|\nabla_{\mathbf{A}}\Psi\right|^2 - \frac{1}{2b\varepsilon^2}\left(2|\Psi|^2 - |\Psi|^4\right) + \frac{b}{\varepsilon^4}\left|\mathrm{curl}\mathbf{A} - 1\right|^2 \right\}.$$

where $\nabla_{\mathbf{A}} := \nabla + i\frac{1}{\varepsilon^2}\mathbf{A}$, $b$ and $\varepsilon$ are positive parameters depending on the material and the applied field, that we assume to be constant throughout the sample. Units have been chosen in such a way that $\frac{1}{\varepsilon^2}$ measures the intensity of the external magnetic field. We consider a model for an infinitely long cylinder of cross section $\Omega \subset \mathbb{R}^2$, a compact simply connected set with regular boundary. The functional is gauge invariant, which implies that the only physically relevant quantities are the gauge invariant ones such as the density $|\Psi|^2$, which provides the local relative density of Cooper pairs. Any minimizing $\Psi$ must satisfy $|\Psi| \leq 1$. A value $|\Psi| = 1$ (respectively, $|\Psi| = 0$) corresponds to the superconducting (respectively, normal) phase where all (respectively, none) of the electrons form Cooper pairs.

In this note we focus on the mixed phase of an extreme type-II superconductor ($\varepsilon \to 0$) in the so called surface superconductivity regime, i.e., for an applied magnetic field between the second and third critical values [FH]. This results in the assumption

$$1 < b < \Theta_0^{-1},$$

where $\Theta_0 < 1$ is some universal number. We denote the GL ground state energy by $E_\varepsilon^{\mathrm{GL}} = \min_{(\Psi,\mathbf{A})} \mathcal{G}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}]$ and by $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ any minimizing pair.

The salient features of the surface superconductivity phase are as follows [FH]:

- $\Psi^{\mathrm{GL}}$ is concentrated in a thin boundary layer of thickness $\sim \varepsilon$; it decays exponentially to zero as a function of the distance from the boundary;
- The applied and induced magnetic fields are very close, i.e., $\operatorname{curl} \mathbf{A} \approx 1$;
- Up to an appropriate choice of gauge and a mapping to boundary coordinates, the GL energy is well approximated by a model 1D energy functional in the direction perpendicular to the boundary.

One of the consequences of the above features is the energy asymptotics

$$(2) \qquad E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1),$$

where $|\partial\Omega|$ is the length of the boundary of $\Omega$, and $E_0^{1D}$ is obtained by minimizing the functional

$$\mathcal{E}_{0,\alpha}^{1D}[f] := \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}\left(2f^2 - f^4\right) \right\}, \tag{3}$$

both with respect to the function $f$ and the real number $\alpha$. Following the partial results in [AH, FHP, Pan] (see also [FH, Theorem 14.1.1]) and using techniques developed in [CRY, CPRY], we proved recently [CR1] that (2) holds in the full surface superconductivity regime, i.e., for $1 < b < \Theta_0^{-1}$. This model problem is related to the GL minimization via the ansatz

$$\Psi^{GL}(\mathbf{r}) \approx f_0(t) \exp\left(-i\alpha_0 \tfrac{s}{\varepsilon}\right) \exp\left(i\phi_\varepsilon(s,t)\right) \tag{4}$$

where $(f_0, \alpha_0)$ is a minimizing pair for (3), $\phi_\varepsilon$ an explicit gauge phase and $(s,t)$ are rescaled boundary coordinates in a tubular neighborhood of $\partial\Omega$ with $\varepsilon t = \mathrm{dist}(\mathbf{r}, \partial\Omega)$. In [CR1, Theorem 2.1] we indeed proved that $|\Psi^{GL}|^2$ is close in $L^2$ to the profile $f_0^2(t)$ for any $1 < b < \Theta_0^{-1}$. A very natural question is whether the above estimate may be improved to a uniform control in $L^\infty$ norm. Indeed an $L^2$ estimate is still compatible with the vanishing of $\Psi^{GL}$ in small regions, e.g., vortices, inside of the boundary layer, whereas an $L^\infty$ bound would rule out such zeros. This problem was formulated as a conjecture in [Pan, Conjecture 1]. An affirmative solution is provided in [CR1, CR2] (and reported on here) for samples with regular boundary (the case with corners is known to require a different analysis [FH, Chapter 15]).

In order to expand the energy to the next order, we introduce a refined model problem in the constant curvature (disc) case:

$$\mathcal{E}_{k,\alpha}^{1D}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\,(1-\varepsilon k t) \left\{ |\partial_t f|^2 + \tfrac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}, \tag{5}$$

where $c_0$ has to be chosen large enough and $k$ is the curvature. We then set

$$E_\star^{1D}(k) := \inf_{\alpha\in\mathbb{R}} \inf_{f\in\mathscr{D}^{1D}} \mathcal{E}_{k,\alpha}^{1D}[f] = \mathcal{E}_{k,\alpha(k)}^{1D}[f_k], \tag{6}$$

i.e., $(\alpha(k), f_k)$ stands for any minimizing pair. Then we split the boundary layer $\mathcal{A}_\varepsilon = \{(s,t) \in [0, |\partial\Omega|] \times [0, c_0|\log\varepsilon|]\}$, into $N_\varepsilon = \mathcal{O}(\varepsilon^{-1})$ cells $\mathcal{C}_n = [s_n, s_{n+1}] \times [0, c_0|\log\varepsilon|]$ of constant side length $s_{n+1} - s_n = \ell_\varepsilon \propto \varepsilon$ in the $s$ direction. We approximate the curvature $k(s)$ by its mean value $k_n$ in each cell:

$$k_n := \ell_\varepsilon^{-1} \int_{s_n}^{s_{n+1}} \mathrm{d}s\, k(s).$$

We also denote for short $f_n := f_{k_n}$ and $\alpha_n := \alpha(k_n)$. The reference profile is then the piecewise continuous function

$$g_{\mathrm{ref}}(s,t) := f_n(t), \qquad \text{for } s \in [s_n, s_{n+1}]. \tag{7}$$

**Theorem 1** (Refined energy asymptotics)**.** *Let $\Omega \subset \mathbb{R}^2$ be any smooth, bounded and simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$, in the limit $\varepsilon \to 0$, it* *holds*

$$E_{\varepsilon}^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}s \, E_{\star}^{\mathrm{1D}}\left(k(s)\right) + \mathcal{O}(\varepsilon|\log\varepsilon|^{\infty}). \tag{8}$$

We now turn to the uniform density estimates that follow from the above theorem. We will compare $|\Psi^{\mathrm{GL}}|$ in $L^{\infty}$ norm to the simplified profile $f_0(t)$, since $f_0(t) - f_k(t) = \mathcal{O}(\varepsilon)$, which is much smaller than the error in the estimate. Also, the result may be proved only in a region where the density is relatively large, namely in $\mathcal{A}_{\mathrm{bl}} := \{\mathbf{r} \in \Omega \, : \, f_0\left(t\right) \geq \gamma_{\varepsilon}\}$, where $\gamma_{\varepsilon} \gg \varepsilon^{1/6}|\log\varepsilon|^{a}$, for some suitably large constant $a > 0$.

**Theorem 2** (Uniform density estimates and Pan's conjecture)**.** *Under the assumptions of the previous theorem, it holds*

$$\left\| \left|\Psi^{\mathrm{GL}}(\mathbf{r})\right| - f_0\left(t\right) \right\|_{L^{\infty}(\mathcal{A}_{\mathrm{bl}})} \leq C\gamma_{\varepsilon}^{-3/2}\varepsilon^{1/4}|\log\varepsilon|^{\infty} \ll 1. \tag{9}$$

*In particular, for any* $\mathbf{r} \in \partial\Omega$ *we have*

$$\left| \left|\Psi^{\mathrm{GL}}(\mathbf{r})\right| - f_0(0) \right| \leq C\varepsilon^{1/4}\left||\log\varepsilon|^{\infty}\right| \ll 1. \tag{10}$$

We now return to the question of the phase of the order parameter. Of course, the full phase cannot be estimated because of gauge invariance but its winding number (a.k.a. phase circulation or topological degree) can: (10) indeed ensures that $\deg\left(\Psi, \partial\Omega\right) \in \mathbb{Z}$ is well-defined.

**Theorem 3** (Winding number of $\Psi^{\mathrm{GL}}$ on the boundary $\partial\Omega$)**.** *Under the previous assumptions, any GL minimizer* $\Psi^{\mathrm{GL}}$ *satisfies as* $\varepsilon \to 0$

$$\deg\left(\Psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\Omega|}{\varepsilon^2} + \frac{|\alpha_0|}{\varepsilon} + \mathcal{O}(\varepsilon^{-3/4}|\log\varepsilon|^{\infty}). \tag{11}$$

Note that the remainder term in (11) is much larger than $\varepsilon^{-1}|\alpha(k) - \alpha_0| = \mathcal{O}(1)$ so that the above result does not allow to estimate corrections due to curvature.

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## Optimal magnetic Sobolev constants in the semiclassical limit

Søren Fournais
(joint work with Nicolas Raymond)

We consider a simply connected, bounded domain $\Omega \subset \mathbb{R}^2$, $p \in [2, +\infty)$, $h > 0$ and a smooth vector potential $\mathbf{A}$ on $\overline{\Omega}$ and define the following nonlinear eigenvalue (or optimal magnetic Sobolev constant):

$$\lambda(\Omega, \mathbf{A}, p, h) = \inf_{\psi \in H_0^1(\Omega), \psi \neq 0} \frac{\mathcal{Q}_{h,\mathbf{A}}(\psi)}{\left(\int_\Omega |\psi|^p \, d\mathbf{x}\right)^{\frac{2}{p}}} = \inf_{\substack{\psi \in H_0^1(\Omega), \\ \|\psi\|_{L^p(\Omega)}=1}} \mathcal{Q}_{h,\mathbf{A}}(\psi). \tag{1}$$

Here the magnetic quadratic form is defined by

$$\forall \psi \in H_0^1(\Omega), \quad \mathcal{Q}_{h,\mathbf{A}}(\psi) = \int_\Omega |(-ih\nabla + \mathbf{A})\psi|^2 \, d\mathbf{x}.$$

The corresponding self-adjoint operator—the magnetic Laplacian—is denoted by $\mathcal{L}_{h,\mathbf{A}}$.

We recall that $\mathbf{B} = \partial_2 A_1 - \partial_1 A_2$ (where $\mathbf{A} = (A_1, A_2)$) is the magnetic field generated by $\mathbf{A}$. There exists a vast literature dealing with the case $p = 2$. In this case $\lambda(\Omega, \mathbf{A}, 2, h)$ is the lowest eigenvalue of the magnetic Laplacian. The reader may consult the books and reviews [4, 5, 8] for more on this subject.

**1. Motivations and context.** A standard argument gives that the infimum in (1) is in fact a minimum. Furthermore, we have the following Euler-Lagrange equation.

**Lemma 1.** *The minimizers (which belong to $H_0^1(\Omega)$) of the $L^p$-normalized version of* (1) *satisfy the following equation in the sense of distributions:*

$$(-ih\nabla + \mathbf{A})^2\psi = \lambda(\Omega, \mathbf{A}, p, h)|\psi|^{p-2}\psi, \qquad \|\psi\|_{L^p(\Omega)} = 1. \tag{2}$$

*In particular (by Sobolev embedding), the minimizers belong to the domain of $\mathcal{L}_{h,\mathbf{A}}$.*

This work is motivated by the seminal paper [3] where the minimization problem (1) is investigated for $\Omega = \mathbb{R}^d$ and with a constant magnetic field (and also in the case of some nicely varying magnetic fields). In particular, Esteban and Lions prove the existence of minimizers by using the famous concentration-compactness method. In the present talk, we want to describe the minimizers as well as the infimum itself in the semiclassical limit $h \to 0$. The naive idea is that, locally, after a rescaling, they should look like the minimizers in the whole plane with a 'model' magnetic field capturing the local behavior of the original field. We will also allow the magnetic field to vanish and this will lead to a variety of minimization problems in the whole plane which are interesting in themselves and for which the results of [3] do not apply. Another motivation to consider the minimization problem (1) comes from the recent paper [1].

**2. Results.** We would like to provide an accurate description of the behavior of $\lambda(\Omega, \mathbf{A}, p, h)$ when $h$ goes to zero. Locally, we can approximate by either a constant magnetic field, or a magnetic field having a zero of a certain order. Therefore, we introduce the following notation.

**Definition 2.** For $k \in \mathbb{N}$, we define

$$\lambda^{[k]}(p) = \lambda(\mathbb{R}^2, \mathbf{A}^{[k]}, p, 1) = \inf_{\psi \in \mathcal{D}(Q_{\mathbf{A}^{[k]}}), \psi \neq 0} \frac{Q_{\mathbf{A}^{[k]}}(\psi)}{\|\psi\|_{L^p}^2}, \tag{3}$$

where $\mathbf{A}^{[k]}(x, y) = \left(0, \frac{x^{k+1}}{k+1}\right)$. Here

$$Q_{\mathbf{A}^{[k]}}(\psi) = \int_{\mathbb{R}^2} |(-i\nabla + \mathbf{A}^{[k]})\psi|^2 \, d\mathbf{x},$$

with domain $\mathcal{D}(Q_{\mathbf{A}^{[k]}}) = \left\{\psi \in L^2(\mathbb{R}^2) \, : \, (-i\nabla + \mathbf{A}^{[k]})\psi \in L^2(\mathbb{R}^2)\right\}$.

In the case $k = 0$ and $p \geq 2$, it is known that the infimum is a minimum (see [3]). We will prove in this talk that, for $k \geq 1$ and $p > 2$, the minimum is also attained, even if the corresponding magnetic field does not satisfy the assumptions of [3].

We can now state our first theorem concerning the case when the magnetic field does not vanish.

**Theorem 3.** *Let $p \geq 2$. Let us assume that $\mathbf{A}$ is smooth on $\overline{\Omega}$, that $\mathbf{B} = \nabla \times \mathbf{A}$ does not vanish on $\overline{\Omega}$ and that its minimum $b_0$ is attained in $\Omega$. Then there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(1 - Ch^{\frac{1}{8}})\lambda^{[0]}(p)b_0^{\frac{2}{p}} h^2 h^{-\frac{2}{p}} \leq \lambda(\Omega, \mathbf{A}, p, h) \leq (1 + Ch^{1/2})\lambda^{[0]}(p)b_0^{\frac{2}{p}} h^2 h^{-\frac{2}{p}}.$$

*Moreover, if the magnetic field is only assumed to be smooth and positive on $\overline{\Omega}$ (with a minimum possibly on the boundary), the lower bound is still valid.*

**Remark 4.** The error estimate in the upper bound in Theorem 3 matches the corresponding bound in the well-known linear case and we expect it to be optimal. However, the relative error of $h^{\frac{1}{8}}$ in the lower bound is unlikely to be best possible. The same remark applies to the error bounds in Theorem 7.

In the following theorem, we state an exponential concentration property of the minimizers.

**Theorem 5.** *Let $p > 2$, $\rho \in \left(0, \frac{1}{2}\right)$, $\varepsilon > 0$ and consider the same assumptions as in Theorem 3. Furthermore, assume that the minimum is unique and attained at $\mathbf{x}_0 \in \Omega$.*

*Then there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $\psi$ solution of (2), we have*

$$\|\psi\|_{L^\infty(\complement D(\mathbf{x}_0, 2\varepsilon))} \leq Ce^{-Ch^{-\rho}} \|\psi\|_{L^\infty(\Omega)},$$

*where $D(\mathbf{x}, R)$ denotes the open ball of center $\mathbf{x}$ and radius $R > 0$.*

**Remark 6.** In Theorem 5, if the minimum of the magnetic field is non-degenerate, we can replace $\varepsilon$ by $h^\gamma$ with $\gamma > 0$ sufficiently small. In Theorem 5, we have the same kind of results in the case of multiple wells. Theorems 3 and 5 are quantitative improvements of [1, Theorem 1.1] in the pure magnetic case. We can notice that, when $p > 2$, we have

$$(-ih\nabla + \mathbf{A})^2\varphi = |\varphi|^{p-2}\varphi, \tag{4}$$

with $\varphi = \lambda(\Omega, \mathbf{A}, p, h)^{\frac{1}{p-2}}\psi$. Thus, we have constructed solutions of (4) which decay exponentially away from the magnetic wells in the semiclassical limit.

The following theorem analyzes the case when the magnetic field vanishes along a smooth curve.

**Theorem 7.** *Let $p > 2$. Let us assume that $\mathbf{A}$ is smooth on $\overline{\Omega}$, that*

$$\Gamma = \{\mathbf{x} \in \overline{\Omega} \,:\, \mathbf{B}(\mathbf{x}) = 0\},$$

*satisfies that $\Gamma \subset \Omega$ is a smooth, simple and closed curve, and that $\mathbf{B}$ vanishes non-degenerately along $\Gamma$ in the sense that*

$$\nabla\mathbf{B}(\mathbf{x}) \neq 0 \text{ for all } \mathbf{x} \in \Gamma.$$

*Assuming that $\mathbf{B}$ is positive inside $\Gamma$ and negative outside, we denote by $\gamma_0 > 0$ the minimum of the normal derivative of $\mathbf{B}$ with respect to $\Gamma$. Then there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(1 - Ch^{\frac{1}{33}})\lambda^{[1]}(p)\gamma_0^{\frac{4}{3p}}h^2h^{-\frac{4}{3p}} \leq \lambda(\Omega, \mathbf{A}, p, h) \leq (1 + Ch^{\frac{1}{3}})\lambda^{[1]}(p)\gamma_0^{\frac{4}{3p}}h^2h^{-\frac{4}{3p}}.$$

**Remark 8.** The case $p = 2$ is treated in [2] (see also [6,7]). In [1], it is only stated that $h^{-2+\frac{2}{p}}\lambda(\Omega, \mathbf{A}, p, h)$ goes to zero when $h$ goes to zero. Moreover, by using the strategy of the proof of Theorem 5, one can establish an exponential decay of the ground states away from $\Gamma$.

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## Sharp inequalities for eigenvalues of the Laplace operator with Robin boundary conditions

Pedro Freitas
(joint work with P. R. S. Antunes and D. Krejčiřík)

Given a domain $\Omega$ in $\mathbb{R}^d$ we consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $n$ is the outer unit normal to $\Omega$ and the boundary parameter $\alpha$ is real.

We are interested in the question of optimising the first eigenvalue of (1) among domains with fixed volume. In the case where the boundary parameter $\alpha$ is positive, it was shown by Bossel in 1986 that the disk is a minimiser in two dimensions [Bos86], in a similar fashion to what happens in the Dirichlet case. This result was extended to higher dimensions by Daners twenty years later [Dan06].

For negative values of $\alpha$ it had been conjectured by Bareket in 1977 that the disk should now become a maximiser, with the switch taking place as the parameter goes through zero when all domains have a first zero eigenvalue (Neumann boundary conditions).

In joint work with David Krejčiřík, we have recently shown that this conjecture cannot hold for all values of the boundary parameter. More precisely, and based on the asymptotic behaviour of the first eigenvalue as $\alpha$ goes to $-\infty$, we prove that spherical shells become better than the ball for large enough (negative) values of the parameter.

**Theorem 1.** *There exists a negative value of $\alpha$, say, $\alpha_1$, for which there exist spherical shells whose first eigenvalue is larger than the first eigenvalue of the ball with the same volume.*

On the other hand, we have been able to show that Bareket's conjecture holds in two dimensions if the boundary parameter is not very large.

**Theorem 2.** *For bounded planar domains of class $C^2$ and fixed area, there exists a negative number $\alpha_2$, depending only on the area, such that the disk maximises the first Robin eigenvalue for all $\alpha \in [\alpha_2, 0]$.*

The above results immediately suggest several questions about which domains become optimisers and how the transition takes place. In joint work with Pedro R.S. Antunes and David Krejčiřík we have carried out a numerical study of the optimisation problem in dimensions two and three which indicates that indeed while the ball becomes a maximiser for small (negative) values of $\alpha$, it is then overtaken by spherical shells which become the optimisers. The bifurcation occurs for a value of the inner radius which is strictly positive, and the radii of these optimal shells change continuously with the value of $\alpha$. We have also obtained other estimates for the first eigenvalue based on the concept of shrinking coordinates previously developed in [FreKre08].

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## Eigenvalue sums of the Heisenberg Laplacian


Bartosch Ruszkowski

(joint work with Hynek Kovařík and Timo Weidl)


The Heisenberg Laplacian is a special type of subelliptic operator, which is connected to the first Heisenberg group $\mathbb{H}$ in a natural way. We study Riesz means of the eigenvalues of the Heisenberg Laplacian with Dirichlet boundary conditions on a cylinder in dimension $d = 3$. We obtain an inequality with a sharp leading term and an additional lower order term for the eigenvalue sum, improving a result by Hansson and Laptev in [1].

We recall that $\mathbb{H}$ is given by the set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ equipped with the group law

$$(x, y, t) \boxplus (u, v, s) \ := \ \Big(x + u, y + v, t + s + \frac{1}{2}(x \cdot v - y \cdot u)\Big).$$

For $x := (x_1, x_2, x_3) \in \mathbb{H}$ we introduce the following vector fields $X_1, X_2, X_3$ in $\mathbb{H}$

$$X_1 \ := \ \partial_{x_1} + \frac{1}{2}x_2\partial_{x_3}, \quad X_2 \ := \ \partial_{x_2} - \frac{1}{2}x_1\partial_{x_3}, \quad X_3 \ := \ \partial_{x_3},$$

which form a basis of the Lie algebra of left-invariant vector fields on the first Heisenberg group, see [2]. $X_1$ and $X_2$ fulfill the Hörmander finite rank condition, see [3], because of $[X_1, X_2] = -X_3$.

Let $\Omega \subset \mathbb{H}$ be an open bounded domain. We consider the Heisenberg Laplacian on $\Omega$ with Dirichlet boundary condition, denoted by

$$\mathrm{A}(\Omega) \ := \ -X_1^2 - X_2^2,$$

where $a, b \in \mathbb{R}$. This self-adjoint operator corresponds to the closure of the semi-bounded quadratic form

$$\mathrm{a}[u] \ := \ \int_\Omega |X_1 u(x)|^2 + |X_2 u(x)|^2 \, \mathrm{d}x,$$

initially defined for $u \in C_0^\infty(\Omega)$. The subelliptic estimate in [4] together with standard Sobolev theory yield that $A(\Omega)$ has discrete spectrum. We denote by $(\lambda_k(\Omega))_{\in\mathbb{N}}$ the non-decreasing and unbounded sequence of the eigenvalues of $A(\Omega)$, where we repeat entries according to their finite multiplicities. In [1] Hansson and Laptev derived universal bounds for

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_-^\gamma \;:=\; \sum_{k=1}^{\infty}(\lambda-\lambda(\Omega)_k)_+^\gamma\,, \qquad \gamma>0.$$

They proved the following inequality for domains of finite measure

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_-^\gamma \;\le\; K_\gamma|\Omega|\lambda^{\gamma+2}, \tag{1}$$

where

$$K_\gamma \;:=\; \begin{cases} \frac{9}{32}\frac{\gamma^\gamma}{(\gamma+2)^{\gamma+2}}, & 0<\gamma\le 1,\\ \frac{1}{16}\frac{1}{(\gamma+1)(\gamma+2)}, & 1\le\gamma. \end{cases}$$

They showed also that the constant $K_\gamma$ is asymptotically sharp for $\gamma \ge 1$ as $\lambda \to \infty$. In addition for $0 \le \gamma < 1$ they proved the following asymptotics

$$\lim_{\lambda\to\infty}\lambda^{-\gamma-2}\sum_{k=1}^{\infty}(\lambda-\lambda_k(\Omega))_+^\gamma \;=\; \frac{\gamma+3}{\gamma+1}K_{\gamma+1}|\Omega|.$$

For $\gamma = 0$ the left-hand side becomes the counting function of $A(\Omega)$. To derive a bound for the eigenvalue sum, we consider inequality (1) for $\gamma = 1$. Applying the Legendre transformation to this inequality leads one to a sharp bound for the eigenvalue sum, see [1],

$$\sum_{k=1}^{n}\lambda_k(\Omega) \;\ge\; \frac{8\sqrt{2}}{3}|\Omega|^{-1/2}n^{3/2}, \qquad \text{for } n\in\mathbb{N}. \tag{2}$$

We call this a Li-Yau inequality, see [5], because they were the first ones who derived such a sharp bound for the eigenvalue sum of the Dirichlet Laplacian on domains with finite volume.

We improve the inequality in (2) by adding a negative lower order term in $\lambda$ for cylindrical domains $\Omega := \tilde{\Omega}\times(a,b)\subset\mathbb{H}$, where $\tilde{\Omega}\subset\mathbb{R}^2$ is a bounded domain; this additional term reflects the geometry of the domain. Our result can be stated as follows:

Let $\Omega := \tilde{\Omega}\times(a,b)\subset\mathbb{H}$ be open bounded and convex. Then the following inequality holds

$$\sum_{k=1}^{n}\lambda_k(\Omega) \;\ge\; \frac{8\sqrt{2}}{3}|\Omega|^{-1/2}n^{3/2}+\frac{5}{576}\frac{1}{\mathrm{R}(\tilde{\Omega})^2}n, \qquad \text{for } n\in\mathbb{N}, \tag{3}$$

where $\mathrm{R}(\tilde{\Omega})>0$ is the in-radius of $\tilde{\Omega}$ in the Euclidean sense in $\mathbb{R}^2$.

To prove inequality (3) we improve the bound for the trace of the eigenvalues in (1) for $\gamma = 1$, because we know that an application of the Legendre transformation always yields an estimate for the eigenvalue sum. To get an estimate for the trace we need two ingredients. First of all we need the spectral decomposition of $A(\mathbb{H})$.

Therefore we compute

$$\mathcal{F}_3 \mathrm{A} \mathcal{F}_3^* \;=\; \left(\mathrm{i}\partial_{x_1} - \frac{1}{2}x_2\xi_3\right)^2 + \left(\mathrm{i}\partial_{x_2} + \frac{1}{2}x_1\xi_3\right)^2 \;=\; \left(\mathrm{i}\nabla_{x'} + \xi_3 \mathrm{A}(x_1,x_2)\right)^2,$$

where $\mathrm{A}(x_1,x_2) := \frac{1}{2}(-x_2,x_1)$ and $\mathcal{F}_3$ is the Fourier transform in the $x_3$-direction. On the right-hand side we obtain the Laplacian with constant magnetic field. We know the Landau levels of this operator and its corresponding orthogonal projections, yielding a spectral decomposition of $A(\mathbb{H})$. Using this property we can derive the same bound as in (1) for $\gamma = 1$. This can then be improved using the technique developed by Kovařík and Weidl in [7], where they derive an improved sharp inequality for the eigenvalue sum of the Dirichlet Laplacian with constant magnetic field on bounded domains. This technique is based on an application of Hardy's inequality combined with the diamagnetic inequality.

Our result can be generalized to bounded domains $\Omega := \tilde{\Omega} \times (a,b) \subset \mathbb{H}$, with $\tilde{\Omega}$ not necessarily convex, if we assume the validity of the Hardy inequality

$$\int_{\tilde{\Omega}} \frac{|u(x)|^2}{\delta(x)^2}\,\mathrm{d}x \le c \int_{\tilde{\Omega}} |\nabla u(x)|^2 \mathrm{d}x \tag{4}$$

for all $u \in C_0^\infty(\tilde{\Omega})$. Here $c$ denotes the optimal constant in this inequality and $\delta(x)$ is the distance from a fixed point $x \in \tilde{\Omega}$ to the boundary of $\tilde{\Omega}$ in the Euclidean sense. For instance (4) is fulfilled if $\tilde{\Omega}$ is simply connected; in this case we know that $4 \le c \le 16$, see [6].

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## Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator

Daniel Daners

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary, and let $\lambda \in \mathbb{R}$. Given $\varphi \in H^{1/2}(\Omega)$ solve the Dirichlet problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega. \tag{1}$$

Problem (1) has a unique solution unless $\lambda$ is one of the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ of the Dirichlet Laplacian on $\Omega$. If $u$ is smooth enough we define the *Dirichlet-to-Neumann operator* $D_\lambda$ by

$$D_\lambda \varphi := \frac{\partial u}{\partial \nu}, \tag{2}$$

where $\nu$ is the outer unit normal to $\partial\Omega$. One can show that $D_\lambda$ extends uniquely to an operator $D_\lambda \in \mathcal{L}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$. The part of $-D\lambda$ in $L^2(\partial\Omega)$ generates an analytic semigroup $(e^{-tD_\lambda})_{t\geq 0}$ on $L^2(\partial\Omega)$ and $C(\partial\Omega)$; see [1,2]. In [1] it is proved that $(e^{-tD_\lambda})_{t\geq 0}$ is a positive semigroup (that is, $e^{tD_\lambda}$ is a positive operator for all $t \geq 0$) whenever $\lambda < \lambda_1$. The aim of the presentation is to explore the positivity properties of $(e^{-tD_\lambda})_{t\geq 0}$ if $\lambda > \lambda_1$. Details are found in [3].

A first example deals with the case of an interval $\Omega = (0, L) \subseteq \mathbb{R}$, where $L > 0$. It turns out that positivity and non-positivity of $(e^{-tD_\lambda})_{t\geq 0}$ alternate at each eigenvalue $\lambda_k = (k\pi/L)^2$ of the Dirichlet Laplacian.

The situation is much more complicated for the unit disk $\Omega = B(0,1)$ in $\mathbb{R}^2$. We find the following phenomena: There are values of $\lambda > \lambda_1$ such that

(i) $e^{-tD_\lambda}$ is not positive for every $t > 0$;
(ii) there exists $t_0 > 0$ such that $e^{-tD_\lambda}$ is positive for every $t \geq t_0$, but not positive for some $t \in (0, t_0)$;
(iii) $e^{-tD_\lambda}$ is positive for all $t > 0$.

In case (ii) we say that the semigroup $(e^{-tD_\lambda})_{t\geq 0}$ is *eventually positive*. An abstract theory of eventually positive semigroups is developed in [2].

The proof is done by a careful analysis of the Fourier series representation of the semigroup. If $\varphi = e^{ik\theta}$, then the solution of (1) is given by

$$u(r,\theta) = \frac{J_{|k|}(\sqrt{\lambda} r)}{J_{|k|}(\sqrt{\lambda})} e^{ik\theta} \qquad \text{and hence} \qquad D_\lambda e^{ik\theta} = \frac{\sqrt{\lambda} J'_{|k|}(\sqrt{\lambda})}{J_{|k|}(\sqrt{\lambda})} e^{ik\theta}, \tag{3}$$

where $J_k$ is the Bessel functions of the first kind. In particular,

$$\mu_k(\lambda) := \frac{\sqrt{\lambda} J'_{|k|}(\sqrt{\lambda})}{J_{|k|}(\sqrt{\lambda})} \tag{4}$$

are the eigenvalues of $D_\lambda$ with eigenfunctions $e^{\pm ik\theta}$ or $\cos k\theta, \sin k\theta$, where $k \in \mathbb{N}$. Given the Fourier series $\varphi(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \in H^{1/2}(\partial\Omega)$ we see that

$$e^{-tD_\lambda}\varphi = \sum_{k=-\infty}^{\infty} c_k e^{-t\mu_k(\lambda)} e^{ik\theta}. \tag{5}$$

A standard result in the theory of positive semigroups asserts that a necessary condition for $(e^{-tD_\lambda})_{t\geq 0}$ to be positive is that the dominating eigenvalue of $D_\lambda$ has a positive eigenfunction. It is clear that the only eigenvalue having a positive eigenfunction is $\mu_0(\lambda)$ with eigenfunction the constant function with value 1. We now discuss the three cases (i)–(iii).

(i) For most $\lambda > \lambda_1$ the eigenvalue $\mu_0(\lambda)$ is not dominant, and hence the semigroup $(e^{-tD_\lambda})_{t\geq 0}$ is not positive.

(ii) If $\varphi \geq 0$, then by Herglotz's theorem its Fourier coefficients form a positive definite sequence; see [4, Section 7.6]. In particuar $c_0 \geq |c_k|$ and $c_{-k} = \bar{c}_k$ for all $k \in \mathbb{N}$. If $\mu_0(\lambda) < \mu_k(\lambda)$ for all $k \in \mathbb{N}$, then

$$\begin{aligned} e^{-tD_\lambda}\varphi = \sum_{k=-\infty}^{\infty} c_k e^{-t\mu_{|k|}(\lambda)} e^{ik\theta} &\geq c_0 e^{-t\mu_0(\lambda)} - 2c_0 \sum_{k=1}^{\infty} e^{-t\mu_k(\lambda)} \\ &= c_0 e^{-t\mu_0(\lambda)} \left(1 - 2\sum_{k=1}^{\infty} e^{-t(\mu_k(\lambda)-\mu_0(\lambda))}\right). \end{aligned}$$

Hence there exists $t_0 > 0$ independent of $\varphi \geq 0$ such that $fe^{-tD_\lambda}\varphi > 0$ for all $t \geq t_0$, hence $(e^{-tD_\lambda})_{t\geq 0}$ is *eventually positive*. An example for a range of $\lambda$ for which this is the case is $\lambda \in (\lambda_3, \lambda_4)$ or $\lambda \in (\lambda_8, \lambda_9)$. It is possible to show that the $e^{-tD_\lambda}$ is eventually positive, but not positive for some $\lambda$; see [2, 3].

(iii) If $\lambda$ is in some left neighbourhood of $\lambda_k$ such that $J_0(\sqrt{\lambda_k}) = 0$ (e.g. $\lambda_3$, $\lambda_9$), then $(e^{-tD_\lambda})_{t\geq 0}$ is positive. To prove this, note that (5) can be written as

$$(e^{-tD_\lambda}\varphi)(\theta) = (G_{\lambda,t} * \varphi)(\theta) := \int_{-\pi}^{\pi} G_{\lambda,t}(\theta - s)\varphi(s)\, ds, \tag{6}$$

where

$$G_{\lambda,t}(\theta) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-td_k(\lambda)} e^{ik\theta} \tag{7}$$

for all $t > 0$. The function $G_{\lambda,t}$ is the "heat kernel" of the semigroup $e^{-tD_k(\lambda)}$. To prove the positivity it is therefore sufficient to show that $G_{\lambda,t} \geq 0$ for all $t > 0$. This is done by representing (7) by means of the Féjer's kernels $K_n \geq 0$, using that $k \mapsto \mu_k(\lambda)$ is a convex function for $k$ large. Then it turns out that

$$G_{\lambda,t}(\theta) = \sum_{n=1}^{\infty} nb_n(\lambda, t)K_{n-1}(\theta), \tag{8}$$

where $b_n(\lambda, t) := e^{-t\mu_{n+1}(\lambda)} + e^{-t\mu_{n-1}(\lambda)} - 2e^{-t\mu_n(\lambda)}$. Due to the convexity of $\mu_k(\lambda)$ as a function of $k$ at most finitely many terms in (8) are possibly negative. If $\mu_0(\lambda)$ is very negative, which is the case if $\lambda \uparrow \lambda_k$, then the total sum is positive as shown in detail in [3].

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## Estimating the eigenvalues of a quantum graph

James Kennedy
(joint work with Pavel Kurasov, Gabriela Malenová and Delio Mugnolo)

A quantum graph is a metric graph – a collection of intervals of varying lengths, connected at a set of vertices – on which a differential operator such as the Laplacian acts. Such objects, apart from modelling diffusion processes on networks, are also sometimes used as toy problems, exhibiting features typical of higher-dimensional problems despite their essentially one-dimensional structure. Together with the parallel theory of discrete graphs and discrete Laplacians, they can also be used to approximate manifolds in various senses. We refer to the monographs [1,6] for more details.

Despite, or perhaps because of, their seemingly simple nature, relatively little seems to be known about the spectral geometry of such graphs, in particular as regards eigenvalue estimates of "isoperimetric" type, giving bounds for the eigenvalues in terms properties of the graph analogous to those usually considered for shape optimisation problems on domains or manifolds: volume, diameter and so on. In this talk, which is based on [4], we explore a prototypical case systematically: we wish to estimate the first non-trivial eigenvalue of the Laplacian with Kirchhoff, or natural, conditions at the vertices, a natural generalisation or analogue of the familiar Neumann condition.

More precisely, denote by $\Gamma$ a compact, connected, nonempty metric graph, i.e., a finite union of edges, each being identifiable with a finite interval equipped with the Euclidean metric in one dimension, connected at a finite set of vertices. We denote by $E \geq 1$ the number of edges, $V \geq 1$ the number of vertices, $L \in (0,\infty)$ the total length (the sum of all the edge lengths), and by $D = \sup\{\text{dist}(x,y) : x,y \in \Gamma\} \in (0,\infty)$ the diameter of $\Gamma$, where the distance between two points is the length of the shortest path connecting them within $\Gamma$. As usual, the eigenvalues of the Kirchhoff Laplacian on $\Gamma$, which we consider as the eigenvalues of the *graph* itself, form a sequence

$$0 = \lambda_0(\Gamma) < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \ldots \to \infty,$$

The first eigenvalue $\lambda_1$, in this case also the spectral gap of the operator, is of interest for a number of reasons; for instance, it determines loosely speaking the rate of convergence of a diffusion process on the graph to the equilibrium. The problem we consider here may be formulated as asking how $\lambda_1$ depends on the four basic quantities $L$, $E$, $V$ and $D$. (Obviously, there are many other possibilites, but in analogy with both the theory of shape optimisation on domains and the "spectral geometry" of finite graphs, we believe these are the most natural.)

All the usual tools, in particular the characterisation of the eigenvalues in terms of the Rayleigh quotient, comparison arguments, test function arguments, and so on, are available in this case, and indeed become more powerful on graphs, since they are essentially one-dimensional objects: it is easy to derive a number of elementary but powerful comparison results describing how modifying the graph affects $\lambda_1$. (Here we speak of performing "surgery": adding or deleting an edge, identifying two vertices and so on.) One can therefore say more than in the case of domains or manifolds, and other questions become interesting: it is for example a subtle question as to which combinations of the aforementioned parameters yield universal upper and lower bounds on $\lambda_1$.

With all this in mind, it is all the more surprising that possibly the only existing result within the framework of our problem is the fundamental "isoperimetric" inequality bounding $\lambda_1$ from below in terms of the length (i.e. volume) of $\Gamma$, originally proved in the 1980s and reproved several times.

**Theorem 1** ( [3,5,7]). *If $\Gamma$ has total length $L > 0$, then*

$$\lambda_1(\Gamma) \geq \frac{\pi^2}{L^2},$$

*with equality if and only if $\Gamma$ has one edge, i.e., $\Gamma$ is an interval of length $L$.*

Interesting here is that although the Kirchhoff vertex condition is morally equivalent to a Neumann boundary condition, the behaviour of the eigenvalues resembles that of a Dirichlet problem; there is no corresponding upper bound in terms of $L$ alone, as a graph with very many short edges, all connected to each other, can have arbitrarily large $\lambda_1$. However, we can recover an upper bound if we fix $E$ (examples show that $V$ cannot help with either an upper or a lower bound):

**Theorem 2.** *If $\Gamma$ has $E \geq 2$ edges and total length $L > 0$, then*

$$\lambda_1(\Gamma) \leq \frac{\pi^2 E^2}{L^2}.$$

The set of maximisers is large, consisting of at least two different "classes" of graphs (so-called "flower" and "pumpkin" graphs).

It turns out that diameter alone is not enough to control $\lambda_1$.

**Theorem 3.** *Given any $D > 0$, there exist sequences of graphs $\Gamma_n$, $\tilde{\Gamma}_n$, all having diameter $D$, such that $\lambda_1(\Gamma_n) \to 0$ and $\lambda_1(\tilde{\Gamma}_n) \to \infty$ as $n \to \infty$: the $\Gamma_n$ may be chosen such that the number of vertices $V(\Gamma_n) = 2$ for all $n$.*

The non-existence of an upper bound seems both non-trivial and non-obvious, and was obtained by introducing a new, special class of graphs which can be compared to a one-dimensional Sturm–Liouville problem. Showing that this class is in a sense "maximising" allows one to recover an upper bound by fixing the number of vertices as well as the diameter.

**Theorem 4.** *If $\Gamma$ has $V \geq 1$ vertices and diameter $D > 0$, then*

$$\lambda(\Gamma) \leq \frac{\pi^2 (V+1)^2}{D^2}.$$

This bound is in general far from optimal, equality holding only if $V = 1$ and $\Gamma$ is a loop, but the key point is the *existence* of an upper estimate, given that there is no corresponding lower bound for this combination of parameters, nor an upper bound for $L$ together with $V$. (Note that specifying $V$ is in a sense weaker than specifying $E$ due to the relation $V \leq E+1$, with equality if $\Gamma$ is simple, i.e., if there is no more than one edge connecting any two vertices.)

If instead we fix $D$ and $E$ simultaneously, or $D$ and $L$ simultaneously, then we can obtain both upper and lower bounds; for example, we have

$$\frac{\pi^2}{D^2E^2} \leq \lambda_1(\Gamma) \leq \frac{4\pi^2E^2}{D^2}$$

as an essentially trivial consequence of the other bounds and/or crude test function arguments. More interestingly, and less trivially, we also have

$$\frac{1}{DL} \leq \lambda_1(\Gamma) \leq \frac{\pi^2}{D^2}\frac{4L-3D}{D};$$

although again far from optimal, at least the lower bound exhibits the correct dependence on $D$ and $L$ (as can be seen via examples). It is an ongoing project to find both the optimising constants and the corresponding graphs in all these cases. Moreover, this is merely our prototype problem: one can ask similar questions for higher eigenvalues, for different operators, in particular for different vertex conditions (as was also done in a recent preprint [2]), and for different geometric, analytic and combinatorial properties of the graph besides our four quantities $L$, $E$, $V$ and $D$.

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## Discrete spectrum of Schrödinger operators with $\delta$-interactions on conical surfaces

Vladimir Lotoreichik

(joint work with Jussi Behrndt and Pavel Exner)

Spectral analysis of multi-dimensional Schrödinger operators with interactions supported on null sets such as points, curves, and surfaces, is a classical topic in mathematical physics. Besides a physical motivation to investigate these operators, originating, in particular, from quantum mechanics, there exist also several purely mathematical motivations, one of which is that these operators exhibit non-trivial interplay between their spectral properties and the geometry of the interaction support. One such interplay is considered in the paper [1] by J. Behrndt, P. Exner, and myself. It was a great pleasure for me to present the results of our paper at the mini-workshop "Eigenvalue problems in surface superconductivity".

In [1] we characterise the spectrum of the self-adjoint lower-semibounded three-dimensional Schrödinger operator $H_{\alpha,\theta}$ with attractive $\delta$-interaction of constant strength $\alpha > 0$ supported on the conical surface

$$\mathcal{C}_\theta := \big\{(x,y,z) \in \mathbb{R}^3 \colon z = \cot(\theta)\sqrt{x^2+y^2}\big\}, \qquad \theta \in \big(0, \pi/2\big].$$

This operator can be rigorously introduced via closed, densely defined, symmetric, and lower-semibounded sesquilinear form

$$\mathfrak{a}_{\alpha,\theta}[u,v] := \big(\nabla u, \nabla v\big)_{L^2(\mathbb{R}^3;\mathbb{C}^3)} - \alpha \int_{\mathcal{C}_\theta} u\overline{v}\,\mathrm{d}\sigma, \quad \mathrm{dom}\,\mathfrak{a}_{\alpha,\theta} := H^1(\mathbb{R}^3).$$

According to the main results of [1], the essential and discrete spectra of $H_{\alpha,\theta}$ are characterised as follows:

- $\sigma_{\mathrm{ess}}(H_{\alpha,\theta}) = \Big[-\dfrac{\alpha^2}{4}, +\infty\Big)$;
- $\sigma_{\mathrm{d}}(H_{\alpha,\theta}) = \emptyset$ if $\theta = \pi/2$;
- $\#\sigma_{\mathrm{d}}(H_{\alpha,\theta}) = \infty$ if $\theta \in (0, \pi/2)$.

The above results remain valid if the $\delta$-interaction is supported on a sufficiently regular local deformation of $\mathcal{C}_\theta$. This fact, in particular, shows that the tip of the conical surface is not "responsible" for the infiniteness of the discrete spectrum and that the corresponding effect is generated exclusively by the shape of the conical surface $\mathcal{C}_\theta$ at infinity.

The key tools in the proofs are Neumann bracketing and variational principles for self-adjoint operators. The test functions for variational principles should be chosen to respect the shape of $\mathcal{C}_\theta$. The crucial difficulty is to prove the infiniteness of the discrete spectrum for all $\theta \in (0, \pi/2)$. For this purpose one has to make the profile of the test functions dependent on the angle $\theta$ in a suitable way. At this point a special family of functions comes into play which is used in [2] to demonstrate sharpness of the celebrated Hardy inequality.

In [1] we obtain also asymptotic upper bounds on the eigenvalues of $H_{\alpha,\theta}$ with $\theta \in (0, \pi/2)$. We expect that the methods of the very recent preprint [3] can be employed to derive the exact spectral asymptotics of $H_{\alpha,\theta}$.

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## Eigenvalues of Robin Laplacians and mean curvature bounds

Nicolas Popoff
(joint work with Konstantin Pankrashkin)

For a regular bounded domain $\Omega \subset \mathbb{R}^n$, we are interested in the following linear boundary value problem:

$$\begin{cases} -\Delta u = \lambda u \quad \text{on } \Omega, \\ \partial_n u - \alpha u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\partial_n$ denotes the outward normal derivative at the boundary and $\alpha \in \mathbb{R}$ is the Robin coefficient. This problem has connections with the study of enhanced surface superconductivity in zero magnetic fields [7]. With this problem we associate the quadratic form

$$q_\alpha : u \mapsto \int_\Omega |\nabla u|^2 dx - \alpha \int_{\partial\Omega} |u|^2 dS, \quad u \in H^1(\Omega),$$

where $dS$ denotes the measure surface of $\partial\Omega$. We denote by $Q_\alpha$ the associated self-adjoint operator, defined on the domain $\{u \in H^2(\Omega), \partial_n u - \alpha u = 0 \text{ on } \partial\Omega\}$. We define $E_j(\alpha, \Omega)$ as the $j$th eigenvalue of $Q_\alpha$. The behavior of $E_j(\alpha, \Omega)$ at first order when $\alpha \to +\infty$ has been investigated in [4,10–12]. It is now well known that for any fixed $j$ there holds

$$E_j(\alpha, \Omega) = -\alpha^2 + o(\alpha^2), \quad \alpha \to +\infty.$$

In the two dimensional case, and when the boundary of $\Omega$ is $\mathcal{C}^4$, the following more precise asymptotics is proved in [5,13] for each fixed $j$:

$$E_j(\alpha, \Omega) = -\alpha^2 - \kappa_{\max}(\Omega)\,\alpha + O(\alpha^{2/3}), \quad \alpha \to +\infty,$$

where $\kappa_{\max}(\Omega)$ denotes the maximum of the curvature of the boundary. A complete asymptotic expansion is given in [8] under additional assumptions on the domain. For the $n$-dimensional spherical shell $\mathcal{S}_{r_1,r_2}$ of outer radius $r_2 > 0$ and inner radius $r_1 \in [0, r_2)$, with $r_1 = 0$ corresponding to a ball, it is proved in [6] that

$$E_j(\alpha, \mathcal{S}_{r_1,r_2}) = -\alpha^2 - \frac{n-1}{r_2}\,\alpha + o(\alpha), \quad \alpha \to +\infty. \tag{1}$$

The main results presented in this talk is [14, Theorem 1]: Assume that $\Omega \subset \mathbb{R}^n$ is $\mathcal{C}^4$, then for any fixed $j \geq 1$ we have the asymptotics:

$$E_j(\alpha, \Omega) = -\alpha^2 - (n-1)H_{\max}(\Omega)\,\alpha + O(\alpha^{1/2}), \quad \alpha \to +\infty \tag{2}$$

where $H_{\max}(\Omega)$ is the maximum of the mean curvature at the boundary of $\Omega$.

A classical problematics is the question of the Faber-Krahn-type inequality: among domains $\Omega$ of fixed volume, which domain optimizes the eigenvalues of the Laplacian? When $\alpha < 0$, it is known that the balls are the minimizers of $E_1(\alpha, \Omega)$, see [2,3], and it was conjectured in [1] that for $\alpha > 0$, the balls are the maximizers of $E_1(\alpha, \Omega)$ (reverse Faber-Krahn inequality). This is proved for $\alpha > 0$ small enough in [6], moreover Freitas and Krejčiřík use the asymptotics (1) to show that the conjecture appears to be false for $\alpha$ large enough: indeed, given a ball $\mathcal{B}$, one can find a spherical shell $\mathcal{S}$ of same volume such that for $\alpha$ large enough: $E_j(\alpha, \mathcal{S}) > E_j(\alpha, \mathcal{B})$.

In order to maximize the eigenvalues $E_j(\alpha, \Omega)$, the asymptotics (2) brings the natural question:

$$\text{How to minimize } H_{\max}(\Omega), \text{ the volume of } \Omega \text{ being fixed?}$$

By using a generalized Minkowski formula [9] we prove in [14] the following result: Assume that $\Omega$ is a star-shaped domain with $\mathcal{C}^2$ boundary. Then

$$H_{\max}(\Omega) \geq \left(\frac{\mathrm{Vol}(\mathcal{B}_n)}{\mathrm{Vol}(\Omega)}\right)^{1/n}, \tag{3}$$

where $\mathcal{B}_n$ is the unit ball of $\mathbb{R}^n$, and the equality holds if and only if $\Omega$ is a ball.

By using (2), we deduce an asymptotic version of the reverse Faber-Krahn inequality: Assume that $\Omega$ is a regular star-shaped domain which is not a ball. Then for all $j \geq 1$, there exists $\alpha_0 \in \mathbb{R}$ such that for all $\alpha \geq \alpha_0$ there holds $E_j(\alpha, \Omega) < E_j(\alpha, \mathcal{B})$, where $\mathcal{B}$ is a ball with same volume as $\Omega$. We address the question of extending (3) to regular domains with connected boundaries.

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## Magnetic WKB Constructions

Nicolas Raymond
(joint work with Virginie Bonnaillie-Noël and Frédéric Hérau)

This talk, based on the paper [2], is devoted to the analysis of the self-adjoint operators on the space $\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$ of the following type

$$\mathfrak{L}_h = (hD_s + A_1(s,t))^2 + (D_t + A_2(s,t))^2, \tag{1}$$

where $A_1$ and $A_2$ are real analytic, $D = -i\nabla$, and $\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$ is equipped with the standard scalar product:

$$\langle \psi_1, \psi_2 \rangle_{\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)} = \int_{\mathbb{R}^m \times \mathbb{R}^n} \psi_1 \overline{\psi_2}\, \mathrm{d}s\,\mathrm{d}t.$$

The corresponding quadratic form is denoted by $\mathfrak{Q}_h$. We would like to describe the lowest eigenpairs (eigenvalues and eigenfunctions) of this operator in the limit $h \to 0$ under elementary confining assumptions. In the electric context such questions are investigated in [6–8] (see also the original reference [1]).

An important motivation to analyze partially semiclassical problems with magnetic fields comes in fact, after homogeneity considerations, from the fully semiclassical case (*i.e.* when the parameter $h$ lies in front of all derivatives). The study of the discrete spectrum of the magnetic Laplacian $(-i\hbar\nabla + \mathbf{A})^2$ has given rise to many contributions in the last twenty years, especially in the semiclassical limit. To have an overview on the subject one may refer to the book by Fournais and Helffer [3], the survey by Helffer and Kordyukov [4] and the lecture notes by Raymond [9].

Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the electro-magnetic Laplacian acting on $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\mathcal{M}_{x,\xi} = (D_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2.$$

Denoting by $\mu(x,\xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the $m$-dimensional pseudo-differential operator:

$$\mu(s, hD_s).$$

We work under the following assumptions. The first assumption states that the lowest eigenvalue of the operator symbol of $\mathfrak{L}_h$ admits a unique and non degenerate minimum and the second one concerns the simplicity of the spectrum of the effective harmonic oscillator.

**Assumption 1.**

- The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ is analytic of type (B) in the sense of Kato [5, Chapter VII].
- For all $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, the bottom of the spectrum of $\mathcal{M}_{x,\xi}$ is a simple eigenvalue denoted by $\mu(x,\xi)$ (in particular it is an analytic function) and associated with a $\mathrm{L}^2$-normalized eigenfunction $u_{x,\xi} \in \mathcal{S}(\mathbb{R}^n)$ which also analytically depends on $(x,\xi)$.

- The function $\mu$ admits a unique and non degenerate minimum $\mu_0$ at point denoted by $(x_0, \xi_0)$ and such that $\liminf_{|x|+|\xi|\to+\infty} \mu(x,\xi) > \mu_0$.
- The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ can be analytically extended in a complex neighborhood of $(x_0, \xi_0)$.

**Assumption 2.** Under Assumption 1, let us denote by $\mathsf{Hess}\, \mu(x_0, \xi_0)$ the Hessian matrix of $\mu$ at $(x_0, \xi_0)$. We assume that the spectrum of $\mathsf{Hess}\, \mu(x_0, \xi_0)(\sigma, D_\sigma)$ is simple.

Assumption 2 is automatically satisfied when $m = 1$. The last assumption is a spectral confinement.

**Assumption 3.** For $R \geq 0$, we let $\Omega_R = \mathbb{R}^{m+n} \setminus \overline{\mathcal{B}(0,R)}$. We denote by $\mathfrak{L}_h^{\mathsf{Dir},\Omega_R}$ the Dirichlet realization on $\Omega_R$ of $(D_t + A_2(s,t))^2 + (hD_s + A_1(s,t))^2$. We assume that there exist $R_0 \geq 0$, $h_0 > 0$ and $\mu_0^* > \mu_0$ such that for all $h \in (0, h_0)$, the first eigenvalue of $\mathfrak{L}_h^{\mathsf{Dir},\Omega_{R_0}}$ satisfies:

$$\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h) \geq \mu_0^*.$$

In particular, due to the monotonicity of the Dirichlet realization with respect to the domain, Assumption 3 implies that there exist $R_0 > 0$ and $h_0 > 0$ such that for all $R \geq R_0$ and $h \in (0, h_0)$:

$$\lambda_1^{\mathsf{Dir},\Omega_R}(h) \geq \lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h) \geq \mu_0^*.$$

**Theorem 1.** *Let us assume Assumptions 1, 2 and 3 and that $A_1$ and $A_2$ are polynomials. For all $n \geq 1$, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ the $n$-th eigenvalue of $\mathfrak{L}_h$ exists and satisfies*

$$\lambda_n(h) = \lambda_{n,0} + \lambda_{n,1} h + o(h),$$

*where $\lambda_{n,0} = \mu_0$ and $\lambda_{n,1}$ is the $n$-th eigenvalue of $\frac{1}{2}\mathsf{Hess}\, \mu(x_0, \xi_0)(\sigma, D_\sigma)$.*

We provide now WKB expansions of the lowest eigenpairs in a pure magnetic case. We reduce here our study to the case when $A_2 = 0$. We therefore focus now on operators of the form

$$\mathfrak{L}_h = D_t^2 + (hD_s + A_1(s,t))^2. \tag{2}$$

Let us state the most important result of this talk.

**Theorem 2.** *We assume that $A_2 = 0$ and that $A_1$ is real analytic. Under Assumptions 1, 2 and 3, there exist a function $\Phi = \Phi(s)$ defined in a neighborhood $\mathcal{V}$ of $x_0$ with $\Re\mathsf{Hess}\,\Phi(x_0) > 0$ and, for any $n \geq 1$, a sequence of real numbers $(\lambda_{n,j})_{j\geq 0}$ such that*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^j,$$

*in the sense of formal series, with $\lambda_{n,0} = \mu_0$. Besides there exists a formal series of smooth functions on $\mathcal{V} \times \mathbb{R}_t^n$*

$$a_n(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^j$$

*with* $a_{n,0} \neq 0$ *such that*

$$(\mathfrak{L}_h - \lambda_n(h)) \left(a_n(.;h)\mathrm{e}^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right)\mathrm{e}^{-\Phi/h}.$$

*Furthermore the functions* $t \mapsto a_{n,j}(s,t)$ *belong to the Schwartz class uniformly in* $s \in \mathcal{V}$. *In addition, if* $A_1$ *is a polynomial function, there exists* $c_0 > 0$ *such that for all* $h \in (0, h_0)$ *there holds*

$$\mathcal{B}\Big(\lambda_{n,0} + \lambda_{n,1}h, c_0 h\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h\right) = \{\lambda_n(h)\},$$

*and* $\lambda_n(h)$ *is a simple eigenvalue.*

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## Ground state energy of the magnetic Laplacian on corner domains

Monique Dauge, Nicolas Popoff
(joint work with Virginie Bonnaillie-Noël)

**1. Introduction.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $A = (A_1, \ldots, A_n)$ be a magnetic potential on $\overline{\Omega}$. The magnetic Laplacian in its "semi-classical" formulation takes the form

$$(-ih\nabla + A)^2 = \sum_j (-ih\partial_j + A_j)^2,$$

with the small parameter $h > 0$, and we denote by $H_h(A, \Omega)$ the operator associated with the corresponding magnetic Neumann boundary condition:

$$(-ih\nabla + A)\psi \cdot n = 0 \text{ on } \partial\Omega,$$

where $n$ denotes the unit normal to the boundary. This operator is self-adjoint, and if $\Omega$ is simply connected, its spectrum depends only on the magnetic field

$B = \operatorname{curl} A$. Moreover $H_h(A,\Omega)$ has compact resolvent and we denote by $\lambda_h(B,\Omega)$ its first eigenvalue. For simplicity we denote by $H(A,\Omega) = H_1(A,\Omega)$ the operator without parameter ($h = 1$).

The study of the spectrum of $H_h(A,\Omega)$ provides equivalent information on the spectrum of $H(\sigma A,\Omega)$ in the limit $\sigma \to +\infty$. This Laplacian with large magnetic field appears as a linearization of the Ginzburg-Landau equation near the third critical field, see [9, 10, 16].

There exists many works around the asymptotics of $\lambda_h(B,\Omega)$ in the semi-classical limit $h \to 0$. In various situations, it has been proved that for $h$ small enough:

$$-Ch^{\kappa_-} \le \lambda_h(B,\Omega) - \mathcal{E}(B,\Omega)h \le Ch^{\kappa_+}, \tag{1}$$

where $\mathcal{E}(B,\Omega) \ge 0$ is a constant (positive if $B$ does not vanish) that is given by the infimum of some local problems on $\Omega$ and on its boundary. Here $\kappa^{\pm} > 1$ are constants depending on the geometry of $\Omega$ and on the variations of the magnetic field. Without being exhaustive, let us cite the works [1, 11, 13, 17, 23] and the book [10] for regular 2d and 3d domains. For polygonal domains in dimension 2, we refer to *e.g.* [2–4, 14, 19]). Much less is known for corner three-dimensional domains, see *e.g.* [19, 22].

This report presents results from our work [5] that provides, in the framework of a unified treatment of smooth and nonsmooth domains, a generalization of former results to domains containing edges, corners and conical points in dimension 3.

**2. Classes of domains.** A cone is an open subset of $\mathbb{R}^n$ invariant by dilation. We define classes of cones and corner domains by recurrence over the dimension as in [8], see also [18]:

**Definition 1.** Let $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$. The class of tangent cones $\mathfrak{P}_n$ and corner domains $\mathfrak{D}(M)$ are defined recursively as follow:

- $\mathfrak{P}_0 = \{0\}$ and $\mathfrak{D}(\mathbb{S}^0)$ is formed by the subsets of $\{-1, 1\}$.
- $\Pi \in \mathfrak{P}_n$ if and only if the section $\Pi \cap \mathbb{S}^{n-1}$ belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$.
- $\Omega \in \mathfrak{D}(M)$ if and only if for all $x \in \overline{\Omega}$, there exists a tangent cone $\Pi_x$ to $\Omega$ at $x$ and $\Pi_x \in \mathfrak{P}_n$.

Here are some examples in lower dimensions:

**Example 2.** In dimension $n = 1$:

- The elements of $\mathfrak{P}_1$ are $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$.
- The elements of $\mathfrak{D}(\mathbb{S}^1)$ are $\mathbb{S}^1$ and all open intervals $I \subset \mathbb{S}^1$ with $\overline{I} \ne \mathbb{S}^1$.

In dimension $n = 2$:

- The elements of $\mathfrak{P}_2$ are $\mathbb{R}^2$ and all sectors with opening $\alpha \in (0, 2\pi)$, including half-spaces ($\alpha = \pi$).
- The elements of $\mathfrak{D}(\mathbb{R}^2)$ are curvilinear polygons with piecewise non-tangent smooth sides. Note that corner angles do not take the values 0 or $2\pi$, and that $\mathfrak{D}(\mathbb{R}^2)$ includes smooth domains.

- The elements of $\mathfrak{D}(\mathbb{S}^2)$ are $\mathbb{S}^2$ and all curvilinear polygons with piecewise non-tangent smooth sides in the sphere $\mathbb{S}^2$.

In dimension $n = 3$:

- The elements of $\mathfrak{P}_3$ are all cones with section in $\mathfrak{D}(\mathbb{S}^2)$. This includes $\mathbb{R}^3$, half-spaces, dihedra and many different cones like octants or axisymmetric cones.
- The elements of $\mathfrak{D}(\mathbb{R}^3)$ are tangent in each point $x$ to a cone $\Pi_x \in \mathfrak{P}_3$. Note that the nature of the section of the tangent cone determines whether the 3d domain has a vertex, an edge, or is regular near $x$.

A cone is said to be polyhedral if its boundary is contained in a finite union of hyperplanes, and a domain is polyhedral if all its tangent cone are polyhedral. As examples, all domains in $\mathfrak{D}(\mathbb{R}^2)$ are polyhedral, but circular cones in $\mathbb{R}^3$ are not polyhedral. Notice that the main fact about a non polyhedral cone is that one of its principal curvatures is unbounded near its vertex. A point $x \in \overline{\Omega}$ with non polyhedral tangent cone $\Pi_x$ is called a conical point of $\Omega$.

**3. Local energies, main result.** To each $x \in \overline{\Omega}$ we associate a tangent magnetic Laplacian $H(A_x, \Pi_x)$, where

$$A_x : \tilde{x} \mapsto (\nabla A)(x) \cdot \tilde{x}, \quad \tilde{x} \in \Pi_x$$

is the linearization of $A$ at $x$. Notice that $\operatorname{curl} A_x = B(x)$ is the magnetic field frozen at $x$.

**Definition 3.** We define the local ground state energy at $x$:

$$E(B_x, \Pi_x), \quad \text{the infimum of the spectrum of} \quad H(A_x, \Pi_x).$$

We now define

$$\mathcal{E}(B, \Omega) = \inf_{x \in \overline{\Omega}} E(B_x, \Pi_x),$$

the infimum of the local energies.

This definition of $\mathcal{E}(B, \Omega)$ coincides with the constant appearing in (1) for particular cases. We show that this quantity provides the semiclassical limit of the first eigenvalue $\lambda_h(B, \Omega)$ in any dimension:

$$\forall \Omega \in \mathfrak{D}(\mathbb{R}^n), \quad \lim_{h \to 0} \frac{\lambda_h(B, \Omega)}{h} = \mathcal{E}(B, \Omega).$$

Our main result consists in estimates with remainders for domains in dimensions $n \leq 3$:

**Theorem 4.** *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $A$ be a regular potential. Then*

$$-C^- h^{11/10} \leq \lambda_h(B, \Omega) - h\mathcal{E}(B, \Omega) \leq C^+ h^{9/8},$$

*where the positive constants $C^-$ and $C^+$ depend only on the domain $\Omega$ and on norms of the magnetic potential $A$, namely:*

$$C^- = C(\Omega)(1 + \|A\|^2_{W^{2,\infty}(\Omega)}) \quad \text{and} \quad C^+ = C(\Omega)(1 + \|A\|^2_{W^{3,\infty}(\Omega)}).$$

*Assume moreover that $\Omega$ is polyhedral. Then we have the following improvements:*

$$-C^{-}h^{5/4} \leq \lambda_h(B,\Omega) - h\mathcal{E}(B,\Omega) \leq C^{+}h^{4/3}.$$

**4. Principles of proof for the lower bounds.** The proof of lower bounds combines standard IMS type formulas and estimates of remainders with original multi-scale covering in order to overcome the unboundedness of curvature near conical points: We make a first covering of $\overline{\Omega}$ with balls of size $h^{\delta_0}$ with $\delta_0 > 0$; Then, in annular regions at distance between $h^{\delta_0}$ and $\mathcal{O}(1)$ of conical points, we replace the first covering by a finer one with balls of size $h^{\delta_0+\delta_1}$. We deduce the following lower bound:

$$(2) \quad \lambda_h(B,\Omega) \geq h\mathcal{E}(B,\Omega) - Ch^2 - C\left(h^{2\delta_0+\frac{1}{2}} + h^{1+\delta_0} + h^{2-2\delta_0}\right) - C\left(h^{\frac{1}{2}+\delta_0+2\delta_1} + h^{1+\delta_1} + h^{2-2(\delta_0+\delta_1)}\right),$$

with $C = C(\Omega)(1+\|A\|^2_{W^{2,\infty}(\Omega)})$, where the error terms come from the linearization of the magnetic potential, of the metrics, and from cut-off errors when using the IMS formula. We optimize these remainder by choosing $\delta_0 = \frac{3}{10}$ and $\delta_1 = \frac{3}{20}$ and the lower bound of the theorem follows.

If there is no conical points, then we use a standard one-scale partition with balls of size $\delta$ and we have a lower bound similar to (2), but involving only the terms with $\delta_0$ alone. By choosing $\delta = \frac{3}{8}$, we deduce the lower bound of the theorem when there is no conical points.

**5. Principles of proof for the upper bounds.** The proof of the upper bound requires a better understanding of tangent operators and of the local ground energy. We review and complete the description of model problems on cones $\Pi$ of $\mathfrak{P}_3$. The case when $\Pi = \mathbb{R}^3$ is well known (Landau modes). We rely on the analysis from [6,12,17] for half-spaces. Particular cases of sectors and wedges are analyzed in [2,14,19,20], and the general case is treated in [21]. For 3d cones without invariance direction, some particular cases are studied in [7,19]. In the general case, we link the bottom of the essential spectrum of $H(A,\Pi)$ with ground state energies of magnetic Laplacians defined on tangent substructures of $\Pi$, see [5, Theorem 6.6].

We then analyze the regularity properties of the ground state energy $x \mapsto E(B_x,\Pi_x)$ with the use of singular chains. A singular chain is a sequence of points $(x_0,x_1,\ldots,x_{\nu-1})$, $\nu \geq 1$, recursively defined, $x_0$ lying in the closure of the corner domain, and the next points belonging to the sections of it tangent cones (see [5, Section 3.4] for the precise definition of singular chains). We define a distance and a partial order on singular chains and we prove a general result: If a real function defined on singular chains is continuous and order preserving, then it is lower semi-continuous. By using the continuity properties of the ground state energy of tangent magnetic Laplacian together with the exhaustive analysis of tangent operators, we prove:

**Theorem 5.** *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $B$ be a continuous magnetic field on $\overline{\Omega}$. Then $x \mapsto E(B_x,\Pi_x)$ is lower semi-continuous on $\overline{\Omega}$.*

In particular there exists $x_0 \in \overline{\Omega}$ such that $E(B_{x_0}, \Pi_{x_0}) = \mathcal{E}(B, \Omega)$. Moreover we combine this result with the diamagnetic inequality to deduce that if $B$ does not vanish on $\overline{\Omega}$, then $\mathcal{E}(B, \Omega) > 0$.

The proof of the upper bounds of Theorem 4 is based on a construction of suitable quasi-modes adapted to the geometry of $\Omega$ near $x_0$. In the standard case, the quasi-modes are constructed from a bounded generalized eigenfunction defined on $\Pi_{x_0}$, then one applies a cut-off and scalings in order to concentrate the support of the quasimodes around $x_0$. We qualify this quasi-mode as *sitting*. When we know the existence of a generalized eigenfunction only on a tangent substructure of $\Pi_{x_0}$ (this corresponds to singular chains of length $\nu \geq 2$), our quasimodes are decentered in some directions provided by the singular chain, have a multiple-scale structure in general, and we qualify it as *sliding*. In dimension $n = 3$, considering chains of length $\nu \leq 3$ is sufficient to conclude. The size of the support of the quasi-modes and the different shifts used depends on the geometry of $\Omega$ near $x_0$. We get remainders similar to those of (2), depending on wether $x_0$ is a conical point or not. We then use particular properties of the initial generalized eigenfunction to improve these remainders. We will adopt different strategies depending on the number $k$ of directions in which it has exponential decay: A Feynman-Hellmann formula if $k = 1$, a refined Taylor expansion of the potential if $k = 2$, and an Agmon decay estimate if $k = 3$.

**6. Conclusion.** Our analysis could be generalized in any dimension $n \geq 4$, provided one has a better knowledge of the model problems, leading to the proof of existence for suitable generalized eigenfunctions on substructures. Let us also mention that the Robin Laplacian with large Dirichlet parameter has similar properties than those of the magnetic Laplacian in the semi-classical limit, see [15].

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## Eigenvalues for the Robin Laplacian in domains with variable curvature: a semi-classical approach

Bernard Helffer

(joint work with Ayman Kachmar)

Let $\Omega \subset \mathbb{R}^2$ be an open domain with a smooth $C^\infty$ and compact boundary $\Gamma = \partial\Omega$. We study the low-lying eigenvalues of the Robin Laplacian in $L^2(\Omega)$ which is defined as

$$\mathcal{P}^\alpha = -\Delta \quad \text{in } L^2(\Omega), \tag{1}$$

with domain, $D(\mathcal{P}^\alpha) = \{u \in H^2(\Omega) \ : \ \nu \cdot \nabla u - \alpha\, u = 0 \quad \text{on } \partial\Omega\}$, where $\alpha > 0$ is a given parameter and $\nu$ is the unit outward normal vector of $\Gamma$.

The operator $\mathcal{P}^\alpha$ is associated with the closed semi-bounded quadratic form, defined on $H^1(\Omega)$ by

$$u \mapsto \mathcal{Q}^\alpha(u) := ||\nabla u||^2_{L^2(\Omega)} - \alpha \int_{\partial\Omega} |u(x)|^2 ds(x)\,. \tag{2}$$

If we denote by $(\lambda_n(\alpha))$ the sequence of min-max eigenvalues of $\mathcal{P}^\alpha$, it is proved In [3, 17, 19] that, for every fixed $n \in \mathbb{N}^*$,

$$\lambda_n(\alpha) = -\alpha^2 - \kappa_{\max}\alpha + o(\alpha) \quad \text{as } \alpha \to +\infty\,, \tag{3}$$

where $\kappa_{\max}$ is the maximal curvature along the boundary. Note that the first term in (3) was obtained previously (see [2, 15] and references therein).

If the domain $\Omega$ is an exterior domain, then the operator $\mathcal{P}^\alpha$ has as essential spectrum $[0, \infty)$. In this case, the asymptotics in (3) show that, for every fixed $n$, if $\alpha$ is selected sufficiently large, the eigenvalue $\lambda_n(\alpha)$ is in the discrete spectrum of the operator $\mathcal{P}^\alpha$. When the domain $\Omega$ is an interior domain, the operator $\mathcal{P}^\alpha$ is with compact resolvent and its spectrum is purely discrete.

Our aim is to improve the asymptotic expansion in (3) and to give the leading term of the spectral gap $\lambda_{n+1}(\alpha) - \lambda_n(\alpha)$. We suppose that the boundary $\partial\Omega$ is parameterized by arc-length $s$ and $\kappa$ is the curvature of $\partial\Omega$. We suppose (Assumption (A)) that the curvature $\kappa$ attains its maximum $\kappa_{\max}$ at a unique point $s = 0$ and that the maximum is non-degenerate, i.e. $k_2 := -\kappa''(0) > 0$. The main result of [8] is:

**Theorem 1.** *Under Assumption (A), for any positive $n$, there exists a sequence $(\mu_{j,n})_{j\in\mathbb{N}}$, such that the eigenvalue $\lambda_n(\alpha)$ has, as $\alpha \to +\infty$, the asymptotic expansion*

$$\lambda_n(\alpha) \sim -\alpha^2 - \alpha\kappa_{\max} + (2n-1)\sqrt{\frac{k_2}{2}}\,\alpha^{1/2} + \sum_{j=0}^{+\infty} \mu_{j,n}\alpha^{-\frac{j}{2}}\,.$$

We present two semi-classical proofs which are either related to the so called harmonic approximation or to the WKB approximation. Compared with what was developed for Schrödinger operators (see Helffer-Sjöstrand [11] and B. Simon [20]), the semi-classical parameter is $h = \alpha^{-2}$, the boundary acts as a potential well and the curvature creates a miniwell [13]. In this last case, the toy model is, in the limit $h \to 0$, the operator (say in $\mathbb{R}^2$ and for some $\kappa > 0$) $-h^2\Delta + x^2(1 + \kappa y^2)$, the well being the line $x = 0$ and the mini well $(0, 0)$. The heuristic idea for this model is to first reduce the problem, inside the well $x = 0$, to the semi-classical analysis of the operator $h^2 D_y^2 + h\sqrt{1 + \kappa y^2}$. The assumption $(A)$ indicates the case of a unique miniwell. In the case considered here, one can roughly say that the bottom of the spectrum of $\mathcal{P}_\alpha$ is obtained by considering at the boundary (parametrized by arc-length coordinate) the operator $H^{bbd} := -\alpha^2 + D_s^2 - \alpha\kappa(s)$, and using for a further analysis the quadratic approximation of $-\kappa(s)$ at the minimum.
As in [11], a natural and interesting question is to discuss the case of multiple maxima. When the curvature attains it maximum on $\Gamma$ at $k$ points $s_i$ many effects can appear depending on the values of the $\kappa''(s_i)$ (as in the case of the Schrödinger operator). In the case of symmetries (see [11, 12] in the case of Schrödinger), the determination of the tunneling effect between the points of maximal curvature is expected to play an important role. A first example is discussed in [10] when the domain $\Omega$ has two congruent corners (at a corner, we can assign the value $\infty$ to $\kappa_{\max}$). In the regular case (typically an ellipse), an interesting step is the

construction of WKB solutions in the spirit of what was done in the context of the Born-Oppenheimer approximation (1927) (see the unpublished analysis of S. Lefebvre (1986) for the model $h^2 D_x^2 + D_y^2 + (1+x^2)y^2$, the general analysis by A. Martinez [16] and a recent work by V. Bonnaillie, F. Hérau and N. Raymond [1]). As in the case of the $(2D)$ model in superconductivity $[1,5,9]$, the idea is that the "one well" eigenfunction is well approximated by the WKB approximation in large domains of the boundary. In the case of an ellipse, we expect (in reference to the model $H^{bd}$) a tunneling in the form

$$\lambda_2 - \lambda_1 \sim \alpha^{-\nu}\big(a_0 + o(1)\big) \exp -\alpha^{\frac{1}{2}} S_0 \,,$$

where $S_0$ is the tangential Agmon distance between the two points of maximal curvature on the boundary associated with the metric $\sqrt{\kappa_{max} - \kappa(s)}\, ds^2$. It could appear strange that the approximation by $H^{bd}$ which determines the first eigenvalues modulo $o(1)$ could predict the size of an exponentially small tunneling effect, but this is what is observed in the case of Born-Oppenheimer (see [16]).

As explained by Giorgi-Smits in [7] the problem is connected with the research of the properties of the minimizers of the $(2D)$-enhanced Ginzburg-Landau functional:

$$G(\psi, \mathbf{A}) := \frac{\hbar^2}{2mb} \int_{\partial\Omega} |\psi|^2 d\sigma_x$$
$$+ \int_\Omega \left[ \frac{\beta_0}{2}|\psi|^4 + \alpha_0|\psi|^2 + \frac{1}{2m}|(\frac{h}{i}\nabla - \frac{2e}{c}\mathbf{A})\psi|^2 + \frac{1}{8\pi}\mu_0|\frac{1}{\mu_0}(\mathrm{curl}\mathbf{A}) - \sigma|^2 \right] dx \,,$$

where $b < 0$ is the parameter modeling the enhanced surface superconductivity, $\beta_0 > 0$, $\alpha_0 < 0$, $\mu_0 > 0$, and $\sigma$ denotes the intensity of the exterior magnetic field ($\sigma = 0$ in the above analysis). Instead of looking at critical fields like in surface superconductivity (see [4] and references therein), one considers instead critical temperatures. When $\sigma \neq 0$, various asymptotic regimes have been analyzed by A. Kachmar (see [14] and further papers). The Robin condition is called in this context the De Gennes condition.

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## Estimates for the first eigenvalue of the magnetic Dirichlet Laplacian

Fabian Portmann
(joint work with Tomas Ekholm and Hynek Kovařík)

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $A \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ a magnetic vector potential with corresponding magnetic field $B := \operatorname{curl} A$. The magnetic Dirichlet Laplacian

$$H^D_{\Omega,B} := (-i\nabla + A)^2$$

is defined through its quadratic form

$$h^D_{\Omega,A}[u] := \int_\Omega |(-i\nabla + A)u(x)|^2 \, dx$$

with form domain $H^1_0(\Omega)$. The main objective of this talk is to provide bounds on the first eigenvalue of $H^D_{\Omega,B}$,

$$\lambda_1(\Omega, B) := \inf \operatorname{spec} H^D_{\Omega,B} = \inf_{u \in H^1_0(\Omega)} \frac{h^D_{\Omega,A}[u]}{\|u\|^2}.$$

There exist two well-known lower bounds for $\lambda_1(\Omega, B)$; the commutator estimate

$$h^D_{\Omega,A}[u] \geq \pm \int_\Omega B(x)|u(x)|^2\, dx \tag{1}$$

and the diamagnetic inequality

$$\lambda_1(\Omega, B) \geq \lambda_1(\Omega, 0). \tag{2}$$

A natural question which arises is whether a combination of (1) and (2) is possible.

As a first result we have the following estimate. Any $u \in H^1_0(\Omega)$ satisfies

$$h^D_{\Omega,A}[u] \geq \pm \int_\Omega B(x)\, |u(x)|^2\, dx + e^{-2S(\Omega,B)}\, \lambda_1(\Omega, 0) \int_\Omega |u(x)|^2\, dx,$$

where

$$S(\Omega, B) := \inf_{\Psi \in \mathcal{F}(\Omega,B)} \operatorname{osc} \Psi,$$

and $\mathcal{F}(\Omega, B)$ denotes the class of super potentials associated to $B$,

$$\mathcal{F}(\Omega, B) := \left\{ \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \,:\, \Delta\Psi = B \text{ in } \Omega \right\}.$$

For a given magnetic field $B$, the choice of optimal $\Psi$ depends very much on the geometry of the domain, and we therefore turn to the special case of constant magnetic field. Let $B(x) = B_0 > 0$ and define

$$\ell(\Omega, 0) := \sup_{x\in\Omega} x_2 - \inf_{x\in\Omega} x_2,$$

the maximal diameter in the $x_2$-direction for a certain orientation of $\Omega$. We then pick a rotation $R = R(\theta) \in SO(2)$, parametrized by an angle $\theta \in [0, 2\pi)$, and set

$$\ell(\Omega, \theta) := \sup_{x\in R(\theta)\Omega} x_2 - \inf_{x\in R(\theta)\Omega} x_2,$$

the maximal $x_2$-distance of the rotated set $R(\theta)\Omega$. The quantity $\ell(\Omega)$ is then defined as follows: $\ell(\Omega) := \inf_{\theta\in[0,2\pi)} \ell(\Omega, \theta)$. From the boundedness of $\Omega$ it follows that $\ell(\Omega)$ is finite. We are then able to show that

$$\lambda_1(\Omega, B_0) \geq B_0 + e^{-\frac{B_0}{4}\,\ell(\Omega)^2}\, \lambda_1(\Omega, 0).$$

If we furthermore assume that the domain $\Omega$ is convex, we can estimate the quantity $\ell(\Omega)$ in terms of the in-radius of the domain: $2R_{\text{in}} \leq \ell(\Omega) \leq 3R_{\text{in}}$.

**The Hardy inequality and the heat equation with magnetic field**

David Krejčiřík

In this extended abstract of works [5] and [1], we are concerned with spectral-threshold properties of the magnetic Schrödinger operator

$$H_B = \big( -i\nabla_x - A(x)\big)^2 - \frac{c_d}{|x|^2} \qquad \text{in} \qquad L^2(\mathbb{R}^d) \tag{1}$$

and the large-time behaviour of the associated heat semigroup

$$\mathrm{e}^{-tH_B}\,. \tag{2}$$

The operator $H_B$ is introduced as the Friedrichs extension of (1) initially defined in $C_0^\infty(\mathbb{R}^d)$. The relationship between the *magnetic potential* (1-form) $A$ and the associated *magnetic tensor* (2-form) $B$ is standard, through the exterior derivative $B = \mathrm{d}A$. The latter is compatible because of the Maxwell equation $\mathrm{d}B = 0$. The dimensional quantity

$$c_d := \left(\frac{d-2}{2}\right)^2 \tag{3}$$

is the best constant in the *classical Hardy inequality*

$$-\Delta_x \geq \frac{c_d}{|x|^2}\,, \tag{4}$$

valid in the sense of quadratic forms, where $-\Delta_x$ should be interpreted as the self-adjoint realisation of the Laplacian in $L^2(\mathbb{R}^d)$. We always assume $d \geq 2$ (the one-dimensional situation is excluded because there is no magnetic field in $\mathbb{R}$).

An important characterisation of the spectral-threshold behaviour of $H_B$ is given by the existence/non-existence of Hardy-type inequalities. In the absence of magnetic field, $H_0$ is critical in the sense that $c_d$ is optimal in (4) and no other non-trivial reminder term can be added on the right hand side of (4). On the other hand, the following *magnetic Hardy inequality* holds whenever $B$ is non-trivial.

**Theorem 1.** *Let $d \geq 2$. Suppose that $B$ is smooth and closed. If $B \neq 0$, then there exists a positive constant $c_{d,B}$ such that for any smooth $A$ satisfying $\mathrm{d}A = B$, the following inequality holds*

$$H_B \geq \frac{c_{d,B}}{1 + |x|^2 \log^2(|x|)}\,. \tag{5}$$

This inequality was first proved by Laptev and Weidl in [7] in $d = 2$ under a flux condition and with a better weight (without the logarithm) on the right hand side of (5). A general version of (5), but with the right hand side being replaced by a compactly supported function in $\mathbb{R}^d$, was given by Weidl in [9]. As the most recent development, Ekholm and Portmann in [2] established (5) in $d = 3$ under an extra assumption on $B$. Since the present version of the magnetic Hardy inequality (in any dimension, with the minimal assumption $B \neq 0$ and with an everywhere positive Hardy weight) does not seem to exist in the literature, we give a proof of Theorem 1 in [1] before proving the main result of the paper.

It is well known that the large-time behaviour of a heat semigroup is determined by spectral-threshold properties of its generator. By the spectral theorem, we have $\|e^{-tH_B}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = e^{-t\lambda_1}$, where $\lambda_1 := \inf\sigma(H_B)$. At the same time, the *diamagnetic inequality*

$$(6)\qquad \left|(\nabla - iA)\psi(x)\right| \geq \left|\nabla|\psi|(x)\right|,$$

valid pointwise for almost every $x \in \mathbb{R}^d$ and any $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, implies $\inf\sigma(H_B) \geq \inf\sigma(H_0) = 0$, so the magnetic field can only improve the large-time behaviour of (2). This is notably evident for non-trivial homogeneous fields, *i.e.* $B(x) = B_0 \neq 0$ for all $x \in \mathbb{R}^d$, when $\lambda_1 > 0$ and the heat semigroup $e^{-tH_{B_0}}$ thus exhibits an exponential decay rate.

In this talk, we are interested in a more delicate situation when $B$ is *local* in the sense that it decays sufficiently fast at infinity, so that

$$(7)\qquad \sigma(H_B) = [0,\infty).$$

Then no extra decay of the heat semigroup is seen at the level above. Although the spectrum as a set is insensitive to this class of magnetic fields, it follows from Theorem 1 that there is a fine difference reflected in the presence of the magnetic Hardy inequality. To exploit this subtle repulsive property of the magnetic field, we introduce a weighted space

$$(8)\qquad L^2_w(\mathbb{R}^d) := L^2(\mathbb{R}^d, w(x)\,\mathrm{d}x), \qquad \text{where} \qquad w(x) := e^{|x|^2/4},$$

and reconsider (2) as an operator from $L^2_w(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. As a measure of the additional decay of the heat semigroup, we then consider the polynomial *decay rate*

$$(9)\qquad \gamma_B := \sup\left\{\gamma \;\middle|\; \exists C_\gamma > 0, \forall t \geq 0,\; \|e^{-tH_B}\|_{L^2_w(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C_\gamma\,(1+t)^{-\gamma}\right\}.$$

It is not difficult to see that $\gamma_0 = 1/2$ for any $d \geq 2$. The primary objective of our work is to study the influence of a local but non-trivial magnetic field $B$ on $\gamma_B$. Our main result reads as follows.

**Theorem 2.** *Let $d \geq 2$. Suppose that $B$ is smooth, closed and compactly supported. Then*

$$(10)\qquad \gamma_B = \begin{cases} \dfrac{1+\beta}{2} & \text{if} \quad d = 2, \\ \dfrac{1}{2} & \text{if} \quad d \geq 3, \end{cases}$$

*where*

$$(11)\qquad \beta := \operatorname{dist}(\Phi_B, \mathbb{Z}), \qquad \Phi_B := \frac{1}{2\pi}\int_{\mathbb{R}^2} {}^*B(x)\,\mathrm{d}x,$$

*with ${}^*B$ denoting the Hodge dual of $B$.*

To prove Theorem 2, we adapt the method of self-similar variables, which was developed for the field-free heat equation by Escobedo and Kavian in [3]. The technique was applied to the magnetic setting by the author in $d = 2$ in [5]. The most recent work [1] can be considered as an extension of [5] to any dimension, but the presence of the inverse-square potential in (1) also invokes [8].

It follows from our method of proof that the dichotomy in (10) is a consequence of topological properties of the sphere $S^{d-1}$. In $d = 2$, the heat semigroup (2) behaves for large times as if the magnetic field degenerated to a singular (Aharonov-Bohm) magnetic field with the same total flux.

**Open Problem 1 (better topology).** The Gaussian weight in (8) is required by our method of proof. However, it is reasonable to expect that Theorem 2 holds with less restrictive weights in (9). It seems natural that the optimal weight should be related to the optimal weight in the Hardy inequality (5).

**Open Problem 2 (transience between local and global fields).** We expect the same decay rates (10) if the assumption about the compact support of $B$ is replaced by a fast decay at infinity only. However, it is quite possible that a slow decay of the field at infinity will improve the decay of the solutions even further. In particular, is there a super-polynomial decay rate for the heat semigroup if $B$ decays to zero very slowly at infinity?

**Open Problem 3 (beyond the polynomial decay rate).** In dimensions $d \geq 3$ or if $\beta = 0$ in $d = 2$, the transient effect of the magnetic field is not observable in the present setting through the polynomial decay rate $\gamma_B$. Anyway, because of the presence of magnetic Hardy inequalities (*cf.* Theorem 1), we expect that there is always an improvement in the decay of the heat semigroup (2) whenever $B \neq 0$.

**Open Problem 4 (abstract conjecture).** More generally, let us recall that we expect that there is *always* an improvement of the decay for the heat semigroup of an operator satisfying a Hardy-type inequality (*cf.* [6, Conjecture in Sec. 6] and [4, Conjecture 1]).

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## Strong confinement limit for the nonlinear Schrödinger equation constrained on a curve

Nicolas Raymond
(joint work with Florian Méhats)

The Dirichlet realization of the Laplacian on tubes of the Euclidean space plays an important role in the physical description of nanostructures. In the last twenty years, many papers were concerned by the influence of the geometry of the tube (curvature, torsion) on the spectrum. For instance, in [10], Duclos and Exner proved that bending a waveguide in dimension two and three always induces the existence of discrete spectrum below the essential spectrum (see also [8]). Another question of interest in their paper is the limit when the cross section shrinks to a point. In particular they prove that, in some sense, the Dirichlet Laplacian on a bidimensional tube, with cross section $(-\varepsilon, \varepsilon)$ is well approximated by Schrödinger operator

$$-\partial_{x_1}^2 - \frac{\kappa^2(x_1)}{4} - \frac{1}{\varepsilon^2}\partial_{x_2}^2,$$

acting on $\mathrm{L}^2(\mathbb{R} \times (-1,1), \mathrm{d}x_1\, \mathrm{d}x_2)$ and where $\kappa$ denotes the curvature of the center line of the tube. Such approximations have been recently considered in [13] or in presence of magnetic fields [12] through a convergence of resolvent method. Concerning this kind of results, one may refer to the memoir by Wachsmuth and Teufel [15] where dynamical problems are analyzed in the spirit of adiabatic reductions.

In this talk, based on the paper [14], we consider the time dependent Schrödinger equation with a cubic non linearity in a waveguide and we would especially like to determine if the adiabatic reduction available in the linear framework can be used to reduce the dimension of the non linear equation and provide an effective dynamics in dimension one. The derivation of nonlinear quantum models in reduced dimensions has been the object of several works in the past years. For the modeling of the dynamics of electrons in nanostructures, the dimension reduction problem for the Schrödinger-Poisson system has been studied in [6, 9] for confinement on the plane, in [4] for confinement on a line, and in [11] for confinement on the sphere. For the modeling of strongly anisotropic Bose-Einstein condensates, the case of cubic nonlinear Schrödinger equations with an harmonic potential has been considered in [1–3, 5, 7].

With the same formalism, we will consider the case of unbounded curves and the case of closed curves. Consider a smooth, simple curve $\Gamma$ in $\mathbb{R}^2$ defined by its normal parametrization $\gamma : x_1 \mapsto \gamma(x_1)$. For $\varepsilon > 0$ we introduce the map

$$\Phi_\varepsilon : \mathcal{S} = \mathbb{M} \times (-1,1) \ni (x_1, x_2) \mapsto \gamma(x_1) + \varepsilon x_2 \nu(x_1) = \mathsf{x}, \tag{1}$$

where $\nu(x_1)$ denotes the unit normal vector at the point $\gamma(x_1)$ such that we have $\det(\gamma'(x_1), \nu(x_1)) = 1$ and where

$$\mathbb{M} = \begin{cases} \mathbb{R} & \text{for an unbounded curve,} \\ \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) & \text{for a closed curve.} \end{cases}$$

We recall that the curvature at the point $\gamma(x_1)$, denoted by $\kappa(x_1)$, is defined by

$$\gamma''(x_1) = \kappa(x_1)\nu(x_1).$$

The waveguide is $\Omega_\varepsilon = \Phi_\varepsilon(\mathcal{S})$ and we work under the assumption that waveguide does not overlap itself and that $\Phi_\varepsilon$ is a smooth diffeomorphism: We assume that there exists $\varepsilon_0 \in (0, \frac{1}{\|\kappa\|_{\mathrm{L}^\infty}})$ such that, for $\varepsilon \in (0, \varepsilon_0)$, $\Phi_\varepsilon$ is injective. We also assume that the function $\kappa$ is bounded, as well as its derivatives $\kappa'$ and $\kappa''$.

We denote by $-\Delta_{\Omega_\varepsilon}^{\mathsf{Dir}}$ the Dirichlet Laplacian on $\Omega_\varepsilon$. We are interested in the following equation:

$$i\partial_t\psi^\varepsilon = -\Delta_{\Omega_\varepsilon}^{\mathsf{Dir}}\psi^\varepsilon + \lambda\varepsilon|\psi^\varepsilon|^2\psi^\varepsilon \tag{2}$$

on $\Omega_\varepsilon$ with a Cauchy condition $\psi^\varepsilon(0;\cdot) = \psi_0^\varepsilon$ and where $\lambda \in \mathbb{R}$ are parameters.

In the sequel, it will be convenient to work in the coordinates $(x_1, x_2)$ and to consider the following change of temporal gauge $\phi^\varepsilon(t; x_1, x_2) = e^{-i\mu_1\varepsilon^{-2}t}\varphi^\varepsilon(t; x_1, x_2)$, where $\mu_1$ is the lowest eigenvalue (associated with the normalized eigenfunction $e_1$) of the Dirichlet realization of $D_{x_2}^2$ on $(-1, 1)$. This leads to the equation

$$i\partial_t\varphi^\varepsilon = \mathcal{H}_\varepsilon\varphi^\varepsilon + (V_\varepsilon - \varepsilon^{-2}\mu_1)\varphi^\varepsilon + \lambda m_\varepsilon^{-1}|\varphi^\varepsilon|^2\varphi^\varepsilon \tag{3}$$

with conditions $\varphi^\varepsilon(t; x_1, \pm 1) = 0$, $\varphi^\varepsilon(0;\cdot) = \phi_0^\varepsilon$ and where

$$\mathcal{H}_\varepsilon = \mathcal{P}_{\varepsilon,1}^2 + \mathcal{P}_{\varepsilon,2}^2 \qquad V_\varepsilon(x_1, x_2) = -\frac{\kappa(x_1)^2}{4(1 - \varepsilon x_2\kappa(x_1))^2}$$

with

$$\mathcal{P}_{\varepsilon,1} = m_\varepsilon^{-1/2} D_{x_1} m_\varepsilon^{-1/2}, \qquad \mathcal{P}_{\varepsilon,2} = \varepsilon^{-1} D_{x_2}, \qquad m_\varepsilon(x_1, x_2) = 1 - \varepsilon x_2\kappa(x_1).$$

Let us introduce the energy associated with (3):

$$\begin{aligned}\mathcal{E}_\varepsilon(\phi) = \frac{1}{2}\int_{\mathcal{S}} |\mathcal{P}_{\varepsilon,1}\phi|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{1}{2}\int_{\mathcal{S}} |\mathcal{P}_{\varepsilon,2}\phi|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \\ + \frac{1}{2}\int_{\mathcal{S}} \left(V_\varepsilon - \frac{\mu_1}{\varepsilon^2}\right)|\phi|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{\lambda}{4}\int_{\mathcal{S}} m_\varepsilon^{-1}|\phi|^4 \,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

Notice that we have substracted the conserved quantity $\frac{\mu_1}{2\varepsilon^2}\|\phi\|_{\mathrm{L}^2}^2$ to the usual nonlinear energy, in order to deal with bounded energies. Indeed, we will consider initial conditions with bounded mass and energy, which means more precisely the following assumption.

**Assumption 1.** There exists two constants $M_0 > 0$ and $M_1 > 0$ such that the initial data $\phi_0^\varepsilon$ satisfies, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\|\phi_0^\varepsilon\|_{\mathrm{L}^2} \leq M_0 \quad \text{and} \quad \mathcal{E}_\varepsilon(\phi_0^\varepsilon) \leq M_1.$$

We will see that (3) is well approximated in the limit $\varepsilon \to 0$ by the following one dimensional equation

$$i\partial_t\theta^\varepsilon = D_{x_1}^2\theta^\varepsilon - \frac{\kappa(x_1)^2}{4}\theta^\varepsilon + \lambda\gamma|\theta^\varepsilon|^2\theta^\varepsilon, \tag{4}$$

with $\gamma = \int_{-1}^{1} e_1(x_2)^4 \, dx_2 = 3/4$ and

$$\theta^\varepsilon(0,x_1) = \theta_0^\varepsilon(x_1) = \int_{-1}^{1} \phi_0^\varepsilon(x_1,x_2) e_1(x_2) \, dx_2 \text{ for } x_1 \in \mathbb{M}.$$

The main result of this talk is the following theorem.

**Theorem 1** ($\mathsf{H}^2$ solutions)**.** *Assume that* $\phi_0^\varepsilon \in \mathsf{H}^2 \cap \mathsf{H}_0^1(\mathcal{S})$ *and that there exist* $M_0 > 0$, $M_2 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$,

$$\|\phi_0^\varepsilon\|_{\mathrm{L}^2} \leq M_0, \qquad \Big\|(\mathcal{H}_\varepsilon - \frac{\mu_1}{\varepsilon^2})\phi_0^\varepsilon\Big\|_{\mathrm{L}^2} \leq M_2. \tag{5}$$

*Then* $\phi_0^\varepsilon$ *satisfies Assumption 1 and*

(i) *The limit problem* (4) *admits a unique solution*

$$\theta^\varepsilon \in C(\mathbb{R}_+; \mathsf{H}^2(\mathbb{M}) \cap C^1\big(\mathbb{R}_+; \mathrm{L}^2(\mathbb{M})\big).$$

(ii) *For all* $\varepsilon \in (0, \varepsilon_1(M_0))$, *the two-dimensional problem* (3) *admits a unique solution* $\varphi^\varepsilon \in C\big(\mathbb{R}_+; \mathsf{H}^2 \cap \mathsf{H}_0^1(\mathcal{S})\big) \cap C^1\big(\mathbb{R}_+; \mathrm{L}^2(\mathcal{S})\big)$.

(iii) *For all* $T > 0$ *there exists* $C_T > 0$ *such that, for all* $\varepsilon \in \big(0, \varepsilon_1(M_0)\big)$, *we have the refined error bound*

$$\sup_{t\in[0,T]} \big\|\varphi^\varepsilon(t) - \theta^\varepsilon(t) e_1\big\|_{\mathrm{L}^2(\mathcal{S})} \leq C_T \, \varepsilon.$$

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