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Mini-Workshop: Dynamical versus Diffraction Spectra in the Theory of Quasicrystals

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Abstract. The dynamical (or von Neumann) spectrum of a dynamical system and the diffraction spectrum of the corresponding measure dynamical system are intimately related. While their equivalence in the case of pure point spectra is well understood, this workshop aimed at an appropriate extension to systems with mixed spectra, building on recent developments for systems of finite local complexity and for certain random systems from the theory of point processes. Another focus was the question for connections between Schrödinger and dynamical spectra.

Mathematics Subject Classification (2010): 37B05, 52C23, 81Q10.

Introduction by the Organisers

The spectral theory of dynamical systems was initiated by Koopman [19] and von Neumann [24], and later developed in various directions. So far, a complete classification result in the realm of measure-theoretic dynamics only exists for dynamical systems with pure point spectrum. This approach is usually formulated via the Halmos–von Neumann theorem [17]. This certainly applies to model sets (also known as cut and project sets), where the corresponding Kronecker factor emerges constructively [8, 4].

The development of (mathematical) diffraction theory [18] admits an alternative approach to measure dynamical systems on locally compact Abelian groups. For systems with pure point spectrum, the equivalence of the two approaches has been established in a series of publications [22, 20, 7, 21]. More recent are first steps towards an extension to systems with mixed spectra [9], with particularly concrete results for systems with finite local complexity (FLC) [2, 5, 10] (compare also the abstracts by D. Lenz, M. Baake, U. Grimm and F. Gähler below).

The essential point here is the insight that, in general, the dynamical spectrum is richer than the diffraction spectrum of a system [23, 10], but not than the collection of diffraction spectra of the system and a suitable family of its factors [9]. In general, one has to expect that such a family is infinite, at least if one restricts to factors of the same complexity type. Astonishingly, in many of the classic examples, one can work with a *finite* family; see [4, 3] and references therein, as well as the abstract by D. Lenz below. What is presently lacking is a classification of those systems where such a finiteness condition applies. Even a useful sufficient criterion is unknown at present.

The primary aim of this mini-workshop was to bring together experts from both ends of the spectrum in order to reach a better understanding of the correct equivalence notion and to take first steps towards a spectral classification beyond the pure point case. To facilitate discussions, the mini-workshop started with four survey talks which set the scene on central topics such as almost periodic measures, dynamical and diffraction spectra, Schrödinger spectra and statistical mechanics approaches; see the abstracts by N. Strungaru, D. Lenz, D. Damanik and A.C.D. van Enter for details. A further three talks addressed various topological aspects of tilings and tiling spaces; compare the abstracts by J. Kellendonk, T. Fernique and A. Clark. The talks of the remaining ten participants discussed specific questions related to spectral properties, as detailed elsewhere in this introduction.

Particularly interesting questions in this context concern systems with singular continuous spectra, which have been studied in the context of Schrödinger operators for some time (see, e.g., [12] and references therein, and compare the abstracts by D. Damanik, A. Gorodetski, W. Yessen and J. Fillman), as well as systems with absolutely continuous spectra, as they appear in the theory of point processes [16, 1]. In the latter case, it quite often occurs that dynamical and diffraction spectra become equivalent again [6]. In fact, this and various heuristic arguments (compare the abstracts by A.C.D. van Enter and H. Kösters) point towards the conjecture that the diffraction spectrum is absolutely continuous relative to the maximal spectral measure of the dynamical spectrum.

A more recent extension of the theory emerged by the use of exact renormalisation techniques for the spectra (compare the abstracts by M. Baake and F. Gähler), which can help significantly to determine the spectral type, and by the study of weak model sets (see the abstracts by C. Richard and C. Huck), which fail to be minimal as dynamical systems. Moreover, they have entropy, but can still show pure point spectrum. This is, in a way, in contrast to highly ordered systems such as the Rudin–Shapiro chain and its generalisations via Hadamard matrices (compare the abstract by N. Frank), which show Lebesgue measures in their dynamical and diffraction spectra.

Another focus of the mini-workshop was to take first steps to clarify the connection between Schrödinger spectra and the dynamical and diffraction spectra discussed above. There is very little understanding of this question in general, aside from some heuristics based on the existing results on the two sides. Namely, based on the known results for some specific classes of examples, it is quite apparent that with increasing disorder of the system, the dynamical and diffraction spectra become more regular (i.e., more continuous), while the Schrödinger spectra become more singular. Finding a direct connection between the two sides, perhaps a suitable duality notion, explaining these tendencies would be highly desirable. Thus, a secondary aim of the proposed mini-workshop was to facilitate and stimulate discussions between those participants working on dynamical and diffraction spectra and those (also) working on Schrödinger operators.

One recent advance in this direction has been obtained in [13, 14], where (for the central model in the context of quasicrystals, the Fibonacci case) the Schrödinger density of states measure, which is the phase average of the spectral measures associated with the Fibonacci Schrödinger operator, was shown to result from the measure of maximal entropy of the associated trace map dynamical system by projection along the stable manifolds of points in the non-wandering set of the trace map. While this result does establish an explicit connection between spectral measures and dynamical measures, it is somewhat special to the Fibonacci case as it makes use of the presence of a hypberbolic basic set for the map implementing the self-similarity. Such a structure is not present in more general settings.

As a result of the discussions that took place during the mini-workshop, several research projects are now under way. One goal is to find a closer connection between those systems that are dynamically pure point and the corresponding Schrödinger spectra. For the most prominent examples, such as the Fibonacci or the period doubling case, it is known that the singular continuous nature of the Schrödinger spectral measure is uniform across the hull [11, 15]. In contrast, no such uniform result is known for cases whose dynamical spectrum is not pure point. It is therefore natural to explore whether there is a connection between these phenomena, and hence to look for further models with pure point dynamical spectrum and uniform singular continuous Schrödinger spectrum, as well as a better understanding of whether models with uniform singular continuous Schrödinger spectrum that do not have pure point dynamical spectrum can or cannot exist.

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Mini-Workshop: Dynamical versus Diffraction Spectra in the Theory of Quasicrystals

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Abstracts

Almost periodic measures and Meyer sets

Nicolae Strungaru
(joint work with Robert V. Moody)

The space $\mathcal{W}AP(G)$ of weakly almost periodic measures contains both the cone of positive definite translation bounded measures [1] and the subspace of translation bounded Fourier transformable measures [2].

Every weakly almost periodic measure μ can be written uniquely as the sum

$$\mu = \mu_{\mathsf{s}} + \mu_0 ,$$

of a strong almost periodic measure μ_{s} and a null weakly almost periodic measure μ_0. We will refer to this decomposition as the Eberlein decomposition.

The Eberlein decomposition is the Fourier dual of the Lebesgue decomposition of a measure into a discrete and continuous measure [1, 2]. More exactly, given a translation bounded Fourier transformable measure μ, the measures μ_{s} and μ_0 are Fourier transformable and

$$\widehat{\mu_{\mathsf{s}}} = (\widehat{\mu})_{\mathsf{pp}} \, ; \quad \widehat{\mu_0} = (\widehat{\mu})_{\mathsf{c}} \, .$$

This result makes the class of weakly almost periodic measures, and the Eberlein decomposition, of special interest for the theory of diffraction.

In general, given a weakly almost periodic measure μ, the supports of μ_{s} and μ_0 can be much larger than the support of μ. If μ is supported inside a lattice, then so are μ_{s} and μ_0. It is intriguing that the same is true for measures supported inside model sets with closed (compact) window [4]: If $(G \times H, \mathcal{L})$ is a cut and project scheme, $W \subset H$ is a compact set, and μ is a weakly almost periodic measure with $\mathrm{supp}(\mu) \subset \Lambda(W)$ then

$$\mathrm{supp}(\mu_{\mathsf{s}}) \, , \, \mathrm{supp}(\mu_0) \subset \Lambda(W) \, .$$

An immediate consequence is that the class of measures supported inside Meyer sets is stable under the Eberlein decomposition: If μ is a weakly almost periodic measure supported inside a Meyer set, then μ_{s} and μ_0 are also supported inside Meyer sets.

This result has important consequences for the diffraction of measures with Meyer set support. If ω is any translation bounded measure, with an autocorrelation measure γ supported inside a Meyer set, then each of the discrete diffraction measure $\widehat{\gamma}_{\mathsf{pp}}$ and continuous diffraction measure $\widehat{\gamma}_{\mathsf{c}}$ is either trivial or has a relatively dense support. In particular, Meyer sets always have a relatively dense set of Bragg peaks, which are highly ordered [4, 3].

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Dynamical versus diffraction spectra in aperiodic order

Daniel Lenz

(joint work with Michel Baake, Aernout C.D. van Enter)

We consider uniquely ergodic one-dimensional subshifts over a finite alphabet consisting of real numbers. Each such dynamical system comes with two spectra: The dynamical spectrum arising from the unitary action on an L^2-space and the diffraction spectrum arising from the autocorrelation of an (arbitrary) element from the system.

A well-known result originally due to Dworkin [1] (and later elaborated on in various works) gives that the diffraction spectrum is part of the dynamical spectrum. At the same time it was shown by van Enter and Miękisz [3] that the inclusion is usually strict as far as spectral types go.

This raises the question 'where the parts of the dynamical spectrum which are missing in the diffraction can be found'. Examples investigated by Baake and van Enter [2] suggested that they may be found in the diffraction of subshift factors. This is indeed the case in a very precise sense as could recently be shown in [4]. In fact, it turns out that it suffices to consider diffraction of conjugate systems. Similar results hold for (uniquely ergodic) Delone dynamical systems with finite local complexity.

These results imply a distinction between those systems where finitely many factors suffice to recover the dynamical spectrum and those where this is not the case. In this context, various questions are open.

To put our results in perspective, we mention that, as far as pure point spectrum goes, there is an equivalence, i.e., pure point diffraction spectrum is equivalent to pure point dynamical spectrum (as could be shown in a collaborative effort over the last fifteen or so years).

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Schrödinger spectrum and quantum dynamics

David Damanik

In this talk we explained how the Schrödinger equation $i\partial_t\psi = H\psi$ describes the time evolution of a quantum state and how the medium is modeled through the potential of the Schrödinger operator H. The relevance of the spectrum as the set of allowed energies, the spectral measures whose type is closely related to the transport behavior of the system, and the actual transport exponents were discussed.

We then focused on the specific case of the Fibonacci Hamiltonian, which is the central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $\ell^2(\mathbb{Z})$,

$$[H_{\lambda,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\,\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)\,\psi(n),$$

where $\lambda > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$, and $\omega \in \mathbb{R}/\mathbb{Z}$.

For this model, we have obtained quite detailed quantitative information about the fractal dimension of the spectrum [1, 2, 6] and of the density of states measure [3, 6], as well as the optimal Hölder exponent of the integrated density of states [4, 6] and the transport exponents [5, 6, 7, 8], particularly the upper transport exponent corresponding to the fastest moving part of the wavepacket.

For all these quantities associated with the Fibonacci Hamiltonian, the precise asymptotics in the regime of small or large coupling have been determined.

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Aperiodicity in statistical mechanics

Aernout C.D. van Enter

In statistical mechanics, one usually considers (ground state or Gibbs) probability measures as the fundamental objects, while the individual configurations (Dirac combs, which at $T = 0$ are typically translation bounded measures) are considered in the quasicrystallography community. Ergodicity arguments often explain why for many global traits μ-almost all, for some measure μ, or even all configurations display the same global behaviour. Spectra are an example in case. Traditionally, the diffraction spectrum was computed for Dirac combs, and the dynamical spectrum as the spectrum of the unitary operator generating translations, acting on the $L^2(\mu)-$space. However, they are intimately connected, and the diffraction spectrum forms a subset of the dynamical spectrum. It turns out that, if the diffraction spectrum is pure point, then so is the dynamical spectrum. This can be interpreted in the sense that the aggregates (the 'molecules') built up from the individual particles (the 'atoms') can be more but not less ordered than these atoms. This leads to the following, more general conjecture.

Conjecture 1. *If the diffraction spectrum has no absolutely continuous component, then neither has the dynamical spectrum.*

As of now, there have been constructed a variety of lattice models whose ground states are ordered in a non-periodic way, but as for Gibbs states, the results are much more modest, see for example [3] and references mentioned there. Indeed, at $T = 0$, for minimal systems it is often not hard to show the existence of interactions for which they are ground state measures. Such arguments go back to Aubry and Radin [1, 9]. Moreover, matching rules for tilings can be translated into nearest-neighbour interactions for which these tilings are ground states. For $T > 0$, only some more or less implicit existence results are known. Some explicit results would be very welome, I discussed what might be expected and in particular I discussed two conjectures.

Conjecture 2. *No quasicrystals in $d = 2$: In two dimensions at positive temperature, all extremal Gibbs measures for finite-range interactions are periodic (that is, quasicrystals do not exist for short-range models).*

Conjecture 3. *Quasicrystals occur in $d = 3$: In three or more dimensions, there exists a translation-invariant Gibbs measure for a short-range interaction which decomposes into non-periodic extremal Gibbs measures.*

At the end of my talk, I discussed how aperiodic sequences and tilings recently have attracted attention in the glass physics community, where issues such as aperiodic order, slowing down, Parisi overlap distributions and chaotic temperature dependence have been discussed. For some of these issues, see e.g. [6, 7, 8, 10, 11, 4, 5, 2].

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Autocorrelation via renormalisation

Michael Baake
(joint work with Natalie Frank, Franz Gähler, Uwe Grimm, E. Arthur Robinson)

Substitution systems of constant length are rather well understood, both combinatorially and spectrally; see [6, 1, 4] and references therein, as well as the recent general approach in [3]. For classic examples such as the Thue–Morse or the Rudin–Shapiro sequence, the spectral properties can be accessed explicitly and constructively via a recursion relation for the autocorrelation coefficients, for instance with weights in $\{\pm 1\}$. These coefficients then constitute a positive definite real-valued function on the group $\mathbb{Z}$. Its Fourier transform is the fundamental part of the diffraction measure of the system, and simultaneously the relevant spectral measure for the discussion of the dynamical spectrum; see [2] and references therein for details on this connection. For the Thue–Morse system, one finds purely singular continuous diffraction, while the Rudin–Shapiro system is one of the rare examples with absolutely continuous diffraction; compare the abstract by N. Frank in this report.

In general, no simple extension of this procedure is known, but one can use an approach via general pair correlation measures, as used in [6] and also in [3] for constant length and lattice substitutions. The new insight comes from the observation that, in the setting with natural tile lengths in one dimension, there is a set of *exact* renormalisation relations for these pair correlation functions that derive from the tile inflation rule together with local recognisability, which holds for all aperiodic cases. These relations comprise two regimes, one of a purely recursive nature and one with a self-consistency structure. The latter determines the solution and can be used to gain insight into the spectral type and other aspects of the system.

For the classic Fibonacci inflation ($a \mapsto ab$, $b \mapsto a$), using a result due to Strungaru [5], see also his abstract in this report, one can give an independent proof of its pure point nature, while for one of the simplest non-PV inflations (namely $a \mapsto abbb$, $b \mapsto a$), one can now show the absence of absolutely continuous spectral components. This means that this example shows, apart from its trivial (constant) eigenfunction, a purely singular continuous spectrum. More generally, one can formulate an algebraic condition for the absence of absolutely continuous diffraction, thus explaining the rarity of examples such as the Rudin–Shapiro chain mentioned above.

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Conjugacies of FLC Delone sets

Johannes Kellendonk

(joint work with Lorenzo Sadun)

Motivated by various results which give a characterisation of repetitive FLC Delone sets in $\mathbb{R}^d$ by means of their associated topological dynamical system (see [1] for an overview), we ask:

(1) Given two Delone sets with topologically conjugate dynamical systems, how are they related? Which geometrical properties are preserved under topological conjugacy?

(2) Given a Delone set, how many other Delone sets have the same dynamical system, up to topological conjugacy?

We require the Delone sets to be of finite local complexity (FLC), as otherwise there will be too many possibilities. It is well known that two Delone sets which are mutually locally derivable have the same dynamical system. Mutual local derivability is therefore considered as the trivial way to obtain Delone sets with the same dynamical system. The key result is the following [2]: *Two FLC Delone sets Λ, Λ' have topologically conjugated dynamical systems whenever there exists a third*

Delone set Λ'' which is mutually local derivable with Λ' and a shape conjugation of Λ.

A shape conjugation of Λ is defined by a map $F\colon \Lambda \to \mathbb{R}^d$ and results in the new set $\Lambda'' = \{x + F(x) \mid x \in \Lambda\}$. F has to satisfy certain conditions which allow for a cohomological interpretation. Therefore, shape conjugations of Λ are, up to mutual local derivability, parametrised by the subgroup of *asymptotically negligible* elements of $H^1(\Lambda, \mathbb{R}^d)$, at least if they are small. Here, $H^1(\Lambda, \mathbb{R}^d)$ is the first cohomology of Λ with values in $\mathbb{R}^d$. The second question above is therefore a question on how big this subgroup is. We provide some general results about it and then specialise to the case of model sets whose internal space is a finite-dimensional real vector space and whose window is a finite disjoint union of convex sets. In that case, we obtain an answer to the first question [3]: *Any shape conjugation is a reprojection*, meaning that the only effect a shape conjugation has is that it changes the direction along which the points in the strip are projected.

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From random to aperiodic tilings

Thomas Fernique

Consider a finite set of polygons of the Euclidean plane, called *tiles*. We call a *tiling* any interior-disjoint covering of the Euclidean plane by translated such tiles. A tile set which forms tilings, but only non-periodic ones, is said to be *aperiodic*. The corresponding tilings are said to be *aperiodic*. The first aperiodic tile set was found in 1964 by R. Berger, and several nice examples have been found since. Aperiodic tilings are widely used to model quasicrystals, and a central question in this context is to understand them beyond isolated examples.

One knows three general methods to find aperiodic tile sets. The first one consists in enforcing a hierarchical structure that prevents periodicity: It has been used for the first examples and later systematised (by S. Mozes in 1990, extended by Ch. Goodman-Strauss in 1998, see also [3]). A second one is the computational way opened by J. Kari in 1995: Tiles are designed to make tilings that simulate computations that themselves enforce aperiodicity (see also [4]). The last one relies on geometrical properties of tilings: It has been opened by L. Levitov in 1988 and continued by several authors (J. Socolar, T. Le, S. Burkov, A. Katz, see also [1]), but no complete characterisation has yet been obtained.

One drawback of an aperiodic tile set as a model of quasicrystals is the systematic existence of *deceptions*, which are finite patches of tiles that cannot be extended to tilings of the whole plane. In other words, attempts to build a tiling by assembling tiles one by one may fail (it remains unknown how often they fail).

Alternatively, people considered so-called *random tilings*: Constraints on the way tiles can be neighbours are relaxed so that not only aperiodic tilings but a huge set of tilings can be formed; in some cases, these tilings are, with high probability, close enough to aperiodic tilings. However, little is known beyond the *dimer case*.

Although random tilings can provide a good model of the first quasicrystals (those obtained by quenching), this approach seems outdated, given the way most of the quasicrystals are now obtained, namely by slow cooling. In this light, we are interested in Markov processes which would gradually transform a random tiling into an aperiodic one (see [2]). This leads to questions about the mixing time of Markov chains defined on rather complicated spaces of tilings.

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Sums of Cantor sets and the square Fibonacci Hamiltonian

Anton Gorodetski

To determine the spectral properties of Laplacian on the graph defined by a Penrose tiling (or any other higher-dimensional model of a quasicrystal) is a famous open problem. One of the ways to get some intuition here is to consider a discrete Schrödinger operator in $\ell^2(\mathbb{Z}^2)$ (or $\ell^2(\mathbb{Z}^d)$) with separable potential, namely

$$[H\psi](m,n) = \psi(m-1,n)+\psi(m+1,n)+\psi(m,n-1)+\psi(m,n+1)+V(m,n)\psi(m,n),$$

where $V(m,n) = V_1(m) + V_2(n)$ with bounded maps $V_{1,2} \colon \mathbb{Z} \to \mathbb{R}$. Consider the associated Schrödinger operators on $\ell^2(\mathbb{Z})$,

$$[H_{1,2}\psi](n) = \psi(n+1) + \psi(n-1) + V_{1,2}(n)\,\psi(n),$$

then the spectrum of H is given by $\sigma(H) = \sigma(H_1) + \sigma(H_2)$. In particular, if one considers the Fibonacci potential

$$V_{1,2}(n) = \lambda_{1,2}\,\chi_{[1-\alpha,1)}(n\alpha + \omega_{1,2} \bmod 1),$$

where $\alpha = \frac{\sqrt{5}-1}{2}$, this construction gives the square Fibonacci Hamiltonian. The spectral properties of the one-dimensional Fibonacci Hamiltonian were studied in detail [3], and it is known that its spectrum is a *dynamically defined* (or *regular*) Cantor set. The question on the structure of the sum of two dynamically defined Cantor sets is classical, and appeared in dynamical systems, harmonic analysis, and number theory.

In [1], we use the results from [2] to prove the following (see [4] for previous restricted results).

Theorem 1. *Let $\{C_\lambda\}$ be a family of dynamically defined Cantor sets of class C^2 (i.e., $C_\lambda = C(\Phi_\lambda)$, where Φ_λ is an expansion of class C^2 both in $x \in \mathbb{R}$ and in $\lambda \in J = (\lambda_0, \lambda_1)$) such that $\frac{d}{d\lambda} \dim_{\mathrm{H}} C_\lambda \neq 0$ for $\lambda \in J$. Let $K \subset \mathbb{R}^1$ be a compact set such that*

$$\dim_{\mathrm{H}} C_\lambda + \dim_{\mathrm{H}} K > 1 \quad \textit{for all} \quad \lambda \in J.$$

Then, $\mathrm{Leb}(C_\lambda + K) > 0$ *for a.e.* $\lambda \in J$.

Theorem 2. *There exists a non-empty open set $U \subset \mathbb{R}^2_+$ such that, for a.e. $(\lambda_1, \lambda_2) \in U$, the density of states measure of the corresponding square Fibonacci Hamiltonian is singular continuous, while the spectrum has positive Lebesgue measure.*

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Tridiagonal substitution Hamiltonians

William Yessen

(joint work with Jake Fillman, May Mei, Yuki Takahashi)

We consider tridiagonal substitution Hamiltonians, namely operators $H\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, given by

$$(H_{\omega,(p,q)}\phi)_n = p_{n-1}\phi_{n-1} + p_n\phi_{n+1} + q_n\phi_n,$$

where $\omega = \cdots\omega_{n-1}\cdots\omega_0\cdots\omega_n\cdots$ is a two-letter primitive invertible substitution sequence (such as, for example, the Fibonacci substitution sequence), and $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ are $\{p,1\}$- and $\{q,0\}$-valued sequences modulated by ω. We allow $q \in \mathbb{R}$ and $p \in \mathbb{R} \setminus \{0\}$ with $(p,q) \neq (1,0)$.

The fractal structure of the spectrum in the cases where $q = 0$ or $p = 1$ is completely understood (see [2]). Our main result in the case $q \neq 0$ *and* $p \neq 1$ states that the spectrum has a richer structure here.

Theorem 1 (Yessen [5], Mei–Yessen [4])**.** *For any ω, if $p \neq 1$ and $q \neq 0$, then the local Hausdorff dimension at x, $LHD(x)$, for x in the spectrum, varies continuously with x and for any x_0 in the spectrum, $LHD(x)$ is nonconstant in any arbitrarily small neighbourhood of x_0.*

Furthermore, unlike in the case where $p = 1$ or $q = 0$, we have

Theorem 2 (Yessen [5], Mei–Yessen [4]). *Given a two-letter primitive invertible substitution sequence ω, there exists a set $D_\omega \subset \{(p,q) \in \mathbb{R}^2 : p \neq 0\}$ of zero measure such that for all $(p,q) \in D$, the Hausdorff dimension of the spectrum of the corresponding tridiagonal Hamiltonian is equal to one (and accumulates at one of the endpoints of the spectrum). On the other hand, for all $(p,q) \notin D$, the Hausdorff dimension of the spectrum is strictly smaller than one.*

The square Hamiltonian can also be considered (see [1] for definitions). In this case, the previous theorem, together with the results from [1], leads to the following result (see [3]).

Theorem 3 (Fillman–Takahashi–Yessen [3]). *There exists a positive measure set of parameters such that the corresponding square Hamiltonian has a spectrum that contains an interval as well as a Cantor set; moreover, the density of states measure contains absolutely continuous as well as singular continuous components.*

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Continuum models of one-dimensional quasicrystals

Jake Fillman
(joint work with David Damanik, Anton Gorodetski)

We discuss some models of one-dimensional quasicrystals, which are studied heavily in the physics literature; for example, see [1, 5, 6, 8, 10, 11, 15, 16]. Our models are given by Schrödinger operators on $L^2(\mathbb{R})$ whose potentials are generated by an underlying ergodic subshift over a finite alphabet and a rule that replaces letters of the alphabet by compactly supported potential pieces. For a survey of the discrete counterparts of these operators, see [3].

We first develop the standard theory that shows that the spectrum and the spectral type are almost surely constant. We discuss applications of cocycle dynamics to the spectral analysis of such operators. In particular, analogues of the theorems of Kotani and Johnson hold for this class of operators. That is, the almost sure absolutely continuous spectrum coincides with the essential closure of the set of energies with vanishing Lyapunov exponent, and the resolvent set coincides with the complement of the uniformly hyperbolic energies, provided the underlying dynamics are minimal [7, 12]. Consequently, the spectrum is a Cantor set of zero Lebesgue measure if the potentials are aperiodic and irreducible and the underlying symbolic dynamics satisfy the Boshernitzan condition [2] — the proof combines work of Damanik–Lenz to deduce absence of nonuniform hyperbolicity and Klassert–Lenz–Stollman to prove absence of absolutely continuous spectrum [4, 9].

We then study the case of the Fibonacci subshift in detail and describe results for the local Hausdorff dimension of the spectrum at a given energy in terms of the value of the associated Fricke–Vogt invariant. In particular, the formalism of Sütő which identifies the spectrum with the set of energies at which the associated trace map is bounded can be worked out in this setting [13, 14].

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Binary bijective block substitutions in d dimensions

Uwe Grimm
(joint work with Michael Baake)

The Thue–Morse substitution and its generalisations generate one-dimensional structures on $\mathbb{Z}$ that, in the case of balanced weights (zero average scattering strength), show purely singular continuous diffraction; see [1, 2] and references therein. The corresponding dynamical spectra contain non-trivial pure point parts, which are absent in the diffraction, but which can be recovered by considering the diffraction of suitable factors, for instance the image under a simple sliding block map. Here, one obtains the period-doubling sequence (with point spectrum on the dyadic integers) and its generalisations as factors; for the precise connection between the diffraction spectra and the relevant spectral measures, see [4].

A natural generalisation to higher dimensions is provided by (non-degenerate) binary bijective block substitutions in d dimensions [6, 7]. Working with the alphabet $\{\pm 1\}$ ensures balanced weights, as both letters are equally frequent due to the bijectivity. The corresponding diffraction measure corresponds to the maximal spectral measure in the orthocomplement of the pure point spectrum; see [4] for details. This measure can then be analysed by an analogous approach to the one used for the Thue–Morse sequence and its generalisations [1], as was first demonstrated for the example of the 'squiral' tiling [3]. The strategy is as follows. Starting from the block substitution, one derives a set of renormalisation equations for the autocorrelation coefficients $\eta(m)$ for $m \in \mathbb{Z}^d$; compare also the abstract by M. Baake in this report. These consist of a finite set of equations that determine a subset of coefficients uniquely in terms of $\eta(0)$, while the remaining relations determine all other coefficients recursively. This suffices to show that the spectral type has to be pure. Then, as it cannot be absolutely continuous by an application of the Riemann–Lebesgue lemma, it is always singular. If the block substitution has trivial height, the diffraction is always purely singular continuous [6], and there is evidence that the only cases with pure point spectrum are those where the block substitution is compatible with a fully periodic structure.

The non-trivial pure point part of the corresponding dynamical spectrum can, once again, be recovered by considering suitable factors, in line with recent general results [4] on the relation between the dynamical and diffraction spectrum; see also [5] for a recent extension to more general lattice substitutions.

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A decorated silver mean tiling with mixed spectrum

Franz Gähler

There are many inflation tilings with a pure-point and a continuous part in their dynamical or diffraction spectrum, also higher-dimensional ones [1, 2]. Most are generated by a constant length inflation, and are thus lattice based. Here, we describe a procedure to construct a mixed-spectrum, almost 2-1 extension of any pure-point inflation tiling, and illustrate it with the well-known silver mean tiling, constructing thus a mixed-spectrum tiling based on a quasiperiodic tiling.

The starting point is the observation that many of the mixed-spectrum examples have a symmetry in their inflation rules [3]. All tiles come in geometrically equal pairs, and tiles within a pair are distinguished by the presence or absence of a bar. Swapping the bar status of all tiles is a symmetry, which commutes with the inflation. Wiping out all bars defines a factor map which is 2-1 almost everywhere. Provided the maximal equicontinuous factors (MEF) of both the barred and the unbarred tiling are the same, the factor map from the barred tiling to its MEF is then 2-1 almost everywhere, which implies that its spectrum is mixed [4].

This picture suggests how to construct mixed-spectrum inflation tilings in a systematic way. Starting with our favourite pure-point inflation tiling, we split each tile type into a pair, one with and one without a bar, and assign the bars in the inflation rule such that i) the bar swap symmetry is observed, ii) the resulting inflation is primitive, and iii) the barred and the unbarred tiling have the same MEF. As there are many ways to assign the bars, often there are such solutions.

We illustrate the procedure with the silver mean tiling, for which we introduce a suitably twisted version with bar swap symmetry. By general arguments, it is in fact true that the spectrum carried by the kernel of an almost 2-1 map to a pure-point factor must be pure. The factor map from the barred to the unbarred tiling is such a map, wherefore the spectrum in the odd sector with respect to the bar swap must be pure, either absolutely continuous or singular continuous. To discriminate between the two, we compute the autocorrelation of the twisted silver mean tiling with a decoration which is odd under the bar swap. This autocorrelation does not tend to zero for a series of distances tending to infinity, which by the Riemann–Lebesgue lemma implies that its Fourier transform, the diffraction spectrum, must have a singular component. As the diffraction spectrum is contained in the dynamical spectrum, the latter thus has a singular continuous component in the odd sector, so that it must be purely singular continuous there.

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Ergodic properties of visible lattice points

Christian Huck
(joint work with Michael Baake)

Recently, the dynamical and spectral properties of square-free integers, visible lattice points and various generalisations have received increased attention; cf. [4, 5, 6, 7]. One reason is the connection with Sarnak's conjecture [8] on the 'randomness' of the Möbius function, another the explicit computability of correlation functions as well as eigenfunctions for these systems. Here, we use the visible points

$$V = \mathbb{Z}^2 \setminus \bigcup_{p \text{ prime}} p\mathbb{Z}^2$$

of the square lattice $\mathbb{Z}^2$ as a paradigm. Clearly, V has holes of arbitrary size and it is classic that the natural density of V is equal to $1/\zeta(2) = 6/\pi^2$. It turns out that V has positive topological entropy equal to its density [7] and one thus might expect to leave the realm of pure point spectrum. However, V has pure point dynamical and diffraction spectrum [3, 1]. In fact, it is a major step to show that the lattice translation orbit closure

$$\mathbb{X}_V = \overline{\{t + V \,|\, t \in \mathbb{Z}^2\}}$$

of V in the product topology on the power set $\mathcal{P}(\mathbb{Z}^2) \simeq \{0,1\}^{\mathbb{Z}^2}$ contains precisely the *admissible* subsets A of $\mathbb{Z}^2$, the latter being defined by the property that, for any prime p, at leat one coset modulo $p\mathbb{Z}^2$ is missing in A, which means $|A/p\mathbb{Z}^2| < p^2$. One can further show that the patch frequencies of V exist and this gives rise to a $\mathbb{Z}^2$-invariant Borel probability measure ν on $\mathbb{X}_V$ that can be seen to be ergodic. Further, V turns out be a ν-generic element of the hull. Our main result is that the measure theoretic dynamical system $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$ is isomorphic to the Kronecker system $\prod_p \mathbb{Z}^2/p\mathbb{Z}^2$, defined by the obvious 'diagonal' action of $\mathbb{Z}^2$ and the normalised Haar measure; see [2] for details. Moreover, both the dynamical and the diffraction spectra are given by the points of $\mathbb{Q}^2$ with square-free denominator.

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On pattern entropy of model sets

Christoph Richard
(joint work with Christian Huck)

Consider $G = \mathbb{R}^d$, a locally compact Abelian Hausdorff group H, and a lattice $\widetilde{L}$ in $G \times H$. Let π_G, π_H denote the canonical projections from $G \times H$ to its factors. Assume that π_G restricted to $\widetilde{L}$ is one-to-one and that $\pi_H(\widetilde{L})$ is dense in H. Consider, for any relatively compact $W \subset H$, the *weak model set* $\Lambda(W) = \{\pi_G(\widetilde{\ell}) \mid \widetilde{\ell} \in \widetilde{L}, \pi_H(\widetilde{\ell}) \in W\}$. Fix Haar measures θ_G on G and θ_H on H such that $\widetilde{L}$ has density 1. A *van Hove sequence* $(A_n)_{n\in\mathbb{N}}$ in G consists of compact sets of positive Haar measure such that for every compact $K \subset G$ we have $\theta_G(\partial^K A_n)/\theta_G(A_n) \to 0$ as $n \to \infty$, where $\partial^K A = (K\overline{A} \cap \overline{A^c}) \cup (K\overline{A^c} \cap \overline{A})$. Fix any van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in G and consider the relative point frequencies

$$f_n(W) = \frac{1}{\theta_G(A_n)} \operatorname{card}\big(\Lambda(W) \cap A_n\big). \tag{1}$$

We extend the density formula [2] for *regular* model sets, which satisfy $\theta_H(\partial W) = 0$. This is done by approximating $\Lambda(W)$ with regular model sets and leads to

Proposition 1 (Generalised density formula [3])**.** *Under the above assumptions,*

$$\theta_H\big(\operatorname{int}(W)\big) \le \liminf_{n\to\infty} f_n(W) \le \limsup_{n\to\infty} f_n(W) \le \theta_H\big(\operatorname{cl}(W)\big).$$

For $A \subset G$ compact and $t \in G$, the finite set $\Lambda(W) \cap tA$ is called an *A-pattern* of $\Lambda(W)$. Let $N_A\big(\Lambda(W)\big)$ denote the number of different A-patterns of $\Lambda(W)$ up to G-translation. The *pattern entropy* $h\big(\Lambda(W)\big)$ of $\Lambda(W)$ is defined as

$$h\big(\Lambda(W)\big) = \limsup_{n\to\infty} \frac{1}{\theta_G(A_n)} \log N_{A_n}\big(\Lambda(W)\big).$$

We can show that the pattern entropy of $\Lambda(W)$ and, more generally, of any coloured Delone set of finite local complexity, is a finite limit for suitable van Hove sequences such as balls or rectangular boxes of diverging inradius, thereby extending earlier results [1]. The following entropy estimate was conjectured by Moody [3].

Theorem 1. *If H is second countable and if all A_n are simply connected, then*

$$h\big(\Lambda(W)\big) \le \theta_H(\partial W) \log 2.$$

This estimate can be seen to hold via analysing V-patterns within fundamental domains of $\widetilde{L}$ of 'arbitrarily thin H-component'. The result then follows by combinatorial standard estimates together with the generalised density formula.

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Substitution $\mathbb{Z}^d$ sequences with non-simple Lebesgue dynamical spectrum

Natalie Priebe Frank

In this talk, we discuss a family of substitution rules in $\mathbb{Z}^d$ whose dynamical systems have a mixed spectrum that includes an absolutely continuous part. Introduced in [1], these substitutions generalize the well-known Rudin–Shapiro sequence to higher dimensions. We explain the method of construction and show how it leads to the destruction of two-point correlations, which ultimately results in a Lebesgue diffraction measure.

The starting point for the construction is a *Hadamard matrix*, which is a square matrix whose entries are ± 1 and whose rows are pairwise orthogonal. Simple examples are $M = \left(\begin{smallmatrix} + & - \\ - & - \end{smallmatrix}\right)$, which generates the original Rudin–Shapiro sequence, and

$$B = \begin{pmatrix} - & - & - & + \\ - & - & + & - \\ - & + & - & - \\ + & - & - & - \end{pmatrix},$$

which allows nontrivial one- or two-dimensional substitutions.

Once the Hadamard matrix is chosen, we use it to define both the alphabet and the size of the substitution. Given that the Hadamard matrix is of size $n \times n$, the alphabet is defined to be $A_n = \{\pm 1, \pm 2, \ldots, \pm n\}$. We also assign a d-dimensional rectangular array with n total entries, denoted I, to serve as the shape of the substitution. For the matrix M, we can take $I = \{0, 1\} \subset \mathbb{Z}$; for the matrix B we can take either $I = \{0, 1, 2, 3\} \subset \mathbb{Z}$ if we want a one-dimensional substitution, or $I = \{(0,0), (1,0), (0,1), (1,1)\} \subset \mathbb{Z}^2$ if we want a two-dimensional substitution.

The next step is to place each entry of I into one-to-one correspondence with $1, 2, \ldots, n$ and decree that any letter $\pm k \in A_n$ can appear only at the position in I associated to k. With these definitions in place, we are ready to define the substitution $S(e)$ for each letter $e \in A_n$. First suppose $e = j$ is a positive element of A_n. We use the jth row of the Hadamard matrix to determine whether to place a $+k$ or a $-k$ at the kth location of I. The result is a word $S(j) \in I^{A_n}$ that we refer to as the substitution of the letter j. We then define $S(-j)$ to be $-S(j)$.

In this way we obtain, for the Hadamard matrix M given above, the substitution $S(1) = 1\,{-2}$, $S(-1) = -1\,2$, $S(2) = -1\,{-2}$, and $S(-2) = 1\,2$, and we see under repeated iteration

$$1 \;\to\; 1\,{-2} \;\to\; 1\,{-2}\,1\,2 \;\to\; 1\,{-2}\,1\,2\,1\,{-2}\,{-1}\,{-2} \;\to\; \cdots,$$

where each block is a word on the alphabet $A_2 = \{\pm 1, \pm 2\}$.

One can begin to see why this construction could result in a Lebesgue component of the dynamical (and diffraction) spectrum as follows. Consider the factor map obtained by forgetting the numbers but keeping the $\pm$ signs. Correlating a substituted block $S(e)$ with another $S(f)$, where $|e| \neq |f|$, will result in a sum of exactly 0, since the rows of the Hadamard matrix associated to e and f are orthogonal. One can prove that iterating the substitution does not change this fact, so that in sequences generated by the substitution there are arbitrarily large subwords that, when correlated, yield exactly 0.

A possible generalization to the construction would be to start with a *Vandermonde matrix*, which is a matrix with entries from a given root of unity, whose rows are pairwise orthogonal. In this situation, it would seem that the dynamical spectrum could contain, in addition to the discrete part, both singular and absolutely continuous parts.

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Diffraction spectra and dynamical spectra of some random point sets

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(joint work with Michael Baake and Robert V. Moody)

Let ω be a stationary and ergodic locally finite random point set in $\mathbb{R}^d$ such that $\mathbb{E}(\omega(B)^2) < \infty$ for any bounded Borel set B. The *autocorrelation measure* γ of ω is given by $\lim_{n\to\infty} \frac{1}{\lambda^d(B_n)}(\omega|_{B_n} * \widetilde{\omega}|_{B_n})$, where B_n is the ball of radius n around the origin, $\widetilde{\omega}$ is the reflection of ω at the origin, and the limit is in the vague topology. The *diffraction measure* $\widehat{\gamma}$ of ω is the Fourier transform of γ. The equivalence class of $\widehat{\gamma}$ is also called the *diffraction spectrum* of ω, while the *dynamical spectrum* of ω is the maximal spectral type of the associated measure-theoretic dynamical system. It is of interest to describe and / or to compare these two spectra.

We discuss two classes of random point sets which allow for some interaction between the points but for which the diffraction spectra and the dynamical spectra may still be determined explicitly. More precisely, we consider *determinantal* and *permanental* point processes (see, e.g., [2] for definitions) with 'nice' kernels.

Theorem 1 ([1]). *Let ω be the determinantal point process associated with the kernel $K(x, y) := \widehat{\varphi}(x - y)$, where φ is a probability density taking values in* $[0, 1]$.

Then

$$\gamma \,=\, \delta_0 + (1 - |\widehat{\varphi}|^2)\, \lambda\!\!\lambda^d \quad \textit{and} \quad \widehat{\gamma} \,=\, \delta_0 + (1 - (\varphi * \varphi_{-}))\, \lambda\!\!\lambda^d\,.$$

Theorem 2 ([1]). *Let ω be the permanental point process associated with the kernel $K(x,y) := \widehat{\varphi}(x-y)$, where φ is a probability density. Then*

$$\gamma \,=\, \delta_0 + (1 + |\widehat{\varphi}|^2)\, \lambda\!\!\lambda^d \quad \textit{and} \quad \widehat{\gamma} \,=\, \delta_0 + (1 + (\varphi * \varphi_{-}))\, \lambda\!\!\lambda^d\,.$$

Here, φ_{-} denotes the reflection of φ at the origin. Observe that, for both classes, the diffraction measure is absolutely continuous and, in fact, equivalent to Lebesgue measure apart from the Bragg peak at the origin. Moreover, an argument due to Soshnikov [3] shows that the maximal spectral type is also equivalent to Lebesgue measure apart from the eigenvalue at the origin. Thus, the diffraction spectrum and the dynamical spectrum coincide in these examples.

These results are perhaps not unexpected in view of the good mixing properties of the point processes under consideration. However, if one moves to more general permanental point processes with a kernel derived from a singularly continuous probability measure, the situation may change. Here, the diffraction spectrum and the dynamical spectrum may additionally contain a singularly continuous part, and they are not necessarily equivalent.

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Topological perspective on the dynamics of tilings

Alex Clark

We describe how, following Freudenthal [4], each compact, metrizable space admits a spectral decomposition into an inverse sequence of compact polyhedra with surjective (even piecewise linear) bonding maps. This spectral decomposition allows one to approximate continuous maps between two compact spaces by using maps between the polyhedral approximating spaces in the towers representing the underlying spaces. There are several variations for how one can find such approximating maps, leading to what are known as 'zig-zag', almost commutative maps of towers. Equally important, appropriately chosen almost commutative zig-zag maps between two towers representing compact metric spaces defines a homeomorphism between the underlying spaces; see, e.g, [5].

Anderson and Putnam [1] and Gähler have found natural ways to express tiling spaces as inverse limits that reflect the underlying translation dynamics. This leads one to consider the possibility of enriching the information encoded in the inverse limit expansion beyond the topological. If the expansions encode the dynamics at the same time as the topology and one only allows maps between towers that 'almost' respect the dynamics in an appropriate sense, one finds that dynamically equivariant maps can be represented as almost commutative maps of towers.

We then revisit the results of [2, 3] from this point of view. It is especially simple in the one-dimensional case where the translation dynamics of a tiling space can be encoded using the length of the circles in the approximating spaces, which correspond to supertiles in the simplest cases. We then show how these ideas can be used to show a theorem from [2]: If two tiling spaces X, X' are based on the same Pisot substitution but with possibly different choices of tile lengths, then (after a linear rescaling of time) the translation dynamics of X and X' are topologically conjugate. Size does not matter in this case.

We then show how using a theorem from [2] describing the continuous eigenvalues of the translation dynamics, one can see that for many substitutions of constant length, the generic choice of lengths for tiles leads to topological weak mixing. Size does matter in this case.

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