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Variational Methods for Evolution

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Abstract. The workshop brought together researchers from geometry, nonlinear functional analysis, calculus of variations, partial differential equations, and stochastics around a common topic: systems whose evolution is driven by variational principles such as gradient or Hamiltonian systems.

The talks covered a wide range of topics, including variational tools such as incremental minimization approximations, Gamma convergence, and optimal transport, reaction-diffusion systems, singular perturbation and homogenization, rate-independent models for visco-plasticity and fracture, Hamiltonian and hyperbolic systems, stochastic models and new gradient structures for Markov processes or variational large-deviation principles.

Mathematics Subject Classification (2010): 35A15, 35Qxx, 49xx, 74xx, 60F10, 60H15, 26Dxx, 82xx.

Introduction by the Organisers

The variational approach to evolutionary systems provides one of the most interesting areas of mathematical research, as it combines geometric information, such as metric or more general dissipation structures, with energy landscapes and functional analysis in infinite-dimensional spaces. Thus it opens up new approaches and mathematical fields for studying evolutionary systems, which already generated a lot of original contributions from various points of view in the pure and applied fields.

In the last decades many problems concerning evolutionary PDEs, modeling of mechanical and biological phenomena, fluid and transport dynamics, stochastic behaviour of many-particle systems, geometric evolution, Hamiltonian, dissipative

and rate-independent flows, have been studied by new and various variational techniques, with a clever combination of well established tools and new ideas.

These investigations have stimulated a fruitful interaction between the classical approaches of PDE's, calculus of variations (such as direct methods, Gamma-convergence and relaxation, iterated minimization schemes, and variational principles), geometric measure theory and nonsmooth analysis (BV functions, motion of interfaces, and analysis in metric-measure spaces), functional analysis (Dirichlet forms, semigroup theory, Gamma-calculus, convex integration, and infinite-dimensional spaces), optimal transport, and calculus of probability (stochastic processes, large-deviation principles, and stochastic perturbations). New beautiful results have arisen and contributed to the advance of each field in a promising interdisciplinary way with interesting and deep connections.

Among the themes presented during the workshop, we mention here:

- gradient flows and large deviations,
- regularity structures and rough paths;
- stochastic homogenization;
- optimal transport techniques and transportation distances, functional inequalities, entropic interpolation;
- discrete interaction systems, evolution on graphs, and their metric-variational interpretation;
- rate-independent problems, quasi-static crack growth, elasto-plasticity;
- singular limit of gradient flows and conservative systems and their asymptotic dynamics, hysteretic phase transitions;
- entropy-entropy dissipation methods, reaction-diffusion systems and their geometric interpretation, discrete approach and structure-preserving numerical methods;
- front propagation and models derived from Allen-Cahn equations;
- variational time discretizations for compressible Euler equations;
- general variational principles and models for evolution;
- evolution in metric spaces and Hamilton-Jacobi equations.

The workshop, organized by Luigi Ambrosio (Scuola Normale Superiore, Pisa), Alexander Mielke (WIAS, Berlin), Mark Peletier (TU Eindhoven), and Giuseppe Savaré (University of Pavia), aimed to present many new, striking and promising achievements in this wide area, thanks to the contribution of 48 participants (16 young researchers) with broad geographic representation from Austria, Czech Republic, France, Germany, Italy, Netherlands, UK and USA, and a variety of research fields, each revealing different methodology, interests, and level of abstraction. Twenty-eight invited talks, mostly of 45 minutes, have been delivered, leaving plenty of time for discussions, which have been greatly stimulated by the diversity of the topics and of the contributions. The friendly atmosphere and the perfect environment of Oberwolfach have also contributed to the success of the meeting.

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Workshop: Variational Methods for Evolution

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A Variational Time Discretization for Compressible Euler Equations

Michael Westdickenberg
(joint work with Fabio Cavalletti and Marc Sedjro)

The compressible Euler equations model the dynamics of compressible fluids such as gases. They form a system of hyperbolic conservation laws

$$\left.\begin{aligned} \partial_t \varrho + \nabla \cdot (\varrho \boldsymbol{u}) &= 0 \\ \partial_t(\varrho \boldsymbol{u}) + \nabla \cdot (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \pi &= 0 \\ \partial_t \varepsilon + \nabla \cdot \big((\varepsilon + \pi)\boldsymbol{u}\big) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbf{R}^d. \tag{1}$$

The unknowns $(\varrho, \boldsymbol{u}, \varepsilon)$ depend on time $t \in [0,\infty)$ and space $x \in \mathbf{R}^d$ and we assume that suitable initial data $(\varrho, \boldsymbol{u}, \varepsilon)(t=0,\cdot) =: (\bar\varrho, \bar{\boldsymbol{u}}, \bar\varepsilon)$ is given. We will consider ϱ as a map from $[0,\infty)$ into the space of nonnegative, finite Borel measures, which we denote by $\mathscr{M}_+(\mathbf{R}^d)$. The quantity ϱ is called the density and it represents the distribution of mass in time and space. The first equation in (1) (the continuity equation) expresses the local conservation of mass, where

$$\boldsymbol{u}(t,\cdot) \in \mathscr{L}^2\big(\mathbf{R}^d, \varrho(t,\cdot)\big) \quad \text{for all } t \in [0,\infty) \tag{2}$$

is the Eulerian velocity field taking values on $\mathbf{R}^d$. The second equation in (1) (the momentum equation) expresses the local conservation of momentum $\boldsymbol{m} := \varrho\boldsymbol{u}$. The quantity ε is the total energy of the fluid and $\varepsilon(t,\cdot)$ is a measure in $\mathscr{M}_+(\mathbf{R}^d)$. The third equation in (1) expresses the local conservation of energy.

The quantity π in the momentum equation is the pressure. It is determined by the thermodynamic properties of the fluid. Three cases are of interest:

- The pressure vanishes (the pressureless gas case). Then the total energy of the fluid is simply the kinetic energy, and the third equation in (1) follows *formally* from the first two, by the chain rule.
- The pressure is a function of the density ϱ only because the thermodynamical entropy is constant throughout time and space (the isentropic case). Again the conservation of total energy follows formally from the continuity and the momentum equation.
- The pressure is a function of the density ϱ and total energy ε (the full Euler case). In this case, there is again an additional conservation law since the thermodynamical entropy *formally* satisfies a transport equation.

Even though system (1) formally conserves the total energy (being a Hamiltonian system), there actually is a dissipation of energy due to the nonsmoothness of the solutions: In the pressureless case, the modeling suggests a concentration of mass (sticky particle dynamics) by which the kinetic energy decreases. In the cases with pressure, solutions may form jump singularities along codimension-one manifolds, which are called shocks. Again total energy is dissipated in the process.

We consider a variational time discretization for the system of conservation laws (1) in the spirit of minimizing movements for curves of maximal slopes on metric spaces. We recall that for certain (possibly degenerate) parabolic equations, such as the porous medium equations, the solutions are curves on the space of nonnegative measures characterized by the requirement that at each time an energy (or entropy) functional is decreased at maximal rate (which also characterizes gradient flows). This comes with a natural time discretization, where in each timestep one tries to find the right balance between minimizing this energy functional and keeping the step short. For the porous medium equation the update length is measured using the Wasserstein distance. For the variational time discretization of (1) we proceed analogously: In each timestep we minimize the sum of the internal energy and of a new functional (which we call the minimal acceleration functional) measuring the deviation of fluid element trajectories from straight paths. We minimize over the closed convex cone of *monotone* transport maps, which in particular guarantees the non-interpenetration of matter. Notice that for the porous medium equation, the relevant transport maps are *cyclically monotone* because those are the maps that solve the optimal transport problem that underlies the Wasserstein distance. In this case, the (cyclical) monotonicity follows implicitly from the choice of metric, whereas for (1) we make monotonicity an explicit constraint. This can be justified by the fact that in each timestep the transport maps are perturbations of the identity map, which is monotone. Since monotone maps enjoy very good properties (they are e.g. of bounded variation locally) one can prove the existence of a minimizer for each timestep. By a suitable interpolation in time, we obtain a family of approximate solutions to (1), parametrized by the timstep $\tau > 0$. We prove that as $\tau \to 0$, these approximate solutions converge (along a subsequence) to a measure-valued solution of (1). An important ingredient is a characterization of the polar cone of the cone of monotone maps: each element in the polar cone can be represented by the distributional divergence of a matrix field taking values in symmetric positive semidefinite matrices. This matrix field, which we call a stress tensor, therefore has exactly the same structure as the matrix field $\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \pi \mathbf{1}$, which appears in the momentum equation (1). The momentum can be shown to be Lipschitz continuous with respect to a suitable Kantorovorich norm.

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Variational problems on graphs and their continuum limits

Dejan Slepčev
(joint work with Xavier Bresson, James von Brecht, Nicolás García Trillos and Thomas Laurent)

We discuss variational problems arising in analysis of data clouds. Many of the tasks of machine learning, including clustering, classification, dimensional reduction, have variational descriptions. One of the standard approaches is to introduce an objective functional which encodes the desirable properties of the object sought and then develop and implement algorithms to find a minimizer. A large class of the approaches, relevant to high-dimensional data, relies on creating a graph out of the data cloud by connecting nearby points (see [4] and references therein). This allows one to leverage the geometry of the data set.

An important question regarding such approaches is how they behave if more data become available. To be precise consider problems for which there exists an (unknown) ground truth given by a probability measure ν, supported on a compact domain D, such that the available data points $X_n = \{x_1, \dots, x_n\}$ are random i.i.d. samples of the measure ν. It is highly desirable if a procedure is such that if more data become available it converges to some well defined ideal object, which corresponds to full information being known. Such property of algorithms is referred to as ***consistency***, [5, 9]. For example if one is interested in partitioning data into two clusters a consistent procedure converges to a ideal continuum partitioning of the measure ν. In other words minimizers of the discrete objective functionals describing discrete partitioning should converge to a minimizer of an objective functional describing the ideal partitioning in the continuum setting. While consistency is one of the key properties of machine learning algorithms relatively few results are available (see [1, 2, 3, 9, 5], and references therin).

To address consistency questions we approach them using tools of applied analysis and calculus of variations. Namely we show Γ-convergence of the discrete functionals considered on random, proximity, graphs towards their continuum counterparts, which along with a compactness result implies the desired convergence of minimizers. A key element is identifying the proper topology with respect to which the Γ-convergence takes place. Let us denote by ν_n the empirical measure associated to the n data points:

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}. \tag{1}$$

The issue is then how to compare functions in $L^1(\nu_n)$ with those in $L^1(\nu)$. More generally we consider how to compare functions in $L^p(\mu)$ with those in $L^p(\theta)$ for arbitrary probability measures μ, θ on D and arbitrary $p \in [1, \infty)$. We set

$$TL^p(D) := \{(\mu, f) \ : \ \mu \in \mathcal{P}(D),\ f \in L^p(D, \mu)\},$$

where $\mathcal{P}(D)$ denotes the set of Borel probability measures on D. For (μ, f) and (ν, g) in TL^p we define the distance

$$d_{TL^p}((\mu,f),(\nu,g)) = \inf_{\pi\in\Gamma(\mu,\nu)} \left(\iint_{D\times D} |x-y|^p + |f(x)-g(y)|^p d\pi(x,y) \right)^{\frac{1}{p}}$$

where $\Gamma(\mu,\theta)$ is the set of all *couplings* (or *transportation plans*) between μ and θ.

An important consideration when investigating consistency of algorithms is how the graphs on X_n are constructed. In simple terms, when building a graph on X_n one sets a length scale ε_n such that edges between vertices in X_n are given significant weights if the distance between vertices is ε_n or less. Taking smaller ε_n is desirable because it is computationally less expensive and gives a better resolution, but there is a price. If ε_n is too small the resulting graph may not represent the geometry of D well and consequently the discrete graph cut may be very far from the desired one. We worked on determining precisely how small ε_n can be taken for the consistency to hold. More precisely consider a kernel $\eta : \mathbb{R}^d \to [0,\infty)$ to be radially symmetric and decaying to zero sufficiently fast. Let $\eta_\varepsilon(z) = \frac{1}{\varepsilon^d}\eta\left(\frac{z}{\varepsilon}\right)$. The edge weights are

$$w_{i,j} = \eta_\varepsilon(x_i - x_j). \tag{2}$$

Given a function $u_n : X_n \to \mathbb{R}$ its (appropriately scalled) *graph total variation* is defined as

$$GTV_{n,\varepsilon}(u_n) = \frac{1}{\varepsilon}\frac{1}{n^2}\sum_{i,j} w_{i,j}|u_n(x_i) - u_n(x_j)|. \tag{3}$$

The role of the perimeter of $Y \subset X_n$ on the graph is played by the graph cut, that is the sum of all edges between Y and Y^c, which is nothing but (a multiple of) the graph total variation of the characteristic function of Y.

To prove consistency of machine learning approaches to clustering, the key ingredient is the variational behavior of graph total variation as $n \to \infty$. This was investigated in [6]:

Theorem [**Γ-convergence and Compactness**] Let $D \subset \mathbb{R}^d$, $d \geq 2$ be a domain with Lipschitz boundary. Let ν be a probability measure on D with continuous density ρ, which is bounded from below and above by positive constants. Let $x_1, \ldots, x_n, \ldots$ be a sequence of i.i.d. random points on D chosen according to measure ν. Let $\varepsilon_n \to 0$ as $n \to 0$ be such that

$$\lim_{n\to\infty} \frac{(\log n)^p}{n}\frac{1}{\varepsilon_n^d} = 0. \tag{4}$$

where $p_d = 1$ if $d \geq 3$ and $p_2 = \frac{3}{2}$. Then, GTV_{n,ε_n}, defined by (3), Γ-converges with respect to TL^1 topology to a constant (explicitly given) multiple of total variation (weighted by ρ^2) on D.

Furthermore for any sequence of functions $u_n \in L^1(D,\nu_n)$: If

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(\nu_n)} + GTV_{n,\varepsilon_n}(u_n) < \infty$$

then $\{u_n\}_{n\in\mathbb{N}}$ is TL^1-relatively compact.

In [8] this result was used to obtain strong results on consistency of graph-based clustering algorithms. Namely given a weighted graph $\mathcal{G}_n = (X_n, W_n)$ the balanced graph cut problems seeks to partition the graph in clusters by cutting few edges (measured according to their weight) while also seeking to obtain clusters of comparable sizes. For simplicity, here the attention is restricted to the two-class case and a particular balance term, corresponding to Cheeger cuts:

(5)
$$E_n(Y) = \frac{\mathrm{Cut}_n(Y, Y^c)}{\min(|Y|, |Y^c|)} := \frac{\sum_{x_i\in Y}\sum_{x_j\in Y^c} w_{ij}}{\min(|Y|, |Y^c|)} \quad \text{over all nonempty } Y \subsetneq X_n.$$

The continuum partitioning problem that corresponds to the discrete problem is the following: Minimize the continuum balanced cut objective functional

$$E(A) = \frac{\mathrm{Per}(A : D)}{\min(\nu(A), \nu(D\backslash A))}, \qquad A \subset D \ \text{ with } 0 < \nu(A) < 1. \tag{6}$$

where $\mathrm{Per}(A : D)$ is the relative perimeter of A in D, weighted by ρ^2. We show that under assumptions of the Theorem above, almost surely, the minimizers, $\{Y_n, Y_n^c\}$, of the balanced cut (5) of the graph $\mathcal{G}_n$, converge in the TL^1 sense (applied to the characteristic functions of the sets) to $\{A, A^c\}$, the minimizer of the problem (6), if such minimizer is unique. Otherwise convergence holds up along subsequences.

In addition to techniques of calculus of variations and analysis the results rely on sharp estimates on the ∞-transportation distance between the measure ν and the empirical measure ν_n of the i.i.d sample $\{x_1, \ldots, x_n\}$, [7, 10, 11].

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From particles to a conservation law: large deviations for the totally asymmetric exclusion process on the half line

Johannes Zimmer
(joint work with Horacio González Duhart Muñoz de Cote and Peter Mörters)

We study the totally asymmetric exclusion process on the lattice $\mathbb{N} := \{1, 2, \dots\}$ with a source of particles at the origin. In an exclusion process, any lattice site is either empty or occupied by at most one particle. Every particle carries an independent Poisson clock with rate 1; when the clock rings, the particle tries to jump to the right (never to the left, hence totally asymmetric) and succeeds in doing so if the site to the right is unoccupied. The source is an infinite supply of particles; it also carries a Poisson clock with rate $\alpha < 1$. The source inserts a particle if the source clock rings and the first lattice site is then empty.

The motivation for the study of this system is twofold: firstly, from the viewpoint of statistical mechanics, the boundary driven TASEP is a prototypical nonequilibrium system. In nonequilibrium, steady states generally exhibit long-range correlations [3]. This is likely to complicate a large deviation analysis, which can yield an effective description in terms of thermodynamics functionals. Secondly, asymmetric exclusion processes lead in the hydrodynamic scaling limit a conservation law, often called Burgers' equation [1]

$$\partial_t \rho(x,t) + \partial_x (f(\rho(x,t)) = 0, \tag{1}$$

with flux function $f(\rho) = \rho(1-\rho)$. More precisely, the empirical measure converges to the unique entropy solution of (1), with boundary data satisfying the Bardos-le Roux-Nédélec boundary conditions [2]. To illustrate the need of such boundary conditions, we simulate the reservoir as Burgers equation on the real line with initial data

$$\rho(x,0) = \begin{cases} \alpha & x \leq 0 \\ 0 & \text{else} \end{cases}.$$

Two very different scenarios can arise. Namely, for $\alpha < 1/2$, the characteristics intersect the half-line $\{(0,t)\ t > 0\}$ in the (x,t) plane in which we wish to prescribe the boundary data, thus the boundary data α is the appropriate choice. For $\alpha > 1/2$, however, this half line lies in a rarefaction fan, and therefore the prescription of suitable boundary data is nontrivial. The aforementioned theory of Bardos, Le Roux and Nédélec [2] addresses this problem. For the particular problem under consideration, one can see that boundary data chosen in $[1/2, 1]$ is admissible [2, 5].

Large deviations are often a powerful tool to describe the macroscopic behaviour of many, say n, particles. Here, let $X_n = \frac{1}{n}\sum_{k=1}^{n} \eta_k$ be the empirical density of a semi-infinite TASEP with injection rate $\alpha \in (0,1)$ at time infinity, starting with an empty lattice ($\eta_k \in \{0,1\}$ is the number of particles at site k). Then X_n satisfies, roughly speaking, a large deviation principle with rate function I if

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in A\} = -\inf_{x\in A} I(x),$$

so I characterises the degree of unlikeliness (with the minimisers of I being the only possible observable states in the limit $n \to \infty$).

Can we understand the different behaviour regarding boundary conditions depending on α from a large deviation argument? The answer is a cautious yes — currently there seems no pathwise large deviation argument for the driven TASEP on the half line to be available. Yet, for the average particle density in the limit $t \to \infty$, we are able to obtain the large deviation rate function. Namely, in the easy case $\alpha \leq \frac{1}{2}$, the rate function is

$$I(x) = x \log \frac{x}{\alpha} + (1-x) \log \frac{1-x}{1-\alpha};$$

this is a well-known result implying that the empirical density converges at α, as one would naïvely expect. For $\alpha > \frac{1}{2}$, the rate function is [6]

$$I(x) = \begin{cases} x \log \dfrac{x}{\alpha} + (1-x) \log \dfrac{1-x}{1-\alpha} + \log\left(4\alpha(1-\alpha)\right) & \text{if} \quad 0 \leq x \leq 1-\alpha, \\ 2\left[x \log x + (1-x)\log(1-x) + \log 2\right] & \text{if} \quad 1-\alpha < x \leq \frac{1}{2}, \\ x \log x + (1-x)\log(1-x) + \log 2 & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

This result can be interpreted as follows. For $\alpha \leq \frac{1}{2}$, the minimiser of I is α, in agreement with the prescribed boundary at $x = 0$ in this case. For $\alpha > \frac{1}{2}$, the minimiser of I is always $\frac{1}{2}$, which is in the range of admissible boundary data. Note that the rate function is non-analytic for $x = 1 - \alpha$, showing a phase transition occurs at this point.

A few words about the proof. The method relies on the so-called Matrix Product Ansatz (MPA). This method to study correlations has been developed in the seminal paper [4]; Großkinsky [7] has used it to characterise the stationary measure in this case (whose existence was shown by Liggett [8]). From these results, it is not difficult to show that the cumulant generating function is given by

$$\Lambda(\theta) = \lim_{n \to \infty} \tfrac{1}{n} \log w^T (e^\theta D + E)^n v - 2 \log 2$$

with D, E infinite matrices and v, w vectors characterised by the matrix product ansatz. The rate function I is in this case, by the Gärtner-Ellis theorem, the Legendre transform of Λ. We compute Λ by obtaining upper and lower bounds for the expression above, and showing that these bounds agree. The upper bound relies on a characterisation of the matrices involved (D, E and v, w are explicitly known), which allows us to determine the spectrum in a suitable weighted space, exploiting the spectral theory of Toeplitz operators. The lower bound is shown by expanding the expression for Λ and focusing on a few relevant coefficients.

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Deficit estimates for the Gaussian logarithmic Sobolev inequality

Max Fathi
(joint work with Emanuel Indrei and Michel Ledoux)

In recent years, the question of stability for functional inequalities has garnered much attention. The problem is as follows: given a functional inequality for which cases of equality exist and are explicitly know, if a function almost achieves equality, is it close in some sense to a case of equality? This brings us to trying to bound from below the deficit in the functional inequality by a function which measures how far we are from a case of equality. Typically, the lower bounds we are looking for involve powers of some metric. Examples of inequalities that have been studied include isoperimetric inequalities [6], Sobolev inequalities [3] and the Brunn-Minkowski inequality [5].

In this talk, I presented a few new results for the Gaussian logarithmic Sobolev inequality

$$\int f \log f d\gamma_n \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} d\gamma_n \tag{1}$$

where γ_n is the standard centered gaussian measure on $\mathbb{R}^n$, and f is any smooth probability density with respect to γ_n. The left-hand side of this inequality is known as the relative entropy of the probability measure $\nu = f\gamma_n$ with respect to γ_n, and usually denoted by $\mathrm{H}(\nu)$. The right-hand side (without the factor $1/2$) is known as the Fischer information $\mathrm{I}(\nu)$

The constant $1/2$ in (1) is optimal in every dimension, and a probability density f achieves equality if and only if it is of the form $f(x) = \exp(b \cdot x + |b|^2/2)$ for some $b \in \mathbb{R}^n$, i.e. if the probability measure $f\gamma_n$ is a non-centered standard gaussian (see [2]). In particular, the only centered probability measure achieving equality is the gaussian measure itself.

Our aim is to obtain lower bounds on the deficit

$$\delta_{\mathrm{LSI}}(\nu) = \frac{1}{2}\mathrm{I}(\nu) - \mathrm{H}(\nu)$$

in terms of transport (or Wasserstein) distances

$$W_{p,q}(\nu,\tilde{\nu}) = \inf_{\pi} \left(\int ||x-y||_q^p d\pi(x,y) \right)^{1/p}$$

where the infimum runs over all couplings π of the probability measures ν and $\tilde{\nu}$.

We obtained two types of results: under additional assumptions on ν, we obtain a lower bound in terms of $W_{2,2}(\nu,\gamma)$ that is dimension-free. For general measures, we obtain an estimate involving $W_{1,1}$, which however seems to behave badly in large dimensions. As of the time of writing, I am not aware of any completely general result that behaves well in large dimension.

Denote by $\mathcal{P}(\lambda)$ the class of probability measures ν on the Borel sets of $\mathbb{R}^n$ satisfying a Poincaré inequality with constant $\lambda > 0$ in the sense that for every smooth $g : \mathbb{R}^n \to \mathbb{R}$ such that $\int g d\nu = 0$,

$$\lambda \int g^2 d\nu \leq \int |\nabla g|^2 d\nu. \tag{2}$$

On $\mathcal{P}(\lambda)$, we established the following improved LSI:

Theorem 1. *For any centered ($\int x d\nu = 0$) probability measure $d\nu = f d\gamma$ in the class $\mathcal{P}(\lambda)$,*

$$\mathrm{H}(\nu) \leq \frac{c(\lambda)}{2}\, \mathrm{I}(\nu),$$

where

$$c(\lambda) = \frac{1-\lambda+\lambda\log\lambda}{(1-\lambda)^2} < 1 \qquad \big(c(1) = \tfrac{1}{2}\big).$$

This improves results previously obtained in [1] and [7]. The constant is sharp on $\mathcal{P}(\lambda)$, as can be seen when taking ν with density $f(x) = \sqrt{\lambda}\, e^{(1-\lambda)x^2/2}$, $\lambda > 0$, on the line. Of course, since the constant $1/2$ in the Gaussian LSI is optimal, such a strengthening can only be expected to hold on a subset of probability measures.

This inequality implies a lower bound on the deficit

$$\delta_{\mathrm{LSI}}(\nu) \geq \left(\frac{1-c(\lambda)}{2} \right) W_{2,2}(\nu,\gamma_n)^2 \tag{3}$$

whenever ν is a centered probability measure in $\mathcal{P}(\lambda)$.

It should be mentioned that one cannot expect

$$\delta_{\mathrm{LSI}}(\nu) \geq c\, \mathrm{W}_2(\nu,\gamma)^2$$

to hold for some $c > 0$ and all probability measures ν. Indeed, such an inequality combined with the HWI inequality of [8] would then imply the logarithmic Sobolev inequality $\mathrm{H}(\nu) \leq \frac{1+c}{2+4c}\, \mathrm{I}(\nu)$ with therefore a constant strictly better than the optimal $1/2$. A complete stability result for the Gaussian LSI therefore requires a distance weaker than W_2. This is the aim of the following result:

Theorem 2. *There is a numerical constant $c > 0$ such that for any centered probability measure $d\nu = f d\gamma$ on $\mathbb{R}^n$ with $f > 0$ locally bounded and positive entropy,*

$$\delta_{\mathrm{LSI}}(\nu) \geq \frac{c}{\mathrm{H}(\nu)} \min\left(\frac{\mathrm{W}_{1,1}(\nu,\gamma)^4}{n^2}, \frac{\mathrm{W}_{1,1}(\nu,\gamma)^2}{n}\right).$$

This result has the advantage of being valid for general measures. However, this estimate does not scale well for product measures in large dimension. For an n-dimensional product measure $\nu^n = \nu^{\otimes n}$, $\delta_{\mathrm{LSI}}(\nu^n) = n\delta_{\mathrm{LSI}}(\nu)$ grows linearly in n. This is also the behavior of

$$\frac{\mathrm{W}_{1,1}(\nu^n,\gamma^n)^2}{n} = n\,\mathrm{W}_{1,1}(\nu,\gamma^1)^2.$$

When $n >> \mathrm{W}_{1,1}(\nu,\gamma^1)^{-2}$, the expected growth is lost.

Both these results are stated for centered measures. However, we can obtain deficits for noncentered probability measures through the following transform: if $d\nu = f d\gamma$ has mean $b \in \mathbb{R}^n$, let

$$d\nu_b(x) = f(x+b)e^{-\left(b\cdot x + \frac{|b|^2}{2}\right)}\, d\gamma(x).$$

Then ν_b is a centered probability measure, and $\delta_{\mathrm{LSI}}(\nu_b) = \delta_{\mathrm{LSI}}(\nu)$. We can therefore apply the previous results to ν_b, and still recover deficit estimates for ν.

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Front propagation in stratified media

Matteo Novaga
(joint work with Annalisa Cesaroni and Cyrill Muratov)

Front propagation is a common phenomenon in nonlinear systems governed by reaction-diffusion mechanisms and their analogs, and arises in many applications, including phase transitions, combustion, chemical reactions, population dynamics, developmental biology, etc. By a *front*, one usually understands a narrow transition region in which the solution of the underlying reaction-diffusion equation changes abruptly between two equilibria. At the core of the phenomenon of propagation is the fact that such fronts may exhibit *wave-like* long-time behavior, whereby the level sets of the solution advance in space with some positive average velocity.

As a prototypical model, we consider the following Allen-Cahn equation in the presence of a heterogeneous forcing term:

$$u_t = \Delta u + \frac{f(u)}{\epsilon^2} + \frac{a(x,u)}{\epsilon}, \tag{1}$$

where $f = -W'$, W is a nonnegative double-well potential with zeroes in $u = 0, 1$, and a is such that $a(x, 0) = 0$. In particular, for ϵ small enough $u = 0$ is a stable equilibrium of (1).

As $\epsilon \to 0$ the solution u converges to the characteristic function of a set evolving by the forced men curvature flow

$$V(x) = \kappa(x) - \frac{g(x)}{c_W}, \tag{2}$$

where V is the (inner) normal velocity, κ is the sum of the principal curvatures, $g(x) = \int_0^1 a(x,u)du$, and $c_W = \int_0^1 \sqrt{2W(u)}du$. We shall consider equation (1) in an infinite cylinder $\Sigma = \Omega \times \mathbb{R}$, under the assumption that the function a depends only on the transverse coordinates of the cylinder. Our main interest is to characterize the speed and the shape of the long time limit of the fronts for (1) and their relation to those for (2).

We now state our assumptions on Ω, a and f:

A) $\partial\Omega$ is of class $C^{2,\alpha}$ for some $\alpha \in (0, 1]$;
B) $a \in C^{\alpha}_{loc}(\overline{\Omega} \times \mathbb{R})$, $a_u \in C^{\alpha}_{loc}(\overline{\Omega} \times \mathbb{R})$, $a(\cdot, 0) = 0$;
C) $f \in C^{1,\alpha}_{loc}(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $W(1) = W(0) = 0$, $W(u) > 0$ for all $u \neq 0, 1$, and $\liminf_{|u|\to\infty} W(u) > 0$;
D) there exists a set $A \subseteq \Omega$ such that

$$\int_A g(y)dy > c_W\, P(A, \Omega), \tag{3}$$

where $g(y) = \int_0^1 a(y,u)du$.

Assumption C states that $W(u)$ is a balanced non-degenerate double-well potential (as a model function one could think of $W(u) = \frac{1}{4}u^2(1-u)^2$). Assumption D ensures that the trivial state $u = 0$ is energetically less favorable for ϵ sufficiently

small, resulting in the existence of the invasion fronts. Notice that (3) implies, in particular, that $\sup_\Omega g > 0$, and is automatically satisfied if

$$\int_\Omega g(y)dy > 0.$$

Our methods are essentially variational. This stems from the basic observation [3, 5] that, when the nonlinearity (1) is translationally invariant along the cylinder axis, the solution of this equation in the reference frame moving with speed $c > 0$ along the cylinder may be viewed as a gradient flow in $L^2_c(\Sigma) = L^2(\Sigma_\epsilon; e^{cz}dx)$ generated by the exponentially weighted Ginzburg-Landau type functional

$$\Phi_c(u) = \int_\Sigma e^{cz}\left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{W(u)}{\epsilon} - A(y,u)\right)dx,$$

where $x = (y,z) \in \Omega \times \mathbb{R} = \Sigma$ and $A(y,u) = \int_0^u a(y,s)ds$. In particular, traveling wave solutions of (1) with speed c that belong to the Sobolev space $H^1_c(\Sigma)$, i.e., the space consisting of all functions in $L^2_c(\Sigma)$ with first derivatives in $L^2_c(\Sigma)$, are stationary points of Φ_c (see [4, 6]).

We extend the results of [6] on existence and uniqueness of traveling waves solutions to (1) with maximal propagation speed.

Theorem 1. *Under Assumptions A–D, there exist positive constants ϵ_0 and C, depending on f, a and Ω, with the property that for all $0 < \epsilon < \epsilon_0$ there exists a unique $c^\dagger_\epsilon > 0$ such that:*

(1) $\Phi^\epsilon_{c^\dagger_\epsilon}$ admits a non-trivial minimizer $\bar{u}_\epsilon \in H^1_{c^\dagger_\epsilon} \cap C^2(\overline{\Sigma})$.

(2) $\Phi^\epsilon_{c^\dagger_\epsilon}(\bar{u}_\epsilon) = 0$, and all non-trivial minimizers of $\Phi^\epsilon_{c^\dagger_\epsilon}$ are translates of $\bar{u}_\epsilon$ along z.

(3) $0 < \bar{u}_\epsilon \leq 1 + C\epsilon$, $(\bar{u}_\epsilon)_z < 0$ in $\overline{\Sigma}$, and

$$\lim_{z\to+\infty} \bar{u}_\epsilon(\cdot,z) = 0 \qquad \lim_{z\to-\infty} \bar{u}_\epsilon(\cdot,z) = v_\epsilon \qquad \text{in } C^1(\overline{\Omega}),$$

where v_ϵ is a stable critical point of

$$E^\epsilon(v) = \int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{W(u)}{\epsilon} - A(y,u)\right)dy,$$

with $E^\epsilon(v_\epsilon) < 0$.

We now introduce the geometric functional

$$\mathcal{F}_c(S) = c_W \int_{\partial^* S \cap \Sigma} e^{cz}\, d\mathcal{H}^{n-1}(x) - \int_S e^{cz}\, g(y)\, dx. \tag{4}$$

As before, travelling waves solution of (2) with speed c belonging to $L^2_c(\Sigma)$ are stationary points of $\mathcal{F}_c$ [2, 1].

We show existence and uniqueness of traveling wave solutions of (2) following the arguments developed in [2].

Theorem 2. *Under Assumptions A–D, there exists a unique $c^\dagger > 0$ such that:*

(1) There exists a function $\psi : \Omega \to [-\infty, \infty)$ such that $(c^\dagger, \psi)$ is a (generalized) traveling wave for the forced mean curvature flow and the set $S_\psi := \{(y, z) \in \Sigma \mid z < \psi(y)\}$ is a minimizer of $\mathcal{F}_{c^\dagger}$.
(2) Letting $\omega = \{\psi > -\infty\}$, the set $\omega \times \mathbb{R}$ is a minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations, and $\psi \in C^2(\omega)$.
(3) ψ is unique up to additive constants on every connected component of ω.
(4) $\partial\omega$ is a solution to the prescribed curvature problem $c_W \kappa = g$.

Following [1], we eventually show that traveling wave solutions of (1) converge to traveling wave solutions of (2) as $\epsilon \to 0$.

Theorem 3. *Let $c^\dagger_\epsilon$, $\bar{u}_\epsilon$ and v_ϵ be as in Theorem 1 and let $c^\dagger$ be as in Theorem 2. There holds*

$$\lim_{\epsilon \to 0} c^\dagger_\epsilon = c^\dagger.$$

Moreover, for every sequence $\epsilon_n \to 0$ there exist a subsequence (not relabeled) and an open set $S \subset \Sigma$ such that

$$\bar{u}_{\epsilon_n} \to \chi_S \qquad \text{in } L^1_{loc}(\Sigma),$$

where S is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$. Moreover,

$$\bar{u}_{\epsilon_n} \to \chi_S \qquad \text{locally uniformly on } \overline{\Sigma} \setminus \partial S,$$

and for every $\theta \in (0, 1)$ the level sets $\{\bar{u}_{\epsilon_n} = \theta\}$ converge to ∂S locally uniformly in the Hausdorff topology.

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The WIDE principle for evolution equations

Ulisse Stefanelli

We present a global-in-time variational approch to the Cauchy problem for the abstract evolution equation

$$\rho u'' + \partial D(u') + \partial E(u) \ni f, \quad u(0) = u_0, \quad \rho u'(0) = u_1. \tag{1}$$

The trajectory $t \in [0, T] \mapsto u(t) \in H$ takes values in the real Hilbert space H (with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$) and the prime denotes time differentiation. The *inertial* term $\rho u''$ is modulated by the nonnegative parameter $\rho \geq 0$ and we assume

to be given the *dissipation* $D : H \to \mathbb{R} \cup \{\infty\}$ and the *energy* $E : H \to \mathbb{R} \cup \{\infty\}$ functionals (the symbol ∂ stands for some suitable notion of gradient). Eventually, $f \in L^2(0,T;H)$ and $u_0, \rho u_1 \in H$ are suitably prescribed initial data.

Problem (1) may arise in connection with the variational formulation of different nonlinear evolution systems including parabolic equations and inequalities, also of degenerate or rate-independent type, semilinear waves, and classes of mixed hyperbolic-parabolic problems.

We coordinate to problem (1) the *Weighted Inertia-Dissipation-Energy* (WIDE) functional $W_\varepsilon : L^2(0,T;H) \to (-\infty, \infty]$ for $\varepsilon > 0$

$$W_\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{\varepsilon^2 \rho}{2} \|u''\|^2 + \varepsilon D(u') + E(u) - (f,u) \right) \mathrm{d}t$$

with domain $\{u \in L^2(0,T;H) \ : \ \rho\|u''\|^2,\, D(u'),\, E(u) \in L^1(0,T)\}$ and address the constrained minimization problem

$$\min\{W_\varepsilon(u) \ : \ u(0) = u_0,\ \rho u'(0) = \rho u_1\}. \tag{2}$$

The relation between minimization (2) and the differential problem (1) is revealed by (formally) computing the Euler-Lagrange system for W_ε which reads

$$\begin{aligned} &\varepsilon^2 \rho u'''' - 2\varepsilon \rho u''' + \rho u'' - \varepsilon \partial^2 D(u') u'' + \partial D(u') + \partial E(u) \ni f \\ &u(0) = u_0, \quad \rho u'(0) = \rho u_1, \quad \rho u''(T) = 0, \quad \varepsilon^2 \rho u'''(T) = \varepsilon \partial D(u'(T)). \end{aligned}$$

Hence, the minimization (2) of the WIDE functional W_ε corresponds to an elliptic-in-time regularization of problem (1). By letting u_ε denote a minimizer of W_ε, the variational resolution of (1) requires to ascertain that

$$u_\varepsilon \to u \quad \text{for some subsequence, where } u \text{ solves (1)} \tag{3}$$

where of course a suitable topology for the convergence has be prescribed. The interest in such a variational resolution relies on the nice structural properties of the functional W_ε. In particular, the WIDE approach (2)-(3) offers a natural frame for the analysis of evolution relaxation, may provide a selection criterion in case (1) shows nonuniqueness, and can be rather easily combined with approximations and discretizations.

The feasibility of the WIDE variational program (2)-(3) has been confirmed in a number of different situations. We provide here a schematic overview of the related literature.

In the case of **gradient flows** ($\rho = 0$ and $D(u') = \|u'\|^2/2$) the WIDE approach has to be traced back at least to Ilmanen [9] who used it in the mean-curvature context. Examples of relaxation related with micro-structure evolution are provided by Conti & Ortiz [7] and the general analysis in the λ-convex E case is in [16]. Still in the convex case Bögelein, Duzaar, & Marcellini [6] address the specific case of quasilinear parabolic equations in divergence form and [27] deals with the mean-curvature flow of cartesian surfaces and, more generally, linear-growth functionals. The non-λ-convex Hilbert-space case $E = \phi_1 - \phi_2$ where ϕ_i are both convex and ϕ_1 *dominates* ϕ_2 is discussed in [4]. The WIDE theory has

been also extended to the case of curves of maximal slope [5] for geodesically-λ-convex functionals in metric spaces in [20, 21], see also [23].

The case of parabolic **doubly-nonlinear** equations, corresponding indeed to a dissipation term of the form $D(u') = \|u'\|^p/p$ for $1 < p \neq 2$, has been tackled in the series of contributions [1, 2, 3]. The **rate-independent** case $p = 1$ has been discussed by Mielke & Ortiz [14], see also [15] for some refined time-discretization results. An application in the context of crack propagation is in Larsen, Ortiz, & Richardson [13].

Moving from Ilmanen's paper, De Giorgi conjectured in [8] that the WIDE variational approach could be implemented in the hyperbolic setting of **semilinear waves** as well, namely for $\rho > 0$, $D = 0$, $E = \int_\Omega (|\nabla u|^2 + |u|^p)\mathrm{d}x$ for $p > 1$. This conjecture has been first settled for finite times in [28] and then by Serra & Tilli [24] in the original infinite-time setting. Extensions to mixed hyperbolic-parabolic semilinear equations [11], lagrangian systems [10], and to some different classes of nonlinear energies are also available [25].

A functional close to WED (with ε fixed though) has been considered by Lucia, Muratov, & Novaga in connection with travelling waves in reaction-diffusion-advection problems [12, 17, 18, 19].

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Renormalized viscosity solution for a class of first order Hamilton-Jacobi equations in metric spaces

Jin Feng

We are interested in possibly singular and discontinuous first order Hamilton-Jacobi equation of isotropic type in a complete metric space with length property. An example of this kind is that Hamiltonian $H(x,p) = \frac{1}{2}p^2 + V(x)$ with V bounded from above but equals to $-\infty$ for a dense subset of the states. Situation like this arises from continuum mechanics, where the state variable lives in space of probability measures and the potential term models attractive Newtonian potential.

We introduce a notion of renormalized viscosity solution by re-defining derivatives using a new base metric that captures singularities in the Hamiltonian. We then establish a well posed-ness theory for a large class of elliptic Hamilton-Jacobi equations in such context.

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Discrete entropy methods for nonlinear diffusive evolution equations

Ansgar Jüngel

The concept of entropy plays a fundamental role in several fields like thermodynamics, kinetic theory, and statistics. In recent years, powerful tools based on entropy-dissipation methods for diffusive problems have been developed to understand their entropy structure and to prove the exponential convergence of the solutions to equilibrium with explicit and sometimes sharp convergence rates. Examples are the approach of Bakry-Emery and its extensions [1], the relation to functional inequalities [6], and the entropy construction method, based on systematic integration by parts [9]. The Bakry-Emery calculus is closely related to geometric properties of metric measure spaces, like geodesic λ-convexity [11] and Ricci curvature [14].

The extension of entropy methods to the discrete setting is of high importance for the design of novel structure-preserving numerical schemes and the understanding of discrete stochastic processes like finite Markov chains and finite graphs. The analysis in discrete spaces started only very recently. For instance, Bonciocat and Sturm [2] developed a notion of rough curvature bounds for discrete spaces, based on optimal mass transportation. Maas [12] presented a discrete counterpart of the Wasserstein gradient flow interpretation of the linear heat flow in the whole space by Jordan, Kinderlehrer, and Otto. Chow et al. [5] related the free energy, Fokker-Planck equations, and stochastic processes on finite graphs. Erbar and Maas [7] proved that the discrete porous-medium equation, where the Laplacian is the generator of a reversible continuous-time finite Markov chain, arises as the gradient flow of a discrete logarithmic entropy with respect to a nonlocal transportation metric. Similarly, Mielke [13] established geodesic λ-convexity of the logarithmic entropy for finite Markov chains. His results are closely related to the Bochner-Bakry-Emery approach of Caputo et al. [3]. Furihata and Matsuo [8] developed the Discrete Variational Derivative Method which allows for the derivation of structure-preserving numerical schemes based on gradient-flow type formulations.

The main interest in these works, however, is not the design of efficient numerical schemes (except [8]) and generally no results on the long-time behavior of the discrete solutions were stated. We have proven the following results:

(1) The time decay of **fully discrete implicit Euler finite-volume approximations** of porous-medium and fast-diffusion equations with Neumann or periodic boundary conditions is proved in the entropy sense [4]. The algebraic or exponential decay rates are computed explicitly. In particular, the numerical scheme dissipates all zeroth-order entropies which are dissipated by the continuous equation. The proofs are based on novel continuous and discrete generalized Beckner inequalities. Furthermore, the exponential decay of some first-order entropies is proved in the continuous and discrete case using systematic integration by parts. Numerical

experiments in one and two space dimensions illustrate the theoretical results and indicate that some restrictions on the parameters seem to be only technical.

(2) New **one-leg multistep time discretizations** of nonlinear evolution equations are investigated in [10]. The main features of the scheme are the preservation of the nonnegativity and the entropy-dissipation structure of the diffusive equations. The key ideas are to combine Dahlquist's G-stability theory with entropy-dissipation methods and to introduce a nonlinear transformation of variables which provides a quadratic structure in the equations. It is shown that G-stability of the one-leg scheme is sufficient to derive discrete entropy dissipation estimates. The general result is applied to a cross-diffusion system from population dynamics and a nonlinear fourth-order quantum diffusion model, for which the existence of semi-discrete weak solutions is proved. Under some assumptions on the operator of the evolution equation, the second-order convergence of solutions is shown. Moreover, some numerical experiments for the population model are presented, which underline the theoretical results.

(3) The dissipation of the discrete entropy for **implicit Runge-Kutta time approximations** of evolution equations was presented. The advantage, compared to the one-leg multistep approach, is that the entropy functional evaluated at the corresponding time step is dissipated and not the more complicated approximate entropy from [10]. The idea is to exploit the concavity property of the difference of the entropies at two consecutive time steps, as a function of the time step size. The computations can be performed using the method of systematic integration by parts of [9].

(4) Work in progress includes the derivation of **discrete Bakry-Emery methods** for nonlinear equations. The results of Mielke [13] can be interpreted as a discrete Bakry-Emery approach for the linear Fokker-Planck equation in one space dimension, discretized by a uniform finite-volume scheme. In collaboration with J.A. Carrillo and M.C. Santos, nonlinear equations like the porous-medium equation are investigated. First results seem to indicate that a generalization of Mielke's results is possible for certain (power-law type) nonlinearities. A second work in progress concerns the development of **higher-order minimizing movement schemes**. The idea is to replace the implicit Euler discretization by a one-leg multistep scheme. This leads to higher-order convergence rates which, for the moment, are verified only numerically in the case of second-order schemes in simple situations. Entropy-dissipating properties are proven in the Hilbert space setting for differential inclusions.

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Singular perturbations of infinite-dimensional gradient flows

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(joint work with Virginia Agostiniani and Giuseppe Savaré)

Let $\mathcal{H}$ be a (separable) Hilbert space and $\mathcal{E} : [0,T] \times \mathcal{H} \to \mathbb{R}$ a time-dependent energy functional. In this note, based on the forthcoming [2], we address the asymptotic analysis as $\varepsilon \downarrow 0$ of the gradient flow driven by $\mathcal{E}$, namely

$$\varepsilon u'(t) + \mathrm{D}_u \mathcal{E}(t, u(t)) = 0 \quad \text{in } \mathcal{H}, \quad \text{for a.a. } t \in (0,T). \tag{1}$$

We assume $\mathcal{E}$ bounded from below and $\mathcal{E}(\cdot, u)$ differentiable, with

$$\exists\, C_0^E,\, C_1^E > 0 \quad \forall\, (t,u) \in [0,T] \times \mathcal{H} : \quad |\partial_t \mathcal{E}(t,u)| \le C_E^0 \mathcal{E}(t,u) + C_E^1. \tag{2}$$

Hereafter we will in fact suppose that $\mathcal{E} \in \mathrm{C}^1([0,T] \times \mathcal{H})$ is Fréchet-differentiable on $\mathcal{H}$ and therefore understand D_u in (1) as the differential of $\mathcal{E}(t,\cdot)$. This will allow us to simplify the exposition, avoiding technicalities and only focusing on the intrinsic difficulties of the *singular perturbation* problem (1). Nonetheless, the analysis in [2] is carried out for nonsmooth energies $\mathcal{E}(t,\cdot)$.

Preliminary considerations. Under the further *coercivity* condition that

$$\text{the map } u \mapsto \mathcal{E}(t,u) \text{ has compact sublevels in } \mathcal{H}, \tag{3}$$

it is well known that for every fixed $\varepsilon > 0$ and for every $u_0 \in \mathcal{H}$ there exists at least a solution $u_\varepsilon \in H^1(0,T;\mathcal{H})$ to (1), fulfilling the Cauchy condition $u_\varepsilon(0) = u_0$. Testing (1) by u_ε', integrating in time, and exploiting the chain rule for $\mathcal{E}$, it is immediate to check that u_ε complies with the *energy identity*

$$\int_s^t \varepsilon |u_\varepsilon'(r)|^2 \,\mathrm{d}r + \mathcal{E}(t,u_\varepsilon(t)) = \mathcal{E}(s,u_\varepsilon(s)) + \int_s^t \partial_t \mathcal{E}(r,u_\varepsilon(r))\,\mathrm{d}r \tag{4}$$

for all $0 \le s \le t \le T$, from which all the a priori estimates on a family $(u_\varepsilon)_\varepsilon$ of solutions are deduced. More specifically, using (2), via the Gronwall Lemma we obtain: (a) The energy bound $\sup_{t\in(0,T)} \mathcal{E}(t,u_\varepsilon(t)) \le C$; (b) The estimate $\int_0^T \varepsilon |u_\varepsilon'|^2 \,\mathrm{d}t \le C'$, for positive constants C, $C' > 0$ independent of $\varepsilon > 0$. While (a) and (3) yield that there exists a *compact* set $K \subset \mathcal{H}$ s.t. $u_\varepsilon(t) \in K$ for all $t \in (0,T)$ and $\varepsilon > 0$, the *equicontinuity* estimate provided by (b) degenerates as $\varepsilon \downarrow 0$. Thus, no Arzelà-Ascoli type result applies to deduce compactness for $(u_\varepsilon)_\varepsilon$.

This major difficulty is not related to the present infinite-dimensional setting but also arises in finite dimension. Likewise, in finite and infinite dimension it can be circumvented by convexity arguments. For example, in $\mathcal{H} = \mathbb{R}^d$ it is possible to show that, if $\mathcal{E} \in \mathrm{C}^2([0,T]\times\mathbb{R}^d)$ and $\mathcal{E}(t,\cdot)$ is uniformly convex, then, starting from any $u_0 \in \mathbb{R}^d$ with $\mathrm{D}_u\mathcal{E}(0,u_0) = 0$ and $\mathrm{D}_u^2\mathcal{E}(0,u_0)$ positive definite, there exists a unique curve $u \in \mathrm{C}^1([0,T];\mathbb{R}^d)$ fulfilling $\mathrm{D}_u\mathcal{E}(t,u(t)) = 0$ for every $t \in [0,T]$, to which the *whole* family $(u_\varepsilon)_\varepsilon$ converge as $\varepsilon \downarrow 0$, uniformly on $[0,T]$.

Therefore, it is significant to focus on the case in which the energy $u \mapsto \mathcal{E}(t,u)$ is allowed to be *nonconvex* (but still supposed smooth for the sake of simplicity). In this context, two problems arise: (1) Prove that, up to the extraction of a subsequence, the gradient flows u_ε converge in $\mathcal{H}$ as $\varepsilon \downarrow 0$ to some limit curve u (pointwise a.e. in $(0,T)$); (2) Describe the evolution of u. Namely, we expect u to be a curve of critical points, jumping at the times $t \in (0,T)$ such that $\mathrm{D}_u^2\mathcal{E}(t,u_-(t)))$ is noninvertible, viz. $u_-(t)$ is a *degenerate* critical point for $\mathcal{E}(t,\cdot)$.

A finite-dimensional result. For the singular perturbation problem (1), a first answer to *(i)–(ii)* was provided in finite dimension (i.e. $\mathcal{H} = \mathbb{R}^d$) in [4], under the assumption that the energy $\mathcal{E} \in \mathrm{C}^3([0,T]\times\mathbb{R}^d)$ (a) has a *finite* number of degenerate critical points; (b) the vector field $\mathrm{F} := \mathrm{D}_u\mathcal{E}$ complies with the *transversality conditions* at every degenerate critical point; (c) a further technical condition. Hence, in [4, Thm. 3.7] it was shown that, starting from a well-prepared datum u_0, there exists a unique piecewise C^2-curve $u : [0,T] \to \mathbb{R}^d$ with a finite jump set $J = \{t_1, \dots, t_k\}$, such that: *(i)* $\mathrm{D}_u\mathcal{E}(t,u(t)) = 0$ with $\mathrm{D}_u^2\mathcal{E}(t,u(t))$ positive definite for all $t \in [t_i, t_{i-1})$ and $i = 1,\dots,k-1$; *(ii)* at $t_i \in J$ $u_-(t_i)$ is a degenerate critical point for $\mathcal{E}(t_i,\cdot)$ and there exists a unique curve $v \in \mathrm{C}^2(\mathbb{R};\mathbb{R}^d)$ such that $\lim_{s\to-\infty} v(s) = u_-(t_i)$, $\lim_{s\to+\infty} v(s) = u_+(t_i)$ and

$$v'(s) + \mathrm{D}_u\mathcal{E}(t_i,v(s)) = 0 \quad \text{for all } s \in \mathbb{R}; \tag{5}$$

(iii) the *whole* sequence $(u_\varepsilon)_\varepsilon$ converge to u uniformly on the compact sets of $[0,T]\backslash J$, and suitable rescalings of u_ε converge to v.

The fact that at each jump point t_i the unique heteroclinic v connecting $u_-(t_i)$ and $u_+(t_i)$ is a gradient flow of the energy $\mathcal{E}(t_i,\cdot)$ bears this mechanical interpretation: The internal scale of the system, neglected in the singular limit $\varepsilon \downarrow 0$, "takes over" and governs the dynamics in the jump regime, which can be in fact viewed as a fast transition between two metastable states.

The structure of the above statement reflects the line of its proof. First, the unique limit curve is *a priori* constructed via the Implicit Function Theorem, also resorting to the *transversality* conditions. Secondly, the convergence of $(u_\varepsilon)_\varepsilon$ is proved. In [2] we extend the result from [4] to the infinite-dimensional (Hilbertian) case. In this context, it is particularly meaningful to disconnect the analysis from smoothness conditions on the energy $\mathcal{E}(t,\cdot)$. Thus, the argument we develop is rather akin to the *variational* approach to (nonsmooth) gradient flows systematized in [1] and recently extended to rate-independent systems in [3]. In particular, first we prove the existence of a limit curve by suitable *compactness* techniques, and then we *variationally* characterize its fast dynamics at jumps.

Our results. The key observation is that (4) equivalently rewrites as

$$\int_s^t \left(\tfrac{\varepsilon}{2}|u_\varepsilon'(r)|^2+\tfrac{1}{2\varepsilon}|\mathrm{D}_u\mathcal{E}(r,u_\varepsilon(r))|^2\right)\,\mathrm{d}r+\mathcal{E}(t,u_\varepsilon(t)) = \mathcal{E}(s,u_\varepsilon(s))+\int_s^t \partial_t\mathcal{E}(r,u_\varepsilon(r))\,\mathrm{d}r$$

from which it is possible to deduce that

$$\int_0^T |u_\varepsilon'(r)||\mathrm{D}_u\mathcal{E}(r,u_\varepsilon(r))|\,\mathrm{d}r \le C. \tag{6}$$

Thus, while no (uniform w.r.t. $\varepsilon>0$) bounds are available on $|u_\varepsilon'|$, estimate (6) suggests that: (1) The limit of the *energy-dissipation integral* $\int_s^t |u_\varepsilon'(r)||\mathrm{D}_u\mathcal{E}(r,u_\varepsilon(r))|\,\mathrm{d}r$ will describe the dissipation of energy (at jumps) in the limit $\varepsilon\downarrow 0$; (2) To extract compactness information from (6), featuring the weight $|\mathrm{D}_u\mathcal{E}(\cdot,u_\varepsilon)|$, it is necessary to suppose that the (degenerate) critical points of $\mathcal{E}$, in whose neighborhood the weight tends to zero, are somehow "well separated" one from each other.

In fact, in [2] we require that for every $t\in[0,T]$ the critical set

$$\mathcal{C}(t) := \{u\in\mathcal{H}:\quad \mathrm{D}_u\mathcal{E}(t,u)=0\} \text{ consists of } \textit{isolated points}. \tag{7}$$

This allows us to prove the following

Theorem: *Let $\mathcal{E}\in \mathrm{C}^1([0,T]\times\mathcal{H};\mathbb{R})$ comply with* (2), (3), *and* (7). *Let $u_0\in\mathcal{H}$ and let $(u_\varepsilon)_\varepsilon$ solve* (1) *with $u_\varepsilon(0)=u_0$. Then, there exist a subsequence $\varepsilon_k\downarrow 0$, a positive Radon measure μ on $[0,T]$, and $u\in L^\infty(0,T;\mathcal{H})$ such that*

(i) *$u_{\varepsilon_k}(t)\to u(t)$ for all $t\in[0,T]$;*

(ii) *$\left(\frac{\varepsilon_k}{2}|u_{\varepsilon_k}'(r)|^2+\frac{1}{2\varepsilon_k}|\mathrm{D}_u\mathcal{E}(r,u_{\varepsilon_k}(r))|^2\right)\mathcal{L}^1 \rightharpoonup^* \mu$;*

(iii) *u and μ fulfill for a.a. $t\in(0,T)$ and a.a. $s\in(0,t)$*

$$-\mathrm{D}_u\mathcal{E}(t,u(t))=0, \tag{8}$$

$$\mu([s,t])+\mathcal{E}(t,u(t)) = \mathcal{E}(s,u(s))+\int_s^t \partial_t\mathcal{E}(r,u(r))\,\mathrm{d}r; \tag{9}$$

(iv) *the jump set J of u coincides with the set of atoms of μ, it is at most countable, and the jump relations*

$$\mu(\{t\}) = \mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) = c(t; u_-(t), u_+(t)) \quad \text{for all } t \in J \tag{10}$$

hold, with the cost $c(t; u_-(t), u_+(t))$ defined as the $\inf \int_0^1 |\theta'(r)||\mathrm{D}_u\mathcal{E}(t, \theta(r))|\,\mathrm{d}r$: $\theta(0) = u_-(t),\ \theta(1) = u_+(t),\ \theta \in \mathcal{A}\}$ *and $\mathcal{A}$ the class of admissible curves, obtained by gluing together a finite number of locally Lipschitz curves connecting points in the critical set $\mathcal{C}(t)$.*

In fact the inf in the definition of $c(t; u_-(t), u_+(t))$ is attained. Hence, from (10) we have at every $t \in J$ that $\mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) = \int_0^1 |\theta'(r)||\mathrm{D}_u\mathcal{E}(t, \theta(r))|\,\mathrm{d}r$ for any optimal jump transition θ. An argument based on the chain rule yields that every locally Lipschitz piece $\theta|_{(\mathsf{t}_i, \mathsf{t}_{i+1})}$ can be reparameterized to a curve $\tilde{\theta} : (\mathsf{s}_i, \mathsf{s}_{i+1}) \to \mathcal{H}$, with $(\mathsf{s}_i, \mathsf{s}_{i+1})$ possibly unbounded, solving the analogue of (5)

$$\tilde{\theta}'(\sigma) + \mathrm{D}_u\mathcal{E}(t, \tilde{\theta}(\sigma)) = 0 \quad \text{in } \mathcal{H}, \quad \text{for a.a. } \sigma \in (\mathsf{s}_i, \mathsf{s}_{i+1}). \tag{11}$$

Thus we retrieve the same mechanical interpretation as for the result in [4].

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Quasi-static crack growth in hydraulic fracture

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(joint work with Stefano Almi and Rodica Toader)

Hydraulic fracture studies the process of crack growth in rocks driven by the injection of high pressure fluids. The main use of hydraulic fracturing is the extraction of natural gas or oil. In these cases, a fluid at high pressure is pumped into a pre-existing fracture through a wellbore, causing the enlargement of the crack.

In the study of hydraulic fracture, all thermal and chemical effects are usually neglected and the fracturing stimulation is performed only by hydraulic forces, not by explosives, thus the inertial effects are negligible. This justifies the use of quasi-static models.

In our model we make the following general hypotheses:

- the rock fills the whole space $\mathbb{R}^3$ and has an initial crack, lying on a plane Σ passing through the origin;
- the rock is linearly elastic and fully impermeable;

- the crack can grow only within Σ;
- the fluid is assumed to be incompressible, and is pumped through the origin into the region between the crack lips;
- at every time t we know the total volume $V(t)$ of the fluid that has been pumped into the crack up to time t.

The mathematical problem is to show that, given the function $t \mapsto V(t)$, we can determine at each time the shape and size of the crack, as well as the fluid pressure $p(t)$.

For the precise formulation of the model we use the following notation:

- $\Gamma(t)$ is the *unknown crack* at time t in the reference configuration $\mathbb{R}^3$;
- $\mathbb{C}(x)\colon \mathbb{M}^{3\times 3}_{sym} \to \mathbb{M}^{3\times 3}_{sym}$ is the known *elasticity tensor* of the rock; it is linear, symmetric, and there exist two constants $0 < \alpha < \beta < +\infty$ such that $\alpha|\mathrm{F}|^2 \le \mathbb{C}(x)\mathrm{F}\cdot\mathrm{F} \le \beta|\mathrm{F}|^2$ for a.e. $x \in \mathbb{R}^3$ and every $\mathrm{F} \in \mathbb{M}^{3\times 3}_{sym}$, so that
$$\frac{1}{2}\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}Eu\cdot Eu\,dx$$
is the stored elastic energy of the rock corresponding to a displacement u, with strain $Eu := (\nabla u + \nabla u^t)/2$;
- $\nu_\Sigma := e_3$ is the upper unit normal vector to Σ and $[u]$ is the *jump* of the displacement u through Σ;
- at time t the *admissible displacements* u of the rock belong to the space $\mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma;\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3\setminus\Sigma;\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3\setminus\Sigma;\mathbb{M}^{3\times 3})\}$ (this incorporates the condition $u = 0$ at infinity in a weak sense), $[u] = 0$ on $\Sigma\setminus\Gamma(t)$ (continuity condition out of the crack), and $[u]\cdot\nu_\Sigma \ge 0$ on $\Gamma(t)$ (non-interpenetration condition on the crack);
- the approximate *volume of the cavity* of the crack determined by an admissible displacement u is $\int_\Sigma [u]\cdot\nu_\Sigma\,d\mathcal{H}^2 = \int_{\Gamma(t)}[u]\cdot\nu_\Sigma\,d\mathcal{H}^2$, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

Given the crack $\Gamma(t) \subset \Sigma$ at time t, the *equilibrium condition* for the rock implies that the displacement $u(t)$ of the rock is the unique minimizer of the elastic energy $\frac{1}{2}\int_{\mathbb{R}^3\setminus\Sigma}\mathbb{C}Eu\cdot Eu\,dx$ among all displacements $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma;\mathbb{R}^3)$ such that $[u] = 0$ on $\Sigma\setminus\Gamma(t)$, $[u]\cdot\nu_\Sigma \ge 0$ on $\Gamma(t)$, and $V(t) \le \int_\Sigma [u]\cdot\nu_\Sigma\,d\mathcal{H}^2$.

- It is possible to prove that $u(t)$ satisfies $V(t) = \int_\Sigma [u(t)]\cdot\nu_\Sigma\,d\mathcal{H}^2$, hence there is no dry region near the crack edge.
- By minimality, we have $-\mathrm{div}(\mathbb{C}Eu(t)) = 0$ on $\mathbb{R}^3\setminus\Gamma(t)$ (*elasticity equation*) and $-(\mathbb{C}Eu(t))\nu_\Sigma = \pm p(t)\nu_\Sigma$ on $\Gamma(t)^\pm$, where $p(t) \ge 0$ is a constant (interpreted as the *pressure* of the fluid), $\Gamma(t)^+$ is the upper face of $\Gamma(t)$ and $\Gamma(t)^-$ is the lower face.

According to *Griffith's theory*, the *energy dissipated* to produce a crack Γ is given by $\kappa\mathcal{H}^2(\Gamma)$, where the constant $\kappa > 0$ is the *toughness* of the rock. The *total* (elastic + dissipated) *energy* corresponding to a displacement u with crack Γ is defined by $\mathcal{E}(u,\Gamma) := \frac{1}{2}\int_{\mathbb{R}^3\setminus\Sigma}\mathbb{C}Eu\cdot Eu\,dx + \kappa\mathcal{H}^2(\Gamma)$. Given a crack Γ and a time t, the *reduced energy* $\mathcal{E}_{min}(\Gamma, V(t))$ is defined as the minimum value of the total

energy $\mathcal{E}(u,\Gamma)$ among all displacements $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\backslash\Sigma;\mathbb{R}^3)$ such that $[u]=0$ on $\Sigma\backslash\Gamma(t)$, $[u]\cdot\nu_\Sigma \geq 0$ on $\Gamma(t)$, and $V(t) \leq \int_\Sigma [u]\cdot\nu_\Sigma \, d\mathcal{H}^2$. We can now define the notion of quasistatic evolution in our context.

Let $T>0$ and $V \in AC([0,T];[0,+\infty))$. We say that $t \mapsto \Gamma(t)$ is an *irreversible quasi-static evolution* of the *hydraulic crack problem* if it satisfies the following conditions:

- *irreversibility*: $\Gamma(\cdot)$ is *increasing*, *i.e.*, $\Gamma(s) \subset \Gamma(t)$ for $0 \leq s \leq t \leq T$;
- *global stability*: $\mathcal{E}_{min}(\Gamma(t),V(t)) \leq \mathcal{E}_{min}(\Gamma,V(t))$ (Griffith's condition) for every $t \in [0,T]$ and every $\Gamma \geq \Gamma(t)$;
- *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_{min}(\Gamma(t),V(t))$ is *absolutely continuous* on the interval $[0,T]$ and $\frac{d}{dt}\mathcal{E}_{min}(\Gamma(t),V(t)) = p(t)\dot{V}(t)$ for almost every $t \in [0,T]$, where $p(t)$ is the pressure introduced above.

Penny-shaped cracks. Assume that the initial crack Γ_0 is a disk centred at the origin and that the rock is *homogeneous* and *isotropic*. Then, by symmetry, it is natural to consider only circular cracks. One can introduce the definition of *irreversible quasi-static evolution* of the *penny-shaped hydraulic crack problem* by assuming that all cracks Γ have the form $B_R := \{(x_1,x_2,0) : x_1^2+x_2^2 \leq R^2\}$.

We proved the following result (see [1]). Let $V \in AC([0,T];[0,+\infty))$ and $R_0>0$. Assume that $\mathcal{E}_{min}(B_{R_0},V(0)) \leq \mathcal{E}_{min}(B_R,V(0))$ for every $R \geq R_0$ (stability at time $t=0$). Then there exists a unique irreversible quasi-static evolution of the penny-shaped hydraulic crack problem, with $\Gamma(0)=B_{R_0}$. It is given explicitly by $\Gamma(t)=B_{R(t)}$, with $R(t)=\max\left\{R_0,\left(\frac{3E_1}{2\kappa\pi}\right)^{1/5}V_*^{2/5}(t)\right\}$, where E_1 is the elastic energy of the displacement corresponding to the crack B_1 and to the volume 1, while $V_*(t)$ is the smallest monotone increasing function which is greater than or equal to $V(t)$, *i.e.*, $V_*(t)=\max\limits_{0\leq s\leq t} V(s)$.

The proof relies on the fact that the function $R \mapsto \mathcal{E}_{min}(B_R,V)$ is convex.

More general cracks. Given $r>0$, we introduce a class $\mathrm{Adm}_r(\Sigma)$ of admissible cracks, whose regularity depends on the parameter r. More precisely, we say that $\Gamma \in \mathrm{Adm}_r(\Sigma)$ if it satisfies:

- $0 \in \Gamma$;
- Γ is a *compact* and *connected* subset of Σ;
- *inner ball condition*: for every $x \in \partial\Gamma$ there exists $y \in \Gamma$ such that $x \in \partial B_r(y)$ and $B_r(y) \subseteq \Gamma$.

One can introduce the definition of *irreversible quasi-static evolution* of the *hydraulic crack problem in* $\mathrm{Adm}_r(\Sigma)$ by assuming that all cracks Γ belong to $\mathrm{Adm}_r(\Sigma)$. In this more general setting we proved the following existence result (see [1]). Let $r>0$, $V \in AC([0,T],[0,+\infty))$, and $\Gamma_0 \in Adm_r(\Sigma)$. Assume that $\mathcal{E}_{min}(\Gamma_0,V(0)) \leq \mathcal{E}_{min}(\Gamma,V(0))$ for every $\Gamma \in \mathrm{Adm}_r(\Sigma)$ such that $\Gamma \supseteq \Gamma_0$ (stability at time $t=0$). Then there exists an *irreversible quasi-static evolution* $t \mapsto \Gamma(t)$ of the *hydraulic crack problem in* $\mathrm{Adm}_r(\Sigma)$, with $\Gamma(0)=\Gamma_0$.

In the proof we use the following properties of the class $\mathrm{Adm}_r(\Sigma)$:

- $\Gamma = \overline{\mathring{\Gamma}}$ for every $\Gamma \in \mathrm{Adm}_r(\Sigma)$;
- $\Gamma_1, \Gamma_2 \in \mathrm{Adm}_r(\Sigma) \Longrightarrow \Gamma_1 \cup \Gamma_2 \in Adm_r(\Sigma)$;
- $\Gamma \in \mathrm{Adm}_r(\Sigma) \Longrightarrow \mathrm{diam}(\Gamma) \leq \frac{8}{\pi r}\mathcal{H}^2(\Gamma) + r$;
- if Γ_k is a sequence in $\mathrm{Adm}_r(\Sigma)$ which converges to Γ in the Hausdorff metric, then $\Gamma \in \mathrm{Adm}_r(\Sigma)$ and $\mathcal{H}^2(\Gamma_k) \to \mathcal{H}^2(\Gamma)$;
- let Γ_0, Γ_k, $\Gamma_\infty \in \mathrm{Adm}_r(\Sigma)$ be such that $\Gamma_0 \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric, and let $V_k, V_\infty \geq 0$ with $V_k \to V_\infty$; if $\mathcal{E}_{min}(\Gamma_k, V_k) \leq \mathcal{E}_{min}(\Gamma, V_k)$ for every $\Gamma \in \mathrm{Adm}_r(\Sigma)$ with $\Gamma \supseteq \Gamma_k$, then $\mathcal{E}_{min}(\Gamma_\infty, V_\infty) \leq \mathcal{E}_{min}(\Gamma, V_\infty)$ for every $\Gamma \in \mathrm{Adm}_r(\Sigma)$ with $\Gamma \supseteq \Gamma_\infty$; moreover, if u_k and u_∞ are the solutions corresponding to (Γ_k, V_k) and $(\Gamma_\infty, V_\infty)$, and p_k and p_∞ are the corresponding pressures, then $u_k \to u_\infty$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \backslash \Sigma)$, $p_k \to p_\infty$, and $\mathcal{E}_{min}(\Gamma_k, V_k) \to \mathcal{E}_{min}(\Gamma_\infty, V_\infty)$.

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Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems

Julian Fischer

Reaction-diffusion equations with mass-action kinetics occur in the mathematical modeling of many phenomena, e.g. in the modeling of chemical reactions or in drift-diffusion models for semiconductors. Consider a single reversible chemical reaction of the form

$$\alpha_1 \mathcal{A}_1 + \ldots + \alpha_S \mathcal{A}_S \rightleftharpoons \beta_1 \mathcal{A}_1 + \ldots + \beta_S \mathcal{A}_S$$

(with $\alpha_i, \beta_i \in \mathbb{N}_0$), where the $\mathcal{A}_i$ denote the different chemical species (i.e. the different types of molecules; e.g. $\mathcal{A}_1$ could be H_2O or O_2). Introduce the notation u_i for the concentration of the chemical species $\mathcal{A}_i$. The simplest corresponding reaction-diffusion equation with mass-action kinetics is then given by

$$\frac{d}{dt}u_i = a_i \Delta u_i + \underbrace{(\beta_i - \alpha_i)\left(c_F \prod_{k=1}^{S} u_k^{\alpha_k} - c_B \prod_{k=1}^{S} u_k^{\beta_k}\right)}_{=:R_i(u)} \quad \forall i \in \{1, \ldots, S\}, \tag{1}$$

where the $a_i > 0$ denote the species-dependent diffusion coefficients and where $c_F, c_B > 0$ denote reaction constants. For this equation, the entropy estimate

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^{S} (\mu_i - 1 + \log u_i) u_i \, dx \leq -c \int_\Omega \sum_{i=1}^{S} |\nabla \sqrt{u_i}|^2 \, dx \tag{2}$$

(which is a consequence of the particular structure of the reaction terms $R_i(u)$ and which is valid for classical solutions e.g. in case of homogeneous Neumann boundary conditions and for appropriately chosen $\mu_i \in \mathbb{R}$, $c > 0$) prevents global

blowup of solutions. Note that formally, the reaction-diffusion equation (1) may even be written as a gradient flow of the entropy [7].

Despite this absence of global blowup, even for the simple reaction-diffusion equation (1) global existence of any kind of solution in general has been an open problem, even for smooth initial data and bounded smooth domains. The key issue in establishing global existence of solutions is to give a meaning to the reaction terms $R_i(u)$: besides the entropy estimate, already on a formal level only an $L^2(\Omega \times [0,T])$ estimate is available for solutions [9]; however, in general higher powers of the solution occur in the reaction terms $R_i(u)$. Thus, *a priori* it is not known whether the reaction terms in (1) even define a distribution. For this reason, previous existence results have been limited to reactions of low degree, i.e. (very) small values of $\sum \alpha_k$ or $\sum \beta_k$, or almost coinciding diffusion coefficients, see e.g. [1, 4, 6, 8] and the references therein.

In the recent paper [5], we propose a notion of renormalized solutions for reaction-diffusion equations of the form

$$\frac{d}{dt}u_i = \nabla \cdot (A_i \nabla u_i) - \nabla \cdot (u_i \vec{b}_i) + R_i(u) \quad \forall i \in \{1, \dots, S\}$$

and establish global existence of solutions. For the reaction rates $R_i(\cdot)$, besides local Lipschitz continuity and the natural condition $R_i(v) \geq 0$ in case $v_i = 0$ (a chemical species that is not present may not be consumed by reactions) our only assumption is that the entropy condition

$$\sum_{i=1}^{S} R_i(v)(\log v_i + \mu_i) \leq 0 \quad \forall v \in (\mathbb{R}_0^+)^S$$

holds for appropriate $\mu_i \in \mathbb{R}$. We would like to emphasize that this entropy condition is satisfied for all reversible reactions with mass-action kinetics; furthermore, it holds for all systems of reversible reactions with mass-action kinetics that are subject to the so-called condition of detailed balance; see e.g. [10].

In general, the concept of *renormalized solutions* for a partial differential equation (introduced by DiPerna and Lions [2, 3]) imposes an evolution equation for nonlinear functions $\xi(u)$ of the actual solution u, whenever ξ belongs to some suitable class of functions; more precisely, $\xi(u)$ is required to satisfy the equation deduced from the original PDE by a formal application of the chain rule. In case of the simplified equation (1), our definition of renormalized solutions consists of requiring that the equation

$$\begin{aligned} \int_\Omega \xi(u)\psi \, dx \Big|_0^T &= -\int_0^T \int_\Omega \sum_{i,j=1}^{S} \psi a_i \partial_i \partial_j \xi(u) \nabla u_i \cdot \nabla u_j \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \sum_{i=1}^{S} a_i \partial_i \xi(u) \nabla u_i \cdot \nabla \psi \, dx \, dt \\ &\quad + \int_0^T \int_\Omega \sum_{i=1}^{S} \partial_i \xi(u) R_i(u) \psi \, dx \, dt \end{aligned} \tag{3}$$

must be satisfied for any $\xi \in C^\infty((\mathbb{R}_0^+)^S)$ with compactly supported derivatives, any test function $\psi \in W^{1,\infty}(\Omega)$, and any $T > 0$. Furthermore, we impose the regularity constraints $u_i \in L^\infty([0,T]; L^1(\Omega; \mathbb{R}_0^+))$ and $\sqrt{u_i} \in L^2([0,T]; H^1(\Omega))$.

In the construction of such renormalized solutions, two major difficulties occur: first, for the family of solutions u^ε to the regularized problems

$$\frac{d}{dt}u_i^\varepsilon = a_i \Delta u_i^\varepsilon + \frac{R_i(u^\varepsilon)}{1+\varepsilon|R(u^\varepsilon)|} \quad \forall i \in \{1,\dots,S\}, \tag{4}$$

compactness in $L^1(\Omega \times [0,T])$ is not immediate: a direct application of the Aubin-Lions Lemma is not possible due to the potential failure of the uniform boundedness of $\frac{d}{dt}u^\varepsilon$ in $L^1([0,T]; (H^m(\Omega))')$ for all m. Introducing a family φ_i^E of multi-variable truncations of the map $v \mapsto v_i$ – the truncation occurring whenever $\sum_{k=1}^S v_k$ exceeds E –, this problem can be overcome by applying the Aubin-Lions Lemma to the family $\varphi_i^E(u^\varepsilon)$ instead.

The key difficulty in the construction of renormalized solutions, however, is the lack of compactness in $L^2(\Omega \times [0,T])$ of the spatial derivatives ∇u^ε of the approximating sequence. Only *weak* convergence of $\nabla\sqrt{u_i^\varepsilon}$ in $L^2(\Omega \times [0,T])$ is implied by the known mathematical energy estimates (which basically consist just of the entropy estimate (2)). The functions u^ε can be shown to satisfy a renormalized formulation of the regularized equation (4) which is completely analogous to (3). However, in the limit $\varepsilon \to 0$ convergence of the first term on the right-hand side of the ε-regularized version of (3) cannot be established directly for our approximating sequence u^ε, as this term contains a product of spatial derivatives.

Instead, we resort to an alternative strategy: we pass to the limit $\varepsilon \to 0$ in the equation satisfied by $\varphi_i^E(u^\varepsilon)$; in this limit, only the oscillations of the sequence $\nabla\sqrt{u_j^\varepsilon}$ on the set $\{E \le \sum_{k=1}^S u_k^\varepsilon \le 2E\}$ may cause a convergence defect in our equation, as only in this range $\varphi_i^E(\cdot)$ is non-affine. From the resulting equation for $\varphi_i^E(u)$ which is only satisfied up to this convergence defect, an equation for $\xi(\varphi^E(u))$ which also only holds up to this convergence defect may be derived by a generalized chain rule. Finally passing to the limit $E \to \infty$, an exact equation for $\xi(u)$ is derived by showing that the convergence defect vanishes in the limit $E \to \infty$. The latter issue is the most delicate part of the proof. It is accomplished by a Fatou-type estimate of the form

$$\sum_{\hat{E}=0}^{\infty} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \chi_{\{\hat{E} \le \sum_{k=1}^S u_k^\varepsilon < \hat{E}+1\}} |\nabla\sqrt{u_i^\varepsilon}|^2 \, dx \, dt \le \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega |\nabla\sqrt{u_i^\varepsilon}|^2 \, dx \, dt$$

(the expression on the right-hand side being finite due to the entropy estimate) which entails the desired decay of oscillations as $E \to \infty$

$$\lim_{E\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \chi_{\{E \le \sum_{k=1}^S u_k^\varepsilon \le 2E\}} |\nabla\sqrt{u_i^\varepsilon}|^2 \, dx \, dt = 0.$$

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Abstract thermodynamics on duals to unital Banach algebras

Tomáš Roubíček

The initial-boundary-value problem for the scalar parabolic equation

$$\dot\vartheta - \Delta\vartheta = f, \qquad \vartheta|_{t=0} = \vartheta_0, \qquad \frac{\partial\vartheta}{\partial\nu} = 0 \text{ on } I\times\partial\Omega$$

on a domain $\Omega\subset\mathbb{R}^d$ and $I=[0,T]$ allows typically for various tests, namely by ϑ or by $\dot\vartheta \equiv \frac{\partial\vartheta}{\partial t}$, or after differentiation in time again by $\dot\vartheta$ or by $\ddot\vartheta$ etc. Such tests always rely on Banach or Hilbert-space structures of underlying function spaces, reflecting its interpretation as a gradient flow governed by the $-\Delta$ operator.

If interpreted as a heat-transfer equation, the natural test leading to an energy balance is however by a constant function 1. The corresponding a-priori L^1-estimate then relies on non-negativity of the solution and on the natural L^1-bounds on the heat source $f\ge 0$ and $\vartheta_0\ge 0$. It reveals that, as a minimal requirement, one needs the ordering and the unity, and the multiplication by unity at disposal.

All this is available at *commutative Banach algebras with unity*, which are embedded into their duals through the Gelfand-triple construction. More specifically, we may use a theory of C*-algebras (based on complex fields and playing still with an involution operator): any C*-algebra has a *Gelfand representation* [3], being homeomorphic with the space of complex-valued continuous functions on a *compact* set $\mathfrak{K}$ consisting of all nontrivial multiplicative functionals on this algebra, equipped with the weak* topology of its dual. The subset of all its selfadjoint elements, let us denote it $\boldsymbol{\Theta}$, is a commutative Banach algebras with unity (called a *unital Banach algebra*) which is homeomorphic with $C(\mathfrak{K})$, the real-valued continuous functions on $\mathfrak{K}$. Ordering of $C(\mathfrak{K})$ by the cone of non-negative functions is then the natural ordering of $\boldsymbol{\Theta}$, and induces also the dual ordering of the dual

space $\boldsymbol{\Theta}^*$ which is isomorphic to the space of Borel measures $\mathrm{Meas}(\mathfrak{K}) \cong C(\mathfrak{K})^*$ on $\mathfrak{K}$. In applications, $\mathfrak{K}$ occurs very explicitly and is typically metrizable and $\boldsymbol{\Theta}$ is separable.

Further, we put $\boldsymbol{\Sigma} = L^2(\mathfrak{K}) :=$ a completion of $C(\mathfrak{K})$ with respect to the norm $\|\boldsymbol{\vartheta}\|_{\boldsymbol{\Sigma}} := (\int_{\mathfrak{K}} \boldsymbol{\vartheta}^2 \,\mathrm{d}x)^{1/2}$ where $\mathrm{d}x$ denotes the measure induced by the constant function 1. We assume $L^2(\mathfrak{K}) \cong L^2(\mathfrak{K})^*$, which unfortunately does not seem to be automatic and is indeed in a position of an assumption, although easily to be satisfied in particular applications we have in mind. Altogether, we thus have

$$\overset{\text{a (real) commutative Banach algebra}}{\searrow} \quad \overset{\text{the unity}}{\nearrow} \mathbf{1} \in \underbrace{\boldsymbol{\Theta} \subset \boldsymbol{\Sigma} \cong \boldsymbol{\Sigma}^* \subset \boldsymbol{\Theta}^*}_{\text{a Gelfand triple}} \quad \overset{\text{an ordered Banach space}}{\swarrow}. \tag{1}$$

Beside the ordering on $\boldsymbol{\Theta}^* \cong \mathrm{Meas}(\mathfrak{K})$, we have at disposal the standard Jordan decomposition of measures and their standard total variation.

The abstract parabolic-like equation is then formulated on such a dual. To define the abstract "heat" flux, we need still an abstract construction of a linear *gradient*-like *operator*: we assume that there exists a family $\{\boldsymbol{L}_p\}_{1\le p\le\infty}$ of Banach spaces with $\boldsymbol{L}_{p'} = \boldsymbol{L}_p^*$ for any $1 \le p < \infty$ and $G \in \mathrm{Lin}(\boldsymbol{\Theta}^*, \boldsymbol{L}_1)$ and define the *abstract Sobolev space* $\boldsymbol{W}_p := \{\boldsymbol{\vartheta} \in \boldsymbol{\Theta}^*;\ G\boldsymbol{\vartheta} \in \boldsymbol{L}_p\}$ with the norm $\|\boldsymbol{\vartheta}\|_{\boldsymbol{W}_p} := \|\boldsymbol{\vartheta}\|_{\boldsymbol{\Theta}^*} + \|G\boldsymbol{\vartheta}\|_{\boldsymbol{L}_p}$, and assume that $G|_{\boldsymbol{W}_p} \in \mathrm{Lin}(\boldsymbol{W}_p, \boldsymbol{L}_p)$ for any $1 \le p \le \infty$, that $\boldsymbol{W}_p \subset \boldsymbol{\Sigma}$ for any $2 \le p \le \infty$, and $\boldsymbol{W}_\infty \subset \boldsymbol{\Theta}$ densely. We will further need a compatibility of the Jordan-type decomposition with $\boldsymbol{W}_2$ in the sense that $\boldsymbol{\vartheta}^+, \boldsymbol{\vartheta}^- \in \boldsymbol{W}_2$ whenever $\boldsymbol{\vartheta} \in \boldsymbol{W}_2$ and a compatibility of G with the Banach-algebra structure $G\mathbf{1} = 0$. The Gelfand triple (1) is thus expanded to the chain

$$\boldsymbol{W}_\infty \subset \boldsymbol{\Theta} \subset \boldsymbol{\Sigma} \subset \boldsymbol{\Theta}^* \subset \boldsymbol{W}_\infty^*. \tag{2}$$

Instead of $[G|_{\boldsymbol{W}_1}]^* : \boldsymbol{L}_1^* = \boldsymbol{L}_\infty \to \boldsymbol{W}_1^*$, we consider the "adjoint gradient" rather as $\mathrm{Div} := -[G|_{\boldsymbol{L}_\infty}]^*|_{\boldsymbol{L}_1} \in \mathrm{Lin}(\boldsymbol{L}_1, \boldsymbol{W}_\infty^*)$.

A certain prototype of the chain (2) is $W^{1,\infty}(\Omega) \subset C(\bar\Omega) \subset L^2(\Omega) \subset \mathrm{Meas}(\bar\Omega) \subset W^{1,\infty}(\Omega)^*$ and $\boldsymbol{L}_p = L^p(\Omega;\mathbb{R}^d)$ with the operators $G = \nabla$ and $\mathrm{Div}=\mathrm{div}$.

Further, we consider an abstract positive-definite "heat-conductivity" operator $K \in \mathrm{Lin}(\boldsymbol{L}_1, \boldsymbol{L}_1)$ and assume also $K|_{\boldsymbol{L}_2} \in \mathrm{Lin}(\boldsymbol{L}_2, \boldsymbol{L}_2)$. Thus the bilinear form $(\mathbf{g}_1, \mathbf{g}_2) \mapsto {}_{\boldsymbol{L}_2}\langle K\mathbf{g}_1, \mathbf{g}_2\rangle_{\boldsymbol{L}_2}$ admits a continuous extension $(\mathbf{g}_1, \mathbf{g}_2) \mapsto {}_{\boldsymbol{L}_1}\langle K\mathbf{g}_1, \mathbf{g}_2\rangle_{\boldsymbol{L}_\infty}$. Then, $\mathrm{Div}(K\mathbf{g}) \in \boldsymbol{W}_\infty^*$ for $\mathbf{g} \in \boldsymbol{L}_1$ is defined by the abstract Green-like formula

$$\forall \mathbf{v} \in \boldsymbol{W}_\infty : \qquad {}_{\boldsymbol{W}_\infty^*}\big\langle \mathrm{Div}(K\mathbf{g}), \mathbf{v}\big\rangle_{\boldsymbol{W}_\infty} := {}_{\boldsymbol{L}_1}\big\langle -K\mathbf{g}, G\mathbf{v}\big\rangle_{\boldsymbol{L}_\infty}.$$

For an external enthalpy source $\mathbf{h}_{\mathrm{ext}} = \mathbf{h}_{\mathrm{ext}}(t)$ valued in $\boldsymbol{\Theta}^*$ we will deal with an initial-value problem for an *abstract heat-transfer equation*:

$$\dot{\boldsymbol{\vartheta}} - \mathrm{Div}\big(KG\boldsymbol{\vartheta}\big) = \mathbf{h}_{\mathrm{ext}}\,, \qquad \boldsymbol{\vartheta}_{t=0} = \boldsymbol{\vartheta}_0, \tag{3}$$

with $\dot{\boldsymbol{\vartheta}}$ denoting the time derivative assumed valued in $\boldsymbol{\Theta}^*$. Approximation by time discretisation ($\tau > 0$ a time step, T/τ integer) leads to:

$$\frac{\boldsymbol{\vartheta}_\tau^k - \boldsymbol{\vartheta}_\tau^{k-1}}{\tau} - \mathrm{Div}\big(KG\boldsymbol{\vartheta}_\tau^k\big) = (\mathbf{h}_{\mathrm{ext}})_\tau^k =: \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \mathbf{h}_{\mathrm{ext},\tau}(t)\,\mathrm{d}t$$

for $k = 1, ..., T/\tau$, with $\boldsymbol{\vartheta}_\tau^0 = \boldsymbol{\vartheta}_0$, and $\mathbf{h}_{\mathrm{ext},\tau}$ a regularization of $\mathbf{h}_{\mathrm{ext}}$ valued in $\boldsymbol{W}_2^*$. Existence of a non-negative $\boldsymbol{\vartheta}_\tau^k \in \boldsymbol{W}_2$ follows by standard monotone-operator technique, assuming $K \in \mathrm{Lin}(\boldsymbol{L}_2, \boldsymbol{L}_2)$ positive definite $\mathbf{h}_{\mathrm{ext}}(t) \geq \mathbf{0}$, $\boldsymbol{\vartheta}_0 \geq \mathbf{0}$.

By $\boldsymbol{\vartheta}_\tau$ and $\overline{\boldsymbol{\vartheta}}_\tau$ we denote the piecewise affine continuous and the piecewise constant left-continuous interpolants, respectively. It is then possible to prove the convergence of the approximate solutions

$$\overline{\boldsymbol{\vartheta}}_\tau \to \boldsymbol{\vartheta} \quad \text{weakly}^* \text{ in } L^\infty_{\mathrm{w}*}(I;\boldsymbol{\Theta}^*) \cong L^1(I;\boldsymbol{\Theta})^*$$

If $K = K(\boldsymbol{\vartheta})$ or for some coupled problems, we need some estimates of the (abstract) gradient $G\overline{\boldsymbol{\vartheta}}_\tau$. To this goal, one can formulate an abstract condition

$$\left.\begin{array}{l}\big(\boldsymbol{\vartheta}_\tau\big)_{\tau>0} \quad \text{bounded in } L^\infty(I;\boldsymbol{\Theta}^*)\\ \big(\dot{\boldsymbol{\vartheta}}_\tau - \mathrm{Div}\big(KG\overline{\boldsymbol{\vartheta}}_\tau\big)\big)_{\tau>0}\\ \qquad\qquad \text{bounded in } L^1(I;\boldsymbol{\Theta}^*)\end{array}\right\} \;\Rightarrow\; \begin{array}{l}\big(G\boldsymbol{\vartheta}_\tau\big)_{\tau>0} \text{ bounded in } L^p(I;\boldsymbol{L}_p)\\ \text{for some } \ 1 \leq p < \infty,\end{array} \tag{4}$$

which implies $\overline{\boldsymbol{\vartheta}}_\tau \to \boldsymbol{\vartheta}$ strongly in $L^p(I;\widetilde{\boldsymbol{\Theta}})$ for and $1 \leq p < \infty$ for any $\boldsymbol{W}_p \subset\subset \widetilde{\boldsymbol{\Theta}}$. In concrete problems, (4) is to be verified by very specific "nonlinear" tests imitating (but not identical) the "energetic" test by **1** as in [2].

Further extension of the above abstract procedure is a coupling with "non-thermal" subsystems. For this, we consider an abstract *free energy* $\boldsymbol{\psi}: \boldsymbol{U} \times \boldsymbol{\Theta} \to \boldsymbol{\Theta}^*$ depending on an abstract "non-thermal" variable $\mathbf{u}$ valued in a Banach space $\boldsymbol{U}$ and an *abstract temperature* $\boldsymbol{\theta}$ valued (rather formally) in $\boldsymbol{\Theta}$. We use a naturally arising convention that *extensive* (resp. *intensive*) *variables* are valued in $\boldsymbol{\Theta}^*$ (resp. in $\boldsymbol{\Theta}$). For the *total free energy* $\Psi : \boldsymbol{U} \times \boldsymbol{\Theta} \to \mathbb{R}$ defined by $\Psi(\mathbf{u}, \boldsymbol{\theta}) := \langle \boldsymbol{\psi}(\mathbf{u}, \boldsymbol{\theta}), \mathbf{1}\rangle$, we define a *specific entropy* $\mathbf{s} \in \boldsymbol{\Theta}^*$ and the *total entropy* $S \in \mathbb{R}$ respectively as

$$\mathbf{s} := -\Psi'_{\boldsymbol{\theta}}(\mathbf{u}, \boldsymbol{\theta}) \quad \text{and} \quad S := \big\langle \mathbf{s}, \mathbf{1}\big\rangle,$$

and further, by an abstract Gibbs' relation, the specific *internal energy* $\boldsymbol{\varepsilon} \in \boldsymbol{\Theta}^*$ as:

$$\boldsymbol{\varepsilon}(\mathbf{u}, \boldsymbol{\theta}, \mathbf{s}) := \boldsymbol{\psi}(\mathbf{u}, \boldsymbol{\theta}) + \mathbf{s} \bullet \boldsymbol{\theta}$$

where we use an abstract "localization" bi-linear mapping $\bullet : \boldsymbol{\Theta}^* \times \boldsymbol{\Theta} \to \boldsymbol{\Theta}^*$, cf. [4, Sec. 5.3.1]. Then, for the *total internal energy* $\mathcal{E}(\mathbf{u}, \boldsymbol{\theta}, \mathbf{s}) :=_{\boldsymbol{\Theta}^*}\langle \boldsymbol{\varepsilon}(\mathbf{u}, \boldsymbol{\theta}, \mathbf{s}), \mathbf{1}\rangle_{\boldsymbol{\Theta}} = \Psi(\mathbf{u}, \boldsymbol{\theta}) +_{\boldsymbol{\Theta}^*}\langle \mathbf{s}, \boldsymbol{\theta}\rangle_{\boldsymbol{\Theta}}$, we have the expression for temperature as expected:

$$\boldsymbol{\theta} = \mathcal{E}'_{\mathbf{s}}(\mathbf{u}, \boldsymbol{\theta}, \mathbf{s}).$$

Furthermore, the *entropy equation* reads as

$$\boldsymbol{\theta} \bullet \dot{\mathbf{s}} - \mathrm{Div}\big(\mathcal{K}(\mathbf{u}, \boldsymbol{\theta})G\boldsymbol{\theta}\big) = \mathbf{r} \tag{5}$$

where the *heat flux* in terms of temperature $\boldsymbol{\theta}$ is governed by the *Fourier law* $-\mathcal{K}(\mathbf{u},\boldsymbol{\theta})G\boldsymbol{\theta}$ and where $\mathbf{r}\in\boldsymbol{\Theta}^*$ denotes the dissipation (i.e. heat production) rate. The *heat-transfer equation* arising from (5) reads as

$$\mathbf{c}_{\rm v}(\mathbf{u},\boldsymbol{\theta})\bullet\dot{\boldsymbol{\theta}}-{\rm Div}(\mathcal{K}(\mathbf{u},\boldsymbol{\theta})G\boldsymbol{\theta})=\mathbf{r}+\boldsymbol{\theta}\bullet\Psi''_{\boldsymbol{\theta}\mathbf{u}}(\mathbf{u},\boldsymbol{\theta})\bullet\dot{\mathbf{u}} \tag{6}$$

with the *heat capacity* $\mathbf{c}_{\rm v}(\mathbf{u},\boldsymbol{\theta})=-\boldsymbol{\theta}\bullet S'_{\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\theta})=-\boldsymbol{\theta}\bullet\Psi''_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\theta})\in\boldsymbol{\Theta}^*$. By using the *enthalpy-like transformation* $\mathbf{C}_{\rm v}(\mathbf{u},\boldsymbol{\theta}):=\int_0^1\mathbf{c}_{\rm v}(\mathbf{u},t\boldsymbol{\theta})\bullet\boldsymbol{\theta}\,{\rm d}t\in\boldsymbol{\Theta}^*$, (6) reads as

$$\dot{\boldsymbol{\vartheta}}-{\rm Div}\big(\mathcal{K}(\mathbf{u},\boldsymbol{\theta})G\boldsymbol{\theta}\big)=\mathbf{r}+\big(\boldsymbol{\theta}\bullet\Psi''_{\boldsymbol{\theta}\mathbf{u}}(\mathbf{u},\boldsymbol{\theta})+[\mathbf{C}_{\rm v}]'_{\mathbf{u}}(\mathbf{u},\boldsymbol{\theta})\big)\bullet\dot{\mathbf{u}}\quad\text{ with }\quad\boldsymbol{\vartheta}=\mathbf{C}_{\rm v}(\mathbf{u},\boldsymbol{\theta}).$$

The simplest coupling arises by considering a (generalized) gradient flow $V\dot{\mathbf{u}}+\Psi'_{\mathbf{u}}(\mathbf{u},\boldsymbol{\theta})=f(t)$ and then $\mathbf{r}=V\dot{\mathbf{u}}\bullet\dot{\mathbf{u}}$ where $V\in{\rm Lin}(\boldsymbol{V},\boldsymbol{V}^*)$ is a coercive operator, $\boldsymbol{V}\supset\boldsymbol{U}$.

For a generalizations for inertial forces and/or rate-independent subsystems and existence of a suitably defined weak solution we refer to [4, Sec. 5.3.2]. For illustration of the abstract scheme by some applications of such a coupled system in continuum thermomechanics of solids like e.g. [1] we refer to [4, Sec. 5.3.3].

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Synchronization and Rhythms: Asymptotic Dynamics for Systems Close to Gradient Flows

Giambattista Giacomin

The driving idea of my presentation can be summed up by: in families of interacting *units* (multicellular organisms, colonies of organisms, components in a circuit,...) synchronization and rhythmic behaviors are often observed. Moreover, rhythms are not mere accidents, but they have profound functional roles. What is then the origin of synchronization and rhythms? The issue is very vast [14], but we can highlight two important classes:

(1) Systems in which there is a pacemaker. One example is the circadian rhythm, where the pacemaker is of course the sun. The pacemaker is at the origin of both synchronization and rhythm.

(2) The interaction of the units leads to (some sort of) synchronization and a rhythm emerges as stable time asymptotic dynamics of the whole family. The main example here is that most of the living organism, even very simple ones, are able to reproduce a rhythm that is very close to the circadian one. This mechanism is definitely more mysterious than the one in (1), but absolutely omnipresent and of profound functional importance.

We focus on the second mechanism – the intrinsically generated rhythms – and the aim is to attack mathematically an idea that has been very much developed in the applied sciences: a system made of very many noisy *excitable units* may behave almost periodically, typically if noise is neither two strong nor too weak [8]. Of course a noisy finite dimensional system cannot behave periodically, unless it is very degenerate (and this is not what we have in mind). One of the key words here is *almost*: periodic behavior appears as a mathematical extrapolation – a law of large numbers – only in the limit of infinitely large families of units. Another word that may be too imprecise at this stage is *excitable unit*: this is actually a vague notion in the literature, but it is typically a finite dimensional dynamical system that models a unit that has a resting position. Under small perturbations the unit relaxes quickly back to the resting state, but it the perturbation is sufficiently strong the unit perfumes a more complex action before getting back to rest (example: a neuron that spikes [8])

Note that an excitable unit has nothing periodic in itself: the periodic behavior is expected to emerge as a result of the interaction between several units that have a non linear dynamics and because of the noise that perturbs the units. We can say that these phenomena are well understood at a numerical level [8], but, in spite of the considerable importance in real world phenomena, mathematical results are extremely meager.

The aim of my presentation has been to introduce a class of models – the active rotator model – proposed by H. Sakaguchi, S. Shinomoto and Y. Kuramoto [9, 10, 12] for which we can rigorously establish the appearance of stable periodic motion. This is really a minimal toy model and it allows to see how challenging in reality the question is. The active rotator model is made up by one dimensional units that are one dimensional: $\varphi \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$:

$$\dot{\varphi}(t) \,=\, U(\varphi(t)), \tag{1}$$

with $U : \mathbb{T} \mapsto \mathbb{R}$ smooth. We assume that $\int_{\mathbb{T}} U \neq 0$, that is U is not the derivative of a function. For the purpose of this abstract it suffices to focus on $U(\varphi) := a\sin(\varphi) - 1$, $a > 1$. Note that the dynamics has one stable fixed point and it is probably easy for the reader to grasp the idea of *excitable unit* by just meditating on this simple dynamics.

We now want to put several units of this type into interaction, we add noise, and the easiest model we can write is [9, 10, 12]

$$\mathrm{d}\varphi_j(t) \,=\, \delta U(\varphi_j(t))\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin\left(\varphi_j(t) - \varphi_i(t)\right)\mathrm{d}t + \sigma\mathrm{d}w_j(t)\,, \tag{2}$$

where $j = 1, 2, \dots N$, $\delta \geq 0$, $K \geq 0$, $\sigma > 0$ and $\{w_j\}_{j=1,2,\dots}$ are independent standard Brownian motions.

At this stage it is important to remark that this stochastic evolution admits only one invariant probability measure but the evolution is *reversible* with respect to this measure if and only if $\delta = 0$. Stochastic reversibility is what in physics is called *detailed balance law* [2].

The Fokker-Planck equation that emerges as evolution equation for the $N \to \infty$ limit of the empirical measure the variables $\{\varphi_j(t)\}_{j=1,\dots,N}$ [2, 9, 10, 12] is

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta \left[p_t(\theta)(J * p_t)(\theta)\right] - \delta\, \partial_\theta \left[p_t(\theta) U(\theta)\right] , \tag{3}$$

where $J(\cdot) = -K \sin(\cdot)$ and $*$ denotes the convolution on $\mathbb{T}$. Of course $p_t(\cdot)$ is a probability density. Let us point out immediately that (3) is a gradient system [2] if and only if $\delta = 0$ (stochastic reversibility becomes a gradient structure in the continuum limit).

It is for (3) that we can establish synchronization and rhythmic behavior, under suitable assumptions on the parameters. Schematically our result is the following: if K/σ^2 is larger than one, but below a threshold that we can compute, for $a > 1$ in a suitable interval (again, that we can make explicit: recall that $U(\cdot) = a \sin(\cdot) - 1$) there exists a (locally) stable pulsating wave for (3) for δ sufficiently small. Our results become sharp in the limit $\delta \searrow 0$ [6] (see [3, 4] for generalizations).

The result can be probably understood better if we recall that the system is a gradient flow of a suitable free energy if $\delta = 0$. For such a PDE all stationary solutions can be computed and one can establish the existence of a stable manifold of stationary solution [5]: these stationary solutions describe coherent or synchronized states. This model in fact is well known in statistical mechanics and it can be explicitly solved [13]. Using dynamical systems technics (robustness of normally contracting manifolds [1, 7, 11]) we can show that such a stable manifold exists also for $\delta \neq 0$, at least if δ is not too large. Moreover a systematic perturbation theory can be developed so that also for $\delta \neq 0$ we can obtain explicit expressions that become more and more accurate the smaller δ is.

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Weak Solutions for a Stochastic Mean Curvature Flow of Two Dimensional Graphs

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(joint work with Martina Hofmanova and Matthias Röger)

We study a stochastically perturbed mean curvature flow for graphs in $\mathbb{R}^3$ over the two-dimensional unit-cube and subject to periodic boundary conditions, giving rise to the SPDE

$$df = \sqrt{1+|\nabla f|^2}\ \mathrm{div}\,\Big(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\Big)\,dt + \sqrt{1+|\nabla f|^2}\circ dW_t,$$

where $\circ\, dW_t$ denotes the formal Stratonvich differential of the noise. This is a special case of a model proposed in the physics literature for the motion of a surface separating the two phases of a binary alloy at critical temperature subject to additional perturbation by a noisy environment [16].

The mathematical difficulties of this model arise from the fact that the equation is only quasilinear and degenerate elliptic in the main part and that the noise is multiplicative in the gradient of the solution.

In a series of papers around 2000 Lions and Souganidis announced a theory of stochastic viscosity solutions for fully nonlinear SPDE which would cover the equation above, yielding well posedness for the model.

In our work we present a full proof of existence of weak martingale solutions on the two dimensional torus by energy estimates, giving an alternative to the announced stochastic viscosity solution approach.

To overcome difficulties induced by the degeneracy of the mean curvature operator and the multiplicative gradient noise present in the model we employ a three

step approximation scheme together with refined stochastic compactness and martingale identification methods.

The crucial energy estimate for the surface area functional depend on an algebraic vanishing theorem which holds true only for two-dimensional matrices. The passage to the limit in the multiplicative noise coefficient is achieved by an extension of compensated compactness arguments to a stochastic setting.

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Analysis of a variational model for nematic shells

Antonio Segatti
(joint work with G. Canevari, M. Snarski and M. Veneroni)

A *nematic shell* is a thin film of nematic liquid crystal coating a rigid and curved substrate Σ represented as a two-dimensional surface. The basic mathematical description of these shells is given in terms of a unit vector field constrained to be tangent to the substrate Σ. In analogy with the nomenclature for liquid crystals on domains, this vector field is named *director* and it expresses at any point of Σ the mean orientation of the nematic molecules. The rigorous mathematical treatment of nematic shells is intriguing since the analysis requires tools from diverse fields such as the calculus of variations, partial differential equations, topology, differential geometry and numerical analysis. On the other hand, the interest in this study is not only academic but rather it is motivated by the vast technological applications of nematic shells. To the best of our knowledge, the study of these structures has been mostly confined to the physical literature (see, among the others, [2, 3, 7]) with the sole exception of [4]. As observed in [7], the liquid crystal ground state (and all its stable configurations, in general) is determined by two competing driving principles: on the one hand the minimization of the "curvature of the texture" penalized by the elastic energy, and on the other the frustration due to constraints of geometrical and topological nature, imposed by anchoring the nematic to the surface of the underlying particle. In [5] and in [6] we have analyzed a new energy model proposed by Napoli and Vergori in [3]. This energy affects the two aspects above, leading to different results with respect to the classical models [2, 4]. To describe our results and to highlight some of the related difficulties, we consider the simplest *one-constant approximation* of the surface energy on a compact two-dimensional surface $\Sigma \subset \mathbb{R}^3$ without boundary (see [5] for the analysis of the full energy):

$$W_{extr}(\mathbf{n}) := \frac{\kappa}{2} \int_\Sigma |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2 \, \mathrm{dVolg}, \tag{1}$$

where $\mathbf{n}$ is tangent vector field with unit norm. Here κ is a positive constant, the symbol D denotes the covariant derivative on Σ, and $\mathfrak{B}$ is the shape operator. We refer to (1) as to the *extrinsic energy* to highlight the difference with the classical energy proposed in [2] which consists only in the Dirichlet part of (1) and thus has an intrinsic character. As a result, the energy in [2, 4] does not take into account how the substrate Σ sits in $\mathbb{R}^3$. In what follows, we will better explain the differences between these two approaches when Σ is the axisymmetric torus. The energy (1) has been derived in [3] via a formal dimension reduction thus suggesting as an interesting open problem its rigorous deduction using the Γ-convergence. Our results address

(a) the relation between the topology of the surface and the functional setting,
(b) the minimization of (1) and the well posedness of its gradient flow on a general genus one surface,
(c) the precise structure of local minimizers on the axisymmetric torus.

The gradient flow of the energy, aside from being an interesting mathematical object on its own, it provides an efficient tool for numerical approximations of minimizers. Furthermore, it can be seen as a first step towards the study of a Leslie-Ericksen type model for nematic liquid crystals on surfaces. Step (a) is necessary to give a rigorous formulation to the problem and to identify for which surfaces the energy (1) is well defined. We thus obtain (see [5]) an H^1 version of the celebrated Hairy Ball theorem or, more generally, that there exist unit norm tangent vector fields on Σ (compact and without boundary) with H^1 regularity if and only if Σ has Euler Characteristic equal to zero. In [1] we extended this result to vector fields and line fields with VMO regularity and to manifolds with boundary. Steps (b) and (c) are two sides of the same story: On the one hand, the analysis in (b) has the advantage of being applicable to any two-dimensional topologically admissible surface and even, up to some technical difficulties, to $(N-1)$-dimensional compact and smooth hypersurfaces embedded in $\mathbb{R}^N$. On the other hand, in (c) we sacrifice generality in order to obtain precise analytical and numerical information on the solutions. Thus, once we have fixed Σ to be an aximmetric torus, we study the energy (1) representing the vector field using a scalar parameter α that measures the angle that $\mathbf{n}$ forms with a given moving orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$, i.e. $\mathbf{n} = \mathbf{e}_1 \cos(\alpha) + \mathbf{e}_2 \sin(\alpha)$. As a result, the regularity issue and the existence of solutions with prescribed winding number become more transparent than working directly on (1). The local existence of the representation above is straightforward, but a global one is in general not possible (even when the topology of Σ allows for H^1-fields). Consequently, a first step in the analysis is to prove that for every H^1-regular unit-norm vector field $\mathbf{n}$ there exists a representation $\alpha \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ defined on the universal covering of Σ. Once we have represented $\mathbf{n}$ via the angle α, the energy (1) takes the form ($Q := [0, 2\pi] \times [0, 2\pi]$)

$$W_{extr}(\alpha) = \frac{\kappa}{2} \int_Q \left\{ |\nabla_s \alpha|^2 + \eta \cos(2\alpha) \right\} \, \mathrm{dvol} + f(\mu), \tag{2}$$

where η is a function which depends only on the geometry of the torus and $f(\mu)$ is a constant depending only on the aspect ratio $\mu := \frac{R}{r}$, with R, r being the major and the minor radii, respectively. This structure, a Dirichlet energy plus a double (modulo 2π) well potential, is well-studied in the context of Cahn-Hilliard phase transitions. In particular, the constant states $\alpha_m \equiv 0$ and $\alpha_p \equiv \pi/2$ (see Figure 1) are equilibrium configurations. Moreover, depending on the torus aspect ratio, the sign of η may not be constant on Q and this forces a smooth transition between the states α_m, where $\eta < 0$, and α_p, where $\eta > 0$. Note that α_p and α_m (and more generally all the constant states) are minimizers for the intrinsic energy, independently of the value of μ. On the other hand the energy (2) is sensitive to changes in the torus aspect ratio due to the potential term. In particular, when $\mu < \mu^* \approx 1.52$ (found numerically) the high bending energy associated to $\alpha = \alpha_p$ in the internal hole of the torus makes the constant solution α_p no longer stable. Another feature is that the new (non constant) solution depicted at the bottom of Figure 1 emerges. This new solution attempts to minimize the effect of the curvature by orienting the director field along the meridian lines $\alpha = \alpha_m$ (which are geodesics on the torus) near the hole of the torus, while near the external equator the director is oriented along the parallel lines $\alpha = \alpha_p$ (which are lines of curvature). It is important to note that this new solution is a consequence of the extrinsic term in the energy (1) and does not show up when using the intrinsic energy.

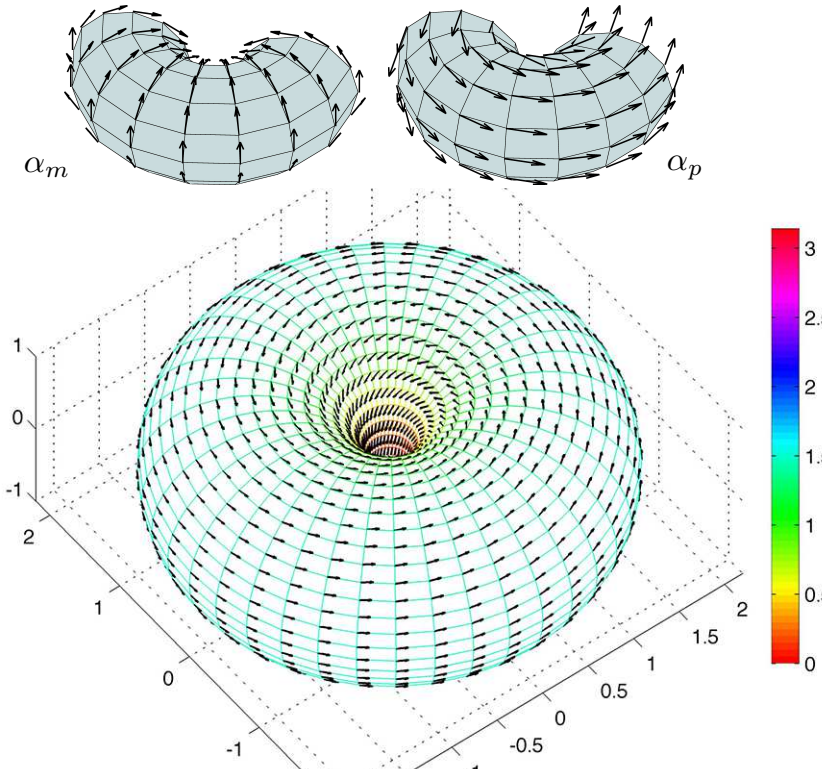

FIGURE 1.

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Dynamics of entropic interpolations

Christian Léonard

Displacement interpolations are flows of probability measures that solve some optimal transport problem, while entropic interpolations are flows of probability measures that solve some Schrödinger problem: an entropy minimization problem addressed by Schrödinger in 1931. These two notions are intimately linked to each other. In several settings of interest, displacement interpolations are limits of entropic interpolations.

Notation. For any measurable space Y, $\mathrm{M}(Y)$ is the set of all positive measures on Y and $\mathrm{P}(Y)$ is the subset of all probability measures. The path space on $\mathcal{X}$ is denoted by $\Omega \subset \mathcal{X}^{[0,1]}$. The canonical process $(X_t)_{t\in[0,1]}$ is defined for each $t \in [0,1]$ and $\omega \in \Omega$ by $X_t(\omega) = \omega_t \in \mathcal{X}$. For any $Q \in \mathrm{M}(\Omega)$ and $0 \le t \le 1$, we denote $Q_t := (X_t)_\# Q := Q(X_t \in \cdot) \in \mathrm{M}(\mathcal{X})$ the law of X_t under Q.

Displacement interpolations. To make things easy, let the state space $\mathcal{X}$ be a Riemannian manifold and d be its Riemannian distance. Consider the problem of minimizing the average kinetic action

$$\int_\Omega dP \int_{[0,1]} |\dot X_t|^2_{X_t}/2\, dt \to \min; \quad P \in \mathrm{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \tag{1}$$

under the constraint that the initial and final marginals of P_0 and P_1 of P are equal to the prescribed probability measures μ_0 and $\mu_1 \in \mathrm{P}(\mathcal{X})$ on $\mathcal{X}$. Suppose for simplicity that there is a unique solution P to this problem. Then P has the form $P(\cdot) = \int_{\mathcal{X}^2} \delta_{\gamma^{xy}}(\cdot)\, \pi(dxdy)$ where $\delta_{\gamma^{xy}}$ is the Dirac mass at γ^{xy}: the unique geodesic between x and y, and $\pi \in \mathrm{P}(\mathcal{X}^2)$ is the unique solution of the optimal transport problem

$$\int_{\mathcal{X}^2} d^2(x,y)\, \pi(dxdy) \to \min; \quad \pi \in \mathrm{P}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1$$

where π_0 and $\pi_1 \in \mathrm{P}(\mathcal{X})$ stand for the first and second marginals of π. The displacement interpolation between μ_0 and μ_1 is the flow of marginals $[\mu_0, \mu_1] := (P_t)_{0\le t\le 1}$ of the solution P of (1). This notion has been introduced by McCann in his PhD Thesis [8]. It is the basis of the development of the theory of lower bounds for the Ricci curvature of geodesic spaces, see the textbooks [1, 10].

Entropic interpolations. Consider the Schrödinger problem of minimizing the relative entropy

$$H(P|R) := \int_\Omega \log(dP/dR)\, dP \to \min; \quad P \in \mathrm{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \tag{2}$$

with respect to the reference path measure $R \in \mathrm{M}(\Omega)$ under the same marginal constraints as in (1). For a survey of basic results about the Schrödinger problem, see [7]. If we choose R to be the law of the reversible Brownian motion on $\mathcal{X}$, we obtain with Girsanov's theory that

$$H(P|R) = H(P_0|\mathrm{vol}) + \int_\Omega dP \int_{[0,1]} |v_t^P(X_t)|^2_{X_t}/2\, dt$$

where v_t^P is Nelson's forward velocity field of the diffusion law P, [9]. When $\mathcal{X} = \mathbb{R}^n$, denoting $E_P[\cdot|\cdot]$ the conditional expectation,

$$v_t^P(x) = \lim_{h\to 0, h>0} \frac{1}{h} E_P[X_{t+h} - X_t \mid X_t = x]. \tag{3}$$

The entropic interpolation between μ_0 and μ_1 is the flow of marginals $[\mu_0, \mu_1]^R := (P_t)_{0\le t\le 1}$ of the unique solution P of (2).

A worthy analogy. The analogy between (1) and (2) is not only formal. Considering the slowed down process $R^\epsilon = (X^\epsilon)_\# R$ which is the law of $X_t^\epsilon = X_{\epsilon t}, 0 \le t \le 1$, it is known that

$$\epsilon H(P|R^\epsilon) \to \min; \quad P \in \mathrm{P}(\Omega) : P_0 = \mu_0, P_1 = \mu_1$$

Γ-converges to (1), see [6]. In particular, the entropic interpolation $[\mu_0, \mu_1]^{R^\epsilon}$ is a smooth approximation of the displacement interpolation $[\mu_0, \mu_1]$.

This kind of convergence also holds for optimal L^1-transport on graphs [4] and Finsler manifolds (instead of optimal L^2-transport on a Riemannian manifold) where diffusion processes must be replaced by random processes with jumps (work in progress).

Dynamics of the interpolations. Unlike entropic interpolations, displacement interpolations lack regularity. Already known results about the dynamics of the displacement interpolations in geodesic spaces with a Ricci curvature bounded from below can be found in [3]. Understanding the dynamics of entropic interpolations could be a first step (before letting ϵ tend to zero) to recover such results. A *formal* representation of the displacement interpolation is given by

$$\dot{X}_t = \nabla\psi(t, X_t), \quad P\text{-a.s.}$$

where P is a solution of (1), ψ is the viscosity solution of the Hamilton-Jacobi equation $\begin{cases} \partial_t \psi + |\nabla \psi|^2/2 = 0 \\ \psi_{t=1} = \psi_1 \end{cases}$ and ψ_1 is in accordance with the endpoint data μ_0 and μ_1. Note that

$$\ddot{X}_t = \nabla[\partial_t \psi + |\nabla \psi|^2/2](t, X_t) = 0, \quad P\text{-a.s.} \tag{4}$$

fitting the standard geodesic picture. Similarly, a *rigorous* representation of the entropic interpolation is given by

$$v_t^P = \nabla \psi(t, X_t), \quad P\text{-a.s.}$$

where v^P is defined at (3), P is the solution of (2) and ψ is the classical solution of the Hamilton-Jacobi-Bellman equation $\begin{cases} \partial_t \psi + \Delta \psi/2 + |\nabla \psi|^2/2 = 0, \\ \psi_{t=1} = \psi_1. \end{cases}$ Iterating time derivations in the spirit of (3) in both directions of time allows to define a relevant notion of stochastic acceleration a^P, see for instance [9, 2]. We obtain the following analogue of (4)

$$a_t^P = \frac{1}{2}\nabla[\partial_t \psi + \Delta \psi/2 + |\nabla \psi|^2/2] + \frac{1}{2}\nabla[-\partial_t \varphi + \Delta \varphi/2 + |\nabla \varphi|^2/2](t, X_t) = 0, \quad P\text{-a.s.}$$

where φ solves some HJB equation $\begin{cases} -\partial_t \varphi + \Delta \varphi/2 + |\nabla \varphi|^2/2 = 0 \\ \varphi_{t=0} = \varphi_0 \end{cases}$ in the other direction of time. As an interesting consequence, we obtain in [5] that along any entropic interpolation $\mu_t := P_t, 0 \leq t \leq 1$ on a Riemannian manifold, the relative entropy $H(\mu_t|\mathrm{vol})$ of μ_t with respect to the volume measure vol, we have

$$\frac{d^2}{dt^2} H(\mu_t|\mathrm{vol}) = \frac{1}{2}\left\{\Gamma_2(\varphi_t) + \Gamma_2(\psi_t)\right\}$$

where φ and ψ are the solutions of the above HJB equations in both directions of time and Γ_2 is the Bakry-Émery operator given by $\Gamma_2(u) = \mathrm{Ric}(\nabla u) + \sum_{i,j}(\partial_i \partial_j u)^2$ with Ric the Ricci tensor. This last formula is a rigorous (in the sense that the second derivative is well defined) analogue of the heuristic formula (obtained with Otto's heuristic calculus)

$$\frac{d^2}{dt^2} H(\mu_t|\mathrm{vol}) = \Gamma_2(\psi_t)$$

which holds for the displacement interpolation (μ_t) and is the basic step for developing the Lott-Sturm-Villani theory of lower bounded Ricci curvature of geodesic spaces, see [10].

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A toy model for hysteretic phase transitions

Michael Herrmann
(joint work with Michael Helmers)

The one-dimensional lattice ODE

$$\dot{u}_j = p_{j+1} - 2\,p_j + p_{j-1} \quad \text{with} \quad p_j = u_j - \operatorname{sign} u_j \tag{1}$$

admits solutions with propagating phase interfaces and provides a microscopic justification for macroscopic hysteresis models. Here, the two *phases* correspond to the sets $\{u < 0\}$ and $\{u > 0\}$, on which the bistable function $u \mapsto u - \operatorname{sign} u$ is strictly increasing.

Microscopic dynamics. For a finite system with $N < \infty$ particles and either periodic or Neumann boundary conditions, equation (1) can be regarded is a microscopic H^{-1}-gradient flow for u. In particular, it satisfies the energy balance

$$\dot{\mathcal{E}}(t) = -\mathcal{D}(t)\,, \quad \mathcal{E} := \tfrac{1}{2}\sum_j p_j^2\,, \quad \mathcal{D} := \sum_j (p_{j+1} - p_j)^2\,,$$

so there is a strong tendency to reach a state with small dissipation. However, due to phase transitions (one of the u_j's changes sign) there exist small time intervals with huge dissipation and strong microscopic fluctuations, see Figures 1 and 2 for an illustration.

Macroscopic dynamics. The parabolic scaling limit

$$\tau := \varepsilon^2 t, \qquad \xi := \varepsilon j$$

has been investigated in [1] for a system with infinitely many particles and under certain assumptions on the microscopic initial data; the main result can be formulated as follows: The discrete p-data converge as $\varepsilon \to 0$ strongly to a limit function P, which is uniquely determined by the hysteretic free boundary problem

$$\partial_\tau\big(P(\tau,\,\xi) + \mu(\tau,\,\xi)\big) = \partial_\xi^2 P(\tau,\,\xi)\,, \qquad \mu(\cdot,\,\xi) = \mathcal{R}\big[P(\cdot,\,\xi)\big]\,. \tag{2}$$

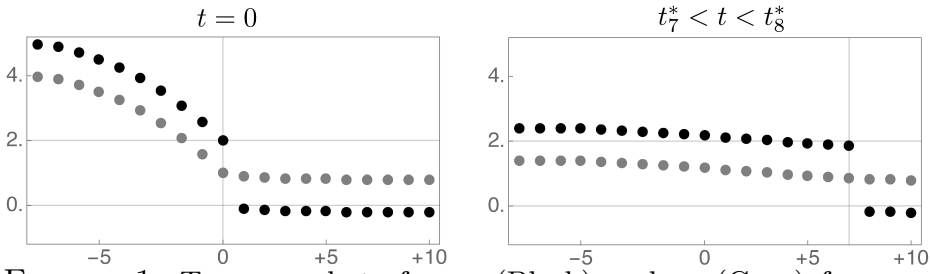

FIGURE 1. Two snapshots for u_j (Black) and p_j (Gray) for a numerical single-interface solution with 20 particles: The phase interface (vertical line) propagates to the right since the particles undergo a phase transition one after another.

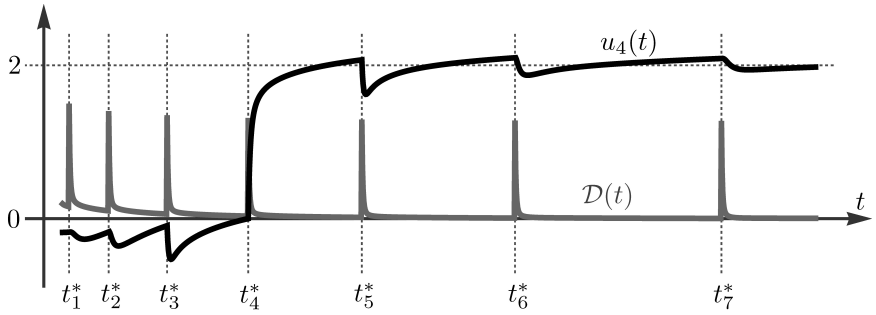

FIGURE 2. Evolution of u_4 and the (rescaled) dissipation $\mathcal{D}$ for the simulation from Figure 1; the particle j undergoes a phase transition at time t_j^*.

Here, $\mathcal{R}$ abbreviates the hysteresis operator from Figure 3 and the limit U of the u-data satisfies $U = P + \mu$. The well-posedness of the initial value problem to (2) has been proven in [2].

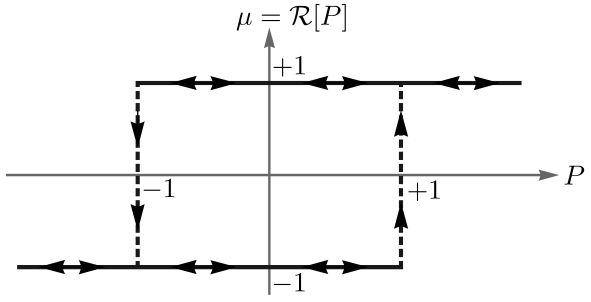

FIGURE 3. The relay operator $\mathcal{R}$ describes the hysteresis of phase interfaces in the macroscopic scaling limit.

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Geodesics convexity of the entropy in discrete interacting systems

Jan Maas
(joint work with Matthias Erbar, Max Fathi and Prasad Tetali)

A celebrated result by McCann [9] asserts that the Boltzmann–Shannon entropy is *displacement convex*, i.e., geodesically convex in the L^2-Wasserstein space of probability measures. This discovery triggered many developments at the interface of analysis, probability and geometry; we refer to [12] for an overview.

If the underlying space is discrete, the notion of displacement convexity becomes meaningless, since the induced L^2-Wasserstein space does not contain any geodesics. However, in independent recent works [2, 8, 10], new metrics have been introduced for probability measures on discrete spaces, which take over the role of the Wasserstein metric in discrete settings. In this talk we present recent geodesic convexity results for entropy functionals with respect to these metrics.

A gradient flow structure for Markov chains. Let $\mathcal{L}$ be the generator of a continuous time Markov chain on a finite set $\mathcal{X}$, thus for functions $\psi : \mathcal{X} \to \mathbb{R}$, the operator $\mathcal{L}$ is of the form $\mathcal{L}\psi(x) = \sum_{y\in\mathcal{X}} Q(x,y)(\psi(y) - \psi(x))$ where $Q(x,y) \geq 0$ for all $x, y \in \mathcal{X}$ with $x \neq y$, and $Q(x,x) = 0$ for all $x \in \mathcal{X}$. We shall assume that there exists a reversible probability measure π on $\mathcal{X}$, which means that the detailed balance equations $\pi(x)Q(x,y) = \pi(y)Q(y,x)$ hold for all $x, y \in \mathcal{X}$. As usual, the relative entropy functional is defined by

$$\mathcal{H}(\rho) = \sum_{x\in\mathcal{X}} \rho(x) \log \rho(x)\pi(x)$$

on the space of probability densities $\mathcal{P}(\mathcal{X}) = \{\rho \in \mathbb{R}_+^{\mathcal{X}} : \sum_x \rho(x)\pi(x) = 1\}$.

The following result has been obtained in several independent works: by Chow, Huang, Li and Zhou [2] in the setting of Fokker-Planck equations on graphs, by Mielke [10] in the setting of reaction-diffusion systems, and by the author [8] in the setting of continuous time Markov chains.

Theorem 1 (Gradient flow structure of Markov chains)**.** *The discrete heat equation $\partial_t \rho = \mathcal{L}\rho$ is the gradient flow equation for the relative entropy $\mathcal{H}$ with respect to a Riemannian structure on the interior of $\mathcal{P}(\mathcal{X})$.*

More formally, this means that the equation $\partial_t \rho = \mathcal{L}\rho$ can be written as $\partial_t \rho = -\nabla_{\mathcal{W}}\mathcal{H}(\rho)$, where the gradient $\nabla_{\mathcal{W}}$ is taken in a suitable Riemannian structure. The associated Riemannian distance, which we denote by $\mathcal{W}$, is given a discrete analogue of the Benamou–Brenier formula for the Wasserstein metric.

Geodesic convexity of the entropy. It is natural to ask whether there are examples of reversible Markov chains $(\mathcal{X}, Q, \pi)$ for which the entropy is geodesically κ-convex in the metric space $(\mathcal{P}(\mathcal{X}), \mathcal{W})$ for some $\kappa \in \mathbb{R}$. This means that

$$\mathcal{H}(\rho_t) \leq (1-t)\mathcal{H}(\rho_0) + t\mathcal{H}(\rho_1) - \frac{\kappa}{2}t(1-t)\mathcal{W}(\rho_0,\rho_1)^2 \tag{1}$$

for any curve $(\rho_t)_{t\in[0,1]}$ in $\mathcal{P}(\mathcal{X})$ with $\mathcal{W}(\rho_s,\rho_t)=|s-t|\mathcal{W}(\rho_0,\rho_1)$. As a consequence of the gradient flows structure, this property has several interesting implications for the associated Markov chain. In particular, if $\kappa>0$, it follows that $\mathcal{L}$ satisfies a modified logarithmic Sobolev inequality, which implies the exponential convergence bound

$$\mathcal{H}(\rho_t)\le e^{-2\kappa t}\mathcal{H}(\rho_0)$$

for solutions to the gradient flow equation $\partial_t\rho=\mathcal{L}\rho$.

The first examples for geodesic convexity of the entropy in reversible Markov chains have been obtained in the recent papers [3, 11]:

- Mielke [11] proved geodesic κ-convexity for one-dimensional birth-death chains under appropriate conditions. He applies this results to discrete approximations of one-dimensional Fokker–Planck equations with κ-convex potential, and shows that the optimal convexity constants for the discrete approximations converge to the desired limit.
- Erbar–Maas [3] obtained a tensorisation result for geodesic κ-convexity: if two Markov chains $(\mathcal{X}_i,Q_i,\pi_i)_{i=1,2}$ satisfy (1), then the associated product chain enjoys the same property with the same constant. This result allows one to obtain dimension independent bounds for high-dimensional product systems.

Discrete interacting systems. Until recently, no results were available for high-dimensional systems without a product structure, except for the elementary example of the complete graph. In this talk we present the first results in this direction.

The Bernoulli-Laplace model. We consider the simple exclusion process on the complete graph: consider k indistinguishable particles distributed over n sites labeled by $[n]=\{1,\dots,n\}$, where $1\le k<n$. Each site contains at most one particle. The state space of the system is the set $\Omega(n,k)=\{x\in\{0,1\}^n\,:\,x_1+\cdots+x_n=k\}$ (or equivalently, the set of all subsets of $[n]$ of size k). The Bernoulli-Laplace model is the continuous time Markov chain described as follows: after random waiting times (independent exponentially distributed with rate $\frac{1}{k(n-k)}$), one particle is selected uniformly at random, and jumps to a free site, selected uniformly at random. The uniform probability measure on $\Omega(n,k)$, given by $\pi_{\mathrm{BL}}(x)=\binom{n}{k}^{-1}$ for all x, is reversible. The following result is proved in [5]:

Theorem 2 (Geodesic κ-convexity for the Bernoulli-Laplace model)**.** *Let $n>1$ and $1\le k\le n-1$, and set $\kappa=\frac{n+2}{2k(n-k)}$. Then the entropy is κ-geodesically convex along $\mathcal{W}$-geodesics in $(\mathcal{P}(\Omega(n,k)),\mathcal{W})$.*

The obtained constant is optimal, possibly up to a factor 2. Moreover, we recover the best known constant in the modified logarithmic Sobolev inequality for the Bernoulli-Laplace model obtained in [1].

The random transposition model. Let $\mathcal{S}_n$ be the group of all permutations of $\{1,\dots,n\}$. We consider simple random walk on the canonical graph structure on $\mathcal{S}_n$, obtained by connecting two permutations if they differ by exactly one transposition. The unique reversible measure is the uniform measure π_{RT} given by $\pi_{RT}(\sigma) = 1/n!$. The following result is proved in [5]:

Theorem 3 (Geodesic κ-convexity for the random transposition model)**.** *Let $n > 1$ and set $\kappa = \frac{4}{n(n-1)}$. Then the entropy is κ-geodesically convex along $\mathcal{W}$-geodesics in $(\mathcal{P}(\mathcal{S}_n), \mathcal{W})$.*

Combining this result with known results on the log-Sobolev inequality we infer that the optimal constant κ satisfies $4/(n^2-n) \le \kappa \le 2/(n-1)$. It remains an open question to determine the correct order.

The results obtained in collaboration with Erbar and Tetali [5] rely on combinatorial methods. In subsequent work with Fathi [6] we reproved the results using a complete different method. This method relies on an extension of the Bochner approach, which has been developed in the setting of discrete functional inequalities by Caputo, Dai Pra and Posta [1]. As another application of this method, we obtain geodesic κ-convexity results for zero-range processes on the complete graph.

Related geodesic convexity results in non-linear settings have been obtained in [7] (for reaction-diffusion systems) and in [4] (for porous medium equations).

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Stochastic PDEs, Regularity structures, and interacting particle system

Hendrik Weber

Over the last few years there has been remarkable progress in the theory of non-linear stochastic partial differential equations (SPDE), mostly by Hairer (see [4] and in particular in the groundbreaking work [3]) and Gubinelli (see [2]). In these lectures I presented some of the ideas that are at the basis of these developments.

Key examples of interest include the KPZ equation and the Φ^4 model. The KPZ equation is given by the SPDE

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + \frac{1}{2}(\partial_x h(t,x))^2 + \xi(t,x)\ , \tag{KPZ}$$

where the spatial variable x is one-dimensional. The term $\xi(t,x)$ is a formal expression denoting space-time white noise. It is a quite irregular random distribution. The irregularity of this term is the main difficulty one has to address. Equation (KPZ) was introduced in [5] in 1986. It models fluctuations around the growth of a flat surface in $1+1$ dimensions.

A second principle example is the dynamic Φ^4_d model. It is formally given by the stochastic partial differential equation (SPDE)

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi^3(t,x) - m\phi(t,x) + \xi(t,x)\ . \tag{Φ^4_d}$$

Here the spatial variable x takes values in a d-dimensional space. We can treat this model in the cases $d = 1, 2, 3$ only.

In both of these examples, the space time white noise ξ is too irregular to solve the equation in a classical sense. More precisely, if we measure the regularity in a "parabolic way", ξ is a random distribution taking values in a space of distributions of regularity below $-\frac{d+2}{2}$. Schauder-like estimates then suggest that one can expect the solution h of the KPZ equation to have regularity below $\frac{1}{2}$ and that the solution ϕ of the (Φ^4_d) takes values in spaces of regularity below $\frac{2-d}{2}$. In both of these cases this is insufficient to interpret the non-linearities $(\partial_x h)^2$ and ϕ^3 (at least if $d \geq 2$ in (Φ^4_d)).

However, it turns out that as soon as such an equation satisfies a certain scaling property (called *subcriticality* in [3]), then it can be interpreted as limits of a renormalisation procedure. Roughly speaking subcriticality means that on small scales the solution behaves like a perturbation of the stochastic heat equation. For example, if h is a solution of (KPZ) and we set $\hat{h}(t,x) = \lambda^{-\frac{1}{2}} h(\lambda^2 t, \lambda x)$, then a formal calculation yields

$$\partial_t \hat{h} = \partial_x^2 \hat{h} + \frac{\lambda^{\frac{1}{2}}}{2}(\partial_x \hat{h})^2 + \hat{\xi}\ .$$

On small scales, i.e. for $\lambda \to 0$, the prefactor $\lambda^{\frac{1}{2}}$ of the non-linear term goes to zero. A similar statement holds true for (Φ^4_d) in space dimensions $d = 1, 2, 3$.

The renormalisation procedure for such equations consists of regularising the solution and subtracting the right diverging counter terms when removing this regularisation. If we let $\xi_\delta = \xi * \rho_\delta$ be a smooth approximation of the white noise,

then the main results of [4, 3] state that there exist choices of diverging C_δ, $\bar{C}_\delta$ such that the solutions of

$$\partial_t h_\delta = \partial_x^2 h_\delta + \frac{1}{2}(\partial_x h_\delta)^2 - C_\delta + \xi_\delta$$
$$\partial_t \phi_\delta = \Delta \phi_\delta - (\phi_\delta^3 - 3\bar{C}_\delta \phi_\delta) + \xi_\delta$$

converge to a non-trivial limit when δ goes to zero.

These limits arise can be obtained as scaling limits for interacting particle systems. In the case of the KPZ equation this was already observed by Bertini and Giacomin [1] in the nineties. There the "infinite constant" in the renormalisation procedure can be interpreted is a shift of reference frame. Recently, Mourrat and myself could establish a similar result for the two dimensional Φ^4 equation [6]. We approximate the solution of (Φ_d^4) by an Ising model with a long interaction range near the critical temperature. The "infinite constant" appears as a shift of this critical temperature.

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Abstract and stochastic homogenization of plasticity equations with the needle problem approach

Ben Schweizer

(joint work with Martin Heida and Marco Veneroni)

With a new approach, the "needle-problem approach" introduced in [6], we derive effective models for plasticity equations in heterogeneous media. Our model are the small strain plasticity equations of von-Mises type with a positive hardening effect. The material parameters are the elasticity tensor, the hardening coefficient and the flow-rule function — all these coefficients are allowed to depend on a parameter $\varepsilon > 0$. In a typical application, the coefficients vary on a space scale of order ε. We treat the following set of equations:

$$\begin{aligned} -\nabla \cdot \sigma^\varepsilon &= f\,, & \sigma^\varepsilon &= C_\varepsilon^{-1} e^\varepsilon\,, \\ \nabla^s u^\varepsilon &= e^\varepsilon + p^\varepsilon\,, & \partial_t p^\varepsilon &\in \partial \Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon)\,. \end{aligned} \tag{1}$$

In these equations, u is the deformation, σ the stress tensor, e and p are the elastic and the plastic strain component of $\nabla^s u = (\nabla u + \nabla u^T)/2$. The equations are posed on a bounded domain $Q \subset \mathbb{R}^n$ and accompanied by an initial condition

for p and a boundary condition for u. All unknowns have the superscript ε, since the coefficients C_ε, B_ε and the convex flow-rule function Ψ_ε depend on ε. In the periodic setting, the homogenization limit for the above system has been firstly stated in [1] and has been verified e.g. in [7] and [8]. Regarding an abstract approach we mention [4].

We say that the coefficients "allow averaging" if, for some abstract strain-stress-map Σ, the following property is satisfied: For an arbitrary simplex $T \subset Q$, for a vanishing force $f \equiv 0$, and for affine boundary data $U : x \mapsto \xi(t) \cdot x$, let $(u^\varepsilon, \sigma^\varepsilon, p^\varepsilon)$ be a solution to (1). Then, as $\varepsilon \to 0$, the stress averages converge:

$$\frac{1}{|T|} \int_T \sigma^\varepsilon(t) \to \Sigma(\xi)(t)\,. \tag{2}$$

The operator Σ associates to an evolution of strains $\xi = \xi(t)$ (note that $\xi(t)$ coincides with the average strain $\nabla^s u^\varepsilon(t)$ of the solution) a stress tensor $\sigma(t_0) = \Sigma(\xi(.))(t_0)$. It is given by a map $\Sigma : H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$ (modulo initial conditions). Due to memory effects in plasticity problems, Σ must depend on the history: the evolution of strains $\xi = \xi(t)$ is mapped to an evolution of stresses $\Sigma(\xi)(t) = \Sigma(\xi(.))(t)$.

In [2], we derived from this averaging assumption a homogenization result: For general domains Q, general boundary data U and general forces f, the effective problem for every limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ reads

$$-\nabla \cdot \Sigma(\nabla^s u) = f \quad \text{ in } Q \times (0,T)\,. \tag{3}$$

We emphasize the locality in space: For every point $x \in Q$, the stress $\sigma(x,t_0) = \Sigma(\nabla^s u(x,.))$ is a function of the history of strains $\nabla^s u(x,.)$, but it depends only on the strain history in the point x.

In [3], we analyzed the plasticity system with stochastic coefficients. The coefficients are constructed in a standard way with an ergodic dynamical system on a probability space. We obtain that, in the stochastic setting, the coefficients allow averaging. This provides (apart from the one-dimensional result in [5]) the first stochastic homogenization result for a plasticity system.

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Systems of diffusion equations as gradient flows in multi-component transportation metrics

Daniel Matthes
(joint work with Jonathan Zinsl)

Consider the following coupled system of evolution equations for the vector-valued density function $\rho : [0,T] \times \mathbb{R} \to J \subset \mathbb{R}^N$ in one space dimension:

$$\partial_t \rho(t;x) = \partial_x \left(\mathbf{M}\big(\rho(t;x)\big)\, \partial_x \left[\frac{\delta E(\rho)}{\delta \rho} \right] (t;x) \right). \tag{1}$$

Here $\delta E/\delta\rho$ is the variational derivative of the *energy functional* E, and $\mathbf{M} : J \to \mathbb{R}^{N\times N}_{spsd}$ is the *mobility matrix*. We assume that the range J of ρ is a compact and convex subset of $\mathbb{R}^N$. Formally, (1) is a gradient flow of E with respect to a multi-component transportation metric $\mathrm{d}_{\mathbf{M}}$. This metric — if it exists — is given in the spirit of Benamou and Brenier [1] as follows:

$$\mathrm{d}_{\mathbf{M}}(\rho_0,\rho_1)^2 = \inf \left\{ \int_0^1 \left[\int_{\mathbb{R}} w_s^T \mathbf{M}(\rho_s)^{-1} w_s \, dx \right] ds \,;\, \partial_s \rho_s + \partial_x w_s = 0 \right\}, \tag{2}$$

where the infimum runs over all weakly continuous curves $\rho : [0,1] \times \mathbb{R} \to J$ that connect ρ_0 to ρ_1; the function $w : [0,1] \times \mathbb{R} \to \mathbb{R}^N$ is the associated momentum. By extending the ideas from the scalar setting [2, 3] to vector-valued densities, it can be shown [7] that the infimum in (2) defines a pseudo-metric on the space $\mathfrak{M}(\mathbb{R};J)$ of measurable J-valued densities on $\mathbb{R}$ if the following holds:

- *(S) smoothness:* $\mathbf{M} : J \to \mathbb{R}^{N\times N}$ is continuous, and is smooth on the interior $\operatorname{int} J$ of J;
- *(P) positivity:* $\mathbf{M}(z)$ is a symmetric positive definite matrix for each $z \in \operatorname{int} J$;
- *(C) concavity:* $\sum_{m,n=1}^N \zeta_m \zeta_n \partial_{z_m} \partial_{z_n} \mathbf{M}(z)$ is a symmetric negative semi-definite matrix for each $z \in \operatorname{int} J$, $\zeta \in \mathbb{R}^N$.

The pseudo-metric is a genuine, geodesically complete metric on each subset of $\mathfrak{M}(\mathbb{R};J)$ consisting of densities with finite $\mathrm{d}_{\mathbf{M}}$-distance to some given reference density. It turns out that condition (C) is quite restrictive, and examples of admissible mobility matrices $\mathbf{M}$ are rare.

Example 1. *A possible generalization of the scalar ($N = 1$) mobility function $\mathbf{m}(z) = z(1-z)$ to $N > 1$ components is the following,*

$$\mathbf{M}(z) = \begin{pmatrix} z_1(1-z_1) & -z_1 z_2 & \cdots & -z_1 z_N \\ -z_1 z_2 & z_2(1-z_2) & \cdots & -z_2 z_N \\ \vdots & \vdots & \ddots & \vdots \\ -z_1 z_N & z_2 z_N & \cdots & z_N(1-z_N) \end{pmatrix},$$

which is defined on the simplex $J = \{z \in \mathbb{R}^N_{\geq 0}; z_1 + \cdots + z_N \leq 1\}$*, satisfies (S)+(P)+(C), and can be interpreted as describing the reduction of particle mobility due to volume filling.*

In the following, we limit ourselves to *fully decoupled* mobility matrices $\mathbf{M}$, defined on $J := [0,1]^N$, which are of the following form:

$$\mathbf{M}(z) = \begin{pmatrix} \mathbf{m}_1(z_1) & & \\ & \ddots & \\ & & \mathbf{m}_N(z_N) \end{pmatrix}.$$

If each of the N mobility functions $\mathbf{m}_n : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$, and smooth, positive and concave on $(0,1)$, then $\mathbf{M}$ satisfies (S)+(P)+(C). To each fully decoupled $\mathbf{M}$, we associate a function $H : J \to \mathbb{R}$ of the form $H(z) = h_1(z_1) + \cdots + h_N(z_N)$, where the $h_n : (0,1) \to \mathbb{R}$ are such that $\mathbf{m}_n h_n'' \equiv 1$.

Proposition 1. *Let* $\mathbf{M}$ *be a fully decoupled mobility matrix on* $J = [0,1]^N$*, with associated pseudo-metric* $\mathrm{d}_{\mathbf{M}}$*.*

- *Let* $\psi \in C_c^\infty(J;\mathbb{R})$*. Then, for every sufficiently small* $\epsilon \geq 0$*, the* internal energy functional $E(\rho) = \int \big(H + \epsilon\psi\big)(\rho)\,dx$ *possesses a* $\mathrm{d}_{\mathbf{M}}$*-geodesically contractive gradient flow, given by*

$$\partial_t \rho = \partial_{xx}\rho + \epsilon \partial_x\big(\mathbf{M}(\rho)\,\partial_x \nabla_z \psi(\rho)\big).$$

- *Let* $\varphi \in C_c^\infty(\mathbb{R};\mathbb{R}^N)$*. Then, for every* $\epsilon > 0$*, the* potential energy functional $E(\rho) = \int \langle \rho, \varphi\rangle\,dx + \epsilon \int H(\rho)\,dx$ *possesses a* $\mathrm{d}_{\mathbf{M}}$*-geodesically* λ*-contractive gradient flow, given by*

$$\partial_t \rho = \partial_x\big(\mathbf{M}(\rho)\,\partial_x \varphi\big) + \epsilon \partial_{xx}\rho, \tag{3}$$

for some $\lambda \geq -C\epsilon^{-1}$*.*

The first result says that the component-wise heat equation and its small perturbations are contractive in $\mathrm{d}_{\mathbf{M}}$. This follows from calculations using the formalism developed of Liero and Mielke [5] on basis of the "Eulerian calculus" of Otto-Westdickenberg. The second result is actually concerned with *scalar* diffusion equations, since the gradient flow decouples into N equations of the form

$$\partial_t \rho_n = \partial_x\big(\mathbf{m}_n(\rho_n)\,\partial_x \varphi_n\big) + \epsilon\,\partial_{xx}\rho_n.$$

λ-contractivity of these flows with $\lambda \geq -C\epsilon^{-1}$ has been proven in [4].

The result that is summarized in Proposition 1 is rather weak in comparison to what is known about convexity in the L^2-Wasserstein metric. Still, the result is sufficient to prove [7] that weak solutions to certain (non-convex) evolution equations of type (1) can be obtained by means of the *minimizing movement scheme* in $\mathrm{d}_{\mathbf{M}}$.

Theorem 1. *Let a convex function* $F \in C^2(\operatorname{int} J)$ *and a potential* $V \in C_c^\infty(\mathbb{R};\mathbb{R}^N)$ *be given. Assume that* $F(z) \to 0$ *for* $z \to 0$*. Define the energy*

$$E(\rho) = \int \big[F(\rho) + \langle \rho, V\rangle\big]\,dx.$$

Further, let an initial condition $\rho^0 \in L^1(\mathbb{R}; J)$ with $E(\rho^0) < \infty$ be given. For every time step $\tau > 0$, define inductively

$$\rho_\tau^0 := \rho^0, \quad \rho_\tau^n := \operatorname*{arg\,min}_{\rho \in L^1(\mathbb{R};J)} \left[\frac{1}{2\tau} \mathrm{d}_{\mathbf{M}}(\rho, \rho_\tau^{n-1})^2 + E(\rho) \right],$$

and let $\bar{\rho}_\tau : [0, \infty) \to L^1(\mathbb{R}; J)$ be the piecewise constant in time interpolation.

Then $\bar{\rho}_\tau$ converges — e.g., strongly in $L^2([0,T] \times \mathbb{R})$ — to a weak solution of

$$\partial_t \rho = \partial_x \big(A(\rho) \partial_x \rho \big) + \partial_x \big(\mathbf{M}(\rho)\, \partial_x V \big), \quad A(z) = \mathbf{M}(z) \nabla_z^2 F(z). \tag{4}$$

The key step in the proof of Theorem 1 is the derivation of two a priori estimates. The first follows from dissipation of $\int H(\rho)\, dx$ and yields a τ-uniform control on $\bar{\rho}_\tau$ in $L^2([0,T]; H^1)$. The second estimate is obtained by perturbing the minimizers ρ_τ^n along the gradient flow (3) and leads to a time-discrete weak formulation of (4); the auxiliary potential φ plays the role of a test function. The contractivity stated in Proposition 1 is essential to make the formal a priori estimates rigorous, using the *flow interchange principle* developed in [6].

Remark 1. *Using the same techniques, one is able to prove a similar existence result for a class of degenerate evolution equations of* fourth order*. This generalizes the results from* [4] *for the scalar Cahn-Hilliard equation to coupled systems.*

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The bi-axial test: a case study for uniqueness in small strain 3D elasto-plasticity

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(joint work with Alessandro Giacomini and Jean-Jacques Marigo)

It is well known by now that, in any well-posed quasi-static evolution, the stress field and correspondingly, the elastic strain field are uniquely determined. Let us recall that, in elasto-plasticity, the (linearized) strain, that is the symmetrized gradient Eu of the displacement field u, is additively decomposed into $e+p$ where e stands for the elastic strain field and p for its plastic counterpart. In the existence theory for such evolutions the plastic strain field is a finite Radon measure (see e.g. [1]). Uniqueness of the plastic strain has thus far never been established, except for a one-dimensional example whose focus was actually regularity, rather than uniqueness [2].

In this talk which is an abridged summary of [3, Section 3], I consider a specific three-dimensional setting, that of the bi-axial test. The domain is a rectangular parallelepiped $(x_1, x_2, x_3) \in \Omega := (0, \ell) \times (0, d) \times (0, \ell)$. A pressure $\bar{p} \geq 0$ is applied to the faces $x_2 = 0, d$, while a displacement $u_3 = tx_3$ is applied to the faces $x_3 = 0, \ell$, which furthermore, are frictionless surfaces. The faces $x_1 = 0, \ell$, are traction free. The yield stress condition is of Von Mises type and it is given by $|\sigma_D| \leq \sqrt{2/3}\sigma_c$ where $\sigma_D := \sigma - \text{tr } \sigma/3i$ is the deviatoric part of the stress field σ, itself related to the elastic strain field through the isotropic Hooke's law $\sigma = \mathbf{E}e$ (the Young's modulus $\mathbf{E}$ is positive and, for simplicity, we assume that the Poisson's ratio ν is 0). It is also assumed that $\bar{p} < \sigma_c$.

It is then proved that the problem admits a spatially homogeneous solution for which the stress field σ remains constant when $t \geq t_{\text{crit}}$, t_{crit} being an explicit function of the data of the problem. Because of stress uniqueness, that stress field is *the* stress field for the evolution.

The investigation focuses on whether that solution is unique. It is shown that such is the case, provided that $\bar{p} \neq \sigma_c/\sqrt{3}$. To our knowledge, this result is the first uniqueness result for a *bona fide* three-dimensional elasto-plastic evolution problem, that is one which does not admit a unidirectional solution.

The case $\bar{p} = \sigma_c/\sqrt{3}$ is more involved. The time derivative of the plastic strain field must then be of the form $\dot{p}(t) = \eta(t)\sigma_D(t_{\text{crit}})$, $\eta \in AC([0,T]; \mathcal{M}_b(\Omega))$, where $\mathcal{M}_b(X)$ stands for the set of bounded Radon measures on X, a locally compact set. Thanks to compatibility, it is then easily shown that

$$\eta(t) = \check{\eta}(t) + \beta(t)x_2,$$

where β is an element of $\mathcal{M}_b([t_{\text{crit}}, \infty))$ while $\check{\eta}(t)$ satisfies the following spatial wave equation:

$$\frac{\partial^2 \check{\eta}}{\partial x_1^2} - \frac{\partial^2 \check{\eta}}{\partial x_3^2} = 0.$$

Consequently, one can in particular seek solutions of the form

$$\beta \equiv 0,\ \check{\eta}(t) = \zeta_-(t, x_1 - x_3) + \zeta_+(t, x_1 + x_3),\ \zeta_\pm \in \mathcal{M}_b^+(\mathbb{R}).$$

From this, the computation of $u(t)$ is rather simple since $E\dot{u}(t) = \check{\eta}(t)\sigma_D(t_{\rm crit})$. In particular, if uniqueness is to hold true, then the boundary conditions in $x_3 = 0, \ell$, should determine both ζ_- and ζ_+. But this cannot be so whenever $d < \ell$ because the domains of dependence of those boundary conditions do not cover all of Ω.

Consequently, an infinite number of plastic strain rates can be obtained. In particular, $\dot{p}(t), t \geq t_{\rm crit}$ can be any Cantor function in x multiplied by any element of $C_c^\infty([t_{\rm crit}, \infty)$! So spurious Cantor plastic strain rates can appear and disappear at will, provided that $t \geq t_{\rm crit}$. This provides a rather dramatic example of non-uniqueness (and non-smoothness) of the plastic strain rate in three-dimensional Von Mises elasto-plasticity.

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Evolutionary Γ-convergence results for rate-independent processes in viscoelastic materials

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(joint work with Riccarda Rossi)

This contribution deals with the mathematical modeling and the analysis of mechanical systems that couple rate-dependent and rate-independent processes. A guiding example is the dynamical deformation of a viscoelastic body, located in a domain $\Omega \subset \mathbb{R}^d$, $1 < d \in \mathbb{N}$, under the influence of further dissipative processes such as plasticity, phase transitions or damage, tracked during a time span $[0, T]$. Assuming small strains, the deformation of the body is described by the displacement field $u : [0, T] \times \Omega \to \mathbb{R}^d$ with $e := \frac{1}{2}(\nabla u + \nabla u^\top)$ the small strain tensor. In the spirit of generalized standard materials the changes of the elastic behavior of the material due to the evolving dissipative process are modelled with the aid of an additional internal variable $z : [0, T] \times \Omega \to D$, $D \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d\times d}\}$. It is assumed that the evolution of z is rate-independent, governed by a convex and positively 1-homogeneous potential $\mathcal{R} : \mathbf{Z} \to [0, \infty]$. In contrast, the viscosity of the material is modelled as a rate-dependent effect, featured by a convex dissipation potential $\mathcal{V} : \mathbf{V} \to [0, \infty)$ of general, superlinear growth, which acts on the rate of the displacements $\dot{u}$ or rather on the strain rate $e(\dot{u})$. The spaces $\mathbf{Z}$ and $\mathbf{V}$ are separable Banach spaces, and $\mathbf{V}$ is also reflexive. The energy functional $\mathcal{F} : [0, T] \times \mathbf{W} \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$, $\mathcal{F}(t, \dot{u}, u, z) := \frac{\varrho}{2}\|\dot{u}\|^2_{\mathbf{W}} + \mathcal{E}(t, u, z)$, consists of the kinetic energy $\frac{\varrho}{2}\|\dot{u}\|^2_{\mathbf{W}}$ defined on a Hilbert space $\mathbf{W}$ with $\mathbf{V} \subset \mathbf{W}$ continuously and densely, and of the stored energy functional $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$, which is weakly sequentially lower semicontinuous on $\mathbf{V} \times \mathbf{Z}$ and coercive on a

separable, reflexive Banach space $\mathbf{U} \times \mathbf{X}$ with $\mathbf{U} \subset \mathbf{V}$ continuously and densely, $\mathbf{U} \subset \mathbf{W}$ compactly, and $\mathbf{X} \subset \mathbf{Z}$ compactly. In this way, the spaces $\mathbf{U}, \mathbf{V}, \mathbf{W}$ satisfy that $\mathbf{U} \subset \mathbf{V} \subset \mathbf{W} = \mathbf{W}^* \subset \mathbf{V}^* \subset \mathbf{U}^*$ with continuous, dense embeddings, so that a coupled rate-independent – rate-dependent system is characterized by the sixtuple $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{F}, \mathcal{V}, \mathcal{R})$.

Now, taking into account the convexity properties of the dissipation potentials $\mathcal{V}$ and $\mathcal{R}$ one may apply the Legendre-Fenchel transform to obtain their conjugates $\mathcal{V}^*(\xi_v) := \sup_{v \in \mathbf{V}} (\langle \xi_v, v \rangle_{\mathbf{V}} - \mathcal{V}(v))$ and $\mathcal{R}^*(\xi_z) := \sup_{z \in \mathbf{Z}} (\langle \xi_z, z \rangle_{\mathbf{Z}} - \mathcal{R}(z))$, where, due to the 1-homogeneity of $\mathcal{R}$, the conjugate functional $\mathcal{R}^*$ in particular is the indicator of a convex set. In view of these relations, and assuming the uniform convexity of $\mathcal{E}$, there are several equivalent ways to formulate the evolution equations of the state variables (u, z), namely,

$$(1) \qquad \begin{cases} \text{Rate equations: for a.e. } t \in (0,T): & \\ -\rho \ddot{u}(t) \in \mathrm{D}_{\dot{u}} \mathcal{V}(\dot{u}(t)) + \partial_u \mathcal{E}(t, u(t), z(t)) & \text{in } \mathbf{V}^*, \\ \dot{z}(t) \in \partial_{\xi_z} \mathcal{R}^*(-\xi_z(t)) & \text{in } \mathbf{Z}^*, \end{cases}$$

$$(2) \qquad \Leftrightarrow \quad \begin{cases} \text{Force balance: for a.e. } t \in (0,T): & \\ \dot{u}(t) \in \partial_{\xi_u} \mathcal{V}^*(-\xi_u(t) - \rho \ddot{u}(t)) & \text{in } \mathbf{V}, \\ 0 \in \partial_{\dot{z}} \mathcal{R}(\dot{z}(t)) + \partial_z \mathcal{E}(t, u(t), z(t)) & \text{in } \mathbf{Z}, \end{cases}$$

$$(3) \qquad \Leftrightarrow \quad \begin{cases} \text{De Giorgi's formulation:} & \\ \partial_{\dot{u}} \mathcal{V}(\dot{u}(t)) + \partial_{\xi_u} \mathcal{V}^*(-\xi_u(t) - \rho \ddot{u}(t)) & = -\langle \xi_u(t) + \rho \ddot{u}(t), \dot{u}(t) \rangle_{\mathbf{V}} \\ \partial_{\dot{z}} \mathcal{R}(\dot{z}(t)) + \partial_{\xi_z} \mathcal{R}^*(-\xi_z(t)) & = -\langle \xi_z(t), \dot{z}(t) \rangle_{\mathbf{Z}} \end{cases}$$

with $\xi_u(t) \in \partial_u \mathcal{E}(t, u(t), z(t))$, $\xi_z(t) \in \partial_z \mathcal{E}(t, u(t), z(t))$ for a.e. $t \in (0,T)$. However, as $\mathcal{R}$ is 1-homogeneous, only, and as $\mathcal{E}$ in general needs not to be uniformly convex, the lines of (1)–(3) that involve $\dot{z}$ may not be well defined. Instead, we use ideas from the fully rate-independent theory, cf. e.g. [4], and introduce a weaker notion of solution by replacing the subdifferential formulation for z (second of (1) or (2)) by a semistability inequality and an upper energy dissipation estimate, i.e. we define an *energetic solution* (u, z) for coupled rate-independent – rate-dependent systems as follows: *A pair* (u, z) *is an energetic solution of* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{F}, \mathcal{V}, \mathcal{R})$ *if*

$$(4) \qquad z \in L^\infty(0,T;\mathbf{X}) \cap \mathrm{BV}(0,T;\mathbf{Z}),\ u \in L^\infty(0,T;\mathbf{U}) \cap W^{1,1}(0,T;\mathbf{V}),\ \ddot{u} \in L^\infty(0,T;\mathbf{W}),$$

and if (u, z) *also satisfies the momentum balance, i.e. the first of* (1)*, for a.a.* $t \in (0,T)$*, and, in addition, for all* $t \in [0,T]$*, the upper energy dissipation estimate*

$$(5) \qquad \begin{aligned} \mathcal{F}(t, q(t)) + \textstyle\int_0^t \mathcal{V}(\dot{u}(s)) + \mathcal{V}^*(-\xi_u(s) - \rho \ddot{u}(s))\, \mathrm{d}s + \mathrm{Diss}_{\mathcal{R}}(z; [0,t]) & \\ \leq \mathcal{F}(0, q(0)) + \textstyle\int_0^t \partial_s \mathcal{E}(s, u(s), z(s))\, \mathrm{d}s & \end{aligned}$$

with $q := (\dot{u}, u, z)$ *and* $\mathrm{Diss}_{\mathcal{R}}(z; [0,t]) := \sup_{\text{all partitions of } [0,t]} \sum_j \mathcal{R}(z(t_j) - z(t_{j-1}))$*, together with the semistability inequality*

$$(6) \qquad \forall \tilde{z} \in \mathbf{Z}: \qquad \mathcal{F}(t, \dot{u}(t), u(t), z(t)) \leq \mathcal{F}(t, \dot{u}(t), u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t)).$$

This notion of solution has been introduced and analyzed in [5] for general rate-independent processes in viscoelastic materials with continuous $\mathcal{R}$ and quadratic $\mathcal{V}$ such that $\mathcal{V}(\dot{u}(s))+\mathcal{V}^*(-\xi_u(s)-\varrho\ddot{u}(s)) = 2\mathcal{V}(\dot{u}(s))$ in (5). Here, we allow for more general growth properties so that this identity must not hold true. Let us also mention that a formulation involving (5) has been used for generalized gradient flows ($\varrho = 0$) in e.g. [1, 2], and for the Vlasov-Fokker-Planck equation in [3].

Moreover, we introduce the following weaker notion of solution: *We say that a pair (u, z) of regularity* (4) *is a* weak *energetic solution of $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{F}, \mathcal{V}, \mathcal{R})$ if, for all $t \in [0, T]$, the pair (u, z) only satisfies the upper energy dissipation estimate* (5) *together with semistability* (6) *(but not the momentum balance).*

Existence of (weak) energetic solutions is obtained via a time-discretization in terms of a minimzing movement scheme. At each time-step, existence of minimizers results from the direct method of the calculus of variations. From these, piecewise constant and linear interpolants are constructed and it has to be shown that they satisfy a time-discrete version of the (weak) energetic formulation. By compactness arguments thanks to the time-discrete upper energy dissipation estimate and a version of Helly's selection principle, see e.g. [4], subsequences suitably converging to a limit (u, z) are extracted and it has to be shown that (u, z) is a (weak) energetic solution of the time-continuous process. Existence proofs for $\mathcal{V}$ of superlinear growth, $\mathcal{R}$ lower semicontinuous, with and without inertial terms and for $\mathcal{E}(t, \cdot, z)$ λ-convex and Fréchet-subdifferentiable, are given in [6, I.].

Hence, already passing from time-discrete to -continuous in the existence proof requires that the defining properties of the notion of solution are preserved during this limit passage. For this, suitable compactness and closedness conditions have to be imposed on the functionals. In the fully rate-independent setting such conditions on evolutionary Γ-convergence have been deduced in [4]. For similar results in the setting of generalized gradient systems we refer to [2]. We now state a closedness result which is obtained in [6, II.] for weak energetic solutions: *Consider a family of systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{F}_k, \mathcal{V}_k, \mathcal{R}_k)_k$ with weak energetic solution (u_k, z_k) for all $k \in \mathbb{N}$, such that $(u_k(t), \dot{u}_k(t), z_k(t)) \rightharpoonup (u(t), \dot{u}(t), z(t))$ in $\mathbf{V} \times \mathbf{V} \times \mathbf{Z}$, $\ddot{u}_k(t) \rightharpoonup \ddot{u}(t)$ in $\mathbf{V}^*$ for a.e. $t \in (0, T)$. Let $\mathcal{R} + \mathcal{E} \le \Gamma\text{-}\liminf_k(\mathcal{R}_k + \mathcal{E}_k)$ and $\mathcal{V} + \mathcal{V}^* \le \Gamma\text{-}\liminf_k(\mathcal{V}_k + \mathcal{V}_k^*)$, as well as $\mathcal{F}_k(0, q_k(0)) \to \mathcal{F}(0, q(0))$. Furthermore, assume that the following three closedness conditions are satisfied:*

$$\text{(7a)}\ \tilde{q}_k \rightharpoonup \tilde{q} \text{ in } \mathbf{V} \times \mathbf{Z} \quad \Rightarrow \quad \partial_t \mathcal{E}_k(t, \tilde{q}_k) \to \partial_t \mathcal{E}(t, \tilde{q})\,,$$

$$\text{(7b)}\ \begin{cases} \text{Closedness of semistable sets:} \\ \text{If } z_k \text{ satisfy (6) for } (\mathbf{Z}, \mathcal{E}_k(t, u_k, \cdot), \mathcal{R}_k),\ (u_k, z_k) \rightharpoonup (u, z) \text{ in } \mathbf{V} \times \mathbf{Z}, \\ \text{then } z \text{ is semistable for } (\mathbf{Z}, \mathcal{E}(t, u, \cdot), \mathcal{R}), \end{cases}$$

$$\text{(7c)}\ \begin{cases} \text{Weak}_{\mathbf{V}}\text{-weak}_{\mathbf{V}^*}\text{-closedness of } \cup_k \partial_u \mathcal{E}_k(t, u_k, z_k): \\ \text{If } u_k \rightharpoonup u \text{ in } \mathbf{V},\ z_k \to z \text{ in } \mathbf{Z},\ \sup_k \mathcal{E}_k(t, u_k, z_k) \le C,\ \xi_k \in \partial_u \mathcal{E}_k(t, u_k, z_k), \\ \text{and } \xi_k \rightharpoonup \xi \text{ in } \mathbf{V}^*, \text{ then } \xi \in \partial_u \mathcal{E}(t, u, z). \end{cases}$$

Then, the limit (u, z) is a weak energetic solution of the system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{F}, \mathcal{V}, \mathcal{R})$.

Note that (7a) allows us to pass to the limit with the right-hand side of (5). Condition (7b) is in analogy to the fully rate-independent setting, see [4], and can

be ensured by an adaption of the *mutual recovery sequence condition*. Moreover, (7c) can also be relaxed to strong $\mathbf{V}$-weak $\mathbf{V}^*$-closedness if $\mathcal{E}_k(t,\cdot,z):\mathbf{V}\to\mathbb{R}\cup\{\infty\}$ are equicoercive in $\mathbf{U}\subset\mathbf{V}$ compactly. Another sufficient condition for (7c) can be stated for energetic solutions: Then, (7c) can be concluded if $(\mathcal{E}_k(t,\cdot,z_k))_k$ Mosco-converge to the limit functional $\mathcal{E}(t,\cdot,z)$ and if the momentum balance of the k-systems allows us to deduce that $\limsup_{k\to\infty}\langle\xi_k(t),u_k(t)\rangle_{\mathbf{V}}\le\langle\xi(t),u(t)\rangle_{\mathbf{V}}$.

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Homogenisation of dislocation dynamics

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(joint work with Maria Giovanna Mora and Mark Peletier)

It is well known that plastic, or permanent, deformation in metals is caused by the concerted movement of many curve-like defects in the crystal lattice, called dislocations. What is not yet known is how to use this insight to create theoretical predictions at continuum scales. It would be natural to take a sequence of systems with increasing numbers of dislocations, and derive an effective description in terms of dislocation densities. A mathematical procedure that proved to be very successful for the micro-to-macro upscaling is based on Γ-convergence, a variational convergence that is well known in the mathematical community and has been already applied to a variety of problems in materials science.

In [5] and [10] we used Gamma-convergence to derive a continuum description of the behaviour of *walls of dislocations* close to an obstacle, starting from a discrete model of the dislocation interactions. Our rigorous approach led to a family of upscaled models that we compared with other theories proposed in the literature, offering a selection criterion to identify the hidden assumptions in some of the previous derivations.

In [7] we extended these results to more general distributions of positive dislocations in the plane, still in the single-slip case with Burgers vector $b=e_1$, but removing the constraint for the dislocations to arrange into vertically periodic structures.

More precisely, for a density of dislocations given by $\mu=\frac{1}{n}\sum_{i=1}^n\delta_{z_i}$, with $z_i\in\Omega\subset\mathbb{R}^2$ for every i, we considered a semi-discrete dislocation energy of the

form

$$F_n(\mu) = \inf_{\beta \in \mathcal{A}_n(\mu)} \left\{ \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C}\beta : \beta \, dx \right\}, \tag{1}$$

where $\Omega_n(\mu) = \Omega \setminus \cup_i B_{\varepsilon_n}(z_i)$, with $\varepsilon_n \ll 1/n$, and the admissible class $\mathcal{A}_n(\mu)$ is defined as

$$\mathcal{A}_n(\mu) := \Big\{ \beta \in L^2(\Omega; \mathbb{R}^{2\times 2}) : \int_{\partial B_{\varepsilon_n}(z_i)} \beta \, \tau \, d\mathcal{H}^1 = \frac{e_1}{n} \text{ for every } i = 1, \dots, n \Big\}.$$

It was shown in [3] that the energy contribution in a small neighbourhood of every dislocation is of order $|\log \varepsilon_n|/n^2$ (hence the total *self* energy is of order $|\log \varepsilon_n|/n$), while every pair of dislocations contributes an interaction energy of order n^{-2} (hence the total *interaction* energy is of order 1). Therefore, depending on how ε_n scales with respect to n, one of the two energy contributions will be dominant. On the other hand, the self energy is a (possibly very large) constant and plays no role in the dislocations interactions, which is our main interest.

For this reason, instead of studying the asymptotic behaviour as $n \to \infty$ of the energy F_n in (1), we first subtracted the self energy contribution from F_n, obtaining an interaction energy $\mathcal{F}_n$. In [7] we proved that the Γ-limit of $\mathcal{F}_n$ with respect to narrow convergence is the functional $\mathcal{F}$ given by

$$\mathcal{F}(\rho) = \iint_{\Omega\times\Omega} V(x,y) d\rho(x) d\rho(y) + \min I_\rho(v), \tag{2}$$

where ρ is the limit of discrete measures μ_n. In (2) the interaction potential V is defined as

$$V(x,y) := \int_\Omega \mathbb{C}K_x(z) : K_y(z) \, dz,$$

where K_x is the canonical strain field generated by a single dislocation at x in $\mathbb{R}^2$, namely it solves

$$\begin{cases} \operatorname{div} \mathbb{C}K(\cdot; x) = 0 & \text{in } \mathbb{R}^2, \\ \operatorname{Curl} K(\cdot; x) = e_1 \delta_x & \text{in } \mathbb{R}^2. \end{cases} \tag{3}$$

The second term in (2) is a boundary term due to the boundedness of the domain Ω, and represents their interactions with $\partial\Omega$.

The continuum limit energy (2) has an expression that is common to many (also unrelated) systems of interacting particles. In particular, from (3) follows that $|V(x,y)| \sim -\log|x-y|$ for x close to y, and potentials with a logarithmic singularity have been studied extensively in the literature (see e.g. [1], [9]-[8]). Our work is an important first step towards the analysis of equilibrium dislocation configurations and patterning.

However, macroscopic plasticity is heavily dependent on dynamic properties of the dislocation curves. This motivated us to go further and try to extend our

results to the dynamical case. In [7] we added to the interaction energy $\mathcal{F}_n$ a time-dependent forcing term

$$\int_\Omega f(x,t)\,d\mu(x)$$

and coupled the total microscopic energy $\widetilde{\mathcal{F}}_n(\mu) := \mathcal{F}_n(\mu) - \int_\Omega f d\mu$ with a dissipation distance of the following form:

$$d(\mu,\nu) := \begin{cases} \inf_{\gamma\in\Gamma(\mu,\nu)} \iint_{\Omega\times\Omega} |x-y|\,d\gamma(x,y) & \text{if } (\pi_2)_\#\mu = (\pi_2)_\#\nu, \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

where $\Gamma(\mu,\nu)$ is a restricted set of couplings of μ and ν,

(5)

$$\Gamma(\mu,\nu) := \big\{\gamma\in\mathcal{P}(\Omega\times\Omega) : \gamma(A\times\Omega) = \mu(A),\ \gamma(\Omega\times A) = \nu(A) \text{ for all Borel sets } A\subset\Omega, \text{ and } \pi_2(x) = \pi_2(y) \text{ for } \gamma\text{-a.e. } (x,y)\in\Omega\times\Omega\big\}.$$

This is the usual 1-Wasserstein or Monge-Kantorovich transport distance on $\mathcal{P}(\Omega)$ [11], except for the additional restriction that $\pi_2(x) = \pi_2(y)$ for γ-a.e. (x,y); this restriction forces the transport to move parallel to $b = e_1$.

We then proved the existence of a rate-independent evolution driven by the microscopic total energy $\widetilde{\mathcal{F}}_n$ and by the dissipation d, and used the static Γ-convergence result illustrated above to obtain a rate-independent limit evolution, using the method in [6]. In strong form, the limit continuum dislocation density ρ satisfies the transport equation

$$\partial_t\rho + \operatorname{div}(\rho v) = 0,$$

where velocity v satisfies $v\cdot e_2 = 0$ and

$$-(v(t,x)\cdot e_1)\,\partial_{x_1}\frac{\delta\widetilde{\mathcal{F}}}{\delta\rho}(\rho(t),t), = |v(t,x)|,$$

where $\partial_{x_1}\frac{\delta\widetilde{\mathcal{F}}}{\delta\rho}(\rho(t),t)$ is the horizontal component of the total mesoscopic force acting on a dislocation at x. Hence either $v(x,t) = 0$ or the force equals ± 1. This means that the force has to reach a *threshold* for dislocations to move.

There are several steps we are going to take in the near future towards more realistic and complex systems, including the boundary-layer analysis for pile-ups of dislocations at an obstacle (see [4]), the analysis of dislocation dipoles, and the multiple slip case.

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From conservative to dissipative systems through quadratic time-rescaling

Yann Brenier

From the Euler equations to the Darcy law. Darcy's law can be recovered in an unusual way from Euler's equations

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\nabla p \tag{1}$$

(where $(\rho, p, v) \in \mathbb{R}^{1+1+3}$ are the density, pressure and velocity fields of a fluid, p being a given function of ρ), through the quadratic time rescaling

$$t \to \theta = t^2/2, \quad \rho(t,x) \to \rho(\theta, x), \quad v(t,x)dt \to v(\theta,x)d\theta, \tag{2}$$

leading to $\partial_\theta \rho + \nabla \cdot (\rho v) = 0$, $\rho v + 2\theta[\partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v)] = -\nabla p(\rho)$, and, as $\theta << 1$, to the the Darcy law and the porous medium equation

$$\rho v = -\nabla p(\rho), \quad \partial_\theta \rho = \triangle(p(\rho)).$$

Quadratic time-rescaling of ODEs. Under the quadratic time-rescaling $t \to \theta = t^2/2$, the ordinary dynamical system $\frac{d^2X}{dt^2} = -\nabla\varphi(X)$ becomes $\frac{dX}{d\theta} + 2\theta\frac{d^2X}{d\theta^2} = -\nabla\varphi(X)$, with two asymptotic regimes as θ is small or large:
the gradient flow $\frac{dX}{d\theta} = -\nabla\varphi(X)$, and the inertial motion $\frac{d^2X}{d\theta^2} = 0$. Consistently, the conservation of energy $\frac{d}{d\theta}[\varphi(X)] + \theta\frac{d}{d\theta}|\frac{dX}{d\theta}|^2 = -|\frac{dX}{d\theta}|^2$ leads to the dissipation of energy in the gradient flow regime. Furthermore, we may compare the respective solutions X and Y of the dynamical system and the gradient flow, with initial conditions $\frac{dX}{dt} = 0$, $Y = X$ at $t = 0$, by monitoring the "modulated energy" (or "relative entropy")

$$\frac{1}{2}|\frac{dX}{dt} - t\frac{dY}{d\theta}|^2 + \varphi(X) - \varphi(Y) - \nabla\varphi(Y) \cdot (X - Y), \tag{3}$$

provided φ is strongly convex with bounded third derivatives. We get

$$|X(t) - Y(t^2/2)|^2 + |\frac{dX}{dt}(t) - t\frac{dY}{d\theta}(t^2/2)|^2 \leq t^4 \exp(t^2 c)c.$$

where c is a constant that depends only on φ and $X(0)$.

[Remark: The quadratic time-rescaling $t \to \theta = t^2/2$ exactly fits with Galileo's experiment, in which a rigid ball descends a rigid ramp of constant slope, with zero initial velocity and constant acceleration G, reaching position $X = x_0 + \frac{Gt^2}{2}$ at time t. So, X is just a linear function of the rescaled time θ, $X = x_0 + \theta G$ and we not only get $\frac{dX}{d\theta} + 2\theta\frac{d^2X}{d\theta^2} = G$ but also *simultaneously* $\frac{dX}{d\theta} = G, \quad \frac{d^2X}{d\theta^2} = 0$, i.e. *both* gradient flow *and* inertial motion.]

Derivation of "Darcy's Magnetohydrodynamics" from Born-Infeld's Electromagnetism. The following model, combining Darcy law and dissipative Magnetohydrodynamics:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \rho v = \nabla \cdot (\eta B \otimes B) - \nabla p \tag{4}$$

$$\partial_t B + \nabla \times (B \times v) + \nabla \times (\mu \nabla \times (\nu B)) = 0 \tag{5}$$

(where $(\rho, p, v, B) \in \mathbb{R}^{1+1+3+3}$ are the density, pressure, velocity and magnetic fields, (μ, ν, η, p) being given functions of ρ) looks very far from "first principles". Nevertheless, in the *special* case $\mu = \nu = \eta = \rho^{-1} = -p$ (which involves the "Chaplygin pressure law", sometimes used in Cosmology, with sound speed $(\frac{dp}{d\rho})^{1/2} = \rho^{-1}$), this model can be directly derived, through the simple quadratic time-rescaling $t \to t^2/2$, from the very "pure" Born-Infeld equation, designed in 1934 [1] as a nonlinear substitute to the Maxwell equation and still used in String Theory [3]. In general, the Born-Infeld theory involves a $d+1$ dimensional Lorentzian space-time manifold of metric $g_{ij}dx^i dx^j$ and vector potentials $A = A_i dx^i$ that are critical points of the (fully covariant) "action" $\int \sqrt{-\det(g + dA)}$. Here, we concentrate on the 3+1 Minkowski space of special relativity (as Max Born and Leopold Infeld did in 1934). Then, the Born-Infeld equations read, using classical electromagnetic notations,

$$\partial_t B + \nabla \times (\frac{B \times (D \times B) + D}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}}) = 0, \quad \nabla \cdot B = 0$$

$$\partial_t D + \nabla \times (\frac{D \times (D \times B) - B}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}}) = 0, \quad \nabla \cdot D = 0$$

Using Noether's theorem, we get 4 extra conservation laws

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v - \frac{B \otimes B - D \otimes D}{\rho}) = \nabla(\rho^{-1}),$$

$$v = \frac{D \times B}{\rho}, \quad \rho = \sqrt{1 + D^2 + B^2 + (D \times B)^2}$$

Following [2], we observe that it is consistent (and much simpler) to consider (B, D, ρ, v) as *independent* variables solving the 10×10 *augmented* system (which includes the 4 extra conservation laws)

$$\partial_t B + \nabla \times (B \times v + \rho^{-1} D) = 0, \;\; \partial_t D + \nabla \times (D \times v - \rho^{-1} B) = 0,$$

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \; \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v - \frac{B \otimes B - D \otimes D}{\rho}) = \nabla(\rho^{-1}),$$

while ignoring $v = \frac{D \times B}{\rho}$, $\rho = (1 + D^2 + B^2 + (D \times B)^2)^{1/2}$. (Indeed, these algebraic constraints are preserved by smooth solutions of the ABI system.) We now compute the time-rescaled (augmented) BI equations

$$t \to \theta = t^2/2, \quad \rho, B, v, D \; \to \; \rho, B, v \frac{d\theta}{dt}, D \frac{d\theta}{dt},$$

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta B + \nabla \times (B \times v + \rho^{-1} D) = 0,$$

$$D + 2\theta[\partial_\theta D + \nabla \times (D \times v)] = \nabla \times (\rho^{-1} B),$$

$$\rho v + 2\theta[\partial_\theta (\rho v) + \nabla \cdot (\rho v \otimes v - \frac{D \otimes D}{\rho})] = \nabla \cdot (\frac{B \otimes B}{\rho}) + \nabla(\rho^{-1}).$$

We get, as $\theta << 1$, the desired model of Darcy MHD, with "constitutive laws" $\mu = \nu = \eta = -p = 1/\rho$, while, for very large times $\theta >> 1$, we get

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta D + \nabla \times (D \times v) = 0, \quad \partial_\theta (\rho v) + \nabla \cdot (\rho v \otimes v - \frac{D \otimes D}{\rho}) = 0$$

which actually describes a continuum of vibrating strings.

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Variational Analysis of a reduced Allen-Cahn action functional

Annibale Magni

(joint work with Matthias Röger)

We consider systems in a bounded domain $\Omega \subset \mathbf{R}^n$ having two stable states and admitting a mean field description in terms of a phase function $u : \Omega \to \mathbf{R}$ and with a free energy of the form

$$\mathcal{F}(u) = \int_\Omega \Big(\frac{\epsilon}{2} |\nabla u|^2 + W(u) \Big) dx \,, \tag{1}$$

where $W : \mathbf{R} \to \mathbf{R}$ is a suitable double well potential, and $\epsilon > 0$.
While the *zero–temperature* dynamics of such a system is given by the gradient

flow of (1), the evolution at a temperature $\gamma > 0$ is described by means of the Allen-Cahn equation perturbed by an additive white noise η, i.e.

$$\epsilon\partial_t u = \epsilon\Delta u - \frac{1}{\epsilon}W'(u) + \sqrt{2\gamma}\,\eta\,. \tag{2}$$

Large deviation principles for stochastically perturbed Allen–Cahn equations have been considered among others by [1],[2],[3] and the corresponding action functional for a time $T > 0$ has been computed to be

$$\mathcal{S}_\epsilon(u) = \int_0^T \int_\Omega \Big(\epsilon(\partial_t u)^2 + \frac{1}{\epsilon}(-\epsilon\Delta u + \frac{1}{\epsilon}W'(u))^2\Big)\,dx\,dt\,. \tag{3}$$

For given initial and final states of the system, an action minimizer represents a most likely connecting path between the two states and the value of the minimum of the action is related to the probability that the transition between the two states takes place in the given time T.
In [4] we have studied the minimization of the sharp interface limit of $\mathcal{S}_\epsilon$ for $\epsilon \to 0$, with prescribed initial and final states. The limit functional, also called *reduced* action functional has been computed in [3], [6] and reads

$$\mathcal{S}_0(\Sigma) = \int_0^T \int_{\Sigma_t} \Big(|\vec{v}(x,t)|^2 + |\vec{\mathrm{H}}(x,t)|^2\Big) d\mathcal{H}^n(x)\,dt + 4\sum_{i\in J}\mathcal{H}^n(\Sigma^i)\,, \tag{4}$$

where $\Sigma := (\Sigma_t)_{t\in[0,T]}$ is a family of smoothly evolving smooth hypersurfaces (with normal speed $\vec{v}$ and mean curvature $\vec{\mathrm{H}}$) out of a discrete set of times $J \subset [0,T]$ at which new components can be nucleated.
The straightforward application of the direct method of the Calculus of Variations to the functional (4) in the class of evolving integral varifolds with speed and mean curvature in L^2 fails. This failure is due to the impossibility of keeping track of the initial and final data along a minimizing sequence, which in turn is a consequence of the fact that a bound on (4) ensures only a control in BV((0,T)) for the total area of the evolving varifolds. To overcome this problem, we complemented the evolution of varifolds with a phase evolution according to the following definitions

Definition 1. *Consider a family $\boldsymbol{\mu} := (\mu_t)_{t\in(0,T)}$ of Radon measures on $\mathbf{R}^{n+1}$ and set $\mu := \mu_t \otimes \mathcal{L}^1$. We call $\boldsymbol{\mu}$ an L^2-flow if (for almost all $t \in (0,T)$) μ_t is an integral $n-$varifold with mean curvature $\vec{H} \in L^2(\mu_t;\mathbf{R}^{n+1})$, $\boldsymbol{\mu}$ has generalized normal speed $\vec{v} \in L^2(\mu;\mathbf{R}^{n+1})$ (see* [5]*, Definition 3.1) and $\sup_{0<t<T}\mu_t(\mathbf{R}^{n+1}) < \infty$.*

Definition 2. *Given $T > 0$ and two open sets $\Omega(0), \Omega(T) \subset \mathbf{R}^{n+1}$ with finite perimeter, let $\mathcal{M} = \mathcal{M}(T,\Omega(0),\Omega(T))$ be the class of pairs $(\boldsymbol{\mu},\boldsymbol{u})$, with $\boldsymbol{\mu} := (\mu_t)_{t\in(0,T)}$ and $\boldsymbol{u} := (u_t)_{t\in(0,T)}$, such that the evolution $\boldsymbol{\mu}$ is an L^2-flow, for almost all $t \in (0,T)$*

$$u(\cdot,t) \in BV(\mathbf{R}^{n+1},\{0,1\}), \quad \textit{with} \quad |\nabla u(\cdot,t)| \le \mu_t \tag{5}$$

hold, the evolution of phases $\boldsymbol{u}$ *satisfies* $u \in C^{\frac{1}{2}}([0,T];L^1(\mathbf{R}^{n+1}))$, $\boldsymbol{u}$ *attains the initial and final data*

$$u(\cdot,0) = \chi_{\Omega(0)}, \quad u(\cdot,T) = \chi_{\Omega(T)}, \tag{6}$$

and

$$\int_{\mathbf{R}^{n+1}\times(0,T)} \partial_t\eta(x,t)u(x,t)\,dx\,dt = \int_{\mathbf{R}^{n+1}\times(0,T)} \eta(x,t)\vec{v}(x,t)\cdot\nu(x,t)\,d|\nabla u(\cdot,t)|\,dt \tag{7}$$

holds for all $\eta \in C_c^1(\mathbf{R}^{n+1}\times(0,T))$, *where* $\vec{v}$ *is the generalized velocity of* $\boldsymbol{\mu}$ *and where* $\nu(\cdot,t)$ *denotes the generalized inner normal on* $\partial^*\{u(\cdot,t)=1\}$.

In the class $\mathcal{M}$ of generalized evolutions, we have given a generalized definition for the action functional taking into account also the phase evolution $\boldsymbol{u}$.

Definition 3. *For* $\boldsymbol{\Sigma} \in \mathcal{M}$, $\boldsymbol{\Sigma} = (\boldsymbol{\mu}, \boldsymbol{u})$ *as above, we define*

$$\mathcal{S}(\boldsymbol{\Sigma}) := \mathcal{S}_+(\boldsymbol{\Sigma}) + \mathcal{S}_-(\boldsymbol{\Sigma}), \tag{8}$$

$$\begin{aligned}\mathcal{S}_+(\boldsymbol{\Sigma}) := \sup_\eta \Big[&2|\nabla u(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) \\ &+ \int_{\mathbf{R}^{n+1}\times(0,T)} -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v}-\vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big],\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{S}_-(\boldsymbol{\Sigma}) := \sup_\eta \Big[&-2|\nabla u(\cdot,T)|(\eta(\cdot,T)) + 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) \\ &+ \int_{\mathbf{R}^{n+1}\times(0,T)} 2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v}+\vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big],\end{aligned} \tag{10}$$

where the supremum is taken over all $\eta \in C^1(\mathbf{R}^{n+1}\times[0,T])$ *with* $0 \le \eta \le 1$.

For sufficiently regular evolutions, the generalized action functional agrees with (4), as proven in the following

Theorem 4. *Let* $\boldsymbol{\Sigma}$ *be given by an evolution* $(\Omega(t))_{t\in[0,T]}$ *of open sets* $\Omega(t) \subset \mathbf{R}^{n+1}$, *which means*

$$u(\cdot,t) = \chi_{\Omega(t)} \qquad \text{and} \qquad \mu_t := \mathcal{H}^n \llcorner \partial\Omega(t).$$

Assume that $(\partial\Omega(t))_{t\in[0,T]}$ *represents, outside of a set (possibly empty) of singular times* $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$, *a smooth evolution of smooth hypersurfaces. Then*

$$\mathcal{S}(\boldsymbol{\Sigma}) = \int_0^T \int_{\partial\Omega(t)} (|\vec{v}(\cdot,t)|^2 + |\vec{\mathrm{H}}(\cdot,t)|^2)\,d\mathcal{H}^n\,dt + 2\sum_{j=0}^{k+1} \sup_\psi |\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)|, \tag{11}$$

where the supremum is taken over all $\psi \in C^1(\mathbf{R}^{n+1})$ *with* $|\psi| \le 1$, *and where we have set* $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(0)$ *for* $t < 0$ *and* $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(T)$ *for* $t > T$.

Within the setting of generalized evolutions, we finally were able to apply the direct method and prove existence of global minimizers for $\mathcal{S}$.

Theorem 5. *Let $T > 0$ and let $\Omega(0) \subset \mathbf{R}^{n+1}$, $\Omega(T) \subset \mathbf{R}^{n+1}$ be two given open bounded sets with finite perimeter. Consider a family of evolutions $(\mathbf{\Sigma}_l)_{l\in\mathbb{N}} \subset \mathcal{M}(T, \Omega(0), \Omega(T))$ with*

$$\mathcal{S}(\mathbf{\Sigma}_l) \leq \Lambda \quad \text{for all } l \in \mathbb{N}, \tag{12}$$

where $\Lambda > 0$ is a fixed constant.
Then there exists a subsequence $l \to \infty$ (not relabelled) and a limit evolution $\mathbf{\Sigma} \in \mathcal{M}(T, \Omega(0), \Omega(T))$, such that

$$u^l \to u \qquad \text{in } L^1(Q_T) \cap C^0([0,T]; L^1(\mathbf{R}^{n+1})), \tag{13}$$

$$\mu_t^l \to \mu_t \quad \text{for almost all } t \in (0,T) \text{ as integral varifolds on } \mathbf{R}^{n+1}, \tag{14}$$

$$\mu^l \to \mu \quad \text{as Radon measures on } Q_T, \tag{15}$$

and such that $u \in C^{0,1/2}([0,T]; L^1(\mathbf{R}^{n+1}))$ and $\mu \ll \mathcal{H}^{n+1}$.
Moreover it holds

$$\mathcal{S}(\mathbf{\Sigma}) \leq \liminf_{l\to\infty} \mathcal{S}(\mathbf{\Sigma}_l)\,. \tag{16}$$

In particular, the minimum of $\mathcal{S}$ in $\mathcal{M}(T, \Omega(0), \Omega(T))$ is attained.

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Minimizers of Interaction Energies

José A. Carrillo

The aggregation equation is given by the mean field limit of the system of ODEs

$$\frac{dx_i}{dt} = -\sum_{j\neq i} \nabla W(x_i - x_j)\,.$$

given by

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \\ u = -\nabla W * \rho \end{cases}$$

The aggregation equation has a natural associated Liapunov functional defined by

$$E[\rho] = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(x-y)\ \rho(x)\ \rho(y)\ dx\, dy\,.$$

The first result presented in this talk concerns local minimizers of this interaction energy functional. They are the candidates to be the long time asymptotics of the aggregation equation. We showed in [1] that the dimensionality of the support of local minimizers of the interaction energy $E[\rho]$ depends on the repulsion at the origin of the potential. If the potential is essentially C^2 smooth at the origin, numerical simulations show that there is concentration on points. We showed that the dimension of the support has to be zero if of integer value for smooth potentials. For more singular potentials, we proved a bound from below on the Haussdorf dimension of the support. More precisely, if the potential behaves like the power $-|x|^b/b$ at the origin, with $2-d \leq b < 2$, $d \geq 2$, then the dimension of the support of local minimizers is larger or equal than $2-b$. Therefore, as the potential gets more and more repulsive at the origin the support of the minimizers gets larger and larger in dimension.

The second result concerned the question of global minimizers of the interaction energy. The existence of compactly supported global minimisers is shown in [2] under almost optimal hypotheses. The main assumption on the potential is that it is catastrophic, or not H-stable, which is the complementary assumption to that in classical results on thermodynamic limits in statistical mechanics. The proof is based on a uniform control on the local mass around each point of the support of a global minimiser, together with an estimate on the size of the "gaps" it may have. The class of potentials for which we prove existence of global minimisers includes power-law potentials and, for some range of parameters, Morse potentials, widely used in applications. We also showed that the support of local minimisers is compact under suitable assumptions.

Finally, the third result was dealing with the case of very repulsive potentials at the origin but integrable, that is, when the potential behaves like the power $-|x|^b/b$ at the origin, with $d < b \leq 2-d$, $d \geq 2$. We showed in [3] that the repulsion strength at the origin also determines the regularity of local minimizers of the interaction energy in this range. If this repulsion is like Newtonian or more singular than Newtonian (but still locally integrable), then the local minimizers must be locally bounded densities (and even continuous for more singular than Newtonian repulsion). We proved this (and some other regularity results) by first showing that the potential function associated to a local minimizer solves an obstacle problem and then by using classical regularity results for such problems.

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