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Geometric Methods of Complex Analysis

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Abstract. The purpose of this workshop was to discuss recent results in Several Complex Variables, Complex Geometry and Complex Dynamical Systems with a special focus on the exchange of ideas and methods among these areas. The main topics of the workshop included Pluripotential Theory and the Monge-Ampère equation, Complex Dynamics, Almost Complex Geometry, Geometric Questions of Complex Analysis (including Theory of Foliations) and Applications, the $\bar{\partial}$-equation and Geometry.

Mathematics Subject Classification (2010): 32xx, 53xx, 14xx, 37Fxx.

Introduction by the Organisers

The workshop *Geometric Methods of Complex Analysis* attracted 49 researchers from 16 countries. Both, leading experts in the field and young researchers (including three Ph. D. students and two postdocs) were well represented in the meeting and gave talks. A rather wide spectrum of topics related to Complex Analysis (and this was one of the aims of the workshop) was covered by the talks and informal discussions. All 22 lectures presented on the meeting can be conditionally divided into the following groups.

Pluripotential Theory and the Monge-Ampère equation was represented by talks of S. Kołodziej, D. Witt Nyström, T. Harz and H. C. Lu. Kołodziej presented a stability result for solutions of the Monge-Ampère equation on a compact Hermitian manifold. As an application of this result one can show, in particular, Hölder continuity of the solution. Witt Nyström discussed regularity of solutions to the complex homogeneous Monge-Ampère equation. He also gave examples whose perturbed solutions are not "almost smooth" which contradict the results of X. X.

Chen and G. Tian. Harz discussed some properties of the *core* of a domain, {i.e. the set of all points of this domain such that the Levi form of every bounded above plurisubharmonic function has degeneration at this point}. In particular, he gave a characterization of the core of a pseudoconvex domain under the additional assumption that this core has a structure of the product with some $\mathbb{C}^k$. Lu presented results on the regularity of the maximal Kähler-Ricci flow.

Complex Dynamics was represented by the talks of H. Peters and E. Bedford. Peters explained that there exist polynomial maps of $\mathbb{C}^2$ and holomorphic endomorphisms of $\mathbb{P}^2$ with a wandering Fatou component. Bedford discussed the dynamical expansion of a polynomial automorphism of $\mathbb{C}^2$ and related to its geometry.

Almost Complex Geometry was represented by the talk of J.-P. Demailly who discussed embedding results for compact almost complex manifolds into complex algebraic varieties thus giving a partial answer to a problem from 1996 by Bogomolov.

Geometric Questions of Complex Analysis (including Theory of Foliations) and Applications was represented by the talks of N. Sibony, F. Lárusson, M. Tsukamoto, J. Winkelmann, J.-M. Hwang, K.-T. Kim, J. Globevnik, E. Wold and F. Forstnerič. Sibony presented results on value distribution theory for parabolic Riemann surfaces motivated by the Green-Griffiths conjecture. Lárusson discussed the interpolation property and convex interpolation property from Stein spaces into affine toric varieties. Tsukamoto calculated the mean dimension (in the sense of Gromov) of the space of all Brody curves in $\mathbb{C}P^n$ thus giving an answer to a problem of Gromov from 1999. Winkelmann explained that a compact Kähler manifold is rationally connected (and hence also projective) if there is a map from $\mathbb{C}^n$ to this manifold of small growth. Hwang discussed how the local differential geometry of the given web-structure on a projective manifold affects its global algebraic geometry. Kim presented semi-continuity results for automorphism groups of domains in $\mathbb{C}^2$ with D'Angelo finite type boundary. Globevnik explained the construction of a complete closed hypersurface immersed in the unit ball of $\mathbb{C}^n$. This solves a long standing open problem of Yang from 1977. Wold gave sharp estimates of the squeezing function on a strictly pseudoconvex domain. Forstnerič presented a characterization of the minimal hull of a compact set in $\mathbb{R}^3$ analogous to the classical characterization by Poletsky of the polynomial hull by sequences of holomorphic discs.

The $\bar\partial$-equation and Geometry were represented by the talks of Z. Błocki, X. Zhou, J. Ruppenthal, B.-Y. Chen, E. Wulcan and T. Ohsawa. Błocki presented lower bounds for the Bergmann kernel on the diagonal of a pseudoconvex domain in terms of the Azukawa indicatrix and the Kobayashi indicatrix of its pluricomplex Green function. Zhou discussed the proof of Demailly's strong openness conjecture, Demailly-Kollár's conjecture and Jonsson-Mustaţă's conjecture. Ruppenthal explained how the Koppelman type formulas recently developed by Andersson and Samuelsson can be used in the study of L^2-cohomologies of isolated singularities, in particular of A_1-singularities. Chen presented results describing complex analytic properties of Riemannian surfaces in terms of their Riemannian geometry. Wulcan explained how one can define generalized cycles and how they are related to the local intersection numbers. Ohsawa gave a survey on recent developments in the problem of holomorphic extensions, both effective and noneffective.

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Workshop: Geometric Methods of Complex Analysis

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Abstracts

Value Distribution Theory for Parabolic Riemann Surfaces

Nessim Sibony
(joint work with Mihai Păun)

A conjecture by Green-Griffiths states that if X is a projective manifold of general type, then there exists an algebraic proper subvariety of X which contains the image of all holomorphic curves from the complex plane to X. To our knowledge, the general case is far from being settled. We question here the choice of the complex plane as a source space.

Let $\mathcal{Y}$ be a parabolic Riemann surface, i.e bounded subharmonic functions defined on $\mathcal{Y}$ are constant. The results of Nevanlinna's theory for holomorphic maps f from $\mathcal{Y}$ to the projective line are parallel to the classical case when $\mathcal{Y}$ is the complex line except for a term involving a weighted Euler characteristic. Parabolic Riemann surfaces could be hyperbolic in the Kobayashi sense.

Let X be a manifold of general type, and let A be an ample line bundle on X. It is known that there exists a holomorphic jet differential P (of order k) with values in the dual of A. If the map f has infinite area and if $\mathcal{Y}$ has finite Euler characteristic, then f satisfies the differential relation induced by P. As a consequence, we obtain a generalization of Bloch Theorem concerning the Zariski closure of maps f with values in a complex torus. We then study the degree of Nevanlinna's current $T[f]$ associated to a parabolic leaf of a foliation $\mathcal{F}$ by Riemann surfaces on a compact complex manifold. We show that the degree of $T[f]$ on the tangent bundle of the foliation is bounded from below in terms of the counting function of f with respect to the singularities of $\mathcal{F}$, and the Euler characteristic of $\mathcal{Y}$. In the case of complex surfaces of general type, we obtain a complete analogue of McQuillan's result: a parabolic curve of infinite area and finite Euler characteristic tangent to $\mathcal{F}$ is not Zariski dense. That requires some analysis of the dynamics of foliations by Riemann Surfaces.

Extending holomorphic maps from Stein manifolds into affine toric varieties

Finnur Lárusson
(joint work with Richard Lärkäng)

In Oka theory we study approximation and interpolation problems for holomorphic maps from Stein spaces into a complex manifold. We are interested in manifolds for which there are only topological obstructions to solving such problems. This is the class of *Oka manifolds*.

To make this more precise, say that a complex manifold Y has the *interpolation property* (IP) if a holomorphic map to Y from a subvariety S of a reduced Stein space X has a holomorphic extension to X if it has a continuous extension.

Define the ostensibly much weaker *convex interpolation property* (CIP) by taking $X = \mathbb{C}^n$ and S to be a contractible submanifold.

By a deep and fundamental theorem of Forstnerič, CIP implies IP. The manifold Y is said to be an *Oka manifold* if it satisfies CIP, IP, or any of a dozen other nontrivially equivalent properties.

Our basic question is: *What if Y is allowed to be singular?*

The targets Y that we study are affine toric varieties, always assumed irreducible but *not necessarily normal* ("nnn"). Our main results may be roughly summarised as follows.

- Every nnn affine toric variety satisfies a weakening of the interpolation property that is much stronger than the convex interpolation property.
- The full interpolation property fails for most nnn affine toric varieties, even for a source as simple as the product of two annuli embedded in $\mathbb{C}^4$.

In particular, the implication CIP $\Rightarrow$ IP fails for singular targets, even for smooth sources.

More precisely, our main results are the following four theorems.

Theorem 1. *Let Y be a nnn affine toric variety. Let S be a factorial subvariety of a reduced Stein space X such that $H^p(X,\mathbb{Z}) \to H^p(S,\mathbb{Z})$ is surjective for $p = 0, 1, 2$. Then every holomorphic map $S \to Y$ extends to a holomorphic map $X \to Y$.*

As a very particular consequence, every nnn affine toric variety satisfies the convex interpolation property. The key to the proof of the theorem is a new notion of a *twisted factorisation* of a nondegenerate holomorphic map into Y. We call a holomorphic map $f : S \to Y \subset \mathbb{C}^n$ *nondegenerate* if the image by f of each irreducible component of S intersects the torus in Y. Equivalently, no component of f is identically zero on any component of S. (We take a nnn affine toric variety Y in $\mathbb{C}^n$ to be embedded in $\mathbb{C}^n$ as the zero set of a prime lattice ideal. Then the torus in Y is $Y \cap (\mathbb{C}^*)^n$.)

Recall that S is factorial if the stalk $\mathscr{O}_x$ of the structure sheaf of S is a factorial ring for every $x \in S$. Then S is normal and hence locally irreducible, so its connected components and irreducible components are the same. Factoriality is a strong property, not much weaker than smoothness, but it is a natural assumption in our work. The key consequence of factoriality is that Weil divisors and Cartier divisors are the same.

Theorem 2. *Let Y be a nnn affine toric variety of dimension d in $\mathbb{C}^n$ with $0 \in Y$.*

(a) *Suppose that the normalisation of Y is $\mathbb{C}^d$. If S is a normal subvariety of a reduced Stein space X, then every nondegenerate holomorphic map $S \to Y$ extends to a holomorphic map $X \to Y$.*

(b) *Suppose that the normalisation of Y is not $\mathbb{C}^d$. There is a smooth surface S in $\mathbb{C}^4$, biholomorphic to the product of two annuli, and a nondegenerate holomorphic map $S \to Y$ that does not extend to a holomorphic map $\mathbb{C}^4 \to Y$.*

Here, 0 denotes the origin in $\mathbb{C}^n$. A variety Y as in the theorem is contractible, so the extension problem has no topological obstruction.

The following dichotomy refines the first case of the previous one. Note that case (a) refers to arbitrary, possibly degenerate holomorphic maps. We have an example showing that for (b) to hold in general, the source $\mathbb{C}\times\{0\}\cup\{(0,1)\}$ needs to be disconnected.

Theorem 3. *Let Y be a nnn affine toric variety of dimension d in $\mathbb{C}^n$ with $0\in Y$. Suppose that the normalisation of Y is $\mathbb{C}^d$.*

(a) *Suppose that Y is locally irreducible. If S is a seminormal subvariety of a reduced Stein space X, then every holomorphic map $S\to Y$ extends to a holomorphic map $X\to Y$.*

(b) *Suppose that Y is not locally irreducible. There is a degenerate holomorphic map $\mathbb{C}\times\{0\}\cup\{(0,1)\}\to Y$ that does not extend to a holomorphic map $\mathbb{C}^2\to Y$.*

The following examples illustrate the three kinds of varieties in Theorems 2 and 3.

- The map $\mathbb{C}^2\to\mathbb{C}^4$, $(s,t)\mapsto(s,t^2,t^3,st)$, is the normalisation of its image Y. The map induces a homeomorphism $\mathbb{C}^2\to Y$, so Y is locally irreducible.
- Whitney's umbrella in $\mathbb{C}^3$, defined by the equation $x^2y=z^2$, has normalisation $\mathbb{C}^2\to Y$, $(s,t)\mapsto(s,t^2,st)$, and is not locally irreducible.
- The cone in $\mathbb{C}^3$ defined by the equation $xy=z^2$ is normal but of course not isomorphic to $\mathbb{C}^2$.

Finally, in the normal case, we have the following result.

Theorem 4. *Let Y be a singular nondegenerate normal affine toric variety.*

(a) *Let S be a connected factorial subvariety of a connected reduced Stein space X such that $H^2(X,\mathbb{Z})\to H^2(S,\mathbb{Z})$ is surjective. Then every holomorphic map $S\to Y$ extends to a holomorphic map $X\to Y$.*

(b) *There is a smooth surface S in $\mathbb{C}^4$, biholomorphic to the product of two annuli, and a nondegenerate holomorphic map $S\to Y$ that does not extend to a holomorphic map $\mathbb{C}^4\to Y$.*

A normal affine toric variety may be factored as $Y\times(\mathbb{C}^*)^k$, where the normal affine toric variety Y has no torus factors, and is thus said to be nondegenerate, and Y is naturally embedded in some $\mathbb{C}^n$ so as to contain the origin. Then Y is contractible, and Y is singular unless Y is isomorphic to some $\mathbb{C}^m$. The extension problem into $\mathbb{C}^*$ is of course fully understood: the hypothesis on H^1 in Theorem 1 is only there to take care of torus factors. Thus Theorem 4 follows from Theorems 1 and 2.

We also have the following further results.

It is well known in Oka theory that interpolation yields approximation.

Proposition 5. *Let Ω be a Runge domain of finite embedding dimension in a factorial Stein space X, such that $H^p(\Omega,\mathbb{Z})=0$ for $p=1,2$. Let Y be a nnn* *affine toric variety, and let $f : \Omega \to Y$ be a holomorphic map. Then f can be approximated uniformly on compact subsets of Ω by holomorphic maps $X \to Y$.*

Interpolation on a discrete subset is always possible.

Proposition 6. *Let S be a discrete subset of a reduced Stein space X. Let Y be a nnn affine toric variety. Then every map $S \to Y$ extends to a holomorphic map $X \to Y$.*

The following result is known, but we offer a new proof.

Proposition 7. *A factorial affine toric variety is smooth.*

Finally, meromorphic extensions are readily obtained.

Proposition 8. *Let S be a factorial subvariety of a reduced Stein space X. Let $Y \subset \mathbb{C}^n$ be a nnn affine toric variety with $0 \in Y$. Let $f : S \to Y$ be a nondegenerate holomorphic map. Then f extends to a meromorphic map $X \to Y$.*

We expect that our results could be extended to arbitrary toric varieties without much difficulty. On the other hand, what Oka theory for general singular targets might look like is entirely unclear. It remains to be seen what sort of general theorems, if any, one could expect. Clearly, allowing the targets to have even a single, mild, isolated singularity makes a big difference.

Finally, many thanks to the organisers for the invitation to speak on this work and for organising such a pleasant and stimulating workshop.

The complex Monge-Ampère equation on compact Hermitian manifolds

Sławomir Kołodziej
(joint work with N. C. Nguyen)

This is joint work with N.C. Nguyen. The main result is the stability of solutions of the Monge-Ampère equation on a compact Hermitian manifold (X, ω).

Theorem 1. Let $0 < c_0 \leq f, 0 \geq g \in L^p(X)$, be such that there exist continuous ω-psh solutions of the equation

$$(\omega + dd^c u)^n = f\omega^n; \quad (\omega + dd^c v)^n = g\omega^n,$$

with $\sup_X u = 0 = \sup_X v$. Fix $0 < \alpha < 1/(n+1)$. Then, there exists $C = C(\omega, \alpha, c_0, ||f||_p)$ such that

$$||u - v|| \leq ||f - g||_p^\alpha.$$

One can use the stability to show that under the above assumptions for f the solution of the equation is Hölder continuous.

Theorem 2. Suppose $0 < c_0 \leq f \in L^p(X)$, and consider ω-psh solution of the Monge-Ampère equation

$$(\omega + dd^c u)^n = f\omega^n,$$

on a compact Hermitian manifold. Then u is Hölder continuous for any Hölder exponent smaller than $2/[p^* n(n+1)+1]$, with p^* being the conjugate of p.

The second application of the stability result is an extension of a theorem of Székelyhidi and Tosatti to the case of compact Hermitian manifolds.

Theorem 3. Let (X, ω) be a compact n-dimensional Hermitian manifold. Suppose that u is bounded and ω-psh solution of

$$(\omega + dd^c u)^n = \exp(-F(u, z)),$$

in the weak sense of currents, where F is smooth. Then u is smooth.

Mean dimension of the dynamical system of Brody curves

Masaki Tsukamoto

Mean dimension is a topological invariant for dynamical systems with infinite dimension and infinite entropy, introduced by Gromov [1]. Our main purpose here is to show that this notion reveals a new fundamental structure in holomorphic curve theory.

Let $z = x + \sqrt{-1}y$ be the standard coordinate in $\mathbb{C}$. Let $f = [f_0 : f_1 : \cdots : f_N] : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic curve. We define the spherical derivative $|df|(z) \geq 0$ by

$$|df|^2(z) := \frac{1}{4\pi}\Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2) \quad \left(\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We call f a **Brody curve** if $|df| \leq 1$ all over the plane. We define $\mathcal{M}(\mathbb{C}P^N)$ as the space of all Brody curves $f : \mathbb{C} \to \mathbb{C}P^N$ endowed with the compact-open topology. This is compact and the group $\mathbb{C}$ continuously acts on it by

$$\mathbb{C} \times \mathcal{M}(\mathbb{C}P^N) \to \mathcal{M}(\mathbb{C}P^N), \quad (a, f(z)) \mapsto f(z+a).$$

This is an infinite dimensional and infinite entropy system. We denote by $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$ the mean dimension of the $\mathbb{C}$-action on $\mathcal{M}(\mathbb{C}P^N)$. This is a nonnegative real number which counts the number of parameters in $\mathcal{M}(\mathbb{C}P^N)$ "per unit area of the plane $\mathbb{C}$".

Main Problem (Gromov 1999). Calculate the mean dimension $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. We define its **energy density** $\rho(f)$ by

$$\rho(f) := \lim_{R\to\infty}\left(\frac{1}{\pi R^2}\sup_{a\in\mathbb{C}}\int_{|z-a|<R}|df|^2 dxdy\right).$$

We define $\rho(\mathbb{C}P^N)$ as the supremum of $\rho(f)$ over $f \in \mathcal{M}(\mathbb{C}P^N)$.

Main Theorem (arXiv:1410.1143).

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) = 2(N+1)\rho(\mathbb{C}P^N).$$

A key ingredient is an information theoretic version of mean dimension introduced by Lindenstrauss–Weiss [2]. We also use the analytic machineries developed in [3].

References

[1] M. Gromov, *Topological invariants of dynamical systems and spaces of holomorphic maps: I*, Math. Phys. Anal. Geom. **2** (1999), 323–415.
[2] E. Lindenstrauss, B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24.
[3] S. Matsuo, M. Tsukamoto, *Brody curves and mean dimension*, Journal of the American Mathematical Society **28** (2015), 159–182.

Rationality and growth conditions

Jörg Winkelmann
(joint work with Frédéric Campana)

This is joint work with Frédéric Campana ([4]).

Let X be a Kähler compact complex manifold and let $f : \mathbb{C}^n \to X$ be a differentiably non-degenerate meromorphic map. Our goal is to relate algebraic-geometric properties of X to the existence of such maps of small growth. One easily sees that for every unirational projective manifold X there is a rational such map, hence in particular a map of very small growth. We look for a result in the opposite direction. The main result is:

If there exists such a map of order $\rho_f < 2$, then X must be rationally connected. In particular, X is projective.

Here the order ρ_f is a tool to measure the growth of a meromorphic map $f : \mathbb{C}^n \to X$. For an algebraic map we have necessarily $\rho_f = 0$. On the other hand $\rho_\tau = 2$ for the universal covering map $\tau : \mathbb{C}^n \to T$ of a n-dimensional compact complex torus.

The order is defined in the following way: Let ω be a Kähler form on X and let α be the euclidean Kähler form on $\mathbb{C}^n$, i.e., $\alpha = dd^c||z||^2$. Define the characteristic function for $f : \mathbb{C}^n \to X$ as

$$T_f(r) = \int^r \frac{dt}{t^{2n-1}} \int_{B_t} (f^*\omega) \wedge \alpha^{n-1}.$$

Then the order is defined as

$$\limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

A compact complex manifold is called "rationally connected" (RC) if for any two points there exists a chain of rational curves linking these two points ([1]). RC Kähler manifolds are automatically projective.

Given a projective complex manifold X, there exists an RC-quotient $\phi : X \to Y$ ([1],[6]). This is a meromorphic map such that generic fibers are maximal RC subvarieties of X. The quotient has only few rational curves, more precisely it is not uniruled ([5]).

Not being uniruled is equivalent to the canonical bundle being pseudoeffective ([2]). In this way a projective manifold X which is not RC obtains (by pulling back the canonical bundle from Y) an invertible pseudoeffective subsheaf of some Ω_X. For a line bundle pseudoeffectivity is equivalent to admitting a singular hermitian metric with semipositive curvature which in turn implies that $\log ||s||$ is plurisubharmonic for every holomorphic section s. This plurisubharmonicity is then used to deduce a lower bound for $||Df||$ from which one may conclude that $\rho_f \geq 2$ unless X is RC. This is the key line of reasoning in the case where X is projective. To show that the statement for general compact Kähler manifolds, one observes the following: Kodairas argument combined with Hodge theory imply that every non-projective compact Kähler manifold admits non-zero holomorphic 2-forms, which can be used to deduce a lower bound on the growth of f, following the reasoning in [7].

This result improves on earlier work of Campana, Păun ([3]), Noguchi and Winkelmann ([7]).

The result does not hold without the Kähler assumption, in fact there is non-degenerate map $f : \mathbb{C}^2 \to S$ for a Hopf surface S with $\rho_f = 1$ although Hopf surfaces do not contain any rational curves ([7]).

References

[1] Campana, F.: *Connexité rationnelle des variétés de Fano.* Ann. Sci. Ec. Norm. Sup **25**, 539–545 (1992)

[2] Campana, F.; Demailly, J.P.; Peternell, Th.: *Rationally Connected Manifolds and semipositivity of the Ricci Curvature.* arxiv:1210.2092

[3] Campana, F.; Păun, M.: *Une généralisation du théorème du Kobayashi-Ochiai.* Manu. math., **125**, no. 4, 411–426 (2008)

[4] Campana, F.; Winkelmann, J.: *Rational connectedness and order of non-degenerate meromorphic maps from* $\mathbb{C}^n$. (submitted; arXiv:1412.4381)

[5] Graber, T.; Harris, J.; Starr, J.: *Rationally connected varieties.* J. Alg., **1**, 429–448 (1992)

[6] Kollár, J.; Miyaoka, Y.; Mori, S.: *Rational connectedness and boundedness of Fano manifolds.* J. Differential Geom. **36** (1992), no. 3, 765–779

[7] J. Noguchi, J. Winkelmann: *Order of Meromorphic Maps and rationality of the Image Space.* J. Math. Soc. Japan **64**, (4), 1169–1180 (2012)

Algebro-differential embeddings of compact complex structures

Jean-Pierre Demailly
(joint work with Hervé Gaussier)

The goal of this talk was to present an embedding theorem for compact almost complex manifolds into complex algebraic varieties, cf. [DG14] for details. An almost complex manifold of dimension n is a pair (X, J_X), where X is a C^∞ real manifold of dimension $2n$ and J_X a C^∞ section of $\mathrm{End}(TX)$ such that $J_X^2 = -\mathrm{Id}$.

Let Z be a complex projective manifold of dimension N. Such a manifold carries a natural integrable almost complex structure J_Z. Now, assume that we are given an algebraic distribution $\mathcal{D}$ in TZ, namely a holomorphic subbundle $\mathcal{D} \subset TZ$. Every fiber $\mathcal{D}_x$ of the distribution is then invariant under J_Z, i.e. $J_Z\mathcal{D}_x \subset \mathcal{D}_x$ for every $x \in Z$. Here, the distribution $\mathcal{D}$ is not assumed to be integrable. We recall that $\mathcal{D}$ is integrable in the sense of Frobenius (i.e. stable under the Lie bracket operation) if and only if the fibers $\mathcal{D}_x$ are the tangent spaces to leaves of a holomorphic foliation. More precisely, $\mathcal{D}$ is integrable if and only if the torsion operator θ of $\mathcal{D}$, defined by

$$\theta : \mathcal{O}(\mathcal{D}) \times \mathcal{O}(\mathcal{D}) \longrightarrow \mathcal{O}(TZ/\mathcal{D}),\ (\zeta, \eta) \longmapsto [\zeta, \eta] \mod \mathcal{D} \tag{1}$$

vanishes identically. As is well known, θ is skew symmetric in (ζ, η) and can be viewed as a holomorphic section of the bundle $\Lambda^2\mathcal{D}^* \otimes (TZ/\mathcal{D})$.

Let M be a real submanifold of Z of class $\mathcal{C}^\infty$ and of real dimension $2n$ with $n < N$. We say that M is *transverse to* $\mathcal{D}$ if for every $x \in M$ we have

$$T_xM \oplus \mathcal{D}_x = T_xZ. \tag{2}$$

One can in fact assume more generally that the distribution $\mathcal{D}$ is singular, i.e. given by a certain saturated subsheaf $\mathcal{O}(\mathcal{D})$ of $\mathcal{O}(TZ)$. Then $\mathcal{O}(\mathcal{D})$ is actually a subbundle of TZ outside an analytic subset $\mathcal{D}_{\mathrm{sing}} \subset Z$ of codimension ≥ 2, and we further assume in this case that $M \cap \mathcal{D}_{\mathrm{sing}} = \emptyset$. When M is transverse to $\mathcal{D}$, one gets a natural $\mathbb{R}$-linear isomorphism

$$T_xM \simeq T_xZ/\mathcal{D}_x \tag{3}$$

at every point $x \in M$. Since $TZ/\mathcal{D}$ carries a structure of holomorphic vector bundle (at least over $Z \smallsetminus \mathcal{D}_{\mathrm{sing}}$), the complex structure J_Z induces a complex structure on the quotient and therefore, through the above isomorphism (3), an almost complex structure $J_M^{Z,\mathcal{D}}$ on M. Bogomolov made the following basic observation.

Proposition 1. *When $\mathcal{D}$ is a foliation (i.e. $\mathcal{O}(\mathcal{D})$ is an integrable subsheaf of $\mathcal{O}(TZ)$), then $J_M^{Z,\mathcal{D}}$ is an* integrable *almost complex structure.*

In general, the Nijenhuis tensor $N_J(\xi, \eta) = [\xi^{0,1}, \eta^{0,1}]^{1,0}$ of $J = J_M^{Z,\mathcal{D}}$ can be expressed by

$$N_J \in C^\infty(X, \Lambda^{0,2}TX \otimes T^{1,0}X), \quad N_J(\zeta, \eta) = 4\,\theta(\overline{\partial}_J f \cdot \zeta, \overline{\partial}_J f \cdot \eta), \tag{4}$$

for all vector fields $\xi, \eta \in C^\infty(X, T^{0,1}X)$. This shows again that $\theta \equiv 0$ implies $N_J \equiv 0$. Moreover, assuming $\overline{\partial} f$ injective, integrable complex structures are obtained precisely when the image of $\overline{\partial} f$ lies in the (algebraic) isotropic locus

$$\mathrm{Iso}_{Z,\mathcal{D}} := \big\{S \in \mathrm{Gr}(\mathcal{D}, n)\,;\ \theta_{|S\times S} = 0\big\} \subset \mathrm{Gr}(\mathcal{D}, n)$$

in the Grassmannian bundle of n-dimensional subspaces associated with $\mathcal{D}$. The next interesting question is to understand what happens when one considers a variation of transverse embeddings.

Proposition 2. *Assume again that $\mathcal{D}$ is a foliation. If $f_t : X \to (Z, \mathcal{D})$, $t \in [0, 1]$ is a smooth isotopy of transverse embeddings, i.e. a smooth family such that each f_t is an embedding of X onto a transverse submanifold $M_t = f_t(X)$, then the induced complex structures (X_t, J_{f_t}) are all biholomorphic through a smooth family of diffeomorphisms in $\mathrm{Diff}^0(X)$, i.e. the identity component in the group of diffeomorphisms.*

The following basic question somehow suggests that every compact complex manifold should admit such an algebraic realization, starting from algebraic data Z, $\mathcal{D}$, and a (topological) isotopy class α of transverse embeddings.

Basic question 3 (Bogomolov [Bog96])**.** *For any compact complex manifold (X, J), does there exist a triple $(Z, \mathcal{D}, \alpha)$ formed by a smooth complex projective variety Z, an algebraic foliation $\mathcal{D}$ on Z and an isotopy class α of transverse embeddings $X \to Z$, such that $J = J_f$ for some $f \in \alpha$?*

There are indeed many examples of Kähler and non Kähler compact complex manifolds which can be embedded in that way (the case of projective ones being of course trivial): compact complex tori, Hopf and Calabi-Eckman manifolds, and more generally all manifolds given by the so-called LVMB construction [Bos01], due to López de Medrano, Verjovsky, Meersseman and Bosio. In the non integrable case, one can further obtain a precise formula for the variation of induced almost complex structures.

Theorem 4. *For $r \in [1, +\infty[$, let $\mathcal{J}^r(X)$ be the space of C^r almost complex structures on X, and let $\widetilde{\Gamma}^r(X, Z, \mathcal{D})$ be the space of transverse embeddings $X \hookrightarrow (X, \mathcal{D})$ of class C^r that are transversally C^{r+1}. Then*

(i) *$\mathcal{J}^r(X)$ is a Banach manifold whose tangent space at a point J is $C^r(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX))$, and $\widetilde{\Gamma}^r(X, Z, \mathcal{D})$ is also a Banach manifold. Its tangent space at a point $f : X \to Z$ is*

$$C^r(X, f^*\mathcal{D}) \oplus C^{r+1}(X, TX).$$

(ii) *The natural map $f \mapsto J_f$ sends $\widetilde{\Gamma}^r(X, Z, \mathcal{D})$ in $\mathcal{J}^r(X)$;*

(iii) *The differential dJ_f of $f \mapsto J_f$ on $\widetilde{\Gamma}^r(X, Z, \mathcal{D})$ is a continuous morphism*

$$C^r(X, f^*\mathcal{D}) \oplus C^{r+1}(X, TX) \longrightarrow C^r(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX)), \quad (u, v) \longmapsto 2i\big(\theta(\overline{\partial} f, u) + \overline{\partial} v\big).$$

If $\overline{\partial} f$ is injective and such that $u \mapsto \theta(\overline{\partial} f(x) \bullet, u)$ defines a surjection from $\mathcal{D}_x$ to $\mathrm{End}_{\overline{\mathbb{C}}}(T_x X))$ at every point $x \in X$, then the implicit function theorem implies that $f \mapsto J_f$ is a submersion. This requires $\mathrm{rank}(\mathcal{D}) = N - n \geq n^2$, hence the dimension of Z must be at least quadratic in n. It turns out that there actually exist universal embedding spaces for this problem. They are in some sense a combination of Grassmannians and twistor spaces.

Theorem 5. *For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $N = 2k + 2(k^2 + n(k - n))$ possessing a real* *structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution* $\mathcal{D}_{n,k} \subset TZ_{n,k}$ *of codimension* n*, for which every compact* n*-dimensional almost complex manifold* (X, J) *admits an embedding* $f : X \hookrightarrow Z^{\mathbb{R}}_{n,k}$ *transverse to* $\mathcal{D}_{n,k}$ *and contained in the real part of* $Z_{n,k}$*, such that* $J = J_f$*. Moreover,* f *can be chosen to depend in a simple algebraic way on the almost complex structure* J *selected on* X*, and* J_f *is integrable if and only if the image of* $\overline{\partial}_{J_f}$ *lies in the isotropic locus* $\mathrm{Iso}_{Z_{n,k},\mathcal{D}_{n,k}} \subset \mathrm{Gr}(\mathcal{D}_{n,k}, n)$*.*

The choice $k = 4n$ yields the explicit embedding dimension $N = 38n^2 + 8n$, and we have seen that a quadratic bound $N = O(n^2)$ is indeed optimal. In case (X, J) carries a J-compatible symplectic structure ω of type $(1, 1)$, the embedding theorem of Tischler and Gromov for symplectic structures allows to construct similar universal spaces equipped with an additional transverse Kähler structure β, such that $f^*\beta = \omega$. They have dimensions growing linearly in $b = h^{1,1}(X)$ and quadratically in n. Finally, there is an interesting connection between Bogomolov's conjecture and a natural approximation problem for holomorphic foliations.

Proposition 6. *Assume that holomorphic foliations can be approximated by Nash algebraic foliations uniformly on compact subsets of any polynomially convex open subset of* $\mathbb{C}^N$*. Then every compact complex manifold can be approximated by compact complex manifolds that are embeddable in the sense of Bogomolov into foliated projective manifolds.*

References

[Bos01] Bosio, F., *Variétés complexes compactes : une généralisation de la construction de Meersseman et López de Medrano-Verjovsky*, Ann. Inst. Fourier, Grenoble **51** (2001), 1259–1297.

[Bog96] Bogomolov, F., *Complex Manifolds and Algebraic Foliations*, Publ. RIMS-1084, Kyoto, June 1996 (unpublished).

[DG14] Demailly, J.-P., Gaussier, H., *Algebraic embeddings of smooth almost complex structures*, preprint hal-01092620 and arXiv: math.CV/1412.2899, 21 pages.

Webs of curves and their web-structures

Jun-Muk Hwang

A (1-dimensional) *web-structure* on a complex manifold U is a collection of line subbundles $\{W_i \subset T(U), 1 \leq i \leq d\}$ such that $W_i \cap W_j = 0$ if $i \neq j$. We say that the web-structure is *bracket-generating* if the distribution $\sum_{i=1}^d W_i$ on U is bracket-generating in the sense of distribution. We say that the web-structure is *pairwise nonintegrable* if for each W_i, there exists some $W_j, j \neq i$, such that the distribution $W_i + W_j$ on U is not integrable. Given a web-structure $\{W_i \subset T(U), 1 \leq i \leq d\}$ on a complex manifold U and a web-structure $\{W'_i \subset T(U'), 1 \leq i \leq d\}$ on a complex manifold U', we say that a biholomorphic map $\varphi : U \to U'$ is an *equivalence of web-structures* if $\mathrm{d}\varphi(\cup_{i=1}^d W_i) = \cup_{j=1}^d W'_j$.

We are interested in web-structures arising from families of algebraic curves covering a projective variety in the following way. A (not necessarily irreducible) family $\mathcal{W}$ of algebraic curves covering a projective variety X is called a *web of curves* if there are only finitely many members of $\mathcal{W}$ through a general point of X. Given a general point of X, we can find a Euclidean neighborhood U of the point contained in the smooth locus of X such that members of $\mathcal{W}$ give a web-structure on U, to be denoted by $\mathcal{W}|_U$. We say that $\mathcal{W}$ is *bracket-generating* (resp. *pairwise nonintegrable*) if the web-structure $\mathcal{W}|_U$ is bracket-generating (resp. pairwise nonintegrable).

We are interested in how the local differential geometry of the web-structure $\mathcal{W}|_U$ affects the global algebraic geometry of X. Our main result is the following.

Theorem 1 Let $\mathcal{W}$ (resp. $\mathcal{W}'$) be a web of curves on a projective variety X (resp. X'), which is bracket-generating and pairwise nonintegrable. Let $\varphi : U \to U'$ be a biholomorphic map between connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ which is an equivalence of $\mathcal{W}|_U$ and $\mathcal{W}'|_{U'}$. Then φ can be extended to a generically finite algebraic correspondence between X and X', i.e., there exists a projective variety $\Gamma \subset X \times X'$ which is generically finite over both X and X' such that $\mathrm{Graph}(\varphi) \subset \Gamma$.

The original motivation for this theorem is to prove the following.

Theorem 2 Let $X, X' \subset \mathbf{P}^N$ be two projective manifolds covered by lines of projective space $\mathbf{P}^N$. Assume that $b_2(X) = b_2(X') = 1$. Let $\varphi : U \to U'$ be a biholomorphic map between two connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ such that φ (resp. φ^{-1}) sends pieces of lines in U (resp. U') to pieces of lines in U' (resp. U). Then there exists a biholomorphic map $\Phi : X \to X'$ such that $\varphi = \Phi|_U$.

This was proved in my joint paper with Ngaiming Mok, *Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1* (Journal Math. Pures Appl. **80** (2001) 563-575) under the assumption that the family of lines passing through a general point of X and X' has positive dimension. The remaining part of Theorem 2 is exactly when the families of lines on X and X' form webs of curves. One can use the condition $b_2(X) = b_2(X') = 1$ to check that these webs of curves are both bracket-generating and pairwise nonintegrable. Thus Theorem 1 gives an extension of φ to a generically finite algebraic correspondence between X and X'. Once this is done, we use a more delicate structure of the family of lines covering X and X' to show that the correspondence actually determines a biholomorphic map between X and X'.

Lower Bounds for the Bergman Kernel

Zbigniew Błocki

We discuss lower bounds for the Bergman kernel on the diagonal in terms of pluripotential theory. For $n = 1$ the main result is the following estimate, conjectured by Suita [4] and proved in [1]:

$$K_\Omega(w,w) \geq \frac{1}{\pi} c_\Omega(w)^2,$$

where $c_\Omega(w)$ is the logarithmic capacity of $\mathbb{C} \setminus \Omega$ w.r.t. w. In arbitrary dimension a stronger estimate from [2] is the following one:

$$K_\Omega(w,w) \geq \frac{1}{e^{-2nt}\lambda(\{G_\Omega(\cdot,w) < t\})},$$

where $\Omega \subset \mathbb{C}^n$ is pseudoconvex, G_Ω is the pluricomplex Green function and $t \leq 0$. This estimate is especially interesting when $t \to -\infty$. Using this one can get the following bound from [3]:

$$K_\Omega(w,w) \geq \frac{1}{\lambda(I^A_\Omega(w))},$$

where

$$I^A_\Omega(w) = \{X \in \mathbb{C}^n \colon \limsup_{\zeta \to 0}(G_\Omega(w + \zeta X, w) - \log|\zeta|) \leq 0\}$$

is the Azukawa indicatrix, and for convex Ω

$$K_\Omega(w,w) \geq \frac{1}{\lambda(I^K_\Omega(w))},$$

where

$$I^K_\Omega(w) = \{\varphi'(0) \colon \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$$

is the Kobayashi indicatrix.

References

[1] Z. Błocki, *Suita conjecture and the Ohsawa-Takegoshi extension theorem*, Invent. Math. 193 (2013), 149–158

[2] Z. Błocki, *A lower bound for the Bergman kernel and the Bourgain-Milman inequality*, Geometric Aspects of Functional Analysis, Israel Seminar (GAFA) 2011-2013, eds. B. Klartag, E. Milman, Lect. Notes in Math. 2116, pp. 53–63, Springer, 2014

[3] Z. Błocki, W. Zwonek, *Estimates for the Bergman kernel and the multidimensional Suita conjecture*, arXiv:1404.7692

[4] N. Suita, *Capacities and kernels on Riemann surfaces*, Arch. Ration. Mech. Anal. 46 (1972), 212–217

Harmonic Discs of Solutions to the Complex Homogeneous Monge-Ampère Equation

David Witt Nyström
(joint work with Julius Ross)

I will report on joint work with Julius Ross (for more details see the preprint arXiv:1408.6663).

Let (X, ω) be a compact Kähler manifold of dimension n and $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. Consider boundary data consisting of a family $\omega + dd^c\phi(\cdot, \tau)$ of Kähler forms where $\phi(\cdot, \tau)$ is a smooth function on X for $\tau \in \partial\mathbb{D}$. The Dirichlet problem for the complex Homogeneous Monge Ampère equation (HMAE) asks for a function Φ on $X \times \overline{\mathbb{D}}$ such that

$$\begin{aligned} \Phi(\cdot, \tau) &= \phi(\cdot, \tau) \text{ for } \tau \in \partial\mathbb{D}, \\ \pi^*\omega + dd^c\Phi &\geq 0, \\ (\pi^*\omega + dd^c\Phi)^{n+1} &= 0. \end{aligned}$$

We say Φ is a *regular solution* if it is smooth and $\omega + dd^c\Phi(\cdot, \tau)$ is a Kähler form for all $\tau \in \overline{\mathbb{D}}$. By an example of Donaldson [7], we know there exist smooth boundary data for which there does not exist a regular solution (see also the work of Lempert-Vivas [9] and Darvas-Lempert [6] on geodesic segments). Nevertheless, the equation always has a unique weak solution, which by the work of Chen [4] with complements by Błocki [3] we know is at least "almost" $C^{1,1}$ (so in particular $C^{1,\alpha}$ for any $\alpha < 1$). See [8] for a recent survey.

A more subtle aspect of the regularity of solutions to the HMAE is the question of existence and distribution of harmonic discs.

Definition. Let $g \colon \mathbb{D} \to X$ be holomorphic. We say that the graph of g is a *harmonic disc* (with respect to Φ) if Φ is $\pi^*\omega$-harmonic (i.e. $\pi^*\omega + dd^c\Phi$ vanishes) along this graph.

As is well known, a regular solution to the HMAE yields a complex foliation of $X \times \overline{\mathbb{D}}$ whose leaves restrict to harmonic discs in $X \times \mathbb{D}$. Even when the solution is not regular, the existence of such harmonic discs is important; for instance along such a harmonic disc the density of the varying measure $\omega^n_{\phi(\cdot,\tau)}$ is essentially log-subharmonic (see [1] [5] [2, Sec 3.2]).

It was hoped that any weak solution would enjoy a weaker form of regularity, so that a dense open subset of $X \times \mathbb{D}$ would be foliated by harmonic disc, but as we will see this is not always the case.

I will describe a correspondence between on the one hand the HMAE when $X = \mathbb{P}^1$ and the boundary data has a certain kind of symmetry and on the other hand the so-called "Hele-Shaw" flow in the plane. As a result we see that the set of harmonic discs is determined by the topology of the flow.

To state precise results, let ω_{FS} denote the Fubini-Study form on $\mathbb{P}^1$ and ϕ be a smooth Kähler potential, i.e. a smooth function on $\mathbb{P}^1$ such that $\omega_{FS} + dd^c\phi$

is Kähler. Let ρ denote the usual $\mathbb{C}^\times$-action on $\mathbb{P}^1$ which acts by multiplication on $\mathbb{C} \subset \mathbb{P}^1$. We consider the function $\phi(z,\tau) := \phi(\rho(\tau)z)$ as boundary data over $\mathbb{P}^1 \times \partial\mathbb{D}$, so for each $\tau \in \partial\mathbb{D}$ we have a Kähler form $\omega_{FS} + dd^c\phi(\cdot,\tau)$. We show that the solution Φ to the Homogeneous Monge-Ampère equation with this boundary data is intimately connected to the Hele-Shaw flow

$$\Omega_t := \{z : \psi_t(z) < \phi(z)\}$$

where

$$\psi_t := \sup\{\psi : \psi \text{ is usc and } \psi \leq \phi \text{ and } \omega_{FS} + dd^c\psi \geq 0 \text{ and } \nu_0(\psi) \geq t\}.$$

By this we mean the supremum is over all upper semicontinuous (usc) functions from $\mathbb{P}^1$ to $\mathbb{R} \cup \{-\infty\}$ with these properties, and $\nu_0(\psi)$ denotes the order of the logarithmic singularity (Lelong number) of ψ at $0 \in \mathbb{C} \subset \mathbb{P}^1$. In fact, we show that the solution Φ and the family ψ_t are related via a Legendre transform.

Using this we prove the following:

Theorem 1. *Let Φ be the solution to the HMAE with boundary data ϕ and $g\colon \mathbb{D} \to \mathbb{P}^1$ be holomorphic. Then the graph of g is a harmonic disc of Φ if and only if either*

1. *$g \equiv 0$, or*
2. *$g(\tau) = \tau^{-1}z$ for some fixed $z \in \mathbb{P}^1 \setminus \Omega_1$, or*
3. *$\tau \mapsto \tau g(\tau)$ is a Riemann mapping for a simply connected Hele-Shaw domain Ω_t that maps $0 \in \mathbb{D}$ to $0 \in \Omega_t$.*

The Hele-Shaw flow Ω_t has a physical interpretation as describing the expansion of a liquid in a medium with permeability inversely proportional to $\Delta(\phi + \ln(1+|z|^2))$. Guided by this one can rather easily find potentials ϕ for which at some time t the flow domain Ω_t becomes multiply connected.

This then translates into an obstruction to the presence of harmonic discs of the associated solution to the HMAE:

Theorem 2. *There exist smooth boundary data $\phi(\cdot,\tau)$ for which the solution to the Dirichlet problem for the HMAE has the following property: there exists an open set U in $\mathbb{P}^1 \times \overline{\mathbb{D}}$ meeting $\mathbb{P}^1 \times \partial\mathbb{D}$, such that no harmonic disc intersects U.*

Next we address the question whether generic boundary data give rise to solutions with a weak form of regularity. The following theorem answers that question negatively.

Theorem 3. *There exist smooth boundary data $\phi(\cdot,\tau)$ for which the following is true: there exist a nonempty open set U' in $\mathbb{P}^1 \times \mathbb{D}$ and an $\epsilon > 0$ such that if $\phi'(z,\tau)$ is any smooth boundary data with*

$$\|\phi' - \phi\|_{C^2(\mathbb{P}^1 \times \partial\mathbb{D})} < \epsilon$$

and Φ' is the associated solution to the HMAE then no harmonic disc (associated to Φ') passes through U'.

The first theorem gives solutions that are not "partially smooth" and the second examples whose perturbed solutions are not "almost smooth", in apparent contradiction with [5, Thm. 1.3.2] and [5, Thm. 1.3.4], respectively.

References

[1] E Bedford and D Burns *Holomorphic mapping of annuli in* $\mathbf{C}^n$ *and the associated extremal function.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 6 (1979), no. 3, 381–414.

[2] R Berman and B Berndtsson *Convexity of the K-energy on the space of Kähler metrics* (2014) Preprint arXiv:1405.0401

[3] Z Błocki *On geodesics in the space of Kahler metrics* Advances in Geometric Analysis, ed. S. Janeczko et al., Advanced Lectures in Mathematics 21, pp. 3-20, International Press, 2012.

[4] X X Chen *The Space of Kähler Metrics* J. Differential Geom. Volume 56, Number 2 (2000), 189–234.

[5] X X Chen and G Tian *Geometry of Kähler metrics and foliations by holomorphic discs.* Publ. Math. Inst. Hautes Études Sci. No. 107 (2008), 1–107.

[6] T Darvas and L Lempert *Weak geodesics in the space of Kähler metrics* Math. Res. Lett. 19 (2012), no. 5, 1127–1135.

[7] S K Donaldson *Holomorphic discs and the complex Monge-Ampère equation.* J. Symplectic Geom. 1 (2002), no. 2, 171–196.

[8] Vincent Guedj (Editor) *Complex Monge-Ampère Equations and Geodesics in the Space of Kähler Metrics* Lecture Notes in Mathematics 2038, Springer 2012

[9] L Lempert and L Vivas *Geodesics in the space of Kähler metrics* Duke Math. J. 162 (2013), no. 7, 1369–1381.

[10] J Ross and D Witt Nyström *The Hele-Shaw flow and moduli of holomorphic discs* (2012) Preprint arXiv:1212.2337.

A two-dimensional polynomial mapping with a wandering Fatou component

Han Peters

(joint work with Matthieu Astorg, Xavier Buff, Romain Dujardin and Jasmin Raissy)

Consider a holomorphic self-map of a complex manifold. The Fatou set is the largest open set on which the family of iterates is locally normal. Its components are called Fatou components.

In 1982 Dennis Sullivan proved that every Fatou component of a rational function is periodic or pre-periodic, which completed the classifications of Fatou components on the Riemann sphere. Here we show that there exist polynomial maps in 2 complex dimensions for which there are Fatou components that are not periodic or pre-periodic, so called wandering Fatou components. Our examples extend to holomorphic endomorphisms of $\mathbb{P}^2$.

To be more precise, our examples are two-dimensional polynomial skew-products of the form

$$F(z,w) = \left(f(z) + \frac{\pi^2}{4}w, g(w)\right),$$

where $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of the form

$$f(z) = z + z^2 + az^3 + O(z^4),$$

and the polynomial g satisfies

$$g(w) = w - w^2 + O(w^3).$$

The constant a is chosen in a small disk with the point 1 in its boundary, so that the *Lavaurs map* of f has an attracting fixed point. We use tools developed for the study of one-dimensional parabolic implosion to prove the existence of the wandering Fatou components.

The existence of Fatou components for polynomials was considered by Krastio Lilov in 2004, who proved that in a neighborhood of a super-attracting fiber there cannot exist wandering Fatou components. More recently Liz Vivas and Peters showed that in an attracting fiber there can exist Fatou *disks* if resonance occurs between the rate of the attraction and the rate of repulsion of a repelling fixed point in the invariant fiber. In the current work there is also some sort of resonance between the parabolic behaviors in both directions.

This is joint work with Matthieu Astorg, Xavier Buff, Romain Dujardin and Jasmin Raissy, and is based on an original idea of Misha Lyubich.

L^2 extensions and multiplier ideal sheaves

Xiangyu Zhou

In this talk, we'll present our recent work on the multiplier ideal sheaves, namely, our proof of Demailly's strong openness conjecture, immediate consequences of the conjecture, solutions of further problems including a conjecture of Demailly-Kollár, a conjecture of Jonsson-Mustaţă, effectiveness in the strong openness conjecture, and the structure of the multiplier ideal sheaves associated to the psh functions with Lelong number 1.

1. Introduction

1.1. **Definition.** Associated to a given plurisubharmonic function φ on a complex manifold X, the multiplier ideal sheaf is the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ of germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-\varphi}$ is locally integrable near $x \in X$.

1.2. **First properties.**

- $\mathcal{I}(\varphi)$ is a coherent analytic sheaf.
- $\mathcal{I}(\varphi)$ is integrally closed.
- Nadel vanishing theorem.

2. Demailly's strong openness conjecture and corollaries

2.1. **Statement and result.**

Denote by $\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi)$.
Strong openness conjecture ([3], [4]): *Let φ be a psh function on X. Then*

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

Remark. Assuming further that $\mathcal{I}(\varphi) = \mathcal{O}_X$, it was proved by Berndtsson [1].

Theorem 2.1 ([7], [8])**.** *Demailly's strong openness conjecture holds.*

The main idea of the proof is also presented during the talk, which is a combining use of Ohsawa-Takegoshi L^2 extension theorem, curve selection lemma and a key observation in one complex variable.

2.2. **Corollaries.**

Corollary 2.2. *Let $(L, e^{-\varphi})$ be a pseudo-effective line bundle (i.e., φ is psh) on a compact Kähler manifold X of dimension n, Then*

$$H^p(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $p \geq n - nd(L, \varphi) + 1$.

This was conjectured in [2].
Remark. When the bundle is a big line bundle on X (i.e., the curvature current Θ of the singular Hermitian metric is a Kähler current: $\exists \epsilon > 0$, s.t., $\Theta \geq \epsilon\omega$), it's known that $nd(L) = n$. Therefore the theorem reduces to Nadel vanishing theorem.

Corollary 2.3. *For the big line bundle L, the equality*

$$\mathcal{J}(||mL||) = \mathcal{J}(h^m_{\min}) \tag{1}$$

holds for every integer $m > 0$.

This was conjectured in [5].

Corollary 2.4. *One has $\{\mathcal{I}(\varphi)\} = \{\mathcal{I}(\varphi_A)\}$, where φ is a psh function and φ_A is a psh function with analytic singularities.*

3. Further problems

We mention also the following results.

- A conjecture of Demailly-Kollár ([6]) is proved in [9];
- A conjecture of Jonsson-Mustaţă ([13]) is proved in [9];
- Effectiveness conditions in the strong openness conjecture are found in [11];
- The structure of the multiplier ideal sheaves of the psh functions with Lelong number 1 is obtained in [12].

In the proofs of the first three results, some idea and method about sharp L^2 extension theorem in [10] are used. In the proof of the fourth result, the solution of Demailly's strong openness conjecture and Siu's analyticity theorem and decomposition theorem are used.

References

[1] B. Berndtsson, *The openness conjecture for plurisubharmonic functions*, arXiv:1305.5781.

[2] J.Y. Cao, *Numerical dimension and a Kawamata-Viehweg-Nadel type vanishing theorem on compact Kähler manifolds*, arXiv:1210.5692.

[3] J-P. Demailly, *Multiplier ideal sheaves and analytic methods in algebraic geometry.* School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[4] J.-P. Demailly, *Analytic Methods in Algebraic Geometry*, Higher Education Press, Beijing, 2010.

[5] J-P. Demailly, L. Ein, and R. Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. 48 (2000), 137–156.

[6] J-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds.* Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.

[7] Q.A. Guan and X.Y. Zhou, *Strong openness conjecture for plurisubharmonic functions*, arXiv:1311.3781.

[8] Q.A. Guan and X.Y. Zhou, *A proof of Demailly's strong openness conjecture*, Ann. of Math., accepted, 2015.

[9] Q.A. Guan and X.Y. Zhou, *Strong openness conjecture and related problems for plurisubharmonic functions*, arXiv:1401.7158.

[10] Q.A. Guan and X.Y. Zhou, *A solution of an L^2 extension problem with optimal estimate and applications*, arXiv:1310.7169. Ann. of Math., 181 (2015), 1139–1208.

[11] Q.A. Guan and X.Y. Zhou, *Effectiveness of Demailly's strong openness conjecture and related problems*, Invent. Math. published online, 2015.

[12] Q.A. Guan and X.Y. Zhou, *Classification of multiplier ideal sheaf with Lelong number one weight*, arXiv:1411.6737.

[13] M. Jonsson and M. Mustaţă, *An algebraic approach to the openness conjecture of Demailly and Kollár*, J. Inst. Math. Jussieu (2013), 1–26.

[14] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. 195 (1987), 197–204.

[15] Y.T. Siu, *Multiplier ideal sheaves in complex and algebraic geometry.* Sci. China Ser. A 48 (2005), suppl., 1–31.

L^2-cohomology of isolated singularities, integral representation formulas and canonical singularities

Jean Ruppenthal

The L^2-$\overline{\partial}$-cohomology of isolated singularities of complex spaces can be described in terms of a resolution of singularities. This has been achieved recently in [OV] and [R1], where resolutions of singularities are used to give smooth models for the L^2-cohomology with respect to different $\overline{\partial}$-operators. By use of L^2-Serre duality for $\overline{\partial}$-cohomology classes, [R2], one obtains an almost complete picture.

However, several questions remain open. For example, if a $\overline{\partial}$-equation is solvable at an isolated singularity, then it is desirable to give an explicit solution operator. Here, the L^2-methods from [OV], [R1] and [R2] are not really helpful and integral representation formulas for singular complex spaces come into play. Also in this area, big progress has been made recently by Andersson–Samuelsson [AS] who introduced Koppelman formulas (i.e., $\overline{\partial}$-homotopy formulas) for singular complex spaces. Clearly, these $\overline{\partial}$-homotopy formulas cannot hold in the L^2-sense in general because we know that there exist obstructions to solving the $\overline{\partial}$-equation in the L^2-sense. Actually, the integral operators of Andersson–Samuelsson are defined in general only as principal value integrals which must be applied to smooth forms (and something derived from smooth forms, the so-called $\mathcal{A}$-sheaves). They cannot be applied to L^2-forms on arbitrary varieties.

But, if the $\overline{\partial}$-equation is solvable in the L^2-sense, then there is a good hope that the solution can be provided explicitly by the Andersson–Samuelsson operators. Here singularities as appearing in the minimal model program come into play. It is shown in [R2] that the $\overline{\partial}$-equation is solvable in the L^2-sense for $(0,q)$-forms and (n,q)-forms at canonical singularities (let n be the dimension of the complex space). Moreover, canonical singularities have also a "good" influence on the singularity of the integral kernels in the Andersson–Samuelsson operators. Besides the BMK-part, $\|\zeta - z\|^{1-2n}$, the singularity of the integral kernels consists of the so-called structure form which measures the "badness" of the singularity. These structure forms seem to be less harmful for canonical singularities. So, it appears reasonable to study the mapping properties of the Andersson–Samuelsson operators for canonical singularities.

In a joint project with Richard Lärkäng, [LR], we pursued this idea for the A_1-singularity, i.e., the isolated singularity at the origin of the cone $X = \{z^2 = xy\} \subset \mathbb{C}^3$. For this singularity, it is shown in [LR] that the Andersson–Samuelsson operators map continuously from $L^p(X)$ to $L^p(X)$ for $p > 4/3$, and from $L^\infty(X)$ to $C^0(X)$. Using these (and some even stronger) mapping properties, it is also shown that the Koppelman formulas from [AS] provide actually $\overline{\partial}$-homotopy formulas in the L^2-sense (for all the different $\overline{\partial}$-operators we are interested in). It seems possible to extend the methods from [LR] to treat arbitrary canonical isolated singularities. That is work in progress with Richard Lärkäng, Håkan Samuelsson-Kalm and Elizabeth Wulcan.

All in all, the principle idea behind this research is the following: It seems that for singularities as appearing in the Minimal Model Program, i.e., canonical singularities, the L^2-theory for the $\overline{\partial}$-operator could be possible as in the smooth case. Unfortunately, this cannot be achieved by classical L^2-methods at the moment. But, the gaps could be filled by the Andersson–Samuelsson formulas. Moreover, as usually, these integral representation formulas have several other applications. They allow for example to treat other function spaces (L^p for $p \neq 2$, C^α) besides just L^2, and they could provide further regularity results for the $\overline{\partial}$-equation.

There is research from a completely different direction in Complex Geometry which substantiates the idea that canonical singularities are nice with respect to the L^2-theory for the $\overline{\partial}$-operator. In [GKKP] it is shown that holomorphic p-forms behave very well on complex spaces with canonical singularities: reflexive differential forms extend holomorphically over the exceptional set of a resolution of singularities. From that one can deduce the following statement: Consider a holomorphic differential form ϕ on the regular part of a complex projective variety. A priori, ϕ could have arbitrary growth at the singular set, but if the variety has only canonical singularities, then ϕ is square-integrable.

References

[AS] M. Andersson, H. Samuelsson, A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas, *Invent. Math.* **190** (2012) 261–297.

[GKKP] D. Greb, S. Kebekus, S. Kovács, T. Peternell, Differential Forms on Log Canonical Spaces, Publ. Math. de l'IHÉS, **114** (2011), 87–169.

[LR] R. Lärkäng, J. Ruppenthal Koppelman formulas on the A_1-singularity, Preprint 2014, arXiv:1407.5703, submitted.

[OV] N. Øvrelid, S. Vassiliadou, L^2-$\overline{\partial}$-cohomology groups of some singular complex spaces, *Invent. Math.* **192** (2013), no.2, 413–458.

[R1] J. Ruppenthal, L^2-theory for the $\overline{\partial}$-operator on compact complex spaces, *Duke Math. J.* **163** (2014), 2887–2934.

[R2] J. Ruppenthal, Serre duality and L^2-vanishing theorems on singular spaces, Preprint 2014, arXiv:1401.4563, submitted.

The Bergman metric and isoperimetric inequalities

Bo-Yong Chen

Let M be an orientable surface. By means of patching up together local metrics through a partition of unity, one can construct many Riemannian metrics on M. Thanks to the famous Korn-Lichtenstein theorem, every Riemannian metric ds^2 induces a complex structure on M. Thus there exist two seemingly complete different theories on the same target, one is Riemannian geometry, the other is complex analysis, both initiated by Riemann himself. The purpose of this research is to study complex analysis via Riemannian geometry. More precisely, we are interested in the following problem:

Problem 1. *Let M be an open Riemann surface with a complex structure induced by some complete Riemannian metric ds^2. How does the (Riemannian) geometry of ds^2 influence the behavior of the Bergman kernel or metric?*

Popular conditions in Riemannian geometry are curvature, volume, etc. However, these conditions are not strong enough for studying Bergman analysis, certain *global* condition is still needed.

Definition 1. *Let M be a complete Riemannian manifold. For each $0 < \nu \le \infty$, the $\nu-$dimensional isoperimetric constant of M is defined by*

$$I_\nu(M) = \inf |\partial\Omega|/|\Omega|^{1-1/\nu}$$

where the infimum is taken over all precompact domains $\Omega \subset M$ with a smooth boundary, and $|\cdot|$ stands for the volume.

Theorem 1. *Let M be a complete Riemannian surface. Let d denote the corresponding distance. Suppose either of the following conditions is verified:*

(1) *$I_\nu(M) > 0$, for some $2 < \nu < \infty$;*

(2) *$I_\infty(M) > 0$, the Gaussian curvature is bounded below by a constant, and*

$$\inf_{x\in M} |B_1(x)| > 0,$$

where $B_r(x)$ stands for the geodesic ball with center x and radius r.

Then the Bergman distance d_B satisfies

$$d_B(x, y) \geq \text{const.}\, d(x, y)$$

for all $x, y \in M$ with $d(x, y) \geq 2$. In particular, M is Bergman complete.

Theorem 2. *Let M be a complete Riemannian surface such that the Gaussian curvature is bounded below by a constant.*

(1) *If $I_\infty(M) > 0$, then for any number $\alpha < 1/36$ there exists a constant $C > 0$ such that*

$$|K_M(x, y)| \leq \frac{C}{\sqrt{|B_1(x)|}\sqrt{|B_1(y)|}} \exp\left\{-\alpha I_\infty(M)^2 d(x, y)\right\}$$

for all $x, y \in M$.

(2) *If $I_\nu(M) > 0$ for some $2 < \nu < \infty$, then for any number $\alpha < 1/16$ there exists a constant $C > 0$ such that*

$$|K_M(x, y)| \leq C\, d(x, y)^{\alpha(2-\nu)}, \quad \forall\, x, y \in M.$$

Finally, recall a fundamental concept from Riemannian geometry as follows:

Definition 2. *A sequence (M_j, ds_j^2) of complete Riemannian manifolds is said to converge in the sense of Cheeger-Gromov to a complete Riemannian manifold (M, ds^2) if there exist*

(1) *a sequence of points $p_j \in M_j$ and a point $p \in M$;*

(2) *a sequence of precompact open sets $\Omega_j \subset M$ exhausting M, with $p \in \Omega_j$ for each j;*

(3) *a sequence of smooth maps $\phi_j : \Omega_j \to M_j$ which are diffeomorphic onto their image and satisfy $\phi_j(p) = p_j$;*

such that $\phi_j^*(ds_j^2) \to ds^2$ in the sense that for all compact subsets $K \subset M$, the tensor $\phi_j^*(ds_j^2) - ds^2$ and its covariant derivatives of all orders (with respect to any fixed background connection) each converge uniformly to zero on K.

Theorem 3. *Let (M_j, ds_j^2) be a sequence of complete Riemannian surfaces which converge in the sense of Cheeger-Gromov to a complete Riemannian surface (M, ds^2) and satisfy*

$$I_\infty(M) > 0 \quad \text{and} \quad \inf_j I_\infty(M_j) > 0.$$

Let $ds^2_{B,j}$ (resp. ds^2_B) denote the Bergman metric of the Riemann surface M_j (resp. M). Then $ds^2_{B,j}$ converges to ds^2_B in the sense that for all compact subsets $K \subset M$, the tensor $\phi_j^*(ds^2_{B,j}) - ds^2_B$ and its covariant derivatives of all orders (with respect to any fixed background connection) converge uniformly to zero on K.

Dynamical Expansion and its Geometry

Eric Bedford

First we consider a polynomial $p(z) : \mathbf{C} \to \mathbf{C}$, and then we consider a polynomial automorphism (Hénon map) $f : \mathbf{C}^2 \to \mathbf{C}^2$. In both cases, the chaotic part of the dynamics is carried by the Julia set J. We will discuss the philosophical idea that

Dynamical expansion of the iterates of the map on J
can be seen in the geometry of the Julia sets J, J^+ and J^-.

In dimension one, the most straightforward sort of expansion is *uniform expansion* or *hyperbolicity*, which means that there are $c > 0$ and $\lambda > 1$ such that $|p^n(z)'| \geq c\lambda^n$ for all $z \in J$. In this case, J will have the *fractal* property that the general shape of J at large scale is repeated "everywhere" at arbitrarily small scales. More precisely, if U is a (small) open set with $U \cap J \neq \emptyset$, then there is a number $N = N_U$ such that $p^N(J \cap U) = J$. By the Koebe Distortion Theorem, all geometry of J is contained already in the small piece $U \cap J$.

Thus uniform expansion on J implies a fractal-like geometry on J. Conversely, are there geometric properties of J that will imply some sort of expansion? We discuss a result of [CJY] which brings together expansion properties of a polynomial $p(z)$ and geometric properties of J.

The first of these is an expansion property called *semi-hyperbolicity*. For this, let $B(z, r)$ denote the disk of radius r centered at a point $z \in J$, and let B_n denote any connected component of $p^{-n}(B(z, r)$. Thus $p^n|_{B_n} : B_n \to B(z, r)$ is proper. The map p is *semi-hyperbolic* if there is a number $M < \infty$ such that the mapping degree of $p^n|_{B_n} : B_n \to B(z, r)$ is no greater than M.

Let $A_\infty = \{z \in \mathbf{C} : p^n(z) \to \infty, \text{ as } n \to \infty\}$ be the basin of infinity. Thus $J = \partial A_\infty$. Let γ be a path in A_∞ which lands at a point $z_0 \in \partial A_\infty$. For $\epsilon > 0$, we define the ϵ-*carrot* about γ to be $C(\gamma, \epsilon) = \bigcup_{z \in \gamma} B(z, \epsilon|z - z_0|)$. We say that A_∞ is a *John domain* if there is an $\epsilon > 0$ such that for each $z_0 \in \partial A_\infty$ there is a path γ connecting z_0 to ∞ and the ϵ-carrot is contained in A_∞, i.e., $C(\gamma, \epsilon) \subset A_\infty$.

In case p has a parabolic fixed point z_0, it is known that the Julia set has a cusp at z_0, and there can be no ϵ-carrot inside A_∞, landing at z_0.

Theorem [CJY]**.** The following are equivalent:

1. p is semi-hyperbolic.
2. p has no parabolic points, and for each critical point c, $c \notin \omega(c)$, which is to say that the orbit of c is not recurrent to itself.

3. A_∞ is a John domain.
4. J is fractal.

An essential ingredient is the following result, as formulated in [CJY]:

Theorem (Multi-Valent Distortion). Suppose that $\mathcal{D} \subset \mathbf{C}$ is simply connected, and $F : \mathcal{D} \to \Delta$ is a proper map to the unit disk, which is M-to-1. If ρ denotes the Poincaré metric, then there is C depending only on M such that:

$$\{w \in \Delta : \rho_\Delta(F(z_0), w) \leq C^{-1}\} \subset F\{z \in \mathcal{D} : \rho_{\mathcal{D}}(z, z_0) \leq 1\} \subset \{w \in \Delta : \rho_\Delta(F(z_0), w) \leq 1\}.$$

While the conclusion of this result is not as sharp as the Koebe Distortion Theorem, it may be applied to maps that are not conformal, and may be applied to semi-hyperbolic maps and is what is needed for Theorem [CJY].

Now we attempt to give a reformulation of all of this in complex dimension 2. We consider complex Hénon maps, which are polynomial diffeomorphisms of $\mathbf{C}^2$. Familiar objects are the sets $K^\pm$ where forward/backward orbits are bounded and $J^\pm := \partial K^\pm$. We use the pluri-complex Green function G^+ for the set K^+.

A map f is said to be *(uniformly) hyperbolic* if $J := J^+ \cap J^-$ is a hyperbolic set. This means that there is a splitting $E^s \oplus E^u$ of the tangent space over J such that Df is uniformly contracting/expanding in the stable/unstable directions E^s/ E^u. The Stable Manifold Theorem then yields *laminations* $\mathcal{W}^s$/ $\mathcal{W}^u$ of J^+/ J^-. Thus we see that:

Uniform expansion and contraction yields
geometric structures (laminations) of J^- and J^+.

We want to define "semi-hyperbolicity" in dimension 2, but we do not have "critical points" since f is a diffeomorphism. In fact, "tangencies" will be the two-dimensional replacement for critical points. We can use the Green function G^+ to define a canonical metric on the unstable subspaces at saddle points.

Let $\mathcal{S}$ denote the set of periodic points of saddle type, and let $J^* := \overline{\mathcal{S}}$ denote its closure. If $q \in \mathbf{C}^2$ is a periodic saddle point, we let $W^u(q)$ denote the unstable manifold of q. There is a uniformization $\xi_q : \mathbf{C} \to W^u(q) \subset \mathbf{C}^2$. We normalize these maps so that $\xi_q(0) = q$ and $\max_{|\zeta|\leq 1} G^+(\xi_q(\zeta)) = 1$. For each $q \in \mathcal{S}$ there is $\lambda_q \in \mathbf{C}$ such that

$$f \circ \xi_q(\zeta) = \xi_{f(q)}(\lambda_q \zeta), \quad \zeta \in \mathbf{C}$$

We define a canonical metric $||\cdot||_q^\#$ on E_q^u by the condition that $||\xi_q(0)'||_q^\# = 1$.

Theorem [BS8]**.** The following are equivalent, and if they hold, we say that f is quasi-expanding:

1. There exists $\kappa > 1$ such that $|\lambda_q| \geq \kappa$ for all periodic points q.
2. The set of normalized maps $\Xi = \{\xi_q : q \in \mathcal{S}\}$ is a normal family of entire functions.
3. There exists $\kappa > 1$ such that $||Df|_{E_q^u}||_q^\# \geq \kappa$ for all $q \in \mathcal{S}$.

We have defined quasi-expansion by the condition that Df is uniformly expanding with respect to the metric $||\cdot||^\#$ in the tangent directions E_q^u. Recall that q runs over the countable (non-closed) set $\mathcal{S}$, so it is unclear whether $||\cdot||^{\#}$ is equivalent to the Euclidean metric.

Now let us consider the (unparametrized) unstable manifolds $W^u(q)$. Let $(W^u(q)\cap B(q,r))_q$ denote the connected component of $W^u(q)\cap B(q,r)$ which contains q. We say that the family $\mathcal{W}^u(\mathcal{S}) := \{W^u(q) : q \in \mathcal{S}\}$ is *locally proper* if there is an $r > 0$ such that $(W^u(q) \cap B(q,r))_q$ is closed in $B(q,r)$ for all $q \in \mathcal{S}$. We say that the family $\mathcal{W}^u(\mathcal{S})$ has *locally bounded area* if $Area((W^u(q) \cap B(q,r))_q) < M$ for all $q \in \mathcal{S}$.

If $\mathcal{W}^u$ is part of a lamination, then it satisfies both the locally proper and locally bounded area conditions. The converse is not true.

Theorem [BS8]**.** f is quasi-expanding if and only if the family $\mathcal{W}^u(\mathcal{S})$ satisfies the proper, locally bounded area condition.

As was the case with [CJY], this result needs a Multi-Valent Distortion Theorem. The version we need is:

Theorem [BS8]**.** Let $A < \infty$ and $\chi > 1$ be given. Then there exist $\rho > 0$ and $a > 0$ with the following property: If $\mathcal{D} \subset \mathbf{C}$ is a simply connected domain containing the origin, and if $\phi : \mathcal{D} \to B(0,1)$ is a proper holomorphic mapping with $\phi(0) = 0$ and $Area(\phi(\mathcal{D})) \le A$, then for some r the component D_0 of $\phi^{-1}(B(0,\rho))$ containing the origin satisfies:

$$\{|\zeta| < ar\} \subset D_0 \subset \{|\zeta| < r\} \subset \{|\zeta| < \chi r\} \subset \mathcal{D}.$$

If f is quasi-expanding, then by Bishop's Theorem and [LP], $\mathcal{W}^u(\mathcal{S})$ may be extended to a family of manifolds $\{W^u(x) : x \in J^*\}$. These are unstable manifolds, but they may or may not fit together as part of a lamination. The geometry of transverse laminar structure in fact yields a dynamical consequence of uniform hyperbolicity:

Theorem [BS8]**.** Suppose that $\mathcal{W}^{u/s}(\mathcal{S})$ extend to a laminations of $J^{-/+}$. If these laminations are transverse at J, then f is uniformly hyperbolic on J.

We say that f is *quasi-hyperbolic* if both f and f^{-1} are quasi-expanding. It is known that $J^* \subset J$ for all Hénon maps f, but we point out a fundamental question:

If f is quasi-hyperbolic, is $J^* = J$?

The answer is known to be "Yes" when f is uniformly hyperbolic.

The next result says that within the class of quasi-hyperbolic maps, we can see uniform hyperbolicity by the non-existence of tangencies.

Theorem. Suppose that f is quasi-hyperbolic, then f is uniformly hyperbolic on J^* if and only if there is no point of tangency between $\mathcal{W}^u(\mathcal{S})$ and $\mathcal{W}^s(\mathcal{S})$.

This Theorem is ongoing work with John Smillie.

References

[BS8] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$. VIII: Quasi-expansion. *Amer. J. Math.* 124 (2002), no. 2, 221–271.

[CJY] L. Carleson, P. Jones and J-C Yoccoz, Julia and John, *Bol. Soc. Mat.* Vol. 25, 1–30 (1994)

[LP] M. Lyubich and H. Peters, Classification of invariant Fatou components for dissipative Hénon maps, *Geom. Funct. Anal.* Vol. 24 (2014) 887–915.

Stably-interior points and the Semicontinuity theorems for Automorphism Groups

Kang-Tae Kim

The semicontinuity phenomenon under consideration is, roughly speaking, as follows:

> *If a sequence Ω_j of domains converge in some sense to a domain Ω_0 as $j \to \infty$, then there exists $N > 0$ such that Aut $(\Omega_j) \hookrightarrow$ Aut (Ω_0) for every $j > N$.*

The line of study along this has quite a deep root. Perhaps the beginning was in the following observation in the Euclidean space: when a sequence (K_j) of compact convex solids in a Euclidean space converges, in the sense of Hausdoff set convergence for instance, to another compact solid K_0, then the symmetry group of K_j is isomorphic to a subgroup of the symmetry group of the limit solid K_0 for j sufficiently large.

This, in more generality, was handled in the theorem by Montgomery and Zippin [5] which states: *For a Lie group G and a compact subgroup H, there exists an open neighborhood U of H such that any subgroup K with $K \subset U$ admits an element $g \in G$ such that $g^{-1}Kg \subset H$.*

Then D. Ebin [1] showed that *on a compact Riemannian manifold M, if a sequence (γ_j) of $\mathcal{C}^\infty$-smooth Riemannian metrics converges to a $\mathcal{C}^\infty$-smooth Riemannian metric γ_0 then there exists $N > 0$ such that, for every $j > N$, a diffeomorphism $\psi_j : M \to M$ exists to satisfy $\psi_j^{-1} \circ Isom_{\gamma_j}(M) \circ \psi_j \subset Isom_{\gamma_0}(M)$.*

Then in 1982 and 1985, Greene and Krantz established the following

Theorem 1 ([3], [4]). *In the collection of bounded strongly pseudoconvex domains with $\mathcal{C}^\infty$ smooth boundary in $\mathbb{C}^n$ with $n \geq 2$, if a sequence (Ω_j) converges in $\mathcal{C}^\infty$-topology to Ω_0 with a compact holomorphic automorphism group, then there exists a constant $N > 0$ such that, for every $j > N$, there exists a diffeomorphism $\psi_j : \Omega_0 \to \Omega_j$ satisfying $\psi_j^{-1} \circ$ Aut $(\Omega_j) \circ \psi_j \subset$ Aut (Ω_0).*

Their proof depended upon many excellent analytic/geometric theorems concerning the bounded strongly pseudoconvex domains.

The main aim of this presentation is the following theorem. For instance let $\mathcal{F}$ be the collection of bounded pseudoconvex domains in $\mathbb{C}^2$ with D'Angelo finite type boundary.

Theorem 2 (Greene & Kim, [2]). *If a sequence (Ω_j) in $\mathcal{F}$ converges in $\mathcal{C}^\infty$-topology to Ω_0 with a compact holomorphic automorphism group, then there exists* *a constant $N > 0$ such that, for every $j > N$, Aut (Ω_j) is Lie isomorphic to a Lie subgroup of Aut (Ω_0)*

Then the method of proof is of an elementary nature so that it also yields the same conclusion for the following collections:

- the collection of bounded strongly pseudoconvex domains in $\mathbb{C}^n$ $(n \geq 2)$ with $\mathcal{C}^2$ boundary equipped with the $\mathcal{C}^2$-topology.
- the collection of bounded convex domains in $\mathbb{C}^n$ with $\mathcal{C}^1$ boundary equipped with the $\mathcal{C}^1$-topology.

It is not known whether the theorem holds for the case of domains with finite type boundary if the dimension is higher than 2.

References

[1] D. Ebin: On the space of Riemannaian metrics, *Bull. Am. Math. Soc.* 74: 1001–1003 (1968)
[2] R. E. Greene, K.-T. Kim: Stably-interior points and the semicontinuity of the automorphism group, *Math. Z.* 277: 909–916 (2014)
[3] R. E. Greene, S. G. Krantz: The automorphism groups of strongly pseudoconvex domains, *Math. Ann.* 261: 425–446 (1982)
[4] R. E. Greene, S. G. Krantz: Normal families and the semicontinuity of isometry and automorphism groups, *Math. Z.* 190: 455–467 (1985)
[5] D. Montgomery, L. Zippin: A theorem on Lie groups, *Bull. Am. Math. Soc.* 48: 448–452 (1942)

A complete complex hypersurface in the ball of $\mathbf{C}^N$

Josip Globevnik

In 1977 P. Yang asked whether there exist complete immersed complex submanifolds $\varphi\colon M^k \to \mathbf{C}^N$ with bounded image [Y1, Y2]. The first answer was obtained by P. Jones [J] who constructed a bounded complete immersion $\varphi\colon \Delta \to \mathbf{C}^2$ and a complete proper holomorphic embedding $\varphi\colon \Delta \to \mathbf{B}_4$. Since then there has been a series of results on bounded complete holomorphic curves $(k = 1)$ *immersed* in $\mathbf{C}^2$ [MUY, AL1, AF] the most recent being that every bordered Riemann surface admits a complete proper holomorphic immersion to $\mathbf{B}_2$ and a complete proper holomorphic embedding to $\mathbf{B}_3$ [AF]. The more difficult complete *embedding* problem for $k = 1$ and $N = 2$ has been solved only recently by A. Alarcón and F. J. López [AL2] who proved that every convex domain in $\mathbf{C}^2$ contains a complete, properly embedded complex curve.

In the present talk we are interested primarily in the higher dimensional case $(k > 1)$ where there are partial answers which are easy consequences of the results for complete curves. For instance, it is known that for any $k \in \mathbb{N}$ there are complete bounded embedded complex k-dimensional submanifolds of $\mathbf{C}^{2k}$ and it is an open question whether, in this case, $N = 2k$ is the minimal possible dimension [AL2]. We consider the case where φ is a proper holomorphic embedding. In this case $\varphi(M^k)$ is a closed submanifold. We restate the definition of completeness for this case:

Definition *A closed complex submanifold M of $\mathbf{B}_N$ is complete if every path $p\colon [0,1) \to M$ such that $|p(t)| \to 1$ as $t \to 1$ has infinite length.*

Note that this coincides with the standard definition of completeness since the paths $p\colon [0,1) \to M$ such that $|p(t)| \to 1$ as $t \to 1$ are precisely the paths that leave every compact subset of M as $t \to 1$.

Our main result is **Theorem** *Let $N \geq 2$. There is a holomorphic function f on $\mathbf{B}_N$ such that $\Re f$ is unbounded on every path of finite length that ends on $b\mathbf{B}_N$.* So our function f has the property that if $p\colon [0,1] \to \overline{\mathbf{B}_N}$ is a path of finite length such that $|p(t)| < 1$ $(0 \leq t < 1)$ and $|p(1)| = 1$ then $t \to \Re\big(f(p(t)\big)$ is unbounded on $[0,1)$. The following corollary answers the question of Yang in all dimensions k and N by providing properly embedded complete complex manifolds. **Corollary** *For each $k, N,\ 1 \leq k < N$, there is a complete, closed, k-dimensional complex submanifold of $\mathbf{B}_N$.* **Proof.** We first prove the corollary for $k = N-1$ (that is, we first prove the existence of the hypersurface, mentioned in the title). Let f be the function given by Theorem 1.1. By Sard's theorem one can choose $c \in \mathbf{C}$ such that the level set $M = \{z \in \mathbf{B}_N \colon f(z) = c\}$ is a closed submanifold of $\mathbf{B}_N$. Let $p\colon [0,1) \to M$ be a path such that $p(t) \to b\mathbf{B}_N$ as $t \to 1$. Assume that p has finite length. Then there is a point w on $b\mathbf{B}_N$ such that $\lim_{t\to 1} p(t) = w$. By the properties of f, $\Re f$ is unbounded on $p([0,1))$. On the other hand, $f((p(t)) = c$ $(0 \leq t < 1)$, a contradiction. So p must have infinite length. This proves that M is complete and so completes the proof of the corollary for $k = N-1$. Assume now that $1 \leq k \leq N-2$. By the first part of the proof, there is a complete, closed, $k-$dimensional complex submanifold M of $\mathbf{B}_{k+1} \subset \mathbf{B}_N$. Clearly M is a complete, closed $k-$dimensional manifold of $\mathbf{B}_N$. This completes the proof. In the talk we describe the proof of Theorem which is based on an old idea of the speaker and E.L.Stout [GS], and on a new result from convex geometry obtained by the author which is needed for the proof. The paper with complete proofs is available on arXiv: 14013135, in a paper with the same title as the title of this talk.

References

[AL1] A. Alarcón and F. J. López: Null curves in $\mathbf{C}^3$ and Calabi-Yau conjectures.Math. Ann. 355 (2013) 429-455

[AL2] A. Alarcón and F. J. López: Complete bounded complex curves in $\mathbf{C}^2$.To appear in J. Europ. Math. Soc. , Arxiv:1305.2118v2 , May 2013

[AF] A. Alarcón and F. Forstnerič: Every bordered Riemann surface is a complete proper curve in a ball. Math. Ann. 357 (2013) 1049-1070

[GS] J. Globevnik and E. L. Stout: Holomorphic functions with highly noncontinuable boundary behavior. J. Anal. Math. 41 (1982) 211-216

[J] P. W. Jones: A complete bounded complex submanifold of $\mathbf{C}^3$.Proc. Amer. Math. Soc. 76 (1979) 305-306

[MUY] F. Martin, M. Umehara and K. Yamada: Complex bounded holomorphic curves immersed in $\mathbf{C}^2$ with arbitrary genus. Proc. Amer. Math. Soc. 137 (2009) 3437-3450

[Y1] P. Yang: Curvature of complex submanifolds of $\mathbf{C}^n$. J. Diff. Geom. 12 (1977) 499-511

[Y2] P. Yang: Curvature of complex submanifolds of $\mathbf{C}^n$. In: Proc. Symp. Pure. Math. Vol. 30, part 2, pp. 135-137. Amer. Math. Soc., Providence, R. I. 1977

Generalized cycles and local intersection numbers

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(joint work with M. Andersson, D. Eriksson, H. Samuelsson Kalm, and A. Yger)

Let $\mathcal{Z}(\mathbb{P}^N)$ denote the group of analytic cycles on $\mathbb{P}^N$, i.e., formal finite sums

$$Z = \sum_j \alpha_j Z_j,$$

where Z_j are irreducible subvarieties of $\mathbb{P}^N$. If Z is irreducible itself, then at each point $x \in Z$ there is a well-defined positive integer $\operatorname{mult}_x Z$, the multiplicity of Z at x. Roughly speaking one takes a generic plane through x of complimentary dimension, moves it slightly and counts the number of intersection points close to x. There is also a positive integer $\deg Z$ which is the total number of intersection points with a generic such plane. These two numbers extend to arbitrary cycles by linearity.

Let Z be the cusp $\{x_1^3 - x_2^2 x_0 = 0\}$ in $\mathbb{P}^2$. Then $p = [1, 0, 0]$ is the only non-smooth point. We have that

$$\operatorname{mult}_x Z = 1, x \in Z \setminus p, \ \operatorname{mult}_p Z = 2, \quad \deg Z = 3.$$

Analytically we can represent the cycle Z by the corresponding Lelong current $[Z]$

$$[Z].\xi = \int_{\mathbb{P}^N} [Z] \wedge \xi, \quad \xi \in \mathcal{E}(\mathbb{P}^N).$$

Clearly Z is determined by its Lelong current, and this representation makes it possible to give analytic definitions of multiplicity and degree: We have that

$$\operatorname{mult}_x Z = \ell_x[Z],$$

where the right hand side is the Lelong number of the current $[Z]$ at x, this is a measure of the mass concentration at x. Furthermore,

$$\deg Z = \int_{\mathbb{P}^N} [Z] \wedge \omega^{\dim Z}$$

(provided that Z has pure dimension), where $\omega = dd^c \log |x|^2$ is the Fubini-Study metric form.

If $Z, W \in \mathcal{Z}(\mathbb{P}^N)$ have pure dimensions and $\dim(Z \cap W) = \dim Z + \dim W - N$, then there is a well-defined cycle

$$Z \cdot W = \sum \alpha_\ell V_\ell,$$

called the proper intersection, where V_ℓ are the irreducible components of the set-theoretical intersection V of Z and W, and α_ℓ are integers. The classical definition is geometric and/or algebraic, but by means of the Lelong current representation we have

$$[Z \cdot W] = [Z] \wedge [W],$$

where the product on right hand side is defined by choosing suitable regularizations of the currents and go to the limit. For instance, the proper intersection of the cusp and a generic line through p is equal to $2\{p\}$, whereas the intersection with the line $x_2 = 0$ is $3\{p\}$.

In the classical non-proper case, see [4], the intersection product $Z \cdot W$ is a certain Chow class on V of dimension $\dim Z + \dim W - N$; this means that it is represented by a cycle on V that is determined only up to rational equivalence. In particular, it has a well-defined degree and the Bezout equality

$$\deg(Z \cdot W) = \deg Z \cdot \deg W$$

holds, provided that $\dim Z + \dim W \geq N$; otherwise $Z \cdot W$ is zero.

For instance, the self-intersection of the cusp Z above is represented by the set of 9 points obtained by taking one of the Z and move it slightly so that one gets a proper intersection. (More precisely any divisor of a generic section of the line bundle $\mathcal{O}(3)$ restricted to Z is a representative.)

In the 90's Tworzewski, [6], Gaffney-Gassler, [5], and Achilles-Manaresi, [1], independently introduced integers

$$\epsilon_k(Z, W, x), \ \ k = 0, 1, \dots, \dim V,$$

called the *local intersection numbers* or *Segre numbers* at x, where k describes the complexity of the local intersection at x on dimension k. The definition in [6] and [5] is geometric and relies on a local variant of the so-called Stückrad-Vogel procedure, [7], whereas the definition in [1] is algebraic. In [3] we found an analytic definition as the Lelong numbers of certain currents.

If the intersection is proper, then $\epsilon_k(Z, W, x) = \mathrm{mult}_x(Z \cdot W)$ for $k = \dim V$ and 0 otherwise. If $Z = W$ is the cusp, then

$$\epsilon(Z, Z, x) = (0, 1), \ \ x \in Z \setminus \{p\}, \quad \epsilon(Z, Z, p) = (3, 2),$$

that is, at the point p we have the local intersection number 3 on dimension 0 and 2 on dimension 1.

It is clear that no representative of the self-intersection $Z \cdot Z$ of the cusp can represent the local intersection numbers. Tworzewski, [6], proved however that there is a unique analytic cycle $Z \circ W$ such that (lower index denotes component of dimension k)

$$\sum_k \mathrm{mult}_x(Z \circ W)_k = \sum_\ell \epsilon_\ell(Z, W, x).$$

For instance, if Z is the cusp, then

$$Z \circ Z = Z + 3\{p\}.$$

Notice however that $\deg(Z \circ Z) = 6 \neq 9 = 3 \cdot 3 = (\deg Z)^2$ so the Bezout equality is not fufilled (and clearly there is no cycle at all with the right multiplicities that also satisfies the Bezout equality in this case).

We introduce, for any subvariety X of $\mathbb{P}^N$, a group $\mathcal{B}(X)$ of currents that we call generalized cycles on X. If we identify classical cycles with the associated Lelong currents we get an inclusion $\mathcal{Z}(X) \subset \mathcal{B}(X)$. We also have a natural inclusion $\mathcal{B}(X) \subset \mathcal{B}(X')$ if $X \subset X'$, and $\mathcal{B}(X)$ is precisely the subgroup of the μ in $\mathcal{B}(\mathbb{P}^N)$ whose support $|\mu|$ is contained in X. Each generalized cycle μ has a natural decomposition $\mu = \mu_0 + \mu_1 + \cdots$, where μ_k has dimension k. It turns out that $\operatorname{mult}_x \mu := \ell_x \mu$ and

$$\deg \mu := \int \mu \wedge \omega^{\deg \mu}$$

are integers. Intuitively generalized cycles are obtained as certain mean values of classical cycles. For instance,

$$\omega_p := dd^c \log(|x_1|^2 + |x_2|^2)$$

is a generalized cycle that is singular only at the point p. It has degree 1 and the multiplicity at p is 1; at each other point the multiplicity is zero. In fact, ω_p is a mean value of all lines through p.

Our main result is the following:

Theorem 1. *There is a bilinear pairing* $\mathcal{B}(X) \times \mathcal{B}(X') \to \mathcal{B}(X \cap X')$, $(Z, W) \mapsto Z \bullet W$ *with the following properties:*
(i) $\operatorname{mult}_x(Z \bullet W)_k = \epsilon(Z, W, x)$ *for all* x *and* k
(ii) $\deg(Z \bullet W) = \deg Z \cdot \deg W$ *provided that* $\dim(|Z| \cap |W|) \geq \dim Z + \dim W - N$,
(iii) $Z \bullet W$ *coincides with* $Z \cdot W$ *on "cohomology level".*

It follows that $Z \bullet W = Z \cdot W$ if the intersection is proper. If ℓ is a line, then $\ell \bullet \ell = \ell$. If Z is the cusp above in $\mathbb{P}^2$, then

$$Z \bullet Z = Z + 3\{p\} + \mu,$$

where μ is a generalized cycle on Z of dimension 0 and total mass 3, intuitively meaning 3 points that move around on Z. Notice that the total degree of $Z \bullet Z$ is 9 as expected.

The formal definition of $\mathcal{B}(X)$ is the following: Let $f \colon Y \to X$ be any proper holomorphic mapping and let α be a product of Chern forms of Hermitian line bundles over Y. Then $\mu = f_* \alpha$ defines a generalized cycle with support on X and $\mathcal{B}(X)$ is defined so that the element is independent of the choice of Chern forms (Hermitian metrics). For instance, if $i \colon Z \to X$ is an inclusion, then $[Z] = i_* 1$. Let $\pi \colon \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at p and let α be minus the Chern form of the exceptional divisor. Then $\omega_p = \pi_* \alpha$.

References

[1] Achilles, R. & Manaresi, M.: *Multiplicities of a bigraded ring and intersection theory*, Math. Ann. **309** (1997), 573–591

[2] Achilles, R. & Rams, S.: *Intersection numbers, Segre numbers and generalized Samuel multiplicities*, Arch. Math. (Basel) **77** (2001), 391–398

[3] M. Andersson, H. Samuelsson, W. Wulcan, A. Yger: *Local intersection numbers and a generalized King formula*, arXiv:1009.2458, Crelle, to appear

[4] Fulton, W.: *Intersection theory, second edition*, Springer-Verlag, Berlin-Heidelberg (1998)
[5] Gaffney, T. & Gassler, R.: *Segre numbers and hypersurface singularities*, J. Algebraic Geom. **8** (1999), 695–736
[6] Tworzewski, P.: *Intersection theory in complex analytic geometry*, Ann. Polon. Math. **62** (1995), 177–191
[7] Stückrad, J.; Vogel, W.: *An algebraic approach to the intersection theory*, Queen's Papers in Pure and Appl. Math. **61** (1982), 1–32

On pluripolarity of cores of pseudoconvex domains

Tobias Harz
(joint work with N. Shcherbina and G. Tomassini)

In [1] the following notion of the core of a complex manifold is introduced.

Definition. Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. The set

$$\mathfrak{c}(\Omega) := \big\{z \in \Omega : \text{every smooth plurisubharmonic function on } \Omega \text{ that is bounded from above fails to be strictly plurisubharmonic in } z\big\}$$

is called the *core* of Ω.

It is well-known that for every bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary there exists a smooth strictly plurisubharmonic function φ defined on an open neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$. The function φ is called a global defining function for Ω. Easy examples show that in general it is not possible to extend the same result to the case of unbounded domains.

Example. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and

$$\Omega := \big\{(z, w) \in \mathbb{C}^2 : \log|w - f(z)| + C_1\big(|z|^2 + |w|^2\big) < C_2\big\} \subset \mathbb{C}^2,$$

where C_1 and C_2 are constants and $C_1 > 0$. For almost all constants C_2, Ω is an unbounded strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^2$ containing the complex line $L := \{(z, f(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\}$. Let φ be a plurisubharmonic function defined on a neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$. Then φ is subharmonic and bounded from above on L, hence it is a constant by Liouville's theorem. In particular, φ is not strictly plurisubharmonic at the points of L.

Thus, in the setting of unbounded domains $\Omega \subset \mathbb{C}^n$, it is necessary to allow for some degenerecies of the Levi form of global defining functions inside Ω. This is the initial motivation for introducing the concept of the core. The Main Theorem from [1] states that the core is the only obstruction for existence of strictly plurisubharmonic defining functions of unbounded strictly pseudoconvex domains.

Main Theorem. *Every strictly pseudoconvex domain Ω with smooth boundary in a complex manifold $\mathcal{M}$ admits a bounded global defining function that is strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$. Moreover, $\mathfrak{c}(\Omega)$ is closed in $\mathcal{M}$.*

By the above theorem, the study of global defining functions is reduced to the study of the core. Much of the work in [1] is devoted to study properties of the core. In particular, questions on existence of analytic structure and of Liouville type properties of $\mathfrak{c}(\Omega)$ are addressed. The strongest general result that is obtained is the following.

Theorem 1. *Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Then $\mathfrak{c}(\Omega)$ is 1-pseudoconcave in Ω. In particular, $\mathfrak{c}(\Omega)$ is pseudoconcave in Ω if $\dim_{\mathbb{C}} \mathcal{M}$ is two.*

However, there are many natural questions on the structure of the core which are still open. For example, a general open problem is to understand how properties of Ω are related to properties of $\mathfrak{c}(\Omega)$. One instance of this problem, which is motivated by the structure of available examples, is to understand if, and if applicable how, pseudoconvexity of Ω is related to pluripolarity of $\mathfrak{c}(\Omega)$. The goal of this talk is to present a partial result on this question, namely, under the additional assumption that the core possesses a certain product structure. In particular, we discuss the following theorem.

Theorem 2. *Let $n \geq 2$. The following assertions hold true for domains $\Omega \subset \mathbb{C}^n$ with coordinates $(z_1, z_2, \ldots, z_n)$, $z_j = x_j + iy_j$:*

(1) *There exists a domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^{n-1}$, where $E \subset \mathbb{C}$ is the set $E = [0,1] \times \mathbb{R}_{y_1}$.*
(2) *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for some $k \in \{1, 2, \ldots, n-1\}$ and some set $E \subset \mathbb{C}^{n-k}$. Then either E is locally complete pluripolar or E is open. In the later case $\Omega = E \times \mathbb{C}^k$.*
(3) *Let $k \in \{1, 2, \ldots, n-1\}$ be arbitrary but fixed. Then there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for a set $E \subset \mathbb{C}^{n-k}$ if and only if E is closed and complete pluripolar.*

References

[1] T. Harz, N. Shcherbina and G. Tomassini: *On defining functions for unbounded pseudoconvex domains.* arXiv:1405.2250.

An estimate of the Squeezing function on Strictly Pseudoconvex Domains

Erlend F. Wold

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The squeezing function, which was introduced in [1], inspired by [5], [6] and [8], measures how much a domain looks like the unit ball observed from a given point z. More precisely it is defined as follows: For a given injective holomorphic map $f : \Omega \to \mathbb{B}^n$ satisfying $f(z) = 0$ we set

$$S_{\Omega,f}(z) := \sup\{r > 0 : r\mathbb{B}^n \subset f(\Omega)\},$$

and then we set

$$S_\Omega(z) := \sup_f \{S_{\Omega,f}(z)\},$$

where f ranges over all injective holomorphic maps $f : \Omega \to \mathbb{B}^n$ with $f(z) = 0$. Using the method of exposing points from [2] and the method from [3], it was proved in [1] that

$$\lim_{z \to b\Omega} S_\Omega(z) = 1$$

if Ω is a $\mathcal{C}^2$-smooth strictly pseudoconvex domain, and it was proved in [4] that the squeezing function is bounded on any bounded convex domain. Our goal is to improve this estimate when the boundary has higher regularity, and to give an application to invariant metrics.

Theorem 1. *Let $\Omega = \{\delta < 0\} \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with a defining function δ of class $\mathcal{C}^k$ for $k \geq 3$. The squeezing function $S_\Omega(z)$ for Ω satisfies the estimate*

$$S_\Omega(z) \geq 1 - C \cdot \sqrt{|\delta(z)|}$$

for a fixed constant C. If we even have $k \geq 4$, then there exists a constant $C > 0$ such that the squeezing function $S_\Omega(z)$ for Ω satisfies

$$S_\Omega(z) \geq 1 - C \cdot |\delta(z)|$$

for all z

Combining with a theorem due to D. Ma [7] and a result of Deng, Guan and Zhang [1], an immediate consequence is a sharp estimate for invariant metrics near the boundary of a strictly pseudoconvex domain. Before we state the result, we briefly recall the definitions of some invariant metrics. Let Δ denote the unit disc, and let $\mathcal{O}(M, N)$ denote the holomorphic maps from M to N.

- Kobayashi metric $K_\Omega(p, \xi)$. We define

  $$K_\Omega(p, \xi) = \inf\{|\alpha|; \exists f \in \mathcal{O}(\Delta, \Omega)\ f(0) = p, \alpha f'(0) = \xi\}.$$

- Carathéodory metric $C_\Omega(p, \xi)$. We define

  $$C_\Omega(p, \xi) = \sup\{|f'(p)(\xi)|; \exists f \in \mathcal{O}(\Omega, \Delta)\ f(p) = 0\}.$$

- Sibony metric $S_\Omega(p, \xi)$. We define
  $S_\Omega(p, \xi) = \sup\{(\sum_{i,j} \frac{\partial^2 u(p)}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j)^{1/2}, u(p) = 0, 0 \leq u < 1,$ u is $\mathcal{C}^2$ near p and $\ln u$ is plurisubharmonic in $\Omega\}$.

- Azukawa metric $A_U(p, \xi)$. We define

  $$A_\Omega(p, \xi) = \sup_{u \in P_\Omega(p)} \{\limsup_{\lambda \searrow 0} \frac{1}{|\lambda|} u(p + \lambda\xi)\}$$

  where

  $$\begin{aligned} P_\Omega(p) &= \{u : \Omega \to [0, 1), \ln u \text{ is plurisubharmonic and} \\ \exists &\ M_u > 0, r_u > 0 \text{ such that} \\ &\ \mathbb{B}^n(p, r) \subset \Omega, u(z) \leq M\|z - p\|, z \in \mathbb{B}^n(p, r)\} \end{aligned}$$

Theorem 2. *Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain of class $\mathcal{C}^3$, let $p \in b\Omega$, and let δ be a defining function for Ω near p, such that $\|\nabla\delta(z)\| = 1$ for all $z \in b\Omega$. Then if $F_\Omega(z,\zeta)$ is either the Carathéodory, Sibony or Azukawa metric, there exists a constant $C > 0$ such that*

$$(1 - C\sqrt{|\delta(z)|})\left[\frac{L_{\pi(z)}(\xi_T)}{|\delta(z)|} + \frac{\|\xi_N\|}{4\delta(z)^2}\right]^{1/2} \leq F_\Omega(z,\xi)$$
$$\leq (1 + C\sqrt{|\delta(z)|})\left[\frac{L_{\pi(z)}(\xi_T)}{|\delta(z)|} + \frac{\|\xi_N\|}{4\delta(z)^2}\right]^{1/2}$$

for all z near p, and all $\xi = \xi_N + \xi_T$, where π is the orthogonal projection to $b\Omega$, ξ_N is the complex normal component of ξ at $\pi(z)$ and ξ_T is the complex tangential component, and L is the Levi form of δ.

Ma's result is the corresponding statement for the Kobayashi metric, and the result is sharp in the sense that one cannot in general do better than the square root of the boundary distance. Theorem 2 is a direct consequence of Ma's result combined with Theorem 1 in the light of the fact that

$$S_\Omega(z) \cdot K_\Omega(z,\xi) \leq F_\Omega(z,\xi) \leq K_\Omega(z,\xi). \tag{1}$$

References

[1] Deng, F., Guan, Zhang, L.: Some properties of squeezing functions. arXiv.1202.4896

[2] Diederich, K., Fornæss, J. E. Wold, E. F.; Exposing Points on the Boundary of a Strictly Pseudoconvex or a Locally Convexifiable Domain of Finite 1-Type. *J. Geom. Anal.* **24** (2014), no. 4, 2124–2134.

[3] Fridman, B. and Ma, D.: On exhaustion of domains. *Indiana Univ. Math. J.* **44** (1995), no. 2, 385–395.

[4] Kim, Kang-Tae and Zhang, L.; On the uniform squeezing property and the squeezing function. http://arxiv.org/pdf/1306.2390.

[5] Liu, K., Sun, X. and Yau, S. T.; Canonical metrics on the moduli space of Riemann Surfaces I. *J. Differential Geom.* **68** (2004) 571–637.

[6] Liu, K., Sun, X. and Yau, S. T.; Canonical metrics on the moduli space of Riemann Surfaces II. *J. Differential Geom.* **69** (2005) 163–216.

[7] Ma, D.: Sharp estimates for the Kobayashi metric near strongly pseudoconvex points. *Contemp. Math.* **137** (1992), 329–338.

[8] Yeung, S. K.; Geometry of domains with the uniform squeezing property, *Adv. Math.* **221** (2009) 547–569.

A survey on effective and noneffective extension theorems

Takeo Ohsawa

The theory of extending functions from S to Y to those from X to Y has developed since Newton and Langange, for $(X,S,Y) = (\mathbb{R},\mathbb{Z},\mathbb{R})$ until Cartan and Oka for (Stein manifolds, closed analytic subsets, $\mathbb{C}$). Induction on dimension works often in virtue of extension theorems of this kind. Extensions with growth conditions, whose instance is as follows, yielded more applications recently.

Theorem 1 (cf. [O-T]) Let D be a pseudoconvex domain in $\{z \in \mathbb{C}^n; |z_n| < 1\}$ and let $D' = D \cap \{z_n = 0\}$. Then there exists a constant $C(\leq 1620\pi)$ such that, for any plurisubharmonic function φ and for any holomorphic function f on D' satisfying

$$\int_{D'} e^{-\varphi}|f|^2 < \infty,$$

there exists a holomorphic function $\tilde{f}$ on D satisfying $\tilde{f}|_{D'} = f$ and

$$\int_D e^{-\varphi}|\tilde{f}|^2 \leq C \int_D' e^{-\varphi}|f|^2.$$

One can see from this theorem an estimate for the Bergman kernel $K_D(z) \gtrsim \delta(z)^{-2}$ for bounded and C^1-smooth D (a conjecture of S. Bergman). Other important applications appeared since 1992 (cf. [D]).

After the remarkable solution of a conjecture of Suita [S], first by Błocki [B] and later by Guan and Zhou [G-Z] in a more general formulation on Riemann surfaces, further effective versions of Theorem 1 have been obtained.

In [O-1], a simple proof of such an extension theorem in [G-Z] is given, by exploiting the Poincaré metric on the punctured disc.

For the case (X, S, Y) = (compact manifold M, support of effective divisor D, holomorphic vector bundle over M), a noneffective extension theorem was obtained in [O-2] by refining an L^2 estimate of Hörmander [H].

References

[B] Błocki, Z., Suita conjecture and the Ohsawa-Takegoshi extension theorem, Invent. Math. 193 (2013), 149-158.

[D] Demailly, J.-P., Regularization of closed positive currents and intersection theory, J. Algebraic Geom. 1 (1992), 361-409.

[G-Z] Guan, Q.-A. and Zhou, X.-Y., A solution of an L^2 extension problem with optimal estimate and applications, Ann. of Math. 181 (2015), 1139-1208.

[H] Hörmander, L., L^2 estimates and existence theorems for the $\overline{\partial}$ operator, Acta Math., 113 (1965), 89-152.

[O-1] Ohsawa, T., Application and Simplified Proof of a Sharp L^2 Extension Theorem, to appear in Nagoya Math. J.

[O-2] —, A remark on Hörmander's isomorphism theorem, to appear in Proc. KSCV 10.

[O-T] Ohsawa, T. and Takegoshi, K., On the extension of L^2 holomorphic functions, Math. Z., 195 (1987), 197-204.

[S] Suita, N., Capacities and kernels on Riemann surfaces, Arch. Rational Mech. Anal., 46, (1972), 212-217.

The weak Kähler-Ricci flow

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(joint work with Eleonora Di Nezza)

Let X be a compact Kähler manifold of dimension n and $\alpha_0 \in H^{1,1}(X,\mathbb{R})$ a Kähler class. The Kähler-Ricci flow is the following:

$$\frac{\partial \omega_t}{\partial t} = -Ricci(\omega_t)\ ,\ \omega_t|_{t=0} = T_0, \tag{1}$$

where T_0 is a fixed closed positive $(1,1)$-current in α_0. When T_0 is a Kähler form, it is well-known (see [1], [5], [4]) that the flow admits a unique smooth solution on a maximal interval $[0, T_{\max})$, where

$$T_{\max} := \sup\{t \geq 0 \,|\, tK_X + t\{\eta\} + \alpha_0 \ \text{ is nef}\}.$$

One can rewrite the Kähler-Ricci flow at the level of potentials. Fix ω a Kähler form in α_0. Let φ_0 be a global potential of T_0, i.e. $T_0 := \omega + dd^c\varphi_0$. Set

$$\chi := -Ricci(\omega),\ \theta_t := \omega + t\chi,$$

and consider the following equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\omega^n}\right],\ \varphi_t \to \varphi_0 \text{ as } t \to 0. \tag{2}$$

If φ_t solves (2) then a straightforward computation shows that $\omega_t := \theta_t + dd^c\varphi_t$ solves the flow (1). Conversely, if ω_t solves the flow (1) then it follows from the dd^c-lemma that we can write

$$\omega_t = \theta_t + dd^c\varphi_t,$$

where φ_t solves the parabolic Monge-Ampère equation (2).

The maximal Kähler-Ricci flow. Fix $T_0 = \omega + dd^c\varphi_0$ a closed positive $(1,1)$ current in the class α_0. The integrability index of T_0 (or φ_0) is defined by

$$c(T_0) = c(\varphi_0) := \sup\left\{\lambda > 0 \;\middle|\; e^{-2\lambda\varphi_0} \in L^1(X)\right\}.$$

Assume that $1/2c(T_0) < T_{\max}$. Let $\varphi_{0,j}$ be a sequence of smooth ω_0-psh functions decreasing to φ_0. Let $\varphi_{t,j}$ be the unique solution of the parabolic equation (2) with initial data $\varphi_{0,j}$. As shown by Guedj and Zeriahi in [2], as $j \to +\infty$ the sequence $\varphi_{t,j}$ decreases to φ_t which satisfies the following:

- For each $t > 0$, φ_t is a θ_t-psh function. *Moreover, if $t > 1/2c(T_0)$, φ_t is smooth on X and solves (2) in the classical sense.*
- φ_t converges in capacity to φ_0 as $t \to 0$.

The assumption $1/2c(\varphi_0) < T_{\max}$ is necessary to insure that the maximal scalar solution φ_t is well-defined. Without this condition the sequence $\varphi_{t,j}$ could decrease to $-\infty$. The flow φ_t constructed as above is called the maximal Kähler-Ricci flow.

The regularity of this flow was obtained for t not too small and nothing was known when $t < 1/2c(T_0)$. Example 6.4 in [2] suggests that there might be no regularity at all due to the presence of positive Lelong numbers. However, as in Demailly's regularization theorem, one can expect that the regularizing effect happens outside some analytic subset. Our first result shows that it is indeed the case.

Theorem 1. [3] *Assume that* $1/2c(T_0) < T_{\max}$. *Then the maximal Kähler-Ricci flow starting from* T_0 *is smooth in a Zariski open subset of* X.

The Zariski open subset in Theorem 1 is described by the complement of Lelong superlevel sets of T_0:

$$D_s := \{x \in X \mid \nu(T_0, x) \geq s\}, \quad s > 0.$$

These are analytic subsets as follows from Siu's theorem.

The result in Theorem 1 is optimal in the sense that any maximal Kähler-Ricci flow starting from currents with positive Lelong numbers has positive Lelong numbers in short-time.

Our second result is the stability of the weak Kähler-Ricci flow starting from currents with zero Lelong numbers.

Theorem 2. [3] *Assume that* $c(T_0) = +\infty$. *Then the following holds:*

- **Uniqueness:** *Any weak Kähler-Ricci flow starting from* T_0 *is maximal. In other words the flow is unique.*
- **Stability:** *Assume that* $T_{0,j}$ *is a sequence of positive closed* $(1,1)$*-currents converging to* T_0 *in the* L^1 *topology. Then the corresponding maximal Kähler Ricci flow* $\omega_{t,j}$ *converges to* ω_t *in the following sense: for each* $t \in (0, T_{\max})$, $\omega_{t,j}$ *converges in* $\mathcal{C}^\infty(X)$ *to* ω_t *as* $j \to +\infty$.

References

[1] H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359-372.

[2] V. Guedj, A. Zeriahi, *Regularizing properties of the twisted Kähler-Ricci flow*, Preprint arXiv:1306.4089.

[3] E. Di Nezza, H. C. Lu, *Uniqueness and short time regularity of the weak Kähler-Ricci flow*, arXiv:1411.7958.

[4] G. Tian, Z. Zhang, On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192.

[5] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), no. 1, 123–133.

Minimal hulls of compact sets in $\mathbb{R}^3$

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(joint work with Barbara Drinovec Drnovšek)

When discussing hulls in various geometries, one typically deals with dual sets of objects. Given a set $\mathcal{P}$ of real functions on a manifold X, the $\mathcal{P}$-hull of a compact subset $K \subset X$ is

$$\widehat{K}_{\mathcal{P}} = \{x \in X : f(x) \leq \sup_K f \quad \forall f \in \mathcal{P}\}. \tag{1}$$

Suppose that $\mathcal{G}$ is a class of geometric objects in X (for example, submanifolds or subvarieties) such that the restriction $f|_C$ satisfies the maximum principle for every $f \in \mathcal{P}$ and $C \in \mathcal{G}$. Then $C \subset \widehat{K}_{\mathcal{P}}$ for every $C \in \mathcal{G}$ with boundary $bC \subset K$, and the main question is how closely is the hull described by such objects. A basic example is the convex hull $\mathrm{Co}(K)$ of a compact set in an affine space $X \cong \mathbb{R}^n$; here $\mathcal{P}$ is the class of all affine linear functions on X and $\mathcal{G}$ is the collection of straight line segments in X. A classical example is that of the polynomially convex hull of a compact set in $\mathbb{C}^n$; this is the hull with respect to the set of plurisubharmonic functions.

We introduce and study a suitable notion of the *minimal hull*, $\widehat{K}_{\mathfrak{M}}$, of a compact set K in $\mathbb{R}^n$. The idea is that $\widehat{K}_{\mathfrak{M}}$ should contain every bounded 2-dimensional minimal surface $M \subset \mathbb{R}^n$ with boundary bM contained in K and hopefully not much more. Any such minimal surface is a solution of the *Plateau problem with free boundary* in K; for a closed Jordan curve K we have the classical Plateau problem. We define $\widehat{K}_{\mathfrak{M}}$ by using the class of *minimal plurisubharmonic functions.* We obtain three characterizations of the minimal hull in $\mathbb{R}^3$: by sequences of conformal minimal discs, by minimal Jensen measures, and by Green currents. The only reason for restricting to $\mathbb{R}^3$ is that the main technical tool (the approximate solution of the Riemann-Hilbert boundary value problem for conformal minimal discs) is currently only available in dimension 3.

An upper semicontinuous function $u\colon \omega \to \mathbb{R} \cup \{-\infty\}$ on a domain $\omega \subset \mathbb{R}^n$ is said to be *minimal plurisubharmonic* if the restriction of u to any affine 2-dimensional plane $L \subset \mathbb{R}^n$ is subharmonic on $L \cap \omega$ (in any isothermal coordinates on L). The set of all such functions is denoted by $\mathfrak{M}\mathrm{Psh}(\omega)$. For every $u \in \mathfrak{M}\mathrm{Psh}(\omega)$ and every conformal minimal disc $f\colon \mathbb{D} \to \omega$ the composition $u \circ f$ is a subharmonic function on $\mathbb{D}$, so minimal surfaces form a class of objects which is dual to the class of minimal plurisubharmonic functions. It is easily seen that a $\mathscr{C}^2$ function u is minimal plurisubharmonic if and only if the sum of the two smallest eigenvalues of its Hessian is nonnegative at every point; hence $\mathscr{C}^2$ minimal plurisubharmonic functions are exactly 2-*plurisubharmonic functions* studied by Harvey and Lawson (*p-convexity, p-plurisubharmonicity and the Levi problem,* Indiana Univ. Math. J. **62** (2013) 149–169.)

We define the minimal hull, $\widehat{K}_{\mathfrak{M}}$, of a compact set $K \subset \mathbb{R}^n$ as the hull (1) with respect to the family $\mathcal{P} = \mathfrak{M}\mathrm{Psh}(\mathbb{R}^n)$.

In analogy with the classical theorem of E. Poletsky (*Holomorphic currents,* Indiana Univ. Math. J., **42** (1993) 85–144) and Bu-Schachermayer (*Approximation of Jensen measures by image measures under holomorphic functions and applications*, Trans. Amer. Math. Soc. **331** (1992) 585–608) we characterize the minimal hull of a compact set $K \subset \mathbb{R}^3$ by sequences of conformal minimal discs whose boundaries converge to K in the measure theoretic sense; here is the precise result.

Theorem 1. *Let K be a compact set in $\mathbb{R}^3$, and let $\omega \Subset \mathbb{R}^3$ be a bounded open convex set containing K. A point $p \in \omega$ belongs to the minimal hull $\widehat{K}_{\mathfrak{M}}$ of K if and only if there exists a sequence of conformal minimal discs $f_j\colon \overline{\mathbb{D}} \to \omega$ such* *that for all* $j = 1, 2, \ldots$ *we have* $f_j(0) = p$ *and*

$$\big|\{t \in [0, 2\pi] : \mathrm{dist}(f_j(e^{\imath t}), K) < 1/j\}\big| \geq 2\pi - 1/j.$$

Theorem 1 is also used to characterize the minimal hull by limits of Green currents supported on conformal minimal discs.

Theorem 1 is a corollary to the following main result which gives an effective way of constructing minimal plurisubharmonic functions on domains in $\mathbb{R}^3$. Given a domain $\omega \subset \mathbb{R}^n$ and a point $x \in \omega$ we denote by $\mathfrak{M}(\mathbb{D}, \omega, x)$ the set of all conformal minimal immersions $f : \overline{\mathbb{D}} \to \omega$ with $f(0) = x$.

Theorem 2. *Let* ω *be a domain in* $\mathbb{R}^3$ *and let* $\phi \colon \omega \to \mathbb{R} \cup \{-\infty\}$ *be an upper semicontinuous function on* ω*. Then the function*

$$u(x) = \inf\Big\{ \int_0^{2\pi} \phi(f(\mathrm{e}^{\imath t}))\, \frac{dt}{2\pi} : \ f \in \mathfrak{M}(\mathbb{D}, \omega, x)\Big\}, \quad x \in \omega, \tag{2}$$

is minimal plurisubharmonic on ω *or identically* $-\infty$*; moreover,* u *is the supremum of the minimal plurisubharmonic functions on* ω *which are not greater than* ϕ*.*

To obtain Theorem 1 we apply Theorem 2 to the function ϕ which equals -1 in a small open neighbourhood U of K and equals 0 elsewhere. Clearly we have $u = -1$ on $\widehat{K}_{\mathfrak{M}}$. Given a point $p \in \widehat{K}_{\mathfrak{M}}$, Theorem 2 provides a conformal minimal disc centered at p which has most of its boundary contained in U.

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