Mathematisches Forschungsinstitut Oberwolfach

Report No. 6/2015

DOI: 10.4171/OWR/2015/6

Mini-Workshop: Modern Applications of s-numbers and Operator Ideals

Organised by
Fernando Cobos, Madrid
Dorothee D. Haroske, Jena
Thomas Kühn, Leipzig
Tino Ullrich, Bonn

8 February – 14 February 2015

Abstract. The main aim of this mini-workshop was to present and discuss some modern applications of the functional-analytic concepts of s-numbers and operator ideals in areas like Numerical Analysis, Theory of Function Spaces, Signal Processing, Approximation Theory, Probability on Banach Spaces, and Statistical Learning Theory.

Mathematics Subject Classification (2010): Primary: 47B06, 47L20; Secondary: 41A65, 41A63, 60G15, 68T05, 94A20.

Introduction by the Organisers

The mini-workshop was devoted to modern applications of s-numbers and operator ideals in a various more applied areas. It was attended by 16 mathematicians from Germany (10), Spain (2), Austria (1), Canada (1), Finland (1) and France (1), the participants were a mixture of experienced senior scientists and younger researchers, with different mathematical backgrounds.

The theories of s-numbers and operator ideals, which are both closely related to geometry and local theory of Banach spaces, and also to probability on Banach spaces, were already developed in the 1970s and 1980s, with main contributions due to Albrecht Pietsch. During the last 15 years these by now almost classical abstract functional-analytic concepts appeared quite naturally in several, more applied branches of mathematics. In particluar, they have found important applications in areas such as

- Compressed Sensing and Image Processing (Gelfand numbers, Johnson-Lindenstrauss lemma)
- Numerical Analysis and Information-based Complexity (approximation and entropy numbers, 2-summing operators, Banach spaces of type)
- Function Spaces (various s-numbers, operator ideal techniques)
- Approximation Theory (abstract approximation spaces)
- Small Deviations of Gaussian Processes (entropy numbers)
- Statistical Learning Theory (covering numbers)

The main aims of the Mini-Workshop were to

- bring together experienced functional analysts and younger researchers from applied areas,
- present some modern applications of s-numbers and operator ideals in the above-mentioned areas,
- discuss open problems and identify directions for future research,
- initiate exchange and co-operation between different communities.

In order to achieve these goals the mini-workshop was organized as follows. Each participant gave a 50-minutes talk on her/his field of research, pointing out in particular the use of s-number and operator ideal techniques, open questions and relations to other fields. In this way the participants from different communities could learn from each other, and the ground was laid for further discussions.

The first talk was given by Albrecht Pietsch, who presented an overview over important problems that have been left open in the area of s-numbers and operator ideals itself. Let us mention just one, which is related to the famous counterexample by Enflo, who showed that there are Banach spaces without the approximation property. In the language of operators, the problem is to quantify the gap between compact and approximable operators, that means to determine the smallest entropy ideal that contains non-approximable operators.

The four organizers gave survey talks on the role of s-numbers and operator ideals in the theory of function spaces (Dorothee Haroske), approximation theory (Fernando Cobos), signal processing and numerical analysis (Tino Ullrich) and Gaussian processes (Thomas Kühn). In the talks of the remaining participants several other interesting topics were presented, e.g. Khintchine-type inequalities (Hermann König, Gilles Pisier), entropy inequalities (Nicole Tomczak-Jaegermann), tractability problems in information-based complexity (Stefan Heinrich), polynomials on Banach spaces (Andreas Defant), entropy numbers in statistical learning theory (Ingo Steinwart), singular traces (Albrecht Pietsch).

Moreover, apart from these talks which were already scheduled in advance, we spontaneously organized an informal session on Thursday afternoon, in which Hermann König lectured on new developments concerning the famous Grothendieck constant. Since the works of Krivine in 1975 in the real case and Haagerup in 1986 in the complex case, there has been no progress for many years. Only very recently Assaf Naor introduced new averaging processes which led to an improvement. But still the problem of the exact value of the Grothendieck constant is wide open.

This talk was followed by a problem session, where several really important and quite challenging problems were presented and discussed. These problems covered a wide range, e.g. Schur multipliers on Schatten p-classes, cotype of projective tensor products, tractability of star-discrepancy, approximation vs. sampling numbers, entropy numbers in learning theory.

During the whole week of the mini-workshop there was an intensive scientific interaction between the participants from different communities. There were lot of discussions in smaller groups on specific problems, which led to several new mathematical contacts and to first plans for concrete projects of future co-operation. Throughout the mini-workshop the atmosphere was very inspiring.

As usual, on Wednesday afternoon we had an excursion, this time consisting of a walk to Oberwolfach and a visit of the MiMa. The guided tour through the museum with so many beautiful minerals and the interactive mathematics part was very interesting and enjoyable, thanks to our expert tour guide Stephan Klaus.

Last but not least we would like to thank – on behalf of all participants – the director, administration and staff of the MFO for their excellent professional work and kind support before and during the mini-workshop. This made it possible to create the fruitful scientific atmosphere which is so typical for Oberwolfach meetings, and to make our mini-workshop a full success, an impression shared by all participants.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Mini-Workshop: Modern Applications of s-numbers and Operator Ideals

Table of Contents

Abstracts

Open problems concerning s-numbers and operator ideals

Albrecht Pietsch

Operator ideals on the class of all Banach spaces have been studied since about 50 years. One of the most important methods to construct those object is via s-numbers. In many ways we can assign to every bounded linear operator T from a Banach space X into a Banach space Y a sequence $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$, which satisfies some axioms and measures the 'size' of T. Then, for $0 < p < \infty$, the components of a (quasi-Banach) operator ideal are given by

$$\mathfrak{L}_p^{(s)}(X,Y) := \big\{T : \sum_{n=1}^{\infty} s_n(T)^p < \infty\big\}.$$

We present several challenging problems from the following idea groups:

(1) Comparing operator ideals generated from different s-numbers.

(2) Which ideals $\mathfrak{L}_p^{(s)}$ contain non-approximable operators?

(3) In which quasi-Banach ideals $\mathfrak{L}_p^{(s)}$ are the finite rank operators dense?

(4) Approximation properties with respect to various ideals $\mathfrak{L}_p^{(s)}$.

(5) Traces on quasi-Banach operators ideals.

(6) How many operator ideals are there on the separable infinite-dimensional Hilbert space?

Here are my favorite problems:

(A) Is it true that $\mathfrak{L}_q^{\text{hilb}} \subseteq \mathfrak{L}_p^{\text{app}}$ whenever $\frac{1}{q} \geq \frac{1}{p} + 1$?

app: approximation numbers, hilb: Hilbert numbers.

(B) For which p are all operators in $\mathfrak{L}_p^{\text{ent}}$ approximable?

YES: $0 < p \leq \frac{2}{3}$, NO: $2 < p < \infty$, OPEN: $\frac{2}{3} < p \leq 2$, ent: entropy numbers.

(C) Let $\mathfrak{A}$ be any quasi-Banach operator ideal supporting a trace τ that coincides with the usual trace on the finite rank operators. Then all operators $T \in \mathfrak{A}^2(X)$ (products of two operators in $\mathfrak{A}$) have an absolutely summable sequence of eigenvalues $\lambda_n(T)$. Does the trace formula $\text{trace}(T) = \sum_{n=1}^{\infty} \lambda_n(T)$ hold for those T's.

References

[1] K.-D. Kürsten and A. Pietsch, *Non-approximable compact operators*, Archiv der Math. **103** (2014), 473–480.

[2] A. Pietsch, *Operator Ideals*, Deutscher Verlag Wiss., Berlin, 1978; North–Holland, Amsterdam–London–New York–Tokyo, 1980.

[3] A. Pietsch, *Eigenvalues and s-Numbers*, Geest & Portig, Leipzig, and Cambridge Univ. Press, 1987.

[4] M. C. White, *Analytic multivalued functions and spectral trace*, Math. Ann. **304** (1996), 669–683.

Best constants in the the Khintchine-Steinhaus inequality

Hermann König

The best constants in the Khintchine inequality have been determined for all $0 < p < \infty$ by U. Haagerup [H]. For $p = 1$ the constant was derived by Szarek [Sz] before. We consider the complex case of the inequality, i.e. for the Steinhaus variables: Let $S_j : (\Omega, P) \to S^1 \subset \mathbb{C}$ be an i.i.d. sequence of Steinhaus random variables, i.e. variables which are equally distributed on the circle S^1. These are the complex analogues of the Rademacher variables. We discuss the best constants a_p in the Khintchine-Steinhaus inequality

$$a_p \ ||x||_2 \leq (\mathbb{E}|\sum_{j=1}^{n} x_j \, S_j|^p)^{1/p} \leq b_p \ ||x||_2 \qquad ; \ x = (x_j)_{j=1}^n \in \mathbb{C}^n$$

which were not known for $0 < p < 1$. They are given by

$$a_p = \min \ \left(\Gamma(\frac{p}{2}+1)^{1/p} \ , \ \sqrt{2} \ \left(\Gamma(\frac{p+1}{2}) \Big/ [\Gamma(\frac{p}{2}+1)\sqrt{\pi}] \right)^{1/p} \right) .$$

Both expressions are equal for $p = p_0 \simeq 0.4756$. For $p = 1$ the best constant a_1 was given by Sawa [S], for $1 < p < \infty$ the constants a_p and b_p were determined by König, Kwapien [KK] and independently by Baernstein, Culverhouse [BC]. For $p_0 \leq p < \infty$ the vectors $\frac{1}{\sqrt{n}}(1, \cdots, 1)$ are asymptotically optimal, for $0 < p \leq p_0$ equality for a_p is attained by the vector $\frac{1}{\sqrt{2}}(1, 1, 0, \cdots, 0)$.
The result implies for a norm 1 sequence $x \in \mathbb{C}^n$, $||x||_2 = 1$, that

$$\mathbb{E} \ln |(S_1 + S_2)/\sqrt{2}| \ \leq \ \mathbb{E} \ln |\sum_{j=1}^{n} x_j S_j| \ ,$$

answering a question of A. Baernstein and R. Culverhouse [BC]. The proof relies on Haagerup's integral formula from [H] and a technique using distributions functions developed by Nazarov, Podkorytov [NP], using estimates for Bessel functions.

References

[BC] A. Baernstein, R.C. Culverhouse, *Majorization of sequences, sharp vector Khintchine inequalities, and bisubharmonic functions*, Studia Math. **152** (2002), 221 – 248.

[H] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. **70** (1982), 231 – 283.

[KK] H. König, S. Kwapień, *Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors*, Positivity **5** (2001), 115 – 152.

[NP] F.L. Nazarov, A.N. Podkorytov, *Ball, Haagerup, and distribution functions*, in: Complex Analysis, Operators and Related Topics, ed. V. Havin, N. Nikolski, Birkhäuser, 2000, 247 – 267.

[S] J. Sawa, *On the best constant in the Khintchine inequality for complex Steinhaus variables, the case* $p = 1$, Studia Math. **81** (1985), 107 – 126.

[Sz] S. Szarek, *On the best constants in the Khintchine inequality*, Studia Math. **58** (1976), 197 – 208.

The non-commutative Khintchine inequalities for $0 < p < \infty$

Gilles Pisier

This is based on joint work with Éric Ricard (Université de Caen Basse-Normandie). We give a proof of the Khintchine inequalities in non-commutative L_p-spaces for all $0 < p < 1$. This case remained open since the first proof given by Françoise Lust-Piquard in 1986 in [7] for $1 < p < \infty$. These inequalities are valid for the Rademacher functions or Gaussian random variables, but also for more general sequences, e.g. for the analogues of such random variables in free probability.

The Khintchine inequalities for non-commutative L_p-spaces play an important rôle in the recent developments in non-commutative Functional Analysis, and in particular in Operator Space Theory, see [10]. Just like their commutative counterpart for ordinary L_p-spaces, they are a crucial tool to understand the behavior of unconditionally convergent series of random variables, or random vectors, in non-commutative L_p ([13]). The commutative version is closely related to Grothendieck's Theorem see [12]. Moreover, in the non-commutative case, Random Matrix Theory and Free Probability provide further ground for applications of the non-commutative Khintchine inequalities. For instance, they imply the remarkable fact that the Rademacher functions (i.e. i.i.d. ± 1-valued independent random variables) satisfy the same inequalities as the freely independent ones in non-commutative L_p for $p < \infty$. See [3] for a recent direct simple proof of the free version of these inequalities, which extend to $p = \infty$.

In the most classical setting, the non-commutative Khintchine inequalities deal with Rademacher series of the form

$$S = \sum\nolimits_k r_k(t) x_k$$

where (r_k) are the Rademacher functions on the Lebesgue interval (or any independent symmetric sequence of random choices of signs) where the coefficients x_k are in the Schatten q-class or in a non-commutative L_q-space associated to a semifinite trace τ. Let us denote simply by $\|.\|_q$ the norm (or quasi-norm) in the latter Banach (or quasi-Banach) space, that we will denote by $L_q(\tau)$. When τ is the usual trace on $B(\ell_2)$, we recover the Schatten q-class. By Kahane's well known results, S converges almost surely in norm iff it converges in $L_q(dt; L_q(\tau))$. Thus to characterize the almost sure norm-convergence for series such as S, it suffices to produce a two sided equivalent of $\|S\|_{L_q(dt;L_q(\tau))}$ when S is a finite sum, and this is precisely what the non-commutative Khintchine inequalities provide:
For any $0 < q < \infty$ there are positive constants α_q, β_q such that for any finite set $(x_1, \ldots, x_n)$ in $L_q(\tau)$ we have

$$(\beta_q)^{-1} |||(x_k)|||_q \leq \left(\int \|S(t)\|_q^q dt \right)^{1/q} \leq \alpha_q |||(x_k)|||_q$$

where $|||(x_k)|||_q$ is defined as follows:
If $2 \leq q < \infty$

$$|||(x_k)|||_q \stackrel{\text{def}}{=} \max \left\{ \left\| \left(\sum x_k^* x_k \right)^{\frac{1}{2}} \right\|_q, \left\| \left(\sum x_k x_k^* \right)^{\frac{1}{2}} \right\|_q \right\} \tag{1}$$

and if $0 < q \leq 2$:

$$|||x|||_q \stackrel{\text{def}}{=} \inf_{x_k = a_k + b_k} \left\{ \left\| \left(\sum a_k^* a_k \right)^{\frac{1}{2}} \right\|_q + \left\| \left(\sum b_k b_k^* \right)^{\frac{1}{2}} \right\|_q \right\}. \tag{2}$$

Note that $\beta_q = 1$ if $q \geq 2$, while $\alpha_q = 1$ if $q \leq 2$ and the corresponding one sided bounds are easy. The difficulty is to verify the other side.

The inequalities (1) (2) actually hold for more general sequences than the Rademacher functions, for instance, for free Haar unitaries in the sense of [14] or to the "$Z(2)$-sequences" considered in [5].

As we already mentioned, the case $1 < q < \infty$ goes back to [7]. The case $q = 1$ was proved (in two ways) in [8], together with a new proof of $1 < q < \infty$. This also implied the fact (independently observed by Junge) that $\alpha_q = O(\sqrt{q})$ when $q \to \infty$, which yielded an interesting subGaussian estimate. Later on, Buchholz proved in [1] a sharp version valid when $q > 2$ is any even integer, the best α_q happens to be the same as in the commutative (or scalar) case.

The case $q < 2$ of the Khintchine inequalities has a more delicate formulation, but this case can be handled easily when $1 < q < 2$ using a suitable duality argument. The case $q = 1$ is closely related to the "little non-commutative Grothendieck inequality" in the sense of [12] (first proved in [9]): actually, one of the proofs given for that case in [8] shows that it is essentially "equivalent" to it. More recently, Haagerup and Musat ([4]) gave a new proof that yields the best constant (equal to 2) for $q = 1$ for the complex analogue (namely Steinhaus random variables) of the Rademacher functions. Their proof starts from the analysis of the (rather easy) case $q = 4$ and then deduces $q = 1$ from it.

In [11] the first named author proved by an extrapolation argument that the validity of this kind of inequalities for some $1 < q < 2$ implies their validity for all $1 \leq p < q$, but the case $q < 1$ remained open. However, very recently Éric Ricard noticed that the method proposed in [11] actually works in this case too. The latter method reduced the problem to a certain form of Hölder type inequality which could not be verified because the arguments (duality or triangular projection) that proved it became seemingly unavailable for $0 < q < 1$. In [11] a certain very weak form of the required Hölder type estimate was identified as sufficient to complete the case $q < 1$. It is this form that Ricard was able to establish by an a priori ultraproduct argument. Although his argument failed to produce explicitly a quantitative estimate, it showed that some estimate does exist. In this talk we describe an explicit estimate, and a reasonably self-contained proof of the case $q < 1$. In fact, it turns out that a certain version of Hölder's inequality (perhaps of independent interest) does hold, thus we can produce an explicit estimate, similar to the case $q \geq 1$ but with unexpected exponents.

Theorem. *Let $0 < p < q < s \le \infty$. Let $\alpha = 1/p - 1/s$ and let $0 < \theta < 1$ be such that $\frac{1}{q} = \frac{(1-\theta)}{p} + \frac{\theta}{s}$. Then for any $0 < R < p$ there is a constant C such that for any $x \in L_s(\tau)$ and $f \in L_1(\tau)^+$ with $\|f\|_1 = 1$, for any choice of sign ± 1 we have*

$$\|xf^{\alpha(1-\theta)} \pm f^{\alpha(1-\theta)}x\|_q \le C\|xf^{\alpha} \pm f^{\alpha}x\|_p^{\frac{R}{2}(1-\theta)}\|x\|_s^{1-\frac{R}{2}(1-\theta)}. \tag{3}$$

This inequality is related to the theory of means developed in [6].

The proof uses crucially estimates for the complex uniform convexity of $L_p(\tau)$ for $0 < p \le 2$, a notion that originates in [2].

References

[1] A. Buchholz, Optimal Constants in Khintchine Type Inequalities for Fermions, Rademachers and q-Gaussian Operators. *Bull Polish. Acad.* N. Vol. 53, No. 3, (2005) 315-321.

[2] W. J. Davis, D.J.H. Garling and N. Tomczak-Jaegermann, The complex convexity of quasi-normed linear spaces. *J. Funct. Anal.* **55** (1984), 110–150.

[3] S. Dirksen and É. Ricard, Some remarks on noncommutative Khintchine inequalities. *Bull. Lond. Math. Soc.* 45 (2013), no. 3, 618–624.

[4] U. Haagerup and M. Musat, On the best constants in noncommutative Khintchine-type inequalities, *J. Funct. Anal.* **250** (2007), 588–624.

[5] A. Harcharras, Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets, *Studia Math.* **137** (1999), no. 3, 203–260.

[6] F. Hiai and H. Kosaki, Means of Hilbert space operators. Lecture Notes in Mathematics, 1820. Springer-Verlag, Berlin, 2003.

[7] F. Lust-Piquard, Inégalités de Khintchine dans C_p $(1 < p < \infty)$, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), no. 7, 289–292.

[8] F. Lust-Piquard and G. Pisier, Noncommutative Khintchine and Paley inequalities, *Ark. Mat.* **29** (1991), no. 2, 241–260.

[9] G. Pisier, Grothendieck's theorem for noncommutative C^*-algebras, with an appendix on Grothendieck's constants, *J. Funct. Anal.* **29** (1978), no. 3, 397–415.

[10] G. Pisier, Introduction to operator space theory, *London Mathematical Society Lecture Note Series, 294*, Cambridge University Press, Cambridge, 2003. viii+478 pp.

[11] G. Pisier, Remarks on the non-commutative Khintchine inequalities for $0 < p < 2$. *J. Funct. Anal.* 256 (2009), 4128–4161.

[12] G. Pisier, Grothendieck's Theorem, past and present, *Bull. Amer. Math. Soc.* 49 (2012), 237–323.

[13] G. Pisier and Q. Xu, Non-commutative L^p-spaces, *Handbook of the geometry of Banach spaces*, Vol. 2, 1459–1517, North-Holland, Amsterdam, 2003.

[14] D. Voiculescu, K. Dykema and A. Nica, Free random variables, *CRM Monograph Series, Vol. 1*, Amer. Math. Soc., Providence, RI.

On the use of s-numbers and other functional analytic tools in the theory of optimal algorithms

Stefan Heinrich

In the first part of this talk we give a short introduction to information-based complexity theory (IBC) - a branch of (theoretical) numerical analysis, that is concerned with optimal algorithms, see [3, 4] for details. We discuss the basic quantities of this theory – the n-th minimal errors – in various settings: nonadaptive, adaptive, deterministic, randomized, linear information, standard information. We explain the connection of these quantities to s-numbers: either they are s-numbers themselves, or they exhibit very similar features - which turns out to be useful for their analysis in IBC. More details on the relation of minimal errors and s-numbers can be found in [1].

In the second part we survey some recent results on the n-th minimal errors for definite and indefinite integration, as well as for initial value problems for systems of ordinary differential equations. We also discuss the parameter-dependent versions of these numerical problems and view them as Banach space valued problems. Here further functional analytic tools - the notion of type and related ones - turn out to be useful. We illustrate this by presenting a recent result on the complexity of Banach space valued definite integration:

Let $d \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $Q = [0,1]^d$ be the d-dimensional unit cube, let $B_{C^r(Q,X)}$ be the unit ball of the space of r-times continuously differentiable functions with values in a Banach space X. For $f \in B_{C^r(Q,X)}$ we want to compute (approximately) $S^{\mathrm{int},X} f = \int_Q f(t)dt$ using X-valued information $\{\delta_t, t \in Q\}$. Let $e_n^{\mathrm{ran}}(S^{\mathrm{int},X}, B_{C^r(Q,X)})$ denote the n-th minimal error in the randomized setting. The following was shown in [2].

Theorem. Let $1 \le p \le 2$. Then the following are equivalent:

(i) X is of equal norm type p.

(ii) There is a constant $c > 0$ such that for all $n \in \mathbb{N}$

$$e_n^{\mathrm{ran}}(S^{\mathrm{int},X}, B_{C^r(Q,X)}) \le c n^{-r/d-1+1/p}.$$

I am happy to mention that this paper [2] is dedicated to one of the participants of the workshop – Albrecht Pietsch – on the occasion of his 80th birthday.

References

[1] S. Heinrich, Ultrastability of n-th minimal errors, J. Complexity **28** (2012), 440–458.

[2] S. Heinrich, A. Hinrichs, On the randomized complexity of Banach space valued integration, Studia Math. **223** (2014), 205–215.

[3] E. Novak, *Deterministic and Stochastic Error Bounds in Numerical Analysis*, Lecture Notes in Mathematics 1349, Springer, 1988.

[4] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, *Information-Based Complexity*, Academic Press, New York, 1988.

Metric Entropy Inequalities

Nicole Tomczak-Jaegermann

For lack of space we do not recall standard definitions or notations, which can be found in all our references.

We prove "entropy extension-lifting theorem". It consists of two inequalities for covering numbers of two symmetric convex bodies. The first inequality, provides upper estimates in terms of entropy of sections of the bodies. To emphasize a similarity to the extension property of ℓ_∞ (which follows from the Hahn-Banach theorem) it is natural to see it as "entropy extension theorem." The second estimate, which can be called "entropy lifting theorem", provides estimates in terms of entropies of projections. These results were strongly motivated by a result from [4], and proved in [3].

In the second part of the talk we discuss the duality conjecture formulated by A. Pietsch (1972): *Do there exist numerical constants $a, b \geq 1$ such that for any dimension n and for any two symmetric convex bodies K, L in $\mathbb{R}^n$ one has*

$$b^{-1} \log N(L^\circ, aK^\circ) \leq \log N(K, L) \leq b \log N(L^\circ, a^{-1}K^\circ)?$$

(Here K° and L° denote the polar bodies for K and L respectively.) Studies of this conjecture gave rise (in [2]) to a new concept of "convexified separation" which clarified the concept of duality.

For a set K and a symmetric convex body L we define

$$\begin{aligned} \hat{M}(K, L) \quad := \quad & \sup\{N : \exists\, x_1, \ldots, x_N \in K \quad \text{such that} \\ & \forall j \ (x_j + \mathrm{int} L) \cap \mathrm{conv}\{x_i, i < j\} = \emptyset\}, \end{aligned}$$

where "int" is the interior of a set. The main result in this direction is ([2]):

For any pair of convex symmetric bodies $K, L \subset \mathbb{R}^n$ one has

$$\hat{M}(K, L) \leq \hat{M}(L^\circ, K^\circ/2)^2.$$

Therefore, to prove for example the result that establishes the duality when one of spaces is a Hilbert space (see [1]), it is enough to show that:

Let $K \subset \mathbb{R}^n$ be a convex symmetric body and let B_2^n be the unit Euclidean ball. Then one has:

$$\log M(B_2^n, K) \leq \beta \log \hat{M}(B_2^n, K/2),$$

where $\beta > 0$ is a universal constant.

These and other results can be found in [2, 1] and references therein.

References

[1] S. Artstein, V. Milman, S. J. Szarek, *Duality of metric entropy*, Ann. Math. **159** (2004), 1313–1328.

[2] S. Artstein, V. Milman, S. J. Szarek, N. Tomczak-Jaegermann, *On convexified packing and entropy duality*, Geom. Funct. Anal. **14** (2004), 1134–1141.

[3] A. E. Litvak, V. Milman, A. Pajor, N. Tomczak-Jaegermann, *Entropy extension*, Russian, translation in Funct. Anal. Appl. **40** (2006), 298–303.

[4] A. E. Litvak, A. Pajor and N. Tomczak-Jaegermann, *Diameters of Sections and Coverings of Convex Bodies*, J. Funct. Anal., **231** (2006), 438–457.

Unconditionality in Spaces of Polynomials

Andreas Defant

Let X be a Banach sequence space (i.e., $\ell_1 \subset X \subset c_0$ such that the e_k's form a 1-unconditional basis) and Λ a finite subset of multi indices (finite sequences $\alpha = (\alpha_1, \ldots, \alpha_N, 0, \ldots)$ with entries in $\mathbb{N}_0$). As usual the αth monomial on X is given by $z^\alpha(x) = x_1^{\alpha_1} \ldots x_N^{\alpha_N}$, $x \in X$. Define $\chi\left((z^\alpha)_{\alpha\in\Lambda}; X\right)$ to be the best constant $c > 0$ such that for every finite choice of $c_\alpha \in \mathbb{C}$, $\alpha \in \Lambda$ we have $\|\sum_\Lambda |c_\alpha| z^\alpha\|_\infty \leq c\|\sum_\Lambda c_\alpha z^\alpha\|_\infty$, where $\|\cdot\|_\infty$ stands for the sup norm with respect to the unit ball B_X of X. In other words, $\chi\left((z^\alpha)_{\alpha\in\Lambda}; X\right)$ is the unconditional basis constant of the monomial basic sequence $(z^\alpha)_{\alpha\in\Lambda}$ within the Banach space $C(B_X)$. Our main interest is to find "good" constants $C_1, C_2 > 0$ and an "optimal" exponent $\lambda > 0$ such that

$$C_1|\Lambda|^\lambda \leq \chi\left((z^\alpha)_{\alpha\in\Lambda}; X\right) \leq C_2|\Lambda|^\lambda .$$

Our motivation and orientation comes from four concrete projects originally initiated by the four authors Sean Dineen, Harold Boas, Hervé Queffélec, and Harald Bohr:

- Dineen: Can the Banach space $\mathcal{P}(^m X)$ of all m-homogeneous polynomials on X ever have an unconditional basis?
- Boas: What is the precise asymptotic order of the n-dimensional Bohr radius of the unit ball in ℓ_r^n?
- Queffélec: What is the precise asymptotic order of the Sidon constant of all finite Dirichlet polynomials $\sum_{n=1}^x a_n \frac{1}{n^s}$ of length x?
- Bohr: How big is the set in B_X on which every holomorphic function on B_X is represented by its monomial series expansion?

For information on the historical background, as well as complete and partial solutions see [4], [1, 6, 7], [6], and [2, 3, 5], respectively. The techniques used involve local Banach space theory (operator ideals), complex analysis, probability theory, and number theory.

References

[1] F. Bayart, D. Pellegrino, and J. B. Seoane-Sepúlveda. The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$. *Advances in Math.*, 264:726–746 (2014)

[2] F. Bayart, A. Defant, L. Frerick, M. Maestre, and P. Sevilla-Peris. Monomial series expansion of H_p-functions and multipliers of $\mathcal{H}_p$-dirichlet series. *preprint*, 2014.

[3] F. Bayart, A. Defant, and S. Schlüters. Monomial convergence for holomorphic functions on ℓ_r. *preprint*, 2015.

[4] A. Defant and N. Kalton. Unconditionality in spaces of m-homogeneous polynomials. *Quart. J. Math.*, 56:53–64, 2005.

[5] A. Defant, M. Maestre, and C. Prengel. Domains of convergence for monomial expansions of holomorphic functions in infinitely many variables. *J. Reine Angew. Math.*, 634:13–49, 2009.

[6] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip. The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive. *Ann. of Math. (2)*, 174(1):485–497, 2011.

[7] A. Defant and L. Frerick. The Bohr radius of the unit ball of ℓ_p^n. *J.reine angew. Math.*, 660:131–147, 2011.

The benefit of s-numbers in the study of compact embeddings of function spaces, and applications

Dorothee D. Haroske

An essential motivation to study entropy and certain s-numbers, like approximation numbers, of embeddings of function spaces comes from spectral theory: one would like to estimate the asymptotic behaviour of eigenvalues, say, of a fractal drum in the sense of [12], or describe the negative spectrum of appropriate operators, cf. [4, 6]. This is based on asymptotically sharp estimates for the entropy or approximation numbers of corresponding compact embeddings of function spaces, either on bounded domains or in the weighted setting, and on the famous Carl's inequality [1, 2]. Nowadays the study of compact embeddings of quite general function spaces has gained some life of its own: one wants to determine the ' degree of compactness' characterised by the behaviour of entropy or s-numbers.
In this survey talk we illustrate the nowadays standard approach to such questions by an example related to entropy numbers and weighted Besov spaces,

$$e_k \left(\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^n) \right),$$

where $s_1 \geq s_2$, $0 < p_1, p_2, q_1, q_2 \leq \infty$, $\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0$, and w an appropriate weight function like $w(x) = (1 + |x|^2)^{\beta/2}$, $\beta > 0$. While the used decomposition techniques in [3] essentially admitted complete results in non-limiting cases $\delta \neq \beta/p_1$ merely (though first promising new phenomena were detected for $\delta = \beta/p_1$ already), the remaining gaps could be sealed in [9] only. The approach in [9, 10], based on wavelet decomposition techniques and operator ideal arguments, cf. [11], can be seen as standard method nowadays.
In [5, 7] we followed the described scheme and studied weights from the Muckenhoupt class $\mathcal{A}_\infty$, whereas very recent questions concern compact embeddings of so-called smoothness spaces of Morrey type like $\mathcal{N}^s_{p,u,q}$ or $B^{s,\tau}_{p,q}$. Some first partial results were obtained in [8, 13].

References

[1] B. Carl. Entropy numbers, s-numbers and eigenvalue problems. *J. Funct. Anal.*, 41:290–306, 1981.

[2] B. Carl, H. Triebel. Inequalities between eigenvalues, entropy numbers and related quantities in Banach spaces. *Math. Ann.*, 251:129–133, 1980.

[3] D. Haroske, H. Triebel. Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators I. *Math. Nachr.*, 167:131–156, 1994.

[4] D. Haroske, H. Triebel. Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators II. *Math. Nachr.*, 168:109–137, 1994.

[5] D.D. Haroske, L. Skrzypczak. Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, I. *Rev. Mat. Complut.*, 21(1):135–177, 2008.

[6] D.D. Haroske, L. Skrzypczak. Spectral theory of some degenerate elliptic operators with local singularities. *J. Math. Anal. Appl.*, 371(1):282–299, 2010.

[7] D.D. Haroske, L. Skrzypczak. Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, II. General weights. *Ann. Acad. Sci. Fenn. Math.*, 36(1):111–138, 2011.

[8] D.D. Haroske, L. Skrzypczak. Embeddings of Besov-Morrey spaces on bounded domains. *Studia Math.*, 218:119–144, 2013.
[9] Th. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak. Entropy numbers of embeddings of weighted Besov spaces. *Constr. Approx.*, 23:61–77, 2006.
[10] Th. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak. Entropy numbers of embeddings of weighted Besov spaces II. *Proc. Edinburgh Math. Soc. (2)*, 49:331–359, 2006.
[11] A. Pietsch. *Eigenvalues and s-numbers.* Akad. Verlagsges. Geest & Portig, Leipzig, 1987.
[12] H. Triebel. *Fractals and spectra.* Birkhäuser, Basel, 1997.
[13] W. Yuan, D.D. Haroske, S.D. Moura, L. Skrzypczak, D. Yang. Limiting embeddings in smoothness Morrey spaces, continuity envelopes and applications. *J. Approx. Theory*, 192:306–335, 2015.

Approximation spaces

Fernando Cobos

Let $(X, \|\cdot\|_X)$ be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of subsets in X satisfying that $G_0 = \{0\}, G_n \subseteq G_{n+1}, \lambda G_n \subseteq G_n$ and $G_n + G_m \subseteq G_{n+m}$ for any $n, m \in \mathbb{N}$ and any scalar λ. Given any $f \in X$ and $n \in \mathbb{N}$, we put $E_n(f) = \inf\{\|f - g\|_X : g \in G_{n-1}\}$ for the error of best approximation to f by the elements of G_{n-1}. For $0 < \alpha < \infty$ and $0 < q \leq \infty$, the approximation space X_q^α consists of all $f \in X$ such that the sequence $(E_n(f))$ belongs to the Lorentz sequence space $\ell_{1/\alpha,q}$. The quasi-norm on X_q^α is given by

$$\|f\|_{X_q^\alpha} = \|(E_n(f))\|_{\ell_{1/\alpha,q}} = \Big(\sum_{n=1}^{\infty}(n^\alpha E_n(f))^q n^{-1}\Big)^{1/q}$$

(as usual, the sum should be replaced by the supremum if $q = \infty$). See the papers [1, 7, 5, 8, 9].

Spaces X_q^α are modeled on the operator ideals $\mathcal{L}_{1/\alpha,q}^{(a)}$ defined by the approximation numbers and the sequence space $\ell_{1/\alpha,q}$. In fact, if we take $X = \mathcal{L}(E, F)$, the space of all bounded linear operators T from the Banach space E into the Banach space F, and $G_n = \mathcal{F}_n(E, F)$, the subset of operators with rank less than or equal than n, then $E_n(T)$ coincides with the n-th approximation number of T and $X_q^\alpha = \mathcal{L}_{1/\alpha,q}^{(a)}(E, F)$. If $X = L_p(\mathbb{T})$, the Lebesgue space of periodic measurable functions defined on the unit circle $\mathbb{T}$, and G_n is the set T_n of all trigonometric polynomials of degree less than or equal to n, then X_q^α is the Besov space $B_{p,q}^\alpha$.

In the papers by Pietsch [8, 9], the theory of approximation spaces is developed and applications are given to distribution of Fourier coefficients and eigenvalues, as well as to tensor products of sequences, functions and operators.

Limiting approximation spaces $X_q^{(0,\gamma)}$ have been also studied (see [4, 3, 6]). They are defined as X_q^α but taking $\alpha = 0$ and inserting the weight $(1 + \log n)^\gamma$ with the sequence $(E_n(f))$

$$X_q^{(0,\gamma)} = \Big\{f \in X : \|f\|_{X_q^{(0,\gamma)}} = \Big(((1 + \log n)^\gamma E_n(f))^q n^{-1}\Big)^{1/q} < \infty\Big\}.$$

Here $-1/q \leq \gamma < \infty$ because if $-\infty < \gamma < -1/q$ then $X_q^{(0,\gamma)} = X$. Note that if $\gamma = 0$ then the space $X_q^{(0,0)}$ corresponds to the choice $\alpha = 0$ in X_q^α, but even in this simple case, the theory of spaces $X_q^{(0,0)}$ does not follow by taking $\alpha = 0$ in the theory of spaces X_q^α (see, for example, [4]). If $X = L_p(\mathbb{T})$ and $G_n = T_n$ then $X_q^{(0,\gamma)}$ coincides with the Besov space $B_{p,q}^{0,\gamma}$ which has zero classical smoothness and logarithmic smoothness with exponent γ (see [5]).

In the talk we review all these notions, describing also some recent results with O. Domínguez [2] on reiteration of approximation constructions and their applications to problems on Besov spaces. In particular we show that if $1 \leq p \leq 2, 1/p' + 1/p = 1, 0 < q \leq \infty$ and $\gamma > -1/q$, then the sequence of Fourier coefficients of any $f \in B_{p,q}^{0,\gamma}$ belongs to the Lorentz-Zygmund sequence space $\ell_{p',q}(\log \ell)_{\gamma + \frac{1}{\max\{p',q\}}}$, which improves a previous result of DeVore, Riemenschneider and Sharpley [5, Corollary 7.3/(i)].

References

[1] P.L. Butzer and K. Scherer, *Approximationsprozesse und Interpolationsmethoden*, Mannheim/Zürich, 1968.

[2] F. Cobos and O. Domínguez, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory **189** (2015), 43–66.

[3] F. Cobos and M. Milman, *On a limit class of approximation spaces*, Numer. Funct. Anal. Optim. **11** (1990), 11–31.

[4] F. Cobos and I. Resina, *Representation theorems for some operator ideals*, J. London Math. Soc. **39** (1989), 324–334.

[5] R.A. DeVore, S.D. Riemenschneider and R.C. Sharpley, *Weak interpolation in Banach spaces*, J. Funct. Anal. **33** (1979), 58–94.

[6] F. Fehér and G. Grässler, *On an extremal scale of approximation spaces*, J. Comp. Anal. Appl. **3** (2001), 95–108.

[7] J. Peetre and G. Sparr, *Interpolation of normed abelian groups*, Ann. Mat. Pura Appl. **92** (1972), 217–262.

[8] A. Pietsch, *Approximation spaces*, J. Approx. Theory **32** (1981), 115–134.

[9] A. Pietsch, *Tensor products of sequences, functions, and operators*, Arch. Math. (Basel) **38** (1982), 335–344.

Besov spaces of logarithmic smoothness, characterizations and embeddings

Oscar Domínguez

Besov spaces $B_{p,q}^s$ arise in a natural way in approximation theory. However, to solve some natural questions as compactness in limiting embeddings or sharp embeddings, we need to consider Besov spaces of generalized smoothness where smoothness of functions is considered in a more delicate manner than in $B_{p,q}^s$. In particular, if we take classical smoothness $s \geq 0$ and additional logarithmic smoothness with exponent $b \in \mathbb{R}$, we are dealing with Besov spaces of logarithmic smoothness $\mathbf{B}_{p,q}^{s,b}$. These spaces which are defined via differences were already introduced by DeVore, Riemenschneider and Sharpley [6] in 1979. In this talk, we consider characterizations of Besov spaces in terms of approximation errors in the periodic case which complement previous results by Pietsch [8]. If $0 < p, q \leq \infty, s \geq 0$ and $b \in \mathbb{R}$, then

$$\|f\|_{\mathbf{B}^{s,b}_{p,q}(\mathbb{T})} \sim \left(\sum_{n=1}^{\infty} [n^s (1+\log n)^b E_n(f)_p]^q n^{-1} \right)^{1/q}$$

where $E_n(f)_p$ denotes the error of approximation of $f \in L_p(\mathbb{T})$ by trigonometric polynomials of degree less than or equal to $n-1$. We also show characterizations of Besov spaces as real interpolation spaces. Given $0 \leq \theta < 1, 0 < p, q, r \leq \infty, s > 0$ and $b \in \mathbb{R}$, we have that

$$(L_p, \mathbf{B}^s_{p,r})_{(\theta,b),q} = \mathbf{B}^{\theta s, b}_{p,q}$$

with equivalence of quasi-norms and for $1 \leq p \leq \infty, 0 < q \leq \infty, 0 \leq s < k \in \mathbb{N}$ and $b \in \mathbb{R}$, then

$$(L_p, W^k_p)_{(s/k,b),q} = \mathbf{B}^{s,b}_{p,q}$$

with equivalence of quasi-norms, where $(\cdot,\cdot)_{(\theta,b),q}$ is the real interpolation space (limiting if $\theta = 0$) considered by Gomez and Milman [7] and Cobos, Fernández-Cabrera, Kühn and Ullrich [5].

Sobolev-type embeddings from $\mathbf{B}^{s,b}_{p,q}$ into Lorentz-Zygmund spaces were considered in [6] but their method only works for the Banach case. Using the previous characterizations and the limiting real interpolation method, we extend to the quasi-Banach setting the embeddings given in [6] in the critical case when $s = 1/p$ and in the subcritical case $0 < s < 1/p$. The limiting case when $s = 0$ has been studied by Caetano, Gogatishvili and Opic [1], however the restriction $1 \leq p < \infty$ is essential in their approach. To overcome this obstruction, we consider the class $Y_{p,r,b}, 0 < p < \infty, 0 < r \leq \infty$ and $b > -1/r$, formed by all measurable functions f having a finite quasi-norm

$$\|f\|_{Y_{p,r,b}} = \left(\int_0^{2\pi} \left[(1+|\log t|)^b \left(\int_0^t f^*(s)^p ds \right)^{1/p} \right]^r \frac{dt}{t} \right)^{1/r}.$$

Let $0 < p < \infty, 0 < q \leq \infty, b > -1/q, q \leq r$ and $\gamma = b + 1/q - 1/r$. Then

$$\mathbf{B}^{0,b}_{p,q} \hookrightarrow Y_{p,r,\gamma}.$$

Our method not only allows us to deal with the quasi-Banach case but also improves the result given in [1] in the setting of r.i. spaces. We also compare $\mathbf{B}^{0,b}_{p,q}$ with the corresponding spaces $B^{0,b}_{p,q}$ defined by using the Fourier transform. In particular, we show that $\mathbf{B}^{0,b}_{2,2} = B^{0,b+1/2}_{2,2}$ for $b > -1/2$.

The talk is based on joint work with F. Cobos [2, 3, 4].

References

[1] A.M. Caetano, A. Gogatishvili, B. Opic, *Sharp embeddings of Besov spaces involving only logarithmic smoothness*, J. Approx. Theory **152** (2008), 188–214.

[2] F. Cobos, O. Domínguez, *Embeddings of Besov spaces of logarithmic smoothness*, Studia Math. **223** (2014), 193–204.

[3] F. Cobos, O. Domínguez, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory **189** (2015), 43–66.
[4] F. Cobos, O. Domínguez, *On Besov spaces of logarithmic smoothness and Lipschitz spaces*, J. Math. Anal. Appl. **425** (2015), 71–84.
[5] F. Cobos, L.M. Fernández-Cabrera, T. Kühn, T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.
[6] R.A. DeVore, S.D. Riemenschneider, R.C. Sharpley, *Weak interpolation in Banach spaces*, J. Funct. Anal. **33** (1979), 58–94.
[7] M.E. Gomez, M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. Lond. Math. Soc. **34** (1986), 305–316.
[8] A. Pietsch, *Approximation spaces*, J. Approx. Theory **32** (1981), 115–134.

Entropy Numbers, Eigenvalues, and the Analysis of Statistical Learning Algorithms

Ingo Steinwart

Statistical learning algorithms become increasingly important for the analysis of high-dimensional, complex data. One of the currently most successful classes of learning algorithms is based on a reproducing kernel Hilbert space (RKHS) approach. To be more precise, given a data set $D = ((x_1, y_1), \ldots, (x_n y_n)) \in (X \times \mathbb{R})^n$, these learning algorithm obtain their decision function $f_{D,\lambda}$ by solving the optimization problem

$$f_{D,\lambda} = \arg\min_{f \in H} \lambda ||f||_H^2 + \mathcal{R}_{D,L}(f)\,. \tag{1}$$

Here, H is an arbitrary RKHS over X, λ is a user-specified regularization parameter, and L is a convex loss, e.g. the least squares loss.

For the analysis of these learning algorithms, see e.g. [2, 6] for a detailed account, various properties of the used RKHS H and the data generating measure P on $X \times \mathbb{R}$ have to be understood. We discuss a few of these aspects including the entropy numbers of the embedding $I_k : H \to L_2(P_X)$ and eigenvalues of the integral operator $T_k : L_2(P_X) \to L_2(P_X)$, where k denotes the reproducing kernel of H. In particular, we present a result from [5] that established a relationship between these two quantities and expected entropy numbers of the form

$$\mathbb{E}_{D \sim P^n} e_i(I_k : H \to L_2(D))\,,$$

where $L_2(D)$ denotes the L_2-space with respect to the empirical measure described by D. We further show how this result in combination with interpolation spaces between H and $L_2(P_X)$ was used in [7] to obtain minimax-optimal learning rates for the learning algorithms (1) and compare these rates with those found by [4, 1, 3]

Finally, we discuss a few open questions regarding entropy numbers and eigenvalues for the so-called Gaussian RBF kernels.

References

[1] A. Caponnetto and E. De Vito. *Optimal rates for regularized least squares algorithm*, Found. Comput. Math. **7** (2007), 331–368.

[2] F. Cucker and D. X. Zhou. *Learning Theory: An Approximation Theory Viewpoint.* Cambridge University Press, Cambridge, 2007.

[3] S. Mendelson and J. Neeman. *Regularization in kernel learning*, Ann. Statist. **38** (2010), 526–565.

[4] S. Smale and D.-X. Zhou. *Learning theory estimates via integral operators and their approximations*, Constr. Approx. **26** (2007), 153–172.

[5] I. Steinwart. *Oracle inequalities for SVMs that are based on random entropy numbers*, J. Complexity **25** (2009), 437–454.

[6] I. Steinwart and A. Christmann. *Support Vector Machines.* Springer, New York, 2008.

[7] I. Steinwart, D. Hush, and C. Scovel. *Optimal rates for regularized least squares regression*, In S. Dasgupta and A. Klivans, editors, Proceedings of the 22nd Annual Conference on Learning Theory 2009, 79–93.

Entropy and s-numbers in numerical analysis and signal processing

Tino Ullrich

In recent years sparsity has become an important concept in applied mathematics, especially in mathematical signal and image processing. The key idea is that many types of functions and signals arising naturally in these contexts can be described by only a small number of significant degrees of freedom. The novel theory of Compressive Sensing takes advantage of this observation and predicts, quite surprisingly, that sparse high-dimensional signals $x \in \mathbb{R}^N$ can be recovered accurately and efficiently from what was previously considered as highly incomplete linear measurements. In practice, signals are not perfectly sparse. A reasonable sparsity model is given by the non-convex unit ball $K = B_p^N := \{x \in \mathbb{R}^N : \|x\|_p \leq 1\}$ where $p < 1$. The concept of Gelfand numbers

$$c_m(K, X) := \inf_{A \in \mathbb{R}^{m \times N}} \sup_{v \in K \cap \ker A} \|v\|_X \,, \quad , \quad m < N \,,$$

represent a natural frontier of what is possible with "m linear measurements". In other words, for a fixed measurement matrix A we ask for the maximal error between two instances which can not be distinguished by A. In this talk we will present a sharp lower estimate for $c_m(B_p^N, \ell_2^N)$ which complements the celebrated results by Kashin [4], Garnaev, Gluskin [3] and Carl, Pajor, Tomczak-Jaegermann [1], [8] from the 1970/80s. The proof techniques rely on modern methods from Compressive Sensing. These results are based on a joint work with Foucart, Pajor and Rauhut [2].

In the second part of the talk we aim at approximating N-variate functions from the periodic Sobolev classes

$$\|f\|^2_{H^{s,p}(\mathbb{T}^N)} := \sum_{k \in \mathbb{Z}^N} (1 + \sum_{j=1}^N |k_j|^p)^{2s/p} |\hat{f}(k)|^2 \,.$$

The parameter s represents the smoothness, whereas the parameter p induces a "sparse variable dependency". We are interested in the approximation numbers of these classes embedded in $L_2(\mathbb{T}^N)$ and observe the behavior

$$a_n(H^{s,p}(\mathbb{T}^N), L_2(\mathbb{T}^N)) \asymp e_{\lfloor \log n \rfloor}(B_p^N, \ell_\infty^N)^s \,, \tag{1}$$

where the numbers e_k represent the dyadic entropy numbers of the embedding ℓ_p^N in ℓ_∞^N. Entropy in this situation is completely understood [9, 5] and with (1) we determine the behavior of the approximation numbers a_n explicitly in N and n. This is joint work with Mayer and Kühn [6]. In addition, in [6, 7] we were able to determine the "asymptotic constants" via the identity

$$\lim_{n\to\infty} n^{s/N}\, a_n(H^{s,p}(\mathbb{T}^N), L_2(\mathbb{T}^N)) = \mathrm{vol}(B_p^N)^{s/N} \asymp N^{-s/p} \,.$$

References

[1] B. Carl and A. Pajor, Gel'fand numbers of operators with values in a Hilbert space, *Invent. Math.*, **94**(3):479–504, 1988.

[2] S. Foucart, A. Pajor, H. Rauhut, and T. Ullrich, The Gelfand widths of ℓ_p-balls for $0 < p \leq 1$. *J. Complexity*, **26**:629–640, 2010.

[3] A. Y. Garnaev and E. D. Gluskin, On widths of the Euclidean ball, *Sov. Math., Dokl.*, **30**:200–204, 1984.

[4] B. S. Kashin, Diameters of some finite-dimensional sets and classes of smooth functions. *Math. USSR, Izv.*, **11**:317–333, 1977.

[5] T. Kühn. A lower estimate for entropy numbers. *J. Approx. Theory*, **110**(1):120–124, 2001.

[6] T. Kühn, S. Mayer and T. Ullrich, Counting via entropy–new preasymptotics for the approximation numbers of Sobolev embeddings, Preprint, 2015.

[7] T. Kühn, W. Sickel and T. Ullrich, Approximation numbers of Sobolev embeddings–sharp constants and tractability. *J. Complexity*, **30**:95–116, 2014.

[8] A. Pajor and N. Tomczak-Jaegermann, Subspaces of small codimension of finite-dimensional Banach spaces, *Proc. Amer. Math. Soc.*, **97**, 1986.

[9] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, *J. Approx. Theory* , **40**, 121–128, 1984.

Reconstruction of ridge functions from function values

Sebastian Mayer

(joint work with Benjamin Doerr, Daniel Rudolph, Tino Ullrich, Jan Vybíral)

We are interested in reconstructing an unknown multivariate ridge function $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}, x \mapsto g(a \cdot x)$ from a limited number of function values. The univariate function g is called the *profile* and the vector $a \in \mathbb{R}^d$ the *ridge direction.*

In [2] we studied the reconstruction of ridge functions which are defined on the Euclidean unit ball $\Omega = B_2^d := \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$. For $\alpha = s + \beta$, $s \in \mathbb{N}$, $0 < \beta \leq 1$ and $0 < p \leq 2$, consider the class of ridge functions

$$\mathcal{R}_d^{\alpha,p} := \{x \in \Omega \mapsto g(a \cdot x) : g \in \mathrm{Lip}_\alpha([-1,1]), \|g\|_\alpha \leq 1, \|a\|_p \leq 1\},$$

where $\|g\|_\alpha := \max\{\|g\|_\infty, \|g^{(1)}\|_\infty, \ldots, \|g^{(s)}\|_\infty, |g^{(s)}|_\beta\}$ and $|\cdot|_\beta$ denotes the Hölder constant with exponent β. Let $\mathcal{S}_n^{\mathrm{det}}$ be the class of all deterministic, adaptive sampling algorithms using at most n function values. For the deterministic worst-case error $\mathrm{err}_{\alpha,p}(n,d) := \inf_{S\in\mathcal{S}_n^{\det}} \sup_{f\in\mathcal{R}_d^{\alpha,p}} \|f - Sf\|_\infty$ we established the following characterization in terms of the *entropy numbers.*

Theorem 1 ([2, Section 4, Prop. 4.1, 4.2])**.** *Let $\alpha > 0$, $0 < p \le 2$ and $p' = \frac{1}{1-1/\max\{1,p\}}$. Then we have*

$$\varepsilon_n(\mathbb{S}_p^{d-1}, \ell_2^d)^{2\alpha} \lesssim \mathrm{err}_{\alpha,p}(n,d) \lesssim \varepsilon_{n/\binom{d+s}{s}}(B_2^d, \ell_{p'}^d)^\alpha,$$

Let $\Omega = [-1,1]^d$ be the unit cube. In [1] we consider the class of ridge functions

$$\mathcal{R}_d^{\alpha,p} := \{x \in \Omega \mapsto g(a\cdot x) : g \in \mathrm{Lip}_\alpha([-1,1]), \|g\|_\alpha \le 1, \|a\|_p \le 1\}$$

and the *probabilistic worst-case error* $\mathrm{err^{prob}}_{\alpha,p}(n,d) := \inf_{S\in\mathcal{S}_n} \sup_{f\in\mathcal{R}_d^{\alpha,p}} e^{\mathrm{prob}}(S,f)$ where $e^{\mathrm{prob}}(S,f) = \inf\{\varepsilon > 0 : \; P(\|f - Sf\| \le \varepsilon) \ge 1/2\}$ and $\mathcal{S}_n$ is the class of all randomized, adaptive algorithms using at most n function values.

Theorem 2. *Let $\alpha > 1$ and $0 < p \le 1$. For*

$$\mathrm{err^{prob}}_{p,\alpha}(n,d) \lesssim \begin{cases} \left[1/\log(n)\right]^{\alpha(1/p-1)} & : n \le 2^d, \\ (2^d/n)^\alpha & : n > 2^d. \end{cases}$$

Theorem 3. *Let $\alpha > 0$ and $0 < p \le 1$. Then we have*

$$\mathrm{err^{prob}}_{p,\alpha}(n,d) \gtrsim \left[1/\log(n)\right]^{\alpha(1/p-1)}$$

for $n \le 2^{d/8}/4$.

References

[1] B. Doerr, S. Mayer, and D. Rudolph. *Tractability of capturing ridge functions on the cube.* Work in progress.

[2] S. Mayer, T. Ullrich, and J. Vybiral. *Entropy and sampling numbers of classes of ridge functions.* Constr. Approx. (2014), 1–34.

Weyl Numbers of Embeddings of Tensor Product Besov Spaces

Winfried Sickel

(joint work with Kien van Nguyen)

Weyl numbers have been introduced by Pietsch [4]. They belong to the family of s-numbers as, e.g., Kolmogorov, Gelfand and approximation numbers. In [5] Pietsch started to investigate the asymptotic behaviour of Weyl numbers of the embedding operator $id : B^t_{p_1,q_1}(0,1) \to L_{p_2}(0,1)$ if $1 \le p_1, p_2, q_1 \le \infty$ and $t > \max(0, 1/p_1 - 1/p_2)$. As usual, $B^t_{p_1,q_1}(0,1)$ denotes the Besov space with smoothness t, integrability p_1 and fine index q_1 on the interval $(0,1)$. Finally, it was proved by Lubitz [2], but see also König [1], that there exists some $\alpha = \alpha(p_1,p_2,t)$ (explicitly known but we omit details) such that

$$(1) \qquad x_n(id : B^t_{p_1,q_1}(0,1) \to L_{p_2}(0,1)) \asymp n^{-\alpha}, \qquad n \in \mathbb{N},$$

(except two limiting situations in which the behaviour of the x_n is still unknown). Hence, the behaviour is polynomial in n. We continue this program by replacing the Besov space $B^t_{p_1,q_1}(0,1)$ by a certain tensor product of $B^t_{p_1,p_1}(0,1)$. More exactly, we define

$$S^t_{p_1,p_1}B((0,1)^d) := B^t_{p_1,q_1}(0,1) \otimes_{\delta_{p_1}} \ldots \otimes_{\delta_{p_1}} B^t_{p_1,q_1}(0,1) \qquad \text{(d-fold)}\,,$$

where in case $1 < p_1 < \infty$ the symbol δ_{p_1} refers to the p_1-nuclear norm and in case $p_1 = 1$ the symbol δ_{p_1} stands for the projective norm. It turns out that there exists some $\beta = \beta(p_1, p_2, t)$ such that

$$x_n(id : S^t_{p_1,p_1}((0,1)^d) \to L_{p_2}((0,1)^d)) \asymp n^{-\alpha} \log^{(d-1)\beta} n\,, \qquad n \geq 2\,.$$

(as in (1) except some limiting situations). Here α and β are always explicitly known and in addition, α is as in (1). By switching from the space $B^t_{p_1,p_1}(0,1)$ to the tensor product space $S^t_{p_1,p_1}B((0,1)^d)$ the main term in the behaviour of the x_n remains unchanged, the dimension d shows up in the exponent of the logarithm only. For all details we refer to the recent preprint [3].

References

[1] H. König, *Eigenvalue distribution of compact operators.* Birkhäuser, Basel, 1986.

[2] C. Lubitz, Weylzahlen von Diagonaloperatoren und Sobolev-Einbettungen. *Bonner Math. Schriften* **144**, phd-thesis, Bonn 1982.

[3] K. van Nguyen and W. Sickel, Weyl Numbers of Embeddings of Tensor Product Besov Spaces, Preprint, arXiv:1409.7565.

[4] A. Pietsch, Weyl numbers and eigenvalues of operators in Banach spaces. *Math. Ann.* **247** (1980), 149–168.

[5] A. Pietsch, Eigenvalues of integral operators. I. *Math. Ann.* **247** (1980), 169–178.

Sliceable numbers and applications to stochastic differential equations

Stefan Geiss

(joint work with Juha Ylinen)

We introduce general Besov spaces $\mathbb{B}^\Phi_p$ on the Wiener space and investigate variational properties of Backward Stochastic Differential Equations (BSDEs). The Besov spaces are based on decoupling of the underlying Gaussian structure. A BSDE is a stochastic differential equation of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

where $W = (W_t)_{t\in[0,T]}$ is a standard Brownian motion, and the generator $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is predictable and satisfies

$$|f(s,\omega,y_0,z_0) - f(s,\omega,y_1,z_1)| \leq L_Y|y_0 - y_1| + L_Z[1 + |Z_0| + |Z_1|]^\theta |Z_0 - Z_1|$$

where $\theta \in [0,1]$ is the degree of being non-Lipschitz. Given a terminal condition $\xi \in L_2$, one looks for adapted processes $(Y_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ as solution processes. Lipschitz BSDEs ($\theta = 0$) were initiated in [1] and [5], quadratic BSDEs ($\theta = 1$) in [4]. The knowledge of the L_p-variation, $p \in [2,\infty)$, of $(Y_t)_{t\in[0,T]}$ is important for various reasons. To obtain bounds for this variation, we proceed in

[3] as follows: (a) Firstly we check the fractional smoothness of (ξ, f) in terms of spaces of type $\mathbb{B}_p^\Phi$, (b) secondly we deduce the fractional smoothness of the solution processes Y and Z, (c) and finally we deduce upper bounds for the L_p-variation of Y. In case of non-Lipschitz BSDEs, i.e. $\theta > 0$, one can estimate the L_p-variation of Y from above by the fractional smoothness of (ξ, f), measured *in L_p as well*, only for large $p \in (p_0, \infty)$, where $p_0 \in [2, \infty)$ is some threshold. Estimates for small p are unknown. To obtain estimates for this threshold p_0 sliceable numbers are used. The concept of sliceable BMO-martingales in connection with BSDEs was used in [2]. For a BMO-martingale M we define the n-th sliceable number as

$$sl_n(M) := \inf_{0=\tau_0 \le \cdots \le \tau_n = T} \|{}^{\tau_{i-1}}M^{\tau_i}\|_{\mathrm{BMO}},$$

with the infimum taken over sequences of stopping times, and obtain generalized s-numbers. It turns out that the sliceable numbers of the fractional process $(\int_0^t |Z_s|^\theta dW_s)_{t\in[0,T]}$ give an explicit expression for this threshold p_0.

References

[1] J.M. Bismut: Théorie probabiliste du contrôle des diffusions. *Mem. AMS* 176, 1973.

[2] F. Delbaen and S. Tang: Harmonic analysis of stochastic equations and backward stochastic differential equations. *Prob. Theory Relat. Fields* 146:291-336, 2010.

[3] S. Geiss and J. Ylinen: Decoupling on the Wiener space and applications to BSDEs. arXiv:1409.5322.

[4] M. Kobylanski: Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Prob.* 28:558-602, 2000.

[5] E. Pardoux and S. Peng: Adapted solutions of backward stochastic equations. *System and Control Lett.* 14:55-61, 1990.

Metric entropy and Gaussian processes

Thomas Kühn

This is a survey talk on relations between Gaussian measures on Banach spaces (resp. Gaussian processes), operator ideals and metric entropy. In particular, the following topics will be discussed:

- Description of Gaussian measures by operators
- Relations between γ-summing operators and entropy ideals
- Small ball probabilities via metric entropy

In 1974 Linde and Pietsch [7] introduced the ideals $\mathcal{P}_\gamma$ of γ-summing and $\mathcal{R}_\gamma$ of γ-Radonifying operators, see also [8]. These operator ideals can be used to characterize Gaussian measures on Banach spaces. By Dudley's and Sudakov's famous inequalities there is a close connection between Gaussian measures/processes and metric entropy; a proof in the language of operator ideals was given in [4]. Combined with Tomczak-Jaegermann's result on duality of entropy numbers [10] this implies, for any Hilbert space H and any Banach space E,

$$\mathcal{L}_{2,1}^{(e)}(H,E) \subset \mathcal{R}_\gamma(H,E) \subset \mathcal{P}_\gamma(H,E) \subset \mathcal{L}_{2,\infty}^{(e)}(H,E)$$

where $\mathcal{L}^{(e)}_{p,q}$ denotes the ideal of operators whose entropy numbers belong to the Lorentz space $\ell_{p,q}$. The fine indices 1 and ∞ of the entropy ideals in the above inclusions are optimal. More results on Gaussian measures and Banach space geometry can be found in Pisier's monograph [9].

In 1993 Kuelbs and Li [3] discovered a tight relation between small ball probabilities of Gaussian measures and metric entropy, further results in this direction are contained in [6] and [2]. As examples of this relation, I will describe two recent applications concerning small deviation probabilities of certain smooth Gaussian processes, see [5] and [1]. The motivation of [5] was to improve a result of Zhou [11] on covering numbers, which is important in learning theory. This is related to Ingo Steinwart's talk on the use of entropy numbers in statistical learning theory.

References

[1] F. Aurzada, F. Gao, T. Kühn, W. V. Li and Q. M. Shao, *Small deviations for a family of smooth Gaussian processes*, J. Theoret. Probab. **26** (2013), 153–168.

[2] F. Aurzada, I. A. Ibragimov, M. A. Lifshits and H. van Zanten, *Small deviations of smooth stationary Gaussian processes* (Russian) Teor. Veroyatn. Primen. **53** (2008), 788–798; translation in Theory Probab. Appl. **53** (2009), 697–707.

[3] J. Kuelbs and W. V. Li, *Metric entropy and the small ball problem for Gaussian measures*, J. Funct. Anal. **116** (1993), 133–157.

[4] T. Kühn, *γ-Radonifying operators and entropy ideals*, Math. Nachr **107** (1982), 53–58.

[5] T. Kühn, *Covering numbers of Gaussian reproducing kernel Hilbert spaces*, J. Complexity **27** (2011), 489–499.

[6] W. V. Li and W. Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures*, Ann. Probab. **27** (1999), 1556–1578.

[7] W. Linde and A. Pietsch, *Mappings of Gaussian measures of cylindrical sets in Banach spaces*, (Russian) Teor. Verojatnost. Primenen. **19** (1974), 472–487.

[8] A. Pietsch, *Operator ideals*, VEB Deutscher Verlag der Wissenschaften, Berlin 1978, and North-Holland Amsterdam-New York, 1980.

[9] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge University Press 1989.

[10] N. Tomczak-Jaegermann, *Dualité des nombres d'entropie pour des opérateurs à valeurs dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 299–301.

[11] D. X. Zhou, *The covering number in learning theory*, J. Complexity **18** (2002), 739–767.

Complexity of Multivariate Integration

Aicke Hinrichs

In this talk we discuss results on the deterministic and randomized complexity of multivariate integration.

In particular, we discuss negative tractability results from [2, 3] for some small classes of smooth functions in the deterministic setting. An exemplary result is the following theorem for functions from the unit ball of r times continuously differentiable functions

$$\mathcal{C}_d^r = \{f \in C^r(\mathbb{R}^d) \mid \|D^\beta f\| \le 1 \quad \text{for all} \quad |\beta| \le r\},$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, with non-negative integers β_j, $|\beta| = \sum_{j=1}^{d} \beta_j$, and D^β denotes the operator of β_j times differentiation with respect to the jth variable for $j = 1, 2, \ldots d$. By $\| \cdot \|$ we mean the sup norm, $\|D^\beta f\| = \sup_{x \in \mathbb{R}^d} |(D^\beta f)(x)|$. The information complexity $n(\varepsilon, F)$ of the integration problem for a class F of continuous functions is the minimal number n of points needed to approximate the integral of all $f \in F$ with a deterministic algorithm using n function values with an error at most ε.

Theorem 1. *The curse of dimensionality holds for the classes $\mathcal{C}_d^r$ with the* super-exponential *lower bound on the information complexity*

$$n(\varepsilon, \mathcal{C}_d^r) \, \geq \, c_r \, (1 - \varepsilon) \, d^{\, d/(2r+3)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0,1),$$

where $c_r \in (0,1]$ depends only on r.

We also discuss positive tractability results from [1] for the randomized setting in a rather general reproducing kernel Hilbert space context based on the domination theorem for 2-summing operators. Using change of density arguments from Banach space theory it is shown that the integration problem on reproducing kernel Hilbert spaces with a nonnegative kernel is strongly polynomial tractable, that is the information complexity does not depend on the dimension and only depends polynomially on ε^{-1}. The importance sampling density is derived from a Pietsch measure via the Pietsch Domination Theorem.

Since the proof of the general Pietsch Domination Theorem uses a Hahn-Banach argument, the question arises how the density can be found explicitly. We show how in some important cases of tensor product spaces this can be done via sharp Sobolev type inequalities.

References

[1] A. Hinrichs, *Optimal Importance Sampling for the Approximation of Integrals*, J. Complexity **26** (2010), 125–134.

[2] A. Hinrichs, E. Novak, M. Ullrich and H. Woźniakowski, *The curse of dimensionality for numerical integration of smooth functions*, Math. Comp. **83** (2014), 2853–2863.

[3] A. Hinrichs, E. Novak, M. Ullrich and H. Woźniakowski, *The curse of dimensionality for numerical integration of smooth functions*, J. of Complexity **30** (2014), 117–143.

Traces and residues of pseudo-differential operators on the torus

Albrecht Pietsch

Let $\mathfrak{A}(H)$ be an ideal in the ring $\mathfrak{L}(H)$ constituted by all bounded linear operators S on the separable infinite-dimensional complex Hilbert space H. A linear form τ on $\mathfrak{A}(H)$ is called a *trace* if $\tau(SA) = \tau(AS)$ for all $S \in \mathfrak{A}(H)$ and $A \in \mathfrak{L}(H)$. Using techniques from Banach space theory, we are able to construct those traces quite easily. This new approach is demonstrated in the case of Dixmier traces on the ideal $\mathfrak{L}_{1,\infty}(H)$, which can be obtained as follows:

Recall that the *Hilbert–Schmidt norm* of any finite rank operator S is given by

$$\|S|\mathfrak{L}_2\| := \Big(\sum_{m=1}^{\infty} \|Se_m\|^2\Big)^{1/2}$$

and does not depend on the underlying orthonormal basis (e_m). By definition, the ideal $\mathfrak{L}_{1,\infty}(H)$ consists of all operators $S \in \mathfrak{L}(H)$ admitting a representation

$$S = \sum_{k=0}^{\infty} S_k \quad \text{such that } \operatorname{rank}(S_k) \leq 2^k \text{ and } \|S_k|\mathfrak{L}_2\| = O(2^{-k/2}).$$

If $\operatorname{trace}(S_k)$ denotes the usual trace of the finite rank operator S_k, then we have $|\operatorname{trace}(S_k)| \leq 2^{k/2}\|S_k|\mathfrak{L}_2\|$. Hence $\big(\operatorname{trace}(S_k)\big)$ is a bounded scalar sequence, and it follows that, for any *shift-invariant* linear form λ on ℓ_∞, the expression

$$\tau(S) := \lambda\big(\operatorname{trace}(S_k)\big)$$

is well-defined and yields a trace on the ideal $\mathfrak{L}_{1,\infty}(H)$. The map $\lambda \mapsto \tau$ is one-to-one, and all traces can be obtained in this way.

We apply the construction above to *classical* pseudo-differential operators living on the d-dimensional torus and prove an extended version of *Connes' trace theorem*, which relates normalized traces of those operators to their *Wodzicki residues*. The same approach works in the setting of closed Riemannian manifolds.

References

[1] A. Connes, *Noncommutative Geometry*, Acad. Press, New York etc., 1994.
[2] S. Lord, F. Sukochev, and D. Zanin, *Singular Traces*, De Gruyter, Berlin, 2012.
[3] A. Pietsch, *Eigenvalues and s-Numbers*, Geest & Portig, Leipzig, and Cambridge Univ. Press, 1987.
[4] A. Pietsch, *Traces on operator ideals and related linear forms on sequence ideals* (*part I*), Indag. Math. **25** (2014), 341–365.
[5] A. Pietsch, *Traces on operator ideals and related linear forms on sequence ideals* (*part II*), Integr. Equ. Oper. Theory **79** (2014), 255–299.
[6] A. Pietsch, *Traces on operator ideals and related linear forms on sequence ideals* (*part III*), J. Math. Anal. Appl. **421** (2015), 971–981.

Reporter: Thomas Kühn

Participants

Prof. Dr. Fernando Cobos
Depto. de Analisis Matematico
Facultad de Matematicas
Universidad Complutense de Madrid
28040 Madrid
SPAIN

Prof. Dr. Andreas Defant
Institut für Mathematik
Carl von Ossietzky Universität
Postfach 2503
26015 Oldenburg
GERMANY

Oscar Dominguez
Depto. de Analisis Matematico
Facultad de Matematicas
Universidad Complutense de Madrid
28040 Madrid
SPAIN

Prof. Dr. Stefan Geiß
Department of Mathematics & Statistics
P.O. Box 35
40014 University of Jyväskylä
FINLAND

Prof. Dr. Dorothee Haroske
Institut für Mathematik
Friedrich-Schiller-Universität
Ernst-Abbe-Platz 1-4
07743 Jena
GERMANY

Prof. Dr. Stefan Heinrich
Fachbereich Informatik
Universität Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Prof. Dr. Aicke Hinrichs
Institut für Analysis
Johannes-Kepler-Universität Linz
4040 Linz
AUSTRIA

Prof. Dr. Hermann König
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
24098 Kiel
GERMANY

Prof. Dr. Thomas Kühn
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04109 Leipzig
GERMANY

Sebastian Mayer
Hausdorff Center for Mathematics
Institute for Numerical Simulation
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Albrecht Pietsch
Mathematisches Institut
Universität Jena
Ernst-Abbe-Platz 2-4
07743 Jena
GERMANY

Prof. Dr. Gilles Pisier
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Winfried Sickel
Fakultät f. Mathematik & Informatik
Friedrich-Schiller-Universität
Ernst-Abbe-Platz 1-4
07743 Jena
GERMANY

Prof. Dr. Nicole Tomczak-Jaegermann
Dept. of Mathematical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

Prof. Dr. Ingo Steinwart
Fachbereich Mathematik
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Dr. Tino Ullrich
Hausdorff Center for Mathematics
Institute for Numerical Simulation
Endenicher Allee 60
53115 Bonn
GERMANY