Mathematisches Forschungsinstitut Oberwolfach

Report No. 7/2015

DOI: 10.4171/OWR/2015/7

Mini-Workshop: Discrete p-Laplacians: Spectral Theory and Variational Methods in Mathematics and Computer Science

Organised by
Matthias Hein, Saarbrücken
Daniel Lenz, Jena
Delio Mugnolo, Hagen

8 February – 14 February 2015

Abstract. The p-Laplacian operators have a rich analytical theory and in the last few years they have also offered efficient tools to tackle several tasks in machine learning. During the workshop mathematicians and theoretical computer scientists working on models based on p-Laplacians on graphs and manifolds have presented the latest theoretical developments and have shared their knowledge.

Mathematics Subject Classification (2010): 47B39, 05C63, 60J20, 94A08.

Introduction by the Organisers

The mini-workshop *Discrete p-Laplacians: Spectral Theory and Variational Methods in Mathematics and Computer Science* organized by Matthias Hein, Daniel Lenz, and Delio Mugnolo was involved with a field common to both mathematics and computer science in recent investigations around the so-called discrete p-Laplacian. The participants came from various parts of mathematics, including geometry and analysis, and computer science with a slight majority of participants from mathematics. A central aim was to make the involved researchers aware of methods and objectives of the 'other' community. Accordingly, the talks were mostly introductory in nature and presented a wide range of topics within the field. Special attention was paid to

- classical $p = 2$ theory for graphs (as the starting point for the case of $p \neq 2$),
- classical $p = 2$ and $p \neq 2$ theory for manifolds,

- Cheeger cuts in both mathematics and computer science,
- discretization procedures to get from continuum models to discrete models and vice versa.

By giving a brief introduction into the topic, we will now put these points in perspective. The *p-Laplacian* operators

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \qquad p > 1 \ ,$$

have been studied since the 1960s in order to model special diffusive systems. One decade later Yamasaki proposed a discrete version of them: because the signed incidence matrix $\mathcal{I}$ of a directed graph can be seen as a discrete version of the divergence operator, a *discrete p-Laplacian* operator is naturally defined by

$$\mathcal{L}_p f := \mathcal{I}(|\mathcal{I}^T f|^{p-2} \mathcal{I}^T f), \qquad p > 1 \ .$$

The special case $p = 1$ where the discrete p-Laplacian becomes multi-valued is of particular interest.

It was mostly the potential theoretical features of this family of difference operators that initially motivated their study. However, it was observed in the early 1990s by Perona, Malik, P.-L. Lions and other authors that the parabolic equation associated with Δ_p be conveniently used for image processing. Given a picture, i.e., a function $u_0 : \Omega \to \mathbb{R}^k$ ($k = 1$ or $k = 3$ for a b/w or rgb picture, respectively), the rationale behind the choice of parameter p relies upon the modeling purposes, as diffusion-driven smoothing of input pictures will be stronger in regions of *low* gradient for $1 \leq p < 2$, but in regions of *high* gradient for $2 < p < \infty$: this suggests applications to denoising or segmentation ($p \approx 1$) or morphing ($p \approx \infty$), respectively. Observe that

$$\frac{\partial f}{\partial t}(t, \mathsf{v}) = -\mathcal{L}_p f(t, \mathsf{v}), \qquad t \geq 0, \ \mathsf{v} \in \mathsf{V} \ , \tag{1}$$

turns into a partial differential inclusion for $p = 1$, as $\mathcal{L}_1$ is multivalued. Plugging a noisy picture as the initial data of (1) for $1 \leq p < 2$ and letting the system evolve with respect to the fictive time variable t will expectedly deliver pictures that are less and less blurry.

These considerations have paved the road for the celebrated Rudin–Osher–Fatemi model of image denoising, which is essentially an optimization problem for the energy functional

$$\mathcal{E}_p : u \mapsto \frac{1}{p} \int_\Omega |\nabla u|^p dx$$

associated with Δ_p, for $p \approx 1$. Analogous considerations hold for the discrete p-Laplacians.

The discrete p-Laplacians are used for similar reasons for clustering purposes: given a set of data, i.e., of vectors in $\mathbb{R}^d$, a graph is built upon determining an adjacency structure and hence a graph by means of a similarity function. Mirroring the structure of $\mathcal{L}_p$ for $p \to 1$, its eigenvectors will be strongly localized: the supports of its positive and negative parts will deliver meaningful clusters of the graph and thus is used to detect structures in a graph in an unsupervised way in machine learning.

If the graph is finite, then 0 is an eigenvalue of $\mathcal{L}_p$ for all $p \in [1,\infty)$. Strictly positive eigenvalues yield interesting information about the Cheeger constant, which is defined as

$$h_\rho(\mathsf{G}) := \min_{S\subset\mathsf{V}} \frac{|\partial S|}{\min\{|S|,|S^C|\}} ,$$

where ∂S denotes the set of edges with one endpoint in a subset S of the vertex set V and the other in its complement S^C, and $|\cdot|$ is the measure of a set with respect to a given node weight ρ. The famous isoperimetric inequality relating the second eigenvalue of the classical Laplacian ($p = 2$) has first been established by Cheeger for Riemannian manifolds and then for graphs by Alon and Milman. In computer science the so called spectral relaxation of the Cheeger cut problem – which is known to be NP-hard – has been used for clustering the vertices of a graph. Subsequently, similar inequalities have been established for the p-Laplacian both for the continuous and discrete problem. The case $p = 1$ is particularly interesting as the second eigenvalue of the discrete 1-Laplacian is equal to the Cheeger constant and the second eigenvector is the indicator vector of the optimal partition.

We quickly summarize the talks and their relation to the above subjects.

The relation of discrete and continuous graph Laplacian has been discussed in talks by Y. Kurylev, M. Gerlach and D. Slepcev. Y. Kurylev showed that eigenvalues and eigenvectors of the Laplace-Beltrami operator of a compact Riemannian manifold can be approximated by the corresponding objects of the discrete graph Laplacian built on a ϵ-net of the manifold. M. Gerlach showed that this approximation works as well for the case where the discretization is built from an i.i.d. sample of the manifold. D. Slepcev discussed Gamma-convergence of the Cheeger cut corresponding to the the second eigenvalue of the 1-Laplacian of a neighborhood graph built from an i.i.d. sample of a compact Riemannian manifold to the corresponding Cheeger cut of the manifold. J. Giesen gave a talk on spectral embeddings via particularly constructed graphs and their application in exploratory data analysis.

The relation of Cheeger cuts and discrete p-Laplacian and generalizations of the classical Cheeger inequality were discussed by M. Hein, S. Liu and D. Zhang. M. Hein discussed the relation of the spectrum of the p-Laplacian and (higher-order) Cheeger cuts and discussed generalizations to directed graphs and hyerpgraphs with applications in machine learning. S. Liu discussed generalization of the classical case $p = 2$ to signed graphs and more general magnetic Laplacians and gave Cheeger inequalities both for continuous and discrete case. D. Zhang discussed the graph 1-Laplacian and its properties in particular also of higher-order eigenvectors. M. Keller showed how powerful Cheeger inequalities for general p could be obtained for infinite graphs with unbounded degree via intrinsic metrics. J. Kerner extended these ideas to the case of p-Laplacians on quantum graphs, a possible relaxation of the usual combinatorial graph setting.

The topic of intrinsic metrics in the description of diffusion processes was taken up by D. Lenz and put in the context of general Dirichlet forms, which cover Laplacians on graphs and manifolds. The classical topic of the Feller property for diffusion was presented for manifolds by A. Setti and for graphs by R. Wojciechowski. S. Golénia showed how rich the theory of such a simple object as the diagonal matrix of vertex degrees can be and developed a comprehensive operator theory thereof. D. Mugnolo studied well-posedness, long-time behaviour and regularity of the solutions of the parabolic differential equation associated with discrete p-Laplacians on graphs and hypergrahs by means of a nonlinear extension of the theory of Dirichlet forms. B. Kawohl presented an overview on the theory of eigenvalues, eigenfunctions and nodal domains of p-Laplacians on domains, while P. Pucci offered an invitation to recent results on Kirchoff-type evolution equations associated with the fractional p-Laplacians.

The great atmosphere of Oberwolfach lead to numerous discussions and to a fruitful mutual exchange of ideas and concepts of the participants which had a quite heterogeneous background in mathematics and computer science. Thus we think that the mini-workshop has been very successful in partially initiating and partially strengthening the interaction between mathematics and computer science in all aspects around continuous and discrete aspects of the p-Laplacian. The organizers would like to thank the administration and staff members of the Oberwolfach institute for their hospitality and the great support before and during the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Mini-Workshop: Discrete p-Laplacians: Spectral Theory and Variational Methods in Mathematics and Computer Science

Table of Contents

Abstracts

Spectral convergence of random geometric graphs

Moritz Gerlach
(joint work with Matthias Hein)

We present a first result on the convergence of the spectrum of an unnormalized graph Laplacian to the spectrum of the Laplace-Beltrami operator.

Given a finite sequence of samples $X = \{X_1, \dots, X_n\}$ drawn independently and uniformly at random from an m-dimensional compact submanifold M of $\mathbb{R}^d$, we construct a graph with vertex set X deterministically by connecting two samples by an edge if their distance is smaller than a neighborhood parameter h depending on n. Then each edge (X_i, X_j) is endowed with the weight $w_{i,j} = 1/(h^{m+2}n^2)$ and we consider the unnormalized graph Laplacian

$$(\Delta u)(X_i) = \frac{1}{nh^{m+2}} \sum_{X_j \sim X_i} (u(X_j) - u(X_i)).$$

In order to prove convergence of the spectrum of Δ as the sample size n tends to infinity while h goes to zero, we make use of a recent work by Burago, Ivanov and Kurylev [1] on discretization and approximation of the Laplace-Beltrami operator. Their discretization is based on a partition of the manifold with measurable sets $V_1, \dots, V_n$ such that each V_i is sufficiently close to the sample point X_i. On each of these sets, a function $f \in L^2(M)$ is then approximated by its mean

$$\frac{1}{\mathrm{vol}(V_i)} \int_{V_i} f d\mathrm{vol}$$

and, vice versa, a vector $u \in L^2(X)$ is extended to the piecewise constant function

$$\sum_{i=1}^{n} u(X_i) 1_{X_i}$$

on M.

In order to apply this method in our random setting, we need to ensure the following two conditions with sufficiently high probability.

- The balls of a certain radius $\varepsilon > 0$ centered at the sample points $X_1, \dots, X_n$ cover M.
- There exists a partition $V_1 \cup \dots \cup V_n$ of M with measurable sets V_i of equal volume such that each V_i belongs to the ball of radius ε centered at X_i.

In this talk, we present one possibility to obtain this. Due to [1], this yields convergence of the spectrum of Δ to that of the Laplace-Beltrami operator in probability.

References

[1] D. Burago, S. Ivanov, and Y. Kurylev, *A graph discretization of the Laplace–Beltrami operator.*, Journal of Spectral Theory **4(4)** (2014), 675–714.

Exploratory Analysis of Graphs via Spectral Embeddings

Joachim Giesen

(joint work with Claudia Dahl, Philipp Lucas)

Introduction. Given a graph $G = (V, E)$ with a finite vertex set V and edge set $E \subseteq \binom{V}{2}$, exploratory analysis aims at identifying characteristic features of G like

(1) clusters, i.e., a partitioning of V into dissimilar groups of similar vertices,
(2) central vertices, and
(3) motifs, i.e., subgraphs of G that are more frequent than expected when compared to some given random graph model.

Starting point for these exploratory tasks is often a similarity matrix S that stores a similarity value for each pair of vertices. Examples for similarity matrices include

(1) $S = A^T A = A^2$, where A is the adjacency matrix of G. Here two vertices are considered similar if they have many neighbors in common.
(2) $S_{ij} = \exp(-\lambda\, d(i,j)^2)$, where $\lambda > 0$ and $d(i,j)$ is the graph distance between the vertices i and j, i.e., the number of edges on a shortest path connecting i and j, or ∞ if such a path does not exist.
(3) $S_{ij} = \exp(-\lambda\, c(i,j)^2)$, where $\lambda > 0$ and $c(i,j)$ is the min-cut value for the vertices i and j, i.e.,
$$c(i,j) = \min_{V' \neq \emptyset, V,\, i \in V',\, j \in V \setminus V'} \left|\left\{\{v,u\} \in E \,|\, v \in V' \wedge u \in V \setminus V'\right\}\right|.$$

Note that in all three examples the matrix S is symmetric and positive semi-definite, and the entries of S are non-negative. In the following we assume that these properties hold for S. Additionally, we assume that S is connected, i.e., there is a path with strictly positive edge weights connecting any pair of vertices if we consider S as a weighted adjacency matrix of a graph on the vertex set V.

Spectral embedding. We want to use well developed Euclidean techniques based on computing Euclidean distances and angles for exploratory graph analysis. For doing so we embed the vertex set V into Euclidean space $\mathbb{R}^k$, where $k \leq n = |V|$, such that the Euclidean distance between two points is a good approximation of the similarity of the corresponding vertices.

Nadler et al. [1] have suggested diffusion maps as an embedding technique where the Euclidean distance in the embedding space has an interpretation as diffusion distance in the graph whose weighted adjacency matrix is given by the similarity matrix S. For the definition of a diffusion map they consider the matrix $D^{-1}S$, where D is the diagonal matrix with $D_{ii} = \sum_{j=1}^n S_{ij}$. The matrix $D^{-1}S$ is stochastic, i.e., its row sums are 1. The eigenvalues, and left- and right eigenvectors of $D^{-1}S$ can be obtained from the eigenvalues $\lambda_0 \geq \dots \lambda_{n-1}$ and the eigenvectors $v_0, \dots, v_{n-1}$ of the adjoint matrix $D^{-1/2}SD^{-1/2}$. The left eigenvectors are given as $\phi_i = v_i D^{1/2}$ and the right eigenvectors are given as $\psi_i = v_i D^{-1/2}$, respectively, for the eigenvalues λ_i. The diffusion map (applied to vertex x) is then given as

$$\psi_t^{(k)}(x) = \left(\lambda_1^t \psi_1(x), \dots, \lambda_k^t \psi_k(x)\right).$$

The diffusion distance between two vertices is defined as

$$d_t^2(x_i, x_j) = \|p(t, x|x_i) - p(t, x|x_j)\|_{\phi_0}^2 = \sum_{x=1}^{n} \left(p(t, x|x_i) - p(t, x|x_j)\right)^2 \phi_0(x)^{-1},$$

where $p(t, x|x_i) = e_i(D^{-1}S)^t$, i.e., multiplying the i'th standard basis vector from the left to the t'th power of the stochastic matrix $D^{-1}S$, which corresponds to t steps in Markov chain that corresponds to $D^{-1}S$ with the whole probability mass concentrated at vertex i at $t = 0$. A simple calculation shows that $\lim_{t\to\infty} p(t, x|y) = \phi_0(x)$ regardless of y since the Markov chain is irreducible and aperiodic. The diffusion distance thus compares the probability mass distribution of the Markov chain for the initial mass distribution concentrated at x_0 and x_1, respectively, after t time steps.

Nadler et al. prove that

(1) $d_t^2(x_i, x_j) = \|\psi_t^{(n-1)}(x_i) - \psi_t^{(n-1)}(x_j)\|^2$, and
(2) $\left|d_t^2(x_i, x_j) - \|\psi_t^{(k)}(x_i) - \psi_t^{(k)}(x_j)\|^2\right| \leq \lambda_{k+1}^2 \left(\frac{1}{\phi_0(x_i)} + \frac{1}{\phi_0(x_j)}\right)$.

That is, the Euclidean distance of the mapped vertices either recovers their diffusion distance, or it is a good approximation if the eigenvalues decay quickly.

Exploratory analysis. Let $p_i = \psi_t^{(\ell)}(x_i)$ be the point that corresponds to the i'th vertex for some fixed ℓ and t.

Clustering. The well known k-means clustering method now becomes applicable to graphs. The method determines k cluster centers $c_1, \dots, c_k \in \mathbb{R}^\ell$ as the solution of the following optimization problem,

$$\min_{c_1,\dots,c_k \in \mathbb{R}^\ell} \sum_{i=1}^{n} \|p_i - \hat{c}_i\|^2,$$

where $\hat{c}_i = \operatorname{argmin}_{c \in \{c_1,\dots,c_k\}} \|x_i - c\|$. For the clustering each point p_i is assigned to its closest cluster center.

Ordinary and extraordinary vertices. For comparing vertices or local subgraphs we compute first a local neighborhood of a vertex p_i as the set $N_k(p_i)$ of its k-nearest neighbors in the point cloud $\{p_1, \dots, p_n\}$. From the neighborhood we can compute a local coordinate system by considering the eigenvectors of the covariance matrix $C_i = \sum_{p \in N_k(p_i)} (p - p_i)(p - p_i)^T$. The eigenvalues of C_i are indicative of the shape of the local neighborhood (spherical, ellipsoidal, disc like). One can identify the points p_i and thus the vertices of the graph with the sorted eigenvalue vectors $(\nu_1^{(i)}, \dots, \nu_\ell^{(i)})$, i.e., we assume $\nu_j^{(i)} \geq \nu_{j+1}^{(i)}$. Accumulation points in the space of these vectors can be considered as (shape-)motifs (or prototypical ordinary vertices) and outliers can be considered as extraordinary vertices. In experiments we have observed that extraordinary vertices often maximize well known centrality measures.

Approximate symmetries. One can try to approximately match the (eigenvalue weighted) local coordinate systems at the points p_i under Euclidean transformations (rotations, translations, and reflections). Every approximate match provides a vote for the corresponding transformation. Accumulation points of the votes in the space of all possible symmetries can be considered as approximate local or global symmetries of the point cloud and thus also of the graph.

References

[1] B. Nadler, S. Lafon, R. R. Coifman, I. G. Kevrekidis, *Diffusion Maps - a probabilistic interpretation for spectral embedding and clustering algorithms* in *Principal Manifolds for Data Visualization and Dimension Reduction, A.N. Gorban, B. Kegl, D.C. Wunsch and A. Zinovyev (Eds)*, Springer, 2007

Eigenvalue asymptotics for Schrödinger operators on sparse graphs

Sylvain Golénia

(joint work with Michel Bonnefont, Matthias Keller)

The spectral theory of discrete Laplacians on finite or infinite graphs has drawn a lot of attention for decades. One important aspect is to understand the relations between the geometry of the graph and the spectrum of the Laplacian. Often a particular focus lies on the study of the bottom of the spectrum and the eigenvalues below the essential spectrum.

In this talk we focus on sparse graphs to study discreteness of spectrum and eigenvalue asymptotics. In a moral sense, the term sparse means that there are not 'too many' edges.

The techniques used in [1] owe on the one hand to considerations of isoperimetric estimates, e.g., [2], as well as a scheme developed in [3] for the special case of trees. In particular, we show that a notion of sparseness is a geometric characterization for an inequality of the type

$$(1-a)\deg - k \leq \Delta \leq (1+a)\deg + k$$

for some $a \in (0,1)$, $k \geq 0$, which holds in the form sense. The moral of this inequality is that the asymptotic behavior of the Laplacian Δ is controlled by the vertex degree function deg (the smaller a the better the control).

Furthermore, such an inequality has very strong consequences which follow from well-known functional analytic principles. These consequences include an explicit description of the form domain, characterization for discreteness of spectrum and eigenvalue asymptotics.

References

[1] M. Bonnefont, M. Keller, and S. Golénia *Eigenvalue asymptotics for Schrödinger operators on sparse graphs*, to appear in the Annales de l'institut Fourrier.

[2] M. Keller, *Essential spectrum of the Laplacian on rapidly branching tessellations*, Math. Ann. **346**, (2010) 51–66.

[3] S. Golénia, *Hardy inequality and eigenvalue asymptotic for discrete Laplacians*, J. Funct. Anal. **266** (2014), no. 5, 2662–2688.

Discrete graph *p*-Laplacians, Cheeger cuts and Extensions

Matthias Hein

(joint work with T. Bühler, L. Jost, S. Rangapuram, S. Setzer and F. Tudisco)

In computer science the so called Cheeger cut or sparsest cut is a quality measure to partition an undirected, weighted graph $G = (V, E)$ into two sets $(C, \overline{C})$ such that the cut is small and the size of the sets $|C|$ and $|\overline{C}|$ is roughly equal. The optimal Cheeger cut ϕ^* is defined as

$$\phi^* = \min_{\emptyset \neq C \subset V} \frac{\mathrm{cut}(C, \overline{C})}{\min\{|C|, |\overline{C}|\}},$$

where $\mathrm{cut}(C, \overline{C}) = \sum_{i \in C, j \in \overline{C}} w_{ij}$ and w_{ij} is the non-negative weight of the edge between vertices i and j.

This problem is known to be NP-hard and thus typically relaxations into continuous problems are used, which can be computed globally optimal. The most often employed relaxation in machine learning is the spectral one based on the second eigenvector of the graph Laplacian. In recent years, the nonlinear generalization based on the graph p-Laplacian has become of great interest as Amghibech [1], see also [4], has shown that the Cheeger cut $\phi_{p-SPECTRAL}$ of the partition obtained by optimal thresholding of the second eigenvector of the graph p-Laplacian satisfies,

$$\phi^* \;\leq\; \phi_{p-SPECTRAL} \;\leq\; p(\max_i d_i)^{\frac{p-1}{p}}\,(\phi^*)^{\frac{1}{p}}.$$

The inequality becomes tight as $p \to 1$. An analogous result for the continuous case has been obtained by Kawohl and Fridman [9]. This has led to the technique of p-spectral clustering in machine learning [4]. In [5] we could show that in the case $p = 1$ the second eigenvalue of the graph 1-Laplacian is equal to the optimal Cheeger cut and the second eigenvector is the indicator vector of the optimal partition. Moreover, we propose a nonlinear inverse power method to compute the second eigenvectors of the 1-Laplacian. While the method is guaranteed to converge to an eigenvector of the 1-Laplacian, there is not guarantee that one achieves the second one [8]. However, in practice the results are superior to the standard relaxation corresponding to the case $p = 2$, see also [11].

This result can be extended in various ways. From a machine learning point of view the Cheeger cut is used to find clusters. A cluster is a group of vertices which is strongly connected inside the group but only weakly connected to the rest of the graph. In [6, 2] we show how to integrate prior knowledge into the clustering via must-link and cannot-link constraints [6] and seed and size constraints [2]. A very general relaxation of the discrete ratio problem is treated in [2] where the following theorem is shown.

Theorem 1.1. *Let* $\hat{R}, \hat{S} : 2^V \to \mathbb{R}$ *be non-negative set functions and* $R, S : \mathbb{R}^n \to \mathbb{R}$ *their Lovasz extensions. Then,*

$$\min_{C \subset V} \frac{\hat{R}(C)}{\hat{S}(C)} = \min_{f \in \mathbb{R}^n_+} \frac{R(f)}{S(f)} .$$

If in addition $\hat{R}(V) = \hat{S}(V) = 0$*, then* $\min_{C \subset V} \frac{\hat{R}(C)}{\hat{S}(C)} = \min_{f \in \mathbb{R}^n} \frac{R(f)}{S(f)}$*.*
Moreover,

$$\frac{R(f)}{S(f)} \geq \min_{i=1,\dots,n} \frac{\hat{R}(C_i)}{\hat{S}(C_i)} ,$$

where $C_i = \{j \in V \,|\, f_j > f_i\}$*. If* $\hat{R}(V) = \hat{S}(V) = 0$*, the inequality holds for all* $f \in \mathbb{R}^n$*.*

Further constraints can be integrated via a penalty-based approach. This result allows us also to tackle the Cheeger cut problem for hypergraphs [7] and directed graphs (work in progress).

Finally, we discuss current work on a nodal domain theorem for the graph p-Laplacian. We show an analogous result to the one obtained for the graph Laplacian [3]. Moreover, based on nodal domains one can also obtain a higher-order Cheeger inequality for a decomposition of the vertices into k non-empty disjoint sets $C_1, \dots, C_k$ according to the criterion,

$$\min_{C_1,\dots,C_k} \max_{l=1,\dots,k} \frac{\mathrm{cut}(C_l, \overline{C_l})}{|C_l|}.$$

A corresponding higher-order Cheeger inequality becomes tight as $p \to 1$ if there exists an eigenfunction which has k strong nodal domains.

References

[1] S. Amghibech, *Eigenvalues of the discrete p-Laplacian for graphs*, Ars Combinatoria, **67** (2003), 283–302.

[2] T. Bühler, S.S. Rangapuram, S. Setzer, M. Hein, *Constrained fractional set programs and their application in local clustering and community detection*, Proc. of the 30th Int. Conf. on Machine Learning (ICML), (2013), 624–632.

[3] E.B. Davies, G.M.L. Gladwell, J. Leydold, P.F. Stadler, *Discrete nodal domain theorems*, Linear Algebra and its Applications, **336** (2001), 51–60.

[4] T. Bühler, M. Hein, *Spectral Clustering based on the graph p-Laplacian*, Proc. of the 26th Int. Conf. on Machine Learning (ICML), (2009), 81 – 88.

[5] M. Hein, T. Bühler, *An Inverse Power Method for Nonlinear Eigenproblems with Applications in* 1*-Spectral Clustering and Sparse PCA*, Advances in Neural Information Processing Systems 23 (NIPS), (2010), 847 – 855.

[6] M. Hein, S. Setzer, *Beyond Spectral Clustering - Tight Relaxations of Balanced Graph Cuts*, Advances in Neural Information Processing Systems 24 (NIPS), (2011), 2366–2374.

[7] M. Hein, S. Setzer, L. Jost, S.S. Rangapuram, *The Total Variation on Hypergraphs - Learning on Hypergraphs Revisited*, Advances in Neural Information Processing Systems 26 (NIPS), (2013), 2427–2435.
[8] L. Jost, S. Setzer, M. Hein, *Nonlinear Eigenproblems in Data Analysis: Balanced Graph Cuts and the RatioDCA-Prox*, in "Extraction of Quantifiable Information from Complex Systems", eds. S. Dahlke, W. Dahmen, M. Griebel, W. Hackbusch, K. Ritter, R. Schneider, C. Schwab, H. Yserentant, 2014, 263–279.
[9] B. Kawohl, V. Fridman, *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant*, Comm. Math. Univ. Carol., **44** (2003), 659–667.
[10] S.S. Rangapuram, M. Hein, *Constrained* 1*-Spectral Clustering*, In Journal of Machine Learning Research Workshop and Conference Proceedings (AISTATS), 22 (2012), 1143–1151.
[11] A. Szlam, X. Bresson, *Total variation and Cheeger cuts*, In Proc. of the 27th Int. Conf. on Machine Learning (ICML), (2010), 1039–1046.

First eigenvalues of the indiscrete p-Laplacian

Bernd Kawohl

For $\Omega \subset \mathbb{R}^n$ a bounded domain and nontrivial $v \in W^{1,p}(\Omega)$ with $p \in (1, \infty)$ we can define the Rayleigh-quotient

$$R_p(v) := \frac{\int_\Omega |\nabla v|^p, dx}{\int_\Omega |v|^p\, dx}.$$

Any critical point of this functional satisfies the Euler-equation

$$-\Delta_p u := -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}u \qquad \text{in } \Omega. \tag{1}$$

My lecture treats first eigenvalues of (1) under Dirichlet and Neumann boundary conditions and some closely related issues.

Dirichlet eigenvalues and eigenfunctions

If we minimize R_p on $W_0^{1,p}(\Omega) \setminus \{0\}$ then a minimizer u_p exists for any $p \in (1, \infty)$, and it satisfies Dirichlet's boundary condition $u_p = 0$ on $\partial\Omega$ as well as $-\text{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = \lambda_p |u_p|^{p-2}u_p$ in Ω. Observe that the index p refers to the exponent of integration, so that λ_2 is the first eigenvalue of the (linear) 2-Laplacian, λ_3 the first eigenvalue of the 3-Laplacien etc. Without loss of generality there is a nonnegative minimizer, since $R_p(u) = R_p(|u|)$. But then the right hand side in 1 is nonnegative and due to a Harnack inequality the solution is positive in Ω. This observation is crucial in proving the next result.

Theorem 1.1. *For any $p \in (1, \infty)$ the first eigenfunction u_p is simple, in other words any eigenfunction of λ_p is a multiple of u_p.*

Proof. Since [10] is not easily available I present the short proof, which reveals a hidden convexity of $R_p(u)$ in terms of u^p. Suppose there are two eigenfunctions u and w, both positive. Then I set

$$v := \left[\tfrac{1}{2}\left(u^p + w^p\right)\right]^{\frac{1}{p}}$$

and calculate

$$\begin{aligned}\nabla v &= \frac{1}{2^p}\left(u^p+w^p\right)^{\frac{1}{p}-1}\left(|u|^{p-2}u\nabla u+|w|^{p-2}w\nabla w\right)\\ &= v\left(\frac{u^p}{u^p+w^p}\nabla\log u+\frac{w^p}{u^p+w^p}\nabla\log w\right),\end{aligned}$$

where $(\ldots)$ is now a convex combination of $\nabla\log u$ and $\nabla\log w$. Therefore Jensen's inequality and the definition of v implies

$$|\nabla v|^p \le v^p\left(\frac{u^p}{u^p+w^p}|\nabla\log u|^p+\frac{w^p}{u^p+w^p}|\nabla\log w|^p\right)=\frac{1}{2}\left(|\nabla u|^p+|\nabla w|^p\right),$$

so that $R_p(v)\le\lambda_p=\inf R_p$. Consequently the inequality must be an equality and $\nabla\log u=\nabla\log w$ a.e.in Ω or $\nabla(u/w)=0$, i.e. $u=const.w$. □

What happens as $p\to 1$? If we look for nonnegative eigenfunctions in $W_0^{1,1}(\Omega)$ we have trouble proving existence of a minimizer, so R_1 is extended to functions of bounded variation. In terms of their level sets $\Omega_t:=\{x\in\Omega\ ;\ u(x)>t\}$ one can express the denominator of R_1 as

$$\int_\Omega u\ dx=\int_0^\infty|\Omega_t|\ dt$$

(this is known as Cavalieri's principle) and the enumerator as

$$\int_0^1|\partial\Omega_t|\ dt,$$

by the coarea formula for BV-functions. So level sets try to minimize the Cheeger quotient $|\partial A|/|A|$ of perimeter over area among subsets A of Ω, and any set that minimizes this quotient will be called Cheeger set of Ω and denoted by Ω_c. It turns out that the Cheeger constant $h(\Omega)=|\partial\Omega_c|/|\Omega_c|$ equals $\lambda_1(\Omega)$.

Theorem 1.2. *If Ω_c is Cheeger set of Ω then the function* $\chi_{\Omega_c}(x)=\begin{cases}1 & \text{if } x\in\Omega_c\\ 0 & \text{if } x\notin\Omega_c\end{cases}$ *minimizes $R_1(v)$ in $BV(\Omega)$. Moreover this function solves the eigenvalue problem* $-\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)=\lambda_1\left(\frac{u}{|u|}\right)$ *in Ω, $u=0$ on $\partial\Omega$, in the sense of viscosity solutions.*

For a proof I refer to [5] and [11]. In two dimensions Cheeger sets of convex polygons are easy to calculate. There $\partial\Omega_c\cap\Omega$ consists of circular arcs with radius $1/\lambda_1(\Omega)$, see [9]. In general Cheeger sets are not unique. Nodal patterns of second eigenfunctions of the p-Laplacian were calculated in [6]. For higher eigenvalues see also [1, 12, 13] and [14]

Neumann eigenvalues and eigenfunctions

A minimization of R_p on $W^{1,p}(\Omega)\setminus\{0\}$ provides constant functions as eigenfunctions to the eigenvalue zero. The first nontrivial eigenvalue $\nu_p>0$ is obtained by minimizing R_p on $W^{1,p}(\Omega)\setminus\{0\}\cap\{\int|u|^{p-2}u\ dx=0\}$, and the minimizers u_p solve $-\Delta_p up=\nu_p|u-p|^{p-2}u_p$ in Ω, $|\nabla u_p|^{p-2}\frac{\partial u_p}{\partial\nu}=0$ on $\partial\Omega$. If $p=1$ the side constraint $\int_\Omega u/|u| = 0$ implies that the sets $\Omega_+ = \{x \in \Omega \mid u(x) > 0\}$ and Ω_- have equal area, so Ω is equipartitioned.

Theorem 1.3. *For plane convex domains we have $\nu_1(\Omega) \le \nu_1(\Omega^*)$, where Ω^* denotes the disc of same area as Ω.*

This theorem answers a conjecture of Pólya from 1958 and is an analogue to the Szegö-Weinberger inequality $\nu_2(\Omega) \le \nu_2(\Omega^*)$ which holds even for nonceonvex Ω in arbitrary dimension. For details see [3].

What happens for $p \to \infty$?

Theorem 1.4. *As $p \to \infty$, $\nu_p^{1/p} \to \Lambda_\infty = \frac{2}{diam\ \Omega}$, and $u_\infty = \lim_{p\to\infty} u_p$ solves*

$$\begin{cases} \min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in } \Omega \cap \{u > 0\} \\ -\Delta_\infty u = -\sum_{i,j=1}^n u_{x_i} u_{x_i x_j} u_{x_j} = 0 & \text{in } \Omega \cap \{u = 0\} \\ \min\{-|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in } \Omega \cap \{u < 0\} \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of viscosity solutions.

This theorem provides the analogue of the Szegö-Weinberger inequality $\Lambda_\infty(\Omega) \le \Lambda_\infty(\Omega^*)$ in the case $p = \infty$. For the proof and other consequences see [4]. There is no smaller eigenvalue than Λ_∞, and if Ω is convex, u_∞ attains its max and min over $\overline{\Omega}$ in those boundary points that have maximal distance from each other, e.g. in diagonal corners of a rectangle. Moreover, along this diameter u_∞ has constant slope. Thus u_∞ satisfies the so called "hot spot conjecture", which states that in convex domains u_2 should attain its max and min on the boundary. But diffusion equations are a different subject, and I just refer the reader to [2] and [7].

References

[1] J. Benedikt, P. Drábek, P. Girg, *The second eigenfunction of the p-Laplacian on the disk is not radial.* Nonlinear Anal. **75** (2012), 4422-4435.

[2] K. Does, *An evolution equation involving the normalized p-Laplacian.* Commun. Pure Appl. Anal. **10** (2011), 361-396.

[3] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, C. Trombetti, *The longest shortest fence and sharp Poincaré Sobolev inequalities* Arch. Ration. Mech. Anal. **206** (2012), 821-851.

[4] L. Esposito, B. Kawohl, C. Nitsch, C. Trombetti, *The Neumann eigenvalue problem for the ∞-Laplacian*, arXiv: 1405.3535, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., to appear (2015)

[5] V. Fridman, B. Kawohl, *Isoperimetric estimates for the first eigenvalue of the p- Laplace operator and the Cheeger constant,* Comment. Math. Univ. Carolinae **44** (2003), 659-667.

[6] J. Horák, *Numerical investigation of the smallest eigenvalues of the p-Laplace operator on planar domains.* Electron. J. Differential Equations **2001** (2011), No. 132, 1–30.

[7] P. Juutinen, B. Kawohl, *On the evolution governed by the infinity Laplacian*, Math. Annalen **335** (2006), 819-851.

[8] B. Kawohl, S. Krömer, J. Kurtz, *Radial eigenfunctions for the game-theoretic p-Laplacian on a ball.* Differential and Integral Equations **27** (2014) 659-670.

[9] B. Kawohl,T. Lachand-Robert, *Characterization of Cheeger sets for convex subsets of the plane,* Pacific J.Math. **225** (2006), 103-118

[10] B. Kawohl, P. Lindqvist, *Positive eigenfunctions for the p-Laplace operator revisited*, Analysis **26** (2006), 539-544.
[11] B. Kawohl, F. Schuricht, *First eigenfunctions of the 1-Laplacian are viscosity solutions.* Commun. Pure Appl. Anal. **14** (2015), 329-339.
[12] S. Littig, F. Schuricht, *Convergence of the eigenvalues of the p-Laplace operator as p goes to 1.* Calc. Var. Partial Differential Equations **49** (2014), 707-727.
[13] Z. Milbers, F. Schuricht, *Existence of a sequence of eigensolutions for the 1-Laplace operator.* J. Lond. Math. Soc. (2) **82** (2010), 74-88.
[14] E. Parini, *The second eigenvalue of the p-Laplacian as p goes to 1.* Int. J. Differ. Equ. (2010), Art. ID 984671, 23 pp.

On Cheeger's inequality for graphs

Matthias Keller
(joint work with Frank Bauer/Radosław Wojciechowski and Delio Mugnolo)

In 1969 Jeff Cheeger [4] proved his famous inequality

$$\frac{h_M^2}{4} \leq \lambda_1(\Delta_M),$$

where $\lambda_1(\Delta_M)$ is the first non-trivial eigenvalue of the Laplace Beltrami operator $\Delta_M \geq 0$ on $L^2(M, \mathrm{vol})$ of a compact manifold M and the Cheeger constant h_M is defined as

$$h_M = \inf \frac{\mathrm{Area}(\partial S)}{\mathrm{vol}(S) \wedge \mathrm{vol}(M \setminus S)},$$

where the infimum runs over all $S \subseteq M$ with sufficiently smooth boundary. On non-compact manifolds the bottom of the spectrum can also be estimated by $h_M^2/4$.

In order to even formulate an analogous inequality for graphs, one has to define the corresponding quantities first. While for manifolds the area, the volume and the Laplace Beltrami operator arise in a canonical way from the metric, for graphs these choices are non-obvious and as it turns out not uniquely determined.

Let us be more specific. Let X be the *vertex* set of a graph and if $x, y \in X$ are connected by an *edge* (x, y) we write $x \sim y$ and assume $x \sim y$ if and only if $y \sim x$. We denote the *degree* of a vertex x by $d(x) = \#\{y \in X \mid y \sim x\}$.

Let $C(X) = \{X \to \mathbb{R}\}$. We consider the quadratic form

$$Q(f) = \frac{1}{2} \sum_{x \sim y} (f(x) - f(y))^2, \quad f \in C(X).$$

To define a Laplacian, we need to restrict Q to a Hilbert space. This boils down to the question: What is a suitable a volume measure for graphs? Two natural choices are to count volume by the number of vertices or by the number of edges.

Let us first consider the volume measure vol_1 that counts vertices, i.e.,

$$\mathrm{vol}_1(W) = \sum_{x \in W} 1 = \#W, \quad W \subseteq X.$$

The positive selfadjoint operator Δ_1 on $\ell^2(X, \mathrm{vol}_1) = \ell^2(X)$ arising from Q restricted to $\{f \in \ell^2(X) \mid Q(f) < \infty\}$ acts as

$$\Delta_1 f(x) = \sum_{y \sim x} (f(x) - f(y))$$

on the domain $D(\Delta_1) = \{f \in \ell^2(X) \mid \Delta_1 f \in \ell^2(X)\}$ cf. [12]. Observe that Δ_1 is bounded (by $2D$) if and only if $D := \sup_{x \in X} d(x) < \infty$. Of course, $D(\Delta_1) = \ell^2(X) = C(X)$ whenever X is finite.

In 1984 the following analogue of Cheeger's inequality was proven independently by Alon-Milman [1] for finite graphs and by Dodziuk [5] for infinite graphs

$$\frac{h_1^2}{2D} \leq \lambda_1(\Delta_1) \quad \text{with } h_1 = \inf_{W \subseteq X} \frac{\#\partial W}{\#W \wedge \#(X \setminus W)}$$

where $\lambda_1(\Delta_1)$ is the first non-trivial eigenvalue in the case of finite graphs and the bottom of the spectrum in the case of infinite graphs and $\partial W = \{(x, y) \in W \times X \setminus W \mid x \sim y\}$. However, as Dodziuk/Kendall put it in their paper [6]: "The results are somewhat unsatisfactory. For example, the lower bound for the bottom of the spectrum depended not only on the isoperimetric constant (as it does in Cheeger's inequality) but also on the number $D = \sup_x d(x)$ [...], it is intuitively obvious that such a bound is unnecessary."

Dodziuk-Kendall solved this issue in 1986 by modifying the Hilbert spaces associated to a graph, that is they modified the notion of volume. Instead of counting vertices they considered the measure $\mathrm{vol}_d = d$ which counts edges and is given by

$$\mathrm{vol}_d(W) = \sum_{x \in W} d(x) = \#E_W + \#\partial W,$$

where $E(W)$ are the edges in $W \times W$. The form Q is always bounded on $\ell^2(X, \mathrm{vol}_d) = \ell^2(X, d)$ and the corresponding positive selfadjoint Laplacian Δ_d acts as

$$\Delta_d f(x) = \frac{1}{d(x)} \sum_{y \sim x} (f(x) - f(y))$$

on $\ell^2(X, d)$. For this operator, Dodziuk/Kendall [6] proved the inequality

$$\frac{h_d^2}{2} \leq \lambda_1(\Delta_d) \quad \text{with } h_d = \inf_{W \subseteq X} \frac{\#\partial W}{d(W) \wedge d(X \setminus W)}$$

with $\lambda_1(\Delta_d)$ being the first non-trivial eigenvalue in the finite case and the bottom of the spectrum in the infinite case.

Although this result recovers the original form of Cheeger's inequality for Δ_d, the problem addressed by Dodziuk/Kendall for Δ_1 remained open. In addition there were various other results, where Δ_d provided the correct analogues to the case of manifolds while the corresponding results for Δ_1 failed to be true.

In 2009 Frank/Lenz/Wingert [7] introduced the concept of intrinsic metrics for general regular Dirichlet forms. For strongly local Dirichlet forms which include manifolds as a special case, this concept had already been proven very effective, but it was not available for non-local forms including graphs until 2009. These metrics served as a remedy to resolve disparities that appeared in results for manifolds as opposed to results for general graph Laplacians, see e.g. [10] for a survey.

To introduce these metrics let us consider general weighted graphs. Given a discrete countable set X, a weighted graph is a symmetric map $b : X \times X \to [0, \infty)$ with zero diagonal and $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$. Consider the form

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2$$

on the finitely supported functions. For a measure $m : X \to (0, \infty)$, its closure in $\ell^2(X, m)$ gives rise to a positive selfadjoint operator L on $\ell^2(X, m)$ acting as

$$Lf(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)).$$

For $b : X \times X \to \{0, 1\}$, we get Δ_1 for $m \equiv 1$ and Δ_d for $m = d$ as special cases.

According to Frank/Lenz/Wingert [7] a pseudo metric ρ on X is *intrinsic* if

$$\sum_{y \in X} b(x, y)\rho^2(x, y) \le m(x), \quad \text{for all } x \in X.$$

It can be seen that a pseudo metric is intrinsic if and only if the corresponding 1-Lipshitz functions are included in the set of functions whose “discrete gradient” is bounded by one. An example of an intrinsic metric was given by Huang [8]

$$\rho_0(x, y) = \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=0}^{n-1} \left(\mathrm{Deg}(x_i) \vee \mathrm{Deg}(x_{i+1})\right)^{-1/2},$$

with $\mathrm{Deg}(x_i) = \frac{1}{m(x_i)} \sum_{z \in X} b(x_i, z)$. Observe that whenever $b : X \times X \to \{0, 1\}$, we have $\mathrm{Deg}(x) = d(x)$ for $m \equiv 1$ and $\mathrm{Deg}(x) = 1$ for $m = d$. In particular, ρ_0 is the combinatorial graph distance d_{comb} in the case of the operator $L = \Delta_d$. This explains why many results for Δ_d parallel the results for manifolds directly.

Now, in order to prove a general Cheeger inequality in its original form, one needs to see where d_{comb} already enters the definition of h_d which yields the correct result. Clearly, the denominator is determined by the volume, and, hence, is not subject to change. So, we take a look at the definition of the enumerator

$$\#\partial W = \sum_{(x,y) \in \partial W} 1 = \sum_{(x,y) \in \partial W} d_{\mathrm{comb}}(x, y).$$

For an intrinsic metric ρ this suggest the following definition of the area

$$\mathrm{Area}(\partial W) = \sum_{(x,y) \in \partial W} b(x, y)\rho(x, y).$$

This idea was used in [2] to solve the issue addressed Dodziuk/Kendall in 1986, [6] and recovered Cheeger's inequality in its original form

$$\frac{h^2}{2} \le \lambda_1(L), \qquad \text{with } h = \inf_{W \subseteq X} \frac{\mathrm{Area}(\partial W)}{m(W) \wedge m(X \setminus W)},$$

where $\lambda_1(L)$ is the first non-trivial eigenvalue for finite graphs and the bottom of the spectrum for infinite graphs.

For the p-Laplacian arising from the convex functional

$$E_p(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))^p.$$

for $p \in (1, \infty)$, similar considerations can be made. Here, one is interested in $\lambda_1^{(p)}$ being the infimum of $E_p(f)$ for functions $\|f\|_p \leq 1$ and $f \perp 1$ for finite graphs and $\|f\|_p \leq 1$ only for infinite graphs. Let ρ be a pseudo-metric such that

$$\sum_{y \in X} b(x,y)\rho^q(x,y) \leq m(x) \quad \text{for all } x \in X$$

with q such that $1/p + 1/q = 1$ and define h_p with ρ as above. Then,

$$2^{p-1} \frac{h_p^p}{p^p} \leq \lambda_p,$$

which is shown in [11] paralleling the results of Kawohl/Fridman [9] for the p-Laplacian of manifolds and improving the results of Bühler/Hein [3] for graphs.

References

[1] N. Alon, V. D. Milman, λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory Ser. B **38** (1985), 73–88.

[2] F. Bauer, M. Keller, and R. Wojciechowski, *Cheeger inequalities for unbounded graph Laplacians*, J. Eur. Math. Soc. (JEMS), (2015) to appear.

[3] T. Bühler, M. Hein, *Spectral clustering based on the graph p- Laplacian*, In Proc. 26th Annual Int. Conf. Mach. Learning, (2009), 81–88.

[4] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis (1969), Princeton Univ. Press, 195–199.

[5] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.

[6] J. Dodziuk, W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From local times to global geometry, control and physics, Pitman Res. Notes Math. Ser. **150**, Longman Sci. Tech., Harlow, 1986, 68–74.

[7] R. L. Frank, D. Lenz, D. Wingert, *Intrinsic metrics for non-local symmetric Dirichlet forms*, J. Funct. Anal. **266** (2014), 4765–4808.

[8] X. Huang, *On stochastic completeness of weighted graphs*, Ph.D. thesis, 2011.

[9] B. Kawohl, V. Fridman, *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant*, Comment. Math. Univ. Carolin **44** (2003), 659–667.

[10] M. Keller, *Intrinsic metrics on graphs – a survey*, In D. Mugnolo, editor, Proc. Math. & Stat. **128**, (2015) to appear.

[11] M. Keller, D. Mugnolo *Cheeger inequalities for general p-Laplacian on infinite graphs*, (2015).

[12] R. K. Wojciechowski, *Stochastic completeness of graphs*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–City University of New York.

Equitable partitions, clustering and a bit of Cheeger

Joachim Kerner
(joint work with Delio Mugnolo)

In this first part of the talk we restrict ourselves to finite, undirected, unweighted d-regular graphs $G = (V, E)$. It is well-known that a classical approach to the clustering of such combinatorial graphs employs a Cheeger-type inequality. Namely, an "optimal" partition $\pi = \{\Sigma_i\}$ of the vertex set with $|\pi| = k$ is defined to be a minimizer of

$$\max_{i=1,\dots,k} \frac{|\partial\Sigma_i|}{d|\Sigma_i|} , \tag{1}$$

where $|X|$ refers to the number of vertices in X and $|\partial X|$ to the number of edges with one end point in X and the other in X^c. Its minimal value $h_G(k)$ is then called the Cheeger constant of the graph G, fulfilling (as shown in [3])

$$\frac{\lambda_k}{2} \leq h_G(k) \leq c\sqrt{\lambda_k} , \tag{2}$$

where $c > 0$ is some constant and $\{\lambda_k\}_{k=1,\dots,|V|}$ are the eigenvalues of the normalised Laplacian $L = I - \frac{1}{d}A$. Here $A \in \mathbb{R}^{|V|\times|V|}$ is the adjacency matrix of the graph. Due to its resemblance to the famous Cheeger inequality for manifolds [1], (2) is often also called a Cheeger inequality.
However, it seems that at least in some situations, other clusters might seem more "natural" than the clusters obtained by minimizing (1). For example, imagine one starts with k disjoint, complete graphs C_k with $|C_k| \gg 1$, then adding a *few* edges to obtain a connected graph. In this scenario, two immediate question arise: Under what circumstances is $\pi = \{C_k\}$ an optimal partiton in terms of Cheeger? Also, how is one able to "see" (or measure for that matter) that our graph is somehow "close" to the initial scenario of k disjoint and complete graphs? Given one knew this, each C_k would then form a natural cluster.
Generalizing this setting, one is led to the notion of equitable partitions of graphs. Let $\pi = \{\Sigma_i\}_{i=1,\dots,k}$ be a partition such that *each* vertex $v \in \Sigma_i$ has exactly c_{ii} neighbours in Σ_i and c_{ij} neighbours in Σ_j. Then this partition π is called equitable and one associates a matrix $C \in \mathbb{R}^{k\times k}$ (the so called quotient matrix) with π, i.e., $(C)_{ij} := c_{ij}$. Note that in general, $C \neq C^T$. Now, from a spectral point of view, the following theorem is crucial and (partly) explains why looking at equitable partitions is interesting in the first place.

Theorem 1.1. [2] *Let π be an equitable partition of a graph G with corresponding quotient matrix C. Then the characteristic polynomial of C divides the characteristic polynomial of A. In particular, $\sigma(C) \subset \sigma(A)$.*

Note that, for any partition $\pi = \{\Sigma_i \subset V\}$, one can define a characteristic matrix $P \in \mathbb{R}^{|V|\times|\pi|}$ whose entries are either zero or one. More precisely, $(P)_{ij} = 1$ if and only if the vertex i is Σ_j. Regarding the existence of an equitable partition, one then has the following result.

Theorem 1.2. [2] *Let π be any partition of a graph with characteristic matrix $P \in \mathbb{R}^{|V|\times|\pi|}$. Then π is equitable if and only if there exists a matrix $C \in \mathbb{R}^{|\pi|\times|\pi|}$ such that $AP = PC$. In this case, C is the quotient matrix associated with π.*

As shown in [2], equitable partitions can also be identified as those partitions for which the interlacing of eigenvalues is tight. More precisely, for *each* partition π, one can define the (symmetric) matrix $B := (P^T P)^{-1} P^T AP \in \mathbb{R}^{|\pi|\times|\pi|}$. This matrix is then shown to interlace the eigenvalues of the adjacency matrix A, i.e., $\lambda_{|V|-|\pi|+j}(A) \leq \lambda_j(B) \leq \lambda_j(A)$ for $j = 1, ..., |\pi|$. Note that the eigenvalues are ordered here in a descending manner. Furthermore, the interlacing is called tight if there exists an index i such that $\lambda_j(B) = \lambda_j(A)$ for all $j \leq i$ and $\lambda_j(B) = \lambda_{|V|-|\pi|+j}(A)$ for $j > i$.

Theorem 1.3. [2] *Let π be any partition of a graph with characteristic matrix P. Then the eigenvalues of $B = (P^T P)^{-1} P^T AP$ interlace the eigenvalues of A. If the interlacing is tight, the partition π is equitable.*

Further results regarding relaxed notions of equitability can be found in [4]. Finally, it is interesting to note that Theorem 1.2 can be strengthened for a particular class of graphs that allow for equitable partitions, namely, the class of distance-regular graphs. They are defined by the fact that for any pair of vertices $x, y \in V$ at a given distance d, the number of $z \in V$ that have distance d_x to x and d_y to y, is *independent* of x and y. One can show that each distance-regular graph allows for an equitable partition which is obtained as follows: one picks any vertex v_0 which then constitutes Σ_1. The set Σ_{n+1} is then defined to contain all vertices which are at distance n from v_0.

Theorem 1.4. [5] *Let G be a finite, connected, distance-regular graph with corresponding quotient matrix C. Then $\sigma(C) = \sigma(A)$.*

Based on Theorem 1.4 and Theorem 1.1 we ask the following question: is it possible to define a "meaningful" spectral distance function $\mu_{\sigma(A),\sigma(C)} : X_d \to \mathbb{R}$ on the set X_d of d-regular graphs which measures "how far" the graph is away from having an equitable partition with associated quotient matrix C? For example, one is aiming at estimates of the form

$$F_1[\mu_{\sigma(A),\sigma(C)}(G)] \leq N_{\min} \leq F_2[\mu_{\sigma(A),\sigma(C)}(G)] \; , \tag{3}$$

where $F_i : \mathbb{R}_+ \to \mathbb{R}_+$ are (possibly) continuous and strictly positive for $x > 0$ with $F_i(0) = 0$. Also, $N_{\min}$ is defined to be the minimal number of edges that have to be removed or added to yield the desired property. Note that, by definition, $\mu_{\sigma(A),\sigma(C)}(G) = 0$ if and only if the property is fulfilled.

In the second part of the talk, we want to discuss a Cheeger-type inequality for finite *metric* graphs $\Gamma = (V, E)$ as obtained in [6] for $p = 2$ and provide an extension to the nonlinear setting, i.e., the case of general $p \in (1, \infty)$. Intuitively, a metric graph is obtained from a combinatorial graph by associating an interval $I_e := [0, l_e]$ to each edge $e \in E$ which then allows to define functions and hence differential operators on graphs. More precisely, in this talk we are interested in the p-Laplacian which acts (on suitably regular functions) component-wise via

$$(-\Delta_p f)_e = -\nabla(|\nabla f_e|^{p-2}\nabla f_e) \ . \tag{4}$$

On a variational level, the p-Laplacian is associated with the functional

$$\mathcal{E}_p[f] := \frac{\int_\Gamma |\nabla f|^p \ \mathrm{d}x}{\int_\Gamma |f|^p \ \mathrm{d}x} \ , \tag{5}$$

defined for all $f \in W^{1,p}(\Gamma)$. Note that $W^{1,p}(\Gamma)$ consists of all functions whose components are elements of $W^{1,p}(I_e)$.

Also, for a metric graph, the connectivity can be expressed in analogy to the combinatorial setting via a Cheeger constant $h(\Gamma)$. For this, let $Y \subset \Gamma$ be an open subset with $Y \neq \{\emptyset, \Gamma\}$. Denoting the number of points in the boundary as $|\partial Y|$ and the volume of Y as $|V| = \int_\Gamma 1_Y \ \mathrm{d}x$, one defines

$$h(\Gamma) := \inf_Y \frac{|\partial Y|}{\min\{|Y|, |Y^c|\}} \ . \tag{6}$$

The main result then reads as follows.

Theorem 1.5. *For $p \in (1,\infty)$, let Γ be a connected metric graph and $f \in W^{1,p}(\Gamma)$ a function that is continuous across the vertices. If f changes sign, one has*

$$\mathcal{E}_p[f] \geq p^{-p} h^p(\Gamma) \ . \tag{7}$$

The key ingredient in the proof is to consider the positive part of f (this defines the corresponding open set Y), then using Hölder inequality and finally the coarea formula (see also [7]).

Finally we want to mention that the p-Laplacian is interesting concerning spectral clustering tasks as discussed in [8]. In particular, recognizing that the estimates (2) are usually rather crude, one concludes from (7) that the p-Laplacian in the limit $p \to 1$ might present better estimates on the connectivity of the graph.

References

[1] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis (R. C. Gunning, ed.), Princeton Univ. Press (1970), 195-199.

[2] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag (2001).

[3] J. R. Lee, S. O. Gharan, L. Trevisan, *Multi-way spectral partitioning and higher-order Cheeger inequalities*, Proc. of 44th ACM STOC (2012), 1117–1130.

[4] J. J. Molitierno, *Applications of combinatorial matrix theory to Laplacian matrices of graphs*, Taylor & Francis Group (CRC Press) (2012).

[5] B. Mohar and M. Omladič, *Divisors and the spectrum of infinite graphs*, Linear Algebra and its Applications (91) (1987), 99–106.

[6] S. Nicaise, *Spectre des réseaux topologiques finis*, Bull. Sci. Math. (2), (1987), 401–413.

[7] O. Post, *Spectral analysis of metric graphs and related spaces.*, Limits of graphs in group theory and computer science (2008) , 109–140.

[8] T. Bühler and M. Hein, *Spectral clustering based on the graph p-Laplacian*, Proceedings of the 26th Annual International Conference on Machine Learning. ACM, (2009).

$\rho-$Laplacian on metric-measure spaces

Yaroslav Kurylev
(joint work with D. Burago and S. Ivanov)

1. Definition

Let (X, d, μ) be a compact metric-measure space (we sometimes write $\mu = \mu_X$, $d = d_X$.) For any $\rho > 0$ define the associated ρ-Laplacian,

$$\Delta_X^\rho u(x) = \frac{1}{\mu(B_\rho(x))\rho^2} \int_{d(x,y)<\rho} [u(x) - u(y)] d\mu(y). \tag{1}$$

This is a (bounded) non-negative self-adjoint operator in $L^2(X, \mu^\rho)$:

$$\langle u, v \rangle_{L^2(X,\mu^\rho)} = \int_X \left[\mu(B_\rho(x))\rho^2\right] u(x)v(x) d\mu(x). \tag{2}$$

The Dirichlet form corresponding to (1) is given by

$$D[u] = \int \int_{d(x,y)<\rho} |u(x) - u(y)|^2 d\mu(x) d\mu(y). \tag{3}$$

In this talk, we discuss some spectral properties of this operator, in particular, its approximation of the Laplacian on a compact Riemannian manifold and the spectral stability under proper variation of distance and measure.

2. Approximation of a Riemannian Laplacian

Let (M, g) be a compact Riemannian manifold with Δ_M its Laplacian. Assume that $X = \{x_n\}_1^N \subset M$ be an $\varepsilon-$net. Let μ_n be discrete measures associated with x_n which enjoy the following

Condition A There is subdivision $M = \cup V_n$, $\mu(V_n \cap V_k) = 0$, $k \neq n$ such that $\mu_n = \mu(V_n)$.

Consider $\rho-$Laplacian associated to (X, μ_n) and assume that $\rho > 3i$, $i-$being the injectivity radius of M.

Theorem 2.1. *Denote by $\lambda_k(x)$, $\lambda_k(M)$ the eigenvalues of Δ_X^ρ, Δ_M, respectively. There are $C_d, c_d > 0$ such that if $\lambda_k(M)\rho < c_d$, $K\rho^2 < c_d$,*

$$|\lambda_k(M) - 2(d + 2)\lambda_k(X)| \leq C_d(\varepsilon/\rho + K\rho^2)\lambda_k(M) + C_d\rho\lambda_k(M)^{3/2}. \tag{4}$$

Here d is the dimension of M and K is the bound for its sectional curvature.

Moreover, the corresponding eigenfunctions, ϕ_k and ϕ_k^ρ of Δ_M and Δ_X^ρ are also close. For simplicity, we consider only a non-degenerate $\lambda_k(M)$ and the projection of ϕ_k in $L^2(X)$ (see [1] for the general result).

Let

$$(P\phi_k)(x_n) = \frac{1}{\mu_n \rho^{(d+2)/2}} \int_{V_n} \phi_k(x) d\mu.$$

Then

$$||P\phi_k - \phi_k^\rho||_{L^2(X)} \leq C_{\mathcal{M},k}\delta^{-1}(\varepsilon/\rho + \rho). \tag{5}$$

Here $\mathcal{M}$ is the Gromov class of $d-$dimensional Riemannian manifolds with sectional curvature and diameter uniformly bounded and injectivity radius uniformly bounded below while $\delta = \min\{\lambda_k(M) - \lambda_{k-1}(M),\ \lambda_{k+1}(M) - \lambda_k(M)\}$.

3. Spectral stability of $\rho-$Laplacians

Definition 1. *(X, d_x, μ_x) and $(Y, d_y, \mu_y))$ are $(\varepsilon, \delta)-$ Gromov-Prohorov close if there are $\tilde{\mu}_x, \tilde{\mu}_y$ with*

$$\varepsilon^{-\delta} < \frac{\tilde{\mu}_x}{\mu_x},\ \frac{\tilde{\mu}_y}{\mu_y} < \varepsilon^{\delta},$$

such that $(X, d_x, \tilde{\mu}_x)$, $(Y, d_y, \tilde{\mu}_y)$ are $\varepsilon-$Gromov-Wassertein close. The last condition means that (X, d_x), (Y, d_y) are $\varepsilon-$Gromov-Hausdorff close and, denoting by Z the corresponding disjoint union of X and Y, the Wasserstein distance, $W^{\infty}(\tilde{\mu}_x,\ \tilde{\mu}_y) < \varepsilon$. Here we treat $\tilde{\mu}_x$, $\tilde{\mu}_y$ as push-forwards to Z of $\tilde{\mu}_x$, $\tilde{\mu}_y$.

Theorem 3.1. *Let (X, d_x, μ_x), (Y, d_y, μ_y) are $(\varepsilon, \delta)-$Gromov-Prohorov close. There is a constant C, which depends on the volume growth conditions for X and Y such that*

$$\frac{1}{1 + C(\varepsilon/\rho + \delta)} \leq \frac{\lambda_k(X)}{\lambda_k(Y)} \leq 1 + C(\varepsilon/\rho + \delta), \quad \lambda_k < C^{-1}\rho^{-2}.$$

Here $\lambda_k(X, Y)$ are the eigenvalues $\Delta^{\rho}_{X,Y}$.

We note that, up to $c\rho^{-2}$, the spectra of $\Delta^{\rho}_{X,Y}$ is the discrete one.

References

[1] D. Burgao , S. Ivanov, Y. Kurylev, *A graph discretization of the Laplace-Beltrami operator*, J. Spectral Theory, **4**, 675-714 (2014).

Intrinsic metrics for regular Dirichlet forms

Daniel Lenz

(joint work with Rupert Frank, Daniel Wingert)

Regular Dirichlet forms provide an analytic way to describe symmetric Markov processes with ‘nice’ paths in continuous time. As such they serve as a common umbrella for the study of Laplacians on manifolds and of Laplacians on graphs (and many more Laplacians). In fact, in this context the study of the Laplacian on manifolds can be generalized to the study of strongly local Dirichlet forms. In the corresponding investigations the concept of an intrinsic metric has played a major role starting with the seminal work of Sturm [4]. Recently, various similar results could successfully be proven for graphs as well due to the tool of intrinsic metrics for general regular Dirichlet forms provided in [2]. A survey on results on graphs obtained using intrinsic metrics is given in [3]. Here, we discuss the background on intrinsic metrics from [2].

Let X be a locally compact metric space and m a Radon measure on X and $L^2(X,m)$ the real Hilbert space of square integrable real-valued functions on X. Then, a *non-negative form* over (X,m) is a bilinear map

$$Q : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R} \text{ with } Q(f) := Q(f,f) \geq 0$$

for all f in the subspace $\mathcal{D}$ of $L^2(X,m)$. If $\mathcal{D}$ is complete with respect to the norm

$$\|f\|_Q := \left(\|f\|^2 + Q(f)\right)^{1/2}$$

such a form is *closed*. To any closed form there corresponds a unique selfadjoint operator $L \geq 0$ with domain contained in $\mathcal{D}$ and

$$\langle f, Lg\rangle = Q(f,g)$$

for all g in the domain of L and all $g \in \mathcal{D}$. A closed form is called a *Dirichlet form* if it furthermore satisfies

$$Q(Cf) \leq Q(f)$$

for all $f \in \mathcal{D}$ and all normal contractions C (i.e all $C : \mathbb{R} \longrightarrow \mathbb{R}$ with $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$). Such a form corresponds to a Markov Process (X_t) via the formula

$$e^{-tL}f(x) = \mathbb{E}_x(f(X_t)).$$

A Dirichlet form is called *regular* if $\mathcal{D} \cap C_c(X)$ is dense in $\mathcal{D}$ w.r.t. $\|\cdot\|_Q$ and in $C_c(X)$ w.r.t. $\|\cdot\|_\infty$. Regular Dirichlet forms are in one-to-one correspondence to Markov processes whose paths are right continuous and have limits from the left (cadlag processes). For details on the preceding considerations and general background on Dirichlet forms we refer to [1].

Whenever Q is a Dirichlet form then there exist a unique bilinear map

$$\Gamma : \mathcal{D} \times \mathcal{D} \longrightarrow \text{Radon measures on } X$$

and a unique measure k on X with

$$Q(f,g) = \int_X d\Gamma(f,g) + \int_X fg dk$$

for all $f, g \in C_c(X) \cap \mathcal{D}$. The map Γ is called the *energy measure*. As shown in [2] the map Γ can be extended to a certain space $\mathcal{D}^*_{loc}$. This extension will also be denoted by Γ. With this extension comes the set

$$\mathcal{A}_m := \{f \in C(X) \cap \mathcal{D}^*_{loc} : \Gamma(f,g) \leq m\}$$

and the function $d : X \times X \longrightarrow [0,\infty]$ defined by

$$d(x,y) := \sup\{f(x) - f(y) : f \in \mathcal{A}_m\}.$$

Clearly, d is symmetric, vanishes on the diagonal and satisfies the triangle inequality.

Example - Riemannian manifold M. The Laplace Beltrami Δ operator on a Riemannian manifold M is associated to the regular Dirichlet form

$$Q_M(f,g) := \int_M \langle \nabla f, \nabla g\rangle dx$$

on $\mathcal{D} = H_0^1(M)$. In this case

$$\Gamma(f,g) = \langle \nabla f, \nabla g \rangle dx$$

and (under suitable completeness assumptions) d is the geodesic distance.

Example - Graph. A weighted graph (b,c) graph over the discrete measure space (X,m) gives rise to the regular Dirichlet form $Q_{b,c}$ with

$$Q_{b,c}(f,g) = \frac{1}{2} \sum_{x,y} b(x,y)(f(x) - f(y))(g(x) - g(y)) + \sum_x c(x) f(x) g(x)$$

for $f, g \in C_c(X)$. Then $\mathcal{D}$ is the closure of $C_c(X)$ with respect to $\| \cdot \|_Q$ and the associated operator is the graph Laplacian. Here, $\Gamma(f,g)$ is the measure whose mass at the point $x \in X$ is given as

$$\Gamma(f,g)(x) = \frac{1}{2} \sum_{y \in X} (f(x) - f(y))(g(x) - g(y)).$$

Definition. A symmetric function $\varrho : X \times X \longrightarrow [0, \infty]$ which vanishes on the diagonal and satisfies the triangle inequality is called an *intrinsic metric* if

$$\mathrm{Lip}_{1,\varrho} \subset \mathcal{A}_m$$

holds, where $\mathrm{Lip}_{1,\varrho}$ denotes the set of Lipshitz functions with constant not exceeding one with respect to ϱ.

Remarks. (a) This definition is slightly more general than the definition in [2]. However, for the applications to strongly local forms and graphs it agrees with the definition given there.

(b) Note that an intrinsic metric is automatically continuous as

$$[\varrho(x,y) - \varrho(x',y')| \leq \varrho(x,x') + \varrho(y,y')$$

and both $\varrho(x,\cdot)$ and $\varrho(y,\cdot)$ are continuous (as they are Lipshitz functions with constant 1 and hence belong to $\mathcal{A}_m$ by definition of an intrinsic metric).

Example - Strongly local forms. A Dirichlet form is called *strongly local* if $Q(f,g) = 0$ holds whenever f is constant on the support of g. Obviously, the example of Dirichlet form on a manifold given above is strongly local. For a strongly local form the function d is an intrinsic metric if it is continuous [4, 2]. It is this intrinsic metric that has been successfully employed starting with the work of Sturm [4].

Example - jump processes. Graphs are special case of jump processes. For a jump process with absolutely continuous kernel j on the measure space (X,m) the map ϱ is an intrinsic metric if and only if

$$\int \varrho(x,y)^2 j(x,y) dm(y) \leq 1$$

holds for m-almost every $x \in X$ [2].

There is a clear distinction between local and non-local Dirichlet forms in terms of intrinsic metrics [2]: In the strongly local case the metric d is the maximal intrinsic metric (if it is continuous). For a non-local Dirichlet form on the other hand there is in general no maximal intrinsic metric. In fact, for rather general regular Dirichlet forms Q with d as above the following assertions are equivalent:

(i) There exists a maximal intrinsic metric.
(ii) The function d is an intrinsic metric.
(iii) The equality $\mathrm{Lip}_{1,d} = \mathcal{A}_m$ holds.
(iv) The set $\mathcal{A}_m$ is closed under taking suprema.

This can essentially be inferred from [2]. Details will be discussed elsewhere.

References

[1] M. Fukushima, Y. Ōshima, M. Takeda, Dirichlet forms and symmetric Markov processes, de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 1994.

[2] L. Frank, D. Lenz, D. Wingert, Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory, J. Funct. Anal. **266** (2014), no 8, 4765–4808. (ArXive:1012.5050)

[3] M. Keller, Intrinsic metrics on graphs - A survey to appear in Mathematical Technology of Networks (Proc. Bielefeld 2013), Proc. Math. & Stat. Springer-Verlag, New York.

[4] K.-T. Sturm, Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p Liouville properties, J. Reine Angew. Math **456** (1994), 173–196.

Parabolic *p*-Laplace equations on graphs

Delio Mugnolo

Let G be a graph with node set V, edge set E, and node and edge weights μ, ν, respectively. I consider the evolution equation

$$\frac{df}{dt}(t, \mathsf{v}) = -\mathcal{L}_p f(t, \mathsf{v}), \qquad t \geq 0,\ \mathsf{v} \in \mathsf{V}\ , \tag{1}$$

associated with the *discrete p-Laplacian* $\mathcal{L}_p$ on a graph G, formally defined by

$$\mathcal{L}_p f := \frac{1}{\nu} \mathcal{I} \mathcal{M} (|\mathcal{I}^T f|^{p-2} \mathcal{I} f), \qquad p \geq 1\ ,$$

where $\mathcal{I}$ is the signed incidence matrix of an arbitrary orientation of G defined by

$$\iota_{\mathsf{ve}} := \begin{cases} +1 & \text{if } \mathsf{v} \in \mathsf{V} \text{ is terminal endpoint of } \mathsf{e} \in \mathsf{E} \\ -1 & \text{if } \mathsf{v} \in \mathsf{V} \text{ is initial endpoint of } \mathsf{e} \in \mathsf{E} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{M}$ is the diagonal matrix whose e-th entry is the weight $\mu(\mathsf{e})$. A simple computation shows that while $\mathcal{I}$ depends on the orientation of G, $\mathcal{L}_p$ does not.

The operator $\mathcal{L}_p$ is nonlinear: existence and uniqueness of a solution of the Cauchy problem for (1) can be deduced from the Picard–Lindelöf Theorem if the graph is finite. More advanced methods are needed in the case of infinite graphs: they have been the topic of my talk, which is based on the results in [6, 5].

My analysis is based on the properties of the convex energy functional

$$\mathcal{E}_p : f \mapsto \frac{1}{p} \sum_{\mathsf{e} \in \mathsf{E}} \mu(\mathsf{e}) |\mathcal{I}^T f(\mathsf{e})|^p \tag{2}$$

for all $p > 1$. If the graph is infinite, defining the domain of $\mathcal{E}_p$ becomes of utmost importance. The largest possible domain is the *discrete Sobolev space*

$$w^{1,p,2}_{\mu,\nu} := \{ f \in \ell^2_\nu(\mathsf{V}) : \mathcal{I}^T f \in \ell^p_\mu(\mathsf{E}) \} \ .$$

We consider Sobolev spaces of mixed order in order to enforce the embedding

$$w^{1,p,2}_{\mu,\nu}(\mathsf{V}) \hookrightarrow \ell^2_\nu(\mathsf{V}) \ .$$

In this way we can study (1) as a gradient flow in the Hilbert space $\ell^2_\nu(\mathsf{V})$, applying a well developed abstract theory.

One may define $\mathcal{E}_p$ on smaller Banach spaces as well, as long as their norm is equivalent to

$$f \mapsto \left(\mathcal{E}_p(f) + \|f\|^2_{\ell^2_\nu(\mathsf{V})} \right)^{\frac{1}{p}} \ ,$$

and hence to the norm of $w^{1,p,2}_{\mu,\nu}(\mathsf{V})$; in particular, one may close up the space of finitely supported functions $c_{00}(\mathsf{V})$ in the norm of $w^{1,p,2}_{\mu,\nu}(\mathsf{V})$ and restrict $\mathcal{E}_p$ to this closure $\mathring{w}^{1,p,2}_{\mu,\nu}(\mathsf{V})$. Depending on the geometry of the graph, and in particular on the weights μ, ν, the two spaces $\mathring{w}^{1,p,2}_{\mu,\nu}(\mathsf{V})$ and $w^{1,p,2}_{\mu,\nu}(\mathsf{V})$ may coincide; this is e.g. the case if the weighted degree function

$$\deg_{\mu,\nu} : \mathsf{v} \mapsto \frac{\sum_{\mathsf{e} \in \mathsf{E}} \mu(\mathsf{e}) |\iota_{\mathsf{ve}}|}{\nu(\mathsf{v})}$$

is bounded. (Observe that $\deg_{\mu,\nu}(\mathsf{v})$ is the number of neighbours of v in the unweighted case of $\mu \equiv 1$ and $\nu \equiv 1$.)

We study the p-Laplacian $\mathcal{L}_p$ as the Fréchet derivative of $\mathcal{E}_p$ in the Hilbert space $\ell^2_\nu(\mathsf{V})$ for $p > 1$; in fact, any restriction of $\mathcal{E}_p$ to a closed subspace $D(\mathcal{E}_p)$ of $w^{1,p,2}_{\mu,\nu}(\mathsf{V})$ that contains $\mathring{w}^{1,p,2}_{\mu,\nu}(\mathsf{V})$ induces a different Fréchet derivative: one may thus find different versions of the p-Laplacian which can, like in the classical case of the Laplacian on bounded domains, be regarded as realisations of the same operator with different *boundary conditions* – the boundary of a graph consisting of its *points at infinity*, in a certain sense made precise in [4].

(In the case of $p = 1$ the same holds for the multivalued operator $\mathcal{L}_1$, which is well-defined as the *subdifferential* of the convex functional $\mathcal{E}_1$: this is not Fréchet differentiable, but still lower semicontinuous.)

The general theory of gradient systems [1] yields that under boundedness assumptions on μ, ν the Cauchy problem associated with (1) is well-posed, i.e., for each initial data $f_0 \in D(\mathcal{E}_p)$ there is a unique solution f. This can be found both by a time discretisation based on the Crandall–Liggett Theorem or by a space exhaustion scheme that boils down to the classical Galerkin scheme. Due to a strong maximum principle satisfied by (1), the solutions of the localised equations are monotonically increasing as the underlying graphs grow and eventually exhaust the original graph. The theory of *nonlinear Dirichlet forms* [2] can be applied and yields that the solutions are given by *nonlinear sub-Markovian semigroups*, i.e., strongly continuous families of nonlinear $\ell^2_\nu(\mathsf{V})$-contractions that satisfy the semigroup law, are positivity preserving and, as long as $\inf_{\mathsf{v}\in\mathsf{V}} \nu(\mathsf{v}) > 0$, are also contractive with respect to all $\ell^q_\nu(\mathsf{V})$-norms, $1 \leq q \leq \infty$.

We can a priori only deduce from the general theory that the unique solution is of class $W^{1,2}(\mathbb{R}_+;\ell^2_\nu(\mathsf{V})) \cap L^\infty(\mathbb{R}_+;D(\mathcal{E}_p))$, but in our discrete setting the atomic nature of the measure space V allows for much better regularity results: (1) is solved not only almost everywhere like in the case of the p-Laplacians on domains, but in fact pointwise. Accordingly, a bootstrap argument based on the differentiability properties of the function $x \mapsto |x|^{p-2}x$ yields that for any $\mathsf{v} \in \mathsf{V}$

- $f(\cdot,\mathsf{v}) \in C^\infty(\mathbb{R}_+)$ if p is an even integer,
- $f(\cdot,\mathsf{v}) \in C^{p-1,1}(\mathbb{R}_+)$ if p is an odd integer, and
- $f(\cdot,\mathsf{v}) \in C^{\lfloor p\rfloor,p-\lfloor p\rfloor}(\mathbb{R}_+)$ if $p \in (1,\infty) \setminus \mathbb{N}$.

This is remarkable in the light of the available regularity results for the counterpart

$$\frac{\partial u}{\partial t}(t,x) = \Delta_p u(t,x), \qquad t \geq 0,\ x \in \Omega\ , \tag{3}$$

of (1) on domains $\Omega \subset \mathbb{R}^d$, for whose solutions only Hölder-C^α-regularity for $\alpha < 1$ is known, no matter how large p is [3]. Unfortunately, no convergence schemes of $-\mathcal{L}_p$ towards the p-Laplacian Δ_p on a domain are currently available, so that it is impossible to deduce higher regularity of solutions of (3) by space discretisation, solving (1) and the letting the corresponding mesh become finer and finer.

Several qualitative properties of solutions of (1) are known that strongly resemble analogous features of (3). As a rule of thumb, defining $\mathcal{E}_p$ on the maximal space $w^{1,p,2}_{\mu,\nu}(\mathsf{V})$ (as in the case of finite graphs) leads to the discrete counterpart of (3) with Neumann boundary conditions, whereas Dirichlet boundary conditions can be reproduced on fast growing graphs by defining $\mathcal{E}_p$ on $\mathring{w}^{1,p,2}_{\mu,\nu}(\mathsf{V})$; by *fast growing* we mean that a *d-dimensional isoperimetric inequality*

$$\nu(\mathsf{V}_0)^{\frac{d-1}{d}} \leq C_d \mu(\partial \mathsf{V}_0) \qquad \forall\ \mathsf{V}_0 \subset \mathsf{V} \text{ finite,} \tag{IS$_\mathrm{d}$}$$

is satisfied for some $d > 2$ and $0 < C_d < \infty$, where $\partial\mathsf{V}_0$ is the set of edges of G with exactly one endpoint in V_0, $\nu(\mathsf{V}_0) := \sum_{\mathsf{v}\in\mathsf{V}_0} \nu(\mathsf{v})$ and $\mu(\partial\mathsf{V}_0) := \sum_{\mathsf{e}\in\partial\mathsf{V}_0} \mu(\mathsf{e})$. In this way, conservation of mass and convergence to a constant function can be proved for "Neumann boundary conditions" and $p > 2$, whereas extinction in finite time holds for "Dirichlet boundary conditions" and $p \in (1,2)$.

Finally, I have briefly considered p-Laplacians on hypergraphs. A hypergraph $\mathsf{H} = (\mathsf{V},\mathsf{E})$ consists of a set V of vertices and a set E of *hyperedges*, i.e., of subsets of V: unlike in the case of a graph's edges, no restriction on the cardinality of hyperedges is imposed. An orientation of a hypergraph is imposed by partitioning each hyperedge e into a set $\mathsf{e}_{\mathrm{init}}$ of *initial endpoints* and a set $\mathsf{e}_{\mathrm{term}}$ of *terminal* *endpoints*. In this way a signed incidence matrix $\mathcal{I} = (\iota_{\mathsf{ve}})$ can be defined by

$$\iota_{\mathsf{ve}} := \begin{cases} +1 & \text{if } \mathsf{v} \in \mathsf{V} \text{ belongs to the terminal endset } \mathsf{e}_{\text{term}} \text{ of } \mathsf{e} \in \mathsf{E} \\ -1 & \text{if } \mathsf{v} \in \mathsf{V} \text{ belongs to the initial endset } \mathsf{e}_{\text{init}} \text{ of } \mathsf{e} \in \mathsf{E} \\ 0 & \text{otherwise ,} \end{cases}$$

A convex, Fréchet differentiable energy functional can again be defined by (2) upon replacing the incidence matrix of a graph by that of a hypergraph. Again, for all $p > 1$ this functional can be differentiated in $\ell^2_\nu(\mathsf{V})$ and yields a p-Laplacian $\mathcal{L}_p$. The associated evolution equation (1) is well-posed again, with its solutions being given by a semigroup of nonlinear $\ell^2_\nu(\mathsf{V})$ contractions. However, as soon as H is a genuine hypergraph, i.e., as soon as it contains at least one hyperedge with cardinality larger than 2, this semigroup will not be sub-Markovian.

References

[1] R. Chill and E. Fašangová. *Gradient Systems*. MatFyzPress, Prague, 2010.

[2] F. Cipriani and G. Grillo. Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces. *J. Reine Ang. Math.*, 562:201–235, 2003.

[3] E. DiBenedetto, U. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular equations*. Monographs in Mathematics. Springer-Verlag, New York, 2012.

[4] A. Georgakopoulos, S. Haeseler, M. Keller, D. Lenz, and R. Wojciechowski. Graphs of finite measure. arXiv:1309.3501.

[5] B. Hua and D. Mugnolo. Time regularity and long-time behavior of parabolic p-Laplace equations on infinite graphs. arXiv:1410.1584.

[6] D. Mugnolo. Parabolic theory of the discrete p-Laplace operator, *Nonlinear Anal., Theory Methods Appl.* **87** (2013), 33–60.

Combined effects in Kirchhoff fractional elliptic problems with lack of compactness

Patrizia Pucci

Recently, a great attention has been drawn to the study of fractional and nonlocal operators of elliptic type. These operators arise in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of *Lévy* processes, see e.g. [1, 5]. The literature on nonlocal operators and on their applications is interesting and quite large, we refer the reader to the references given in [15].

The talk is focused on recent results concerning existence, multiplicity and asymptotic behavior of positive solutions of some *Kirchhoff* type problems, involving fractional integro-differential elliptic operators and presenting also difficulties due to intrinsic lacks of compactness, which arise from different reasons. The problems presented are highly nonlocal because of the presence of the fractional integro–differential elliptic operators and of the *Kirchhoff* coefficients. The proof techniques should therefore overcome the nonlocal nature of the problems as well as the lack of compactness, and the suitable strategies adopted depend of course on the problem under consideration.

First, we discuss the existence of entire solutions of the stationary *Kirchhoff* type equations driven by the fractional p–Laplacian operator

$$M\left([u]_{s,p}^p\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda\omega(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N,$$
$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy,$$

where $(-\Delta)_p^s$ is the fractional p–Laplacian operator, which (up to normalization factors) may be defined along any $\varphi \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\, dy$$

for $x \in \mathbb{R}^N$ and $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. Furthermore, $0 < s < 1 < p < \infty$, $ps < N$, $0 < \lambda < \infty$, $1 < q < r$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous *Kirchhoff* function and $V, \omega, h : \mathbb{R}^N \to \mathbb{R}^+$ are three weights, as assumed in [15]. We prove multiplicity results depending on λ and according to the integrability properties of the ratio $\omega^{(r-1)/(r-q)}/h^{(q-1)/(r-q)}$. The existence of infinitely many pairs of entire solutions $\{\pm u_k\}_{k=1}^\infty$ in which the critical values $I_\lambda(\pm u_k) < 0$ is also obtained. Obviously, I_λ denotes the underlying energy functional of the variational problem under consideration. The results of [15] extend the previous recent work of [16] from the case of variable exponent elliptic problems to the case of fractional p–Laplacian problems of *Kirchhoff* type. The main theorems of [15] also generalize the works [12, 3, 4]. In particular, they weaken the condition $2 < q < \min\{r, 2^*\}$, $2^* = 2N/(N-2)$, assumed in [12, 3, 4] into the simple request that $1 < q < r$, as first treated and extended in [16]. More interestingly, the results in [15] cover a main feature of *Kirchhoff* type problems which is the fact that the *Kirchhoff* function M can be zero at zero, that is that the *Kirchhoff* problem is *degenerate*. Hence the results are completely new.

On one hand, in the context of fractional quantum mechanics, a nonlinear fractional *Schrödinger* equation was first proposed by *Laskin* in [9, 10] as a result of expanding the *Feynman* path integral, from the *Brownian*–like to the *Lévy*–like quantum mechanical paths. In the last years, there has been a great interest in the study of the fractional *Schrödinger* equation

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $(-\Delta)^s = (-\Delta)_2^s$ and the nonlinearity f satisfies some general conditions. For standing wave solutions of fractional *Schrödinger* equations in $\mathbb{R}^N$ we mention e.g. [6, 4] and the references therein. Models governed by the fractional p–Laplacian and unbounded potentials are investigated e.g in [17, 13, 14]. For basic properties on the first eigenvalue and eigenfunction of the fractional p–Laplacian we refer to [11], while of the classical p–Laplacian to [8] and to the references in [8, 11], as well as to the Abstract in this volume, *First eigenvalues of the indiscrete p–Laplacian*, by B. Kawhol.

In the second part of the talk, we present the main results given in [14], that is the existence of multiple solutions for the nonhomogeneous fractional p–Laplacian equations of *Schrödinger–Kirchhoff* type

$$M\left([u]_{s,p}^{p}\right)(-\Delta)_{p}^{s}u+V(x)|u|^{p-2}u=f(x,u)+g(x)\quad\text{in }\ \mathbb{R}^{N},$$

where $0<s<1<p<\infty$ and $ps<N$, the coefficient $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a continuous *Kirchhoff* function, $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies the *Ambrosetti–Rabinowitz* type condition, $V:\mathbb{R}^N\to\mathbb{R}^+$ is a potential function and $g:\mathbb{R}^N\to\mathbb{R}$ is a perturbation term. In particular, in [14] we first establish *Batsch–Wang* type and *Strauss–Lions* type compact embedding theorems for the fractional *Sobolev* spaces. Then multiplicity results are obtained by using the *Ekeland* variational principle and the Mountain Pass Theorem.

In the very recent paper [7], *Fiscella* and *Valdinoci* provide a detailed discussion about the physical meaning underlying the fractional *Kirchhoff* problems and their applications. Indeed, they construct in the Appendix of [7] a stationary *Kirchhoff* variational problem, which models, as a special significant case, the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. In [7] the problem

$$(\mathcal{P}_\lambda)\qquad M\left([u]_{s,2}^{2}\right)(-\Delta)^{s}u=\lambda f(x,u)+|u|^{2_s^*-2}u\quad\text{in }\Omega,\qquad u=0\quad\text{in }\mathbb{R}^N\setminus\Omega,$$

was introduced for the first time in the literature. Here $\Omega\subset\mathbb{R}^N$ is a bounded domain, $2s<N$, $s\in(0,1)$, the number $2_s^*=2N/(N-2s)$ is the critical exponent of the fractional Sobolev space $H^s(\mathbb{R}^N)$, the function f is a subcritical term and satisfies the *Ambrosetti–Rabinowitz* condition, and finally λ is a positive parameter. *Fiscella* and *Valdinoci* prove in Theorem 1 of [7] the existence of a nontrivial nonnegative solution of $(\mathcal{P}_\lambda)$ for any $\lambda\geq\lambda^*$, where $\lambda^*>0$ is an appropriate threshold. They assume that the continuous *Kirchhoff* function M is also increasing in $\mathbb{R}_0^+$, with $M(0)>0$.

Finally, in the third part of the talk we comment the main results contained in [2]. The first goal of [2] is to complete the picture given in [7] and to cover in Theorem 1.1 the *degenerate* case $M(0)=0$, *without requiring any monotonicity assumption on* M, but under natural and general growth conditions on M. In Theorem 1.2 of [2] we treat the non–degenerate case of $(\mathcal{P}_\lambda)$, in other words we assume that $\inf_{t\in\mathbb{R}_0^+}M(t):=a>0$. Theorem 1.2 of [2] extends Theorem 1 of [7] in the *non–degenerate* case. Indeed, the *Kirchhoff* function

$$M(t)=(1+t)^m+(1+t)^{-1},\quad t\in\mathbb{R}_0^+,\quad m\in(0,1),$$

for which $M(0)=2$ and $a=m^{-m/(m+1)}(1+m)<2$ shows that Theorem 1.2 of [2] can be applied even when neither M is increasing in $\mathbb{R}_0^+$, nor $M(0)=a$, as required in Theorem 1 of [7].

References

[1] D. Applebaum, *Lévy processes – from probability to finance quantum groups*, Notices Amer. Math. Soc. **51** (2004), 1336–1347.

[2] G. Autuori, A. Fiscella, P. Pucci, *Stationary Kirchhoff problems involving a fractional operator and a critical nonlinearity*, submitted for publication, 24pp.

[3] G. Autuori, P. Pucci, *Existence of entire solutions for a class of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. **20** (2013), 977–1009.
[4] G. Autuori, P. Pucci, *Elliptic problems involving the fractional Laplacian in* $\mathbb{R}^N$, J. Differential Equations **255** (2013), 2340–2362.
[5] L. Caffarelli, *Nonlocal equations, drifts and games, Nonlinear Partial Differential Equations*, Abel Symposia **7** (2012), 37–52.
[6] P. Felmer, A. Quaas, J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1237–1262.
[7] A. Fiscella, E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. **94** (2014), 156–170.
[8] B. Kawohl, P. Lindqvist, *Positive eigenfunctions for the p–Laplace operator revisited*, Analysis **26** (2006), 539–544.
[9] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), 298–305.
[10] N. Laskin, *Fractional Schrödinger equations*, Phys. Rev. E **66** (2002), 056108 – 7pp.
[11] E. Lindgren, P. Lindqvist, *Fractional eigenvalues*, Calc. Var. Partial Differential Equations **49** (2014), 795–826.
[12] P. Pucci, V. Rădulescu, *Combined effects in quasilinear elliptic problmes with lack of compactness*, Rend. Lincei Mat. Appl. **22** (2011) 189–205, special volume dedicated to the memory of Prof. G. Prodi.
[13] P. Pucci, S. Saldi, *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators*, Rev. Mat. Iberoam. to appear.
[14] P. Pucci, M. Xiang, B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p–Laplacian in* $\mathbb{R}^N$, submitted for publication, pages 23.
[15] P. Pucci, M. Xiang, B. Zhang, *Existence of entire solutions for Kirchhoff type equations involving the fractional p–Laplacian*, in preparation, pages 36.
[16] P. Pucci, Q. H. Zhang, *Existence of entire solutions for a class of variable exponent elliptic equations*, J. Differential Equations **257** (2014) 1529–1566.
[17] S. Secchi, *Ground state solutions for the fractional Schrödinger in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), 031501– 17pp.

Stochastic properties of weighted Laplacians and PDEs

Alberto G. Setti
(joint work with G. Pacelli Bessa, Stefano Pigola)

We consider a weighted Riemannian manifold $M_f = (M, \langle , \rangle, f)$ where $(M, \langle , \rangle)$ is a Riemannian manifold, and $f : M \to \mathbb{R}$ is a smooth function on M, which induces a weighted measure $d\mathrm{vol}_f = e^{-f} d\mathrm{vol}$, $d\mathrm{vol}$ being the Riemannian measure of M, a Dirichlet energy form

$$E_f(u) = \int_M |\nabla u|^2 d\mathrm{vol}_f.$$

and the associated f-Laplacian $\Delta_f u = \mathrm{div}_f(\nabla u) := e^f \,\mathrm{div}\left(e^{-f}\nabla u\right) = \Delta u - \langle \nabla f, \nabla u\rangle$, which is self adjoint on $L^2(M, d\mathrm{vol}_f)$ and essentially selfadjoint on $C_c^\infty(M)$.

The asymptotic behavior of the heat kernel of a Riemannian manifold gives rise to the classical concepts of parabolicity, stochastic completeness (or conservative property) and Feller property (or C^0-diffusion property).

Both parabolicity and stochastic completeness have been the subject of a systematic study, which led to discovering sharp geometric conditions for their validity and to a rich array of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas'minskii criterion), comparison techniques etc....

Our aim is to describe a similar apparatus for the Feller property for the semigroup generated by the weighted Laplacian, to review geometric conditions that imply its validity, and to describe the consequences of the Feller property on the behavior of solutions of PDE's involving the weighted Laplacian. We say that the Feller property holds for Δ_f is the heat semigroup P_t^f generated by Δ_f maps the space C_0 of continuous functions which tend to zero at infinity into itself.

The best know geometric condition implying the validity of the Feller property for the standard Laplacian is due to E. Hsu, [5], and it is expressed in terms of Ricci curvature lower bounds. It uses a probabilistic approach that relies on a result by R. Azencott, [1], according to which M is Feller if and only if, for every compact set K and for every $t_0 > 0$, the probability that Brownian motion X_t issuing from x_0 enters K before the time t_0 tends to zero as $x_0 \to \infty$.

There are several notions of curvature in the context of weighted manifolds that play the role of the Ricci curvature for unweighted manifolds. In particular, we have the family of Bakry-Emery modified Ricci curvatures

$$\mathrm{Ric}_{f,q} = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{q} df \otimes df \quad q \in (0, \infty)$$

and the limit case for $q \to +\infty$

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\,(f)\,.$$

Imposing lower bounds on the modified Ricci curvature one has the following generalizations of Hsu's results.

Theorem 1.1. *Let M_f be a complete weighted Riemannian manifold. Then M_f is Feller provided one of the following conditions holds:*
(i) ([5], [6]) $\mathrm{Ric}_{f,q}(x) \geq -G^2(r(x))$ where G is a positive, continuous increasing function satisfying $\int^{+\infty} \frac{1}{G(r)} = +\infty$.
(ii) ([9]) $\mathrm{Ric}_f \geq -k^2$ for some constant $k \geq 0$,
(iii) ([9]) $\mathrm{Ric}_f(x) \geq -k_1^2(r(x))$, $|\nabla f|(x) \leq k_2$, where $k_i(r)$ are continuous non-decreasing functions satisfying $k_i(r) \to +\infty$ as $r \to +\infty$ and $1/\sqrt{k_1^2(t) + k_2^2(t)} \notin L^1(+\infty)$.

Note also that (i) above is precisely the condition on the Ricci curvature that ensures the stochastic completeness of M. So one may be led to believing that, as in the case of stochastic completeness, "big volumes"are an obstruction to the Feller property. This is not the case, and, in some sense, the obstruction is given by "small volumes". Indeed we have the following:

Theorem 1.2 ([1])**.** *If M is a Cartan-Hadamard manifold (complete, simply connected with nonpositive sectional curvature), then M is Feller.*

Moreover, a model manifolds $M = \mathbb{R}^m$ with metric $g = dt^2 + \rho(t)^2 d\theta^2$, is Feller if and only if either
(1)

$$(a)\,\frac{1}{\mathrm{vol}\,(\partial B_r)} \in L^1\,(+\infty) \;\text{ or }\; (b)\,\frac{1}{\mathrm{vol}\,(\partial B_r)} \notin L^1\,(+\infty) \;\text{ and }\; \frac{\mathrm{vol}\,(M \setminus \partial B_r)}{\mathrm{vol}\,(\partial B_r)} \notin L^1\,(+\infty)$$

([1], [8]). In particular, an infinite volume model manifold is always Feller.

Finally, using results by A. Grigor'yan, [4], and G. Carron, [3], one shows ([8]) that a manifold which supports an L^2 Sobolev inequality of the form of the form

$$||\nabla u||_{L^2} \geq S_{2,p}||u||_{L^{\frac{2p}{p-2}}},\ \forall u \in C^1_c(M) \tag{2}$$

is Feller. In particular minimal submanifolds of Cartan-Hadamard manifolds are Feller.

The connection between the Feller property and PDEs is given by the following equivalent characterization obtained by Azencott [1]: M_f is Feller if and only if unique the minimal solution to the exterior problem

$$\begin{cases} \Delta_f h = \lambda h \text{ on } M\backslash\bar{\Omega} \\ h = 1 \text{ on } \partial\Omega \\ h > 0 \text{ on } M\backslash\Omega, \end{cases} \tag{3}$$

which is obtained by an exhaustion procedure and satisfies $0 \leq h \leq 1$ by the maximum principle, tends to zero at infinity.

Combining this with stochastic completeness, in the equivalent form of the validity of the weak maximum principle at infinity, namely, for every function u bounded above and every $\epsilon > 0$, $\inf\{\Delta_f u(x) : u(x) > \sup u - \epsilon\} \leq 0$, one obtains

Theorem 1.3 ([2]). *Let M_f be stochastically complete, and let $u \geq 0$ be a bounded solution of*

$$\Delta_f u \geq \lambda u$$

outside a smooth domain $\Omega \subset\subset M$ If $h > 0$ is the minimal solution of (3)*, then there is a constant $c > 0$ such that*

$$u\,(x) \leq ch\,(x)\,,\ on\ M\backslash\Omega.$$

In particular, if M_f is Feller, $u(x) \to 0$ as $x \to \infty$.

This results has applications to solutions of PDEs defined in exterior domains. For example, we have the following generalization of the classical result asserting that there are no bounded minimal submanifolds in $\mathbb{R}^n$.

Theorem 1.4. *Let $(M, \langle\,,\rangle)$ be a stochastically complete and Feller Riemannian manifold. Assume that, outside a compact set $\Omega \subset M$, there exists a bounded isometric immersion $f : M\backslash\Omega \to \mathbb{B}_R\,(O) \subset \mathbb{R}^n$. Then the mean curvature $\mathbf{H}$ of f satisfies*

$$\sup_{M\backslash\Omega} |\mathbf{H}|\,R \geq 1.$$

Finally, using results of P. Pucci, M. Rigoli and J. Serrin, [7], we obtain

Theorem 1.5. *Let* $(M, \langle , \rangle)$ *be a complete and stochastically complete, Cartan-Hadamard manifold. Let* $u \geq 0$ *be a bounded solution of*

$$\Delta u \geq f(u), \text{ on } M \backslash \Omega \tag{4}$$

for some domain $\Omega \subset\subset M$ *and for some non-decreasing function* $f : [0, +\infty) \to [0, +\infty)$ *satisfying the following conditions:*

$$\text{(a) } f(0) = 0; \text{ (b) } f(t) > 0 \ \forall t > 0; \text{ (c) } \liminf_{t \to 0+} \frac{f(t)}{t^{\xi}} > 0, \tag{5}$$

for some $0 \leq \xi < 1$. *Then* u *has compact support.*

References

[1] R. Azencott, Behavior of diffusion semi-groups at infinity. Bull. Soc. Math. France **102** (1974), 193–240.

[2] G. P. Bessa, S. Pigola, A. G. Setti, Spectral and stochastic properties of the f-Laplacian, solutions of PDE's at infinity, and geometric applications. Rev. Mat. Iberoam. **29** (2013), no. 2, 579–610.

[3] G. Carron, Inégalités isopérimétriques et inégalités de Faber-Krahn. Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, 63–66.

[4] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bulletin of Amer. Math. Soc. **36** (1999) 135-249.

[5] E.-P. Hsu, Heat semigroup on a complete Riemannian manifold. Ann. Probab. **17** (1989), no. 3, 1248–1254.

[6] Li, X.-D. *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds.* J. Math. Pures Appl. **84** (2005), 1295–1361.

[7] P. Pucci, M. Rigoli, J. Serrin Qualitative properties for solutions of singular elliptic inequalities on complete manifolds. JDE **234** (2007), 507–543.

[8] S. Pigola, A.G. Setti, The Feller property on Riemannian manifolds. Jour. Funct. Anal. **262** (2012), no. 5, 2481–2515. .

[9] Qian, Z., *On conservation of probability and the Feller property.* Ann. Probab. **24** (1996), 280–292.

Cheeger constants with signatures and spectral clustering via quotient space metrics

Shiping Liu

(joint work with Carsten Lange, Norbert Peyerimhoff, Olaf Post)

We propose a Cheeger constant on a finite graph $G = (V, E)$ with a signature σ and prove the related (higher order) Cheeger inequalities [4]. In this process, we develop multi-way spectral clustering algorithms via metrics in lens spaces and complex projective spaces. Such discussions on discrete graphs help us to further establish the corresponding estimates for magnetic Laplacians on closed Riemannian manifolds.

A signature σ of G is a map $\sigma : E^{or} \to \Gamma$ from the set of oriented edges to a group Γ, such that

$$\sigma_{uv} = (\sigma_{vu})^{-1}, \quad (u, v) \in E^{or}.$$

We restrict ourselves to the case that Γ is the cyclic group S^1_k of order k, and the case that Γ is the unitary group $U(1)$. We consider the following operator Δ^σ: for any function $f : V \to \mathbb{C}$ and any vertex $u \in V$,

$$\Delta^\sigma f(u) = \frac{1}{d_u} \sum_{v, v\sim u} (f(u) - \sigma_{uv} f(v)),$$

where d_u is the vertex degree of u. This operator is an extension of the discrete 2-Laplacian. Note that when $\Gamma = U(1)$, it is the so-called discrete magnetic Laplacian. It has $N := \sharp V$ real eigenvalues

$$0 \leq \lambda_1^\sigma \leq \lambda_2^\sigma \leq \cdots \leq \lambda_N^\sigma \leq 2.$$

For any nonempty subset $S \subseteq V$, we define the frustration index $\iota^\sigma(S)$ as

$$\iota^\sigma(S) := \min_{\tau:S\to\Gamma} \sum_{\{u,v\}\in E, u,v\in S} |1 - \tau(u)^{-1}\sigma_{uv}\tau(v)|.$$

We denote

$$\phi^\sigma(S) := \frac{\iota^\sigma(S) + |E(S, \overline{S})|}{\mathrm{vol}(S)},$$

where $|E(S, \overline{S})|$ is the number of edges connecting S with its complement, and $\mathrm{vol}(S)$ is the summation of all $d_u, u \in S$.

Definition 1 ([4]). *The n-way Cheeger constant h_n^σ is defined as*

$$h_n^\sigma := \min_{\{S_i\}_{i=1}^k} \max_{1\leq i\leq k} \phi^\sigma(S_i),$$

where the minimum is taken over all possible nontrivial n-subpartitions $\{S_i\}_{i=1}^k$ of V.

It is the 2-way Cheeger constant h_2^σ that extends the classical Cheeger constant.

The above definition is motivated by a joint work of Atay and the speaker [1] about a spectral perspective of Harary's structural balance theory [2] on a signed graph, i.e. a graph G with a signature $\sigma : E^{or} \to \{+1, -1\}$. In that case, the constants $\{h_n^\sigma\}_{n=1}^N$ reduce to the so-called signed Cheeger constants [1] defined via Harary's blance theorem. In particular, the index $\iota^\sigma(S)$ reduces to twice of the minimal number of edges that need to be removed from the induced subgraphs of S to make it balanced, which is the original definition of frustration index of Harary [3]. We remark that the constants $\{h_n^\sigma\}_{n=1}^N$ unify the multi-way Cheeger [5] and dual Cheeger constants [6] on an unsigned graph.

Theorem 1 ([4]). *There exists an absolute constant $C > 0$ such that for any finite graph G with a signature σ and all $1 \leq n \leq N$, we have*

$$\frac{1}{2}\lambda_n^\sigma \leq h_n^\sigma \leq Cn^3\sqrt{\lambda_n^\sigma}. \tag{1}$$

For any nonzero function $f : V \to \mathbb{C}$, we denote $V^f(t) := \{u \in V \mid |f(u)| \geq t\}$. When $n = 1$, this estimate is essentially the following coarea inequality related to the frustration index:

$$\int_0^\infty (\iota^\sigma(V^f(\sqrt{t})) + |E(V^f(\sqrt{t}), \overline{V^f(\sqrt{t})})|)dt \leq 2 \sum_{\{u,v\}\in E} |f(u) - \sigma_{uv} f(v)|(|f(u)| + |f(v)|).$$

For the proof of the higher order Cheeger inequalities, we have to employ a multi-way spectral clustering algorithm via quotient space metrics, extending previous ideas in [6, 1]. We first map the set V to the sphere $\mathbb{S}^{2n-1}$ via the first n eigenfunctions of Δ^σ. The traditional spectral clustering algorithms then use the Riemannian metric on the sphere to cluster the vertices in order to find out useful substructures [5]. But here for our purpose, we need to use the quotient metric of $\mathbb{S}^{2n-1}/\Gamma$, where $\Gamma \subset \mathbb{C}$ acts on $\mathbb{S}^{2n-1}$ by scalar multiplication. Recall that $\mathbb{S}^{2n-1}/\Gamma$ is the lens space when $\Gamma = S^1_k$, and the complex projective space when $\Gamma = U(1)$.

On a closed Riemannian manifold, we can define the analogue Cheeger constants via a proper frustration index and prove the higher order Cheeger inequalities (analogue to the upper bound estimates of the Cheeger constants in (1)) for magnetic Laplacians [4], by observing a terminology dictionary between Harary's structural balance theory [2, 3] and the gauge invariance of magnetic potentials (see e.g. [7]).

References

[1] F. M. Atay, S. Liu, Cheeger constants, structural balance, and spectral clustering analysis for signed graphs, arXiv: 1411.3530, November 2014.

[2] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. **2** (1953), no. 2, 143-146.

[3] F. Harary, On the measurement of structural balance, Behavioral Sci. **4** (1959), 316-323.

[4] C. Lange, S. Liu, N. Peyerimhoff, O. Post, Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians, arXiv: 1502.06299, Feburary 2015.

[5] J. R. Lee, S. Oveis Gharan, L. Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, STOC'12-Proceedings of the 2012 ACM Symposium on Theory of Computing, 1117-1130, ACM, New York, 2012; J. ACM **61** (2014), no. 6, 37:1-30.

[6] S. Liu, Multi-way dual Cheeger constants and spectral bounds of graphs, Adv. Math. **268** (2015), 306-338.

[7] I. Shigekawa, Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemannian manifold, J. Funct. Anal. **75** (1987), no. 1, 92-127.

Variational problems on graphs and their continuum limits

Dejan Slepčev

(joint work with Xavier Bresson, James von Brecht, Nicolás García Trillos, Thomas Laurent)

We discuss variational problems arising in analysis of data clouds. One of the standard approaches to machine learning tasks such as clustering, classification, dimensional reduction is to introduce an objective functional which encodes the desirable properties of the object sought and then develop and implement algorithms to find a minimizer. A large class of the approaches, relevant to high-dimensional data, relies on creating a graph by connecting nearby points of the data cloud (see [5] and references therein). This allows one to explore and utilize the geometry of the data set.

An important and desirable property of an machine learning approach is its *consistency* as the number of data points available increases. To be precise consider problems for which there exists an (unknown) ground truth given by a probability measure ν, supported on a compact domain D, such that the available data points $X_n = \{x_1, \ldots, x_n\}$ are random i.i.d. samples of the measure ν. It is highly desirable if a procedure is such that if more data become available it converges to some well defined ideal object, which corresponds to full information being known. For example if one is interested in partitioning data into two clusters a consistent procedure converges to a ideal continuum partitioning of the measure ν. In other words minimizers of the discrete objective functionals describing discrete partitioning should converge to a minimizer of an objective functional describing the ideal partitioning in the continuum setting. While consistency is one of the key properties of machine learning algorithms relatively few results are available (see [1, 2, 3, 4, 6, 10], and references therein).

To address consistency questions we approach them using tools of applied analysis and calculus of variations. Namely we show Γ-convergence of the discrete functionals considered on random geometric graphs towards their continuum counterparts. Along with a compactness result this implies the desired convergence of minimizers and thus the consistency of the algorithms studied. A key element is identifying the proper topology with respect to which the Γ-convergence takes place. Let us denote by ν_n the empirical measure associated to the n data points:

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}. \tag{1}$$

The issue is then how to compare functions in $L^1(\nu_n)$ with those in $L^1(\nu)$. More generally we consider how to compare functions in $L^p(\mu)$ with those in $L^p(\theta)$ for arbitrary probability measures μ, θ on D and arbitrary $p \in [1, \infty)$. We set

$$TL^p(D) := \{(\mu, f) \ : \ \mu \in \mathcal{P}(D),\ f \in L^p(D, \mu)\},$$

where $\mathcal{P}(D)$ denotes the set of Borel probability measures on D. For (μ, f) and (ν, g) in TL^p we define the distance

$$d_{TL^p}((\mu, f), (\nu, g)) = \inf_{\pi \in \Gamma(\mu, \nu)} \left(\iint_{D \times D} |x - y|^p + |f(x) - g(y)|^p d\pi(x, y) \right)^{\frac{1}{p}}$$

where $\Gamma(\mu, \theta)$ is the set of all *couplings* (or *transportation plans*) between μ and θ.

An important consideration when investigating consistency of algorithms is how the graphs on X_n are constructed. In simple terms, when building a graph on X_n one sets a length scale ε_n such that edges between vertices in X_n are given significant weights if the distance between vertices is ε_n or less. Taking smaller ε_n is desirable because it is computationally less expensive and gives a better resolution, but there is a price. If ε_n is too small the resulting graph may not represent the geometry of D well and consequently the discrete graph cut may be very far from the desired one. We worked on determining precisely how small ε_n can be taken for the consistency to hold. More precisely consider a kernel $\eta : \mathbb{R}^d \to [0, \infty)$ to be radially symmetric and decaying to zero sufficiently fast. Let $\eta_\varepsilon(z) = \frac{1}{\varepsilon^d}\eta\left(\frac{z}{\varepsilon}\right)$. The edge weights are

$$w_{i,j} = \eta_\varepsilon(x_i - x_j). \tag{2}$$

Given a function $u_n : X_n \to \mathbb{R}$ its (appropriately scalled) *graph total variation* is defined as

$$GTV_{n,\varepsilon}(u_n) = \frac{1}{\varepsilon}\frac{1}{n^2}\sum_{i,j} w_{i,j}|u_n(x_i) - u_n(x_j)|. \tag{3}$$

The role of the perimeter of $Y \subset X_n$ on the graph is played by the graph cut, that is the sum of weights of all edges between Y and Y^c, which is nothing but (a multiple of) the graph total variation of the characteristic function of Y.

To prove consistency of machine learning approaches to clustering, the key ingredient is the variational behavior of graph total variation as $n \to \infty$. This was investigated in [7]:

Theorem [Γ-convergence and Compactness] Let $D \subset \mathbb{R}^d$, $d \geq 2$ be a domain with Lipschitz boundary. Let ν be a probability measure on D with continuous density ρ, which is bounded from below and above by positive constants. Let $x_1, \ldots, x_n, \ldots$ be a sequence of i.i.d. random points on D chosen according to measure ν. Let $\varepsilon_n \to 0$ as $n \to 0$ be such that

$$\lim_{n\to\infty} \frac{(\log n)^p}{n}\frac{1}{\varepsilon_n^d} = 0. \tag{4}$$

where $p_d = 1$ if $d \geq 3$ and $p_2 = \frac{3}{2}$. Then, GTV_{n,ε_n}, defined by (3), Γ-converges with respect to TL^1 topology to a constant (explicitly given) multiple of total variation (weighted by ρ^2) on D.

Furthermore for any sequence of functions $u_n \in L^1(D, \nu_n)$: If

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(\nu_n)} + GTV_{n,\varepsilon_n}(u_n) < \infty$$

then $\{u_n\}_{n\in\mathbb{N}}$ is TL^1-relatively compact.

In [9] this result was used to obtain strong results on consistency of graph-based clustering algorithms. Namely given a weighted graph $\mathcal{G}_n = (X_n, W_n)$ consider balanced graph cuts. For simplicity, here the attention is restricted to the two-class case and a particular balanced cut, corresponding to Cheeger cuts:

(5)

$$E_n(Y) = \frac{\mathrm{Cut}_n(Y, Y^c)}{\min(|Y|, |Y^c|)} := \frac{\sum_{x_i\in Y}\sum_{x_j\in Y^c} w_{ij}}{\min(|Y|, |Y^c|)} \quad \text{over all nonempty } Y \subsetneq X_n.$$

The continuum partitioning problem that corresponds to the discrete problem is the following: Minimize the continuum balanced cut objective functional

$$(6) \qquad E(A) = \frac{(A:D)}{\min(\nu(A), \nu(D\backslash A))}, \qquad A \subset D \ ; \ \text{with } 0 < \nu(A) < 1,$$

where $(A : D)$ is the relative perimeter of A in D, weighted by ρ^2. We show that under assumptions of the Theorem above, almost surely, the minimizers, $\{Y_n, Y_n^c\}$, of the balanced cut (5) of the graph $\mathcal{G}_n$, converge in the TL^1 sense (applied to the characteristic functions of the sets) to $\{A, A^c\}$, the minimizer of the problem (6), if such minimizer is unique. Otherwise convergence holds up along subsequences.

In addition to techniques of calculus of variations and analysis the results rely on sharp estimates on the ∞-transportation distance between the measure ν and the empirical measure ν_n of the i.i.d sample $\{x_1, \ldots, x_n\}$, [8, 11, 12].

References

[1] E. Arias-Castro and B. Pelletier. *On the convergence of maximum variance unfolding*, The Journal of Machine Learning Research, 14 (2013) 1747–1770.

[2] E. Arias-Castro, B. Pelletier, and P. Pudlo. *The normalized graph cut and Cheeger constant: from discrete to continuous*, Advances in Applied Probability, 44 (2012) 907–937.

[3] M. Belkin and P. Niyogi. *Convergence of Laplacian eigenmaps*, Advances in Neural Information Processing Systems (NIPS), (2006) 19:129.

[4] M. Hein, J.-Y. Audibert, and U. Von Luxburg. From graphs to manifolds–weak and strong pointwise consistency of graph Laplacians. In *Learning theory*, pages 470–485. Springer, 2005.

[5] U. von Luxburg. *A tutorial on spectral clustering*, Statistics and Computing, 17 (2007) 395–416.

[6] U. von Luxburg, M. Belkin, and O. Bousquet *Consistency of spectral clustering*, The Annals of Statistics, 36 (2008) 555–586.

[7] N. García Trillos and D. Slepčev. *Continuum limit of total variation on point clouds*, preprint, (2014).

[8] N. García Trillos and D. Slepčev. *On the rate of convergence of empirical measures in ∞-transportation distance*, preprint, (2014).

[9] N. García Trillos, D. Slepčev, J. Von Brecht, T. Laurent, and X. Bresson, *Consistency of Cheeger and ratio graph cuts*, preprint, (2014).

[10] D. Pollard. *Strong consistency of k-means clustering*, Annals of Statistics, 9 (1981), 135–140.

[11] P. W. Shor and J. E. Yukich. *Minimax grid matching and empirical measures*, Ann. Probab., 19 (1991), 338–1348.

[12] M. Talagrand, *The generic chaining*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

Aspects of the Feller Property on Graphs

Radosław Krzysztof Wojciechowski

Introduction. We present some aspects related to the vanishing of solutions of the heat equation at infinity for the discrete Laplacian on infinite weighted graphs. We mostly summarize the results found in [8] which follows the recent work of Pigola and Setti on Riemannian manifolds [7].

The Feller property and general criteria. Let $G = (X, b, m)$ denote an infinite, weighted, locally finite, connected graph. In particular, X is the set of *vertices*, $b : X \times X \to [0, \infty)$ is the *edge weight* and $m : X \to (0, \infty)$ is the *vertex measure*. The edge weight is symmetric, has zero diagonal and, for every $x \in X$, $b(x, y)$ is non-zero for only finitely many y. The vertex measure, which can be extended to all subsets by additivity, will play a crucial role as will be seen below. See [6] for more details on definitions.

We let $\widetilde{L}$ denote the *formal Laplacian* which acts on real-valued functions defined on vertices as

$$\widetilde{L}f(x) = \frac{1}{m(x)} \sum_{y} b(x, y)(f(x) - f(y))$$

and let L denote the minimal self-adjoint restriction of this operator to the Hilbert space $\ell^2(X, m)$. L is called the *Dirichlet Laplacian.* Under some additional conditions, such as completeness with respect to an intrinsic path metric, it can be shown that L is the only self-adjoint restriction of $\widetilde{L}$, see [4] for more details.

For $t \geq 0$, let $P_t = e^{-tL}$ denote the *heat semigroup* acting on $\ell^2(X, m)$ which can also be seen to act on $\ell^\infty(X)$. Furthermore, let $C_c(X)$ denote the space of finitely supported functions on X and $C_0(X) = \overline{C_c(X)}^{\|\cdot\|_\infty}$ denote the space of functions *vanishing at infinity.*

Definition 1.1. *G is called* Feller *if $P_t : C_0(X) \to C_0(X)$ for all $t \geq 0$.*

Note, by general principles, that it suffices to check that the *heat kernel* $p_t(x, y)$ which is connected to the semigroup via $P_t f(x) = \sum_y p_t(x, y) f(y) m(y)$ satisfies $p_t(x, \cdot) \in C_0(X)$ for all $x \in X$ and $t \geq 0$.

We now present the first general criterion which involves only the measure.

Theorem 1.2. *If $\sum_n m(x_n) = \infty$ for all sequences satisfying $x_n \to \infty$ as $n \to \infty$, then G is Feller.*

Here, $x_n \to \infty$ means that the sequence of vertices leaves every finite set, never to return. In particular, we see that the measure must decay at infinity in some direction for the graph to be non-Feller.

A second general criterion can be stated in terms of the *weighted degree* which is defined as $\operatorname{Deg}(x) = \frac{1}{m(x)} \sum_y b(x, y)$. This criterion can be seen to be a counterpart to a result of Yau [9] stating that if the Ricci curvature on a Riemannian manifold is uniformly bounded from below, then the manifold is Feller. It can be proven using standard maximum principle arguments as developed by Dodziuk in the setting of Riemannian manifolds, see [2].

Theorem 1.3. *If $\operatorname{Deg}(x) \leq K$ for all $x \in X$, then G is Feller.*

Note that the assumption above is equivalent to the boundedness of the operator L, see [5].

The weakly spherically symmetric case. For $x_0 \in X$, let $\rho(x) = d(x, x_0)$ where d denotes the combinatorial graph metric which counts the number of edges in the shortest path connecting two vertices and let $S_r(x_0) = \{x \mid \rho(x) = r\}$. We can then define the *outer* and *inner curvatures* with respect to x_0 as follows:

$$\kappa_{\pm}(x) = \frac{1}{m(x)} \sum_{\rho(y)=\rho(x)\pm 1} b(x,y).$$

That is, $\kappa_+(x)$ is the weighted outer degree of x and $\kappa_-(x)$ is the weighted inner degree of x with respect to x_0. The connection of these two quantities to the Laplacian is that $\widetilde{L}\rho(x) = \kappa_-(x) - \kappa_+(x)$.

We call a weighted graph *weakly spherically symmetric* or *model* if it contains a vertex x_0 such that $\kappa_{\pm}(x) = \kappa_{\pm}(\rho(x))$. That is, if the inner and outer curvatures are constant on $S_r(x_0)$. We call, x_0 the *root* for the model and denote quantities such as $S_r(x_0)$ by S_r.

In this case, we can characterize the Feller property by giving an analogue to results of Azencott [1] as follows. Let

$$\partial B(r) = \sum_{x \in S_r} \sum_{y \in S_{r+1}} b(x,y) = \kappa_+(r) m(S_r)$$

and let $B_r^c = \{x \mid \rho(x) > r\}$.

Theorem 1.4. *If G is model, then G is Feller if and only if either*

(1) $\sum_r \frac{1}{\partial B(r)} < \infty$
or
(2) $\sum_r \frac{1}{\partial B(r)} = \infty$ *and* $\sum_r \frac{m(B_r^c)}{\partial B(r)} = \infty$.

Note that the first condition is equivalent to the transience of models, see [3]. Therefore, all transient models are Feller. From the preceding results, one can see that a graph can be Feller for two distinct reasons: strong growth of the graph, as in the case of transient models, which forces the heat to infinity where it dissipates and slow growth, as in the case of the bounded Laplacian, which prevents the heat from getting too far.

Comparison results. We use standard maximum principle techniques to give comparison results for the Feller property. Namely, we compare a general graph to a model one as in the work of Pigola and Setti [7].

We say that a graph has *stronger curvature growth* than a model if the outer curvature is greater and the inner curvature is smaller than that of the model. More specifically, let $\widetilde{\kappa}_{\pm}$ denote the curvatures of the model $\widetilde{G}$ with root $\widetilde{x}_0$ and let $\kappa_{\pm}$ denote the curvatures on the general graph G. We say G has *stronger curvature growth* than the model $\widetilde{G}$ if there exists a vertex x_0 in G such that the curvature defined with respect to x_0 satisfy $\kappa_+(x) \geq \widetilde{\kappa}_+(r)$ and $\kappa_-(x) \leq \widetilde{\kappa}_-(r)$ for all $x \in S_r(x_0)$. We say that G has *weaker curvature growth* than $\widetilde{G}$ if the opposite inequalities hold.

Theorem 1.5. *If G has stronger curvature growth than a model graph $\widetilde{G}$ which is Feller, then G is Feller. If G has weaker curvature growth than a model graph $\widetilde{G}$ which is not Feller, then G is not Feller.*

Stability. In general, the Feller property is not very stable. For example, the result of gluing together a Feller and a non-Feller model graph at a single vertex will not be Feller as the heat kernel will not vanish on the non-Feller part and, hence, not on the entire graph. A more subtle question is if the gluing of infinitely many graphs to a non-Feller graph can produce a Feller graph. Here we only mention an example.

Example 1.6. *Let $X = \mathbb{N}_0$ with $b(x,y) = 1$ if and only if $|x-y| = 1$ and 0 otherwise. Choose m such that $m(n) \sim n^{-3}$ so that $\sum_r m(B_r^c) < \infty$ where the ball is centered at 0 so that the resulting graph is a non-Feller model.*

Now, create a new graph by attaching to each vertex $n \in \mathbb{N}_0$ a single vertex x_n. It turns out that the new graph can either be Feller or non-Feller. For example, if $b(n, x_n) \sim cn^{-1}$ for a constant $c \geq 2$ and $m(x_n) \sim n^{-2}$, then the graph is Feller. On the other hand, if $0 \leq c < 2$ and $m(x_n) \sim n^{-(2+\epsilon)}$ for any $\epsilon > 0$, then the resulting graph is not Feller.

References

[1] R. Azencott, *Behavior of diffusion semi-groups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240.

[2] J. Dodziuk, *Maximum principle for parabolic inequalities and the heat flow on open manifolds*, Indiana Univ. Math. J. **32** (1983), no. 5, 703–716,

[3] X. Huang, *Escape rate of Markov chains on infinite graphs*, J. Theoret. Probab. **27** (2014), no. 2, 634–682.

[4] X. Huang, M. Keller, J. Masamune, and R. K. Wojciechowski, *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. **265** (2013), no. 8, 1556–1578.

[5] M. Keller, D. Lenz, *Unbounded Laplacians on graphs: basic spectral properties and the heat equation*, Math. Model. Nat. Phenom. **5** (2010), no. 4, 198–224.

[6] M. Keller, D. Lenz, *Dirichlet forms and stochastic completeness of graphs and subgraphs*, J. Reine Angew. Math. **666** (2012), 189–223.

[7] S. Pigola, A. G. Setti, *The Feller property on Riemannian manifolds*, J. Funct. Anal. **262** (2012), no. 5, 2481–2515.

[8] R. K. Wojciechowski, *The Feller property for infinite graphs*, arXiv:1411.0639 [math.PR].

[9] S. T. Yau, *On the heat kernel of a complete Riemannian manifold*, J. Math. Pures Appl. (9) **57** (1978), no. 2, 191–201.

Some remarks on the 1-Laplacian and Cheeger Cut

Dong Zhang

Let $G = (V, E)$ denote an undirected and unweighted graph with vertex set $V := \{1, 2, \cdots, n\}$ and edge set E. Let A and A' be two nonempty subsets of V. We use

$$E(A, A') = \{\{i,j\} \in E : i \in A, j \in A'\}$$

to denote the set of edges between A and A'. The edge boundary of A is $\partial A := E(A, V \setminus A)$ and the volume of A is defined to be $vol(A) = \sum\limits_{i \in A} d_i$. The number

$$h(G) = \min_{S \subset V, S \notin \{\emptyset, V\}} \frac{|\partial S|}{\min\{vol(S), vol(S^c)\}}$$

is called the **Cheeger constant**, where $S^c = V\backslash S$ and $|\partial S|$ is the cardinality of the set ∂S. A partition (S, S^c) of V is called a **Cheeger cut** of G if

$$\frac{|\partial S|}{\min\{vol(S), vol(S^c)\}} = h(G).$$

An interesting result about the connectedness of each part of a Cheeger cut is as follows.

Theorem [5] If $G = (V, E)$ is connected, then the following statements hold.

- If (A, A^c) is a Cheeger cut, and A is a disjoint union of two nonempty sets A_1 and A_2 satisfying $E(A_1, A_2) = \emptyset$, then $vol(A) \le vol(A^c)$, and (A_1, A_1^c) and (A_2, A_2^c) are also Cheeger cuts with $\frac{|\partial A_i|}{vol(A_i)} = h(G), i = 1, 2$.
- If (A, A^c) is a Cheeger cut, then one of A and A^c is connected.
- There exists a Cheeger cut (A, A^c) such that A and A^c are both connected.

Now we turn to the definition of 1-Laplacian. Let B be the incidence matrix of G and $D = diag(d_1, d_2, \cdots, d_n)$ be the diagonal matrix of degrees d_i, $i = 1, 2, \cdots, n$. For each $x \in \mathbb{R}^n$, the set-valued map:

$$\Delta_1 : x \mapsto B^T Sgn(Bx),$$

is called the 1-**Laplacian** on G, where $Sgn : \mathbb{R}^n \to (2^{\mathbb{R}})^n$ is a set-valued mapping:

$$Sgn(y) = (Sgn(y_1), Sgn(y_2), \dots, Sgn(y_n)), \forall y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n,$$

and

$$Sgn(t) = \begin{cases} 1, & \text{if } t > 0, \\ -1, & \text{if } t < 0, \\ [-1, 1], & \text{if } t = 0. \end{cases}$$

Let

$$S(G) := \{x \in \mathbb{R}^n \setminus \{0\} : \exists \mu \in \mathbb{R} \ s.t.\ \Delta_1 x \cap \mu D Sgn(x) \neq \emptyset\}$$

be the set of eigenvectors of Δ_1, and K be the set of critical points of the nonsmooth Dirichlet function:

$$I(x) = \sum_{i \sim j} |x_i - x_j|,$$

under the nonsmooth constraint:

$$X := \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n d_i |x_i| = 1 \right\}.$$

The relationship between the set $S(G)$ of eigenvectors and the set K of critical points is

Theorem [1] The eigenvectors of Δ_1 are the critical points of I under the constraint X, i.e., $S(G) = K$.

The Liusternik-Schnirelmann theory is extended to study the multiplicity of the critical points for the even function $I(x)$. The notion of genus due to Krasnoselski is introduced, see for instance, [2] and [3].

Let $T \subset \mathbb{R}^n \backslash \{0\}$ be a symmetric set, i.e., $-T = T$ satisfying $0 \notin T$. The genus of T is defined to be:

$$\gamma(T) = \begin{cases} 0, & \text{if } T = \emptyset, \\ \min\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous } h : T \to S^{k-1}\}, & \text{otherwise.} \end{cases}$$

Obviously, the genus is a topological invariant.

Let us define

$$c_k = \inf_{\gamma(T) \geq k} \max_{x \in T \subset X} I(x), \quad k = 1, 2, \cdots n.$$

It can be proved that these c_k's are critical values of I under the constraint X. One has

$$c_1 \leq c_2 \leq \cdots \leq c_n,$$

and if

$$c = c_{k+1} = \cdots = c_{k+l}, \ 0 \leq k \leq k + l \leq n,$$

then $\gamma(K_c) \geq l$.

Lee [4] studied the relationship between the k-th eigenvalue of the standard Laplacian and the higher-order Cheeger constant

$$h_k = \min_{S_1, S_2, \cdots, S_k \text{ disjoint}} \max_{i=1,2,\cdots,k} \frac{|\partial S_i|}{vol(S_i)}.$$

An important result in [4] is $\frac{\lambda_k}{2} \leq h_k \leq O(k^2)\sqrt{2\lambda_k}$ for $k = 1, 2, \cdots$.

As an analogue of c_k and h_k, we have

Theorem. $c_k \leq h_k$, $k = 1, 2 \cdots, n$.

In the above theorem, the equalities $c_1 = h_1 = 0$ and $c_2 = h_2$ always hold. But for $k \geq 3$, the equality $c_k = h_k$ may not hold. A counterexample for $k = 3$ is as follows:

Example. Let K_n be the complete graph with n vertices. Then $c_3(K_5) = \frac{3}{4} < 1 = h_3(K_5)$.

At last, we provide a connectedness property of another definition of graph cut.

Theorem. For $k \in \{1, 2, 3\}$, if $A_1, \cdots, A_k$ satisfy

$$(A_1, A_2, \cdots, A_k) = argmin_{(A_1, A_2, \cdots, A_k) \in P_k} \sum_{i=1}^{k} \frac{|\partial A_i|}{|A_i|},$$

then $A_1, \cdots, A_k$ are all connected.

The above result is not true for $k \geq 4$. A counterexample for $k = 4$ is as follows:

Example. For given integers $m > 2n > 20$, we define 5 sets $V_i = \{v_{i1}, \cdots, v_{in}\}$, $i = 1, 2$, and $V_j = \{v_{j1}, \cdots, v_{jm}\}$, $j = 3, 4, 5$. Let $V = \cup_{i=1}^{5} V_i$, and

$$\begin{aligned} E = &\{\{u, v\} : u \neq v, \exists i \in \{1, 2, 3, 4, 5\} \text{ such that } \{u, v\} \subset V_i\} \cup \\ &\{\{v_{11}, v_{31}\}, \{v_{11}, v_{41}\}, \{v_{11}, v_{51}\}, \{v_{21}, v_{31}\}, \{v_{21}, v_{41}\}, \{v_{21}, v_{51}\}\}. \end{aligned}$$

Consider the connected graph $G = (V, E)$. We can easily verify that

$$(V_1 \cup V_2, V_3, V_4, V_5) = argmin_{(A_1,A_2,A_3,A_4)\in P_4} \sum_{i=1}^{4} \frac{|\partial A_i|}{|A_i|}.$$

However, $V_1 \cup V_2$ is not connected.

References

[1] K. C. Chang, Spectrum of the 1-laplacian and Cheeger's constant on graphs, arXiv: 1412.1150 [J. Graph Theor. (to be published)].

[2] K. C. Chang, Critical Point Theory and Its Applications (in Chinese), Shanghai Science and Technology Press, 1985.

[3] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications in Differential Equations, Shanghai Science and Technology Press, 1986.

[4] James R. Lee, Shayan Oveis Gharan, Luca Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, arXiv 1111.1055.

[5] K. C. Chang, Sihong Shao, Dong Zhang, The 1-Laplacian Cheeger cut: Theory and Algorithms [submitted].

Reporter: Joachim Kerner

Participants

Dr. Moritz Gerlach
FR 6.1 - Mathematics & Computer
Science
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Prof. Dr. Joachim Giesen
Fakultät für Mathematik & Informatik
Friedrich-Schiller-Universität Jena
Ernst-Abbe-Platz 2
07743 Jena
GERMANY

Dr. Sylvain Golénia
Institut de Mathématiques de Bordeaux
351, cours de la Libération
33405 Talence Cedex
FRANCE

Prof. Dr. Matthias Hein
FR 6.1 - Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
GERMANY

Prof. Dr. Bernd Kawohl
Mathematisches Institut
Universität zu Köln
50923 Köln
GERMANY

Dr. Matthias Keller
Mathematisches Institut
Universität Jena
Ernst-Abbe-Platz 2-4
07743 Jena
GERMANY

Dr. Joachim Kerner
Fachbereich Mathematik
Fernuniversität zu Hagen
58095 Hagen
GERMANY

Prof. Dr. Jaroslav Kurylev
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Daniel Lenz
Institut für Mathematik
Friedrich-Schiller-Universität Jena
07737 Jena
GERMANY

Dr. Shiping Liu
Dept. of Mathematical Sciences
Durham University
Science Laboratories
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Delio Mugnolo
Lehrgebiet Analysis
Fakultät für Mathematik & Informatik
Fernuniversität zu Hagen
58084 Hagen
GERMANY

Prof. Dr. Patrizia Pucci
Università degli Studi di Perugia
Dipto di Matematica ed Informatica
Via Vanvitelli, 1
06123 Perugia
ITALY

Prof. Dr. Alberto G. Setti
Dipartimento di Scienza e Alta
Tecnologia
Università dell'Insubria
Via Valleggio, 11
22100 Como
ITALY

Prof. Dr. Dejan Slepcev
Department of Mathematical Sciences
Carnegie Mellon University
5000 Forbes Ave.
Pittsburgh, PA 15213-3890
UNITED STATES

**Prof. Dr. Radoslaw K.
Wojciechowski**
York College, CUNY
Dept. of Mathematics & Computer
Science
94 - 20 Guy R. Brewer Blvd.
Jamaica NY 11451
UNITED STATES

Dong Zhang
Department of Mathematics
Beijing University
No. 5 Yiheyuan Rd., Haidian District
Beijing 100 871
CHINA