Mathematisches Forschungsinstitut Oberwolfach

Report No. 12/2015

DOI: 10.4171/OWR/2015/12

Control Theory: A Mathematical Perspective on Cyber-Physical Systems

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22 February – 28 February 2015

Abstract. Control theory is an interdisciplinary field that is located at the crossroads of pure and applied mathematics with systems engineering and the sciences. Recently the control field is facing new challenges motivated by application domains that involve networks of systems. Examples are interacting robots, networks of autonomous cars or the smart grid. In order to address the new challenges posed by these application disciplines, the special focus of this workshop has been on the currently very active field of Cyber-Physical Systems, which forms the underlying basis for many network control applications. A series of lectures in this workshop was devoted to give an overview on current theoretical developments in Cyber-Physical Systems, emphasizing in particular the mathematical aspects of the field. Special focus was on the dynamics and control of networks of systems, distributed optimization and formation control, fundamentals of nonlinear interconnected systems, as well as open problems in control.

Mathematics Subject Classification (2010): 93xx.

Introduction by the Organisers

Control theory is an interdisciplinary field that is located at the crossroads of pure and applied mathematics with systems engineering and the sciences. Traditionally, the interaction with systems engineering and signal processing has been particularly strong. More recently, deep interactions are emerging with new application areas, such as network sciences, robotics and information technology. The field therefore covers a wide variety of topics, ranging from fundamental mathematical

aspects and new control paradigms in the sciences to real world engineering applications of industrial relevance. In particular, it has deep connections to different branches of pure and applied mathematics, including e.g. ordinary and partial differential equations, operator theory, real and complex analysis, probability theory, numerical analysis, discrete mathematics and graph theory, as well as algebraic and differential geometry.

The Oberwolfach workshop "Control Theory: A Mathematical Perspective on Cyber-Physical Systems" brought together 56 internationally active researchers from Australia, Austria, Canada, China, Germany, Israel, Italy, Japan, The Netherlands, Russia, Sweden, Switzerland, United Kingdom, and the United States, with both a mathematical and systems engineering background. Cyber-Physical Systems (CPS) is a new field which offers an enormous potential for applications of pure and applied mathematics. Thus special focus of this workshop has been on the interaction between mathematical systems and control theory and cyber-physical systems. This was enhanced by nine survey lectures on recent developments in CPS and complemented by an open discussion session on mathematical aspects of cyber-physical systems. Topics of these lectures included the foundational aspects of cyber-physical systems, algorithmic aspects of cyber-physical networks, control of rigid formations, chemical reaction networks, distributed optimization, hybrid control synthesis for multi-agent systems, distributed randomized algorithms in social and sensor networks, and data-driven cyber-physical model estimation. To complement these survey talks by challenging mathematical and systems engineering topics, a series of lectures was devoted to the control of interconnected systems, another current research topic that is of very strong interest to the systems engineering community. In all these talks, the interaction of mathematical methods from nonlinear dynamics and control with those from discrete mathematics (esp. graph and information theory) played a crucial role. Although several fundamental mathematical questions in cyber-physical systems are still unanswered or even unasked, it became evident through the workshop that the appropriate combination of mathematical tools will be instrumental for further success in this area.

The program comprised more than 20 stimulating talks on the theory and applications of control theory. The survey talks had a length of 45 minutes with 15 minutes discussion time, while the other special topics lectures were thirty-five minutes long, with at least 5 minutes discussion time. In addition to these lectures and the very active discussions throughout the workshop there was an open discussion session on mathematical aspects of cyber-physical systems and a Tuesday evening open problem session, in which participants presented six open mathematical problems in control. On Wednesday evening a small informal meeting took place on four challenging research topics, with focus on broad mathematical issues in systems and control theory. On Thursday evening there was poster session with about ten contributions. This session was very well attended and was a great success. Snow prevented the traditional Wednesday afternoon walk to St. Roman. Thus the excursion went to Wolfach, where we visited the excellent Museum on Mathematics and Mineralogy and enjoyed the famous black forest cake in a

nearby Cafe. As an additional social event, Brian Anderson and Matthias Müller delighted the workshop participants by a performance of works by Georg Philipp Telemann and Franz Schubert for violin and piano.

Altogether the workshop must be seen as a great success, in that many new ideas and solution approaches have been stimulated for the field of control of cyber-physical systems. A special feature of this workshop was the close interaction between mathematicians and engineers that has been very fruitful. Oberwolfach workshops in the area of control theory have the reputation of being the most prestigious and worthwhile to attend meetings in this field. This is the reason why this workshop attracted the leading researchers in the field, that have been brought together with young promising junior scientists. Many of the participants, including the senior people, commented that this has been the most interesting and rewarding scientific event they ever(!) attended. Quite a remarkable statement from researchers who have witnessed hundredth of conferences and workshops in their life.

The organizers would like to thank the Mathematical Research Institute, and especially its great staff, for the opportunity to spend a most fruitful week of scientific interaction there, and for the marvelous atmosphere that is being provided. This institute is a jewel whose positive influence on mathematics and beyond cannot be emphasized enough.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Brian D.O. Anderson in the "Simons Visiting Professors" program at the MFO.

Control Theory: A Mathematical Perspective on Cyber-Physical Systems

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Abstracts

Multi agent formation operations with restricted sensing

Brian D. O. Anderson
(joint work with Bomin Jiang, Mohammad Deghat, Mengbin Ye, Changbin Yu)

Formations of mobile agents, including unmanned airborne vehicles, may often be used to localize objects in the environment. Typically, simultaneous measurements of the target?s range and/or bearing are obtained by the agents in the formation, and the measurements are then aggregated to determine the object?s location. There are often optimum formation shapes for target localization, depending on the sensing technologies. Operating formations this way requires an ability not just to control and maintain their shape, but also to move the agents in the formation with a common velocity, thereby maintaining the formation shape. In turn, this requires the agents to be able to sense the relative position (i.e. range and direction) of other agents in the formation, or at least other agents in close proximity, known as neighbors. This talk reviews algorithms for formation shape control, and for achieving velocity consensus of formation agents; in these algorithms, it is assumed that each agent has access to the range and bearing (relative position) of its neighbors in its own coordinate basis. The talk moves on to consider how these tasks can be achieved when there is limited sensing, i.e. the formation agents may be able to sense either the bearing of their neighbors, or the range of their neighbors, but not the relative position of their neighbors. We show that by superimposing limited periodic motion on top of the general formation motion, it becomes possible to dispense with much sensing and still achieve both shape control and velocity consensus.

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The Isomorphism problem for Kinematic Chains

Roger Brockett

Central question: Let the variables x_i and y_i range over the real line. Given an ordered set of n-by-n matrices $A_1, A_2, \cdots A_m$ characterize the collection of all ordered sets $B_1, B_2, \cdots B_m$ such that

$$\mathcal{I}me^{A_1x_1}e^{A_2x_2}\cdots e^{A_mx_m} = \mathcal{I}me^{B_1y_1}e^{B_2y_2}\cdots e^{B_my_m}$$

We will assume that A_i is the basis for a Lie Algebra and in that case the image of a neighborhood of $x = 0$ in $\mathbb{R}^m$ will be a neighborhood of the identity in the group. If the B_i are also a basis for the same Lie algebra the image of a neighborhood of $y = 0$ will also be a neighborhood of I in the group. Thus the issues are of a global nature. It is possible to refine this question in a number of ways, adapting it to problems relating to the study of kinematic chains. We give particular results applicable to the case where there is, or is not, a diffeomorphism f such that

$$e^{A_1x_1}e^{A_2x_2}\cdots e^{A_mx_m} = e^{B_1f_1(x)}e^{B_2f_2(x)}\cdots e^{B_mf_m(x)}$$

holds globally. In some cases of interest the matrices $A_1, A_2, \cdots A_m$ do not form a basis for the Lie algebra but do generate under bracketing a m-dimensional Lie algebra. In this case the existence of a local diffeomorphism f can be investigated rather easily. It is the global question that presents the main challenge.

Often kinematic studies consider kinematic chains with both prismatic and rotary joints but in this talk we only consider kinematic chains with rotary joints. The configuration of a rotary joint is, of course, necessarily a 2π periodic function of the angle of rotation. Thus we limit discussion to the case where the functions $e^{A_ix_i}$ are periodic of period 2π.

Lemma (The adjoint action): Let $\mathcal{G}$ be a Lie group and let $\mathcal{L}$ be the corresponding Lie algebra. Then for $G \in \mathcal{G}$ and $L \in \mathcal{L}$, $GLG^{-1} \in \mathcal{L}$.

Proof: By definition $e^{\epsilon L} \in \mathcal{G}$ and so $Ge^{\epsilon L}G^{-1} \in \mathcal{G}$ but

$$Ge^{\epsilon L}G^{-1} \approx G(I + \epsilon L)G^{-1} = I + \epsilon GLG^{-1}$$

Because $I + \epsilon GLG^{-1}$ is the first order approximation to an element of $\mathcal{G}$ it follows that GLG^{-1} must belong to $\mathcal{L}$.

The basic relationship between a Lie group and its Lie algebra is the fact the the exponential of an element of the algebra gives an element of the group. The algebra is a vector space and is therefore incorporates all the ideas of linear algebra such as the existence of a basis, linear span, etc. The exponential map provides a way to make use of some form of these ideas at the group level. The following lemma describes a relationship between canonical coordinates of the first kind, appearing here on the right, to canonical coordinates of the second kind, appearing here on the left. It is related to the Cambell- Baker-Hausdorff expansion.

Lemma: (The quadratic approximation) The product $e^{A_1x_1}e^{A_2x_2}\cdots e^{A_mx_m}$ can be expressed as

$$e^{A_1x_1}e^{A_2x_2}\cdots e^{A_mx_m} = e^{\sum A_ix_i} + \frac{1}{2}\sum_{i<j}[A_i, A_j]x_ix_j + \eta(x)$$

with η being third order in x.

Proof: Correct to second order we have

$$\Phi(x) = e^{A_1x_1}\cdots e^{A_mx_m} \approx (I + A_1x_1 + \frac{1}{2}A_1^2x_1^2)\cdots(I + A_mx_m + \frac{1}{2}A_m^2x_m^2)$$

Rearranging terms, we have, correct to second order,

$$\Phi(x) \approx I + \sum A_ix_i + \frac{1}{2}\sum_{i,j} A_iA_jx_ix_j$$

By comparison,

$$e^{\sum A_ix_i} = I + \sum_i A_ix_i + \frac{1}{2}\left(\sum_{ij} A_iA_jx_ix_i\right)^2$$

again correct to second order. Comparing these gives the formula of the lemma.

Theorem: (Standard form) Let $\{A_1, A_2, \cdots, A_m\}$ be a set of real n-by-n matrices belonging to a Lie algebra $\mathcal{L}$ and let $\mathcal{G}$ be the corresponding matrix Lie group. Assume that $T_i \in \mathcal{G}$. Then there exists $B_i \in \mathcal{L}$ and $T_B \in \mathcal{G}$ such that

$$e^{A_1x_1}T_1e^{A_2x_2}T_2\cdots e^{A_mx_m}T_m = e^{B_1x_1}e^{B_2x_x}\cdots e^{B_mx_m}T_B$$

The ordered set $\{B_1, B_2, \cdots, B_m\}$ and the group element T_B are unique provided that for $i = 1, 2, \cdots m-1$ the brackets $[B_i, B_{i+1}]$ are nonzero.

Proof: The rewriting implied by the idendity

$$e^{A_1x_1}T_1e^{A_2x_2}\cdots e^{A_mx_m}T_m = e^{A_1x_1}e^{T_1A_2T_2^{-1}}e^{T_1T_2A_3T_2^{-1}T_1^{-1}}\cdots$$

shows that it is possible to express the product on the left as a product of exponentials with the factor $\hat{T} = T_1T_2\cdots T_m$on the right. To address the uniqueness question, suppose that there are two such representations, $e^{B_1x_1}e^{B_2x_x}\cdots e^{B_mx_m}T_1 = e^{C_1x_1}e^{C_2x_x}\cdots e^{C_mx_m}T_2$. Letting all the $x_i = 0$ we see that $T_1 = T_2$. Letting all the x_is except x_a be zero we see that if $e^{B_ax_a}$ appears on the left, there must be an identical C term on the right. The ordering of the exponential factors can be fixed as follows. Using the assumption $[B_i, B_{i+1}] \neq 0$ the quadratic expansion identity shows that the order must be preserved.

We refer this reduction process as the *reduction to exponential form.*

Theorem: (Shift theorem) Let $\{A_1, A_2, \cdots, A_m\}$ be a set of real n-by-n matrices. belonging to a Lie algebra $\mathcal{L}$ and let $\mathcal{G}$ be the corresponding Lie group. Consider $\Phi(x) = e^{A_1x_1}\cdots e^{A_mx_m}$. Then there exists $\Psi(x)$ of the form $\Psi(x) = e^{B_1x_1}\cdots e^{B_mx_m}$ such that

$$\Phi(x + a) = \Psi(x)\Phi(a)$$

Moreover, for all $i = 1, 2, \cdots, m$ $B_i = G_i A_i G_i^{-1}$ for some $G_i \in \mathcal{G}$ and, in particular, $B_i = A_i$ for $i = 1$ and $i = m$.

Proof: Expand $\Phi(x + a)$ as

$$\Phi(x + a) = e^{A_1(x_1+a_1)} \cdots e^{A_m(x_m+a_m)} = e^{A_1 x_1} e^{A_1 a_1} \cdots e^{A_m x_m} e^{A_m a_m}$$

Proceeding with the reduction, described in the proof of the standard form theorem, $B_1 = A_1$, $B_2 = e^{A_1 a_1} B_2 e^{-A_1 a_1}$, etc. Thus $\Phi(x + a)$ can be expressed as $e^{B_1 x_1} \cdots e^{B_m x_m} \Phi(a)$, as required.

Example: Say $m = 4$ and observe that if $e^{A_1 a_1} e^{A_2 a_2} e^{A_3 a_3} e^{A_4 a_4} = I$ then

$$e^{A_1(x_1+a_1)} e^{A_2(x_2+a_2)} e^{A_3(x_3+a_3)} e^{A_4 a_(x_4+a_4)} =$$

$$e^{A_1 x_1}(e^{A_1 a_1} e^{A_2 x_2} e^{-A_1 a_1})(e^{A_1 a_1} e^{A_2 a_2} e^{A_3 x_3} e^{-A_2 a_2} e^{-A_1 a_1})((e^{A_1 a_1} e^{A_2 a_2} e^{A_3 a_3} e^{A_4 x_4}$$

so that

$$B_1 = A_1, B_2 = e^{A_1 a_1} A_2 e^{-A_1 a_1}, B_3 = e^{A_1 a_1} e^{A_2 a_2} A_3 e^{-A_2 a_2} e^{A_1 a_1},$$

which can also be written as

$$B_1 = A_1, B_2 = e^{A_1 a_1} A_2 e^{-A_1 a_1}, B_3 = e^{-A_4 a_4} A_3 e^{A_3 a_3}, B_4 = A_4$$

Definition: We will say that the expressions $\Phi(x) = e^{A_1 x_1} \cdots e^{A_m x_m} T_A$ and $\Psi(x) = e^{B_1 x_1} \cdots e^{B_m x_m} T_B$ are *shift equivalent* if for some $r \in \mathbb{R}^m$ the expression $e^{B_1 x_1} \cdots e^{B_m x_m} T_B$ is the reduced form of $\Phi(x + r)$. Clearly this defines an equivalence relation on expressions of the form $e^{A_1 x_1} \cdots e^{A_m x_m} T_A$.

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Dynamic coupling design for nonlinear output agreement and time-varying flow control

Claudio De Persis
(joint work with Mathias Bürger)

We consider a network of dynamical systems defined on a connected, undirected graph $\mathcal{G} = (V, E)$. Each node represents a nonlinear system

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i, w_i) \\ y_i &= h_i(x_i, w_i), \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{r_i}$ is the state, and $u_i, y_i \in \mathbb{R}^p$ are the input and output, respectively. Each system (1) is driven by the time-varying signal $w_i \in \mathbb{R}^{q_i}$, representing, e.g., a disturbance or reference. We assume that the exogenous signals w_i are generated by systems of the form

$$\dot{w}_i = s_i(w_i), \quad w_i(0) \in \mathcal{W}_i, \tag{2}$$

where $\mathcal{W}_i$ is a compact set. The dynamics of the exosystems (2) satisfy

Assumption 1. *The vector field $s_i(w_i)$ satisfies for all w_i, w_i' the inequality*

$$(w_i - w_i')^T (s_i(w_i) - s_i(w_i')) \leq 0. \tag{3}$$

Vectorizing the systems above, the overall system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, u, w) \\ y &= h(x, w) \end{aligned} \tag{4}$$

is obtained, with state space $\mathcal{W} \times \mathcal{X}$ and $\mathcal{X}$ a compact subset of $\mathbb{R}^{r_1} \times \ldots \times \mathbb{R}^{r_n}$. The control objective is to reach output agreement of all nodes in the network, independent of the exact representation of the time-varying external signals. We aim to achieve this control objective by a suitable design of dynamic couplings between any pair of neighboring nodes, i.e., on any edge of $\mathcal{G}$, a dynamical system (in the following called *"controller"*) is placed, taking the form

$$\begin{aligned} \dot{\xi}_k &= F_k(\xi_k, v_k) \\ \lambda_k &= H_k(\xi_k, v_k), \quad k = 1, 2, \ldots, m, \end{aligned} \tag{5}$$

with state $\xi_k \in \mathbb{R}^{l_k}$, input $v_k \in \mathbb{R}^p$ and output $\lambda_k \in \mathbb{R}^p$. Together, the controllers (5) give raise to the overall controller

$$\begin{aligned} \dot{\xi} &= F(\xi, v) \\ \lambda &= H(\xi, v), \end{aligned} \tag{6}$$

where $\xi \in \Xi$, a compact subset of $\mathbb{R}^{l_1} \times \ldots \times \mathbb{R}^{l_m}$. Systems (5) and (6) are interconnected via the relations

$$v = -(B^T \otimes I_p) y, \quad u = (B \otimes I_p) \lambda, \tag{7}$$

where B is the $(n \times m)$ signed incidence matrix of the graph $\mathcal{G}$.
We are now ready to formally introduce the output agreement problem.

Definition 1 (Output Agreement Problem)**.** *The* output agreement problem *is solvable for the process (4) under the interconnection relations (7) if there exists controllers (6), such that every solution* $(w(t), x(t), \xi(t))$ *originating from* $\mathcal{W} \times \mathcal{X} \times \Xi$ *is bounded and satisfies* $\lim_{t\to\infty} (B^T \otimes I_p)\, y(t) = \mathbf{0}$.

For the problem to be solvable, the following must hold:

Proposition 1. *If the output agreement problem is solvable, then, for every* w *solution to* $\dot{w} = s(w)$ *originating in* $\mathcal{W}$*, there must exist solutions* (x^w, u^w, ξ^w) *such that the equations*

$$\begin{aligned} \dot{x}^w &= f(x^w, u^w, w) \\ \mathbf{0} &= (B^T \otimes I_p) h(x^w, w) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \dot{\xi}^w &= F(\xi^w, \mathbf{0}) \\ u^w &= (B \otimes I_p) H(\xi^w, \mathbf{0}). \end{aligned} \tag{9}$$

are satisfied.

In the most general form, the existence of a feedforward controller (9) is equivalent to the constraint that there exist an integer d and maps $\tau : \mathcal{W} \to \mathbb{R}^d$, $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and $\psi : \mathbb{R}^d \to \mathbb{R}^{mp}$ satisfying

$$\begin{aligned} &\frac{\partial \tau}{\partial w} s(w) = \phi(\tau(w)) \\ &\lambda_p^w + \lambda_0^w = \psi(\tau(w)),\ \lambda_0^w \in \mathcal{N}\big(B \otimes I_p\big). \end{aligned} \tag{10}$$

To design a controller that decomposes into controllers on the edges of $\mathcal{G}$, we introduce a vector $\eta_k \in \mathbb{R}^d$ for each edge $k = 1, \ldots, m$, and denote with ψ_k the entries of the vector valued function ψ corresponding to the edge k. Each edge is now assigned a controller of the form

$$\dot{\eta}_k = \phi(\eta_k),\ \lambda_k = \psi_k(\eta_k), \quad k = 1, 2, \ldots, m. \tag{11}$$

With the stacked vector $\eta = [\eta_1^T, \ldots, \eta_m^T]^T$, and vector valued functions $\bar{\phi}(\eta) = [\phi(\eta_1)^T, \ldots, \phi(\eta_m)^T]^T$, $\bar{\psi}(\eta) = [\psi_1(\eta_1)^T, \ldots, \psi_m(\eta_m)^T]^T$, the overall controller (11) is $\dot{\eta} = \bar{\phi}(\eta)$, $\lambda = \bar{\psi}(\eta)$.
We can now introduce the following result which characterizes sufficient conditions for the solvability of the output agreement problem.

Theorem 1. *Consider the network* $\mathcal{G}$ *with dynamics on the nodes* (4)*. Suppose all exosystems satisfy* (3)*, the regulator equations* (8) *hold, and all node dynamics are incrementally passive. Consider the controllers*

$$\begin{aligned} \dot{\eta} &= \bar{\phi}(\eta, v) \\ \lambda &= \bar{\psi}(\eta) + \nu \end{aligned} \tag{12}$$

where $\bar{\phi}$ *and* $\bar{\psi}$ *are the stacked functions of* $\phi_k(\eta_k, v_k)$ *and* $\psi_k(\eta_k)$*, and* ν *is an additional input to be designed. Suppose the controllers have the internal model*

property and are incrementally passive when $\nu = \mathbf{0}$[1]. *Then, the controller* (12) *with the interconnection structure (7) and* $\nu := v = -(B^T \otimes I_p)y$ *solves the output agreement problem, that is every solution starting from* $\mathcal{W} \times \mathcal{X} \times \Xi$ *is bounded and* $\lim_{t\to+\infty}(B^T \otimes I_p)y(t) = \mathbf{0}$.

This result has a number of important implications in problems of optimal time-varying flow control ([1]), possibly in the presence of constraints ([2]), and in problems of optimal frequency regulation in power networks with time-varying voltages ([3]).

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Formal methods based planning and control of distributed hybrid systems

Dimos V. Dimarogonas

(joint work with Jana Tumova, Meng Guo, Dimitris Boskos)

Current control applications necessitate in many cases the consideration of systems with multiple interconnected components. These components/agents may need to fulfill high-level tasks at a discrete planning layer and also coupled constraints at the continuous control layer. Towards this end, the need for combined decentralized control at the continuous layer and planning at the discrete layer becomes apparent. While there are approaches that handle the problem in a top-down centralized manner, decentralized bottom-up approaches have not been pursued to the same extent. We present here some of our initial results for the problem of combined, hybrid control and task planning from high-level specifications for multi-agent systems in a bottom-up manner. In the first part, we consider a purely discrete setup where agents are assigned individual tasks in the form of Linear Temporal Logic (LTL) formulas. These are enriched with coupled constraints in the second part where a combined decentralized continuous control and task planning strategy is presented. In the last part we present some initial results on extending the necessary notion of abstractions to multi-agent systems.

The first part of the talk involves the introduction of an efficient, iterative limited horizon planning technique in the context of bottom-up control strategy synthesis for multi-agent systems from local LTL specifications. To our best knowledge, such an approach has not been taken to address the distributed multi-agent

[1]For a definition of incrementally passive systems and of the internal model property, we refer the interested reader to [1].

planning problem and its extreme computational demands before. In our solution, we repetitively build a finite discrete plan fragment, i.e., several next steps to be taken and services to be provided by the individual agents, using ideas from automata-based verification. We discuss the correctness of the solution and find assumptions, under which the proposed iterative algorithm leads to provable eventual satisfaction of the desired specifications. The solution was designed under the assumption that the agents synchronize after every discrete step, however we show that it is enough to synchronize and recompute the finite plan fragments only upon certain events. This allows the agents to execute their finite plans to a large extent independently, in an asynchronous manner. As a result, although each agent follows its finite plan, the real collective team behavior might deviate from the planned one due to different time durations of agents' discrete steps. Our algorithm is adaptive in that sense that even if the real behavior of the team is not as planned, the event-based synchronization and replanning still guarantees the satisfaction of all the missions. This feature can be especially beneficial in heterogeneous multi-robot motion and task planning problems, where individual robots traverse their common environment at different speeds.

We next tackle the multi-agent control problem under local LTL tasks from the bottom-up perspective under continuous-time constraints. Even though the local tasks are assumed here to be mutually independent, the agents within a multi-agent group are often more than a collection of stand-alone systems. Instead, they are subject to dynamic constraints with their neighboring agents and in such a case, integration of the continuous motion control with the high-level discrete network structure control is essential. Particularly, the agents are subject to relative-distance constraints that need to be satisfied at all times, which is closely related to the connectivity of the multi-agent network in robotic tasks. In particular, maintaining this connectivity is of great importance for the stability, safety and integrity of the overall team. Very often the connectivity of underlying interaction graphs is imposed by assumption rather than treated as an extra control objective. By adding this coupling constraint in our formulation, the team of agents becomes competitive as each agent has to satisfy its local task and at the same time cooperative as they have to maintain the relative distance within the team. We propose a fully decentralized and communication-free solution that is applicable, e.g., to low-cost robotic systems equipped with range and angle sensors, but without communication capabilities. This solution consists of four ingredients: an initial discrete plan synthesis algorithm, a decentralized potential-field-based motion controller with two different control modes, a switching strategy between these two control laws and finally a real-time plan adaptation algorithm. Three different cases are considered where the agents have their local task specifications given as syntactically only co-safe or general LTL formulas, or a mix of these two.

The last part of the talk focuses on distributed abstractions of the multi-agent system. In particular, in order to accomplish high-level plans, we need to specify a finite abstraction of our original system, namely a system that preserves some properties of interest of the initial system, while ignoring detail. In our framework,

we are interested in multi-agent systems and assume that the agents' dynamics consist of feedback interconnection terms, which ensure that certain system properties as for instance connectivity or (and) invariance are preserved, and free input terms, which provide the ability for motion planning under the coupled constraints. Towards this goal, we aim at quantifying admissible space-time discretizations of our system's behaviour which enable us to capture reachability properties of the original system. In these first results we provide sufficient conditions which establish that the abstraction of our original system is well posed. The latter implies that the finite transition system which serves as an abstract model of the multi-agent system has at least one outgoing transition for each discrete state.

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From Algorithms to Architectures in Cyber-Physical Networks

Magnus Egerstedt

Cyber-physical systems (CPS) are, at their core, characterized by fundamentally different models of computation. On the physical side, the Laws of Physics apply, i.e., differential equations describe the dynamics of the systems. On the cyber side, discrete models dictate the evolution of the computations. The result is a hybrid dynamic system, and, by now, a rich body of work exists for characterizing, modeling, designing, and analyzing such systems, thus providing a general model for CPS. (For a representative sample, see [5, 10] and references therein.)

However, one aspect of CPS that has not yet received the same *systematic* treatment is the fact that such systems are oftentimes interconnected, e.g., as is the case in power grids, precision agriculture infrastructure, smart building controls, and mobile sensor and communication networks, just to name a few, [1, 4, 8]. There certainly is a vast literature on networked systems in terms of coordinated controls, e.g., [3, 7, 9], but an explicit focus on what the *cyber* and what the *physical* aspects of such networks entail has been somewhat absent. The purpose of this paper is to highlight one key feature of such networks, where physical interconnections between physical nodes have to co-exist with an overlaid computational, information-exchange network, thus creating a network (or really a network of networks) that also must be characterized by different computational models. We call such networks *Cyber-Physical Networks*, or CPN, and this short paper is to be understood as a small step towards a general theory of CPN, as

opposed to a complete treatment of the subject; such a treatment does not yet exist.

A CPN is comprised of (at least) two interacting networks, G_P and G_C, where $G_P = V_P \times E_P$, with V_P being the set of physical nodes, and $E_P \subseteq V_P \times V_P$ encodes the existence of physical couplings between the nodes. The cyber-part of the network, $G_C = V_C \times E_C$, encodes the information flow among computational nodes, i.e., the edges in this graph denotes communication channels between cyber agents – as opposed to dynamical coupling terms. The way these two networks come together to form a CPN, G_{CP}, is through the coupling between cyber-nodes and physical nodes. And, there are two distinctly different ways in which these two types of nodes can interact, namely through sensing and actuation. As such, we define two more edge sets, $\tilde{E}_a \subseteq V_C \times V_P$ and $\tilde{E}_s \subseteq V_P \times V_C$, where the subscripts denote *sensing* and *actuation*, respectively. The interpretation is that cyber-node i can influence (directly) physical node j if and only if $(i,j) \in \tilde{E}_a$, while it can sense physical node j if and only if $(j,i) \in \tilde{E}_s$. The resulting CPN is obtained through the union of these constituent components, i.e.,

$$G_{CP} = (V_P \cup V_C, E_P \cup E_C \cup \tilde{E}_a \cup \tilde{E}_s).$$

Now, associate a state $x_i,\ i = 1, \ldots, N_P,\ (|V_P| = N_P)$, with each physical node and use $x_P = [x_1, \ldots, x_{N_P}]^T$ to denote the aggregate. Moreover, let $u_j,\ j = 1, \ldots, N_C,\ (|V_C| = N_C)$, be a decision variable/control signal associated with the cyber nodes, the physical constraints can be written on the form

$$\dot{x} = F(x,u),\ G(x,u) = 0.$$

But, the differential coupling constraints must respect the sparsity pattern of the underlying network, since they encode pairwise dynamic couplings, and we denote this physical sparsity pattern by

$$F \in \text{sparse}_P(G_{CP}),$$

which means that the phsyical nodes can only "affect" each other directly if they form an edge in E_P, while the decision variables can only "affect" the physical node states if they form an edge in $\tilde{E}_a$. Examples of such couplings are the Kuramoto coupled oscillator models [6] or the Bergen-Hill power exchange model [2], just to name a few.

This CPN model captures the ways in which a physical network interacts with a cyber-network through actuators and sensors. The dynamic coupling constraints as well as the physical interaction network are typically given *a priori* since the Laws of Physics are what they are, and the design task is to construct effective ways of controlling and coordinating such networks, i.e., design the cyber part. But, one can easily include other, more architectural questions, and what this means is really that we have only begun to scratch the surface of CPN, and significant work remains to be done in order to fully harness their expected utilities.

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The Beurling-Lax theorem and the control of networks of linear systems

Paul A. Fuhrmann

The purpose of this paper is to outline a, seemingly new, approach to a wide variety of optimal control problems for linear, causal, time-invariant systems. This approach has the advantages of not being restricted to finite-dimensional systems, and has extensions to optimization problems for various classes of transfer functions, including positive real and bounded real functions. The technique used is based on translation semigroups and their Fourier transforms, invariant subspaces, intertwining maps and realizations based on model operators. In the rational case, the Beurling-Lax theorem is used to derive state space formulas based on solutions to various Riccati equations. It is a pleasure to acknowledge that the circle of ideas exposed here owes much to cooperations with R. Ober, of which [7] is but one example, and one with U. Helmke, culminating in [6].

The setting in which we work is as follows: For the input and output function spaces, we take the Hilbert spaces $L^2(-\infty,\infty;\mathbb{C}^m)$ and $L^2(-\infty,\infty;\mathbb{C}^p)$ respectively. The space of past inputs is $L^2(-\infty,0;\mathbb{C}^m)$ while the space of future outputs is $L^2(0,\infty;\mathbb{C}^p)$. Denote by P_-, P_+ the orthogonal projection of $L^2(-\infty,\infty;\mathbb{C}^m)$ onto $L^2(-\infty,0;\mathbb{C}^m)$ and $L^2(0,\infty;\mathbb{C}^m)$ respectively. The external description of

our system is given by the **input/output map** $\Phi : L^2(-\infty,\infty;\mathbb{C}^m) \longrightarrow L^2(-\infty,\infty;\mathbb{C}^p)$ is given by the convolution integral

$$y(t) = (\Phi(u))(t) = \int_{-\infty}^{t} K(t-\tau)u(\tau)d\tau, \qquad t \geq 0.$$

Causality is expressed by having $K(t) = 0$ for $t < 0$ or, equivalently, by $\Phi L^2(0,\infty;\mathbb{C}^m) \subset L^2(0,\infty;\mathbb{C}^p)$. The $p \times m$ matrix function $K(t)$, which we assume to be in $L^1(-\infty,\infty)^{p\times m}$, is called the **impulse response** of the system. The **restricted input/output map** $\phi : L^2(-\infty,0;\mathbb{C}^m) \longrightarrow L^2(0,\infty;\mathbb{C}^p)$ is given by

$$y(t) = (K * u)(t) = \int_{-\infty}^{0} K(t-\tau)u(\tau)d\tau, \qquad t \geq 0.$$

In $L^2(-\infty,\infty;\mathbb{C}^m)$, we have the unitary group $\{U(t)\}_{-\infty}^{\infty}$ acting as $(U(t)f)(x) = f(x-t)$. We refer to $U(t)$ as the left translation group. The adjoint of $U(t)$ is given by the right translation group $(U(t)^*f)(x) = f(x+t)$. The orthogonal direct sum decomposition $L^2(-\infty,\infty;\mathbb{C}^m) = L^2(0,\infty;\mathbb{C}^m) \oplus L^2(-\infty,0;\mathbb{C}^m))$ transforms into $L^2(i\mathbb{R}) = H^2_- \oplus H^2_+$, where $H^2_\pm$ are the Hardy spaces of the left and right half planes. There are two ways of looking at these Hardy spaces. We can think of H^2_+ as the space of analytic functions in the open left half plane equipped with the norm $\|f\|^2 = \sup_{x>0} \int_{-\infty}^{\infty} \|f(x+it)\|^2 dt$, or as the subspace of $L^2(i\mathbb{R};\mathbb{C}^p)$ of non-tangential boundary values, which, by a theorem of Fatou, exist a.e. on the imaginary axis. Similarly for H^2_-. The analytic representation of elements of H^2_+ are given by

$$\hat{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-st}dt, \qquad \Re s > 0.$$

The translation group transforms, under the Fourier-Plancherel transform, into $(U(t)\hat{f})(i\omega) = e^{i\omega t}\hat{f}(i\omega)$. Since $\mathcal{F}(K*u)(s) = G(s)u(s)$, where $G(s) = \int_0^\infty K(t)e^{-st}dt \in H^\infty$, the input/output map transforms into the map $\Phi : L^2(i\mathbb{R},\mathbb{C}^m) \longrightarrow L^2(i\mathbb{R},\mathbb{C}^p)$, given by $\Phi f = Gf$, for $f \in L^2(i\mathbb{R},\mathbb{C}^m)$, whereas the restricted input/output map is given, in the frequency domain, by the Hankel operator $H_G : H^2_+ \longrightarrow H^2_-$, defined, for $f \in H^2_+$, by $H_G f = P_+ Gf$. Using Hardy space analogs of polynomial techniques, described and developed in [6], opens up the possibility of addressing control problems of networks of a class of infinite dimensional systems whose transfer functions are strictly noncyclic analytic functions. Functions of this class are "close" to rational functions and can be characterized by the Hankel operators induced by them having "large" kernels and "small" range, all these in a sense that can be made precise. This class of functions was studied in [2] in the scalar case and was extended in [3] to the multivariable case. Strictly noncyclic analytic functions can also be characterized by having special coprime factorizations over H^∞_+ of the form

$$G(s) = N_r(s)M_r(s)^{-1} = M_\ell(s)^{-1}N_\ell(s), \tag{1}$$

where $M_r(s), M_\ell(s) \in H^\infty_+$ are square inner functions. There are two notions of coprimeness we can use. Weak left coprimeness of $M_\ell(s), N_\ell(s)$ is defined by the

nonexistence of a nontrivial, common left inner factor, whereas strong left coprimeness of $M_\ell(s), N_\ell(s)$ is defined by the existence of $V_r(s), U_r(s) \in H_+^\infty$ solving the Bezout equation $M_\ell(s)V_r(s)+N_\ell(s)U_r(s) = I$. Similarly for right coprimeness.

Note that, by the assumption of causality, the kernel of the restricted I/O map is invariant under the translation semigroup $\{U(t)\}_0^\infty$, restricted to $L^2(0,\infty)$. In frequency domain terms, this means that $\operatorname{Ker} H_G$ is invariant under multiplication by all $e^{st}, t \geq 0$, hence invariant under multiplication by all H_+^∞ functions. Such invariant subspaces have been characterized in [1] in the discrete time case and in [8] for the continuous-time case. The Beurling-Lax theorem states that a subspace of full range $\mathcal{V} \subset (H_+^2)^m$ is H_+^∞-invariant and only if it hs a representation $\mathcal{V} = MH_+^2$, where $M(s) \in H_+^\infty$ is an inner function, that is it is analytic and contractive in the right half-plane and its nontangential limiting boundary values are unitary a.e., i.e., $M(i\omega)^* = M(i\omega)^{-1}$. Assuming weak coprimeness, the factorizations (1) are equivalent to

$$\left\{ \begin{array}{l} \operatorname{Ker} H_G = M_r H_+^2 \\ \overline{\operatorname{Im} H_G} = H_-(M_\ell^*) = H_-^2 \ominus M_\ell^* H_-^2, \end{array} \right. \tag{2}$$

with the invariant subspaces $M_r H_+^2$ and $M_\ell^* H_-^2$ having full range. However, if the factorizations (1) are strongly coprime, then the second equality in (2) is replaced by the stronger statement $\operatorname{Im} H_G = H_-(M_\ell^*) = H_-^2 \ominus M_\ell^* H_-^2$.

The coprime factorizations (1) can be rewritten as the intertwining relation $N_\ell(s)M_r(s) = M_\ell(s)N_r(s)$, and assuming the strong coprimeness conditions are satisfied, there exists a doubly unimodular embedding:

$$\left(\begin{array}{cc} V_\ell & U_\ell \\ -N_\ell & M_\ell \end{array} \right) \left(\begin{array}{cc} M_r & -U_r \\ N_r & V_r \end{array} \right) = \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array} \right). \tag{3}$$

Now, the solutions to the two Bezout equations

$$V_\ell(s)M_r(s) + U_\ell(s)N_r(s) = I, M_\ell(s)V_r(s) + N_\ell(s)U_r(s) = I, \tag{4}$$

are anything but unique. This nonuniqueness is parametrized by an H_+^∞ function $Q(s)$. In fact, if U_0, V_0 solves the second Bezout equation, then the general solution is given by

$$\left(\begin{array}{c} -U_r \\ V_r \end{array} \right) = \left(\begin{array}{c} -U_0 \\ V_0 \end{array} \right) + \left(\begin{array}{c} M_r \\ N_r \end{array} \right) Q.$$

Multiplying on the left by $\left(\begin{array}{cc} M_r^* & 0 \end{array} \right)$, we obtain $M_r^* U = M_r^* U_r - Q$. Denoting by Q_S the proper stable part of $M_r^* U_r$, and defining

$$\left(\begin{array}{c} -U_S \\ V_S \end{array} \right) := \left(\begin{array}{c} -U_r \\ V_r \end{array} \right) + \left(\begin{array}{c} M_r \\ N_r \end{array} \right) Q_S, \tag{5}$$

it follows that $R_S^* := M^* U_S$ is indeed in H_-^∞ and strictly proper. Since the Hankel operator, induced by an H_+^∞ is trivial, it follows that the Hankel operator $H_{M_r^* U}$ is independent of the particular solution of the Bezout equation, and thus depends only on the normalized coprime factorization, which is essentially unique.

Using shift based realization theory, as in [4], we obtain a, not necessarily finite dimensional, state space model. The model we take uses $H(M_\ell)$ as state space. Rather than using the infinitesimal generator A of the semigroup, we use the semigroup itself, i.e., for $f \in H(M_\ell)$, defined by $e^{At}f = P_{H(M_\ell)}e^{ts}f$. To this system, we can associate a reachability map $\mathcal{R}_{(M_\ell,N_\ell)} : H^2_+ \longrightarrow H(M_\ell)$ that has the frequency domain representation in the form

$$\mathcal{R}_{(M_\ell,N_\ell)}u = P_{H(M_\ell)}N_\ell u. \tag{6}$$

This shows that it is closely related to the Hankel operator H_G. In fact, we have $\mathcal{R}_{(M_\ell,N_\ell)} = M_\ell H_G$. The reachability map is not only linear but actually is, as is easily checked, an H^∞_+-homomorphism. It is not invertible as it has a large kernel given by $\operatorname{Ker}\mathcal{R}_{(M_\ell,N_\ell)} = M_r H^2_+$. In case we assume that M_ℓ, N_ℓ are strongly left coprime it follows that the reachability map is surjective. Using $\mathcal{R}_{(M_\ell,N_\ell)} = M_\ell H_G$, it follows that $\operatorname{Ker} H_G = \operatorname{Ker}\mathcal{R}_{(M_\ell,N_\ell)}$ and from (2), we conclude that $\operatorname{Ker}\mathcal{R}_{(M_\ell,N_\ell)} = M_r H^2_+$. The reachability map $\mathcal{R}_{(M_\ell,N_\ell)}$ induces the intertwining map $Z : H(M_r) \longrightarrow H(M_\ell)$ which is actually invertible. Using any unimodular embedding (3), the inverse of Z is computed, for $g \in H(M_\ell)$, to be

$$u_* = Z^{-1}g = P_{H(M_r)}U_r g. \tag{7}$$

Thus, $u_*(s)$ is the Fourier-Plancherel transform of the time control function that steers the system from rest in the remote past to the state g at time zero. Any other steering controller $u(s)$ has the representation $u = u_* + M_r f$, for $f \in M_r H^2_+$. The orthogonal, direct sum representation $H^2_+ = H(M_r) \oplus M_r H^2_+$ now implies, using the fact that $M_r(s)$ is inner, the following equality:

$$\|u\|^2 = \|u_0\|^2 + \|M_r f\|^2 = \|u_0\|^2 + \|f\|^2 \geq \|u_0\|^2.$$

This shows that u_* is the optimal, minimum norm, controller that steers to $g(s)$.

In the infinite dimensional case, the solvability of the Bezout equation, although existing by a theorem of Carleson, is a formidable task. However, in the rational case, the unimodular embedding of the coprime factorizations (3) can be computed by solving a homogeneous Riccati equation or, an equivalent, Lyapunov equation. This leads directly to the solution of the optimal control problem. This has been done in [7] and we quote the result. Assume that $G(s) \in H^\infty_-$ is strictly proper and has a minimal realization $G(s) = C(sI - A)^{-1}B$, with $C \in \mathbb{C}^{p\times n}, A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$, then a state space realization of the normalized right coprime factors in (3) is given by

$$\begin{pmatrix} M_r & -U_r \\ N_r & V_r \end{pmatrix} = \left(\begin{array}{c|cc} A - BB^*X & B & YC^* \\ \hline -B^*X & I & 0 \\ C & 0 & I \end{array}\right) \tag{8}$$

where X is the unique positive definite solution of the homogeneous Riccati equation $A^*X + XA = XBB^*X$, and Y is the unique positive definite solution of the homogeneous Riccati equation $AY + YA^* = YC^*CY$.

We end this outline by indicating some directions for future work. Of course, the case of systems with a strictly noncyclic, antistable transfer function $G(s) \in H^\infty_-$

is rather special. In order to address other classes of optimal control problems, one can use coprime factorizations normalized with respect to different metrics, see [7], in conjunction with associated characteristic functions to reduce the analysis to the case we handled. For characteristic functions, see [5]. Our intention is to extend the method described here to some optimal control problems, like LQG control as well as more specialized problems for the classes of positive real and bounded real functions. The assumption that a realization is strongly reachable, namely that the Hankel operator H_G has closed range, is rather restrictive. However, our hope is that by using model reduction techniques, (AAK, balanced truncation, rational approximation), in combination with unimodular embedding, one can solve the problem of steering optimally to a prescribed neighborhood of a required state.

Key words: Beurling-Lax theorem, optimal control, realization theory, Bezout equation and invertibility of intertwining maps.

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Detection of coordinated cyber attacks on state estimationin power systems

Hideaki Ishii

Cyber security is of critical importance in large-scale control systems in view of the growing roles that communication networks play in such systems. In this talk, we focus on security issues in power systems and introduce system theoretic approaches for detection of cyber attacks. The class of attacks considered are data manipulation in the measurement signals transmitted over networks.

First, we introduce the classic problem of static state estimation at the transmission grid level. The phase angles and voltage magnitudes of the buses in the grid are to be estimated based on measurements of power flows over transmission lines and power injections at the buses. Due to the large size of the problem, least squares techniques are often employed after linearization. It has however been

noticed that the bad data detection algorithms, typically relying on the residues in the estimation, are not useful if data manipulations are carried out in a coordinated manner (see, e.g., [1] and references therein).

We extend this problem setting and study the scenario of malicious attacks on the data of grid topology and/or transmission line parameters [2, 3]. Such attacks may be realized if the attacker has access to the database at the control center with sufficient knowledge of the grid. The consequence of such attacks is changes in the Jacobian matrix or the measurement function of the state estimation, which are even more difficult to handle than measurement manipulations. We approach this problem based on the robust estimation technique of least trimmed squares (LTS). Attack scenarios are outlined considering the number of attacked Jacobian elements and decomposition of the grid to maximize robustness. Stealthy attacks that stay undetected with respect to the robust LTS are also studied.

State estimation and fault detection in power systems have become active research topics in the controls area. We present our work on distributed randomized algorithms on state estimation where the estimators communicate to each other based on gossip protocols [4]. Moreover, recent advances in the sensors in power grids have motivated us to apply fault detection and identification techniques to the dynamic model of the grid [5].

We also demonstrate how cyber security brings new perspectives to the area. In multi-agent consensus problems, malicious agents which behave arbitrarily with malicious intentions to prevent other agents to form consensus can be harmful [6]; this problem has interesting ties with studies in computer science. Privacy of measurement data is another issue closely related to security. In [7], we have studied an observability problem of an ensemble of linear dynamical systems whose outputs are anonymous in the sense that they are not indexed and hence not linked to the individual systems.

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Boundary Value Filtering in Discrete Time

Arthur J. Krener

The Aegis Combat System [1] is used to protect US Navy vessels from planes and missiles. to counter short- and medium-range ballistic missiles threats of the variety typically employed by a number of potential opponent states. It works by firing projectiles to intercept the incoming missiles. Critical to the success of this system are real time accurate estimates of the trajectories of the incoming missiles. This requires filtering of the noisy measurements but what distinguishes it from standard filtering is that we know where the missiles are heading. (If they are not heading toward the ship or the target that the ship is protecting then we don't care about them.) In effect we are interested in filtering a system that has boundary constraints. Hence we consider the problem of filtering a linear system that satisfies boundary conditions rather than initial conditions.

Consider the discrete time boundary value linear system

$$x^+ = A(t)x + B(t)u, \quad y = C(t)x + D(t)w, \quad v = V^0 x(0) + V^T x(T)$$

where $t = 0, 1, 2, \ldots, T$, $x^+(t) = x(t+1)$, $x \in I\!R^{n\times 1}$, $u \in I\!R^{m\times 1}$, $y \in I\!R^{p\times 1}$, $w \in I\!R^{p\times 1}$, $v \in I\!R^{n\times 1}$ and the matrices are sized compatibly. We assume that $A(t)$ and $D(t)$ are invertible.

Such systems in continuous time were introduced in [3]. We assume that this system is well-posed, i.e., for every input sequence $u(t)$ and every vector v, there exists a unique solution to the boundary value problem.

If $u(t)$, $w(t)$ are independent standard white Gaussian noises and v is an independent Gaussian vector with mean $\hat{v}$ and covariance P then the solution $x(t)$ to (1) is the Gaussian process.. In general this process is not Markov but it is reciprocal in the sense of S. Bernstein [2], that is, if $t \in (t_1, t_2)$ and τ is not in $[t_1, t_2]$ then $x(t)$ is conditionally independent of $x(\tau)$ given $x(t_1)$, $x(t_2)$. Every Markov process is reciprocal but not vice versa. In general the measurement process $y(t)$ is neither Markov nor reciprocal. . The optimal unbiased filter for continuous time boundary value systems was presented in [4] and preliminary results for discrete time processes can be found in [5].

Consider a simple example of a double accumulator process

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w$$

where u and w are standard white Gaussian noise processes.

We can impose the deterministic boundary conditions

$$x_1(0) = 0, \qquad x_1(20) = 10$$

or deterministic initial conditions

$$x_1(0) = 0, \qquad x_2(0) = 0.5$$

Figure 1 shows ten sample paths of the x_1 coordinate of the two processes. The mean trajectories of these processes are the same, $\mu(t) = [t/2, 1/2]'$ but their distributions are very different. Which is a better model for one space coordinate

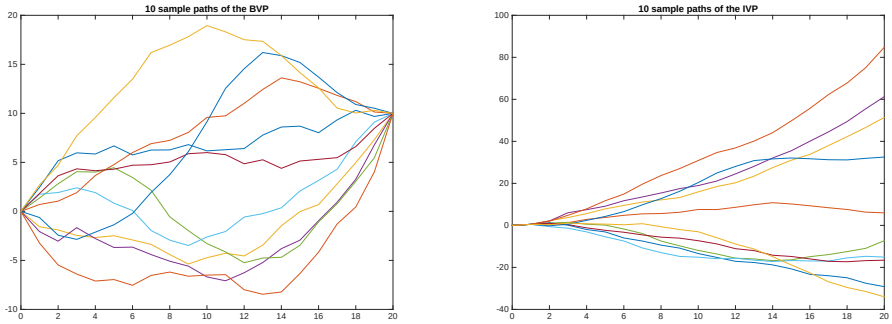

FIGURE 1. Ten Sample Paths of the Boundary Value and the Initial Value Processes

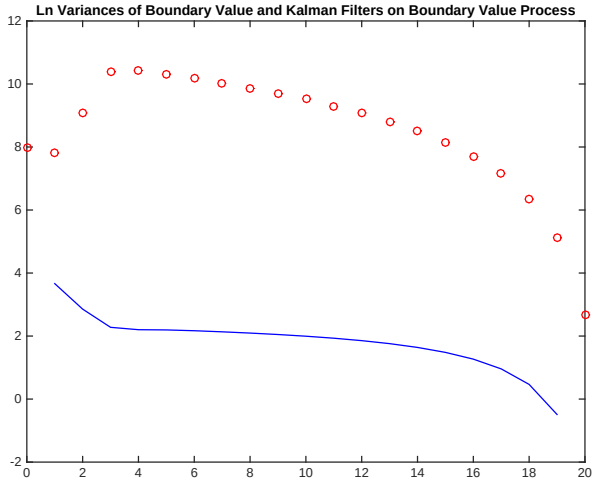

FIGURE 2. Ln Variances of the Boundary Value (blue, solid) and the Kalman (red, dotted) Filters on the Boundary Value Process

of missile that is launched from $x_1 = 0$ under the control of an intelligent adversary who wishes to attack a ship at $x_1 = 10$?

We derived the optimal unbiased filter for this boundary value process and the Kalman filter for the initial value process and compared their performance on the boundary value process. Figure 2 show the natural logarithm of their variances in estimating $x_1(t)$. As you can see the boundary value filter substantially outperformed the Kalman filter.

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Fundamental Challenges in Mechanisms and Applications of Cyber-physical Systems

Panganamala R. Kumar

The first two generations of control systems were analog control and digital control [1]. Due to advances in computing, both hardware and software, and communication, a third generation platform is emerging – cyber-physical systems [2]. This new technology integrates communication, control and computing, and poses many mathematical challenges both for operation of these systems and applications. We present three illustrative mathematical results in the hope that this will stimulate further work on important emerging problems.

First we address the problem of communication over a probabilistic environment between one central node and several peripheral nodes, with hard deadlines [3]. At every discrete time instant, one packet may be sent from the central node to some one of the peripheral nodes. Such a packet, if sent to node i, will reach its intended destination with probability p_i or fail to do so with probability (1-p_i). Packets arrive to the central node, one for each peripheral node, with period T. If such a packet is not successfully delivered to its destination within T time units, then it is dropped. Each node i requires a throughput of q_i packets per unit time of over the infinite time horizon. What tuples (p_i,q_i for $1 \leq i \leq N$), T are feasible? We provide a sharp characterization. This addresses the problem of real-time communication with deadline guarantees over an unreliable medium such as wireless.

The second problem addresses the problem of hybrid systems. These are systems consisting of interacting differential controlled equations and boolean dynamical systems. The goal is to design control laws and provably possess desired properties. As an exemplar, we consider the problem of automated transportation systems with provable safety [4]. There is a system of roadways, each road consisting of multiple lanes, and traffic intersections. Cars are modeled as unicycles. Cars can communicate with each other and with the intersection infrastructure. How should the cars be controlled so that all cars are safe in that they do not collide with each other or run off the roadway, and traverse intersections safely? We develop control laws at several levels for which it is provable that the overall system behaves correctly and safely. This theory allows for unbounded numbers of cars, each with a continuum state space, in contrast to approaches such as model checking that allow only a finite total number of states.

The third problem we consider is a provable theory of security [5]. We consider a finite number of nodes, labeled good or bad. The good nodes announce a set of rules that they follow, with the goal of forming a functioning multi-hop wireless communication network where the flows optimize a given utility function. The

bad nodes wish to disrupt both the formation of such a network as well as its operation. The bad nodes know who the good nodes are and are also capable of perfect cooperation with each other. The good nodes do not know which other nodes are good. They only start with a published protocol for their operation. This gives rise to a dynamic zero-sum game between protocols followed by the good nodes and Byzantine behaviors of the bad nodes. The fundamental result is that this problem admits a saddle point, more precisely sup-inf = inf-sup. We describe an epsilon optimal max-min protocol. This formulation and result can be contrasted with traditional approaches to secure systems that plug holes as vulnerabilities are discovered, and do not offer any provable guarantees.

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Decision making under uncertainty: Using scenarios in power systems

John Lygeros

Uncertainty is very often an important element in decision making problems for large scale cyber-physical systems. More often than not, this uncertainty is only partially characterised, for example through historical data of its past realisations. One then has to decide how to work these uncertainty samples in the decision making process. Several methods for doing this have been proposed in recent years; some have exposed interesting connections to other classes of sample-based decision making problems, for example in machine learning. This talk attempted to explore one such connection, between randomised optimisation through scenario programs and compression learning theory. It also provided an application of the resulting methods to complex optimisation problems that arise in the procurement of reserves in electrical energy transmission networks. The results presented were inspired by and draw on joint work with Kostas Margellos and Maria Prandini for the theoretical developments [1] and with Kostas Margellos, Maria Vrakopoulou, and Goran Andersson for the application to power networks [2].

Optimal decision making in the presence of uncertainty is important for the efficient and economic operation of systems affected by endogenous, or exogenous

uncertainties. One approach to deal with uncertainty is through robust optimisation. In this case a decision is made such that the constraints are satisfied for all admissible values of the uncertainty [3]. In many cases, however, we are only provided with data, e.g. historical values of the uncertainty, requiring the development of a data driven decision making paradigm. Under such a set-up, an alternative to robust optimisation is the so-called scenario based optimisation, which involves solving an optimisation problem that makes use of only a finite number of uncertainty instances, as opposed to information about the uncertainty distribution, or its support. Interestingly, even though scenario programs do not require specific assumptions on the distribution of the uncertainty, the decisions that result from their solution come with probabilistic performance guarantees that generalise their properties to unseen uncertainty instances. For problems that are convex with respect to the decision variables the so called scenario approach offers an already mature theoretical framework for analysing the properties of the optimal solution in terms of constraint satisfaction [4, 5], or optimal value [6]. In the non-convex case, tools from statistical learning theory [7] offer guarantees on the probability that any feasible solution of a scenario program satisfies the constraints of the original program [8, 9].

The talk explored the links between an area in learning theory known as compression learning [10] and scenario based optimisation. Compression learning algorithms rely on an assumption known as consistency of the learning problem. Consistency requires that, when one applies the learning algorithm to a large number of samples, the final result can be characterised through a sub-sample of fixed cardinality; the remaining samples are automatically consistent with the decision. In other words, for problems that enjoy the consistency property, if one draws a large number of samples and makes a decision based on them, they can then keep only a handful of the samples to encode the decision and throw the rest away! The catch of course is that one cannot know a-priori which samples will be contained in this essential subsample and which not. Even though drawing samples only to discard them at a later stage may seem like wasted effort it is not: These additional samples are what provides confidence about the performance of the final decision.

Our results show that many optimisation problems based on scenarios enjoy such consistency properties. The classical example is the scenario approach [4, 5]; here the role of the compression scheme is taken over by the so-called Helly dimension (that bounds the number of support constraints for the optimal decision) or variants such as the S-rank [11, 12]. However, the link we forge between compression learning and randomised optimisation offers a way to unify the treatment of a number of randomised optimisation methods (for example, enclosing set methods [13]) and can inspire the development of novel methods (for example, methods for cascades problems treated in [1]).

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Control of Rigid Formations

A. Stephen Morse

The use of the classical concept of graph rigidity [1, 2, 3] for maintaining formations of mobile autonomous agents {eg robots} was proposed in [4] more than a decade ago. Much has happened since then. We will review several results based on graph rigidity theory which overview the state of the art. We will talk briefly about certain special classes of "directed" formations for which there is a moderately complete methodology. Noteworthy among these, are findings for graphs with cycles which explain the behavior of directed triangular formations [5]. Unfortunately for graphs with cycles, existing results are quite limited. For acyclic formations, the situation is much better. We will briefly summarize several results concerned with acyclic formations in which each agent has at most two co-leaders [6, 7, 8].

Next we turn to "undirected" formations which is the main focus of this presentation. By an *undirected rigid formation* of mobile autonomous agents is meant a formation based on "graph rigidity" in which each pair of "neighboring" agents i and j is responsible for maintaining the prescribed distance d_{ij} between them. Recent research by several different groups has led to the development of an elegant potential function based theory of formation control which provides gradient laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space assuming all agents are described by kinematic point models [9]. This particular methodology is perhaps the most comprehensive currently in existence for maintaining undirected formations based on graph rigidity. The main purpose of this talk is to explain what happens if neighboring agents i and j using such gradient controls have slightly different understandings of what the desired distance d_{ij} between them is suppose to be [10, 11]. The question is relevant because no two positioning controls can be expected to move agents to precisely specified positions because of inevitable imprecision in the physical comparators used to compute the positioning errors. The question is also relevant because it is mathematically equivalent to determining what happens if neighboring agents have differing estimates of what the actual distance between them is. In either case, what one would hope for would be a gradual distortion of the formation from its target shape as discrepancies in desired or sensed distances increase. While this is observed for the gradient laws in question, something else quite unexpected happens at the same time. In this talk we will describe what occurs and explain why. The robustness issues raised here have broader implications extending well beyond formation maintenance to the entire field of distributed optimization and control. In particular, this research illustrates that when assessing the efficacy of a particular distributed algorithm, one must consider the consequences of distinct agents having slightly different understandings of what the values of shared data between them is suppose to be. For without the protection of exponential stability/convergence, it is likely that such discrepancies will cause significant misbehavior to occur.

Finally we will talk about several recent efforts to fix the robustness problem identified in [10, 11]. Among these is recent work reported in [12, 13] which tries to estimate the distance errors and then to take corrective action. This particular approach is very much in the spirit of classical parameter adaptive control.

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A Nonstochastic Theory of Information for Networked Estimation and Control

Girish Nair

In 1948, Shannon proposed a theory of information based on a probabilistic description of uncertainty [1]. In his framework, the *mutual information* shared by two random variables (rv's) X, Y, with joint probability density function (pdf) $f_{X,Y}$ and marginal pdf's f_X, f_Y, is defined as $I[X;Y] := \int f_{X,Y}(x,y) \log_2 \left(\frac{f_{X,Y}(x,y)}{f_X(x) f_Y(y)} \right) \mathrm{d}x\mathrm{d}y$ bits.

This construct possesses many natural properties, but it became important in communications theory mainly because it yielded an intrinsic characterisation of *(ordinary) channel capacity* C. The ordinary capacity is defined in operational terms, as the highest block coding rate that permits arbitrarily small probability of decoding errors, and Shannon showed in the *channel coding theorem* that this coincided with the maximum mutual information rate across the channel, maximised over all input distributions.

Probabilistic concepts of uncertainty and information are natural in communications, for several good reasons. Firstly, communication systems are largely electrical or photonic, and disturbances in these domains, such as thermal and shot noise, are well-modelled by statistical physical laws. Furthermore, in most digital communication systems, the bit periods T_b are of the order of 10^{-5}s or less, and in this regime the standard deviation of these noise terms ($\propto \sqrt{T_b}$) after lowpass or matched filtering can be noticeable compared to detected signal amplitudes ($\propto T_b$). Finally, system performance is often specified in an average or

high-probability sense, since the occasional dropped call or corrupted data packet is acceptable in most consumer applications.

In contrast, in parts of control engineering there is a long tradition of modelling disturbances as deterministic unknowns, and the justifications are equally good. Most obviously, control systems often feature mechanical or chemical components, and the dominant disturbances may not be governed by known, well-defined statistical laws. For instance in mechanical systems, the main disturbance may be vibrations at frequencies determined by machine dimensions and material properties. Even if electrical or photonic components are present, the sampling/bit periods are typically of the order of 10^{-3}s or larger, and in this regime shot and thermal noise may be negligible compared to detected signal amplitudes. Finally, for safety or mission objectives, control systems are often required to satisfy hard performance guarantees every time they are used, not just most times.

Consequently, mutual information has not been as central a concept in control as it is communications. However, both communications and control are key features in the area of cyber-physical systems. It is therefore natural to ask whether there is an operationally relevant analogue of Shannon's theory for cyber-physical systems subject to nonrandom disturbances and worst-case performance criteria.

In this talk, several pre-existing concepts are combined with new ones to yield an operationally meaningful theory of nonstochastic information that possesses several natural properties [2]. While other non-probabilistic theories of information have been proposed previously [3, 4, 5], their operational relevance for systems with noisy communication channels has not been established.

To begin, define an *uncertain variable (uv)* X to be a mapping from an underlying, uncountable space Ω to a space $\mathbb{X}$. Each $\omega \in \Omega$ may represent a specific combination of noise/input signals into a system, and X may represent a state or output variable. For a given ω, the *realisation* of X is $X(\omega) \equiv x$. This matches the textbook definition of an rv, except that no measure is imposed on Ω. Consequently any subset of Ω is a valid event, i.e. the σ-algebra is the full power-set 2^{Ω}.

As uv's have no distributions, prior uncertainty is captured by their ranges. Let the range of uv X be denoted $[\![X]\!] := \{X(\omega) : \omega \in \Omega\} \subseteq \mathbb{X}$. Similarly, $[\![X, Y]\!] \subseteq \mathbb{X} \times \mathbb{Y}$ denotes the joint range of the uv pair (X, Y), and $[\![X|y]\!]$ denotes the *conditional range* $\{X(\omega) : Y(\omega) = y, \omega \in \Omega\}$. In the absence of statistical structure, the joint range fully characterises the relationship between X & Y. Note $[\![X, Y]\!] = \bigcup_{y \in [\![Y]\!]} [\![X|y]\!] \times \{y\}$, i.e. the joint range is given by the conditional and the marginal.

Call uv's X, Y *(mutually) unrelated* if $[\![X, Y]\!] = [\![X]\!] \times [\![Y]\!]$, i.e. if the joint range is a Cartesian product of the marginals. Equivalently, $[\![X|y]\!] = [\![X]\!]$, $\forall y \in [\![Y]\!]$, i.e. conditioning does not change the marginal. Unrelatedness is a weaker condition than statistical independence. It is equivalent to X and Y inducing *qualitatively independent partitions* [6] of Ω, when Ω is finite.

Next, let X, Y, Z be said to form a *Markov uncertainty chain* $X - Y - Z$ if $[\![X|y,z]\!] = [\![X|y]\!]$, $\forall (y,z) \in [\![Y,Z]\!]$. Equivalently, $[\![X,Z|y]\!] = [\![X|y]\!] \times [\![Z|y]\!]$, $\forall y \in [\![Y]\!]$, i.e. X and Z are *conditionally unrelated given* Y.

Finally, a nonstochastic information index is constructed. Let $[\![X|Y]\!] := \{[\![X|y]\!] : y \in [\![Y]\!]\}$ be the conditional range family of X given Y. Two points $x, x' \in [\![X]\!]$ are called $[\![X|Y]\!]$-*overlap-connected* if $\exists$ a sequence of sets $\mathbb{B}_1, \ldots, \mathbb{B}_n \in [\![X|Y]\!]$ s.t. *i)* $x \in \mathbb{B}_1$ and $x' \in \mathbb{B}_n$, and *ii)* $\mathbb{B}_i \cap \mathbb{B}_{i+1} \neq \emptyset$, $\forall i \in [1 : n-1]$. It is easy to see that overlap connectedness is an equivalence relation on $[\![X]\!]$, induced by $[\![X|Y]\!]$. Let the *overlap partition* $[\![X|Y]\!]_*$ of $[\![X]\!]$ denote the equivalence classes, and define

$$I_*[X;Y] := \log_2 |[\![X|Y]\!]_*| \geq 0.$$

(Interestingly, in the definition of *topological entropy* for dynamical systems, the information gained about X from Y is instead essentially measured by the minimal subcover log-cardinality of $[\![X|Y]\!]$; this would generally yield a larger value than above.)

It can be shown that I_* is *monotonic*, i.e. $I_*[X;Y] \leq I_*[X;Y,Z]$; this is almost trivial, because the conditional range family $[\![X|Y,Z]\!]$ refines $[\![X|Y]\!]$. Less obvious properties are *symmetry*, i.e. $I_*[X;Y] = I_*[Y;X]$, and *data processing*: $I_*[X;Z] \leq I_*[X;Y]$ for any Markov uncertainty chain $X - Y - Z$.

The intuitive meaning of I_* arises because it can be shown that there is an indexing $[\![X|Y]\!]_* \equiv \{\mathbb{B}_w\}_w$ and $[\![Y|X]\!]_* \equiv \{\mathbb{C}_w\}_w$, of the overlap partitions so that $X \in \mathbb{B}_w$ iff $Y \in \mathbb{C}_w$, $\forall w$. In other words, the index $W \equiv f(X) \equiv g(Y)$ is a *common variable*, on which two agents observing X and Y separately can both agree. Furthermore, it can be shown that W is the *maximal common variable*, in the sense that if $W' \equiv f'(X) \equiv g'(Y)$ is any other common variable, then there is a mapping h such that $W' = h(W)$. In summary, $I_*[X;Y]$ is the log-cardinality of the range of the maximal variable common to both X and Y. It is worth remarking that this is not equivalent to assuming uniform distributions in $I[X;Y]$, or to extremising $I[X;Y]$ over all joint distributions with given support. Note also that the notion of a maximal common (random) variable was first proposed by Shannon himself, in a much-overlooked brief paper [7]. However, the context was probabilistic, and it was not observed that this notion could be used to define a nonstochastic information measure.

The operational relevance of I_* arises in two ways. Firstly, in analogy with the channel coding theorem, it gives an intrinsic, nonstochastic information-theoretic characterisation of the *zero-error capacity* C_0 of a discrete memoryless noisy channel. The zero-error capacity C_0 is defined operationally as the highest rate over all block codes that yield exactly zero decoding errors [8], and is typically smaller than C. In the uv framework, it can be shown that

$$C_0 = \sup_{n \geq 0, X} \frac{I_*[X_{0:n}; Y_{0:n}]}{n+1} = \lim_{n \to \infty} \sup_X \frac{I_*[X_{0:n}; Y_{0:n}]}{n+1},$$

where the supremum is over all uv sequences X going into the channel.

Secondly, by using this characterisation it can be shown that a necessary and sufficient condition to be able to uniformly estimate an unstable linear time-invariant plant via a noisy communication channel is that C_0 exceed the sum of the logarithm of the unstable eigenvalue magnitudes of the open-loop plant. This condition had been previously derived in [9], using volume-partitioning arguments and a law of large numbers that relied on a random initial state and channel. In the present work, no randomness is assumed, and the same bound is derived using nonstochastic information. See [10] for preliminary results that connect the recently introduced notion of *nonstochastic directed information* to the *zero-error feedback capacity* and the problem of uniform stabilisation over a noisy channel.

Future work will focus on handling disturbances that are bounded not in magnitude but in (time-averaged) power, and formulating a nonstochastic network information theory for three or more users.

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Distributed Optimization over Networks

Angelia Nedić

We will overview some of the distributed optimization frameworks developed over the recent years for optimizing in time-varying networks, where the network agents have limited access to information. Each agent has an objective function, which is private information not to be directly accessed by any other agents. The communications between nodes are described by a time-varying sequence of (directed) graphs, which is uniformly strongly connected over time. For such communication

networks, we will discuss distributed algorithms for solving a network problem, whereby the agents want to solve the sum of their objective functions under the limited access to the information about their individually own objectives. Several algorithmic approaches will be discussed for optimizing over undirected and directed networks in both static and time-varying scenarios, including ADMM [3, 10] and consensus-based approaches [1], [2], [4]–[7]. The efficiency of the algorithms will also be discussed in terms of their scaling with number of iterations and the size of the network. (The related list of literature is substantially abbreviated.)

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Synchronization and scaling in networks with time-delay coupling

Henk Nijmeijer

(joint work with Isaac Castanedo Guerra)

We consider networks of time-delayed diffusively coupled systems and relate conditions for synchronization of the systems in the network to the topology of the network. First we present sufficient conditions for the solutions of the time-delayed coupled systems to be bounded. Next we give conditions for local synchronization and we show that the values of the coupling strength and time-delay for which there is local synchronization in any network can be determined from such conditions for a network of two bi-directionally coupled systems[1]. In doing so, all eigenvalues of the Laplacian of the connection graph appear in the conditions for

network synchronization. In addition we present results on global synchronization in relation to the network topology for networks of a class of nonlinear systems. We illustrate our results with examples of synchronization in networks with FitzHugh-Nagumo model neurons and Hindmarsh-Rose neurons. In the second part of the paper we consider the converse problem, namely to what extend is it possible to extract conditions for synchronization of two bi-directionally time-delayed coupled cells given that a network of time-delayed coupled cells exhibits synchronization. This converse problem is potentially much harder and only for a few particular cases- that is, simply structured networks- so far an answer has been obtained. On the other hand, from a experimental point of view a solution of the problem is relevant, as in most cases slices of brain tissue containing (a few) hundred cells provide measured data[2].

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Discrete variational formulations for the optimal control and multirate integration of dynamical systems

Sina Ober-Blöbaum

Discrete variational formulations can be used to construct variational integrators. These form a special class of geometric integrators with the goal to capture the dynamical system's behavior in a most realistic way. Their use for solving optimal control problems ensures that there is no accuracy loss in the approximated optimal solution in contrast to, for example, the use of Runge-Kutta methods.

Discrete variatonal formulations. Let Q be an n-dimensional configuration manifold with tangent bundle TQ and cotangent bundle T^*Q. Consider a system with time dependent configuration $q(t) \in Q$ and velocity $\dot{q}(t) \in T_{q(t)}Q$, $t \in [0,T]$, whose dynamical behavior is described by the Lagrangian $L : TQ \to \mathbb{R}$ and a force $f : TQ \times U \to T^*Q$ that depends on a time dependent control $u(t) \in U \subseteq \mathbb{R}^m$. The equations of motion can be derived via the Lagrange-d'Alembert principle that seeks curves q with fixed initial and final values $q(0)$ and $q(T)$ satisfying

$$\delta \int_0^T L(q,\dot{q})\,dt + \int_0^T f(q,\dot{q},u)\cdot\delta q\,dt = 0 \tag{1}$$

for all variations δq. This yields the forced Euler-Lagrange equations

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) + f(q,\dot{q},u) = 0. \tag{2}$$

For a discrete variational formulation the state space TQ is replaced by $Q \times Q$ and a time step h and the discrete positions and controls $q_k \in Q$ and $u_k \in$

U, $k = 0, \dots, N$ ($Nh = T$, $N \in \mathbb{N}$) are approximations to $q(kh)$ and $u(kh)$, respectively. The two integrals in (1) are approximated by a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, $L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))\ dt$, and by discrete forces $f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t), u(t)) \cdot \delta q(t)\ dt$, where $f_k^\pm$ depend on (q_k, q_{k+1}, u_k). The discrete Lagrange-d'Alembert principle seeks discrete paths $\{q_k\}_{k=0}^N$ such that for all variations $\{\delta q_k\}_{k=0}^N$ with $\delta q_0 = \delta q_N = 0$, it is true that

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} \left(f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \right) = 0. \tag{3}$$

This results in the forced discrete Euler-Lagrange equations

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- = 0 \tag{4}$$

with $k = 1, ..., N-1$ and D_i denotes the derivative w.r.t. the i-th argument.

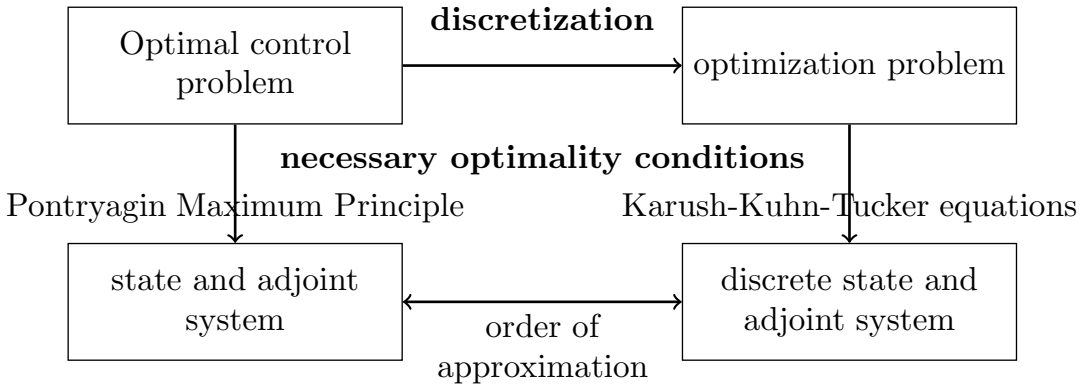
Note that (4) provides a discretization scheme for the Euler-Lagrange equations (2) which is called variational integrator [3] and belongs to the class of geometric integrators. In particular, variational integrators are symplectic and momentum consistent, i.e. the symplectic structure and the momentum maps induced by symmetries are consistent with the control forces in the discrete solution [3, 4].

Variational Integrators and Optimal Control. In optimal control, a control $u(t)$ for (2) is searched for such that a given objective functional is minimized

$$J(q, u) = \int_0^{\mathrm{T}} C(q(t), \dot{q}(t), u(t))\, \mathrm{d}t + \Phi(q(\mathrm{T}), \dot{q}(\mathrm{T})) \tag{5}$$

with $C : TQ \times U \to \mathbb{R}$ and $\Phi : TQ \to \mathbb{R}$ being continuously differentiable. For the numerical solution of optimal control problems, direct methods are based on a discretization of the differential equations (2) which serve as equality constraints for the resulting finite dimensional nonlinear optimization problem. It is well known that discretization and optimization (i.e. the fulfillment of necessary optimality conditions, see diagram) do not commute in general. In particular for Runge-Kutta discretizations, the approximation order of the optimal control solution, which depends on the approximation order of the state and the adjoint scheme, is reduced compared to the order of the Runge-Kutta method applied to the state system [1]. However, using special classes of variational integrators for the discretization of the state equation yields the same discrete variational scheme for the adjoint equation due to its symplecticity and thus guarantees the same approximation order for the optimal control solution [4]. That means that discretization and optimization commute for this class of symplectic schemes. For general classes of variational integrators, the commutation property is still an open question.

In recent years, much effort has been put into the analysis and the further development of variational integrators to make them applicable to a broad class of dynamical systems, including systems with dynamics on different time scales resulting in variational multirate integrators [2].



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Data-Driven Cyber-Physical Model Estimation from Observed Equilibria

Ioannis Ch. Paschalidis

(joint work with Dimitris Bertsimas, Vishal Gupta, Qi Zhao, Arion Stettner, Daniel Segrè)

Equilibrium modeling is common in a variety of fields such as game theory, transportation science, and systems biology. The inputs for these models, however, are often difficult to estimate, while their outputs, i.e., the equilibria they are meant to describe, are often directly observable. By combining ideas from *inverse optimization* with the theory of *variational inequalities*, we develop an efficient, data-driven technique for estimating the parameters of these models from observed equilibria.

In essence, we propose to leverage a cyber infrastructure associated with such cyber-physical systems which enables measuring equilibrium quantities and using these measurements to estimate unknown user preferences. These can in turn be used to alter system parameters and steer the system to a more desirable equilibrium.

A distinguishing feature of our approach is that it supports both *parametric* and *nonparametric estimation* by leveraging ideas from statistical learning (kernel methods and regularization operators). Specific applications we described include: (*i*) estimating the utility functions of players in a game (Nash Equilibrium), (*ii*) estimating the unknown demand or congestion function in a transportation network

(Wardrop equilibrium), and (*iii*) estimating the cellular objective in a bacterial metabolic network.

To define the problem more formally, consider a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and a non-empty set $\mathcal{F} \subseteq \mathbb{R}^n$. The *Variational Inequality* problem, denoted VI$(\mathbf{f}, \mathcal{F})$, is to find an $\mathbf{x}^* \in \mathcal{F}$ such that

$$\mathbf{f}(\mathbf{x}^*)'(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathcal{F},$$

where prime denotes transpose. Assume now that $\mathcal{F}$ can be represented as the intersection of a small number of conic inequalities, that is, $\mathcal{F} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathcal{C}\}$, for some cone $\mathcal{C}$, and that $\mathcal{F}$ has at least one interior point (Slater-condition).

The *inverse variational inequality* problem can now be formulated as follows. We are given observations $(\mathbf{x}_j, \mathbf{A}_j, \mathbf{b}_j, \mathcal{C}_j)$ for $j = 1, \ldots, N$, which give rise to sets $\mathcal{F}_j = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_j\mathbf{x} = \mathbf{b}_j, \mathbf{x} \in \mathcal{C}_j\}$. We seek a function $\mathbf{f}$ such that $\mathbf{x}_j$ is an approximate solution to VI$(\mathbf{f}, \mathbf{A}_j, \mathbf{b}_j, \mathcal{C}_j)$ for each j, namely,

$$\mathbf{f}(\mathbf{x}_j)'(\mathbf{x} - \mathbf{x}_j) \geq -\epsilon_j, \qquad \forall \mathbf{x} \in \mathcal{F}_j, \ \forall j.$$

Letting $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)$, we can formulate this inverse VI problem as

$$\begin{aligned} \min_{\mathbf{f}, \boldsymbol{\epsilon}} \quad & \|\boldsymbol{\epsilon}\| \\ \text{s.t.} \quad & \mathbf{x}_j \text{ is an } \epsilon_j\text{-approximate solution to VI}(\mathbf{f}, \mathbf{A}_j, \mathbf{b}_j, \mathcal{C}_j), \quad j = 1, \ldots, N. \end{aligned}$$

We have developed *parametric* and *non-parametric* methods for estimating $\mathbf{f}$. In the parametric case, we assume that $\mathbf{f}$ belongs to a specific parametric family and solve the problem above to find the parameters. It turns out that the corresponding optimization problem is conic, hence, efficiently solved. In the non-parametric case, we assume $\mathbf{f}$ belongs to a *Reproducing Kernel Hilbert Space (RKHS)* and solve a problem that minimizes the norm of $\mathbf{f}$ subject to a norm-constraint on $\boldsymbol{\epsilon}$. We show that the corresponding problem reduces to a quadratic optimization problem and can be solved for any positive semi-definite kernel of the RKHS.

We discussed three specific applications of this framework. In the first application we are given a directed network of nodes and arcs $(\mathcal{V}, \mathcal{A})$, representing the road network of some city. Let $\mathbf{N} \in \{0, 1\}^{|\mathcal{V}| \times |\mathcal{A}|}$ be the node-arc incidence matrix. For certain pairs of nodes $\mathbf{w} = (w_s, w_t) \in \mathcal{W}$, we are also given an amount of flow $d^{\mathbf{w}}$ that must flow from w_s to w_t. The pair $\mathbf{w}$ is referred to as an origin-destination pair. Let $\mathbf{d}^{\mathbf{w}} \in \mathbb{R}^{|V|}$ be the vector which is all zeros, except for a $(-d^{\mathbf{w}})$ in the coordinate corresponding to node w_s and a $(d^{\mathbf{w}})$ in the coordinate corresponding to node w_t. We say that a vector of flows $\mathbf{x} \in \mathbb{R}_+^{|\mathcal{A}|}$ is feasible if $\mathbf{x} \in \mathcal{F}$ where $\mathcal{F} = \{\mathbf{x} : \exists \mathbf{x}^{\mathbf{w}} \in \mathbb{R}_+^{|\mathcal{A}|} \text{ s.t. } \mathbf{x} = \sum_{\mathbf{w} \in \mathcal{W}} \mathbf{x}^{\mathbf{w}}, \ \mathbf{N}\mathbf{x}^{\mathbf{w}} = \mathbf{d}^{\mathbf{w}}, \ \forall \mathbf{w} \in \mathcal{W}\}$. Let now $c_a : \mathbb{R}_+^{|\mathcal{A}|} \to \mathbb{R}_+$ be some "cost" function for arc $a \in \mathcal{A}$. Denote by $\mathbf{c}(\cdot)$ the vector-valued function whose a-th component is $c_a(\cdot)$. A feasible flow $\mathbf{x}^*$ is a *Wardrop equilibrium* if for every origin-destination pair $\mathbf{w} \in W$, and any path connecting (w_s, w_t) with positive flow in $\mathbf{x}^*$, the cost of traveling along that path is less than or equal to the cost of traveling along any other path that connects (w_s, w_t). It is well-known that a Wardrop equilibrium is a solution to VI$(\mathbf{c}, \mathcal{F})$ and we can use our inverse VI setting to estimate congestion functions from observed equilibria.

A second application focused on estimating the utility functions of players in a game (Nash Equilibrium). The third application considered the "reverse engineering" of bacterial metabolic networks. A *metabolic network* is used to describe the process of thousands of enzymatic reactions used to convert nutrients into metabolites and energy. Organisms have different optimal performances (e.g., maximizing growth rate, or ATP generation) under a range of growth conditions. One of the most important methodologies to analyze the metabolic network in steady-state is *Flux Balance Analysis (FBA)* [4]. FBA formalizes the system of equations describing a metabolic network as the dot product of a matrix of the stoichiometric coefficients and the vector of the unsolved fluxes. The optimal performances under different growth conditions are defined as the objective functions in the FBA formulation. One can write this problem as a VI problem. Solving the inverse VI problem allows us to leverage experimental cellular flux measurements to infer the cell's objective function – an important problem in metabolic engineering.

To the best of our knowledge, we are the first to consider *inverse variational inequality* problems. Previous work has examined the problem of estimating parameters for systems assumed to be in equilibrium, most notably the structural estimation literature in econometrics and operations management ([5], [2], [1]). These techniques, however, assume a detailed parametric model for the system in question and often require restrictive probabilistic assumptions on random quantities while leading to non-convex optimization problems for parameter estimation.

By contrast, in our paradigm, we make no assumptions (parametric or nonparametric) about the true mechanics of the system; we treat it is as a "black-box." Our objective is to fit a model – in fact, a VI – that can be used to predict the behavior of the system. We make no claim that this fitted model accurately reflects "reality," merely that it has good predictive power.

In addition to tractable formulations of the inverse VI problem, we also provided generalization probabilistic guarantees (sample complexity results) on the quality of the estimated function $\mathbf{f}$ and the quality of the equilibrium predicted by using this estimated function. A full paper version of the inverse VI theory and discussion of some of the applications can be found in [3].

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Complex stochastic systems admitting a flocking structure

Giorgio Picci
(joint work with Giulio Bottegal)

We propose a new modeling paradigm for large dimensional aggregates of stochastic systems by Generalized Factor Analysis (GFA) models. These models describe the data as the sum of a *flocking* plus an uncorrelated *idiosyncratic* component. The flocking component describes a sort of collective rigid motion which admits a much simpler mathematical description than the whole ensemble while the idiosyncratic component describes weakly correlated noise. The extraction of the dynamic flocking component can be achieved by an operation of space averaging on the whole ensemble which filters out the idiosyncratic component.

Consider an infinite aggregate of random "agents" indexed by a discrete space variable $k \in \mathbb{Z}_+$ each described by a scalar output variable[1] $\mathbf{y}(k,t)$, which evolves randomly in (discrete) time. The overall evolution of the ensemble is then described by an infinite dimensional random process $\mathbf{y} := \{\mathbf{y}(t)\,;\, t \in \mathbb{Z}\}$ with components $\mathbf{y}(k,t)$, an infinite column vector of zero mean random variables of finite variance. We shall assume that the infinite covariance matrix,

$$\Sigma(\tau) := \mathbb{E}\,\mathbf{y}(t+\tau)\mathbf{y}(t)^\top$$

is well-defined, independent of t and of positive type. We shall call $\mathbf{y}$ a *time-stationary random field*. Let $F \in \mathbb{R}^{\infty\times q}$; we shall say that the q columns of F are *strongly linearly independent* if the $n \times n, (n \geq q)$ upper left corner of $FF^\top$ has q nonzero eigenvalues which tend to infinity as $n \to \infty$. This concept is introduced in [1] and cannot be discussed further here for reasons of space.

Definition 1. *A time-stationary random field has a* dynamic GFA representation *of rank q if it can be written as*

$$\mathbf{y}(t) = F\mathbf{x}(t) + \tilde{\mathbf{y}}(t) \tag{1}$$

where the q columns of F are strongly linearly independent, the q dimensional process $\mathbf{x}(t)$, with $\mathbb{E}\mathbf{x}(t)\mathbf{x}(t)^\top = I_q$, is uncorrelated and jointly (weakly) stationary with $\tilde{\mathbf{y}}(t)$ and the covariance matrix $\tilde{\Sigma}(\tau) := \mathbb{E}\tilde{\mathbf{y}}(t+\tau)\tilde{\mathbf{y}}(t)^\top$ is, for all τ, a bounded linear operator in ℓ^2.

Similarly, we shall say that an infinite covariance matrix function $\Sigma(\tau)$ has a GFA decomposition of rank q if it can be decomposed as

$$\Sigma(\tau) = FP(\tau)F^\top + \tilde{\Sigma}(\tau) \tag{2}$$

where $F \in \mathbb{R}^{\infty\times q}$ has strongly linearly independent columns, $P(\tau)$ is a $q \times q$ covariance matrix normalized so that $P(0) = I_q$ and $\tilde{\Sigma}(\tau)$ is, for all τ, a bounded operator in ℓ^2.

The overall covariance of the observed process $\mathbf{y}$ can then be decomposed in the sum of two contributions.

[1]This assumption is done for ease of notation; finite dimensional output variables can be treated in the same way.

- A *long range* correlation structure which describes the component of $\mathbf{y}$ driven by the latent vector $\mathbf{x}(t)$; this is the flocking component. The *long range* property means that the covariance of two variables $\hat{\mathbf{y}}(k)$ and $\hat{\mathbf{y}}(j)$, say $\hat{\sigma}(k,j)$ does not go to zero when $|k-j| \to \infty$.
- A *short range* correlation structure which corresponds to the idiosyncratic component $\tilde{\mathbf{y}}$. The *short range* property means that the covarinace of any two variables $\mathbf{y}(k)$ and $\mathbf{y}(j)$, say $\tilde{\sigma}(k,j) \to 0$ when $|k-j| \to \infty$.

We discuss this decomposition for linear dynamic systems restricting to the case of processes which are stationary with respect to the time variable which is a natural assumption to make in view of statistical inference.

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Disagreement dynamics within co-operative multi-agent control systems

Roy S. Smith

Implementing co-operative multi-agent systems in practice requires careful analysis of the information belonging to individual agents. If multiple agents' actions depend on common variables (physical or otherwise), one must consider how these variables are obtained for each agent. In practice this is usually via estimators and agents' estimates of identical variables can no longer be assumed to be identical. One must also consider the disagreement between the agents' estimators. These disagreements will also have dynamics which can exhibit behaviours not present in the nominal, zero estimation error, system. Graph theory is used to derive the disagreement dynamics for a high performance co-operative control system. This gives insights into stability, performance, and communication reliability as well as the relationship between disagreement and communication topology.

To illustrate how such dynamics arise we consider a formation of N agents, each capable of controlling its own location. In problem domains where the precision of the formation is critical—for example, space-based formations for science instruments—the optimal control will be a function of the positions and velocities of all of the agents. Once a control algorithm has been specified, implementation requires that every agent has knowledge of the positions and velocities of all other agents. Such an assumption is reasonable for small formations. If we assume that every agent is capable of measuring and estimating the positions and velocities of all of the others, then one can easily implement these estimators in parallel and use a certainty equivalent approach to implement the optimal control.

This control and estimation structure contains disagreement dynamics and this can be seen by studying the simplified case where every agent can measure and estimate, via a Kalman filter, the full system dynamic state. In [1] we show that the closed-loop eigenvalues are the union of the eigenvalues of: a) the designed optimal closed-loop system; b) the designed Kalman estimation error eigenvalues; and c) $N-1$ copies of the open-loop controller dynamics. These $N-1$ copies of the controller open-loop are in fact disagreement dynamics; each agents' measurements have different noise corruption leading to differences between the estimates of ostensibly identical quantities in each agent. The eigenvalues corresponding to the differences in agents' estimates are those of the open-loop controllers, which are not explicitly accounted for in most estimator/state-feedback design methods.

Explicitly handling such disagreement requires communication between the agents. We model the communication network via a directed graph and give an expression relating the eigenvalues of the graph Laplacian to the disagreement eigenvalues of the parallel estimator configuration. It immediately follows that one cannot remove the disagreement unless the communication topology contains a rooted spanning tree. This has the implication that communication receivers are critical to the control of disagreement in cooperative control networks.

In [2] we provide a design formulation that enables one to pose an optimization-based design problem that explicitly accounts for the disagreement dynamics as well as the closed-loop system dynamics. This gives a way of designing the receiver and transmitter gains, as well as the state feedback gains and Kalman estimator gains for the complete system. The formulation also gives mean-square stability results for the case where communication links suffer from packet loss. Both simple probabilistic models as well as Markov chain driven models (Gilbert communication models) can be used to describe the packet loss.

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Randomization and Gossiping in Techno-Social Networks

Roberto Tempo

Abstract

In this study, we consider two specific applications: opinion formation in social networks and centrality measures in technological networks. These applications fall under a general framework which aims at the construction of algorithms for distributed computation over networks. The key ingredients of randomization and

time-averaging are exploited, together with a local gossiping communication protocol, to obtain convergence of these distributed algorithms to the global synchronous dynamics. The connections with the mathematics of cyberphysical systems are also outlined.

Opinion Formation and Centrality Measures

We consider opinion formation in social networks and centrality measures in complex networks, with specific attention to the PageRank computation in the Google search engine. Both applications fall under a general framework we have studied in [1], which has the objective to provide ergodicity properties of distributed algorithms based on a combination of randomization and time-averaging techniques.

Recently, randomization has been shown to be a very effective tool when dealing with control of uncertain systems [2]. In the context of networks, randomization has the goal to improve the overall performance of the system. This objective is achieved, for example, when designing distributed and asynchronous algorithms. On the other hand, time-averaging techniques have been widely used to improve the convergence speed of stochastic approximation algorithms. In this study, we consider network dynamics where the nodes interact in randomly chosen pairs, following a so-called gossip protocol. This protocol is particularly natural when dealing with social networks, where agents may discuss various topics in pairs. The combination of randomization, time-averaging and gossiping techniques, in a suitable way and in a unified setting, provides iterative algorithms which enjoy asymptotic convergence properties in mean-square and almost sure sense.

The first application we consider arises in social sciences and it is focused on the underlying mechanisms of opinion formation, which play a key role in many other areas such as economy, finance, biology and epidemiology. The model we consider, denoted as the Friedkin-Johnsen model [3], is based on the concept of stubborn agents, which may lead to a disagreement of opinions. This model is an alternative to other models available in the literature, where, on the contrary, the objective is to reach a consensus of opinions between several individuals [4].

In particular, in this study we consider pairwise randomized dynamics, which represent the interactions between individuals. At each time instant, a randomly chosen pair of agents update its opinion as a convex combination of its own opinion, the opinion of one of its neighbors (randomly selected), and the so-called prejudice (which is the a priori opinion that an individual has). We show that, even though the resulting dynamics oscillates, its average is a stable opinion profile. However, this opinion profile is not a consensus of opinions, but the opinions of individuals may aggregate into clusters.

The second application we analyze deals with the computation of centrality measures. In particular, we discuss various classical measures often used in complex networks: degree centrality, closeness, betweenness and eigenvector centrality [5]. A comparison of these centrality measures has been provided using a simple illustrative example [6]. We remark that the eigenvector centrality is closely related to the notion of PageRank, which has been introduced in the Google search

engine for ranking websites in order of importance [7]. To alleviate the difficulties to compute PageRank in a centralized fashion, various distributed randomized algorithms have been developed [8, 9]. Numerical results testing the performance of these algorithms are provided using a benchmark consisting of 3,756 Web pages and 31,718 links [10].

Conclusions: Research Directions

Two research directions are currently carried on:

1. The development of a new class of randomized algorithms, called incremental diffusion algorithms, for the PageRank computation, and other centrality measures, of temporal networks.
2. Extensions of the Friedkin-Johnsen model to multidimensional correlated opinions, and analysis of their stability properties.

Acknowledgments

This work is based on joint research with Chiara Ravazzi, Paolo Frasca and Hideaki Ishii and it is partially funded by the CNR Joint International Lab COOPS. The current research directions are based on joint work with Keyou You and Li Qiu (direction 1) and Sergey Parsegov, Anton Proskurnikov and Noah A. Friedkin (direction 2).

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A network dynamics approach to chemical reaction networks

Arjan van der Schaft
(joint work with Shodhan Rao, Bayu Jayawardhana)

Network dynamics has been the subject of intensive research in recent years due to the ubiquity of large-scale networks in various application areas. While many advances have been made in the analysis of linear network dynamics, the study of nonlinear network dynamics still poses many challenges, especially in the presence of in- and outflows.

In this talk, we revisit the analysis of chemical reaction networks as a prime example of nonlinear network dynamics, playing an important role in systems biology, (bio-)chemical engineering, and the emerging field of synthetic biology. Apart from being *large-scale* (typical reaction networks in living cells involve several hundreds of chemical species and reactions) a characteristic feature of chemical reaction network dynamics is their intrinsic *nonlinearity*. In fact, mass action kinetics, the most basic way to model reaction rates, leads to polynomial differential equations. On top of this, chemical reaction networks, in particular in a bio-chemical context, usually have inflows and outflows.

The foundations of the structural theory of (isothermal) chemical reaction networks (CRNs) were laid in a series of seminal papers by Horn, Jackson, and Feinberg in the 1970s. The basic starting point of e.g. [9, 8, 5] is the identification of a graph structure for CRNs by defining the chemical complexes, i.e., the combination of chemical species appearing on the left-hand (substrate) and right-hand (product) sides of every reaction, as the vertices of a graph and the reactions as its edges. This enables the formulation of the dynamics of the reaction network as a dynamical system on the graph of complexes. Furthermore, in these papers the philosophy was put forward of delineating, by means of *structural* conditions on the graph, a large class of reaction networks exhibiting the same type of dynamics, irrespective of the precise values of the (often unknown or uncertain) reaction constants. This 'normal' dynamics is characterized by the property that for every initial condition of the concentrations there exists a unique positive equilibrium to which the system will converge. Other dynamics, such as multi-stability or presence of oscillations, can therefore only occur within reaction networks that are violating these conditions. For an overview of results on CRNs, and current research in this direction including the global persistence conjecture, we refer to [1] and the references quoted therein. An important step in extending the framework of CRNs towards *feedback stabilization* has been made in [12]; also setting the stage for further *regulation* questions.

In this talk, the formulation and analysis of mass action kinetics chemical reaction networks is revisited from the point of view of *consensus dynamics* and its nonlinear versions [2, 3, 17]. The consideration of concepts from algebraic graph theory, such as the systematic use of weighted Laplacian matrices, provides a framework for (re-)proving many of the previously obtained results on CRNs in a much simpler and insightful manner. In particular, as shown in [13, 10, 16], under

the assumption of existence of a detailed-balanced equilibrium, or the weaker assumption of existence of a complex-balanced equilibrium (a concept dating back to Horn & Jackson [9]), the weights of the graph of complexes can be redefined in such a way that the resulting Laplacian matrix becomes symmetric (detailed-balanced case) or balanced (complex-balanced case). As a result, the characterization of the set of positive equilibria and their stability as originating in [9, 8, 5] follows in a simple way. Moreover, this formulation allows for a direct port-Hamiltonian interpretation [14], linking CRNs to thermodynamical theory, and leads to new developments such as a theory of structure-preserving model reduction of chemical reaction networks, based on Kron reduction of the Laplacian matrix [13, 10, 11].

As mentioned above, our approach is based on the assumption of existence of a complex-balanced equilibrium, generalizing the classical notion of a detailed-balanced equilibrium. Based on [15] a necessary and sufficient condition is discussed for the existence of a complex-balanced equilibrium based on the Matrix Tree theorem (a theorem going back to the work of Kirchhoff on electrical circuits), which extends the classical Wegscheider conditions for existence of a detailed-balanced equilibrium. We also make a connection with the property of *mass conservation*. Furthermore, we discuss how these results can be 'dualized' to consensus dynamics, providing new insights.

Finally, we discuss how the dynamical analysis can be extended to chemical reaction networks with *inflows* and *outflows*. The extension of the stability theory of *equilibria* for reaction networks without inflows and outflows to that of *steady states* for reaction networks *with* inflows and outflows (called *open networks*) is far from trivial, due to the intrinsic nonlinearity of the reaction dynamics. By revisiting the classical idea of extending the graph of complexes by a 'zero' complex [9, 7], we will show how the results based on complex-balancedness for closed CRNs can be fully extended to CRNs with constant inflows and mass action kinetics outflows. Moreover, it allows to extend the model reduction techniques of [13, 10, 11] to CRNs with constant inflows and mass action kinetics outflows. From a control perspective the steady state analysis of open CRNs opens the possibility of applying the internal model principle (see e.g. [4]) to achieve output regulation for such systems with constant reference signals using proportional-integral controllers, for example, in the control of CSTR or gene-regulatory networks as in [18].

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An asynchronous distributed optimization algorithm

Fabian Wirth

(joint work with Martin Corless, Robert Shorten, Sonja Stüdli, Jia Yuan Yu)

In this talk we discuss a fully distributed algorithm for the approximation of the optimal point of a specific convex optimization problem. This type of problems occurs in particular in distributed resource allocation problems that are characteristic of smart cities applications. Here we will not discuss the motivation or related work in detail. For this we refer to [2].

We stress that the purpose of our study is not to invent a numerical scheme. The problem we will address can be solved much more efficiently. The idea is to have an algorithm that can be implemented on independent agents in such a manner that the agents approach a common optimum with minimal communication and simple rules.

Let $n \in \mathbb{N}$, $C > 0$ and $f_i : [0, C] \to \mathbb{R}$ be strictly convex and continuously differentiable, $i = 1, \ldots, n$. Consider the optimization problem

$$\min \quad \sum_{i=1}^{n} f_i(x_i) \qquad \text{subject to} \quad \sum_{i=1}^{n} x_i = C\,, \quad x_i \geq 0\,. \tag{1}$$

It is well-known that by compactness of the feasible space an optimal solution x^* exists; it is unique by strict convexity. We assume that there exist constants $0 < \lambda_{\min} < 1$ and $\Gamma > 0$ such that for all $r \in [0, C]$ and all $i = 1, \ldots, n$

$$\lambda_{\min} \leq \Gamma \, \frac{f_i'(r)}{r} \leq 1\,, \tag{2}$$

where for $r = 0$ (2) is interpreted for the continuous extension in $x = 0$. In this case x^* is characterized by the existence of a constant $\mu^* \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} x_i^* = C, \quad x_i^* > 0, \quad f_i'(x_i^*) = \mu^*, \quad i = 1, \ldots, n\,. \tag{3}$$

In our approach we will employ a variant of the AIMD (additive increase, multiplicative decrease) algorithm, that we describe now.

Let $e = \sum_{i=1}^{n} e_i$, the vector of all ones and $\beta \in (0, 1)$. In continuous time the AIMD dynamics are determined by the slope of linear increase $\alpha > 0$ used in the AI phase and the multiplicative decrease parameter β used at congestion. For us congestion occurs at a time instant when the constraint $\sum x_i = C$ is met. At this time a central station will send this information and the agents will reduce their individual x_i; with a certain probability.

It is known that the evolution of the AIMD algorithm from congestion event to congestion event can be described by a discrete-time linear positive system. This system is of the form

$$x(k+1) = A(k)x(k) \tag{4}$$

where $A(k)$ takes values in the set of AIMD matrices $\mathcal{A}$, for details see [1]. Here we describe a simplified version of these matrices.

At congestion event k each agent i updates its value to $\beta_i(k)x_i(k)$, where $\beta_i(k) \in \{\beta, 1\}$. The choice between these two options will happen in a random fashion that still needs to be determined. At each congestion event there are therefore 2^n different possibilities for the future evolution. Let

$$B := \left\{ \tilde{\beta} \in \mathbb{R}^n \;\middle|\; \tilde{\beta}_i \in \{\beta, 1\}\,, \quad i = 1, \ldots, n \right\}\,.$$

The *set of AIMD matrices* is then given by

$$\mathcal{A} := \left\{ \operatorname{diag}(\tilde{\beta}) + \frac{1}{n} e(e - \tilde{\beta})^\top \;\middle|\; \tilde{\beta} \in B \right\}\,. \tag{5}$$

Define the long-term average of a solution $x(\cdot)$ by

$$\bar{x}(k) = \frac{1}{k+1} \sum_{\ell=0}^{k} x(\ell)\,. \tag{6}$$

The algorithm is then described in the following manner: at every congestion event k agent i chooses

$$\beta_i(k) := \begin{cases} \beta & \text{with probability } \Gamma\ \frac{f_i'(\bar{x}_i(k))}{\bar{x}_i(k)} \\ 1 & \text{else.} \end{cases} \tag{7}$$

All choices have to be independent of all other events. This induces a probability for the choice of the matrix $A(k)$ in (4) that depends on the complete history of the process, namely on $\bar{x}(k)$. By augmenting the state to include $\bar{x}$ we arrive at a non-homogeneous Markov chain which has a remarkable convergence property.

Theorem. Consider the optimization problem (1) and let x^* be the optimal point. Consider the stochastic system described by (4) and the probability functions (7). For any initial condition $(x_0, \bar{x}_0)$ we have that

$$\lim_{k\to\infty} \bar{x}(k) = x^* \qquad \text{almost surely} \quad \mathbb{P}_{x_0}\,.$$

Our result says that by local modification of the individual probabilities the agents can ensure almost sure convergence to the optimum. Note that agent i only requires knowledge of its own evolution in order to perform the update of the probability (7). Inter-agent communication is not necessary; rather the only information needed is 1-bit intermittent message to all agents that congestion has occurred. This suffices for convergence.

The proof relies on a detailed analysis of the deterministic set-valued system underlying the Markov chain. The Hilbert metric on the relative interior of the constraint set can be employed as Lyapunov function with nice robustness properties in a neighborhood of x^*. Globalization requires different arguments.

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Uncertain Consensus Networks: Robustness and its Connection to Effective Resistance

Daniel Zelazo

(joint work with Mathias Bürger)

The consensus protocol has recently emerged as a canonical model for the study of *networked dynamic systems*. In the linear setting, the consensus protocol, comprised of a collection of single integrator dynamic agents interacting over an information exchange network (the graph), has been studied from both dynamic systems and graph theoretic perspectives [1]. The most basic setting considered in consensus networks assumes an undirected connected graph with non-negative

weights on the edges of the graph. In such a setting, it is well known that the trajectories of all agents in the network converge to a common value. On the other hand, the study of consensus networks with negative weights has received less attention. Such networks can exhibit rich behaviour including trajectories that synchronise, cluster, or are unstable. It is thus of interest to understand how robust consensus networks are to uncertainties in the edge weights.

In this work, we study the robustness of the linear weighted consensus protocol for n agents over the undirected and weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, corrupted by a finite energy exogenous disturbance signal $\mathrm{w} \in \mathcal{L}_2^n[0, \infty]$,

$$\dot{x}_i = \sum_{j \sim i} w_{ij}(x_j - x_i) + \mathrm{w}_i.$$

Here, w_{ij} is the real valued weight assigned to the edge $\{i, j\} \in \mathcal{E}$, and the notation $j \sim i$ indicates that $\{i, j\} \in \mathcal{E}$. In vector notation, this is expressed using the weighted Laplacian matrix, $L(\mathcal{G}) := E(\mathcal{G})WE(\mathcal{G})^T$, where $E(\mathcal{G})$ is the incidence matrix of the graph. We also introduce a controlled variable z, representing the relative states of the system, to quantify the performance of the system, leading to

$$\Sigma(\mathcal{G}) : \left\{ \begin{array}{rcc} \dot{x} & = & -L(\mathcal{G})x + \mathrm{w} \\ z & = & E(\mathcal{G})^T x \end{array} \right. . \tag{1}$$

We now introduce a notion of uncertainty into the edge agreement protocols. We assume that the exact weights of a subset of edges are an uncertain but bounded perturbation about some nominal value. In this direction, let $\mathcal{E}_\Delta \subseteq \mathcal{E}$ denote the set of uncertain edges. The nominal edge weight for an edge $k \in \mathcal{E}_\Delta$ is determined by the weight function $\mathcal{W}$; i.e., the nominal weight of edge k is $\mathcal{W}(k) = w_k$. The uncertainty of the weight on edge k is modelled as an additive perturbation to the nominal edge weight as $w_k + \delta_k$ with $|\delta_k| \leq \overline{\delta}$ for some finite positive scalar $\overline{\delta}$. Thus, we can define the uncertainty set as

$$\mathbf{\Delta} = \{\Delta \,:\, \Delta = \mathbf{diag}\{\delta_1, \ldots, \delta_{|\mathcal{E}_\Delta|}\},\, \|\Delta\| \leq \overline{\delta}\}.$$

The uncertain consensus network can now be expressed as

$$\Sigma(\mathcal{G}, \Delta) : \left\{ \begin{array}{rcc} \dot{x} & = & -E(\mathcal{G})(W + P\Delta P^T)E(\mathcal{G})^T x + \mathrm{w} \\ z & = & E(\mathcal{G})^T x \end{array} \right. . \tag{2}$$

The stability margins of the uncertain consensus network can now be understood in the context of the celebrated *small-gain theorem*. While this can lead to conservative results in general, we find that for certain graph structures, and in particular for the case of uncertainties in only a single edge, the condition becomes exact. Furthermore, it turns out that this condition relates to the notion of the *effective resistance* of a graph [2]. The main result is summarized in the following theorem.

Theorem 1 ([4])**.** *Assume the unperturbed consensus network $\Sigma(\mathcal{G})$ is stable and let $\mathcal{E}_\Delta = \{\{u, v\}\}$ (i.e., $|\mathcal{E}_\Delta| = 1$). Then the uncertain consensus network $\Sigma(\mathcal{G}, \Delta)$*

in (2) is robustly stable for all $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}(\mathcal{G})$*, where* $\mathcal{R}_{uv}$ *is the effective resistance between nodes* u *and* v*.*

Theorem 1 has the physical interpretation of two resistors connected in parallel. The effective resistance between nodes u and v, $\mathcal{R}_{uv}$, represents a single resistance, and the uncertainty in the edge connecting the nodes is a second resistor in parallel. If the second resistance has a negative resistance exactly matched to the value of $\mathcal{R}_{uv}$, then the parallel interconnection creates an effective *open-circuit*. From a graph-theoretic point-of-view, this is analogous to creating a *cut* in the graph separating nodes u and v.

This talk explores the algebraic and graph-theoretic interpretations of the effective resistance of a network and its relationship to the definiteness of the weighted Laplacian and the uncertain consensus network (2). Results are presented for the case of a single uncertain edge. These results are also shown to lead to insights on the *clustering* phenomena in consensus networks. Non-linear extensions including edges with sector-bounded nonlinearities are also discussed.

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Open problem: Is there global convergence to a four-vehicle formation shape?

Brian D. O. Anderson

We record below the equations governing motion of four point agents in the plane, tasked with having to assemble into a certain shape that is defined by prescribed inter-agent distances. The open problem is to show that the attractive convergence properties of the algorithm which apply when the desired shape is rectangular continue to hold for non rectangular shapes, i.e for a general quadrilateral shape or a shape comprising a triangle together with an interior agent.

To be more precise, let $p = [p_1, p_2, p_3, p_4]^\top \in \mathbb{R}^8$ be a vector of the four agent positions in the plane. Let $\bar{d} = [\bar{d}_{12}, \bar{d}_{13}, \bar{d}_{14}, \bar{d}_{23}, \bar{d}_{24}, \bar{d}_{34}]^\top$ be a vector of desired interagent distances that define the desired formation shape, while the vector $d = [d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]$ with $d_{ij} = ||p_i - p_j||$ defines the instantaneous formation shape. Note that six distances between four agents cannot be independent. In fact, for generic positions p_i, knowledge of five distances determines the remaining distance up to a binary ambiguity.

Reference [1] proposes a control law to achieve the desired shape. Each agent compares its actual distance from each neighbour with the desired distance and then moves to correct the value (if correction is needed) using the direction of the neighbor. It is convenient to define errors using squares of the relevant distances:

$$e_{ij} = d_{ij}^2(p) - \bar{d}_{ij}^2. \tag{1}$$

The motion equation for agent i is then

$$\dot{p}_i = \sum_{j \neq i} e_{ij}(p_j - p_i) \tag{2}$$

This algorithm is actually a gradient descent law: with $e(p)$ denoting the vector of $e_{ij}(p)$, define the function

$$V(p) = \frac{1}{2}||e(p)||^2 \tag{3}$$

which quantifies the overall error between the actual and desired formations. Then there holds

$$\dot{p} = -\nabla V(p) \tag{4}$$

Because the law is a gradient law, there is always convergence to some stationary point at which the gradient vector is zero.

Reference [1] included several examples of the use of this law, including one where the desired formation was rectangular. From many initial conditions, the desired rectangular shape was reached. However, an equilibrium point corresponding to a rectangular shape with different side lengths and different ordering of agents around the boundary of the rectangle was also exhibited.

The later work [2] (itself drawing on two intervening works) showed or summarised facts derived in these works that

(1) Associated with a generic desired rectangular shape there are generically *two* incorrect rectangular shapes, but they always correspond to saddle points of the gradient algorithm;
(2) There can be incorrect equilibria in which agents are collocated or are collinear, but these are also saddle points;
(3) There can be further incorrect equilibria again (in which the agent positions span two dimensions), but these are saddle points.

The upshot is that, provided one starts from an initial condition in which the agents are not collinear, in practice one will always converge to the desired equilibrium shape. (Of course, the orientation of the rectangle and the position of its centroid are not encompassed in such a statement).

The open question is: *suppose that the desired formation shape is not a rectangle, but a general quadrilateral or an arrangement of the agents with three agents forming a triangle and the fourth agent inside the triangle. Are all incorrect equilibria again saddle points?*

By way of additional remarks, we note

(1) Four agent formations with five rather than six defining edges are straightforward to handle. However, with prescribed distances, there are generically two noncongruent shapes consistent with the distances. (Imagine a quadrilateral with a diagonal, and then observe one can flip one of the triangles across the diagonal axis to get a noncongruent shape with the same interagent lengths). The ambiguity might be undesirable in practice. Specification of all six distances rather than five means there is a unique shape i.e. all formations with the desired lengths are congruent.
(2) There is an extensive analysis of equilibrium points for four agent formations constrained to be collinear in [3] and of four agent formations in $\mathbb{R}^3$, corresponding to a tetrahedron, in [4].
(3) There are elegant formulas for the Hessian matrix and alternative ways of viewing the gradient descent equations which are set out in [2]

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Open problem: Languages for Hybrid Systems

Roger Brockett

Inspired by the idea going back to Rabin and Scot [1] of finite state languages defined in terms of accepting states for suitable finite state automaton, the problem posed here is that of specifying suitable languages for controlling hybrid systems. Hybrid systems are a formalization of a wide class of real world systems whose inputs are pairs (u, v) with u being a (possibly vector valued) function defined on a finite interval an v being a symbol string. The following example of a hybrid system is taken from my 1993 paper [2], is gaining new currency in view of the interest in driverless cars.

Example. Consider a simplified model for the engine-transmission system of an automobile with a manual transmission. We view the system as having a throttle position (u) and gear shift lever position (v) as inputs and engine rpm and ground speed as outputs. The throttle position and the gear shift position are "analog" and "digital", respectively. The gear shift is necessary because the torque-speed relationship of an engine is such that little power can be developed at very low, or very high, engine rpm. To circumvent this difficulty the system is equipped with an accelerator that controls engine speed and a finite set of gear configurations that determine the gear ratio between the engine and the drive wheels. The function of the (discretely) variable gear ratio transmission is to let the driver get adequate

power from the motor at a given ground speed by making it possible to operate the engine at an advantageous segment of the speed-torque curve. A crude, but illustrative (completely nondimensionalized) hybrid model is provided by

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = (-a(x_2(t)) + u(t))/(1 + v\lfloor p \rfloor)$$
$$\dot{p}(t) = k$$

This can be thought of as a system that satisfies Newton's second law of motion in rotational form. The $a(\ )$ term models the relationship between engine speed and engine torque. The term $(1 + v\lfloor p \rfloor)$ represents the fact that the effective inertial seen by the engine changes when the gear ratio changes.

An important practical question that arises in treating hybrid systems involves the tradeoff between the complexity of the input signals and the expressiveness of the language. Because the language must be interpreted it should not be too complex; but because the system is to be flexible language restrictions should not impose too many constraints on the expressiveness. Can one find a limited set of input strings based on an instruction set (U, V) that will allow a good approximation to any desired set of motions with adequate robustness. A regulation problem might involve finding a string of instructions that will maintain a desirable relationship between inputs and outputs without requiring more information than is available through the observations.

Because the elements of v are used directly in the evolution equation, the achievable resolution in the space of x-trajectories is directly related to the cardinality of the set V. At the same time, The real-valued inputs represented by u may be limited as would be the case, for example, if they were required to be third order splines with 16 bit coefficients. Our idea here is that one may specify a language for hybrid systems by asking that elements of the language should transfer the hybrid system to a particular state, or set of states while satisfying some variational principle such as a combination of minimum length and minimum L_2 norm. The set of accepting states might, for example, be the set of all equilibrium states of the system.

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Open problem: Strict Dissipativity and the Turnpike Property

Lars Grüne

This open problem considers discrete time optimal control problems of the form

$$\underset{u(\cdot)}{\text{minimize}}\ J_N(x(0),u(\cdot)), \qquad J_N(x(0),u(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k),u(k))$$

s.t.

$$x(k+1) = f(x(k),u(k)), \qquad x(k) \in \mathbb{X},\ u(k) \in \mathbb{U}.$$

Here $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$ is the dynamics, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the stage cost and $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are the state and control constraint set, respectively, which for siplicity of exposition we assume to be compact. Optimal trajectories (which we neither assume to exist nor to be unique) will be denoted by $x^*(\cdot)$

The turnpike property now demands that there exists a point $x^e \in \mathbb{X}$ such that any optimal trajectory, regardless of its initial value, stays in a neighborhood of this point $x^e \in \mathbb{X}$ for a time which is independent of N. Formally this can be expressed as follows.

Turnpike Property. There exists $x^e \in \mathbb{X}$ such that for any $\varepsilon > 0$ there exists $P \geq 0$ such that for all $N \geq P$ and all optimal trajectories $x^*(\cdot)$ of length N the inequality

$$\|x^*(k) - x^e\| > \varepsilon$$

holds for at most P time indices $k = 0, \ldots, N$.

Turnpike properties have been investigated at least since the seminal work by von Neumann in [7]. The name "turnpike property" goes back to Dorfman et al. [3] and the form presented here is the discrete time variant of the version found in Carlson et al. [2]. They have recently gained renewed interest in the context of economic model predictive control [4, 5].

The second property we are investigating goes back to Willems [8, 9].

Strict Dissipativity. There exists an equilibrium $x^e \in \mathbb{X}$ with corresponding control value $u^e \in \mathbb{U}$ (i.e., $f(x^e,u^e) = x^e$) and a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$ and $\rho \in \mathcal{K}_\infty$ such that the inequality

$$\ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)) \geq \rho(\|x - x^e\|) \tag{1}$$

holds for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$.

Like the turnpike property, strict dissipativity has also turned out to be very useful for analysing economic model predictive control schemes [1, 4, 5]. Particularly, it was shown in Theorem 5.3 of [4] (which is essentially a discrete time version of a result in [2]), that under a suitable controllability assumption and if λ is bounded on $\mathbb{X}$, then the implication

$$\text{strict dissipativity} \ \Rightarrow\ \text{turnpike property} \tag{2}$$

holds. The open problem now is:

Under which assumptions does the converse implication to (2) hold?

It should be noted that a partial answer can possibly be obtained using the results from chapter 4 of [6], however, this reference does not use the turnpike property but the related notion of optimal operation at steady state and it does not show that this property implies strict dissipativity but only dissipativity, i.e., (1) with "0" in place of "ρ". Nevertheless, the techniques used in this reference might also be useful for answering the open problem.

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Open problem: Stabilization of nonlinear controllable systems by using trigonometric polynomials

Alexander Zuyev

This talk addresses the stabilization problem for nonlinear control systems of the form

$$\dot{x} = f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in U \subset \mathbb{R}^m, \tag{1}$$

where x is the state, u is the control, $f(x,u)$ is smooth in the domain $D \times U$, $(0,0) \in D \times U$, and $f(0,0) = 0$.

In contrast to the theory of linear control systems, it is a well-known fact that the controllability of system (1) does not imply its stabilizability by a regular state feedback law. In particular, the equilibrium $x = 0$ of a nonholonomic system

$$\dot{x} = \sum_{j=1}^{m} u_j f_j(x), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n \tag{2}$$

cannot be made asymptotically stable by a differentiable feedback law $u = k(x)$, $k(0) = 0$ if the vectors $f_1(0)$, $f_2(0)$, ..., $f_m(0)$ are linearly independent [2]. This negative result also holds for the class of discontinuous state feedback laws provided that the solutions of the closed-loop system are defined in Filippov's sense [7].

Thus, two main strategies can be used to prove stabilizability of general controllable systems. In the first strategy, the equilibrium of controllable system (1) can be stabilized by means of *discontinuous* feedback laws by specifying "exit rule" on the singular set [8] (in the analytic case) or by defining the solutions ("π-trajectories") in the sense of sampling [3]. The second strategy is based on the use of *time-varying* feedback laws $u = k(x,t)$. It was shown by J.-M. Coron [4, Theorem 11.28] that if system (1) satisfies the Lie algebra rank condition at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$, $x = 0$ is locally continuously reachable in small time, and $n \notin \{2,3\}$, then system (1) is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws. It should be emphasized that the proofs of these stabilizability results are not constructive, so that there is no universal control design scheme available for general controllable systems.

In this talk, we raise the question whether it is possible *to construct* a stabilizing controller $u = k(x,t)$ as a trigonometric polynomial with respect to t with coefficients depending on x, provided that the conditions of Coron's theorem are satisfied. For a particular class of driftless control-affine systems, this problem can be formulated as follows.

Open problem. *Assume that*

$$Lie_{x=0}\{f_j \,|\, 1 \le j \le m\} = \mathbb{R}^n.$$

Is it possible to construct a time-varying feedback law

$$u_j = \sum_{k=-N}^{N} v_{kj}(x) e^{i\omega k t}, \quad j = 1, 2, ..., m, \tag{3}$$

for some $N \ge 0$ and $\omega > 0$, such that the trivial solution of the closed-loop system (2), (3) is asymptotically stable? The functions $v_{kj}(x)$ are assumed to be piecewise smooth, $v_{kj}(x) = \overline{v_{-kj}}(x)$ for all $x \in D$, and $v_{kj}(x) \to 0$ as $x \to 0$.

We treat the construction of controller (3) as an algorithm that computes $v_{kj}(x)$ in terms of solutions of auxiliary algebraic systems whose coefficients depend on $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ and their Lie brackets at a point $x \in D$.

To justify the choice of parameterization in (3), we note that sinusoidal inputs appear naturally as optimal controls for the nonholonomic integrator with the L^2-norm cost function [1]. Fast oscillating open-loop controls are known to be effective in the motion planning problem for nonholonomic systems (see, e.g., [6, 5]).

In this talk, we illustrate the possibility of *feedback design* in the form (3) for a class of systems (2) satisfying the controllability condition with the first order Lie brackets [9]. To the best of our knowledge, the problem of constructing such stabilizing controllers remains open for nonlinear systems under higher order controllability conditions.

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