Mathematisches Forschungsinstitut Oberwolfach

Report No. 14/2015

DOI: 10.4171/OWR/2015/14

Homotopy Theory

Organised by
Paul Goerss, Evanston
John Greenlees, Sheffield
Stefan Schwede, Bonn

8 March – 14 March 2015

Abstract. This workshop was a forum to present and discuss the latest results and ideas in algebraic topology and homotopy theory, with a special emphasis on connections to other branches of mathematics, such as algebraic geometry, representation theory, group theory, and the algebraic topology of manifolds.

Mathematics Subject Classification (2010): 55P.

Introduction by the Organisers

Homotopy theory and algebraic topology have seen tremendous growth and transformation over the past decade, with new points of view, new questions, new techniques, and—as a result of all this—remarkable progress. Much of this change has been driven by a new generation of mathematicians, many still within a few years of their doctoral degrees. The main thrust of this workshop was to highlight this transformation and the mathematicians responsible, while keeping in mind some of the classical questions that define the field.

If topology can be defined as the study of phenomena of topological spaces invariant under invertible continuous transformations, then homotopy theory is the study of phenomena that remain invariant a weaker notions, such as homotopy equivalence or even weak homotopy equivalence. In the early postwar period, the field grew of out the observation that many problems of geometric or topological interest are best addressed by first solving a homotopy theory question and, indeed, in some cases the homotopy theory is the deep part of the problem. A recent example of this was the Kervaire Invariant One problem for framed manifolds solved by Hill, Hopkins, and Ravenel. Once the field was established, the flow information went the other way as well. For example the emergence of derived algebraic geometry in homotopy theory has given us new ways to think about manifold invariants with such tools as factorization homology.

The workshop emphasized four themes: the algebraic topology of manifolds, equivariant stable homotopy theory, chromatic stable homotopy theory, and algebraic K-theory, including the algebraic K-theory of structured ring spectra. These themes have considerable overlap; for example, algebraic K-theory is integral to the statement of Novikov conjecture, which is a question about manifolds with group action, which itself leads to equivariant homotopy theory, which in turn is an important technique in chromatic stable homotopy, which itself leads to the red-shift conjecture in K-theory.

In the first area, the algebraic topology of manifolds, we had three very different presentations. David Ayala talked on his work with John Francis and Nick Rozenblyum on factorization homology and new ways to construct and think about invariants. As mentioned above, this theory uses in an integral way the the new methods from ∞- categories and derived algebraic geometry; indeed, factorization homology is adapted from the chiral homology developed by Beilinson and Drinfeld. Wolfgang Lück talked about the Farrell-Jones conjecture, the most basic of the assembly conjectures, and one with many geometric implications. The results presented here, joint with Holger Reich, John Rognes, and Marco Varisco, are the capstone on the work of many people, proving a very general case. We also had a short talk by Martin Palmer of homological stability for diffeomorphism groups of high-dimensional manifolds. This an especially active subfield at the moment, as only recently have researchers developed techniques to get beyond surfaces.

At the core of the results presented by Lück was a discussion of the algebraic K-theory of group rings, especially of large discrete groups. Algebraic K-theory is a powerful invariant, containing a great deal of information, but also very hard to compute. We had a number of talks about K-theory, some discussing its universal properties, some offering calculations. Andrew Blumberg, for example, talked on joint work with Michael Mandell that gave a calculation of the K-theory of the sphere spectrum in terms of the K-theory of the integers and certain Thom spectra. Birgit Richter gave a talk demonstrating that higher topological Hochschild homology, which quite recently seemed quite mysterious, can now be completely calculated for the field with p elements. David Gepner reported on joint work with Antieau and Barthel that effectively *disproves* the existence of a certain higher chromatic generalization of Quillen's localization sequence for algebraic K-theory. On a related theme, Oliver Röndigs gave a talk in motivic homotopy theory. This area, which might be called the homotopy theory of smooth schemes, has its origins in Voevodsky's solution of some of the famous conjectures in K-theory.

A number of talks centered on equivariant stable homotopy theory, a subfield undergoing a renaissance. These included the talks developing foundations, establishing remarkable universal properties and showing the way the framework can offer great insights into classical problems. Mike Hill explained the intricate network of equivariant notions of commutativity, bringing clarity where confusion has been an obstacle to progress. Thomas Nikolaus showed how orbispectral approaches to global homotopy theory illuminates the role of stability, and outlined the vision of how this applies to elliptic cohomology. Niko Naumann gave an account of stratifications of prime spectra of cohomology theories which elegantly unifies (through the notion of nilpotence) and considerably extends a collection of results that were previously only related in form. Irakli Patchkoria showed how to give a a theory of proper-equivariant stable homotopy theory illuminating the relationships between various notions of cohomological dimension, with the foundations established by an elegant and natural model structure on orthogonal spectra.

Chromatic stable homotopy theory was, in some sense, our most classical topic, having its roots in the work of Quillen and Morava in the 1970s and with a steady record since then. Agnès Beaudry talked about her work on the Chromatic Splitting Conjecture, a proposal for assembling a homotopy type from its chromatic layers. In particular, she argued that a strong form of the conjecture cannot be true at the prime 2. Tyler Lawson and Vesna Stojanoska talked about topological modular forms and its generalizations and Craig Westerland spoke about higher chromatic analogs of the J-spectrum.

We did not constrain ourselves to these four themes, and we extended our scope with talks on other subjects as well. David Benson explained stratification in modular representation theory and how equivariant homotopy theory allowed the extension to compact Lie groups. Steffen Sagave gave an account of elegant foundations of the theory of Thom spectra and its applications to topological Hochschild homology. Akhil Mathew described the use of ∞-categorical foundations for powerful applications in several areas described by other speakers. Our speakers came from every career stage, including a graduate student (Mathew) giving an hour talk.

On Wednesday morning we also had a session of shorter talks. We found this way for the newer members of our community to introduce themselves not only quick but also remarkably effective. A list for the speakers in this part of the program is at the end of the report.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Robert Bruner in the "Simons Visiting Professors" program at the MFO.

Workshop: Homotopy Theory

Table of Contents

Abstracts

Factorization homology from higher categories

David Ayala
(joint work with John Francis and Nick Rozenblyum)

Abstract: We construct a pairing, which we call factorization homology, between framed manifolds and higher categories. The essential geometric notion is that of a vari-framing of a stratified manifold, which is a framing on each stratum together with a coherent system of compatibilities of framings along links of strata. Our main result constructs labeling systems on disk-stratified vari-framed n-manifolds from (∞, n)-categories. These (∞, n)-categories, in contrast with the literature to date, are not required to have adjoints. The core calculation supporting this result is a homotopy equivalence between the space of conically smooth diffeomorphisms of a disk-stratified manifold and its space of vari-framings. This allows the following conceptual definition: the factorization homology

$$\int_M \mathcal{C}$$

of a framed n-manifold M with coefficients in an (∞, n)-category $\mathcal{C}$ is the classifying space of $\mathcal{C}$-labeled disk-stratifications over M. This is spiritually similar to the Blob homology of Morrison and Walker [MW].

The factorization homology of a framed n-manifold M with coefficients in $\mathcal{C}$ should be

$$\int_M \mathcal{C} \;\approx\; \Big|\mathcal{C}\text{-labeled disk-stratifications of } M\Big| \;,$$

the classifying space of a category, an object of which consists of a coherent system of:

- a stratification of M, each closed component of which is a k-disk;
- a k-morphism of $\mathcal{C}$ for k-dimensional component of the stratification of M.

There are several important classes of morphisms.

(1) **refinements/compositions**: a stratum is refined away, forgotten, and the labels are composed.
(2) **creations/units**: a new stratum is created, labeled by identity morphisms.
(3) **coherence**: a stratification is moved via diffeomorphism to another stratification.

This template for making factorization is, however, afflicted by the absence of any known model for (∞, n)-categories which can define such a system of labels. Most models for (∞, n)-categories are constructed in terms of presheaves on a combinatorially defined category, such as $\mathbf{\Theta}_n$ or the n-fold product $\mathbf{\Delta}^n$, and none of these are manifestly suitable for decorating a disk-stratification.

We solve this issue in our setting in three steps. In the first step, we construct an ∞-category of labeling systems for stratifications on framed n-manifolds. In the second step, we show that (∞, n)-categories embeds fully faithfully into labeling systems. In the third step, we define factorization homology with coefficients in the specified labeling systems. We elaborate on these steps below.

First step: In our antecedent work on striation sheaves [AFR], we constructed an ∞-category $\mathsf{c}\mathcal{B}\mathsf{un}$ whose objects are compact conically smooth stratified spaces and whose morphisms include refinements and stratum-creating maps, exactly as in points (1) and (2) above. Now, starting from $\mathsf{c}\mathcal{B}\mathsf{un}$, we restrict to the ∞-subcategory $\mathsf{c}\mathcal{D}\mathsf{isk} \subset \mathsf{c}\mathcal{B}\mathsf{un}$ of objects which are disk-stratified, as above. We then introduce the notion of a variform framing – for short, vari-framing – on a stratified space. A vari-framing consists of a framing on each stratum together with compatibilities between these framings in links of strata. From this, we define $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ as the collection of compact disk-stratified manifolds of dimension less or equal to n and equipped with a vari-framing. Lastly, the ∞-category of labeling systems is

$$\mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}) ,$$

space-valued functors on vari-framed compact disk-stratified n-manifolds.

Second step: We use Rezk's presentation [Re2] of the ∞-category of (∞, n)-categories $\mathfrak{Cat}_{(\infty,n)}$ as a full ∞-subcategory of $\mathsf{PShv}(\mathbf{\Theta}_n)$, presheaves on Joyal's category $\mathbf{\Theta}_n$ of [Jo2]. We construct a cellular realization

$$\mathbf{\Theta}_n^{\mathsf{op}} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$$

from Joyal's category. We prove that this is fully faithful, which is the essential technical result of this paper. The core calculation underlying this fully faithfulness is a natural homotopy equivalence

$$\mathsf{Diff}(\mathbb{D}^k) \simeq \mathsf{vfr}(\mathbb{D}^k)$$

between the space of conically smooth diffeomorphisms of a hemispherically stratified k-disk and the space of vari-framings of the k-disk. The lefthand side is a manner of pseudoisotopy space, so this result can be interpreted as a cancellation between pseudoisotopies and vari-framings.

Third step: Lastly, we left Kan extend from $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}$ to $\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. That is, factorization homology is the composite

$$\int : \mathfrak{Cat}_{(\infty,n)} \longrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces}) \longrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}, \mathcal{S}\mathsf{paces})$$

where the first functor is the fully faithful embedding of the second step, and the second functor is left Kan extension along the inclusion $\mathsf{c}\mathcal{D}\mathsf{isk}_n^{\mathsf{vfr}} \subset \mathsf{c}\mathcal{M}\mathsf{fd}_n^{\mathsf{vfr}}$. Equivalently, the factorization homology

$$\int_M \mathcal{C}$$

is the classifying space of the Grothendieck construction of the composite functor

$$\mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{n/M} \longrightarrow \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_{n} \xrightarrow{\mathcal{C}} \mathcal{S}\mathsf{paces}$$

where the functor $\mathcal{C}$ is the right Kan extension of $\mathcal{C} : \mathbf{\Theta}_n^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ along the cellular realization functor $\mathbf{\Theta}_n^{\mathsf{op}} \to \mathsf{c}\mathcal{D}\mathsf{isk}^{\mathsf{vfr}}_n$.

We now state the main result of the present work.

Theorem. *There is a fully faithful embedding of (∞, n)-categories into space-valued functors of vari-framed n-manifolds*

$$\int : \mathfrak{C}\mathsf{at}_{(\infty,n)} \hookrightarrow \mathsf{Fun}(\mathsf{c}\mathcal{M}\mathsf{fd}^{\mathsf{vfr}}_n, \mathcal{S}\mathsf{paces})$$

in which the value $\int_{\mathbb{D}^k} \mathcal{C}$ is the space of k-morphisms in $\mathcal{C}$, where $\mathbb{D}^k$ is the hemi-spherically stratified k-disk.

In later work, we will apply this higher codimension form of factorization homology to construct topological quantum field theories.

References

[AFR] Ayala, David; Francis, John; Rozenblyum, Nick. A stratified homotopy hypothesis. Preprint, 2015.

[Jo2] Joyal, André. Disks, duality and Theta-categories. Unpublished work, 1997.

[MW] Morrison, Scott; Walker, Kevin. Blob homology. Geom. Topol. 16 (2012), no. 3, 1481–1607.

[Re2] Rezk, Charles. A Cartesian presentation of weak n-categories. Geom. Topol. 14 (2010), no. 1, 521–571.

The Chromatic Splitting Conjecture at $n = p = 2$

Agnès Beaudry

Understanding the homotopy groups of the sphere spectrum S is one of the great challenges of homotopy theory. The ring $\pi_* S$ is extremely complex; there is no hope of computing it completely. However, it carries an amazing amount of structure. A famous theorem of Hopkins and Ravenel states that, for a fixed prime p, the p-local sphere spectrum $S_{(p)}$ is filtered by "simpler" spectra called the chromatic layers. The n'th chromatic layer $L_n S$ is the Bousfield localization of the sphere spectrum at the wedge $K(0) \vee \ldots \vee K(n)$, where $K(m)$ is the m'th Morava K-theory. The chromatic convergence theorem of Hopkins and Ravenel [8, §8.6] states that

$$S_{(p)} \simeq \mathrm{holim}_n \{ \ldots \to L_n S \to L_{n-1} S \to \ldots \to L_0 S \}.$$

How the chromatic layers glue together is mysterious. It is the subject of the chromatic splitting conjecture. There is a homotopy pull back square

$$\begin{array}{ccccc} F_n & \longrightarrow & L_nS & \longrightarrow & L_{K(n)}S \\ \| & & \downarrow & & \downarrow \\ F_n & \longrightarrow & L_{n-1}S & \longrightarrow & L_{n-1}L_{K(n)}S, \end{array}$$

where $F_n \simeq F(L_{n-1}S, L_nS)$ denotes the common fiber. The spectrum L_nS is thus built by gluing the spectra $L_{n-1}S$ and $L_{K(n)}S$ along $L_{n-1}L_{K(n)}S$. The chromatic splitting conjecture is a statement about how these pieces fit together. More precisely, it stipulates that the map $L_{n-1}S \to L_{n-1}L_{K(n)}S$ is the inclusion of a wedge summand, so that

$$L_{n-1}L_{K(n)}S \simeq L_{n-1}S \vee \Sigma F_n. \tag{1}$$

Further, it gives an explicit description of the homotopy type of ΣF_n as a wedge of suspensions of various copies of the lower chromatic layers $L_{n-k}S$ for $0 < k \leq n$. For example, when $n = 2$, the decomposition predicted by the conjecture implies that

$$L_1L_{K(2)}S \simeq L_1S \vee L_1S^{-1} \vee L_0S^{-3} \vee L_0S^{-4}. \tag{2}$$

A more general formulation of the conjecture can be found in [7, Conjecture 4.2].

The chromatic splitting conjecture is due to Hopkins. When $n = 1$, it holds at all primes, a fact which follows from computations of Adams and Mahowald. The conjecture is based on computations of Shimomura and Yabe for $p \geq 5$ at chromatic level $n = 2$, where it is known to hold in its most general form [3, Remark 7.8]. When $n = 2$ and $p = 3$, it was proved by Goerss, Henn and Mahowald in [5].

We have shown recently in [2] that the decomposition (2) does not hold at the prime $p = 2$. Indeed, the decomposition ΣF_2 above is too small. The broad strokes of the argument are as follows. Let $V(0)$ be the mod 2 Moore spectrum, that is, the cofiber of multiplication by 2 on S. Since L_0 is rationalization, $L_0V(0)$ is contractible. Therefore, it would follow from (1) and (2) that

$$L_1L_{K(2)}V(0) \simeq L_1V(0) \vee \Sigma^{-1}L_1V(0).$$

This would imply that $\pi_kL_1L_{K(2)}V(0) = 0$ when k is congruent to 5 modulo 8, a consequence of Mahowald's computation of $\pi_*L_1V(0)$. (In fact, this vanishing can be traced back to the fact that $\pi_kKO = 0$ in degrees $k = 5, 6, 7$.) However, we compute in [2] that $\pi_{5+8t}L_1L_{K(2)}V(0) \neq 0$.

Computations at chromatic level $n = 2$ for the prime $p = 2$ are particularly difficult. A lot of technology is used to prove that $\pi_{5+8t}L_1L_{K(2)}V(0) \neq 0$. Our main tool is the duality resolution at the prime $p = 2$, which is a finite resolution of spectra of the form $\Sigma^kE_2^{hG}$, where E_2 is Morava E-theory and G is a finite subgroup of the Morava stabilizer group $\mathbb{G}_2$. The duality resolution methods were first developed at $p = 3$ by Goerss, Henn, Mahowald and Rezk in [6]. The duality resolution framework has been partially adapted at the prime $p = 2$ in [1] and [4]. Computations using the duality resolution are difficult, and only partial information about the homotopy groups of $L_1 L_{K(2)} V(0)$ is known.

We note that, even though the decomposition (2) of ΣF_2 fails when $p = 2$, there is good evidence that the map $L_1 S \to L_1 L_{K(2)} S$ is the inclusion of a wedge summand. If this is the case, (1) would hold at $n = p = 2$. We do not yet have a conjectured description for ΣF_2. One expects that the fact that the $K(1)$-local Picard group $Pic(\mathcal{L}_1)$ contains exotic elements at $p = 2$ will enter the picture.

On the bright side, the decomposition analogous to (2) for $n > 2$ (see [7, Conjecture 4.2(v)]) is still expected to hold when p is large with respect to n. Giving a plausible description of the homotopy type of ΣF_n at "bad" primes p with respect to n is an open problem. We have hope that there is a systematic description of ΣF_n, even if it must be more complicated than the description originally proposed.

References

[1] A. Beaudry, The algebraic duality resolution at $p = 2$, to appear in *Algebr. Geom. Topol.*, *ArXiv e-prints*, December 2014.

[2] A. Beaudry, The chromatic splitting conjecture at $n = p = 2$, *ArXiv e-prints*, February 2015.

[3] M. Behrens, The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited, *Adv. Math.*, 230(2):458–492, 2012.

[4] I. Bobkova, Resolutions in the $K(2)$-local category at $p = 2$, *PhD thesis*, Northwestern University, June 2014.

[5] P. G. Goerss, H.-W. Henn, and M. Mahowald, The rational homotopy of the $K(2)$–local sphere and the chromatic splitting conjecture for the prime 3 and level 2, *Doc. Math.*, 19:1271–1290, 2014.

[6] P. G. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, A resolution of the $K(2)$-local sphere at the prime 3, *Ann. of Math. (2)*, 162(2):777–822, 2005.

[7] M. Hovey, Bousfield localization functors and Hopkins chromatic splitting conjecture, In *In Proceedings of the Čech centennial homotopy theory conference*, 1993..

[8] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1992.

The derived category of cochains on the classifying space of a compact Lie group

David Benson

Let G be a compact Lie group and k a field of characteristic p (possibly $p = 0$). The goal of this talk is to describe the classification of localising subcategories of the derived category $D(C^*(BG))$ of cochains $C^*(BG) = C^*(BG; k)$ in terms of the prime ideal spectrum of the cohomology ring $H^*(BG)$. This theorem is joint work with John Greenlees [1], and uses the machinery developed in joint work with Srikanth Iyengar and Henning Krause [2, 3, 4], building on the work of Mike Hopkins [5] and Amnon Neeman [6].

If X is a space, we write $C^*(X) = C^*(X; k)$ for the function spectrum from X to the Eilenberg–Mac Lane spectrum of k. This is a commutative ring spectrum, so its derived category $D(C^*(X))$ is a tensor triangulated category. The tensor structure comes from the derived tensor product, which we denote $A \otimes_{C^*(X)} B$, and the derived homs $\mathsf{Hom}_{C^*(X)}(A, B)$ give the function objects. The category $D(C^*(X))$ is compactly generated by the tensor identity $C^*(X)$, and so every localising subcategory is "tensor ideal."

The model for classification of localising subcategories is the case of the stable module category $\mathsf{StMod}\,(kG)$ of a finite group G over k. In this case, not every localising subcategory is tensor ideal, but it makes sense to limit ourselves to classification of those that are. In that case, the theorem of [4] states that there is a natural one to one correspondence between these and sets of non-maximal homogeneous prime ideals in the cohomology ring $H^*(G, k)$. This correspondence is described by the theory of support.

Theorem (Benson–Greenlees [1]). *The theory of support gives a natural one to one correspondence between the localising subcategories of $D(C^*(BG))$ and sets of homogeneous prime ideals in $H^*(BG)$.*

Let us write $\mathsf{Spec}R$ for the spectrum of homogeneous prime ideals in a graded ring R. Then $\mathsf{Spec}H^*(BG)$ was explicitly described by Quillen [7, 8]. The Quillen stratification theorem states that the natural map

$$\varinjlim_{E \in \mathcal{A}_p(G)} \mathsf{Spec}H^*(BE) \to \mathsf{Spec}H^*(BG)$$

is a bijection (but not an isomorphism at the level of schemes) where $\mathcal{A}_p(G)$ is the category whose objects are the elementary abelian subgroups $E \cong (\mathbb{Z}/p)^r$ (or $(S^1)^r$ if $p = 0$) and whose arrows are given by conjugation followed by inclusion inside G.

Next, we need to discuss restriction and induction. If H is a closed subgroup of G then we have a restriction homomorphism $\mathrm{res}_{G,H} \colon C^*(BG) \to C^*(BH)$, which gives rise to a restriction map also denoted $\mathrm{res}_{G,H} \colon H^*(BG) \to H^*(BH)$, and a functor

$$\mathrm{res}^*_{G,H} \colon D(C^*(BH)) \to D(C^*(BG)).$$

This has a left adjoint

$$\mathrm{ind}_{G,H} \colon D(C^*(BG)) \to D(C^*(BH))$$

given by the tensor product $C^*(BH) \otimes_{C^*(BG)} -$ and a right adjoint

$$\mathrm{coind}_{G,H} \colon D(C^*(BG)) \to D(C^*(BH))$$

given by $\mathsf{Hom}_{C^*(BG)}(C^*(BH), -)$. These are related by a Wirthmüller isomorphism

$$\mathrm{coind}_{G,H}(M) \cong \Sigma^d \mathrm{ind}_{G,H}(M)$$

where $d = \dim(G/H)$, provided that the action of H on the tangent space to the identity $eH \in G/H$ by conjugation preserves orientation (or k has characteristic two). Without this condition, there is still an isomorphism, but with a twist. We mention that one consequence of this isomorphism is that (with the same mild condition on G and H) the Hochschild homology and cohomology are related by

$$HH^*(C^*(BG)) \cong HH_{*+\dim G}(C^*(BG)).$$

Note that the right hand side may be computed via $HH_*(C^*(BG)) \cong H^{-*}(\Lambda BG^{\wedge}_p)$, the cohomology of the free loop space on the p-completion of BG.

Our classification of localising subcategories goes via a Chouinard style theorem. Recall that in the representation theory of finite groups, Chouinard's theorem says that a module is projective if and only if its restriction to every elementary abelian subgroup is projective. Our analogue is the following theorem.

Theorem. *If M is an object in $D(C^*(BG))$, and $\mathrm{ind}_{G,E}(M) \simeq 0$ for every $E \in \mathcal{A}_p(G)$ then $M \simeq 0$. Similarly, if $\mathrm{coind}_{G,E}(M) \simeq 0$ for every $E \in \mathcal{A}_p(G)$ then $M \simeq 0$.*

Our proof is quite different from Chouinard's, and can be translated into a proof of Chouinard's original theorem that avoids the use of Serre's theorem on products of Bocksteins.

We remark that in the case where G is a finite p-group, there is an equivalence of categories

$$D(C^*(BG)) \simeq K\mathsf{Inj}\,(kG),$$

where the right hand side is the homotopy category of complexes of injective (= projective) kG-modules. Moreover, there is a recollement

$$\mathsf{StMod}\,(kG) \rightleftarrows K\mathsf{Inj}\,(kG) \rightleftarrows D(kG)$$

This allows transfer of information between the stable module category of an elementary abelian p-group E and $D(C^*(BE))$. So our version of Chouinard's theorem, together with Quillen stratification and information about $\mathsf{StMod}\,(kE)$ (or rather $K\mathsf{Inj}\,(kE)$) from [4] is the raw input for proving the main theorem. This raw input is fed into the stratification machinery developed in [2, 3], and we now describe the basic setup for that machinery.

Let $\mathcal{T}$ be a triangulated category with a set of compact generators and having small coproducts. Write $\mathcal{T}^c$ for the compact objects in $\mathcal{T}$. Denote by $Z(\mathcal{T})$ the graded centre of $\mathcal{T}$, namely in degree n it consists of the natural transformations $\eta\colon \mathrm{Id} \to \Sigma^n$ (where Σ is the shift in $\mathcal{T}$) such that $\eta\Sigma = (-1)^n\Sigma\eta$. We assume that $\mathcal{T}$ comes with a Noetherian graded ring R and a map $R \to Z(\mathcal{T})$. Write $\mathsf{Spec}(R)$ for the set of homogeneous prime ideals of R, with the Zariski topology. A subset $\mathcal{U}$ of $\mathsf{Spec}(R)$ is said to be *specialisation closed* if $\mathfrak{p} \in \mathcal{U}$, $\mathfrak{q} \supseteq \mathfrak{p}$ imply $\mathfrak{q} \in \mathcal{U}$. If $\mathcal{U}$ is specialisation closed, we set

$$\mathcal{T}_{\mathcal{U}} = \{X \in \mathcal{T} \mid \mathsf{Hom}^*_{\mathcal{T}}(C, X)_{\mathfrak{p}} = 0\ \forall C \in \mathcal{T}^c,\ \mathfrak{p} \notin \mathcal{U}\}.$$

This is a localising subcategory of $\mathcal{T}$, and there is a localisation functor $L_{\mathcal{U}}\colon \mathcal{T} \to \mathcal{T}$ and a natural triangle

$$\Gamma_{\mathcal{U}}X \to X \to L_{\mathcal{U}}X$$

such that $\Gamma_{\mathcal{U}}X$ is in $\mathcal{T}_{\mathcal{U}}$, and $L_{\mathcal{U}}X = 0$ if and only if X is in $\mathcal{T}_{\mathcal{U}}$.

Given $\mathfrak{p} \in \mathsf{Spec}(R)$, choose specialisation closed subsets $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{V} = \mathcal{U} \cup \{\mathfrak{p}\}$ and $\mathfrak{p} \notin \mathcal{U}$. Then $\Gamma_{\mathfrak{p}}X$ is defined to be $\Gamma_{\mathcal{V}}L_{\mathcal{U}}X = L_{\mathcal{U}}\Gamma_{\mathcal{V}}X$, and this is independent of the choice of $\mathcal{U}$ and $\mathcal{V}$. The *support* of an object X in $\mathcal{T}$ is then defined to be

$$\mathsf{supp}(X) = \{\mathfrak{p} \in \mathsf{Spec}(X) \mid \Gamma_{\mathfrak{p}}X \neq 0\}.$$

The *local-global principle* says that for X in $\mathcal{T}$, the localising subcategory generated by X is equal to the localising subcategory generated by the set of $\Gamma_\mathfrak{p} X$ with $\mathfrak{p} \in \mathsf{Spec}(R)$. We do not know whether the local-global principal holds in general, but we do know that it holds provided R has finite Krull dimension, which is enough for our purposes. Provided that the local-global principal holds, there is a one to one correspondence between localising subcategories of $\mathcal{T}$ and choices of a localising subcategory of $\Gamma_\mathfrak{p}\mathcal{T}$ for each $\mathfrak{p}$.

We say that $\mathcal{T}$ is *stratified* by R if the local-global principle holds, and each $\Gamma_\mathfrak{p}\mathcal{T}$ is either zero or a minimal localising subcategory of $\mathcal{T}$. Under these circumstances, the localising subcategories of $\mathcal{T}$ are in one to one correspondence with subsets of $\mathsf{supp}(\mathcal{T}) = \{\mathfrak{p} \mid \Gamma_\mathfrak{p}\mathcal{T} \neq 0\}$. The main theorem of [1] states that $D(C^*(BG))$ is stratified by the action of $H^*(BG)$, with $\mathsf{supp}(D(C^*(BG))) = \mathsf{Spec}\, H^*(BG)$.

References

[1] D. J. Benson and J. P. C. Greenlees, *Stratifying the derived category of cochains on BG for G a compact Lie group*, J. Pure & Applied Algebra **218** (2014), 642–650.

[2] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.

[3] ———, *Stratifying triangulated categories*, J. Topology **4** (2011), 641–666.

[4] ———, *Stratifying modular representations of finite groups*, Ann. of Math. **174** (2011), 1643–1684.

[5] M. J. Hopkins, *Global methods in homotopy theory*, Homotopy Theory, Durham 1985, Lecture Notes in Mathematics, vol. 117, Cambridge University Press, 1987.

[6] A. Neeman, *The chromatic tower for D(R)*, Topology **31** (1992), 519–532.

[7] D. G. Quillen, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. **94** (1971), 549–572.

[8] ———, *The spectrum of an equivariant cohomology ring, II*, Ann. of Math. **94** (1971), 573–602.

Localization sequences in the algebraic K-theory of ring spectra

David Gepner

(joint work with Benjamin Antieau, Tobias Barthel)

The localization sequence in algebraic K-theory has been extensively studied and provides, along with trace methods, one of the only known techniques for computation of algebraic K-groups. Apart from results on the level of K_0, one of the earliest and most important instances of the localization sequence is due to Quillen [5]. Namely, if we fix a prime p, and consider the "localization at p" map $\mathbb{Z}_{(p)} \to \mathbb{Q}$, then this induces a fiber sequence of K-theory spectra

$$K(\mathbb{F}_p) \to K(\mathbb{Z}_{(p)}) \to K(\mathbb{Q}).$$

It is the identification of the fiber term which is the difficult part of this theorem, since the fiber of the map on the level of module categories

$$\mathrm{Mod}(\mathbb{Z}_{(p)}) \to \mathrm{Mod}(\mathbb{Q})$$

is not $\mathrm{Mod}(\mathbb{F}_p)$ but rather the full subcategory of $\mathrm{Mod}(\mathbb{Z}_{(p)})$ consisting of those modules on which p acts nilpotently. The fact that K-theory identifies this with $K(\mathbb{F}_p)$ is a consequence of Quillen's devissage theorem [5], a key result which has important, but limited, generalizations in the homotopical setting.

The earliest results in this direction are due to Blumberg-Mandell [3], who use a homotopical version of devissage to construct a localization sequence

$$K(BP\langle 0\rangle) \to K(BP\langle 1\rangle) \to K(E(1)).$$

Here $BP\langle n\rangle$ denotes the truncated Brown-Peterson spectrum and $E(n)$ the *nth* Johnson-Wilson spectrum; that is,

$$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$$

where $|v_i| = 2(p^i - 1)$,

$$\pi_* BP\langle n\rangle = \pi_* BP/(v_{n+1}, v_{n+2}, \dots) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n],$$

and

$$E(n) \simeq BP\langle n\rangle[v_n^{-1}].$$

Based on this (the $n = 1$ case) and Quillen's localization sequence (the $n = 0$ case), this led Rognes and others to expect a localization sequence

$$K(BP\langle n-1\rangle) \to K(BP\langle n\rangle) \to K(E(n))$$

for all $n \geq 0$.

Inspired by work of Lurie [4] on the existence of localizations of ring spectra, the spectral Morita theory of Schwede-Shipley [6], and a result of Antieau-Gepner [2] on the existence of compact generators for certain stable ∞-categories, we establish the following result: Let R be a ring spectrum and let $f \subset \pi_* R$ be a homogeneous element. Then the fiber of the localization map

$$\mathrm{Mod}(R) \to \mathrm{Mod}(R[f^{-1}])$$

is generated by a single compact object $K \simeq R/f$. In particular, the fiber is equivalent to $\mathrm{Mod}(A)$, where $A \simeq \mathrm{End}_R(K)$ is the endomorphism algebra of K. It then follows from the general K-theoretic machinery that we obtain a fiber sequence of K-theory spectra

$$K(A) \to K(R) \to K(R[f^{-1}]).$$

Specializing to $R = BP\langle n\rangle$, $f = v_n$, and $A\langle n-1\rangle = \mathrm{End}_{BP\langle n\rangle}(BP\langle n-1\rangle)$, we obtain a localization sequence

$$K(A\langle n-1\rangle) \to K(BP\langle n\rangle) \to K(E(n)).$$

One then hopes that, as in the $n = 0$ and $n = 1$ cases, devissage comes to the rescue in order to obtain a K-equivalence $K(A\langle n-1\rangle) \simeq K(BP\langle n-1\rangle)$, thus verifying Rognes' conjecture.

Unfortunately, this turns out not the case; rather, the map $K(BP\langle n\rangle) \to K(A\langle n\rangle)$ is not a homotopy equivalence for any $n > 0$ (at any prime p). Using the commutative diagram

$$\begin{array}{ccccc} BGL_1(BP\langle n\rangle) & \longrightarrow & K(BP\langle n\rangle) & \longrightarrow & THH(BP\langle n\rangle) \\ \downarrow & & \downarrow & & \downarrow \\ BGL_1(A\langle n\rangle) & \longrightarrow & K(A\langle n\rangle) & \longrightarrow & THH(A\langle n\rangle) \end{array}$$

and the fact that $\pi_*A\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n, \epsilon]$ is a square-zero extension of $\pi_*BP\langle n\rangle$ by a class ϵ in degree $-2(p^n - 1) - 1$, and therefore gives (again for $n > 0$) classes in $\pi_*BGL_1(A\langle n\rangle)$ which are not in the image of $\pi_*BGL_1(BP\langle n\rangle)$, it suffices to show that such a class maps to a class in $THH(A\langle n\rangle)$ which is not in the image of $THH(BP\langle n\rangle)$. In fact we only need to compute the rationalization of these topological Hochschild homology spectra to see this, and rationally, both $BP\langle n\rangle$ and $A\langle n\rangle$ are equivalent to graded symmetric algebras.

References

[1] B. Antieau, T. Barthel and D. Gepner, *On localization sequences in the algebraic K-theory of ring spectra*. Preprint.

[2] B. Antieau and D. Gepner, *Brauer groups and étale cohomology in derived algebraic geometry*, Geom. Topol. 18 (2014), no. 2, 1149–1244.

[3] A. Blumberg and M. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math. 200 (2008), no. 2, 155–179.

[4] J. Lurie, *Higher algebra*. Preprint.

[5] D. Quillen, *Higher algebraic K-theory, I.* Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[6] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology 42 (2003), no. 1, 103–153.

Flavors of Equivariant Commutativity

Michael A. Hill

(joint work with Andrew J. Blumberg and Michael J. Hopkins)

In this talk, I discussed joint work with Blumberg and with Hopkins which continues the study of what sort of structure Bousfield localization preserves in equivariant stable homotopy.

Localizations

Theorem (H.-Hopkins, McClure [6])**.** *If R is an equivariant commutative ring spectrum and E is an arbitrary equivariant spectrum, then $L_E(R)$ need not be commutative.*

The problem is that if Z is an E-acyclic, then we do not know the E-acylicity of

$$Sym^n(Z) = Z^{\wedge n}/\Sigma_n \simeq (Z^{\wedge n})_{h\Sigma_n} = E_G\Sigma_{n+} \wedge_{\Sigma_n} Z^{\wedge n},$$

where $E_G\Sigma_n$ is the universal space for principal Σ_n-bundles in G-spaces. In $E_G\Sigma_n$, we have cells of the form

$$G \times \Sigma_n/\Gamma_+ \wedge D^k,$$

where $\Gamma = \{(h, f(h))|h \in H \subset G, f\colon H \to \Sigma_n\}$. Understanding what localization does amounts to understanding

$$G \times \Sigma_n/\Gamma_+ \wedge_{\Sigma_n} Z^{\wedge n}, \tag{1}$$

but this is essentially the norm functor studied in Hill-Hopkins-Ravenel [4].

The subgroup Γ above is the graph of a homomorphism $H \to \Sigma_n$. This is equivalent to an H-set structure T on the set $\{1, \ldots, n\}$, and Hill-Hopkins-Ravenel identify the orbit space in Equation 1 as

$$G \times \Sigma_n/\Gamma_+ \wedge_{\Sigma_n} Z^{\wedge n} \simeq G_+ \wedge_H N^T i_H^* Z,$$

where $N^T(-)$ is the indexed smash product over the H-set T. This immediately gives sufficient (and in fact, necessary) conditions for localization to preserve commutative rings.

Theorem. *If for all acyclics Z for a localization L and for all subgroups H, $N_H^G Z$ is acylic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

This is a condition on the category of acyclics: if it is not just a symmetric monoidal category, but also one closed under certain norm maps, then the corresponding localization preserves commutative ring objects. In other words, the category of E-acyclics should be a G-symmetric monoidal category. This concept originated in this work, and it has been greatly expanded by work of Bohmann-Osorno and of Barwick [2, 1]. The prototype is a symmetric monoidal Mackey functor (namely a Mackey functor object in symmetric monoidal categories).

The category of modules over an equivariant commutative ring spectrum has the same structure.

Definition. If R is an equivariant commutative ring spectrum and M is an i_H^*R-module, then let

$${}_R N_H^G M = R \wedge_{N_H^G i_H^* R} N_H^G M,$$

where the $N_H^G i_H^* R$-module action on R is via the counit of the norm-restriction adjunction.

By choosing an orbit decomposition of any finite G-set, this immediately gives us a kind of multiplicative tensoring operation with any finite G-set T, and we will denote that ${}_R N^T$.

Theorem. *If R and E are equivariant commutative ring spectra, then $L_E(R)$ is also an equivariant commutative ring spectrum.*

This follows immediately from the condition on localizations using this internal norm in R-modules.

N_∞ Operads

Since the condition that the category of acyclics be a G-symmetric monoidal subcategory of G-spectra is a rather harsh one, we can ask what structure is preserved. We define a weakening of the notion of a G E_∞ operad.

Definition. An operad $\mathcal{O}$ in G-spaces is an N_∞ operad if (1) $\mathcal{O}_n$ is a universal space for a family of subgroups of $G \times \Sigma_n$ which all intersect Σ_n trivially, (2) $\mathcal{O}_1$ and $\mathcal{O}_0$ are G-equivariantly contractible, and (3) $\mathcal{O}_n^G$ is contractible for all n.

Ordinary G E_∞ operads satisfy this, where for each n, $\mathcal{O}_n$ is the universal space for the family of all subgroups which intersect Σ_n trivially. At the other extreme is the trivial N_∞ operad where $\mathcal{O}_n$ is the universal space for the family of subgroups of G.

As described above, associated to any subgroup Γ of $G \times \Sigma_n$ which intersects Σ_n trivially is a subgroup H of G and an H-set structure T on $\{1, \ldots, n\}$.

Definition. We say that an H-set T is *admissible* for an N_∞ operad $\mathcal{O}$ if the graph of its defining homomorphism is in the family for $\mathcal{O}_{|T|}$.

The assignment G/H to the full subcategory of finite H-sets spanned by the admissible sets for $\mathcal{O}$ gives a symmetric monoidal category valued coefficient system $\mathcal{C}^{\mathcal{O}}$. The fact that $\mathcal{O}$ is an operad shows that $\mathcal{C}^{\mathcal{O}}(G/H)$ is closed under disjoint unions, products, inductions indexed by elements of itself, and subobjects. The structure of $\mathcal{O}$ shows that it contains all trivial sets. We call a full sub-coefficient system of the coefficient system of finite sets that satisfies these properties an indexing system.

Theorem (Blumberg-H.)**.** *The symmetric monoidal coefficient system $\mathcal{C}^{\mathcal{O}}$ determines $\mathcal{O}$ up to homotopy. The functor $\mathcal{C}^{-}$ is a full-faithful embedding of the homotopy category of N_∞ operads into the poset of all indexing systems.*

The prototypes for N_∞ operads are the little disks operads $\mathcal{D}(U)$ and the linear isometries operads $\mathcal{L}(U)$ built on a G-universe U.

Proposition. *A finite G set T is admissible for $\mathcal{D}(U)$ iff T equivariantly embeds into U.*

A finite G set T is admissible for $\mathcal{L}(U)$ iff $\mathbb{R}\{T\} \otimes U$ isometrically embeds into U, where $\mathbb{R}\{T\}$ is the permutation representation with basis T.

The admissible sets also play a vital role in understanding what operations show up.

Proposition. *If T is an admissible G-set and R is an $\mathcal{O}$-algebra, then there is a contractible space of maps*

$$N^T(R) \to R.$$

In particular, for $T = * \amalg *$, we have a coherently commutative multiplication on R, and for $T = G/H$, we have norm maps like the counit adjunction.

If R is an $\mathcal{O}$-algebra for an N_∞ operad $\mathcal{O}$, then we can mirror many of the earlier constructions.

Theorem. *If R is an $\mathcal{O}$-algebra and E is any equivariant spectrum, then $L_E(R)$ is an $\mathcal{O}$-algebra if for every admissible H-set T and for every E-acyclic Z, we have $N^T i_H^* Z$ is $i_H^* E$-acyclic.*

In particular, for the trivial N_∞ operad, the only admissible sets are the sets with trivial action, and thus localization always preserves this structure.

More excitingly, when $\mathcal{O}$ is the linear isometries operad for some universe U, then we have a good theory of S-modules in orthogonal spectra.

Theorem. *If $\mathcal{O} = \mathcal{L}(U)$, then there is a symmetric monoidal category of S-modules built out of $\mathcal{L}(U)$ for which the commutative monoids are exactly the $\mathcal{O}$-algebras. The category of modules over an $\mathcal{O}$-algebra R is therefore symmetric monoidal and has as many norms as $\mathcal{O}$ provided.*

References

[1] C. Barwick. Spectral Mackey functors and equivariant algebraic K-theory (II). arXiv.org: 1505.03098

[2] A. M. Bohmann and A. M. Osorno. Constructing equivariant spectra via categorical Mackey functors. arXiv.org: 1405.6126.

[3] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[4] Michael A Hill, Michael J Hopkins, and Douglas C Ravenel. On the non-existence of elements of Kervaire invariant one. arXiv.org: 0908.3724.

[5] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.

[6] J. E. McClure. E_∞-ring structures for Tate spectra. *Proc. Amer. Math. Soc.*, 124 (1996), no 6. 1918–1922.

Modular surfaces and chromatic height 3

Tyler Lawson

A monic degree-4 polynomial $f(z)$ with no repeated roots determines an equation

$$w^3 = f(z).$$

The set of solutions to this is a plane curve, and the closure inside projective space $\mathbb{P}^2$ (which adds one point at ∞) is a smooth curve $\mathcal{C}$ of genus 3. Moreover, there is an action of the third roots of unity on $\mathcal{C}$ by $(w, z) \mapsto (\zeta w, z)$. These curves have isomorphisms between them determined by maps of the form $z \mapsto z + t$.

Being of genus 3, the curve $\mathcal{C}$ has a 3-dimensional basis of the space of differential forms. In this case, they are

$$\frac{dz}{w}, \frac{zdz}{w^2}, \frac{dz}{w}.$$

There is an induced action of the third roots of unity on these: the first two forms are acted on by $\omega \mapsto \zeta^2\omega$, while the third is acted on by $\omega \mapsto \zeta\omega$. As a result, the curve with its action determines a canonical 1-dimensional split summand of its space of differential forms.

These curves were originally studied by Picard [5], who studied the abelian integrals

$$\int \frac{dz}{\sqrt[3]{f(z)}}.$$

Near any point $p \in \mathcal{C}$ (in particular, the point at ∞) with a coordinate function u, this differential form has a power series expression $\ell'(u)du$, and "integration from p to u" determines a uniformizing function ℓ to $\mathbb{C}$.

The power series $\ell(u) = \int \ell'(u)du$ is the logarithm for a 1-dimensional formal group law $\mathbb{G}(\mathcal{C})$. More, if we choose the point p to be the point at infinity, the coefficients of the formal group law are polynomials in the coefficients of $f(z)$. This provides a source of algebraic formal group laws, whose heights can vary between 0 and 3.

We can, instead, describe what is happening from the eyes of algebraic geometry. Each such curve $\mathcal{C}$ has a Jacobian variety $J(\mathcal{C})$, which is a 3-dimensional abelian variety. The action of the third roots of unity on $\mathcal{C}$ extend to the Jacobian, where they factor through an action of the Eisenstein integers $\mathbb{Z}[\zeta]/(\zeta^2+\zeta+1)$ — the ring of integers in $\mathbb{Q}(\sqrt{-3})$. As the Jacobian of a curve, the resulting object has a canonical structure called a polarization, essentially capturing the intersection pairing on $H_1(\mathcal{C};\mathbb{Z})$. The 1-dimensional formal group law $\mathbb{G}(\mathcal{C})$ described is a canonical split 1-dimensional summand $\mathbb{G}(J)^+$ of the formal group law $\mathbb{G}(J)$ of the Jacobian, determined by the action.

In short, this assignment from monic degree-4 polynomials (mod isomorphism) to formal group laws factors through a moduli of polarized abelian 3-folds with action of a ring of integers (a PEL Shimura variety).

The significance for homotopy theory is this: in joint work with Behrens [1], many maps to these moduli were shown to produce functorial E_∞ ring spectra realizing their formal group laws. The necessary conditions are satisfied in this case when $p \equiv 1 \bmod 3$. For such primes, any of the degree-4 polynomials described above can be brought into the canonical form $z^4 + a_2z^2 + a_3z + a_4$, whose quartic discriminant Δ is assumed to not vanish.

In this case, we obtain the following result. For primes $p \equiv 1 \bmod 3$, there exists a complex oriented E_∞ ring spectrum with graded coefficient ring $\mathbb{Z}[a_2, a_3, a_4, \Delta^{-1}]^\wedge_p$, where $|a_i| = 6i$, whose formal group law is that determined by the above assignment. (This cohomology theory, without the ring structure, could also have been constructed using the Landweber exact functor theorem.) The resulting cohomology theory bears a formal similarity to the cohomology theories eo_{p-1} studied by Gorbunov–Mahowald [2].

However, this result is not optimal. It turns out that, while the curve $\mathcal{C}$ degenerates to a singular curve when $\Delta = 0$, the associated Jacobian $J(\mathcal{C})$ is perfectly well-behaved unless $f(z)$ has triple roots; this corresponds to the degeneration of the curve $\mathcal{C}$ to a singular stable curve obtained by joining a string of curves of genus 1 or genus 2 at points. This allows us to extend to a larger moduli of stable curves, capturing a larger portion of the Shimura variety.

In this next case, we obtain the following more refined result. For primes $p \equiv 1 \bmod 3$, there exists a complex oriented E_∞ ring spectrum whose coefficient ring is a square-zero extension of $\mathbb{Z}_p[a_2, a_3, a_4]$, and whose formal group law is the same. (The resulting spectrum *does not* come from the Landweber exact functor theorem.)

Unfortunately, in this case the square-zero portion is the module

$$\mathbb{Z}[a_2, a_3, a_4]/((a_2^2 + 12a_4)^\infty, (27a_2^3 + 8a_3^2)^\infty)_p^\wedge$$

which introduces a great deal of noise into both positive and negative degrees, making computations less practical. This, in some sense, is a consequence of the fact that the moduli of "degree-4 polynomials without triple roots" is not compact.

However, the situation is very closely similar to that with the ordinary moduli of elliptic curves, where there is a compactification allowing elliptic curves to become degenerate objects. There exist several compactifications of these PEL Shimura varieties, and it seems likely that the methods of [3] might extend to this context. This represents ongoing research.

If so, we would obtain the following. For primes $p \equiv 1 \bmod 3$, there would exist a complex oriented E_∞ ring spectrum whose coefficient ring is a square-zero extension of $\mathbb{Z}_p[a_2, a_3, a_4]$ by $\mathbb{Z}[a_2, a_3, a_4]/(a_2^\infty, a_3^\infty, a_4^\infty)_p^\wedge$. In particular, the "noise" would be concentrated in negative degrees, and the resulting spectrum would admit a well-behaved connective cover with coefficient ring $\mathbb{Z}_p[a_2, a_3, a_4]$.

This geometric moduli of curves is the first example of a *Picard modular surface*, classifying abelian 3-folds with a certain type of action of a ring of integers, which have played an important role as examples in the Langlands program [4]. It is special among the Picard modular surfaces: most of them don't have such a compact description because of the difficulty in describing abelian 3-folds with explicit equations, and it does not yet appear to be easy to describe such 3-folds with action as the Jacobians of curves.

However, even in the absence of such concrete models there are still tools to get at calculations. These surfaces, complex-analytically, are quotients of the ball D^4 in $\mathbb{C}^2$ by explicit groups; the Arthur trace formula gives a concrete expression for the dimensions of cohomology groups; and there is an abundance of special algebraic points with extra symmetry which can serve to help identify appropriate rational and integral points. Our ongoing program is to understand what constructions in homotopy theory are made possible by these and employ them to illustrate calculations at chromatic height 3.

References

[1] M. Behrens and T. Lawson, *Topological automorphic forms*, Mem. Amer. Math. Soc. **204** (2010), no. 958, xxiv+141 pp.

[2] V. Gorbounov and M. Mahowald, *Formal completion of the Jacobians of plane curves and higher real K-theories*, J. Pure Appl. Algebra **145** (2000), no. 3, 293–308.

[3] M. Hill and T. Lawson, *Topological modular forms with level structure*, Invent. Math., to appear.
[4] *The zeta functions of Picard modular surfaces*, edited by Robert P. Langlands and Dinakar Ramakrishnan, Univ. Montréal, Montreal, QC, 1992.
[5] E. Picard, *Sur des fonctions de deux variables indépendantes analogues aux fonctions modulaires*, Acta Math. **2** (1883), no. 1, 114–135.

K-theory of group algebras and topological cyclic homology

Wolfgang Lück
(joint work with Holger Reich, John Rognes, Marco Varisco)

We prove that the Farrell-Jones assembly map for connective algebraic K-theory is rationally injective, under mild homological finiteness conditions on the group and assuming that a weak version of the Leopoldt-Schneider conjecture holds for cyclotomic fields. This generalizes a result of Bökstedt, Hsiang and Madsen [7], and leads to a concrete description of a large direct summand of $K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ in terms of group homology. Since the number theoretic assumption holds in low dimensions, this also computes a large direct summand of $\mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. The proof uses the cyclotomic trace to topological cyclic homology, the functor C due to Bökstedt-Hsiang-Madsen, new general injectivity results about the assembly maps for THH and C and equivariant Chern characters.

The Farrell-Jones Conjecture for a group G and an associative ring with unit R predicts that a certain assembly map

$$EG_{\mathcal{VCYC}} \wedge_{\mathrm{Or}(G)} \mathbf{K}_R \to \mathbf{K}(RG)$$

is a weak equivalence, where $\mathbf{K}$ refers to non-connective algebraic K-theory. There is also an L-theory version. The original version is due to Farrell-Jones [9, 1.6 on page 257]. The Farrell-Jones Conjecture has many applications, for instance, it implies the famous Novikov Conjectures about the homotopy invariance of higher signatures and the Borel Conjecture about the topological rigidity of aspherical manifolds in dimensions greater or equal to five. For a survey about it and its applications we refer for instance to [4, 10, 15, 16].

There is a more general version with coefficients in additive categories and with finite wreath products which allows to consider twisted group rings and orientation characters in L-theory. This general version has nice inheritance properties. In the recent years there was tremendous progress in enlarging the class of groups for which this general version is known, it contains hyperbolic groups, CAT(0)-groups, arithmetic groups, lattices in virtually connected locally compact second countable Hausdorff groups, fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary). See for instance [1, 2, 3, 5, 12, 20].

In this paper we will not be able to consider this general version, we have to restrict our attention to the ring $R = \mathbb{Z}$ and to K-theory. Notice, however, that all the papers above treat groups with a flavor of being non-negatively curved and the proofs are rather geometric using controlled topology and flows. We will achieve results which do only need some homotopy theoretical finiteness conditions (besides the number theoretic assumptions) and the methods of proof are completely different. We will cover prominent groups such as mapping class groups, $\mathrm{Out}(F_n)$ and Thompson groups for which nothing was known beforehand.

Here is our main technical result which we, for simplicity, do not state in its most general form:

Theorem. *Let G be a group. Assume that the following two conditions hold:*

- *For every finite cyclic subgroup $C \subseteq G$ and natural number s the integral group homology $H_s(BZ_GC;\mathbb{Z})$ of the centralizer of C in G is a finitely generated abelian group.*
- *For every finite cyclic subgroup $C \subseteq G$ and natural number t the natural homomorphism*

$$K_t(\mathbb{Z}[\mu_{|C|}]) \otimes_{\mathbb{Z}} \mathbb{Q} \to \prod_{p \text{ prime}} K_t\Big(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_{|C|}]; \mathbb{Z}_p\Big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective, where $\mu_{|C|}$ is the primitive $|C|$-th roots of unity.

Then the restriction of the rationalized Farrell-Jones assembly map

$$\bigoplus_{(C)} \bigoplus_{s+t=n} H_s(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C\Big(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}\Big) \to K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

to the summands for $t \geq 0$ is injective for all $n \geq 0$. Here (C) runs through the conjugacy classes of finite cyclic subgroups of G, Θ_C is an idempotent in the rationalized Burnside ring $A(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ which operates on $K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$ thanks to the Mackey structure, and $W_GC := N_GC/Z_GC$ for N_GC the normalizer of C in G.

The Farrell-Jones Conjecture implies that the rationalized Farrell-Jones assembly map is bijective.

The following two corollaries illustrate the generality of our main result:

Corollary. Let G be a group. Assume that for every finite cyclic subgroup C of G the abelian groups $H_1(BZ_GC;\mathbb{Z})$ and $H_2(BZ_GC;\mathbb{Z})$ are finitely generated. Then the map

$$\operatorname{colim}_{H \subseteq G, |H| < \infty} \mathrm{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Corollary. Let G be a group. Assume that there is a cocompact model for the classifying space of proper actions. Then there is a natural number N depending on its dimension and the order of the finite cyclic subgroups of G such that the K-theoretic assembly map induces for all $n \geq N$ an injection

$$\bigoplus_{(C)} \bigoplus_{s+t=n} H_s(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C\Big(K_t(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}\Big) \to K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

For more information about topological cyclic homology we refer for instance to [8]. More information about the Conjectures due to Leopoldt and Schneider can be found in [19]. A survey on classifying spaces is given in [14]. Equivariant Chern characters are discussed in [13]. Basic input for the proofs of the Main Theorem comes also from [6, 11, 17, 18].

References

[1] A. Bartels, F. T. Farrell, and W. Lück. The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups. *J. Amer. Math. Soc.*, 27(2):339–388, 2014.

[2] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)*, 175:631–689, 2012.

[3] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, 172(1):29–70, 2008.

[4] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones Conjecture and its applications. *Journal of Topology*, 1:57–86, 2008.

[5] A. Bartels, W. Lück, H. Reich, and H. Rüping. K- and L-theory of group rings over $\mathrm{GL}_n(\mathbf{Z})$. *Publ. Math., Inst. Hautes Étud. Sci.*, 119:97–125, 2014.

[6] A. J. Blumberg. Continuous functors as a model for the equivariant stable homotopy category. *Algebr. Geom. Topol.*, 6:2257–2295, 2006.

[7] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[8] B. I. Dundas, T. G. Goodwillie, and R. McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London Ltd., London, 2013.

[9] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K-theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993.

[10] G. Gandini, S. Meinert, and H. Rüping. The Farrell-Jones Conjecture for graph of abelian groups. Preprint, arXiv:1309.6909 [math.GR], 2013.

[11] L. Hesselholt and I. Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[12] H. Kammeyer, W. Lück, and H. Rüping. The Farrell-Jones Conjecture for arbitrary lattices in virtually connected Lie groups. Preprint, arXiv:1401.0876 [math.KT], 2014.

[13] W. Lück. Chern characters for proper equivariant homology theories and applications to K- and L-theory. *J. Reine Angew. Math.*, 543:193–234, 2002.

[14] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[15] W. Lück. K- and L-theory of group rings. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 1071–1098, New Delhi, 2010. Hindustan Book Agency.

[16] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[17] W. Lück, H. Reich, and M. Varisco. Commuting homotopy limits and smash products. *K-Theory*, 30(2):137–165, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part II.

[18] H. Reich and M. Varisco. On the Adams isomorphism for equivariant orthogonal spectra. Preprint, arXiv:1404.4034 [math.AT], 2014.

[19] P. Schneider. Über gewisse Galoiscohomologiegruppen. *Math. Z.*, 168(2):181–205, 1979.

[20] C. Wegner. The Farrell-Jones conjecture for virtually solvable groups. Preprint, arXiv:1308.2432 [math.GT], 2013.

Descent and nilpotence in stable homotopy theory

Akhil Mathew

Let A be a commutative ring, and let B be a commutative A-algebra. Suppose B is *faithfully flat* over A. In this case, the classical theory of *faithfully flat descent* enables one to describe an A-module as a B-module (its base-change) together with an extra piece of structure known as a *descent datum.*

Recall this construction. If M is an A-module, then $B \otimes_A M$ is a B-module N. However, N has an additional structure: the two-base changes of N to $B \otimes_A B$ are canonically isomorphic. That is, there is an isomorphism of $B \otimes_A B$-modules

$$\phi \colon N \otimes_A B \simeq B \otimes_A N,$$

because both are base-changes (to $B \otimes_A B$) of the A-module M. In addition, ϕ satisfies a natural cocycle condition, which states that the two natural isomorphisms that ϕ determines from $N \otimes_A B \otimes_A B$ to $B \otimes_A B \otimes_A N$ (in $B \otimes_A B \otimes_A B$-modules) are equal.

We say that a *descent datum* on a B-module N is an isomorphism ϕ as above which satisfies the cocycle condition. Then the theory of faithfully flat descent, due to Grothendieck, states that there is an equivalence between the category of A-modules and the category of B-modules equipped with a descent datum.

Suppose now that $A \to B$ is a morphism of E_∞-ring spectra. One wishes to study module spectra over A and B, and to set up an analogous theory of descent in suitable situations. The above definition of a descent datum is not suited to homotopy theory, where higher coherences will naturally be required. The theory of ∞-categories enables one to solve these problems. It is possible to define a homotopy coherent version of the category (now an ∞-category) of descent data.

Definition (Lurie)**.** The ∞-category of *descent data* is given by a totalization

$$\mathrm{Desc}_{A \to B} = \mathrm{Tot}\left(\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \substack{\to \\ \to \\ \to} \dots \right).$$

The cosimplicial diagram of module ∞-categories arises from the cosimplicial E_∞-ring

$$B \rightrightarrows B \otimes_A B \substack{\to \\ \to \\ \to} \dots,$$

which is the classical *cobar construction* (e.g., which arises in the Adams spectral sequence).

When one takes A and B to be discrete rings and one takes discrete modules, the above totalization recovers the classical category of B-modules with descent data. As in the classical setting, one has a natural functor

$$\mathrm{Mod}(A) \to \mathrm{Desc}_{A \to B},$$

and the basic descent question asks when this functor is an equivalence.

Definition (Lurie)**.** The morphism $A \to B$ of E_∞-ring spectra is said to be faithfully flat if:

(1) $\pi_0(A) \to \pi_0(B)$ is a faithfully flat morphism of commutative rings.

(2) The map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism.

In this case, one has the following derived version of faithfully flat descent.

Theorem (Lurie). *Suppose $A \to B$ is a faithfully flat morphism of E_∞-rings. Then the natural functor* $\mathrm{Mod}(A) \to \mathrm{Desc}_{A\to B}$ *is an equivalence of symmetric monoidal ∞-categories.*

The purpose of this circle of ideas is to use descent-theoretic statements such as the above to study rings and modules in stable homotopy theory, and more generally in a symmetric monoidal ∞-category.

However, faithfully flat descent has a certain shortcoming here. Let A be an E_∞-ring and let M be an A-module. Then $\pi_*(M)$ is a graded $\pi_*(A)$-module, and it holds significant (though partial) information about M: if N is another A-module, then there is a spectral sequence

$$\mathrm{Ext}^{s,t}_{\pi_*(A)}(\pi_*(M), \pi_*(N)) \implies \pi_{t-s}(\mathrm{Hom}_A(M, N)).$$

It is clear that if $\pi_*(A)$ is homologically simpler (e.g., has finite homological dimension) then that will make the above spectral sequence easier to calculate with. More generally, one should expect that the theory of A-modules will be more "algebraic" if $\pi_*(A)$ is homologically simpler. However, many of the E_∞-rings that one hopes to work with (e.g., KO, TMF) have quite complicated homotopy rings and, furthermore, homological "complications" on π_* will be inherited by any faithfully flat extension of ring spectra. Instead, one wants a type of descent that works for morphisms of ring spectra that are not necessarily faithfully flat.

Definition. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal, stable ∞-category and let $R \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object. Let $I = \mathrm{fib}(\mathbf{1} \to R)$ be the fiber of the unit map. We say that R is *descendable* if the map $I \to \mathbf{1}$ is smash nilpotent, i.e., if there exists n such that $I^{\otimes n} \to \mathbf{1}$ is nullhomotopic. If $A \to B$ is a morphism of E_∞-ring spectra, we say that it is descendable if $B \in \mathrm{CAlg}(\mathrm{Mod}(A))$ is descendable.

The above definition is by no means new; it dates back in various forms to Bousfield, and has been explored by many authors, for instance recently by Balmer [1]. Our main result is an ∞-categorical descent theorem in this setting.

Theorem. *If $R \in \mathrm{CAlg}(\mathcal{C})$ is descendable, then we have an equivalence of symmetric monoidal ∞-categories $\mathcal{C} \simeq \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(R) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(R \otimes R) \substack{\to\\\to\\\to} \dots\right)$.*

It turns out that there are numerous examples of descendable morphisms of E_∞-rings which are far from faithfully flat. For instance:

- The map $L_n S^0 \to E_n$ is descendable by the Hopkins-Ravenel smash product theorem.
- Any faithful Galois extension of E_∞-rings in the sense of Rognes is descendable. Given a faithful Galois extension $A \to B$, one obtains an equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}(A) \simeq \mathrm{Mod}(B)^{hG}$,

which has been previously observed by Gepner, Lawson, and Meier and probably others.
- If G is a finite group, then the map $k^{BG} \to \prod_{A \subset G} k^{BA}$, as A ranges over the *elementary abelian* p-subgroups of G, is descendable.

Suppose given a descendable morphism $A \to B$ of E_∞-ring spectra, where $\pi_*(B)$ is simpler homologically than $\pi_*(A)$. A basic example of this occurs from the complexification map $KO \to KU$, which is actually a faithful C_2-Galois extension. The decompositions given by descent theorems actually offer a practical tool for calculating certain invariants of A-modules, when combined with techniques (such as the Bousfield-Kan spectral sequence) for manipulating large totalizations.

An example of these techniques is found in the calculations of *Picard groups* using descent-theoretic methods, in joint work with V. Stojanoska. Suppose $A \to B$ is a faithful G-Galois extension of E_∞-rings, so that we have an equivalence $\mathrm{Mod}(A) \simeq \mathrm{Mod}(B)^{hG}$. As a result, one obtains that the *Picard spectrum* $\mathfrak{pic}(A)$ (whose π_0 records the Picard group of A) is obtained, up to connective covers, as

$$\mathfrak{pic}(A) \simeq \tau_{\geq 0}\mathfrak{pic}(B)^{hG}.$$

If $\pi_*(B)$ is homologically simple, so that one can compute the Picard group of B directly, this yields an often practical technique for computing the Picard group of A. Using these and similar techniques, one can obtain:

Theorem (M.-Stojanoska; Hopkins)**.** *The Picard group of integral TMF is cyclic, given by $\mathbb{Z}/576$ generated by the suspension. The Picard group of Tmf is $\mathbb{Z} \oplus \mathbb{Z}/24$.*

There are also "coarser" invariants of ring spectra that one can study using these constructions, such as the thick subcategories of perfect modules and the Galois group. We refer to [3] and [4] for some instances of this.

References

[1] Paul Balmer Separable extensions in tt geometry. 2013. arXiv:1309.1808.
[2] Jacob Lurie. Derived algebraic geometry.
[3] Akhil Mathew. The Galois group of a stable homotopy theory. 2014. arXiv:1404.2156.
[4] Akhil Mathew. A thick subcategory theorem for modules over certain ring spectra. 2015. To appear in *Geom. Topol.*
[5] Akhil Mathew and Vesna Stojanoska. The Picard group of topological modular forms via descent theory. 2014. arXiv:1409.7702.
[6] Akhil Mathew, Niko Naumann, and Justin Noel. Derived induction and restriction theory I-II. 2015. In preparation.

Descent and nilpotence in equivariant stable homotopy theory

Niko Naumann
(joint work with Akhil Mathew, Justin Noel)

We report on joint work in progress with A. Mathew (Berkeley) and J. Noel (Regensburg) which puts into a general context classical results like Quillen's Theorem on the mod p cohomology of finite groups and certain results from the character theory of Hopkins-Kuhn-Ravenel.

Fix a finite group G and a family F of subgroups of G. The key notion is the following.

Definition. The thick tensor ideal $F^{Nil} \subseteq Sp_G$ of the category of (genuine) G-Spectra generated by $\{G/H_+\}_{H \in F}$ is called the subcategory of *F-nilpotent spectra.*

Our first result provides various characterizations of these F-nilpotent spectra.

Theorem. *For $M \in Sp_G$, the following are equivalent:*

i) *We have $M \in F^{Nil}$.*
ii) *For every subgroup $K \subseteq G$ not in F, there is an $n \geq 0$ such that*

$$e^n_{\tilde{\rho_K}} \wedge Res^G_K(M) \equiv *,$$

where $e_{\tilde{\rho_K}}$ denotes the Euler class of the reduced regular representation of K.
iii) *The canonical map $M \to \lim_{H \in F} M^{G/H_+}$ is an equivalence, and for every $X \in Sp_G$, the resulting holim spectral sequence*

$$E_2^{s,t} = \lim_{H \in F}{}^s \; \pi_t^G(F(X, M^{G/H_+})) \Rightarrow \pi^G_{t-s}(F(X,M)) \simeq M_G^{s-t}(X)$$

has a horizontal vanishing line at a finite page, uniformly in X.

Item *ii*) is what generates all our examples while item *iii*) is what reproduces results like Quillen's Theorem mentioned above.

We gave a list of examples and applications in the spirit of those classical results. We ended the talk by explaining how these ideas, when coupled with the recent solution by the authors of May's nilpotence conjecture, leads to the following result generalizing previous work of Thomason.

Theorem. *Assume G is a finite group and $A \to B$ is a G-Galois extension in the sense of Rognes satisfying $1 \in im(K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q})$. Then the canonical map of K-theory spectra $K(A) \to K(B)^{hG}$ is an L_n-equivalence for all primes and all $n \geq 0$.*

The Universal Property of Global Spectra and Elliptic Cohomology

Thomas Nikolaus
(joint work with David Gepner)

The talk was about an ongoing research project together with David Gepner. The main goal of this project is to understand variants of elliptic cohomology as equivariant cohomology theories. It turns out that for this purpose it is important to understand global spectra in the sense of Schwede [4] from another angle. To sketch the idea let us start by looking at ordinary cohomology theories differently. We define that a *twisted cohomology theory E on spaces* consists of

- abelian groups $E^V(X)$ for every vector bundle $V \to X$ with metric;
- morphisms $f^* : E^V(Y) \to E^{f^*V}(X)$ for every morphism $f : X \to Y$;
- morphisms $g_* : E^V(X) \to E^W(X)$ for every affine morphism $V \to W$ of vector bundles over X;

satisfiying a list of axioms. For such a cohomology theory E and vector bundles $V, W \to X$ we set

$$E^{V-W}(X) := \ker\left(E^{p^*V}(S^W) \xrightarrow{s^*} E^V(X)\right)$$

where S^W is the fibrewise one point compactification, $p : S^W \to X$ the projection and $s : X \to S^W$ the section at 'infinity'. Then the most important axiom states that a certain canonical morphism $E^V(X) \to E^{V\oplus W-W}(X)$ is an isomorphism. An example of such a twisted cohomology theory on spaces is given for every spectrum E by setting $E^V(X) = E^0(X^{-V}) = \pi_0 \operatorname{map}(X^{-V}, E)$ where X^{-V} is the Thom spectrum of the virtual bundle $-V$ over X. For many examples of cohomology theories one can give more geometric descriptions of the groups $E^V(X)$, for example for K-theory or bordism theories.

Proposition. *Every twisted cohomology theory on (finite) spaces is represented by a spectrum. There are canonical Gysin maps*

$$p_! : E^{T_{M/N}\oplus p^*V}(M) \to E^V(N)$$

for $p : M \to N$ a smooth, proper submersion.

We now want to generalize this description to the equivariant setting. Therefore we replace spaces by stacks. In this context a stack is just a contravariant (pseudo)functor from the category of topological spaces to the (2-)category of groupoids, which satisfies descent. Examples include the stack $\mathbb{B}G$ of G-bundles and quotient stacks $[X/G]$ for a G-space X. Stacks posses an interesting homotopy theory first studied by Gepner-Henriques [1] which generalizes equivariant homotopy theory. The main result of [1] is that the homotopy theory of stacks is equivalent to the homotopy theory of Orbispaces $\mathcal{S}^{\mathrm{Orb}}$. An Orbispace is a space-valued presheaves on the global orbit category Orb which is a topologically enriched category whose objects are $\mathbb{B}G$ for G a compact Lie group.

There is a notion of vector bundles $\mathcal{V} \to \mathcal{X}$ for stacks and every such vector bundle has a fibrewise one point compactification $S^{\mathcal{V}} \to \mathcal{X}$. Then a *cohomology* *theory on stacks* is defined exactly as a twisted cohomology theory for spaces, i.e. we have groups $E^{\mathcal{V}}(\mathcal{X})$ for every vector bundle $\mathcal{V}$ over a stack $\mathcal{X}$ together with the respective functorialities and the same axioms. Geometric examples of such cohomology theories can for example be obtained for K-theory.

We define a topologically enriched category $\mathrm{Orb}_{\mathrm{rep}}$ whose objects are pairs consisting of a compact Lie group G and an orthogonal G-representation V. Then the category of *pre-Orbispectra* is defined to be the category of space valued presheaves on $\mathrm{Orb}_{\mathrm{rep}}$. Among these pre-Orbispectra there are those which satisfy the analogue of being an Ω-spectrum, and these we call *Orbispectra*. The homotopy theory of Orbispectra can now be represented by a Quillen model category $\mathcal{S}p^{\mathrm{Orb}}$ or alternatively as a presentable ∞-category. For every stack $\mathcal{X}$ together with a vector bundle $\mathcal{V} \to \mathcal{X}$ there is a certain Thom-Orbispectrum $\mathcal{X}^{-\mathcal{V}}$. Thus every Orbispectrum E gives us a cohomology theory for stacks by setting

$$E^{\mathcal{V}}(X) := \pi_0 \operatorname{map}_{\mathcal{S}p^{\mathrm{Orb}}}(\mathcal{X}^{-\mathcal{V}}, E)$$

Theorem (Gepner, N.). *(1) Every cohomology theory on (finite) stacks is represented by an Orbispectrum.*

(2) There are canonical Gysin morphisms for $p : M \to N$ a representable, smooth proper submersion of stacks

$$p_! : E^{T_{M/N} \oplus p^* V}(M) \to E^V(N)$$

(3) The Picard group of Orbispectra (i.e. the objects which admit a tensor inverse) is given by $\mathbb{Z}$ represented by the ordinary sphere spectra.

Theorem (Hausmann, N.). *The model category $\mathcal{S}p^{\mathrm{Orb}}$ of Orbispectra is Quillen equivalent to Schwede's global spectra, i.e. orthogonal spectra with the global model structure.*

This shows that Schwede's global spectra model cohomology theories on stacks. Now the question remains, what the abstract role played by this global stable homotopy theory is. In other words, which universal property the process which passes from the homotopy theory of stacks $\mathcal{S}^{\mathrm{Orb}}$ to the homotopy theory of Orbispectra $\mathcal{S}p^{\mathrm{Orb}}$ enjoys. The fact that the ∞-category of Orbispectra has almost no tensor invertible elements shows that it can not be described by universally 'inverting' spheres as for ordinary spectra. The correct description is that one has to 'relatively invert' spheres of vector bundles $\mathcal{V} \to \mathcal{X}$. To describe that less informally we need some definitions. Let $\mathcal{C}$ be the ∞-category describing the homotopy theory of stacks (i.e. the ∞-category underlying $\mathcal{S}^{\mathrm{Orb}}$) or more generally any presentable, locally cartesian closed ∞-category. For another presentable ∞-category $\mathcal{D}$ which is tensored over $\mathcal{C}$ (such as the ∞-category underlying $\mathcal{S}p^{\mathrm{Orb}}$) and an object $\mathcal{X} \in \mathcal{C}$ we define

$$\mathcal{D}_{/\mathcal{X}} := \mathcal{C}_{/\mathcal{X}} \otimes_{\mathcal{C}} \mathcal{D}$$

Here the tensor product is the tensor of presentable ∞-categories introduced by Lurie [3, Section 4.8.1]. For example if $\mathcal{C}$ is the ∞-category of spaces and $\mathcal{D}$ is the ∞-category of spectra then $\mathcal{D}_{/\mathcal{X}}$ is equivalent to the ∞-category of parametrized spectra over $\mathcal{X}$. If $\mathcal{D}$ is pointed (i.e. has an object which is initial and terminal) then for every vector bundle $\mathcal{V} \to \mathcal{X}$ we obtain an endofunctor

$$- \wedge S^{\mathcal{V}} : \mathcal{D}_{/\mathcal{X}} \to \mathcal{D}_{/\mathcal{X}}$$

induced from the the smash product with $S^{\mathcal{V}}$ on $(\mathcal{C}_{/\mathcal{X}})_*$.

Definition. We say that $\mathcal{D}$ is *$\mathcal{C}$-stable* if

(1) $\mathcal{D}$ is pointed
(2) For every vector bundle $\mathcal{V} \to \mathcal{X}$ the induced functor

$$- \wedge S^{\mathcal{V}} : \mathcal{D}_{/\mathcal{X}} \to \mathcal{D}_{/\mathcal{X}}$$

is an equivalence of ∞-categories.

This notion really depends on the notion of vector bundles in $\mathcal{C}$, or more precisely on the spherical fibrations associated to vector bundles. One can more generally define $\mathcal{C}$-stability for a locally cartesian closed ∞-category equipped with a notion of 'spherical fibrations'.

Theorem (Gepner, N.)**.** *The functor $\Sigma^\infty_+ : \mathcal{S}^{\mathrm{Orb}} \to \mathcal{S}p^{\mathrm{Orb}}$ exhibits the ∞-category of Orbispectra as the universal $\mathcal{S}^{\mathrm{Orb}}$-stable ∞-category obtained from $\mathcal{S}^{\mathrm{Orb}}$.*

Finally, after we have obtained that universal description of Orbispectra we can use it to show that variants of equivariant elliptic cohomology as discussed in the literature fit in our framework. For example the equivariant elliptic cohomology discussed by Lurie [2] based on ideas of Gronjowski, Ando and many others. Therefore let us fix an oriented derived elliptic curve $\mathcal{E} \to \mathcal{S}$ over a derived stack $\mathcal{S}$. For example if $\mathcal{S}$ is given by $\mathcal{S}pec(R)$ for an E_∞-ringspectrum R then the datum of $\mathcal{E}$ is a highly structured refinement of endowing R with the structure of an elliptic spectrum. Another important example is where $\mathcal{S}$ is the derived stack of elliptic curves $(\mathcal{M}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$ and $\mathcal{E}$ the universal elliptic curve $(\mathcal{M}_{\mathrm{univ}}, \mathcal{O}^{\mathrm{top}})$.

Theorem (Gepner, N.)**.** *There is an Orbispectrum $\mathrm{Ell}_{\mathcal{E}}$ which describes global equivariant elliptic cohomology for $\mathcal{E}$. The value for S^1 is given by global sections of $\mathcal{E}$ and the underlying non-equivariant spectrum by global sections of $\mathcal{S}$.*

For example for the universal elliptic curve this gives a global equivariant version of TMF.

References

[1] D. Gepner and A. Henriques, *Homotopy theory of orbispaces*, Preprint math.AT/0701916 (2007).
[2] J. Lurie, *A survey of elliptic cohomology*, in Algebraic topology (Springer) (2009), 219–277.
[3] J. Lurie, *Higher algebra*, book project, available at http://www.math.harvard.edu/ lurie/ (2014).
[4] S. Schwede, *Global homotopy theory*, book project, available at http://www.math.uni-bonn.de/people/schwede/ (2015).

Proper equivariant stable homotopy and virtual cohomological dimension

Irakli Patchkoria

Let G be a discrete group unless stated otherwise. We denote by $\underline{E}G$ a universal proper G-space, i.e. a universal G-space for the family of finite subgroups of G. It is characterized up to G-homotopy equivalence by the following properties:

(i) $\underline{E}G$ admits the structure of a G-CW complex.
(ii) The H-fixed point space $(\underline{E}G)^H$ is contractible if H is a finite subgroup of G, and empty otherwise.

The existence of $\underline{E}G$ follows for example from [8]. We say that a G-space X is a *model for* $\underline{E}G$ if X satisfies the conditions (i) and (ii). Any two models for $\underline{E}G$ are G-homotopy equivalent.

An important question in geometric group theory is to find the smallest CW-dimension that a model for $\underline{E}G$ can have. More formally, one is interested in the *geometric dimension for proper actions* of G which is defined as follows:

$$\underline{\mathrm{gd}}(G) = \inf\{\dim(X) \mid X \text{ is a model for } \underline{E}G\}.$$

In order to be able to compute this number, it is useful to consider a certain abelian category and do homological algebra inside it. Let $O_{\mathcal{F}}G$ denote the *orbit category of G with finite isotropy.* The objects of $O_{\mathcal{F}}G$ are the cosets G/H with H finite and morphisms are G-equivariant maps. The category of contravariant functors from $O_{\mathcal{F}}G$ to the category of abelian groups is denoted by Mod-$\mathcal{O}_{\mathcal{F}}G$.

The category Mod-$\mathcal{O}_{\mathcal{F}}G$ is an abelian category with enough projectives. Consider the constant functor $\underline{\mathbb{Z}} \in$ Mod-$\mathcal{O}_{\mathcal{F}}G$ which sends every coset to $\mathbb{Z}$. By classical homological algebra, the projective dimension of $\underline{\mathbb{Z}}$ in Mod-$\mathcal{O}_{\mathcal{F}}G$ is equal to

$$\sup\{n \in \mathbb{N} \mid \exists M \in \text{Mod-}\mathcal{O}_{\mathcal{F}}G : \mathrm{Ext}^n_{\mathcal{O}_{\mathcal{F}}G}(\underline{\mathbb{Z}}, M) \neq 0\}.$$

This number is called the *Bredon cohomological dimension* of G and is denoted by $\underline{\mathrm{cd}}(G)$. Note that the group $\mathrm{Ext}^n_{\mathcal{O}_{\mathcal{F}}G}(\underline{\mathbb{Z}}, M)$ is isomorphic to the Bredon cohomology group $H^n_{G,\mathcal{F}}(\underline{E}G, M)$, hence the name.

The Bredon cohomological dimension is closely related to the geometric dimension for proper actions.

Theorem (Lück, Meintrup [10])**.** *For any discrete group G, the inequalities hold*

$$\underline{\mathrm{cd}}(G) \leq \underline{\mathrm{gd}}(G) \leq \max\{3, \underline{\mathrm{cd}}(G)\}.$$

Since, $\underline{\mathrm{gd}}(G) = 0$ if and only if $\underline{\mathrm{cd}}(G) = 0$ (and if and only if G is finite), and $\underline{\mathrm{cd}}(G) = 1$ if and only if $\underline{\mathrm{gd}}(G) = 1$ by [3], it follows that the invariants $\underline{\mathrm{cd}}(G)$ and $\underline{\mathrm{gd}}(G)$ coincide, except for the possibility that one could have $\underline{\mathrm{cd}}(G) = 2$ but $\underline{\mathrm{gd}}(G) = 3$. That this exception actually occurs in the torsion setting is shown in [1].

Another important algebraic invariant which is related to the Bredon cohomological dimension is the *Mackey cohomological dimension* of a group. Let $\mathcal{B}_{\mathcal{F}}G$ denote the *Burnside category* (or the dual of the *Mackey category* as defined in

[13]). This category is pre-additive. Its objects are all G-sets of the form G/H, where H is finite. We do not give a detailed definition here but point out that there is an obvious inclusion of categories $\mathcal{O}_{\mathcal{F}}G \hookrightarrow \mathcal{B}_{\mathcal{F}}G$ and for any inclusion of finite subgroups $H \leq K$, there are transfer maps $G/K \to G/H$ in $\mathcal{B}_{\mathcal{F}}G$. These satisfy certain natural relations. Most notably, the *Mackey double coset formula* holds. The category of *G-Mackey functors* $\mathrm{Mack}_{\mathcal{F}}G$ is the category of contravariant additive functors from $\mathcal{B}_{\mathcal{F}}G$ to abelian groups (see [13]). It is an abelian category with enough projectives. Consider the Burnside ring Mackey functor $\underline{A}$ which assigns to every coset G/H the Burnside ring $A(H)$. The projective dimension of $\underline{A}$ in $\mathrm{Mack}_{\mathcal{F}}G$ is called the *Mackey cohomological dimension* of G and is denoted by $\underline{\mathrm{cd}}_{\mathcal{M}}(G)$. Again classical homological algebra tells us that the following holds:

$$\underline{\mathrm{cd}}_{\mathcal{M}}(G) = \sup\{n \in \mathbb{N} \mid \exists M \in \mathrm{Mack}_{\mathcal{F}}G : \mathrm{Ext}^n_{\mathcal{B}_{\mathcal{F}}G}(\underline{A}, M) \neq 0\}.$$

Now suppose G is virtually torsion free, i.e. there exists a finite index torsion free subgroup $\Gamma \leq G$. Then the *virtual cohomological dimension* of G, denoted by $\mathrm{vcd}(G)$, is defined to be the classical cohomological dimension of Γ. This definition does not depend on the choice of Γ. The virtual cohomological dimension is an important invariant in geometric group theory. The following theorem by Martínez-Pérez and Nucinkis relates $\mathrm{vcd}(G)$ and $\underline{\mathrm{cd}}_{\mathcal{M}}(G)$ [13]:

Theorem (Martínez-Pérez, Nucinkis)**.** *If G is virtually torsion free, then*

$$\mathrm{vcd}(G) = \underline{\mathrm{cd}}_{\mathcal{M}}(G).$$

The inclusion $\mathcal{O}_{\mathcal{F}}G \hookrightarrow \mathcal{B}_{\mathcal{F}}G$ induces the induction functor ind: $\text{Mod-}\mathcal{O}_{\mathcal{F}}G \to \mathrm{Mack}_{\mathcal{F}}G$ which is left adjoint to the forgetful functor $\mathrm{Mack}_{\mathcal{F}}G \to \text{Mod-}\mathcal{O}_{\mathcal{F}}G$. Since the induction functor sends a projective resolution of $\underline{\mathbb{Z}}$ to a projective resolution of $\underline{A}$ [13], it follows that the Bredon cohomological dimension $\underline{\mathrm{cd}}(G)$ is always greater or equal than the Mackey cohomological dimension $\underline{\mathrm{cd}}_{\mathcal{M}}(G)$. Having in mind the theorem by Lück and Meintrup, a natural question arises whether there is any geometric interpretation for $\underline{\mathrm{cd}}_{\mathcal{M}}(G)$. In other words, we would like to have a certain geometrically defined invariant $\underline{\mathrm{gd}}_{\mathrm{st}}(G)$ which will coincide with $\underline{\mathrm{cd}}_{\mathcal{M}}(G)$ and will be less or equal than the geometric dimension $\underline{\mathrm{gd}}(G)$. By the theorem of Martínez-Pérez and Nucinkis, defining such a geometric invariant will also provide a geometric interpretation of the virtual cohomological dimension for virtually torsion free groups.

For G a finite group, the geometry behind G-Mackey functors is the theory of genuine G-spectra ([7], [12]). So we need a theory of genuine G-spectra with finite isotropy, for G an infinite discrete group.

Definition. Let G be a Lie group (for example a discrete group). An *orthogonal G-spectrum* is an orthogonal spectrum X together with a continuous G-action.

The category of orthogonal G-spectra is denoted by Sp_G. For any orthogonal G-spectrum X and any compact subgroup $H \leq G$, one can consider X as an orthogonal H-spectrum by restricting the G-action to H. Let $\pi_n^H X$ denote the n-th H-equivariant homotopy group of this restricted H-spectrum. A morphism

$f\colon X \to Y$ of orthogonal G-spectra is called a *π_*-isomorphism* if $\pi_n^H(f)\colon : \pi_n^H X \to \pi_n^H Y$ is an isomorphism for any integer n and any compact subgroup H of G. The following theorem is part of a joint project with D. Degrijse, M. Hausmann, W. Lück and S. Schwede.

Theorem (joint with D. Degrijse, M. Hausmann, W. Lück and S. Schwede)**.** *Let G be a Lie group.*

(i) The monoidal category Sp_G of orthogonal G-spectra admits a cofibrantly generated stable monoidal model structure with π_*-isomorphisms as weak equivalences.

(ii) The triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ is compactly generated by the set of compact generators $\{\Sigma_+^\infty G/H : H \leq G,\ H \text{ compact}\}$. The object $\Sigma_+^\infty G/H$ corepresents $\pi_*^H(-)$.

(iii) If G is discrete and H, K are finite subgroups of G, then $[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G$ is isomorphic to the direct sum

$$\bigoplus_{g \in H\backslash G/K} \pi_*^{H \cap {}^g K}(\mathbb{S})$$

over H-K double cosets, where $\pi_*^{H\cap {}^gK}(\mathbb{S})$ are classical equivariant stems for finite groups. In particular, the Burnside category $\mathcal{B}_{\mathcal{F}}G$ fully faithfully embeds into the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$.

Here Σ^∞ is the *suspension spectrum functor* from the category of pointed G-spaces to orthogonal G-spectra and $[-,-]_*^G$ stands for the graded abelian group of maps in $\mathrm{Ho}(\mathrm{Sp}_G)$. We are in particular interested in the trivial G-spectrum $\mathbb{S} = \Sigma^\infty S^0$. This spectrum is not cofibrant in general and its cofibrant replacement in our model structure is for example $\Sigma_+^\infty \underline{E}G$.

Note that if G is finite, then our stable model structure is Quillen equivalent to the one from [12]. Note also that the paper [4] constructs a stable model structure on the category of orthogonal G-spectra via an abstract Bousfield localization procedure. It is different from ours but Quillen equivalent and has the π_*-isomorphisms as weak equivalences. We have a better control over the stable fibrations and in particular, the stably fibrant objects in our model structure are the obviously defined *G-Ω-spectra.* Note that [4] does not develop any further theory.

Next, we say that a discrete group G *has enough bundle representations* if the following condition holds: For every finite subgroup H of G and every finite dimensional H-representation V there exists a G-vector bundle ξ over $\underline{E}G$ such that for some (hence any) H-fixed point $x \in (\underline{E}G)^H$ the representation V is isomorphic to an H-subrepresentation of ξ_x. By a theorem of Lück and Oliver [11], a discrete group has enough bundle representations if it has bounded torsion.

Proposition (joint with D. Degrijse, M. Hausmann, W. Lück and S. Schwede)**.** *Let G be a discrete group with enough bundle representations and X a proper finite G-CW complex. Then $[\Sigma_+^\infty X, \mathbb{S}]_n^G$ is isomorphic to Lück's equivariant stable cohomotopy $\pi_G^{-n}(X)$ defined in* [9].

In the joint work with D. Degrijse, M. Hausmann, W. Lück and S. Schwede we also show existence of Eilenberg-MacLane spectra for G-Mackey functors and prove that these represent Bredon cohomologies.

Now we go back to our original question of a geometric interpretation of the Mackey cohomological dimension and hence of the virtual cohomological dimension. Let G be a discrete group. We define a *stable decomposition of* $\mathbb{S}$ in $\mathrm{Ho}(\mathrm{Sp}_G)$ to be a homotopy colimit presentation

$$\mathbb{S} \simeq \mathrm{hocolim}_n \mathbf{X}^n,$$

where $\mathbf{X}^n = *$ if $n < 0$ and such that one has distinguished triangles in $\mathrm{Ho}(\mathrm{Sp}_G)$

$$\mathbf{X}^{n-1} \to \mathbf{X}^n \to \bigvee_{i \in I_n} \Sigma^n \Sigma^\infty_+ G/H_i \to \Sigma \mathbf{X}^{n-1}$$

with H_i finite for all $i \in I_n$. The smallest number n, for which $\mathbf{X}^{l-1} \to \mathbf{X}^l$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$ for all $l \geq n+1$ is called the dimension of the decomposition. A priori this dimension can be infinite. The *stable geometric dimension for proper actions* of G, denoted by $\underline{\mathrm{gd}}_{\mathrm{st}}(G)$, is by definition the smallest number m such that there exists an m-dimensional stable decomposition of $\mathbb{S}$. Again $\underline{\mathrm{gd}}_{\mathrm{st}}(G)$ can be infinite.

Any model X for $\underline{E}G$ provides a stable decomposition for $\mathbb{S}$ by applying the suspension spectrum functor. The dimension of this decomposition is clearly less or equal than the CW-dimension of X. Hence, we have $\underline{\mathrm{gd}}_{\mathrm{st}}(G) \leq \underline{\mathrm{gd}}(G)$. This inequality can be sharp as illustrated in the example below. In fact, the difference $\underline{\mathrm{gd}}(G) - \underline{\mathrm{gd}}_{\mathrm{st}}(G)$ can be arbitrarily large.

The following theorem which is joint with N. Bárcenas and D. Degrijse answers the question of a geometric interpretation of the virtual cohomological dimension:

Theorem (joint with N. Bárcenas and D. Degrijse)**.** *Let G be a discrete group. Then*

$$\underline{\mathrm{gd}}_{\mathrm{st}}(G) = \underline{\mathrm{cd}}_{\mathcal{M}}(G).$$

In particular, if G is virtually torsion free, then

$$\underline{\mathrm{gd}}_{\mathrm{st}}(G) = \mathrm{vcd}(G).$$

Example. Let A_5 be the alternating group on 5 elements. Let L denote a 2-dimensional acyclic simplicial flag complex constructed by Floyd and Richardson with an admissible simplicial A_5-action and without a global fixed point (see [5], [1]). The 1-skeleton of L is a finite graph whose vertex set is denoted by S and whose set of edges is denoted by $E(L)$. The *right angled Coxeter group* W_L associated to L is the group defined by the presentation

$$W_L = \langle S \mid s^2 \text{ for all } s \in S \text{ and } \; [s,t] \text{ if } (s,t) \in E(L) \rangle.$$

The action of A_5 on L induces an action of A_5 on W_L and hence we can form a semi-direct product $\Gamma = W_L \rtimes A_5$. Using the *Davis complex* of W_L one can see that $\underline{\mathrm{gd}}(\Gamma) = 3$ [1]. A recent result by Leary and Petrosyan [6] shows that $\underline{\mathrm{cd}}(G) = 3$. On the other hand, it follows from standard results on Coxeter groups (see [2]) that $\underline{\mathrm{cd}}_{\mathcal{M}}(\Gamma) = \mathrm{vcd}(\Gamma) = 2$. Hence $\underline{\mathrm{gd}}_{\mathrm{st}}(\Gamma) = 2$. In the joint work with N. Bárcenas and D. Degrijse we construct an explicit two dimensional stable decompostion of $\mathbb{S}$ in $\mathrm{Ho}(\mathrm{Sp}_\Gamma)$. Note that there are nontrivial stable attaching maps in this decomposition which do not exist unstably.

References

[1] N. Brady, I. Leary , and B. Nucinkis, *On algebraic and geometric dimensions for groups with torsion*, J. Lond. Math. Soc., **64** (2) (2001), 489–500.

[2] M. Davis, *The geometry and topology of Coxeter groups*, Princeton University Press, vol. 32 (2008).

[3] M. J. Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. Lond. Math. Soc., **38** (3) (1979), 193–215.

[4] H. Fausk, *Equivariant homotopy theory for pro-spectra,* Geom. Top., **12** (2008), 103-176.

[5] E. Floyd and R. Richardson, *An action of a finite group on an n-cell without stationary points*, Bull. Am. Math. Soc., **65** (2) (1959), 73–76.

[6] I. Leary and N. Petrosyan, *On dimensions of groups with cocompact classifying spaces for proper actions*, arXiv:1504.02704.

[7] L. G., Jr. Lewis, J. P. May, M. Steinberger and J. E. McClure, *Equivariant stable homotopy theory*, Lecture notes in mathematics, **1213**, Springer, Berlin, (1986).

[8] W. Lück, *Survey on classifying spaces for families of subgroups*, Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, Progress in Mathematics vol. 248, Birkhäuser (2005), 269–322.

[9] W. Lück, *The Burnside ring and equivariant stable cohomotopy for infinite groups*, J. Pure and Appl. Math. Quat., **1** (3) (2005), 479–541.

[10] W. Lück and D. Meintrup, *On the universal space for group actions with compact isotropy*, Proc. of the conference "Geometry and Topology" in Aarhus, (1998), 293–305.

[11] W. Lück and B. Oliver, *The completion theorem in K-theory for proper actions of a discrete group*, Topology **40** (2001), 585–616.

[12] M. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc., **159** (2002).

[13] C. Martínez-Pérez and B. Nucinkis, *Cohomological dimension of Mackey functors for infinite groups*, J. Lond. Math. Soc., **74** (2) (2006), 379–396.

Higher topological Hochschild homology of rings of integers in number fields

Birgit Richter

(joint work with Bjørn Dundas, Ayelet Lindenstrauss)

For any strictly commutative ring spectrum A and for any simplicial set X one can define the simplicial commutative ring spectrum $A \otimes X$ as $(A \otimes X)_n = \bigwedge_{x \in X_n} A$. Similarly, if M is an A-module spectrum and X is a pointed simplicial set, we define $(M, A) \otimes X$ by placing M at the basepoint of X and A at all other simplices of X. Important examples of this construction are

- *topological Hochschild homology of A with coefficients in M, $THH(A,M)$,* given by $(M,A)\otimes\mathbb{S}^1$,
- *higher topological Hochschild homology of order n of A with coefficients in M,*

$$THH^{[n]}(A,M) = (M,A)\otimes\mathbb{S}^n, n\geq 1, \text{ and}$$

- *torus homology* where we tensor with $(\mathbb{S}^1)^n = \mathbb{T}^n$.

Ordinary topological Hochschild homology is the target of a trace map from algebraic K-theory. This trace map factors via topological cyclic homology, TC, and the latter is often a very good approximation of algebraic K-theory.

If we consider iterated algebraic K-theory, then we can use an iteration of the trace map and obtain torus homology as the natural target of such a trace map. Using the standard cell structure of an n-dimensional torus gives us a method of calculating torus homology from higher topological Hochschild homology.

An important class of examples are rings of integers in number fields. As a starting point we consider higher THH of the integers with coefficients in the residue field $\mathbb{F}_p$, $THH^{[n]}(\mathbb{Z},\mathbb{F}_p)$. Bökstedt [2] calculated

$$THH_*(\mathbb{Z},\mathbb{F}_p)\cong\mathbb{F}_p[x_{2p}]\otimes_{\mathbb{F}_p}\Lambda_{\mathbb{F}_p}(x_{2p-1}).$$

There is a well-known description of iterated Tor-algebras due to Cartan [4]: If we start with a polynomial algebra over $\mathbb{F}_p$ generated by an element w of even degree, then we call this algebra $B^1_{\mathbb{F}_p}(w)$. Iteratively, we define

$$B^{n+1}_{\mathbb{F}_p}(w) := \mathrm{Tor}^{B^n_{\mathbb{F}_p}(w)}(\mathbb{F}_p,\mathbb{F}_p)$$

for all n. The case $n=2$ immediately gives

$$B^2_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^1_{\mathbb{F}_p}(w)}(\mathbb{F}_p,\mathbb{F}_p)\cong\Lambda_{\mathbb{F}_p}(\varepsilon w)$$

where the degree of εw is one higher than the degree of w.

$$B^3_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^2_{\mathbb{F}_p}(w)}(\mathbb{F}_p,\mathbb{F}_p)\cong\Gamma_{\mathbb{F}_p}(\varrho^0\varepsilon w)$$

where the latter denotes a divided power algebra. As the base field is of characteristic p this algebra splits into a tensor product of truncated polynomial algebras

$$\Gamma_{\mathbb{F}_p}(\varrho^0\varepsilon w)\cong\bigotimes_k\mathbb{F}_p[\varrho^k\varepsilon w]/(\varrho^k\varepsilon w)^p;$$

here $\varrho^k\varepsilon w$ corresponds to the p^kth divided power of $\varrho^0\varepsilon w$. For each of the tensor factors we obtain again a periodic resolution and we get

$$B^4_{\mathbb{F}_p}(w) = \mathrm{Tor}^{B^3_{\mathbb{F}_p}(w)}(\mathbb{F}_p,\mathbb{F}_p)\cong\bigotimes_k\Gamma_{\mathbb{F}_p}(\varphi^0\varrho^k\varepsilon w)\otimes\Lambda_{\mathbb{F}_p}(\varepsilon\varrho^k\varepsilon w).$$

From here on the iteration process yields terms of a form that already occurred before.

Theorem (Dundas-Lindenstrauss-R)**.** *For all $n\geq 1$ and for all primes p:*

$$THH^{[n]}_*(\mathbb{Z},\mathbb{F}_p)\cong B^n_{\mathbb{F}_p}(x_{2p})\otimes_{\mathbb{F}_p}B^{n+1}_{\mathbb{F}_p}(y_{2p-2}).$$

A crucial ingredient in the proof is the following

Lemma. *Let C be a commutative augmented $H\mathbb{F}_p$-algebra and assume that there is an isomorphism of graded commutative $\mathbb{F}_p$-algebras $\pi_*C \cong \Lambda_{\mathbb{F}_p}(x)$ where $|x| = m > 0$. Then there is a zigzag of equivalences of commutative augmented $H\mathbb{F}_p$-algebras between C and $H\mathbb{F}_p \vee \Sigma^m H\mathbb{F}_p$.*

This Lemma was suggested by Mike Mandell. Our proof uses a Postnikov argument in the world of commutative $H\mathbb{F}_p$-algebras. With the help of this result we can split off the bottom Postnikov piece of $THH(\mathbb{Z}, \mathbb{F}_p)$ and obtain an iterated homotopy pushout diagram for $THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)$:

$$\begin{array}{ccccc} THH(\mathbb{Z},\mathbb{F}_p) & \longrightarrow & H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p & \longrightarrow & H\mathbb{F}_p \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ H\mathbb{F}_p & \longrightarrow & E & \longrightarrow & THH^{[2]}(\mathbb{Z},\mathbb{F}_p) \end{array}$$

A Tor-spectral sequence calculation yields that E has the exterior algebra $\Lambda_{\mathbb{F}_p}(z_{2p+1})$ as $\pi_*(E)$ and hence we know that $E \sim H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$. The map f factors via the augmentation and unit and this yields with another Tor-spectral sequence calculation that

$$\pi_* THH^{[2]}(\mathbb{Z}, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(z_{2p+1}) \otimes_{\mathbb{F}_p} \Gamma_{\mathbb{F}_p}(a_{2p}).$$

For the iteration of this argument we use that we can express higher THH via an iterated bar construction. For instance $THH^{[3]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the diagonal of the bisimplicial commutative augmented $H\mathbb{F}_p$-algebra $B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, E), H\mathbb{F}_p)$ and as the module structure of E over $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ reduces to the $H\mathbb{F}_p$-module structure this bar construction splits as a bisimplicial commutative augmented $H\mathbb{F}_p$-algebra into

$$B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, H\mathbb{F}_p), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \underline{E}, H\mathbb{F}_p)$$

where $\underline{E}$ denotes the constant simplicial commutative augmented $H\mathbb{F}_p$-algebra on E. For higher n there is a similar splitting and we get that $THH^{[n+1]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the diagonal of an n-fold reduced iterated bar construction on $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ smashed with an $(n-1)$-fold iterated bar construction on $\underline{E}$. The square zero extensions E and $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ can be modelled as the Eilenberg Mac Lane spectra on a simplicial commutative algebra and this allows for a comparison of the above iterated bar constructions with iterated algebraic bar constructions on exterior algebras. The homology groups of such bar constructions were determined in [1] and this gives the proof of Theorem 1.

Let $\mathcal{O}$ denote the ring of integers in a number field and let P be a non-trivial prime ideal in $\mathcal{O}$ with residue field $\mathcal{O}/P = \mathbb{F}_q$ where $q = p^\ell$ for some prime p. Higher THH detects ramification:

Theorem (Dundas-Lindenstrauss-R). *For all $n \geq 1$:*

$$THH_*^{[n]}(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong B^n_{\mathbb{F}_q}(x) \otimes_{\mathbb{F}_q} B^{n+1}_{\mathbb{F}_q}(y)$$

where

(i) $|x| = 2$ and $|y| = 0$ if A is ramified over $\mathbb{Z}$ at P, and
(ii) $|x| = 2p$ and $|y| = 2p - 2$, if A is unramified over $\mathbb{Z}$ at P.

Lindenstrauss and Madsen determined the topological Hochschild homology groups of rings of integers in [6].

In the unramified case we show that we have an isomorphism

$$THH_*^{[n]}(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong THH_*^{[n]}(\mathbb{Z}_p^\wedge, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_q.$$

This uses the Lindenstrauss-Madsen result and an iterative spectral sequence argument. In the ramified case the important input is that the first Hochschild homology group (and therefore also the first THH-group) is isomorphic to $\mathbb{F}_q$. This fact ensures that the differentials in the Brun spectral sequence [3, p. 30]

$$THH_*(\mathcal{O}_P^\wedge/P, \mathrm{Tor}_{*,*}^{\mathcal{O}_P^\wedge}(\mathcal{O}_P^\wedge/P, \mathcal{O}_P^\wedge/P)) \Rightarrow THH_*(\mathcal{O}_P^\wedge, \mathcal{O}_P^\wedge/P)$$

have to vanish and we obtain that

$$THH_*(\mathcal{O}_P^\wedge, \mathcal{O}/P) \cong \mathbb{F}_q[u] \otimes_{\mathbb{F}_q} \Lambda_{\mathbb{F}_q}(\tau)$$

with $|u| = 2$ and $|\tau| = 1$. From this point on the argument is the same as in the case of the rational integers.

References

[1] Irina Bobkova, Ayelet Lindenstrauss, Kate Poirier, Birgit Richter, Inna Zakharevich, *On the higher topological Hochschild homology of $\mathbb{F}_p$ and commutative $\mathbb{F}_p$-group algebras*, accepted by the proceedings of the BIRS workshop WIT: Women in Topology, arXiv:1312.6378.

[2] Marcel Bökstedt, *The topological Hochschild homology of $\mathbb{Z}$ and of $\mathbb{Z}/p\mathbb{Z}$*, preprint.

[3] Morten Brun, *Topological Hochschild homology of $\mathbb{Z}/p^n$*, J. Pure Appl. Algebra **148** (2000), 29–76.

[4] Henri Cartan, *Détermination des algèbres $H_*(\pi, n; Z_p)$ et $H^*(\pi, n; Z_p)$, p premier impair*, Exp. No. 9, 10 p., Séminaire Henri Cartan, 7 no. 1, 1954–1955, Algèbre d'Eilenberg-Maclane et homotopie (available at `http://www.numdam.org/`).

[5] Bjørn Ian Dundas, Ayelet Lindenstrauss, Birgit Richter, *Higher topological Hochschild homology of rings of integers*, preprint arXiv:1502.02504.

[6] Ayelet Lindenstrauss, Ib Madsen, *Topological Hochschild homology of number rings*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 2179–2204.

Slices and stable motivic homotopy groups

Oliver Röndigs
(joint work with Markus Spitzweck, Paul Arne Østvær)

In joint work with Markus Spitzweck and Paul Arne Østvær, we study the spectral sequence based on Voevodsky's slice filtration. This filtration on the stable homotopy category of $\mathbf{P}^1 = S^{2,1}$-spectra over a field F measures the amount of $\mathbf{G}_m = S^{1,1}$-suspensions or "Tate twists" which are necessary to construct a given $\mathbf{P}^1$-spectrum. More precisely, the localizing subcategory $\mathbf{SH}_F^{\mathrm{eff}}$ generated by suspension $\mathbf{P}^1$-spectra of smooth F-schemes defines the *slice filtration*:

$$\cdots \hookrightarrow \Sigma^{2,1}\mathbf{SH}_F^{\mathrm{eff}} \hookrightarrow \mathbf{SH}_F^{\mathrm{eff}} \hookrightarrow \Sigma^{-2,-1}\mathbf{SH}_F^{\mathrm{eff}} \hookrightarrow \cdots \hookrightarrow \mathbf{SH}_F$$

The associated graded for $\mathsf{E} \in \mathbf{SH}_F$ are the *slices* $s_*\mathsf{E}$.

Work of Levine and Voevodsky shows that the unit maps for the $\mathbf{P}^1$-ring spectra $\mathbf{1}$ (sphere spectrum), MGL (algebraic cobordism), KGL (algebraic K-theory), and $\mathsf{M}\mathbb{Z}$ (motivic cohomology) induce an isomorphism on the zeroth slice. Moreover, also the other slices can be described. In the case of the sphere spectrum, the higher slices are determined by the E^2-page of the topological Adams-Novikov spectral sequence.

We use this input to obtain information on the first stable motivic homotopy groups of spheres over fields of characteristic zero, which is compatible with conjectures of Asok and Fasel on unstable motivic homotopy groups of punctured affine spaces. More precisely, we obtain a short exact sequence

$$0 \to K^{\mathrm{Milnor}}_{-n+2}/24 \to \pi_{n+1,n}\mathbf{1}^\wedge_\eta \to \pi_{n+1,n}\mathsf{KQ}_{n+1,n}$$

in which the last homomorphism (induced by the unit map $\mathbf{1} \to \mathsf{KQ}$ for hermitian K-theory) is surjective for $n \geq -3$ (but not always). The proof requires essentially two ingredients. The first ingredient is a convergence result for the slice spectral sequence of cellular $\mathbf{P}^1$-spectra of finite type. It is based on a comparison of the slice completion with the η-completion, where $\eta\colon S^{1,1} \to S^{0,0}$ is the first Hopf map. The second ingredient concerns the identification of the first slice differential in terms of motivic Steenrod operations, which in turn employs corresponding results for hermitian K-theory.

Graded units of ring spectra and *R*-module Thom spectra

Steffen Sagave
(joint work with Samik Basu and Christian Schlichtkrull)

Classically one can form the Thom spectrum associated with a continuous map $f\colon X \to BO$ to the classifying space of the orthogonal group $O = \bigcup_{n\in\mathbb{N}} O(n)$ or with a continuous map $f\colon X \to BF$ to the classifying space of stable spherical fibrations. In the language of structured ring spectra, BF may be identified with $B\mathrm{GL}_1\mathbb{S}$, the classifying space of the units of the sphere spectrum, and a spectrum may be viewed as an $\mathbb{S}$-module spectrum. Replacing the sphere spectrum $\mathbb{S}$ by a general A_∞ ring spectrum R, Ando, Blumberg, Gepner, Hopkins, and Rezk [1] generalized the classical construction of Thom spectra by building R-module Thom spectra associated with continuous maps $f\colon X \to B\mathrm{GL}_1 R$. They also provide a variant of their Thom spectrum functor that respects actions of E_∞ operads.

The aim of the present project is to implement an R-module Thom spectrum functor in the context of symmetric spectra and to define more general R-module Thom spectra associated with continuous maps to a suitable classifying space of the *graded* units of a symmetric ring spectrum R. The graded units $\mathrm{GL}_1^{\mathcal{J}} R$ of a commutative symmetric ring spectrum R were introduced in [3]. By definition, $\mathrm{GL}_1^{\mathcal{J}} R$ is a a space-valued symmetric monoidal functor on a certain symmetric monoidal indexing category $\mathcal{J}$. The diagram $\mathrm{GL}_1^{\mathcal{J}} R$ is built from the loop spaces $\Omega^{n_2} R_{n_1}$ on the levels of the symmetric spectrum R. In contrast to the ordinary units $\mathrm{GL}_1^{\mathcal{I}} R$ of R which are indexed by the category of finite sets and injections $\mathcal{I}$, the graded units $\mathrm{GL}_1^{\mathcal{J}} R$ also detect units of non-zero degree in the graded ring of homotopy groups $\pi_*(R)$ of R.

Symmetric R-module Thom spectra. (joint with C. Schlichtkrull) If R is a commutative symmetric ring spectrum, we define classifying spaces $B\mathrm{GL}_1^{\mathcal{I}} R$ and $B\mathrm{GL}_1^{\mathcal{J}} R$ of the units and the graded units. These are E_∞ spaces over the Barratt–Eccles operad. Building on this we define R-module Thom spectrum functors

$$T\colon \mathcal{S}/B\mathrm{GL}_1^{\mathcal{I}} R \to R\text{-Mod} \qquad \text{and} \qquad T\colon \mathcal{S}/B\mathrm{GL}_1^{\mathcal{J}} R \to R\text{-Mod}$$

on the categories of spaces augmented over these classifying spaces. These functors have many desirable properties: They send weak equivalences over the classifying spaces to stable equivalences of R-module spectra, they preserve homotopy colimits, and they preserve actions of operads augmented over the Barratt–Eccles operad. In particular, the Thom spectrum associated with a map of topological monoids inherits an associative R-algebra structure.

The Thom spectrum functor for graded units extends the one for ordinary units: There is a natural morphism $\iota\colon B\mathrm{GL}_1^{\mathcal{I}} R \to B\mathrm{GL}_1^{\mathcal{J}} R$ such that the restriction of the Thom spectrum functor for maps to $B\mathrm{GL}_1^{\mathcal{J}} R$ along ι coincides with the one for maps to $B\mathrm{GL}_1^{\mathcal{I}} R$. The map ι turns out to be a 0-connected cover map. The extra information about the non-zero degree units of $\pi_*(R)$ in $\mathrm{GL}_1^{\mathcal{J}} R$ is reflected in $B\mathrm{GL}_1^{\mathcal{J}} R$ by the fact that its monoid of path components is $\mathbb{Z}/d\mathbb{Z}$, where d is the periodicity of R, i.e., d is the smallest positive degree of a unit in $\pi_*(R)$ if there exists a unit of positive degree, and zero otherwise. If for example R is the sphere spectrum, then $B\mathrm{GL}_1^{\mathcal{J}} \mathbb{S}$ has the homotopy type of $\mathbb{Z} \times BF$ and may be viewed as a classifying space of virtual spherical fibrations. Hence our work provides a Thom spectrum functor defined on maps that classify virtual spherical fibrations. Its advantage over a naive extension of the classical Thom spectrum functor by shifting to the zero component and suspending or desuspending the resulting spectrum accordingly is that it preserves E_∞ structures.

Topological Hochschild homology of Thom spectra. (joint with S. Basu and C. Schlichtkrull) Our Thom spectrum functors are set up in a way that allows for an immediate generalization of the main result of Blumberg, Cohen, and Schlichtkrull [2] to R-based topological Hochschild homology: If $f\colon A \to B\mathrm{GL}_1^{\mathcal{J}} R$ is a map of topological monoids with A grouplike and well based, then the R-based topological Hochschild homology $\mathrm{THH}^R(T(f))$ of $T(f)$ is stably equivalent to the Thom spectrum associated with a certain morphism $L^\eta(Bf)$. If f is in addition assumed to be a 3-fold loop map, then $\mathrm{THH}^R(T(f))$ is stably equivalent to $T(f) \wedge (BA)_+$. This provides a new tool for computations of R-based topological Hochschild homology groups.

References

[1] M. Ando, A. J. Blumberg, D. Gepner, M. J. Hopkins, and C. Rezk. Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory. *Journal of Topology*, 2014. DOI:10.1112/jtopol/jtu009.

[2] A. J. Blumberg, R. L. Cohen, and C. Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. *Geom. Topol.*, 14(2):1165–1242, 2010.

[3] S. Sagave and C. Schlichtkrull. Diagram spaces and symmetric spectra. *Adv. Math.*, 231(3-4):2116–2193, 2012.

Invertible Tmf-modules

Vesna Stojanoska

(joint work with Akhil Mathew)

This talk is based on the preprint [1].

An $\mathbf{E}_\infty$-ring spectrum R has an associated symmetric monoidal ∞-category of modules $\mathrm{Mod}(R)$, which in turn has well-behaved invariants of algebraic or algebro-geometric type. One such invariant is the Picard group $\mathrm{Pic}(R)$, i.e. the group of isomorphism classes of invertible R-modules (and $\wedge_R$ as its operation). Employing descent-theoretic techniques in the study of Picard groups naturally leads to the notion of a Picard *spectrum*. Namely, the *space* of invertible objects in $\mathrm{Mod}(R)$ and equivalences between them is a group-like infinite loop space under the smash product, hence it is the zeroth space of a connective spectrum $\mathfrak{pic}(R)$. The zeroth homotopy group $\pi_0\mathfrak{pic}(R)$ is the Picard *group* of R, while the higher homotopy groups are determined by the equivalence $\Omega\,\mathfrak{pic}(R) \simeq \mathfrak{gl}_1(R)$.

The functor $\mathfrak{pic}$ from symmetric monoidal ∞-categories to connective spectra satisfies descent, i.e. commutes with homotopy limits. This can be practically very useful since

1. even-periodic ring spectra have algebraic Picard groups, by work of Baker-Richter [2], and
2. many ring spectra of interest are built as homotopy limits of even-periodic ones, for example, as homotopy fixed points or global sections.

For example, if $A \to B$ is a G-Galois extension of $\mathbf{E}_\infty$-rings in the sense of Rognes [3], there is an equivalence $\mathfrak{pic}(A) \simeq \tau_{\geq 0}(\mathfrak{pic}(B))^{hG}$, and an associated homotopy fixed point spectral sequence

$$H^s(G, \pi_t \mathfrak{pic}(B)) \Rightarrow (\mathfrak{pic}(B))^{hG},$$

whose abutment for $t = s$ is the Picard group of A.

The spectra of topological modular forms do not have non-trivial Galois extensions integrally, but nonetheless are given as homotopy limits of even-periodic rings. From the point of view of Picard groups, the most interesting version is Tmf, the global sections spectrum of the Goerss-Hopkins-Miller sheaf $\mathcal{O}^{\mathrm{top}}$ on the *compactified* moduli stack $\overline{\mathfrak{M}}_{ell}$ of elliptic curves. Descent in this situation gives a spectral sequence

$$E_2^{s,t}(\mathfrak{pic}) = \begin{cases} H^s(\overline{\mathfrak{M}}_{ell}, \mathbb{Z}/2), & t = 0, \\ H^s(\overline{\mathfrak{M}}_{ell}, \mathcal{O}^\times), & t = 1, \\ H^s(\overline{\mathfrak{M}}_{ell}, \omega^{\frac{t-1}{2}}), & t \geq 3 \text{ odd}, \\ 0, & \text{else}, \end{cases}$$

whose abutment is $\pi_{t-s}\Gamma(\mathfrak{pic}(\mathcal{O}^{\mathrm{top}}))$. For $t = s$ we get the Picard group of Tmf. Note that in the range $t > 1$, this $E_2(\mathfrak{pic})$-page coincides with a shift of the E_2-page of the well-known descent spectral sequence

$$E_2^{s,t} = H^s(\overline{\mathfrak{M}}_{ell}, \omega^t) \Rightarrow \pi_{t-s} Tmf.$$

To effectively work with these spectral sequences, we need tools to understand the differentials. We develop two such general tools: comparison with the "additive" descent spectral sequence (i.e. the one before applying the $\mathfrak{pic}$ functor) in a range, and a universal formula for the first differential outside of the comparison range. Specifically, the comparison tool tells us that

$$2 \leq r \leq t-1 \text{ implies } d_r^{s,t}(\mathfrak{pic}) = d_r^{s,t-1}.$$

The universal formula concerns $d_t(\mathfrak{pic})$ on $E_t^{t,t}(\mathfrak{pic}) \cong E_t^{t,t-1}$. If $x \in E_t^{t,t-1}$, let $x_{\mathfrak{pic}}$ denote the corresponding element of $E_t^{t,t}(\mathfrak{pic})$. Then, we show,

$$d_t(\mathfrak{pic})(x_{\mathfrak{pic}}) = (d_t(x) + x^2)_{\mathfrak{pic}}.$$

These tools give us a number of non-zero differentials in the $\mathfrak{pic}$-spectral sequence, and an upper bound on the Picard groups we are studying. The suspension ΣR determines a lower bound; namely ΣR always generates a cyclic subgroup of $\mathrm{Pic}(R)$, of order equal to the periodicity of R (which is infinity when R is not periodic). Sometimes these bounds coincide, but in the case of Tmf they do not, and we need to work more to show that the upper bound is realized. Namely, we construct an explicit invertible Tmf-module unequivalent to any suspension of Tmf, and whose 24-th smash power is a suspension.

Theorem. *The Picard groups of KO and TMF are cyclic of order equal to the respective periodicities, i.e. 8 and 576. By contrast, the Picard group of Tmf is not cyclic and equals $\mathbb{Z} \oplus \mathbb{Z}/24$.*

We hope our methods can be employed in many other examples of interest.

References

[1] Akhil Mathew and Vesna Stojanoska, *The Picard group of topological modular forms via descent theory*, available at `arXiv:1409.7702`.
[2] Andrew Baker and Birgit Richter, *Invertible modules for commutative $\mathbb{S}$-algebras with residue fields,* Manuscripta Math. **118** (2005), no. 1, 99–119.
[3] John Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups,* Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137.

Higher chromatic Thom spectra for $(n-1)$-gerbes, and their orientations

Craig Westerland

This talk was based upon work of the speaker in [3]. A theorem of Snaith [2] identifies the K-theory spectrum as the localisation $K \simeq \Sigma^\infty \mathbb{C}P^\infty_+[\beta^{-1}]$ of the suspension spectrum of $\mathbb{C}P^\infty$ at the Bott class. In [3], we extended these results into the higher-chromatic setting[1]:

Theorem (W)**.** *There is an equivalence of E_∞ ring spectra*

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+[\rho_n^{-1}] \simeq E_n^{hS\mathbb{G}_n^\pm}.$$

Here E_n denotes the (Lubin-Tate) Morava E-theory associated to the Honda formal group law, $S\mathbb{G}_n^\pm < \mathbb{G}_n$ is the *special Morava stabiliser group*, and ρ_n is a certain $K(n)$-local Picard-graded homotopy element which we regard as an analogue of the Bott class. This recovers the p-completion of Snaith's theorem when $n = 1$.

For brevity, we notate the common ring spectra in this theorem as R_n. The theorem suggests that we may regard R_n as being a cohomology theory analogous to K-theory, wherein the role of line bundles is played by $(n-1)$-gerbes (which are classified, up to isomorphism, by maps into $K(\mathbb{Z}, n+1)$). The purpose of this talk, then, was to explore several analogues of K-theoretic constructions (Thom spectra, cannibalistic classes, and the J-homomorphism) in this setting.

The description of R_n as a homotopy fixed point spectrum equips it with a residual action of $\mathbb{Z}_p^\times = \mathbb{G}_n/S\mathbb{G}_n^\pm$. We regard this as an analogue of Adams operations. As p is odd, this group is topologically cyclic, with generator ψ^k; then there is a fibre sequence

$$S \xrightarrow{\eta} R_n \xrightarrow{\psi^k - 1} R_n$$

[1] Throughout, we are working at an odd prime, p.

where $S = L_{K(n)}S^0$, and η is the unit of the ring spectrum. Examining the units of these spectra, we construct the diagram whose top row is a cofibre sequence:

$$\begin{array}{ccccccc} \mathrm{gl}_1 S & \xrightarrow{\mathrm{gl}_1 \eta} & \mathrm{gl}_1 R_n & \longrightarrow & b(S, R_n) & \xrightarrow{\gamma} & \Sigma \mathrm{gl}_1 S \\ & & & \searrow^{\psi^k/1} & \downarrow{\scriptstyle c(\psi^k)} & & \\ & & & & \mathrm{gl}_1 R_n & & \end{array}$$

Here maps $X \to B(S, R_n) = \Omega^\infty b(S, R_n)$ consist of R_n-oriented S-Thom spectra over X, and γ is the map which forgets the orientation (the set of choices of which is a $\mathrm{GL}_1 R_n$-torsor). The map $c(\psi^k)$ then computes the associated k^{th} cannibalistic class of such an oriented Thom spectrum.

Theorem (W)**.** *The map $c(\psi^k)$ is an equivalence on connected covers.*

We may therefore define a map $e : K(\mathbb{Z}, n+1) \to B(S, R_n)_{>0}$ uniquely by the requirement

$$c(\psi^k) \circ e = k^{-1} \frac{\psi^k(j-1)}{j-1} \in R_n^0(K(\mathbb{Z}, n+1))^\times$$

where $j : \Sigma^\infty K(\mathbb{Z}, n+1)_+ \to R_n$ is the localisation map in the first theorem. This is defined in analogy with the cannibalistic class of the tautological line bundle over $\mathbb{C}P^\infty$ in the case $n = 1$. Consequently, for any $(n-1)$-gerbe on X with Dixmier-Douady type characteristic class $H \in H^{n+1}(X)$, we may define an associated R_n oriented Thom spectrum X^H over X by the composite

$$X \xrightarrow{H} K(\mathbb{Z}, n+1) \xrightarrow{e} B(S, R_n)$$

Alternatively, if $E \to X$ is the principal $K(\mathbb{Z}, n+1)$-bundle over X corresponding to the $(n-1)$-gerbe, we may define the Thom spectrum X^H as

$$L_{K(n)}(\Sigma^\infty E_+ \wedge_{\Sigma^\infty K(\mathbb{Z},n)_+} S)$$

where $K(\mathbb{Z}, n)$ acts on S via the composite $\Omega(\gamma \circ e) : K(\mathbb{Z}, n) \to \mathrm{GL}_1 S$.

We conclude with an analogue of the J-homomorphism in this context. The standard notion of the (complex) J-homomorphism is the map $J : ku \to b(S^0, ku)$ which carries a complex vector bundle to its associated Thom spectrum with standard ku-orientation. The analogue in our setting should be a map J from a connective cover of R_n to $b(S, R_n)$. Further, this should extend the construction described above; that is, the composite

$$\Sigma^\infty K(\mathbb{Z}, n+1)_+ \xrightarrow{j} (R_n)_{>0} \xrightarrow{J} b(S, R_n)$$

should equal e. We have been unable to construct such a function J at this stage. However, if R_n is replaced by a suitable connective cover of its monochromatisation, $M_n R_n$, constructing such a map is possible, using a result of Ando-Hopkins-Rezk [1] on the coconnectivity of the "discrepancy spectrum."

As one defines the complex bordism spectrum as the Thom spectrum associated to the usual $J : BU \to B(S^0, K)$, we may in turn define an analogue MX_n

as the Thom spectrum of the monochromatic J-homomorphism in the previous paragraph.

References

[1] M. Ando, M. J. Hopkins, and C. Rezk, *Multiplicative orientations of KO-theory and the spectrum of topological modular forms*, preprint, available at `www.math.uiuc.edu/~mando/papers/koandtmf.pdf`, (2010).

[2] V. P. Snaith, *Algebraic cobordism and K-theory*, Mem. Amer. Math. Soc. **21** (1979), no. 221, vii+152 pp.

[3] C. Westerland, *A higher chromatic analogue of the image of J*, preprint, `arXiv:1210.2472,` (2010).

Short presentations

- Robert Bruner (Wayne State University)
  Demonstration of Ext.2.0, a Steenrod algebra cohomology calculator
- Markus Hausmann (Bonn)
  Equivariant symmetric products and the subgroup lattice
- Magdalena Kedziorek (EPFL)
  An algebraic model for rational $SO(3)$-spectra
- Martin Palmer (Münster)
  Homological stability for configuration spaces on closed manifolds
- Christian Wimmer (Bonn)
  Rational global homotopy theory
- Stephanie Ziegenhagen (Paris 13)
  E_n-cohomology as functor cohomology and operations

Reporter: Markus Hausmann

Participants

Dr. Vigleik Angeltveit
Mathematical Sciences Institute
Australian National University
GPO Box 4
Canberra ACT 0200
AUSTRALIA

Dr. Christian Ausoni
Département de Mathématiques
Institut Galilée
Université Paris XIII
99 av. J.B. Clément
93430 Villetaneuse
FRANCE

Dr. David Ayala
Dept. of Mathematical Sciences
Montana State University
Bozeman, MT 59717
UNITED STATES

Prof. Dr. Clark Barwick
Department of Mathematics
MIT
Cambridge, MA 02139
UNITED STATES

Dr. Agnès Beaudry
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. David J. Benson
Dept. of Mathematical Sciences
University of Aberdeen
Fraser Noble Bldg.
Aberdeen AB24 3UE
UNITED KINGDOM

Prof. Dr. Julie Bergner
Department of Mathematics
University of California
Riverside, CA 92521-0135
UNITED STATES

Prof. Dr. Andrew J. Blumberg
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Carl-Friedrich Bödigheimer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Robert R. Bruner
Department of Mathematics
Wayne State University
656 West Kirby Avenue
Detroit, MI 48202
UNITED STATES

Dr. Emanuele Dotto
Department of Mathematics
Massachusetts Institute of Technology
Cambridge MA 02139-4307
UNITED STATES

Prof. Dr. Daniel K. Dugger
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES

Prof. Dr. Johannes Ebert
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. John Francis
Department of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Dr. David J. Gepner
Department of Mathematics
Purdue University
West Lafayette, IN 47907-1395
UNITED STATES

Prof. Dr. Paul G. Goerss
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Prof. Dr. John Greenlees
School of Mathematics & Statistics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UNITED KINGDOM

Prof. Dr. Jesper Grodal
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
DENMARK

Markus Hausmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Jeremiah Heller
Hausdorff Center for Mathematics
Institute for Numerical Simulation
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Hans-Werner Henn
U.F.R. de Mathématique et
d'Informatique
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Prof. Dr. Kathryn Hess
École Polytechnique Fédérale de
Lausanne
SB MATHGEOM
Station 8
1015 Lausanne
SWITZERLAND

Dr. Mike Hill
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Dr. Marc Hoyois
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Dr. Magdalena Kedziorek
EPFL SV BMI UPHESS
MA B3 465
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Nicholas J. Kuhn
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Prof. Dr. Gerd Laures
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Prof. Dr. Tyler Lawson
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Akhil Mathew
Department of Mathematics
University of California at Berkeley
970 Evans Hall
Berkeley, CA 94720-3840
UNITED STATES

Prof. Dr. Niko Naumann
NWF I - Mathematik
Universität Regensburg
93053 Regensburg
GERMANY

Dr. Thomas Nikolaus
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Robert Oliver
Département de Mathématiques
Institut Galilée
Université Paris XIII
99 Av. J.-B. Clement
93430 Villetaneuse
FRANCE

Prof. Dr. Kyle M. Ormsby
Reed College
Department of Mathematics
3203 SE Woodstock Blvd.
Portland, OR 97202-8199
UNITED STATES

Dr. Martin Palmer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Irakli Patchkoria
Institut for Matematiske Fag
Kobenhavns Universitet
Universitetsparken 5
2100 Kobenhavn
DENMARK

Prof. Dr. Douglas Ravenel
Department of Mathematics
University of Rochester
Rochester, NY 14627
UNITED STATES

Prof. Dr. Birgit Richter
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Prof. Dr. Oliver Röndigs
Fachbereich Mathematik/Informatik
Universität Osnabrück
49069 Osnabrück
GERMANY

Dr. Steffen Sagave
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Christian Schlichtkrull
Department of Mathematics
University of Bergen
Johs. Brunsgate 12
5008 Bergen
NORWAY

Dr. Chris Schommer-Pries
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Stefan Schwede
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Jun. Prof. Dr. Markus Spitzweck
Fachbereich Mathematik/Informatik
Universität Osnabrück
49069 Osnabrück
GERMANY

Dr. Nat Stapleton
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Vesna Stojanoska
Department of Mathematics
University of Illinois at
Urbana-Champaign
273 Altgeld Hall, MC-382
1409 West Green Street
Urbana IL 61801
UNITED STATES

Prof. Dr. Neil P. Strickland
School of Mathematics & Statistics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UNITED KINGDOM

Dr. Hiro Lee Tanaka
Department of Mathematics
Harvard University
Science Center
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Ulrike Tillmann
Mathematical Institute
Oxford University
Woodstock Rd.
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Craig Westerland
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis MN 55455-0436
UNITED STATES

Prof. Dr. Kirsten Wickelgren
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Christian Wimmer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Stephanie Ziegenhagen
Département de Mathématiques
Institut Galilée
Université Paris XIII
99 Av. J.-B. Clement
93430 Villetaneuse
FRANCE