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Arbeitsgemeinschaft: The Kadison-Singer Conjecture

Organised by
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Abstract. The solution to the Kadison–Singer conjecture used techniques that intersect a number of areas of mathematics. The goal of this Arbeitsgemeinschaft was to bring together people from each of these fields to support interactions between these areas. While the majority of the talks centered around topics in polynomial geometry, combinatorics, and real algebraic geometry, participants came from areas such as harmonic analysis, convex geometry, and frame theory.

Mathematics Subject Classification (2010): 05-xx, 14Pxx, 15Axx, 15Bxx, 52Axx.

Introduction by the Organisers

The Arbeitsgemeinshaft on the Kadison–Singer conjecture was organized by Adam W. Marcus. The most notable attribute of the workshop was the wide variety of fields represented by the participants. The solution to the Kadison–Singer conjecure drew from a number of different areas, and because of this, the workshop contained a mix of researchers that might not normally interact.

The talks were intended to cover three areas. The first area was the origin and evolution of the Kadison–Singer conjecture. The original Kadison–Singer conjecture came from a paper in C^*-algebras, and a sequence of papers over 50 years reduced it to a question on finite vector spaces that led to the eventual solution. The first day of talks was dedicated to that reduction.

The second area was an introduction to polynomial geometry, as this was the area that many of the techniques that went into the solution came from. While the area of polynomial geometry has a long history, a number of recent advances were incorporated into the solution of Kadison–Singer. For historical reasons, the more recent developments in the area came from two somewhat independent fields. The theory of stable polynomials, which was born out of statistical physics, had the primary focus of understanding the locations of roots polynomials. The theory of hyperbolic polynomials, which was born out of partial differential equations, had the primary focus of exploiting the convexity properties that these polynomials inherit.The second and third days focused on presenting each of these areas and then showing how techniques from each could be used to solve the Kadison–Singer conjecture.

The final area was an attempt to suggest further areas of exploration across various areas. This included talks on problems in combinatorics, statistical physics, optimization, and computer science which involved the use of either hyperbolic or stable polynomials. The fourth and fifth day were dedicated to these areas. On that day, there was also an open problem session, where the participants of the Arbeitsgemeinschaft could present open problems related to the proof of the Kadison-Singer-Conjecture.

A number of the participants were from areas in harmonic analysis and real algebraic geometry. As many of the previous attempts to solve the Kadison–Singer conjecture were based in harmonic and functional analysis, there was a particular interest in whether (and how) the techniques of polynomial geometry could be used in attacking other problems. Attempts to extend the techniques of polynomial geometry, on the other hand, lead quickly into the area of real algebraic geometry, and so there was also a good interest in that direction. In all it was a good mix of people, spanning the range from those who wished to apply the techniques to those who wished to extend them.

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Workshop: Arbeitsgemeinschaft: The Kadison-Singer Conjecture

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Abstracts

Introduction to the problem

Jianchao Wu

The talk aimed at introducing the original form of the Kadison-Singer problem, as well as some necessary background, in particular the developments in quantum mechanics and operator algebra that led up to the formulation of the problem. The main references of the talk include [1] and [2]. Quantum mechanics differs from classical mechanics in that the observables under study, e.g. momentum, position, energy, etc., are no longer real functions on the (classical) state space —the space of all possible configurations of the system, but are self-adjoint operators on a Hilbert space, or more generally, self-adjoint elements of a C*-algebra $\mathcal{A}$, while the quantum state space takes the form of the space of all bounded linear functionals $f : \mathcal{A} \to \mathbb{C}$ that are

- positive: positive elements are mapped to non-negative real numbers;
- unital: the unit of $\mathcal{A}$ is mapped to 1 (all the C*-algebras in this talk are assumed to be unital).

Such functionals are called *states*. The state space as a whole is a compact convex subset of the dual Banach space of $\mathcal{A}$, equipped with the weak-* topology. Of great importance are the extreme points of this convex set. They are called *pure states*. Moreover, if there is an unital inclusion of C*-algebras $\mathcal{A} \subset \mathcal{B}$, then each state of $\mathcal{A}$ extends to a state of $\mathcal{B}$.

To illustrate the concept of states, we explicitly calculated the state space of $M_2(\mathbb{C})$, the C*-algebra of complex-valued 2×2 matrices, as well as that of its sub-C*-algebra $\mathcal{D} \subset M_2(\mathbb{C})$ consisting of the diagonal matrices (see [1, Section 2]). Note that $\mathcal{D}$ is generated by the unit and the self-adjoint element $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In quantum mechanics, this mathematical setting underpins the Pauli model for the spin of an electron, where the observable σ_1 may represent one of the three directional spin quantum number of the electron.

In this simple example we observe an interesting phenomenon: any pure state of $\mathcal{D}$ extends *uniquely* to a pure state of $M_2(\mathbb{C})$. This fact is of physical significance: it means that after measurement of the observable σ_1 (the generator of $\mathcal{D}$), the state of the system at that moment can be precisely determined.

A natural question is whether such a phenomenon can be found in other pairs of C*-algebras $\mathcal{A} \subset \mathcal{B}$. We are particuarly interested in the case where $\mathcal{B} = B(\mathcal{H})$ is the C*-algebra of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, and $\mathcal{A}$ is commutative. It is not hard to see that in order to get a positive answer, $\mathcal{A}$ must be a *maximal abelian self-adjoint algebra*, and in particular a von Neumann algebra. By the classification of separable abelian von Neumann algebras, we only need to investigate the following three cases:

(1) $\mathcal{D}_n \subset M_n(\mathbb{C})$, $n \in \mathbb{N}$, where $\mathcal{D}_n$ consists of diagonal matrices;
(2) $l^\infty(\mathbb{N}) \cong \mathcal{D}_\mathbb{N} \subset B(l^2(\mathbb{N}))$, where $\mathcal{D}_\mathbb{N}$ consists of diagonal operators;
(3) $l^\infty([0,1]) \subset B(l^2([0,1]))$.

It is straightforward to check that the answer is affirmative for the first case, for any $n \in \mathbb{N}$, i.e. every pure state of $\mathcal{D}_n$ extends uniquely to a pure state of $M_n(\mathbb{C})$. On the other hand, Kadison and Singer [3] showed that the answer is negative in the third case. The second case was left open and became the famous Kadison-Singer problem.

The Kadison-Singer Problem: Does every pure state on the C^*-algebra of bounded diagonal operators on $l^2(\mathbb{N})$ extend *uniquely* to a pure state on the C*-algebra of all bounded linear operators on $l^2(\mathbb{N})$?

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From C* algebras to vector spaces, Ultrafilters

Safdar Quddus

The Kadison-Singer Conjecture was a question in functional analysis about uniqueness of the (pure) state extension of a (pure) state on $\mathbb{D}(l_2)$ to $\mathbb{B}(l_2)$. As a question over infinite dimensional spaces it posed severe difficulties. In this talk we discussed the background to the recent proof [4] which changed the question from one that on infinite dimesional space to problem on matrices. It was through the work of Akemann, Anderson and Weaver ([5] and [1]) that the problem was reduced to one estimating norm of vectors in $\mathbb{C}^n$.

Ultrafilters

My part of the Arbeitsgemeinschaft was to introduce the concept of ultrafilters and to reduce the the Kadison-Singer Conjecture using this tool.

For a non-empty set Y, $\mathcal{F} \subset 2^Y$ satisfying the criterion below is a filter.

- $\phi \notin \mathcal{F}, Y \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ in which exactly one of A, A^c belongs to $\mathcal{F}$ is called an ultrafilter. For example: $\mathcal{U}_j := \{A \subset \mathbb{N} | j \in A\}$ is an ultrafilter over $\mathbb{N}$, infact these ultrafilters are called as principal ultrafilters at j.

It is to note that by using axiom of choice we can construct a non-principal ultrafilter but in ZF setup all the ultrafilters are principal.

Lemma 1. *Every filter is contained in an ultrafilter.*

Proof. By Zorn's lemma on poset of filters on Y containing $\mathcal{F}$ we get a maximal element. We need to check the following:

- The union of chain of filters is a filter. (easy)
- A maximal filter is an ultrafilter. (If not then, let $\widehat{\mathcal{F}}$ be a maximal filter, there exists A such that $A \notin \widehat{\mathcal{F}}$, $A^c \notin \widehat{\mathcal{F}}$. Then $\widehat{\mathcal{F}} \cup \{A\}$ is a filter strictly containing $\widehat{\mathcal{F}}$, a contradiction to maximality.)

□

Hence the maximal filter containing the cofinite filter $\mathcal{F}_{cf} := \{A \subset \mathbb{N} | A^c \text{ is infinite}\}$ is a non-principal ultrafilter as it is not contained in any principal ultrafilter $\mathcal{U}_j$ for $j \in \mathbb{N}$. Infact all non-principal ultrafiters contain $\mathcal{F}_{cf}$.

Ultrafilters on $\mathbb{N}$

Let $\beta\mathbb{N}$ denote the set of all ultrafilters on $\mathbb{N}$. We can topologise $\beta\mathbb{N}$ by declaring that sets of the type $\widehat{A} := \{\mathcal{U} | A \in \mathcal{U}\}$ are the open sets. We can easily check that the following properties hold:

- $\widehat{A}$ is clopen.
- the set $\mathcal{A} := \{\widehat{A} | A \subset \mathbb{N}\}$ is indeed a base for the topology.
- $\beta\mathbb{N}$ is compact.
- $\beta\mathbb{N}$ is Hausdorff.
- Principal ultrafilters are dense in $\beta\mathbb{N}$.

$\mathcal{U} - lim$ and $\mathbb{D}(l_2)$

For any $\mathcal{U} \in \beta\mathbb{N}$ we can define a corresponding $\mathcal{U} - lim$. Formally, for $a \in \mathbb{C}^{\mathbb{N}}$ define $a_{\mathcal{U}}$ to the $\mathcal{U} - lim\ a$ if, for all neighbourhoods S of $a_{\mathcal{U}}$ the set $\{i | a_i \in S\}$ is contained in $\mathcal{U}$.

We can easily check that this limit has the following properties:

- If $a \in l_\infty$ then, $\mathcal{U} - lim\ a$ exists.
- $\mathcal{U} - lim\ a$ is unique.
- If $I \in \mathcal{U}$, then $a_{\mathcal{U}} \in \overline{\{a_i | i \in I\}}$.
- $c \cdot \mathcal{U} - lim\ a + d \cdot \mathcal{U} - lim\ b = \mathcal{U} - lim(ca + db)$. $\forall\ c, d \in \mathbb{C}$.
- $(\mathcal{U} - lim\ a) \cdot (\mathcal{U} - lim\ b) = \mathcal{U} - lim\ (a \cdot b)$.
- $(\mathcal{U} - lim\ a)^* = \mathcal{U} - lim\ (a^*)$. Where $(a^*)_i = (a_i)^*$

Using the diagonal map, $\mathrm{Diag} : \mathbb{D}(l_2) \to l_\infty$, we have an isomorphism of $\mathbb{D}(l_2)$ with l_∞. The following results, give an ultrafilter description of the pure states of $\mathbb{D}(l_2)$.

Theorem 2. *The Banach spaces l_∞ and $C(\beta\mathbb{N})$ are isometric isomorpic.*

Proof. Define a map $\phi : l_\infty \to C(\beta\mathbb{N})$ as $\phi(a) = f_a$, where $f_a(\mathcal{U}) = \mathcal{U} - lim\ a$. Conversely, define $\psi : C(\beta\mathbb{N}) \to l_\infty$ as $\psi(f) = a_f$, where $(a_f)_i = f(\mathcal{U}_i)$. It can be easily seen that these are indeed maps and agree on principal ultrafilters and preserve norms. □

Theorem 3. *The pure states on $\mathbb{D}(l_2)$ are precisely the functionals of the form $f_{\mathcal{U}}$ for $\mathcal{U} \in \beta\mathbb{N}$, where*

$$f_{\mathcal{U}}(z) = \mathcal{U} - lim(Diag(z)).$$

Proof. Let f be pure state on $\mathbb{D}(l_2)$, let $\mathcal{U} := \{A \subset \mathbb{N} | f(P_A) = 1\}$, it can be seen that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. To show that $f = f_{\mathcal{U}}$, it suffcies that they agree on the dense set of projections on $\mathbb{D}(l_2)$. Since pure states are multiplicative we can see that $f(P_A) = 1$ or 0. If $f(P_A) = 1$ then $A \in \mathcal{U}$ hence, $f_{\mathcal{U}}(P_A) = 1$. Else if $f(P_A) = 0$ then $A \notin \mathcal{U}$ so, $f_{\mathcal{U}}(P_A) = \mathcal{U} - lim\ d = 0$.

Conversely, for an ultrafilter $\mathcal{U}$, $f_{\mathcal{U}}$ is multiplicative hence a pure state. □

Hence we have reduced the Kadison-Singer Conjecture to problem on ultrafilters on $\mathbb{N}$. Subsequently in the next lectures in the Arbeitsgemeinschaft we saw how this can be further reduced to problem on finite dimensional matrices.

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Proving the equivalence of the Kadison-Singer problem and Anderson's paving conjecture

Itay Londner

This presentation is essentially based on the paper [4] due to Harvey, sections 4 and 5. We begin by restating of the Kadison-Singer problem.

Problem ([5]). *Is it true that every pure state $f : \mathbb{D}(\ell_2) \to \mathbb{C}$ can be uniquely extended to a state $g : \mathcal{B}(\ell_2) \to \mathbb{C}$?*

Previously we have seen that the C^*-algebra of bounded diagonal operators $\mathbb{D}(\ell_2)$ is isometrically isomorphic to the space ℓ_∞ of infinite bounded sequences. Furthermore, the space of ultrafilters $\beta\mathbb{N}$ was introduced, as well as the concept of U-limits. We have also seen that the space $C(\beta\mathbb{N})$ of continuous complex-valued functions is isometrically isomorphic to ℓ_∞ and to $\mathbb{D}(\ell_2)$. Using the fact that pure states on $C(\beta\mathbb{N})$ are precisely the multiplicative functionals ([5], Theorem 3.4.7 and proposition 4.4.1), we get the following.

Theorem 1. *The pure states on $\mathbb{D}(\ell_2)$ are the functionals of the form*

$$U \mapsto U - \lim (diag(D)),$$

where $U \in \beta\mathbb{N}$, $D \in \mathbb{D}(\ell_2)$ and only those.

One should notice that such states admit an immediate extension to the C^*-algebra of bounded linear operators $\mathcal{B}(\ell_2)$ given by $A \mapsto U - \lim(diag(A))$, hence the issue is around uniqueness of the extension rather than its existence.

Throughout the years the Kadison-Singer problem turned out to be equivalent to many different problems in various areas of mathematics (see, for example [3]). One of these equivalent forms is known as Anderson's paving conjecture ([1, 2]).

Conjecture (Anderson's Infinite-Dimensional Paving conjecture)**.** *For every $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that for every self-adjoint $H \in \mathcal{B}(\ell_2)$ with zero diagonal, there exists a partition $\{A_1, \dots, A_r\}$ of $\mathbb{N}$ such that $\|P_{A_i} H P_{A_i}\| \leq \varepsilon \|H\|$ for every $i \in \{1, \dots, r\}$.*

Here $P_{A_i} : \ell_2 \to \ell_2$ denotes the orthogonal projection onto the coordinates in A_i. The main result of this part is the following:

Theorem 2. *The following are equivalent:*

(1) *The Kadison-Singer problem.*
(2) *Anderson's Infinite-Dimensional Paving conjecture.*
(3) *For every $\varepsilon > 0$ and for every self-adjoint $H \in \mathcal{B}(\ell_2)$ with zero diagonal, there exists $r \in \mathbb{N}$ and a partition $\{A_1, \dots, A_r\}$ of $\mathbb{N}$ such that*

$$-\varepsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \varepsilon P_{A_i}$$

for every $i \in \{1, \dots, r\}$.

We point out that the difference between (2) and (3) above lays only in the uniform choice of r with respect to the subspace of self-adjoint operators, which turned out to be equivalent.

Another equivalent form of the Kadison-Singer problem is the conjecture known as Feichtinger conjecture. For the precise formulation and proof of equivalence see [3].

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From infinite dimensional pavings to finite dimensional pavings

Vadim Alekseev

In our talk we presented the reduction of the infinite-dimensional Anderson paving conjecture (paving matrices in $\mathbb{B}(\mathcal{H})$ to various finite-dimensional paving conjectures, including Weaver's conjecture.

The reduction from infinite-dimensional to finite-dimensional Anderson paving proceeds through paving of increasing sequence of corners of a given matrix; ituses a variant of the Knig's lemma about infinite graphs to establish the existence of the required paving.

Next, we discussed the reduction of the finite-dimensional Anderson paving conjecture to various finite-dimensional paving conjectures. Using the standard relationships between self-adjoint matrices, reflections and projections, it is possible to reduce paving of self-adjoint matrices with zeroes on the diagonal to paving of reflections with zeroes on the diagonal, and finally to paving of projections with 1/2 on the diagonal. By using an amplification trick, one can further soften the paving assumptions. Using the relationship between frames and projections, the Anderson paving problem (and thus the Kadison–Singer conjecture) is finally reduced to a problem about frames known as Weaver's conjecture.

Univariate stable polynomials

Markus Schweighofer

In this talk we tried to give a very basic introduction to univariate stable polynomials and to illustrate their usefulness in combinatorics. More complete introductions into this subject are [7] and [8].

All polynomials will be real or complex polynomials in one variable X. We denote the corresponding algebra of polynomials by $\mathbb{R}[X]$ and $\mathbb{C}[X]$, respectively.

1. Stable polynomials

We call $f \in \mathbb{C}[X]$ *stable* if f is the zero polynomial or f has no complex roots with positive imaginary part, i.e., $f = 0$ or $\forall z \in \mathbb{C} : (\mathrm{Im}(z) > 0 \implies f(z) \neq 0)$.

There are many other notions of stability with the upper half plane replaced by other regions. The term "stable" is motivated by control theory where the stable behavior of a system can often be related to stability (in one sense or the other) of a polynomial.

The Gauß-Lucas Theorem [3, Theorem 2.1.1] says that for a non-constant complex polynomial the zeros of its derivative are convex combinations of its zeros:

$$\forall p \in \mathbb{C}[X] \setminus \mathbb{C} : \{z \in \mathbb{C} \mid p'(z) = 0\} \subseteq \mathrm{conv}\{z \in \mathbb{C} \mid p(z) = 0\}.$$

As a corollary, derivatives of stable polynomials are stable.

2. Real stable polynomials

Stable polynomials in $\mathbb{R}[X]$ are called *real stable* (also *real-rooted* and sometimes *hyperbolic* or *real zero polynomial*). These are the polynomials of the form

$$\lambda(X-a_1)\cdots(X-a_n) \qquad (n\in\mathbb{N}_0,\lambda,a_1,\ldots,a_n\in\mathbb{R}).$$

To see that derivatives of real stable polynomials are real stable, one can simply use Rolle's theorem instead of the Gauß-Lucas Theorem.

If $p=(X-a_1)\cdots(X-a_n)$ with $\lambda_i\in\mathbb{C}$, then its reciprocal $p^*:=X^np\left(\frac{1}{X}\right)=(1-a_1X)\cdots(1-a_nX)=:1-p^\sharp$ has the same coefficients in reversed order and we have a formal identity

$$\sum_{k=1}^\infty\frac{1}{k}\underbrace{(a_1^k+\cdots+a_n^k)}_{=:N_k}X^k=-\log p^*=-\log(1-p^\sharp)=\sum_{k=1}^\infty\frac{1}{k}p^{\sharp k}$$

and therefore the k-th *Newton sum* N_k can be expressed polynomially in the coefficients of p. Hence the *Hermite-matrix*

$$H(p):=\begin{pmatrix} N_0 & N_1 & N_2 & \cdots & N_{n-1}\\ N_1 & N_2 & & & \vdots\\ N_2 & & & & \vdots\\ \vdots & & & & N_{2n-3}\\ N_{n-1} & \cdots & \cdots & N_{2n-3} & N_{2n-2}\end{pmatrix}=V(p)^TV(p)$$

can be written easily in terms of the coefficients of p in contrast to the *Vandermonde-matrix*

$$V(p):=\begin{pmatrix}1 & a_1 & \cdots & a_1^{n-1}\\ \vdots & \vdots & & \vdots\\ 1 & a_n & \cdots & a_n^{n-1}\end{pmatrix}.$$

If p is stable, then $H(p)$ is clearly positive semidefinite. By a theorem of Hermite and Sylvester [4, Theorem 4.57], the converse is also true. Since the positive semidefiniteness of a real symmetric matrix can easily be decided without computing its eigenvalues, this gives a very efficient test for real stability of polynomials.

Since Descarte's rule of signs [4, Theorem 2.33] is exact for real stable polynomials [4, Corollary 2.49, Theorem 2.47], we have: If $p=(X-a_1)\cdots(X-a_n)$ with $a_i\in\mathbb{R}$, then $\#\{i\mid a_i>0\}$ is the number of signs in the coefficient sequence of p (disregarding zero coefficients). As a special case, which is however trivial, a nonzero real stable polynomial has no positive roots if and only if it has no negative coefficients.

By Edrei's equivalence theorem [7, Theorem 4.9], a real non-zero polynomial $c_nX^n + \cdots + c_0$ ($c_i \in \mathbb{R}$) is stable without positive roots if and only if the infinite (lower triangular Toeplitz) matrix

$$\begin{pmatrix} c_0 & & & \\ c_1 & c_0 & & \\ c_2 & c_1 & c_0 & \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is *totally positive*, i.e., all its minors are nonnegative.

3. Newton's inequalities and unimodality

A sequence $(c_0, \ldots, c_n)$ of real nonnegative numbers is called *unimodal* if there exists an $m \in \{0, \ldots, n\}$ such that $c_0 \leq \cdots \leq c_{m-1} \leq c_m \geq c_{m+1} \geq \cdots \geq c_n$. We say it has *no internal zeros* if $\{i \mid a_i \neq 0\}$ is an interval in $\{0, \ldots, n\}$. We say it is *log-concave* if it has no internal zeros and $c_{k-1}c_{k+1} \leq c_k^2$ for all $k \in \{1, \ldots, n-1\}$. One checks easily that the sequence $\left(\binom{n}{0}, \ldots, \binom{n}{n}\right)$ of binomial coefficients is log-concave. We say that $(c_0, \ldots, c_n)$ is *ultra log-concave* if it has no internal zeros and satisfies *Newton's inequalities*

$$\frac{c_{k-1}}{\binom{n}{k-1}} \frac{c_{k+1}}{\binom{n}{k+1}} \leq \left(\frac{c_k}{\binom{n}{k}}\right)^2$$

for $k \in \{1, \ldots, n-1\}$. One checks easily

"ultra log-concave $\implies$ log-concave $\implies$ unimodal".

Looking at 2×2 minors in Edrei's theorem, one sees that a real stable polynomial without negative coefficients has log-concave coefficient sequence. However, it has even ultra log-concave coefficient sequence: This can be seen as follows: Fix three consecutive terms in the given polynomial. Kill all terms of lower degree by taking an appropriate higher derivative. To get rid of terms of higher degree, take the reciprocal and take again a suitable higher derivative. All these operations preserve stability. Now you end up with a real stable quadratic polynomial. Its discriminant must be nonnegative and this gives exactly Newton's inequalities.

As an example of this method, consider the unsigned Stirling numbers of the first kind $\left[{n \atop k}\right] := \#\{\sigma \in S_n \mid \sigma \text{ has exactly } k \text{ cycles}\}$ ($k, n \in \mathbb{N}_0$, $0 \leq k \leq n$). Here 1-cycles, i.e., fixed points of the permutation count. We claim that $\left(\left[{n \atop 0}\right], \ldots, \left[{n \atop n}\right]\right)$ is unimodal. To prove this, we show even that it is ultra log-concave. It suffices to show that $p := \sum_{k=0}^n \left[{n \atop k}\right] X^k$ is stable. But $p = X(X+1)\cdots(X+n-1) =: X^{(n)}$ ("rising factorial"). This can be seen easily by counting permutations whose cycles are colored (thinking of X as the number of colors) in two different ways: One way is by grouping together permutations with the same number of cycles. The other way is by successively deciding for each number between 1 and n whether it should go into a new cycle (in which case a color has to be chosen) or whether it should be inserted in one of the already existing cycles (in which case it has to be inserted at some position in these cycles).

4. Stability preservers

Borcea and Brändén characterized in 2009 all *linear stability preservers* $\mathbb{R}[X] \to \mathbb{R}[X]$ and $\mathbb{C}[X] \to \mathbb{C}[X]$ (an example of which is $p \mapsto p'$). This involves however a notion of multivariate stability and therefore goes beyond the scope of this talk [5]. Brenti [2] proved in 1989 that the restriction of the linear map $\mathbb{R}[X] \to \mathbb{R}[X], X^k \mapsto X^{(k)}$ (the "rising factorial" from above) to polynomials with only nonnegative coefficients preserves stability (this is the correct part of [7, Theorem 4.6], the other part being trivially wrong as the counterexample $(X+1)(X+2)$ shows).

As an example of how to use stability preserving maps, consider a variant of the above Stirling numbers: Define a *cycle* of a *function* $f\colon \{1,\dots,n\} \to \{1,\dots,n\}$ as a connected component of the graph $\{\{x, f(x)\} \mid x \in \{1,\dots,n\}, x \neq f(x)\}$. For $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$, define $\left\{ {n \atop k} \right\}$ as the number of functions $\{1,\dots,n\} \to \{1,\dots,n\}$ with exactly k cycles. We claim that $\left(\left\{ {n \atop 0} \right\}, \dots, \left\{ {n \atop n} \right\}\right)$ is unimodal. Again, we show even that it is ultra log-concave. Without loss of generality $n \geq 1$. Using the formula for the number of rooted forests on n vertices with exactly k trees from [6, end of Chapter 30] (I am grateful to Benjamin Matschke for showing me this formula and relating it to this), it is an exercise to show that $\left\{ {n \atop k} \right\} = \sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} \left[{i \atop k} \right]$ for $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$. Hence

$$p := \sum_{k=0}^n \left\{ {n \atop k} \right\} X^k = \sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} \underbrace{\sum_{k=0}^n \left[{i \atop k} \right] X^k}_{=X^{(i)}}.$$

But p is stable by Brenti's result mentioned above since $q := \sum_{i=1}^n \binom{n-1}{i-1} n^{n-i} X^i = X(X+n)^{n-1}$ is stable.

5. Pólya-Schur multiplier sequences

Already in 1914, Pólya and Schur characterized in their fulminant work [1] all *linear stability preservers* $\mathbb{R}[X] \to \mathbb{R}[X]$ and $\mathbb{C}[X] \to \mathbb{C}[X]$ of the form $X^k \mapsto \lambda_k X^k$ for some sequence $(\lambda_k)_{k \in \mathbb{N}}$. We mentioned the main facts of this beautiful work similarly to the exposition in [7, Section 4.3]. A detailed account of this theory can be found in [3, Sections 5.4 and 5.7]

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Multivariate stable polynomials

Mario Kummer

I talked about multivariate stable polynomials: A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called *stable* if f does not vanish on $\mathcal{H}^n$ where $\mathcal{H} \subseteq \mathbb{C}$ denotes the upper half-plane. A basic and important example of stable polynomials is the following: If $A_0, \ldots, A_n$ are positive definite Hermitian matrices of the same size, then

$$\det(A_0 + A_1 \cdot z_1 + \ldots + A_n \cdot z_n)$$

is a stable polynomial. The main focus of the talk was on operations that preserve the class of stable polynomials. There are some basic operations for which one can easily see that they preserve stability like dilation, permuting variables, specialization, differentiation or taking limits [6, 7]. I presented some results of Borcea and Brändén that characterize all linear operations on the vector space of polynomials that preserve stability [1, 2, 3].

An interesting special case is the one where no variable appears in any monomial with power two or higher. Such polynomials are called *multiaffine* polynomials. Many problems on stable polynomials become easier if one restricts attention to multiaffine polynomials. There is for example a nice characterization of multiaffine, real stable polynomials due to Brändén [4]: A multiaffine polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is stable if and only if the inequality

$$\frac{\partial f}{\partial x_i}(x) \cdot \frac{\partial f}{\partial x_i}(x) \geq \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \cdot f(x)$$

holds for every $x \in \mathbb{R}^n$ and all $1 \leq i, j \leq n$. On the other hand, using an operation called *polarization* which assigns to each polynomial a multiaffine polynomial and which preserves stability one can reduce many problems to the multiaffine case [5]. There is also an interesting and fruitful connection between multiaffine stable polynomials and matroid theory [4, 5].

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Interlacers and mixed characteristic polynomials

Claus Scheiderer

We work towards the proof of the Kadison-Singer theorem, following [1]. Given two real rooted monic polynomials f and g, we say that g interlaces f if $\deg(f) = \deg(g)+1 = n$, and if $\alpha_i \le \beta_i \le \alpha_{i+1}$ holds for $i = 1, \dots, n-1$, where α_i (resp. β_i) is the i-th root of f (resp. of g) ordered in an increasing way. If $f_1, \dots, f_k \in \mathbb{R}[x]$ are monic and real rooted polynomials of the same degree that have a common interlacer, we first show that any convex combination f of $f_1, \dots, f_k$ is real rooted with $\operatorname{maxroot}(f) \ge \min_{j=1,\dots,k} \operatorname{maxroot}(f_j)$.
The mixed characteristic polynomial of a sequence $A_1, \dots, A_m$ of complex $d \times d$ matrices is defined as the polynomial

$$\mu[A_1, \dots, A_m](x) \;:=\; \Big(\prod_{j=1}^{m}(1-\partial_{z_j})\Big) \det\Big(xI + \sum_{j=1}^{m} z_j A_j\Big)\Big|_{z_1=\cdots=z_m=0}$$

in $\mathbb{C}[x]$. The main result is the following. Let $A_1, \dots, A_m$ be a sequence of independent random matrices of rank ≤ 1 with finite support. Then

$$\mathbb{E}\chi[A_1 + \cdots + A_m] \;=\; \mu[\mathbb{E}(A_1),\, \dots,\, \mathbb{E}(A_m)](x),$$

that is, the expected characteristic polynomial of $\sum_j A_j$ is the mixed characteristic polynomial of the expectations $\mathbb{E}(A_1), \dots, \mathbb{E}(A_m)$. As a consequence we find that when the A_j are positive semidefinite, the polynomial $\mu = \mu[\mathbb{E}(A_1),\, \dots,\, \mathbb{E}(A_m)](x)$ is real rooted, and if λ is a zero of μ, then $\sum_j A_j$ has an eigenvalue $\le \lambda$ with positive probability.

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Kadison-Singer via Real Stability

Thorsten Jörgens

This talk completes the topic *Stability* and finishes the proof of the Kadison-Singer Conjecture using facts and properties learned in the previous ones. The talk follows the paper 'Interlacing Families II: Mixed Characteristic Polynomials and The Kadison-Singer Problem' by Marcus, Spielman and Srivastava [1].

First, the following statement about the roots of the mixed characteristic polynomial of matrices $A_1, \ldots, A_m$ is proven:

$$\begin{aligned}\mu[A_1, \ldots, A_m](x) &:= \left(\prod_{i=1}^{m} 1 - \partial_{z_i}\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)\Bigg|_{z_1=\ldots=z_m=0} \\ &= \left(\prod_{i=1}^{m} 1 - \partial_{z_i}\right) \det\left(\sum_{i=1}^{m} z_i A_i\right)\Bigg|_{z_1=\ldots=z_m=x}.\end{aligned}$$

Theorem 1. *Let $A_1, \ldots, A_m$ be Hermitian positive semidefinite matrices satisfying $\sum_{i=1}^{m} A_i = I$ and $tr(A_i) \leq \varepsilon$ for all i.*

Then the largest root of the mixed characteristic polynomial is at most $(1+\sqrt{\varepsilon})^2$.

The idea of the proof is to gain the mixed characteristic polynomial by applying operators of the form $(1 - \partial_{z_i})$ to $\det(\sum z_i A_i)$ iteratively. A 'multivariate upper bound' of the arising polynomials can be observed. A term of a multivariate upper bound is provided by the definition of 'above the roots':

Definition 2. *Let $p(z_1, \ldots, z_n)$ be a multivariate polynomial. We say that $z \in \mathbb{R}^n$ is above the roots of p, $z \in Ab_p$, if*

$$p(z+t) > 0 \textit{ for all } t = (t_1, \ldots, t_m) \in \mathbb{R}^m, t_i \geq 0.$$

Furthermore, by defining a suitable barrier function a sufficient 'safety distance' ('cushion') of a point $z \in \mathrm{Ab}_p$ to the roots of the corresponding polynomial p is guaranteed. The following barrier function is used:

Definition 3. *Let p be a real stable polynomial and $z = (z_1, \ldots, z_n) \in Ab_p$. Our barrier function of p in direction i at the point z is*

$$\Phi_p^i(z) = \partial_{z_i} \log p(z) = \frac{\partial_{z_i} p(z)}{p(z)}.$$

These functions are for all $i = 1, \ldots, n$ non-increasing and convex in every direction.

The proof of Theorem 1 uses the following lemma and the idea of iteratively applying the $(1 - \partial_{z_j})$-operators mentioned above:

Lemma 4. *Let p be real stable, $z \in Ab_p$, $\Phi_p^i(z) < 1$. Then,*

$$z \in Ab_{(1-\partial_{z_i})p}.$$

Furthermore, if $\delta > 0$ and $\Phi_p^j(z) \leq 1 - \frac{1}{\delta}$, then for all i,

$$\Phi^i_{(1-\partial_{z_j})p} p(z + \delta e_j) \leq \Phi_p^i(z).$$

The second part proves the Kadison-Singer Conjecture by showing the equivalent Weaver Conjecture. The following theorem is proven at the start:

Theorem 5. *Let $\varepsilon > 0$ and $v_1, \dots, v_m \in \mathbb{C}^d$ be independent random vectors with finite support so that*

$$\sum_{i=1}^{m} \mathbb{E} v_i v_i^* = I_d \text{ and } \mathbb{E}\|v_i\|^2 \leq \varepsilon, i = 1, \dots, m.$$

Then

$$\mathbb{P}\left(\left\| \sum_{i=1}^{m} v_i {v_i}^* \right\| \leq (1 + \sqrt{\varepsilon})^2 \right) > 0.$$

The theorem provides the existence of vectors $w_1, \dots, w_m$ in the value sets of $v_1, \dots, v_m$ which have the property $\|\sum_{i=1}^m w_i {w_i}^*\| \leq (1+\sqrt{\varepsilon})^2$, e.g., the eigenvalues of $\sum w_i w_i^*$ are bounded by $(1+\sqrt{\varepsilon})^2$. The proof uses, beside Theorem 1, results of the previous talks about mixed characteristic polynomials and interlacing families.

The following corollary states the existence of a certain partition of vectors and forms the last step proof Weaver's Conjecture:

Corollary 6. *Let r be a positive integer, $\delta > 0$ and let $u_1, \dots, u_m \in \mathbb{C}^d$ be vectors such that $\sum_{i=1}^m u_i u_i^* = I$ and $\|u_i\|^2 \leq \delta$, $i = 1, \dots, m$.*
Then there exists a partition $\{S_1, \dots, S_r\}$ of $\{1, \dots, m\}$ such that

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2, \quad \text{for } j = 1, \dots, r.$$

The proof uses a clever construction of independend random vectors $v_1, \dots, v_n$ such that each v_i is uniformly distributed on $w_{i,1} := (u_i, 0_d, 0_d, \dots, 0_d)^t, w_{i,2} := (0_d, u_i, 0_d, \dots, 0_d)^t, \dots \in \mathbb{C}^{rd}$. Theorem 5 can be applied and yields the existence of an assignment of the v_i to the w_i such that $\|\sum_{i=1}^m w_i {w_i}^*\| \leq (1+\sqrt{\varepsilon})^2$. Defining the partition-sets as $S_k := \{i : v_i = w_{i,k}\}$ provides the statement.

The talk ends with the proof of Weaver's Conjecture:

Theorem 7. *There exist universal contants $\eta \geq 2$ and $\theta > 0$ so that the following holds. Let $w_1, \dots, w_m \in \mathbb{C}^d$ satisfy $\|w_i\| \leq 1$ for all i and suppose*

$$\sum_{i=1}^{m} |\langle u, w_i \rangle|^2 = \eta \tag{1}$$

for every unit vector $u \in \mathbb{C}^d$. Then there exists a partition S_1, S_2 of $\{1, \dots, m\}$ so that

$$\sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq \eta - \theta,$$

for every unit vector $u \in \mathbb{C}^d$ and each $j \in \{1, 2\}$.

The Conjecture is shown for the values $\eta = 18$ and $\theta = 2$ and follows by applying the last corollary to $u_i := w_i/\sqrt{\eta}$, $i = 1, \dots, n$. Then, Weaver's Condition (1) becomes $\sum_{i=1}^{m} u_i u_i^* = I$. The existence of the partition is obtained by the corollary above. Finally, $\sum_{i \in S_j} | \langle u, w_i \rangle |^2 \leq 16 = \eta - \theta$ is received back.

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Hyperbolic polynomials

Tom Drescher

The aim of this talk was to define hyperbolic polynomials and hyperbolicity cones, and to prove some basic properties such as convexity.

Definition 1. *A homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ is called hyperbolic with respect to a vector $e \in \mathbb{R}^n$ if $h(e) \neq 0$ and if for all $x \in \mathbb{R}^n$ the univariate polynomial*

$$h_x(t) := h(-x + t \cdot e)$$

has only real roots.

h_x is called the characteristic polynomial of x. It is easy to see, that h_x has leading coefficient $h(e)$. Hence, there are functions

$$\lambda_i \colon \mathbb{R}^n \to \mathbb{R}$$

for $i = 1, \dots, n$, such that

$$h_x(t) = h(e) \cdot (t - \lambda_1(x)) \cdot (t - \lambda_2(x)) \cdots (t - \lambda_n(x))$$

and

$$\lambda_{min}(x) := \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$$

for all $x \in \mathbb{R}^n$. From this definition we immediately get the following transformation rule:

$$\lambda_j(s \cdot x + r \cdot e) = \begin{cases} s \cdot \lambda_j(x) + r & , s \geq 0 \\ s \cdot \lambda_{n-j}(x) + r & , s \leq 0. \end{cases}$$

Example. *Let h be a non-degenerate homogeneous quadratic. Then h is linearly isomorphic to a polynomial of the form*

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

(i.e. there exists an invertible matrix T, such that $h(Tx)$ has this form). Now assume, that h is of this form, that $p \leq n - p$, and that $n \geq 3$. Then h is hyperbolic with respect to $e \in \mathbb{R}^n$ if and only if $p = 1$ and $h(e) > 0$.

Example. *Let $M_1, \dots, M_n$ be hermitian matrices, and let*

$$e_1 M_1 + \cdots + e_n M_n$$

be positive definite. Then the polynomial

$$h(x) := \det(x_1 M_1 + \cdots + x_n M_n)$$

is hyperbolic with respect to e. The term characteristic polynomial from above and also other terms regarding hyperbolic polynomials, are inspired by this example.

Definition 2. *Let h be hyperbolic with respect to e. Then we define the following subsets of $\mathbb{R}^n$:*

$$\Lambda_{++}(h,e) := \{x \in \mathbb{R}^n \mid \lambda_{min}(x) > 0\},$$
$$\Lambda_{+}(h,e) := \{x \in \mathbb{R}^n \mid \lambda_{min}(x) \geq 0\}.$$

They are called open and closed hyperbolicity cone of h.

The following proposition gives some basic properties and equivalent definitions of the hyperbolicity cone:

Proposition 3. *Let h be hyperbolic with respect to e. Then*

(1) *$\Lambda_{++}(h,e)$ is basic open semi-algebraic. More precisely, if h has degree d and $D_e^k h$ denotes the k-th derivative of h in the direction e, then*

$$\Lambda_{++}(h,e) = \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{h(x)}{h(e)} > 0, \frac{D_e^k h(x)}{h(e)} > 0 \text{ for } k = 1, \dots, d-1 \right\}.$$

(2) *$\Lambda_{++}(h,e)$ is star-convex at e.*
(3) *$\Lambda_{++}(h,e)$ is the connected component of e in the set $\{x \in \mathbb{R}^n \mid h(x) \neq 0\}$.*
(4) *$\Lambda_{+}(h,e)$ is the closure of $\Lambda_{++}(h,e)$.*

The following theorem is the main step to prove the convexity of hyperbolicity cones:

Theorem 4. *Let h be hyperbolic with respect to e and let $x \in \Lambda_{++}(h,e)$. Then h is hyperbolic with respect to x and $\Lambda_{++}(h,x) = \Lambda_{++}(h,e)$.*

The idea of the proof is to show, that the imaginary parts of the roots of the polynomial

$$r \mapsto h(i\varepsilon \cdot e + r \cdot x + s \cdot y)$$

have the same sign for all $s \geq 0$ and $\varepsilon > 0$. By setting $s = 1$ and letting ε go to 0, we can conclude, that the roots of the polynomial

$$r \mapsto h(r \cdot x + y)$$

must be real, since this is a polynomial with real coefficients. See [2] for details.

With the above theorem at hand, it is not hard to prove the following theorem:

Theorem 5. *Let h be hyperbolic with respect to e. Then $\Lambda_{++}(h,e)$ is convex. Moreover, the function $\lambda_{min} \colon \mathbb{R}^n \to \mathbb{R}$ is concave.*

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Convexity Properties of Hyperbolic Polynomials

Eli Shamovich

The notion of hyperbolic polynomials was first studied in connection with hyperbolic partial differential operators with constant coefficients. The hyperbolic polynomials arise as symbols of such differential operators, see for example [3]. A homogeneous polynomial $f \in \mathbb{R}[x_0, \ldots, x_n]$ of degree m is called hyperbolic with respect to some point $a \in \mathbb{R}^{n+1}$, if $f(a) \neq 0$ and for every $x \in \mathbb{R}^{n+1}$, the roots of the single variable polynomial $f(ta + x)$ are all real. The convexity properties of such polynomials were first considered by L. Gärding in [2]. We write:

$$f(ta + x) = f(a) \prod_{j=1}^{m} (t + \lambda_j(a, x)).$$

Assume without loss of generality that for every x we have $\lambda_1(a, x) \geq \lambda_2(a, x) \geq \cdots \geq \lambda_m(a, x)$. Then we can define the following set:

$$C(a) = \left\{x \in \mathbb{R}^{n+1} \mid f(ta + x) \neq 0 \text{ for } t \geq 0\right\}.$$

It is now obvious that $C(a) = \left\{x \in \mathbb{R}^{n+1} \mid \lambda_m(a, x) > 0\right\}$. Gärding showed that $\lambda_m(a, x)$ is concave and that $C(a)$ is in fact a convex cone and that for every $b \in C(a)$, we have that f is hyperbolic with respect to b and that $C(b) = C(a)$.

The goal of this talk is to present this and other convexity results on homogeneous hyperbolic polynomials. We follow mostly [1] throughout the talk. For a vector $u \in \mathbb{R}^m$ we write $u_{\downarrow}$ for the same vector with the coordinates ordered in a decreasing order. For $U \subset \mathbb{R}^m$ we write $U_{\downarrow} = \{u_{\downarrow} \mid u \in U\}$. In particular for a homogeneous polynomial f of degree m hyperbolic with respect to some $a \in \mathbb{R}^{n+1}$ we have a map $\lambda\colon \mathbb{R}^{n+1} \to \mathbb{R}^m_{\downarrow}$ given by $\lambda(x) = (\lambda_1(a, x), \ldots, \lambda_m(a, x))$. We call λ the characteristic map of f. We define the map $\sigma_k(x) = \sum_{j=1}^{k} \lambda_j(a, x)$ and call σ_m the trace of f.

First we will show that for every symmetric polynomial E_j in m variables the function satisfies that $E_j \circ \lambda$ is a homogeneous polynomial of $n+1$ variables and of degree j hyperbolic with respect to a as well. In particular this implies that $\sigma_m(x)$ is a linear hyperbolic polynomial. Using this fact and some theory of symmetric polynomials we will show that if $q \in \mathbb{R}[y_1, \ldots, y_m]$ is a homogeneous symmetric polynomial of degree k on $\mathbb{R}^m$ hyperbolic with respect to $e = (1, 1, \ldots, 1)$, with characteristic map μ, then $q \circ \lambda$ is a homogeneous polynomial of degree k hyperbolic with respect to a and with characteristic map $\lambda \circ \mu$. Using this result we obtain that for every convex and symmetric function $g\colon \mathbb{R}^m \to [-\infty, \infty]$ we have that $g \circ \lambda$ is convex. We will then see an example in the case of the space of

Hermitian matrices and the determinant and derive some other corollaries. The talk is concluded by stating the generalized Alexander-Fenchel inequalities for an arbitrary homogeneous hyperbolic polynomial proved in [4].

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Kadison-Singer via hyperbolicity (Part II)

Nima Amini

Following Marcus, Spielman and Srivastava's theorem, in particular resolving the Kadison-Singer conjecture, Brändén [1] found a natural extension of the theorem to hyperbolic polynomials making the proof more coherent in its general form. The talk, which is the first out of two on this topic, centers around the statement of Brändén's theorem and to what extent it forms a generalization. A couple of the main ingredients of the proof are developed, culminating in the hyperbolic version of the mixed characteristic polynomial.

The theorem by Marcus, Spielman and Srivastava is a stronger version of the Weaver KS_k-conjecture, which in turn is known to imply a positive solution to the Kadison-Singer problem.

Theorem 1 (Marcus, Spielman, Srivastava). *Let $k \geq 2$ be an integer. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^d$ satisfy $\sum_{i=1}^m \mathbf{v}_i \mathbf{v}_i^* = I$ where I is the identity matrix. If $||\mathbf{v}_i||^2 \leq \epsilon$ for all $1 \leq i \leq m$, then there is a partition of $S_1 \cup \cdots \cup S_k = [m]$ such that*

$$\left|\left|\sum_{i \in S_j} \mathbf{v}_i \mathbf{v}_i^*\right|\right| \leq \frac{(1+\sqrt{k\epsilon})^2}{k},$$

for each $j \in [k]$, where $||\cdot||$ denotes the operator norm.

Definition 2. *Let h be a hyperbolic polynomial w.r.t $\mathbf{e} \in \mathbb{R}^n$. Write*

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x}))$$

where $\lambda_{max}(\mathbf{x}) = \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) = \lambda_{min}(\mathbf{x})$ are called the eigenvalues of $\mathbf{x}$ w.r.t $\mathbf{e}$. The trace, rank and spectral radius of $\mathbf{x} \in \mathbb{R}^n$ w.r.t $\mathbf{e}$ are defined respectively by

$$tr(\mathbf{x}) = \sum_{i=1}^{d} \lambda_i(\mathbf{x}),\ rk(\mathbf{x}) = \#\{i : \lambda_i(\mathbf{x}) \neq 0\},\ ||\mathbf{x}|| = \max_{1 \leq i \leq d} |\lambda_i(\mathbf{x})|.$$

Brändén showed that Theorem 1 can be naturally generalized to the realm of hyperbolic polynomials (in fact with a slightly improved bound) as follows:

Theorem 3 (Brändén). *Let $k \geq 2$ be an integer and $\epsilon > 0$. Suppose h is hyperbolic with respect to $\in \mathbb{R}^n$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \Lambda_+$ be such that*

$$\begin{aligned} &rk(\mathbf{u}_i) \leq 1 \text{ for all } 1 \leq i \leq m, \\ &tr(\mathbf{u}_i) \leq \epsilon \text{ for all } 1 \leq i \leq m, \text{ and} \\ &\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m = \mathbf{e}. \end{aligned}$$

Then there is a partition of $S_1, \cdots, S_k$ of $[m]$ such that

$$\left|\left|\sum_{i \in S_j} \mathbf{u}_i\right|\right| \leq \frac{1}{k}\delta(k\epsilon, m)$$

for each $j \in [k]$, where

$$\delta(\alpha, m) := \left(1 - \frac{1}{m} + \sqrt{\alpha + \frac{1}{m}\left(1 - \frac{1}{m}\right)}\right)^2.$$

Remark. *Note that the hypotheses of Theorem 1 translates into the hypotheses of Theorem 1 by taking $h = \det$ and $\mathbf{e} = I$ so that $\Lambda_+(I)$ is the cone of symmetric positive-definite matrices. Since $\mathbf{u}_i = \mathbf{v}_i \mathbf{v}_i^*$ we see that $rk(\mathbf{u}_i) = 1$ and so in particular $\mathbf{u}_i$ has a single real eigenvalue. This together with the fact that $||\mathbf{u}_i|| = ||\mathbf{v}_i \mathbf{v}_i^*|| = ||\mathbf{v_i}||^2 \leq \epsilon$ implies that $tr(\mathbf{u}_i) \leq \epsilon$.*

As a step towards proving Theorem 1 one generalizes the notion of the mixed characteristic polynomial to the hyperbolic setting and prove that it forms a compatible family - a slightly less general definition than that of interlacing family but one that is sufficient for our purposes.

Definition 4. *Suppose $S_1, \ldots, S_m$ are finite sets. A family of polynomials,*

$$\{f(\mathbf{s}; t)\}_{\mathbf{s} \in S_1 \times \cdots \times S_m}$$

for which all non-zero members are of the same degree and have the same sign of their leading coefficients is called compatible if for all choices of independent random variables $X_1 \in S_1, \ldots, X_m \in S_m$, the polynomial $\mathbb{E} f(X_1, \ldots, X_n; t)$ is real-rooted.

From what we know about polynomials, the roots of a weighted sum of polynomials have in general little to do with the roots of its sum constituents. Perhaps a bit surprisingly then one has the following relationship between the maximal root of the expected polynomial of a compatible family with that of its family members.

Theorem 5. *Let $\{f(\mathbf{s}; t)\}_{\mathbf{s} \in S_1 \times \cdots \times S_m}$ be a compatible family, and let $X_1 \in S_1$, $\ldots, X_m \in S_m$ be independent random variables such that $\mathbb{E} f(X_1, \ldots, X_m; t) \not\equiv 0$. Then there is a tuple $\mathbf{s} = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_m$, with $\mathbb{P}[X_i = s_i] > 0$ for each $1 \leq i \leq m$, such that the largest zero of $f(s_1, \ldots, s_m; t)$ is smaller than or equal to the largest zero of $\mathbb{E} f(X_1, \ldots, X_m; t)$.*

In analogy with the original theorem we now define the following special polynomial

Definition 6. *If* $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ *and* $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n$ *let* $h[\mathbf{v}_1, \dots, \mathbf{v}_m]$ *be the polynomial in* $\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_m]$ *defined by*

$$h[\mathbf{v}_1, \dots, \mathbf{v}_m] = \prod_{j=1}^{m} (1 - y_j D_{\mathbf{v}_j}) h(\mathbf{x}).$$

When the vectors have rank at most one the polynomial takes on a more explicit form

Lemma 7. *If* $\mathbf{v}_1, \dots, \mathbf{v}_m$ *have rank at most one, then*

$$h[\mathbf{v}_1, \dots, \mathbf{v}_m] = h(\mathbf{x} - y_1\mathbf{v}_1 - \cdots - y_m\mathbf{v}_m).$$

Definition 8. *The mixed characteristic polynomial is given by*

$$t \mapsto h[\mathbf{v}_1, \dots, \mathbf{v}_m](t\mathbf{e} + \mathbf{1})$$

where h *is hyperbolic with respect to* $\mathbf{e} \in \mathbb{R}^n$ *and* $\mathbf{1} \in \mathbb{R}^m$ *is the all ones vector, and* $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Lambda_+(\mathbf{e})$ *satisfy* $\mathbf{v}_1 + \cdots + \mathbf{v}_m = \mathbf{e}$ *and* $tr(\mathbf{v}_i) \leq \epsilon$ *for all* $1 \leq i \leq m$.

Finally the following theorem show that the polynomials we have defined indeed form compatible families.

Theorem 9. *Let* $h(\mathbf{x})$ *be hyperbolic with respect to* $\mathbf{e} \in \mathbb{R}^n$. *Let* $V_1, \dots, V_m$ *be finite sets of vectors in* Λ_+ *and let* $\mathbf{w} \in \mathbb{R}^{n+m}$. *For* $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_m) \in V_1 \times \cdots \times V_m$, *let*

$$f(\mathbf{V}; t) := h[\mathbf{v}_1 \dots, \mathbf{v}_m](t\mathbf{e} + \mathbf{w}).$$

Then $\{f(\mathbf{V}; t)\}_{\mathbf{V} \in V_1 \times \cdots \times V_m}$ *is a compatible family.*
In particular if in addition all vectors in $V_1 \cup \cdots \cup V_m$ *have rank at most one, and*

$$g(\mathbf{V}; t) := h(t\mathbf{e} + \mathbf{w} - \alpha_1\mathbf{v}_1 - \cdots - \alpha_m\mathbf{v}_m),$$

where $\mathbf{w} \in \mathbb{R}^n$ *and* $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, *then* $\{g(\mathbf{V}; t)\}_{\mathbf{V} \in V_1 \times \cdots \times V_m}$ *is a compatible family.*

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Kadison-Singer via hyperbolicity (Part II)

Cordian Riener

In this talk we continued to present a generalization of the theorem by Marcus, Spielman and Srivastava to hyperbolic polynomials given by Petter Brändén [2]. Using the nation of hyerbolic polynomials, which was introduced in previous talks, Brändén's Theorem is the following.

Theorem 1 (Brändén)**.** *Let $k \geq 2$ be an integer and $\epsilon > 0$. Suppose h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$, and let $\mathbf{u}_1, \dots, \mathbf{u}_m \in \Lambda_+$ be such that*

$$\begin{aligned} rk(\mathbf{u}_i) &\leq 1 \text{ for all } 1 \leq i \leq m, \\ \operatorname{tr}(\mathbf{u}_i) &\leq \epsilon \text{ for all } 1 \leq i \leq m, \text{ and} \\ \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m &= \mathbf{e}. \end{aligned}$$

Then there is a partition of $S_1, \cdots, S_k$ of $[m]$ such that

$$\left\| \sum_{i \in S_j} \mathbf{u}_i \right\| \leq \frac{1}{k} \delta(k\epsilon, m)$$

for each $j \in [k]$, where

$$\delta(\alpha, m) := \left(1 - \frac{1}{m} + \sqrt{\alpha + \frac{1}{m}\left(1 - \frac{1}{m}\right)}\right)^2.$$

Generalizing the idea of the proof of Marcus, Spielman and Srivastava, Brändén's Theorem can be deduced from the following theorem.

Theorem 2. *Let h be hyperbolic with respect to $\mathbf{e}$ and let $\mathsf{X}_1, \dots, \mathsf{X}_m$ be independent random vectors in the hyperbolicity cone Λ_+ with finite supports, of rank at most one, and such that*

$$\mathbb{E} \sum_{i=1}^m \mathsf{X}_i = \mathbf{e}, \text{ and } \operatorname{tr}(\mathbb{E}\mathsf{X}_i) \leq \epsilon \text{ for all } 1 \leq i \leq m.$$

Then

$$\mathbb{P}\left[\lambda_{\max}\left(\sum_{i=1}^m \mathsf{X}_i\right) \leq \delta(\epsilon, m)\right] > 0.$$

The proof of Theorem 2 relies on the notion of a compatible family, which was introduced in the previous talk. Let V_i be the support of X_i then it was shown previously that for each $1 \leq i \leq m$, the family

$$\{h(t\mathbf{e} - \mathbf{v}_1 - \cdots - \mathbf{v}_m)\}_{\mathbf{v}_i \in V_i}$$

is compatible. Hence there are vectors $\mathbf{v}_i \in V_i$, $1 \leq i \leq m$, such that the largest zero of $h(t\mathbf{e} - \mathbf{v}_1 - \ldots - \mathbf{v}_m)$ is smaller or equal to the largest zero of

$$\mathbb{E}h(t\mathbf{e} - \mathsf{X}_1 - \cdots - \mathsf{X}_m) = \mathbb{E}h[\mathsf{X}_1, \dots, \mathsf{X}_m](t\mathbf{e} + \mathbf{1}) = h[\mathbb{E}\mathsf{X}_1, \dots, \mathbb{E}\mathsf{X}_m](t\mathbf{e} + \mathbf{1}).$$

It is then shown, that the largest zero of the polynomial $t \mapsto h[\mathbf{v}_1, \dots, \mathbf{v}_m](t\mathbf{e} + \mathbf{1})$ is indeed at most $\delta(\epsilon, m)$.
These preparations allowed us to present the proof of Theorem 1 given by Brändén in [2]:
Let $\mathbf{y} = \{x_{ij} : 1 \le i \le k, 1 \le j \le n\}$ and denote $\mathbf{x}^i = (x_{i1}, \dots, x_{in})$. Brändén considers the polynomial

$$g(\mathbf{y}) = h(\mathbf{x}^1)h(\mathbf{x}^2) \cdots h(\mathbf{x}^k) \in \mathbb{R}[\mathbf{y}].$$

Since h is hyperpolic with respect to $\mathbf{e} \in \mathbb{R}^n$ one finds that $g(\mathbf{y})$ is hyperbolic with respect to $\mathbf{e}^1 \oplus \cdots \oplus \mathbf{e}^k$, where $\mathbf{e}^i$ denotes a copy of $\mathbf{e}$ relative to the $\mathbf{x}^i$, for all $1 \le i \le k$. The hyperbolicity cone of g then can be written as $\Lambda_+ := \Lambda_+(\mathbf{e}^1) \oplus \cdots \oplus \Lambda_+(\mathbf{e}^k)$. Let $\mathbf{u}_1^i, \dots, \mathbf{u}_m^i$ denote copies in $\Lambda_+(\mathbf{e}^i)$ of $\mathbf{u}_1, \dots, \mathbf{u}_m$ and take random vectors $\mathsf{X}_1, \dots, \mathsf{X}_m \in \Lambda_+$ such that

$$\mathbb{P}\left[\mathsf{X}_j = k\mathbf{u}_j^i\right] = \frac{1}{k} \text{ for all } 1 \le i \le k \text{ and } 1 \le j \le m.$$

Then

$$\begin{aligned} \mathbb{E}\mathsf{X}_j &= \mathbf{u}_j^1 \oplus \mathbf{u}_j^2 \oplus \cdots \oplus \mathbf{u}_j^k, \\ \mathrm{tr}(\mathbb{E}\mathsf{X}_j) &= k\,\mathrm{tr}(\mathbf{u}_j) \le k\epsilon, \text{ and} \\ \mathbb{E}\sum_{j=1}^m \mathsf{X}_j &= \mathbf{e}^1 \oplus \cdots \oplus \mathbf{e}^k. \end{aligned}$$

Theorem 2 above yields the existence of a partition $S_1 \cup \cdots \cup S_k = [m]$ such that

$$\lambda_{\max}\left(\sum_{i \in S_1} k\mathbf{u}_i^1 + \cdots + \sum_{i \in S_k} k\mathbf{u}_i^k\right) \le \delta(k\epsilon, m).$$

But

$$\begin{aligned} \lambda_{\max}\left(\sum_{i \in S_1} k\mathbf{u}_i^1 + \cdots + \sum_{i \in S_k} k\mathbf{u}_i^k\right) &= k \max_{1 \le j \le k} \lambda_{\max}\left(\sum_{i \in S_j} \mathbf{u}_i^j\right) \\ &= k \max_{1 \le j \le k} \lambda_{\max}\left(\sum_{i \in S_j} \mathbf{u}_i\right), \end{aligned}$$

and one can deduce a proof of Theorem 1.

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Ramanujan Graphs

Christoph Gamm

The purpose of this talk is to present an outline of the proof of the following theorem from [1]:

Theorem 1. *For every $d \geq 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

We will call a d-regular bipartite graph Ramanujan, if $\lambda(G) \leq 2\sqrt{d-1}$, where $\lambda(G)$ is the maximal absolute value of its non-trivial eigenvalues, i.e. the eigenvalues not equal to $-d$ or d.

The basic idea of the proof is to take a Ramanujan graph and show that there is a 2-lift that is again Ramanujan. Here a 2-lift of a graph is a $2:1$ covering graph. To each 2-lift we assign a signing and a signed adjacency matrix. Denote by S the set of all 2-lifts and let f_s be the characteristic polynomial of the signed adjacency matrix corresponding to $s \in S$.

It turns out that the eigenvalues of a 2-lift are exactly the eigenvalues of the original graph together with the eigenvalues of the signed adjacency matrix. In order to proof the theorem we therefore only need to find a lifting s for which the roots of f_s are all bounded in absolute value by $2\sqrt{d-1}$. As bipartite graphs have symmetric spectrum it suffices to find an upper bound for the roots.

The next step is to make use of a relation between the signed characteristic polynomials and the matching polynomial. The matching polynomial μ_G is a real rooted polynomial that has roots bounded by $2\sqrt{d-1}$. The following theorem by Godsil and Gutman implies that our desired statement is at least true in average:

Theorem 2. $\mathbb{E}_{s\in S}[f_s(x)] = \mu_G(x)$

To be able to conclude the statement for a single f_s we need to introduce the notion of interlacing families. The importance of interlacing families lies in the fact that for an interlacing family $f_{s_1,\dots,s_n}$ there is one polynomial $f_{s_1,\dots,s_n}$ for which the maximal root is smaller that the maximal root of the averaged polynomial. All that is left to proof is that the signed characteristic polynomials $\{f_s\}_{s\in S}$ form an interlacing family.

Being an interlacing family is equivalent to the real-rootedness of certain convex combination. Therefore the last statement follows easily from the main theorem of the article:

Theorem 3. *For all $p_1, \dots p_n \in [0,1]$ the following polynomial is real-rooted:*

$$P(x) = \sum_{s\in\{\pm 1\}^n} \left(\prod_{i:s_i=1} p_i \right) \left(\prod_{i:s_i=-1} 1 - p_i \right) f_s(x)$$

This theorem will not be proved in this talk and we will only make some remark on the required methods. The proof makes use the theory of real stable polynomials. In general the idea is to take a real stable polynomial and apply a series of operations that preserve real stability until we end up with the polynomial P

from the main theorem. As real stability and real-rootedness are equivalent in the case of a single-variable polynomial the theorem follows.

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Further Applications: van der Waerden's Conjecture

Michelle Delcourt

Despite the deceptively simple statement, for over 50 years van der Waerden's Conjecture remained unsolved, contributing to the reputation of the difficulty of calculating permanents. The conjecture from 1926 states that the permanent of an $n \times n$ doubly stochastic matrix A satisfies

$$per(A) \geq \frac{n!}{n^n},$$

and furthermore, we have equality if and only if all entries of A are equal to $\frac{1}{n}$ [8].

In 1981 this conjecture was solved by Egorychev [2] and Falikman [4] using a classical inequality of Aleksandrov and Fenchel [1]; subsequently in 1982 Egorychev and Falikman were awarded the Fulkerson Prize for this work. Along the way a number of easier conjectures were formulated. For instance, Erdős and Rényi studied counting perfect matchings in k-regular, bipartite graphs with equal partition size [3], and in the 1960s Erdős and Rényi asked if there exists a real value $\alpha_k > 1$ such that any integer valued $n \times n$ matrix A with all row and column sums equal to k satisfies

$$per(A) \geq \alpha_k^n.$$

Additionally, Erdős and Rényi asked [3]: what is the largest possible value such an α_k can achieve? If we assume van der Waerden's Conjecture to be true, then we see for all positive integers k

$$per(A) = k^n per\left(\frac{1}{k}A\right) \geq k^n \frac{n!}{n^n} \geq \left(\frac{k}{e}\right)^n.$$

In 1966 Wilf [9] showed that

$$\alpha_k \leq \frac{(k-1)^{k-1}}{k^{k-2}},$$

and in a paper from 1998 [7], Schrijver proved that this is in fact the correct value for all positive integers k.

Given the difficulty of the proofs by Egorychev [2] and Falikman [4] as well as by Schrijver, it came as a great surprise when in 2008 Leonid Gurvits published a simple proof [5] of these results using H-stable polynomials. Interestingly, in both cases the bounds can be thought of as being best possible in different asymptotic directions. Note that

$$\inf_k \mu(k,n)^{1/n} = \frac{n!^{1/n}}{n}$$

and

$$\inf_n \mu(k,n)^{1/n} = \left(\frac{k-1}{k}\right)^{k-1}$$

where $\mu(k,n)$ is the minimum permanent of $n \times n$ doubly stochastic matrices with all entries being integer multiples of $\frac{1}{k}$.

In this talk, we explore Gurvits's proofs [5] of both of these results. We follow the exposition from the American Mathematical Monthly article from December 2010 written by Laurent and Schrijver [6].

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Negative dependence II

Guillaume Aubrun

The goal of this talk was to give an introduction to the links between negative dependence and stable polynomials.

I started with defining positive association for probability measures on the boolean hypercube and stating the FKG inequality.

I then defined the property of negative association: a measure μ on 2^S (with S a finite set) is negatively associated if for any pair of increasing functions $F, G : 2^S \to \mathbb{R}$ which depend on disjoint sets of coordinates, we have

$$\int FG \,\mathrm{d}\mu \leq \int F \,\mathrm{d}\mu \int G \,\mathrm{d}\mu.$$

I gave some simple examples of negatively associated meaures: product measures, uniform measure on subsets of a fixed cardinality.

Following [1] and [2], I introduced the (multi-affine) polynomial g_μ associated to a measure μ. I discribed how probabilistic operations such as products, marginals and conditionning affect the corresponding polynomials.

The class of strongly Rayleigh measures is the class of measures for which the corresponding polynomial is stable. I recalled the criterion by Brändén for stability of multi-affine polynomials, which when applied to g_μ implies immediately that μ has pairwise negatively correlated coordinates.

I stated the main theorem: strongly Rayleigh measures are negatively associated. I described some operations on polynomials which preserve stability: homogeneization and symmetrization. This operations are used in the proof of the main theorem.

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Negative dependence II

Jan Hladký

I showed an application of the notion of strongly Rayleigh measure introduced in the previous talk "Negative dependence I" by Guillaume Aubrun to the theory of stochastic processes. The main result of my talk is from [1, Theorem 5.2]. A friendlier presentation of the same result is in the survey paper [2, Section 6].

Given a finite set S, a *state of an exclusion process on the set* S is an arbitrary subset of S. We think of the elements of a state as particles occupying some of the sites S. An *(symmetric) exclusion process on* S is a continuous time Markov process, in which sites i and j exchange their particles at rate λ_{ij} (lets say $\lambda_{ij} = \lambda_{ji}$). Thus, starting with an initial distribution μ_0 on 2^S we denote by μ_t the law of the process at time t.

(More generally, the set S is typically allowed to be countable. However, the problem we consider here becomes non-trivial already for finite sets, and its resolution for finite sets actually implies the countable version by standard arguments.) It was long conjectured that if μ_0 is deterministic then μ_t is negatively associated (see the abstract "Negative dependence I" for definition). To give a feeling to what negative association amounts to in this context, we remark that it would for example conclude that

$$\mathbb{P}_{X\sim\mu_t}[A \subset X \text{ and } B \subset X] \geq \mathbb{P}_{X\sim\mu_t}[A \subset X]\mathbb{P}_{X\sim\mu_t}[B \subset X]$$

for any two disjoint sets $A, B \subset X$. This is a plausible statement which can be understood as "if there are particles in all of A then they are likely to miss B". (Note that there are easy counterexamples for general initial measures μ_0.)

All this is implied by the following result.

Theorem 1. *Suppose that μ_0 is a strongly Rayleigh measure. Then for any $t > 0$, the measure μ_t is strongly Rayleigh.*

Indeed this is a solution of the original conjecture in a stronger form as Dirac measures are easily seen to be strongly Rayleigh, and Rayleigh measures are known to be negatively associated. We reduce the theorem to a seemingly weaker statement.

Proposition 2. *Suppose that μ_0 is a strongly Rayleigh measure. Suppose that we consider an exclusion process in which the only exchanges occur between sites $1, 2 \in S$, i.e., $\lambda_{i,j} = 0$ for $\{i, j\} \neq \{1, 2\}$. Then for any $t > 0$, the measure μ_t is strongly Rayleigh.*

The rigorous proof of the fact that the proposition implies the theorem goes by Lie(-Trotter) product formula applied to the semigroup of the Markov process. We can, however, provide a clear intuition even without introducing these notions, as follows. Let us list all the $\binom{|S|}{2}$ pairs $\{i, j\} \subset S$ as $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{\binom{|S|}{2}}, j_{\binom{|S|}{2}}\}$. Let us divide the time interval into subintervals of length $\frac{1}{N}$, where $N \to \infty$. Within the 1st subinterval, let us consider a process in which only exchanges between i_1 and j_1 are allowed, and at rate $\binom{|S|}{2}\lambda_{i_1 j_1}$. Within the 2nd subinterval, let us consider a process in which only exchanges between i_2 and j_2 are allowed, and at rate $\binom{|S|}{2}\lambda_{i_2 j_2}$. Similarly we proceed with further subintervals, where the same type of exchanges repeats with period of $\binom{|S|}{2}$ subintervals. It is clear that if $N \to \infty$, the law μ_t^N converges to the law of μ_t, for any fixed $t > 0$. At the same time, the proposition tells us that within each subinterval the strongly Rayleigh property was preserved, and thus was preserved globally, implying the theorem.

It thus remains to prove the proposition. We use the one-to-one correspondence between probability measures and their generating functions, as showed in the proceeding talk. That is, let $g_0(x_1, \ldots, x_{|S|})$ be the (real stable, multi-affine) generating polynomial of the measure μ_0. Let g_t be the generating polynomial of the measure μ_t from the above proposition. We need to show that g_t is real stable. To this end, it is key to observe that

$$(1) \quad g_t(x_1, \ldots, x_{|S|}) = (1 - \Theta)g_0(x_1, x_2, x_3, \ldots, x_{|S|}) + \Theta g_0(x_2, x_1, x_3 \ldots, x_{|S|}) \ ,$$

where Θ is the probability that the number of swaps (between 1 and 2, which are the only allowed swaps) is odd. Note that the number of swaps has a Poisson distribution with parameter $t\lambda_{12}$, of which we only need that $\Theta \in (0, 1)$. Thus, (1) can be proved using tools for dealing with real stable polynomials developed in the previous talk.

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Dithering frame vectors with Kadison-Singer

Joseph W. Iverson

To a finite collection of vectors $\Phi = \{u_j\}_{j\in J} \subset \mathbb{C}^n$, we associate the *frame-like operator*

$$S_\Phi = \sum_{j\in J} u_j u_j^*.$$

That is,

$$S_\Phi v = \sum_{j\in J} \langle v, u_j\rangle u_j \qquad (v \in \mathbb{C}^n).$$

We call Φ a *Bessel sequence* if there is a constant $B > 0$ such that $S_\Phi \leq BI$. The constant B is a *Bessel bound.* We say Φ is a *Parseval frame* if $S_\Phi = I$.

Marcus, Spielman, and Srivastava have shown that any Parseval frame can be partitioned into two sets, each of which does approximately half of the frame's work. Explicitly, they have proved the following:

Theorem 1 ([3]). *Let $\Phi = \{u_j\}_{j\in J}$ be a finite Parseval frame for $\mathbb{C}^n$ such that $\|u_j\|^2 \leq \epsilon$ for some fixed $\epsilon > 0$ and all $j \in J$. Then there is a partition of J into two sets J_1 and J_2, such that*

$$\left\| \sum_{j\in J_i} u_j u_j^* - \frac{1}{2}I \right\| \leq 2\sqrt{\epsilon} + \epsilon \qquad (i = 1, 2). \tag{1}$$

Here and throughout, $\|\cdot\|$ is the operator norm. Remarkably, the upper bound in Eqn. (1) does not depend on the dimension n. Theorem 1 was known to imply a positive answer to the Kadison-Singer problem [3], through previous work by Akemann and Anderson [1] and Weaver [6].

The current undertaking is an extension of Theorem 1 due to Akemann and Weaver. Their goal was to replace $\frac{1}{2}I$ in Eqn. (1) with other operators $A \leq I$. The main result is as follows:

Theorem 2 ([2]). *Let $\Phi = \{u_j\}_{j\in J} \subset \mathbb{C}^n$ be a finite Bessel sequence with bound $B = 1$. Suppose there is a fixed $\epsilon > 0$ such that $\|u_j\|^2 \leq \epsilon$ for all $j \in J$. Then for any choice of scalars $t_j \in [0,1]$, $j \in J$, there is a subset $J' \subset J$ such that*

$$\left\| \sum_{j\in J'} u_j u_j^* - \sum_{j\in J} t_j u_j u_j^* \right\| = O(\epsilon^{1/8}).$$

The theorem is proved in stages. Starting from Theorem 1, the operator $\frac{1}{2}I$ in Eqn. (1) is replaced with increasingly generic $A \leq I$, while the hypothesis that $S_\Phi = I$ is slowly relaxed to $S_\Phi \leq I$.

Several interpretations of Theorem 2 are given below. Of these, the second and third appeared in [2].

(1) The numbers $t_j \in [0,1]$ can be viewed as scalings of the vectors in our Bessel sequence: if we let $\tilde{\Phi} = \{t_j^{1/2} u_j\}_{j\in J}$, then

$$S_{\tilde{\Phi}} = \sum_{j\in J} t_j u_j u_j^*.$$

It is as if the Bessel sequence Φ is a vast orchestra of instruments, each of which has its own channel in a recording studio. An engineer carefully sets the level of each instrument by moving the "slider" t_j. The theorem says that, in terms of the frame-like operator, approximately the same effect could have been achieved by just muting some of the instruments and leaving the others at full blast. Hans Feichtinger has remarked that this is also like a dithering: we can approximate the "gray scale" of $\tilde{\Phi}$ using only black and white dots from Φ.

(2) Let $\Omega \subset B(\mathbb{C}^n)$ be the convex hull of the frame-like operators

$$S_{J'} = \sum_{j\in J'} u_j u_j^*$$

as J' ranges over all subsets of J. The theorem says that any point

$$T = \sum_{j\in J} t_j u_j u_j^* \in \Omega \qquad (0 \le t_j \le 1)$$

is close to one of the vertices $S_{J'}$, and we can control the error using only the size of the vectors $u_j \in \Phi$. In other words, the operators $S_{J'}$ do a good job of policing Ω, and we can help them do a better job just by using smaller vectors in Φ.

(3) Denote $[0,1]^J$ for the set of all J-tuples $\mathbf{t} = (t_j)_{j\in J}$ with $t_j \in [0,1]$ for all j. Let Ω be as above, and let $\psi\colon [0,1]^J \to \Omega$ be given by

$$\psi(\mathbf{t}) = \sum_{j\in J} t_j u_j u_j^* \qquad (\mathbf{t} = (t_j)_{j\in J} \in [0,1]^J).$$

Then ψ maps the extreme points of $[0,1]^J$ (the vertices) to the operators $S_{J'}$ described above. The theorem says that the image of the extreme points is *nearly* the entire image of $[0,1]^J$.

A classical theorem of Lyapunov [4] describes a certain affine-linear map $\varphi\colon Q \to V$ from a convex set Q to a linear space V, and concludes that $\varphi(\mathrm{ext}(Q))$ is all of $\varphi(Q)$. In [1], Akemann and Anderson have called any theorem of this form a *Lyapunov theorem.* This is not exactly the situation in Theorem 2, since the image of $[0,1]^J$ is only *approximated* by the image of the extreme points. Nevertheless, it is clear that Theorem 2 belongs roughly in this camp.

We end with an open problem. At the seminar in Oberwolfach, Itay Londner called this "the million dollar question".

Problem. *Find a constructive algorithm for the subset $J' \subset J$ in Theorem 2.*

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Kadison–Singer conjecture for strongly Rayleigh measures

Benjamin Matschke

1. Introduction

Marcus, Spielman and Srivastava [5] proved the following theorem. It implies the long-standing Kadison–Singer conjecture [4, 1, 6], which asserts that every pure state on the abelian von Neumann algebra $D(\ell_2)$ of bounded diagonal operators on ℓ_2 has a unique extension to a pure state on $B(\ell_2)$.

Theorem 1 (MSS). *Let $V_1, \dots, V_k$ be independent random vectors in $\mathbb{R}^d$, each of which take only finitely many values, and let $\varepsilon > 0$ be such that $\sum \mathbb{E}[V_i V_i^t] = \mathrm{id}_d$ and $\mathbb{E}\big[||V_i||^2\big] \leq \varepsilon$ for all $i = 1, \dots, k$. Then*

$$\mathbb{P}\big[|| \sum V_i V_i^t || \leq (1 + \sqrt{\varepsilon})^2\big] > 0.$$

Anari and Oveis Gharan [2] proved a version of the MSS theorem (see Section 3) in which in some sense they managed to weaken the independence assumption for the random vectors $V_1, \dots, V_k$. This allowed them to apply this technique to the Asymmetric Traveling Salesman Problem. In particular they proved a new upper bound for the integrality gap of its natural LP-relaxation.

2. Strongly Rayleigh measures.

Borcea, Brändén and Liggett [3] recently introduced the notion of strongly Rayleigh measures. Let P_n denote the set of all probability measures on $2^{[n]}$. For such a measure $\mu \in P_n$, let $g_\mu := \sum_{S \subseteq [n]} \mu(S) x^S \in \mathbb{R}[x_1, \dots, x_n]$ denote its generating function. We say that $\mu \in P_n$ is homogeneous of degree d if and only if g_μ is. Homogeneous $\mu \in P_n$ of degree 1 are the same as probability measures on $[n]$. There is a product map $P_{n_1} \times P_{n_2} \to P_{n_1+n_2}$, the product of μ_1 and μ_2 being given via $g_{\mu_1 \times \mu_2}(x_1, \dots, x_{n_1+n_2}) := g_{\mu_1}(x_1, \dots, x_{n_1}) \cdot g_{\mu_2}(x_{n_1+1}, \dots, x_{n_1+n_2})$. We call $\mu \in P_n$ strongly Rayleigh if g_μ is a real stable polynomial, i.e. when g_μ has no complex roots $(x_1, \dots, x_n)$ with $\mathrm{im}(x_i) > 0$ for all i. A basic example of homogeneous strongly Rayleigh measures are homogeneous $\mu \in P_n$ of degree 1, and products of such measures.

3. MSS theorem for strongly Rayleigh measures.

Anari and Oveis Gharan [2] proved the following version of the MSS theorem.

Theorem 2 (AO). *Let μ be a homogeneous strongly Rayleigh probability measure on $2^{[m]}$ that satisfies $\mathbb{P}_{S\sim\mu}[i \in S] \leq \varepsilon_1$ for all $i = 1, \dots, m$. Let $v_1, \dots, v_m \in \mathbb{R}^d$ such that $\sum v_i v_i^t = \mathrm{id}_d$ and $||v_i||^2 \leq \varepsilon_2$ for all i. Then*

$$\mathbb{P}_{S\sim\mu}\big[||\sum_{i\in S} v_i v_i^t|| \leq 4(\varepsilon_1 + \varepsilon_2) + 2(\varepsilon_1 + \varepsilon_2)^2\big] > 0.$$

Theorems 1 and 2 are related as follows. The random vectors $V_1, \dots, V_k$ from Theorem 1 have finite supports and can thus be considered as homogeneous measures on $2^{[n_i]}$, respectively. Let $\mu \in P_m$ be their product measure according to the previous section, with $m = \sum n_i$. Thus μ is supported on (some of) the k-subsets of the multiset $\{v_1, \dots, v_m\} := [n_1] \dot\cup \dots \dot\cup [n_k]$. With this correspondance we see that Theorem 2 holds for more general probability measures, but in turn it needs a bound on each $||v_i||$ and not only on some expected norms, and the assertion is also not exactly the analog of the one in Theorem 1.

4. Motivation: Asymmetric Travelling Salesman Problem.

Let $G = (V, E)$ be a directed graph on n vertices with cost function $c : E \to \mathbb{R}_{\geq 0}$. The Asymmetric Travelling Salesman Problem (ATSP) askes for the shortest tour in G that visits each vertex *at least* once. (Equivalently one can write "exactly once" instead of "at least once" if one further requires the triangle inequality for c.) If c is symmetric, $c(u, v) = c(v, u)$, then this is called the Symmetric TSP, for which it is considerably easier to find approximate solutions. On the other hand, the associated decision problems for both ATSP and STSP are NP-complete.

The ATSP has a natural LP relaxation (by Held and Karp '70). The integrality gap is defined as the quotient between the costs of the optimal tours for the LP relaxation and for the original ATSP. It is known that this gap can be at least 2. It is unknown whether it is bounded from above by a constant. The prevous best upper bound was $O(\log(n)/\log\log(n))$, and Anari and Oveis Gharan were able to improve it to $O((\log\log(n))^a)$ for some a. Their approach was via so-called α-spectrally thin trees, which are defined as follows.

Let L_G denote the discrete Laplace operator on G, now regarded as an undirected graph. A matrix representation of L_G is $L_G = \sum_{e\in E} b_e b_e^t \in \mathbb{R}^{n\times n}$, where b_e is the vector $\mathbb{1}_u - \mathbb{1}_v \in \mathbb{R}^n$. Similarly for a spanning tree $T \subseteq G$, define $L_T = \sum_{e\in T} b_e b_e^t \in \mathbb{R}^{n\times n}$. Now, a spanning tree $T \subseteq G$ is called α-spectrally thin, $\alpha \in \mathbb{R}_{>0}$, if $L_T \preceq \alpha L_G$.

A sufficient condition for T being α-spectrally thin is $||\sum_{e\in T} v_e v_e^t|| \leq \alpha$, where $v_e := L_G^{\dagger/2} \cdot b_e$, $L_G^{\dagger/2}$ denoting the square root of the pseudo inverse of L_G. This is precisely a condition that can be obtained from Theorem 2. For this one needs further an adequate probability distribution on the set of spanning trees of G. In [3] it was proved that for any $\gamma : E \to \mathbb{R}$ the measure μ supported on the spanning trees of G and given via $P_\mu(T) \sim \prod_{e\in T} \exp(\gamma(e))$ is a homogeneous and strongly Rayleigh measure in $P_{|E|}$.

5. Mixed characteristic polynomials

Let $\mu \in P_m$ be a homogeneous probability distribution on $2^{[m]}$ of degree d_μ. For m given vectors $v_1, \ldots, v_m \in \mathbb{R}^d$, the mixed characteristic polynomial of μ at $v_1, \ldots, v_m$ is defined as

$$\mu[v_1, \ldots, v_m](x) = \mathbb{E}_{S\sim\mu}\chi\big[\sum_{i\in S} 2v_i v_i^t\big](x^2) \in \mathbb{R}[x].$$

where $\chi[M]$ denotes the ordinary characteristic polynomial of a square matrix M.

Theorem 3 ([2]). *$\mu[v_1, \ldots, v_m](x)$ equals*

$$x^{d-d_\mu} \cdot \big(\prod(1-\partial_{z_i}^2)\big) \cdot \big(g_\mu(x\cdot \mathbb{1} + z)\cdot \det(x\cdot \mathrm{id}_d + \sum z_i v_i v_i^t)\big)\Big|_{z_1=\ldots=z_m=0} \in \mathbb{R}[x].$$

Here, $z_1, \ldots, z_m$ are m further variables. In the formula of the theorem, the differential operators $(1-\partial_{z_i}^2)$ are applied to $g_\mu(\ldots)\det(\ldots)$ *before* the variables z_i are put to zero. Note that both factors $g_\mu(\ldots)$ and $\det(\ldots)$ are are linear in each z_i, whence each operator $\partial_{z_i}^2$ gets "distributed", one ∂_{z_i} for each factor.

This representation of the mixed characteristic polynomial opens the way to apply the theory of stable polynomials. Using the lemmas from [5] the following corollary follows immediately.

Corollary 4. *If μ is strongly Rayleigh, then $\mu[v_1, \ldots, v_m]$ is real rooted.*

6. Interlacing families

Let $\mathcal{F} := \{S \subseteq [n] \mid \mu(S) \neq 0\}$. Let $\{q_S\}_{S\in\mathcal{F}}$ denote the family of polynomials given by $q_S(x) = \mu(S)\cdot\chi\big[\sum_{i\in S} 2v_i v_i^t\big](x^2)$. The characteristic polynomial at $v_1, \ldots, v_m$ is clearly the sum of the q_S. In fact $\{q_S\}_\mathcal{F}$ is a so-called interlacing family (in the sense of [2]; the proof uses the previous corollary), by which one obtains the following theorem.

Theorem 5. *There exists an $S \in \mathcal{F}$ such that the largest root of q_S is less or equal to the largest root of $\mu[v_1, \ldots, v_m](x)$.*

7. Proof scheme for Theorem 2.

By an extension of the so-called multivariate barrier argument of [5], Anari and Oveis Gharan proved that the largest root of $\mu[v_1, \ldots, v_m](x)$ is at most $4(2\varepsilon+\varepsilon^2)$, where $\varepsilon = \varepsilon_1 + \varepsilon_2$. We omit this part, as this is given in large detail in the next talk by Romanos Malikiosis. Then one applies Theorem 5 and obtains the existence of some $S \in \mathcal{F}$ such that all roots of q_S are bounded from above. As q_S is essentially the characteristic polynomial of a matrix $\sum_{i\in S} v_i v_i^t$, this bounds the operator norm of that matrix. And this finishes the proof of Theorem 2.

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The Anari–Oveis-Gharan version of the Marcus–Spielman–Srivastava theorem for strongly Rayleigh measures

Romanos-Diogenes Malikiosis

In this talk we show the idea and some technical details behind the proof of the following [1]: Let $v_1, \ldots, v_m \in \mathbb{R}^d$ be such that

$$\sum_{i=1}^{m} v_i v_i^T = I_d,$$

and let $\mu : 2^{[m]} \longrightarrow \mathbb{R}$ be a homogeneous strongly Rayleigh measure [2] such that $\mathbb{P}_{S\sim\mu}[i \in S] \leq \varepsilon_1$ and $\|v_i\|^2 \leq \varepsilon_2$ for all $i \in [m]$ for some $\varepsilon_1, \varepsilon_2 > 0$. Then, there is some $S \subseteq [m]$ with $\mu(S) > 0$ such that

$$\left\| \sum_{i\in S} v_i v_i^T \right\| \leq 4\varepsilon + 2\varepsilon^2,$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$.

The above norm is the maximal root of the characteristic polynomial of $\sum_{i\in S} v_i v_i^T$ which is real rooted. Furthermore, all such polynomials form an interlacing family, hence there is one such norm which is smaller than the maximal root of the mixed characteristic polynomial $\mu[v_1, \ldots, v_m](x)$ of the measure μ with respect to $v_1, \ldots, v_m$.

The mixed characteristic polynomial is defined as

$$\mu[v_1, \ldots, v_m](x) = \mathbb{E}_{S\sim\mu}\left[\sum_{i\in S} 2 v_i v_i^T\right](x^2),$$

and satisfies

$$\mu[v_1, \dots, v_m](x) = x^{d-d_\mu} \prod_{i=1}^{m} (1 - \partial_{y_i}^2) \left(g_\mu(y) \cdot \det \left(\sum_{i=1}^{m} y_i v_i v_i^T \right) \right) \Bigg|_{y_1 = \cdots = y_m = x},$$

where g_μ is the generating function of the measure μ defined by

$$g_\mu(z) = \sum_{S \subseteq [m]} \mu(S) z^S,$$

and d_μ is the degree of g_μ. Defining

$$Q(y_1, \dots, y_m) = \prod_{i=1}^{m} (1 - \partial_{y_i}^2) \left(g_\mu(y) \cdot \det \left(\sum_{i=1}^{m} y_i v_i v_i^T \right) \right),$$

our task is to bound above the maximal root of $Q(x, \dots, x)$, which as it turns out is real rooted.

In order to bound this maximal root, Anari and Oveis-Gharan [1] refine the techniques of Marcus, Spielman, and Srivastava [3], regarding barrier functions of real stable polynomials. Starting with the polynomial $p(y) = g_\mu(y) \cdot \det \left(\sum_{i=1}^{m} y_i v_i v_i^T \right)$, we observe that the positive orthant is a zero-free region; actually, something stronger is true, namely, that $p(y) > 0$ in this region. Applying the differential operators $1 - \partial_{y_i}^2$ one by one, this zero-free region is moved in the positive direction on all axes. How far it is moved can be controlled by the so called *barrier functions*, defined by

$$\Phi_p^i(z) = \frac{\partial_{z_i} p(z)}{p(z)}, \qquad \Psi_p^i = \frac{\partial_{z_i}^2 p(z)}{p(z)}.$$

This eventually proves the main result. The difference between this version and the original version by Marcus–Spielman–Srivastava [3] is that they are using only the first barrier function $\Phi_p^i(z)$, as the formula for the mixed characteristic polynomial in this case involves first order partial differential operators $1 - \partial_{z_i}$, not $1 - \partial_{z_i}^2$.

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Problem Session

Edited by Adam Marcus

(joint work with Daniel Dadush, Guilaumme Aubrun)

1. Daniel Dadush

(1) Can we get a matrix version of Spencer's "six standard deviations suffice" theorem? That is, given $n \times n$ matrices $A_1, \ldots, A_n$ with $\|A_i\| \leq 1$ does there exist a signing $s_1, \ldots, s_n \in \{\pm 1\}$ such that

$$\left\| \sum_{i=1}^{n} s_i A_i \right\| \leq O(\sqrt{n})?$$

If you allow $m > n$ matrices, the bound should extend to

$$\left\| \sum_{i=1}^{n} s_i A_i \right\| \leq O(\sqrt{n}\sqrt{\log{(m/n)} + 1})$$

(2) A similar conjecture is due to Komlos: let $a_1, \ldots, a_n$ be vectors in $\mathbb{R}^n$ with $\|a_i\| \leq 1$. Do there exist signs $s_i \in \{\pm 1\}$ such that

$$\left\| \sum_{i} s_i a_i \right\|_\infty < C$$

for some universal (not depending on the a_i or n) constant C? Currently, the best that can be proved is

$$\left\| \sum_{i} s_i a_i \right\|_\infty < O(\sqrt{\log n})$$

2. Guilaumme Aubrun

Here is an open problem which shares some flavour with the Kadison–Singer problem: Does every n-dimensional subspace of L^1 linearly embeds with distortion $(1+\varepsilon)$ into ℓ_1^N with $N = O_\varepsilon(n)$?

Given two sets $A, B \subset \mathbb{R}^n$, denote $A + B = \{a + b \,:\, a \in A, b \in B\}$. Given $u \in \mathbb{R}^n$, we denote by $[-u, u]$ the line segment joining u and $-u$ (i.e. the convex hull of $\{u, -u\}$).

A geometric reformulation of the initial problem is as follows: given $\varepsilon > 0$ and a finite family $(u_i)_{1 \leq i \leq k}$ of vectors in $\mathbb{R}^n$, for which value of $N = N(n, \varepsilon)$ can we find a family $(v_j)_{1 \leq j \leq N}$ of vectors in $\mathbb{R}^n$ such that, denoting

$$K = [-u_1, u_1] + \cdots + [-u_k, u_k],$$

$$L = [-v_1, v_1] + \cdots + [-v_N, v_N],$$

the inclusions $K \subset L \subset (1 + \varepsilon)K$ hold?

Convex sets obtain as sums of segments are called zonotopes. The best result is due to Talagrand and achieves $N = (\varepsilon)n \log n$. It is based on the idea of cleverly selecting the (v_j) as a random subfamily of the (u_i). However, as in other Kadison–Singer-like problems, random constructions cannot go below $n \log n$.

It could be interesting to consider the Steiner polynomial: the quantity

$$\mathrm{vol}\,(t_1[-u_1, u_1] + \cdots + t_k[-u_k, u_k])$$

is a polynomial in $t_1, \ldots, t_k \geq 0$.

3. Michelle Delcourt

(1) The Ihara zeta function of a k-regular connected graph is defined as

$$\frac{1}{\zeta(u, G)} = (1 - u^2)^{|E|-|V|} \det \left[I - Au + (k-1)u^2 I\right]$$

where A is the adjacency matrix of G. Ihara showed that G is Ramanujan if and only $\zeta(u, G)$ satisfies a type of "graph Riemann Hypothesis". Can the techniques discussed in this workshop be used to give other information about the Ihara zeta function?

4. Adam Marcus

(1) If $u_1, \ldots, u_n$ are line segments in $\mathbb{R}^n$, is the polynomial

$$p(x_1, \ldots, x_n) = \mathrm{Vol}\left(\sum_i x_i u_i\right)$$

hyperbolic? (Here again the sum is a Minkowski sum).

(2) Many of the methods discussed in this workshop rely on the qualitative property of real-rootedness. Can we find a quantitative version of these? That is, can we find a way to define a "distance from real-rootedness" in a way that the inequalities guaranteed by real-rootedness can be extended by adding a penalty that depends on this "distance"?

(3) While many of the results that come from the method of interlacing polynomials involve bounding norms (which typically requires bounding the largest and smallest eigenvalues), the method itself can only bound one eigenvalue. To achieve two bounds, we essentially have to "cheat" in various ways (using bipartiteness, for example, in finding Ramanujan graphs). Is there a way to extend the method (possibly requiring more structure) to capture two eigenvalues simultaneously?

(4) The current best bounds on the number of blocks needed to pave within a factor of ϵ is that $r = O(1/\epsilon^4)$ suffices. It is known that $r \geq \Omega(1/\epsilon^2)$ is needed. The extra factor of $1/\epsilon^2$ comes in the reduction from partitioning to paving in a place where we have to create two partitions (one to bound the top eigenvalue and one to bound the bottom one). Normally we would try to acoid doing this by cheating (except we have already cheated to get

the partitioning bound). Is there a way to “suspend the cheat” so that we can get both sides of the paving bound as well?

Reporter: David James

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