Mathematisches Forschungsinstitut Oberwolfach

Report No. 29/2015

DOI: 10.4171/OWR/2015/29

Noncommutative Geometry

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14 June – 20 June 2015

Abstract. These reports contain an account of 2015's meeting on noncommutative geometry. Noncommutative geometry has developed itself over the years to a completely new branch of mathematics shedding light on many other areas as number theory, differential geometry and operator algebras. A connection that was highlighted in particular in this meeting was the connection with the theory of II_1-factors and geometric group theory.

Mathematics Subject Classification (2010): 58B34.

Introduction by the Organisers

Noncommutative geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly inter-disciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and theoretical quantum physics. On the basis of ideas and methods from algebraic and differential topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. This includes K-theory, cyclic homology and the theory of spectral triples. Areas of intense research in recent years are related to topics such as index theory, quantum groups and Hopf algebras, the Novikov and Baum-Connes conjectures as well as to the study of specific questions in other fields such as number theory, modular forms, topological dynamical systems, renormalization theory, theoretical high-energy physics and string theory. Many results elucidate important properties of specific classes of examples that arise in many applications. But the properties of many important classes of examples still remain mysterious, and are currently under intense investigation. This meeting concentrated on selected aspects of Noncommutative Geometry. Special emphasis this time was laid on connections to von Neumann algebras and to classification questions for group measure space II_1 factors, as well as to geometric group theory and the study of embeddings of groups into Hilbert and Banach spaces. There are indications for a deep connection between recent progress in that direction and the role of factors, ergodic theory and quantum statistical mechanics in the approach to number theory and L-functions from noncommutative geometry. In addition quite a few other topics of current interest in Noncommutative Geometry were covered as well.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Noncommutative Geometry

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Abstracts

Noncommutative De Leeuw theorems

Martijn Caspers
(joint work with Javier Parcet, Mathilde Perrin, Eric Ricard)

1. Classical De Leeuw theorems

In 1965 Karel de Leeuw [4] proved fundamental theorems about Fourier multipliers acting on L^p-spaces. These theorems play a major role in commutative and noncommutative harmonic analysis and have many applications to for example partial differential equations. In order to state De Leeuw's main results recall the following. Let $m : \mathbb{R}^n \to \mathbb{C}$ be measurable and consider the linear mapping $T_m : L^2(\mathbb{R}) \to L^p(\mathbb{R})$ that is determined by

$$\widehat{T_m f} = m\widehat{f},$$

where $f \mapsto \widehat{f}$ is the Fourier transform.

(1) **Restriction.** Let H be a subgroup of $\mathbb{R}^n$. Suppose that T_m acts boundedly on $L^p(\mathbb{R}^n)$ then the mapping

$$T_{m|_H} : \int_H \widehat{f}(h)\chi_h d\mu(h) \mapsto \int_H m(h)\widehat{f}(h)\chi_h d\mu(h)$$

extends to a $L^p(\widehat{H})$-bounded multiplier for any subgroup $H \subseteq \mathbb{R}^n$ where the χ_h's stand for the characters on the dual group $\widehat{H}$ and μ is the Haar measure.

(2) **Compactification.** Let $\mathbb{R}^n_{\mathrm{Bohr}}$ be the Pontryagin dual of $\mathbb{R}^n_{\mathrm{disc}}$ equipped with the discrete topology. Given $m : \mathbb{R}^n \to \mathbb{C}$ bounded and continuous, the $L^p(\mathbb{R}^n)$-boundedness of T_m is equivalent to the boundedness in $L^p(\mathbb{R}^n_{\mathrm{Bohr}})$ of the multiplier with the same symbol,

$$T_m : \sum_{\mathbb{R}^n_{\mathrm{disc}}} \widehat{f}(\xi)\chi_\xi \mapsto \sum_{\mathbb{R}^n_{\mathrm{disc}}} m(\xi)\widehat{f}(\xi)\chi_\xi.$$

The proof of the compactification theorem proceeds through the restriction theorem. In fact the restriction theorem easily follows from the compactification theorem which therefore is the stronger statement. We also take into consideration periodization and lattice approximation (i.e. Igari's theorem [3]). After De Leeuw's fundamental paper [4] these theorems were soon generalized to nonabelian groups by Saeki [8].

2. Noncommutative De Leeuw theorems

The development of noncommutative integration theory (especially in the second half of the 20th century) naturally raises the question if there are noncommutative De Leeuw theorems. Noncommutative means that $\mathbb{R}^n$ can be replaced by an arbitrary group. In this case the Fourier multipliers T_m act on its Pontryagin dual, which only exists as a so-called quantum group, whose underlying space is the group von Neumann algebra. Very recently a prolific series of papers was devoted to this topic, see [2], [5], [6], [7] and references given there.

In [1] we show to what extent De Leeuw theorems can be generalized to arbitrary locally compact groups. Let G be a locally compact group and let $m : G \to \mathbb{C}$ which we shall always assume to be bounded and continuous. To avoid technicalities in our exposition here, we assume that G is unimodular. Let

$$\mathcal{L}(G) = \{\lambda(f) \mid f \in L^1(G)\}'',$$

be the group von Neumann algebra generated by the left regular representation λ. Let φ be the Plancherel weight on $\mathcal{L}(G)$ which is given by $\varphi(x^*x) = \|f\|_{L^2(G)}$ in case there exists f such that $xg = f * g, g \in L^2(G)$. Otherwise $\varphi(x^*x) = \infty$. As G is unimodular φ is tracial. We may construct noncommutative $L^p(\mathcal{L}(G))$ as the completion of the space $\{x \mid \|x\|_p := \tau(|x|^p)^{1/p} < \infty\}$ with respect to the $\| \; \|_p$ norm. It follows that $C_c(G)^{*2}$ (second convolution power) spans a dense subset of $L^p(\mathcal{L}(G))$ and that we may set

$$T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G)) : \lambda(f) \mapsto \lambda(mf), \qquad m \in C_c(G)^{*2}.$$

We call m an *L^p-Fourier multiplier* in case T_m extends boundedly.

In [1] we prove the following theorem which involves two assumptions. We say that G has small almost invariant neighbourhoods with respect to a subgroup Γ if for every finite subset $F \subseteq \Gamma$ there exists a net of open sets $U_i \to \{e\}$ of G such that for all $s \in F$ we have measure$(U_i \cap sU_is^{-1})/$measure$(U_i) \to 0$. We say that G is approximable by discrete subgroups if there exists a net Γ_i of subgroups of G with fundamental domains shrinking to the identity. Our main results include:

(1) **Restriction.** Let $1 \leq p \leq \infty$. Let $\Gamma \subseteq G$ be a discrete subgroup and suppose that G has small almost invariant neighbourhoods with respect to Γ. Then,

$$\|T_{m|_H} : L^p(\mathcal{L}(H)) \to L^p(\mathcal{L}(H))\| \leq \|T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G))\|,$$

(2) **Compactification.** Let $1 \leq p \leq \infty$. Suppose that G is approximable by discrete subgroups and that G_{disc} with the discrete topology is amenable. Then,

$$\|T_m : L^p(\mathcal{L}(G_{\text{disc}})) \to L^p(\mathcal{L}(G_{\text{disc}}))\| = \|T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G))\|,$$

These theorems recover the classical De Leeuw theorems. The proof strategy is (in a suitable sense) the same as De Leeuw's. However the techniques are totally different and involve an intricate analysis of ucp maps on von Neumann algebras.

3. Final comments

We conclude with three remarks. Firstly in [1] we also prove noncommutative periodization and lattice approximation results as in the classical case. Secondly the unimodularity condition on G can be removed in which case proper noncommutative L^p-spaces of a group von Neumann algebra were defined by Haagerup and Connes-Hilsum. Finally, the above theorems also hold in the operator space setting, meaning that bounds can be replaced by complete bounds. In fact in the operator space setting one gets additional result using the technique of transference to Schur multipliers (found in [6] for discrete groups and generalized in [2] to arbitrary groups).

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The scaling Site

Alain Connes

(joint work with Caterina Consani)

I described in my talk the recent joint work [4] with C. Consani on the Scaling Site. It is the algebraic geometric space obtained from the arithmetic site of [2, 3] by extension of scalars from the Boolean semifield $\mathbb{B}$ to the tropical semifield $\mathbb{R}_{\max}$.

The underlying site inherits from its structural sheaf a natural structure of a tropical curve allowing one to define the sheaf of rational functions and to investigate an adequate version of the Riemann-Roch theorem in characteristic 1. We tested this structure by restricting it to the periodic orbits of the scaling flow, namely the points over the image of $\operatorname{Spec}\mathbb{Z}$ (see [3], §5.1). We found that for each prime p the corresponding circle of length $\log p$ is endowed with a quasi-tropical structure which turns this orbit into the analogue $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ of a classical elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$. In particular rational functions, divisors, etc. all make sense. A new feature is that the degree of a divisor can now be any real number. We determined the Jacobian of C_p : the quotient $J(C_p)$ of the group of divisors of degree 0 by principal divisors and showed that it is a cyclic group of order $p-1$.

For each divisor D we define the corresponding Riemann-Roch problem with solution space $H^0(D)$. We introduce the continuous dimension $\mathrm{Dim}_{\mathbb{R}}(H^0(D))$ of this $\mathbb{R}_{\max}$-module using a limit of normalized topological dimensions and find that $\mathrm{Dim}_{\mathbb{R}}(H^0(D))$ is a real number. Finally, we prove that the Riemann-Roch formula holds true for C_p. The appearance of arbitrary positive real numbers as continuous dimensions in the Riemann-Roch formula is due to the density in $\mathbb{R}$ of the subgroup $H_p \subset \mathbb{Q}$ of fractions with denominators a power of p and the fact that continuous dimensions are obtained as limits of normalized dimensions $p^{-n}\dim_{\mathrm{top}}(H^0(D)^{p^n})$. This outcome is the analogue in characteristic 1 of what happens for modules over matroid C^*-algebras and the type II normalized dimensions as in [5].

One can compare our Riemann-Roch theorem with the tropical Riemann-Roch theorem of [1, 6, 8] and its variants. Let thus C be the elliptic tropical curve given by a circle of length L. In this case, the structure of the group $\mathrm{DivClass}(C)$ of divisor classes is inserted into an exact sequence of the form

$$0 \to \mathbb{R}/L\mathbb{Z} \to \mathrm{DivClass}(C) \stackrel{\text{degree}}{\longrightarrow} \mathbb{Z} \to 0$$

(see [8]). In our case we get the following split exact sequence associated to C_p

$$0 \to \mathbb{Z}/(p-1)\mathbb{Z} \to \mathrm{DivClass}(C_p) \stackrel{\text{degree}}{\longrightarrow} \mathbb{R} \to 0$$

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Analytic cohomology in characteristic $p > 0$.

Guillermo Cortiñas

(joint work with Joachim Cuntz, Ralf Meyer)

Let k be a field of characteristic $p > 0$, $V = W(k)$ the ring of Witt vectors. Thus V is a Noetherian, local domain with principal maximal ideal $\mathfrak{m} = \pi V$ and residue field $V/\pi V = k$, complete in the $\mathfrak{m}$-adic topology. We write K for the field of fractions of V. Our goal is to construct a functor

$$H^{an} : k-\text{algebras} \to ((\mathbb{Z}/2\text{-graded complexes of } K\text{-modules}))$$

which is polynomially homotopy invariant and matrix invariant, which satisfies excision and which for commutative algebras of finite type, recovers Bertherlot's rigid cohomology [3] in the sense that

$$H_n^{an}(A) = \prod_j H_{rig}^{2j-n}(A, K).$$

To explain why having such a theory could be useful, assume for a moment that a functor with the properties above exists, and write

$$H_*^{an}(A, B) = H_*(\mathrm{HOM}(H^{an}(A), H^{an}(B))).$$

It follows from the universal property of algebraic bivariant K-theory [1] and the assumed properties of H^{an}, that there is a Chern character $kk_*(A, B) \to H_*^{an}(A, B)$, compatible with composition. In particular, setting $A = k$, we get a Chern character $KH_*(B) = kk_*(k, B) \to H_*^{an}(B) := H_*^{an}(k, B)$ from Weibel's homotopy algebraic K-theory [6] (and thus also from Quillen's K-theory, using the natural transformation $K \to KH$). Specializing to B commutative of finite type, yields maps $ch_{j,n} : KH_n(B) \to H_{rig}^{2j-n}(B)$.

Next we recall the definition of rigid cohomology for a smooth commutative algebra $k \to A$. By a theorem of Elkik [2], there is a smooth morphism $V \to R$ which reduces to $k \to A$ mod π. Let $R^\dagger$ be the weak completion of Monsky-Washnitzer [4]. Write $\Omega_{R/V}$ for the de Rham complex of Kähler differential forms. The rigid cohomology of A is defined to be [5]

$$H_{rig}^*(A, K) = H^*((\Omega_{R/V} \otimes_R R^\dagger) \otimes_V K).$$

As R is a V-algebra of finite type, it is a quotient of a polynomial ring $V[x_1, \dots, x_m]$ and thus it can be equipped with a filtration coming from the degree filtration on the polynomials. Using this filtration one can define a bornology on $R \otimes_V K$ whose bornological completion is $R^\dagger \otimes_V K$. In particular the latter is a bornological algebra. Our first result relates the periodic cyclic homology of the bornological algebra $R^\dagger \otimes_V K$ to the rigid cohomology of A:

$$HP_n(R^\dagger \otimes_V K) = \prod_j H_{rig}^{2j-n}(A, K).$$

The basic idea for constructing H^{an} is to write $A = TL/I$ as a quotient of the tensor algebra of a free V-module L and then take the periodic cyclic complex of a bornological completion of $TL_K := TL \otimes_V K$ with respect to I: $H^{an}(A) = HP(\widehat{TL_K})$. The highly non-trivial technical point is what bornology to take. The tentative definition of H^{an} we have so far is homotopy invariant. Furthermore, we can prove the following. Let $A = L/\pi L$ be a presentation as a quotient of a unital filtered V-algebra L. Assume that $\mathcal{F}_0 L = V$, that $\mathcal{F}_{n+1}L/\mathcal{F}_n L$ is a free V-module ($n \geq 0$), and that there exists $n > 0$ such that the bimodule $\Omega_V^n L$ of noncommutative differential forms is projective. The algebra $L_K := L \otimes_V K$ carries a natural bornology coming from the filtration and we let $\widehat{L_K}$ be the bornological completion. Assume that $p > 2$, and let $A = L/\pi L$. Then

$$H_*^{an}(A) = HP_*(\widehat{L_K}).$$

As a consequence, we have the following. Let $k \to A$ be smooth commutative and let $V \to R$ as in Elkik's theorem. Assume that there exists L as above such that $\widehat{L_K} \cong R^\dagger \otimes_V K$. Then

$$H_n^{an}(A) = \prod_j H_{rig}^{2j-n}(A, K).$$

An L as above exists, for example, if $A = k[x_1, \ldots, x_m]/f$ is a smooth hypersurface. We do not know whether such an L exists for general smooth algebras $k \to A$.

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Groupoids and pseudodifferential calculus

Claire Debord

(joint work with Georges Skandalis)

We recall how pseudodifferential operators on a groupoid G can be expressed as integrals of kernels on the adiabatic groupoid G_{ad} of G and investigate several generalisations of pseudodifferential operators of the Boutet de Monvel calculus [4, 5].

1. Pseudodifferential operators as integral kernels [4]

A key ingredient here will be the adiabatic groupoid of a groupoid G which is a generalisation of the famous *tangent groupoid* of A. Connes (see [3]).

1.1. **The adiabatic groupoid.** Let $G \rightrightarrows G^{(0)}$ be a smooth groupoid, denote by $\mathfrak{A}G$ its Lie algebroid and $\sharp$ the corresponding anchor map. The *adiabatic groupoid* is the deformation to the normal cone of the inclusion $G(0) \subset G$ (see [7, 8, 9]) :

$$G_{ad} = G \times \mathbb{R}^* \sqcup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R} \ .$$

It can be equipped with a (unique) smooth structure such that its Lie algebroid is the bundle $\mathfrak{A}G_{ad} = \mathfrak{A}G \times \mathbb{R}$ with anchor map :

$$\sharp_{ad} : \mathfrak{A}G \times \mathbb{R} \to TG^{(0)} \times T\mathbb{R} \ ; \ (x, X, t) \mapsto (\sharp(x, tX), (t, 0)) \ .$$

The scaling action of $\mathbb{R}^*$ on $G \times \mathbb{R}^*$ extends to a smooth action of $\mathbb{R}^*$ on G_{ad} which is free and proper outside the units $G^{(0)} \times \mathbb{R}$. For $\lambda \in \mathbb{R}^*$ it is given by :

$$\lambda \cdot (\gamma, t) = (\gamma, \lambda t) \quad \text{for} \quad t \neq 0 \quad \text{and} \; \lambda \cdot (x, X, 0, \lambda) = (x, \frac{1}{\lambda} X, 0) \; .$$

The previous constructions applied to the product groupoid of G with the group $\mathbb{R}$ leads to a *local compactification* of G_{ad} :

$$\overline{G_{ad}} := \big((G \times \mathbb{R})_{ad} \setminus G^{(0)} \times \{0\} \times \mathbb{R}\big)/_{\mathbb{R}^*} = G_{ad} \sqcup G \setminus G^{(0)} \sqcup \mathcal{P}(\mathfrak{A}G) \; .$$

Notice that the map equal to identity on $G \times \mathbb{R}^*$ and which sends $\mathfrak{A}G \times \{0\}$ on $G^{(0)} \times \{0\}$ extends to a proper map $\tau : \overline{G_{ad}} \longrightarrow G \times \mathbb{R}$.

1.2. **Spaces of functions on G_{ad}.** A smooth function f on G_{ad} will be denoted $f = (f_t)_{t \in \mathbb{R}}$ where $f_t \in \mathcal{C}^\infty(G)$ for $t \neq 0$ and $f_0 \in \mathcal{C}^\infty(\mathfrak{A}G)$. We may introduce several spaces of functions on G_{ad} and on its crossed product by the $\mathbb{R}^*$ action:

The Schwartz algebra: $S_c(G_{ad})$ is the restriction to G_{ad} of smooth functions on $\overline{G_{ad}}$ which are ∞-flat (i.e. vanish as well as all the derivatives) outside G_{ad} and whose support is sent by τ on a compact subset of $G \times \mathbb{R}$.

The ideal: $\mathcal{J}_0(G) \subset C_c^\infty(G_{ad})$ of *rapidly decreasing functions* at 0 is made of smooth functions which are ∞-flat outside $G \times \mathbb{R}^*$.

The ideal: $\mathcal{J}(G) \subset S_c(G_{ad})$ is the set of functions $f = (f_t)_{t \in \mathbb{R}}$ which satisfy that for any $g \in C_c^\infty(G)$, $(f_t * g)_{t \in \mathbb{R}^*}$ and $(g * f_t)_{t \in \mathbb{R}^*}$ belong to $\mathcal{J}_0(G)$.

The $\rtimes \mathbb{R}^*$ version : we define similarly $\mathcal{S}_c(G_{ad} \rtimes \mathbb{R}^*)$ and the ideal $\mathcal{J}(G)_\rtimes \subset \mathcal{S}_c(G_{ad} \rtimes \mathbb{R}^*)$.

The ideal $\mathcal{J}(G)$ enables us to recover the pseudodifferential operators on G, precisely we have :

Theorem 1. *For $f = (f_t)_{t \in \mathbb{R}} \in \mathcal{J}(G)$ and $m \in \mathbb{Z}$ (and even $m \in \mathbb{C}$) let*

$$P = \int_0^{+\infty} t^m f_t \frac{dt}{t} \quad \text{and} \quad \sigma : (x, \xi) \in \mathfrak{A}^*G \setminus G^{(0)} \mapsto \int_0^{+\infty} t^m \widehat{f}(x, t\xi, 0) \frac{dt}{t}$$

Then P is a pseudodifferential operator of order $-m$ on G and its principal symbol is σ. Moreover any pseudodifferential operator on G is of this form.

Let us denote by $\mathcal{J}_+(G)$ the image of $\mathcal{J}(G)$ under the restriction of functions to $\overline{G_{ad}}_+ := \tau^{-1}(G \times \mathbb{R}_+)$ and $J_+(G)$ its closure in $C^*(G_{ad})$.

Theorem 2. *A completion of $\mathcal{J}_+(G)$ into a bimodule $\mathcal{E}$ leads to a Morita equivalence between $\Psi_0^*(G)$ and $J_+(G) \rtimes \mathbb{R}_+^*$.*

In the special case of the pair groupoid $G = V \times V$ over a smooth manifold V, this last theorem was proved abstractly by Aastrup-Melo-Monthubert-Schrohe in [1]. In this situation G_{ad} is the tangent groupoid and the previous construction leads to an ideal $\mathcal{J}$ of $\mathcal{S}_c(G_{ad})$ which can be (restricted and) completed into a full $\Psi_0^*(G) = \Psi_0(V)$ module $\mathcal{E}$ which satisfies $\mathcal{K}(\mathcal{E}) \simeq J_+ \rtimes \mathbb{R}_+^*$.

2. The Boutet de Monvel calculus

Let $M = V \times \mathbb{R}_+$ be a manifold with boundary embeded in the smooth manifold $\widetilde{M} = V \times \mathbb{R}$. The aim here is to define a pseudodifferential calculus adapted to M.

Let P be a pseudodifferential operator on $\widetilde{M}$, $f \in \mathcal{C}_c^\infty(M)$ and $\tilde{f}$ the extension of f by 0 on $\widetilde{M}$. The computation $P(\tilde{f})$ gives a function on $M \setminus \partial M$ which may not admit a limit on $\partial M = V \times \{0\}$. This leads to the notion of *transmitting property* : the operator P has the transmitting property when for any smooth function $f \in \mathcal{C}_c^\infty(M)$, $P(\tilde{f})$ coincides on $M \setminus \partial M$ with a smooth function on $\widetilde{M}$. In such a situation we let P_+ be the corresponding operator on $\mathcal{C}_c^\infty(M)$ and we denote by $\mathcal{P}_+^M$ the set of such operators.

When $P,\ Q$ are pseudodifferential operators on $\widetilde{M}$ with the transmitting property it may happen that $P_+Q_+ \neq (PQ)_+$, thus $\mathcal{P}_+^M$ is not an algebra. To solve this problem, Boutet de Monvel defined the algebra $\mathbb{G}_M$ of *singular Green* operators whose typical elements are $P_+Q_+ - (PQ)_+$. The space $\mathcal{P}_+^M + \mathbb{G}_M$ is now an algebra. Moreover he also defined the spaces $\mathcal{K}$ of *singular Poisson* operators $\mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M)$ and $\mathcal{T}$ of *singular Trace* operators $\mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(\partial M)$, adjoint of each other and formed an algebra of 2×2 matrices :

$$\begin{pmatrix} P_+ + G & K \\ T & Q \end{pmatrix} : \mathcal{C}_c^\infty(M) \oplus \mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M) \oplus \mathcal{C}_c^\infty(\partial M)$$

where $P_+ \in \mathcal{P}_+^M$, $G \in \mathbb{G},\ K \in \mathcal{K},\ T \in \mathcal{T}$ and Q is an ordinary pseudodifferential operator on $V = \partial M$. See [2] for the original construction and [6] for a detailed description.

The product of a singular Poisson operator with a singular Trace operator gives a singular Green operator. Thus, forgetting $\mathcal{P}_+^M$, one gets an algebra of 2×2 matrices which (almost) gives a Morita equivalence between the two corners : $\mathbb{G}_M$ and the algebra of pseudodifferential operators on V. The comparison of this phenomenon with the result of Theorem 2 leads to the following [5]:

Theorem 3. *For $f \in \mathcal{J}$ and $F \in \mathcal{J}_\rtimes$:*

- $K_f : \mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M),\ u_0 \mapsto (f_t * u_0)_{t\in\mathbb{R}_+^*}$ *is a singular Poisson operator.*
- $T_f : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(\partial M),\ u \mapsto \int_0^\infty f_t * u_t \, \frac{dt}{t}$ *is a singular Trace operator.*
- $G_F : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)\ ; u \mapsto F * u$ *is a singular Green operator.*

Moreover, we obtain in this way all the singular Green, Trace and Poisson operators of the Boutet de Monvel calculus up to smoothing operators.

This result enables to propose a natural extension of such a calculus.
If $M = V \times \mathbb{R}$ and $G \rightrightarrows V$ is a smooth groupoid on V, the above constructions still make sense : for $f \in \mathcal{J}(G)$ and $F \in \mathcal{J}(G)_\rtimes$ the same formulas give *Poisson type operators* $K_f : \mathcal{C}_c^\infty(G) \to \mathcal{C}_c^\infty(G \times \mathbb{R})$, *Trace type operators* $T_f : \mathcal{C}_c^\infty(G \times \mathbb{R}) \to \mathcal{C}_c^\infty(G)$ and *Green type operators* $G_F : \mathcal{C}_c^\infty(G \times \mathbb{R}) \to \mathcal{C}_c^\infty(G \times \mathbb{R})$.

One can go one step further by considering a groupoid $\mathbb{G} \rightrightarrows M$ which is transverse to a codimension one submanifold V of M. The transversality condition means that (locally) $\mathbb{G}$ is isomorphic to $\mathbb{G}_V^V \times \mathbb{R} \times \mathbb{R}$ around $V \subset \mathbb{G}^{(0)} \subset \mathbb{G}$. By replacing the groupoid $\mathbb{G}$ around V by $(\mathbb{G}_V^V)_{ad} \rtimes \mathbb{R}^*$ one gets a groupoid $\mathbb{G}_{cg} = \mathbb{G}_{M\setminus V}^{M\setminus V} \cup \mathfrak{A}\mathbb{G}_V^V \rtimes \mathbb{R}^*$. We can produce again *Poisson type operators* $K : \mathcal{C}_c^\infty(\mathbb{G}_V^V) \to \mathcal{C}^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V)$, *Trace type operators* $T : \mathcal{C}_c^\infty(\mathbb{G}_V) \to \mathcal{C}_c^\infty(\mathbb{G}_V^V)$ and *Green type operators* $G : \mathcal{C}_c^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V) \to \mathcal{C}^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V)$.

This is a first step in the way to establish generalized Boutet de Monvel index theorems. See [5] for details.

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A complete characterization of connected Lie groups with the Approximation Property

Tim de Laat

(joint work with Uffe Haagerup, Søren Knudby)

Approximation properties provide important information about the structure of locally compact groups. The Approximation Property for groups (AP), as introduced by Haagerup and Kraus in [4], is an important example of such a property in the spirit of Grothendieck's approximation property for Banach spaces. For a locally compact group G, let $A(G)$ denote its Fourier algebra, and let $M_0A(G)$ denote the space of completely bounded Fourier multipliers. It is known that $A(G) \subset M_0A(G)$. The space $M_0A(G)$ is the dual of a certain completion of $L^1(G)$, and we can consider the associated weak-* topology on $M_0A(G)$.

Definition 1. *A locally compact group G has the AP if there is a net (φ_α) in the Fourier algebra $A(G)$ of G such that $\varphi_\alpha \to 1$ in the weak-* topology on $M_0A(G)$.*

Other important examples of approximation properties for locally compact groups are amenability, weak amenability, and the Haagerup property, the first two of which are both strictly stronger than the AP. Moreover, all these properties have natural operator algebraic analogues. We refer to [2] for a thorough account of approximation properties for groups and operator algebras.

The AP has not been considered as much as the other aforementioned properties, probably because until recently, the only examples of groups without the AP followed from the theoretical fact that every discrete group with the AP is exact, as established by Haagerup and Kraus. However, in 2010, Lafforgue and de la Salle provided the first concrete examples of groups without the AP, namely, $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$ and lattices in these groups [8]. In fact, they also proved that $\mathrm{SL}(n,F)$ (with $n \geq 3$) and its lattices do not have the AP for any non-Archimedean local field F. Their results on real Lie groups were put into a more systematic context and were extended by Haagerup and the author of this text, first to connected simple Lie groups with real rank at least 2 and finite center [6], by proving that $\mathrm{Sp}(2,\mathbb{R})$ does not satisfy the AP, and then to all connected simple Lie groups with real rank at least 2 [7], by also considering the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. Indeed, any connected simple Lie group with real rank at least 2 contains a closed subgroup that is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, which, together with the known permanence properties of the AP, implies the general result. From this, it follows that a connected semisimple Lie group, which is always locally isomorphic to a direct product $S_1 \times \cdots \times S_n$ of connected simple Lie groups, has the AP if and only if all these S_i's have real rank 0 or 1.

In a recent joint work with Haagerup and Knudby [5] we consider what happens in the case of non-semisimple Lie groups. Firstly, note that it is straightforward to characterize the AP for more general classes of groups by using the known permanence properties of the AP, but this does not give a satisfactory characterization of connected Lie groups with the AP. However, by using a natural obstruction to the AP that we introduce in [5], it turns out that we do obtain a complete characterization of connected Lie groups with the AP. This obstruction is in fact a strengthening of Kazhdan's property (T) (see [1] for a comprehensive treatment of property (T)), and we call it property (T*).

As mentioned above, property (T*) forms a natural obstruction to the AP, in the sense that a locally compact group having both the AP and property (T*) is necessarily compact. Note that in the same way, property (T) is an obstruction to the Haagerup property. In order to define property (T*), we first need the following result (see [5, Theorem A]).

Theorem 1. *Let G be a locally compact group. Then the space $M_0A(G)$ of completely bounded Fourier multipliers on G carries a unique left invariant mean m. This mean is also right invariant.*

It is known that $M_0A(G)$ is a subspace of the space $W(G)$ of weakly almost periodic functions on G, which is known to have a unique left invariant mean (see e.g. [3]). It follows that the mean on $M_0A(G)$ is the restriction to $M_0A(G)$ of the mean on $W(G)$. The definition of property (T*) is as follows ([5, Definition 1.1]).

Definition 2. *A locally compact group G is said to have property* (T^*) *if the unique left invariant mean m on $M_0A(G)$ is a weak-* continuous functional.*

It is easy to see that compact groups have property (T^*). Also, as mentioned before, any group satisfying both the AP and property (T^*) is compact. Using a powerful result of Veech from [9] and the results of [6] and [7], we are able to prove the following result (see [5, Theorem B]).

Theorem 2. *The groups* $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$, *and the universal covering group* $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ *of* $\mathrm{Sp}(2,\mathbb{R})$ *have property* (T^*).

Property (T^*) satisfies certain permanence properties. One of the essential ones for us is that whenever $\pi\colon H \to G$ is a continuous homomorphism between locally compact groups with dense image and H has property (T^*), then G has property (T^*). Using Theorem 2 and this permanence property, we are able to prove the following theorem (see [5, Theorem C]), which gives a complete characterization of connected Lie groups with the AP. The statement of the theorem uses the Levi decomposition of connected Lie groups, asserting that any connected Lie group G admits a decomposition $G = RS$, where R is a solvable closed normal subgroup of G and S is a semisimple subgroup of G. Note that the subgroup S need not be closed. Recall that a connected semisimple Lie group is always locally isomorphic to a Lie group of the form $S_1 \times \ldots \times S_n$, where the S_i's are connected simple Lie groups.

Theorem 3. *Let G be a connected Lie group, let $G = RS$ be a Levi decomposition, and assume that S is locally isomorphic to the direct product $S_1 \times \cdots \times S_n$ of connected simple factors. Then the following are equivalent:*

(i) the group G has the AP,
(ii) the group S has the AP,
(iii) the groups S_i, where $i = 1, \ldots, n$, have the AP,
(iv) the real rank of the groups S_i, where $i = 1, \ldots, n$, is 0 *or* 1.

The point where property (T^*) is crucial in the proof of the theorem, is in the fact that if G (or S) contains a subgroup that is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, i.e., a subgroup locally isomorphic to a group with property (T^*), then we can use the aforementioned permanence property of property (T^*) in order to obtain a non-compact closed subgroup with property (T^*). Because of the fact that property (T^*) is an obstruction to the AP, this closed subgroup does not have the AP.

It turns out that we can actually generalize Theorem 2 to connected simple higher rank Lie groups with finite center (see [5, Theorem D]).

Theorem 4. *Let G be a connected simple Lie group with real rank at least* 2 *and finite center. Then G has property* (T^*).

We expect that this theorem is also true without the finite center condition.

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Approximation properties for group non-commutative L^p spaces

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(joint work with Tim de Laat, Vincent Lafforgue)

To every von Neumann algebra $\mathcal{M}$ one associates, following Segal and Dixmier (in the tracial case) or Haagerup (general case) a family of non-commutative L^p spaces $L^p(\mathcal{M})$ for $0 < p \leq \infty$. Such a non-commutative L^p space is said to have the completely bounded approximation property (CBAP) if there is a net of finite rank maps $T_\alpha \colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ such that $T_\alpha(x) \to x$ for all $x \in L^p(\mathcal{M})$ and $\sup_\alpha \|T_\alpha\|_{cb} < \infty$, where the completely bounded (cb) norm of T_α is

$$\|T_\alpha\|_{cb} = \sup_n \|T_\alpha \otimes \mathrm{id}\|_{L^p(\mathcal{M}\otimes M_n(\mathbf{C}))\to L^p(\mathcal{M}\otimes M_n(\mathbf{C}))}.$$

An invariant of $\mathcal{M}$ is given by the set of values of $p \in [1,\infty]$ such that $L^p(\mathcal{M})$ has the completely bounded approximation property. So far this invariant has not been so useful because in all cases when it has been computed, it is either (1) $[1,\infty]$, or (2) $(1,\infty)$ or (3) $\{2\}$. For example, among discrete group von Neumann algebras, the von Neumann algebras of weakly amenable groups (e.g. hyperbolic groups) fall into case (1), whereas the von Neumann algebra of $\mathrm{SL}(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$ falls into case (2), and the von Neumann algebra of grous containing $\mathrm{SL}(n,\mathbf{Z})$ for all n falls into case (3). However, there is some recent evidence that this invariant might take other values and distinguish between the von Neumann algebras of lattices in certain algebraic groups, in a similar way as the weak amenability constant allowed Cowling and Haagerup [1] to distinguish the von Neumann algebras of lattices in $\mathrm{Sp}(n,1)$ for different values of n.

The following tabular summarizes, in terms of the group G, what is known about the above invariant for the von Neumann algebra of a lattice in G. I recall that a lattice in a locally compact group G is a discrete subgroup Γ such that the quotient G/Γ has finite Haar measure. In the following $\mathbf{F}$ denotes a *nonarchimedean local field*.

G	$\{p, L^p(\mathcal{L}\Gamma)\text{has CBAP}\}$ for $\Gamma \subset G$ lattice	Reference
$\mathrm{SL}(3,\mathbf{R})$	$\subset [4/3,4]$	[6]
$\mathrm{SL}(3,\mathbf{F})$	$\subset [4/3,4]$	[6]
$\mathrm{SL}(2n+1,\mathbf{F})$	$\subset [2-\frac{2}{n+2}, 2+\frac{2}{n}]$	[6]
$\mathrm{SL}(2n+1,\mathbf{R})$	$\subset [2-\frac{2}{n+2}, 2+\frac{2}{n}]$	[4]
$\mathrm{Sp}(2,\mathbf{R})$	$\subset [\frac{10}{9}, 10]$	[2, 3]
$\mathrm{Sp}(2,\mathbf{F})$	$\subset [4/3,4]$	[7]

The first line says for example that the non-commutative L^p space of the von Neumann algebra of $\mathrm{SL}(3,\mathbf{Z})$ does not have the CBAP for $p > 4$ or $p < 4/3$, but does not say anything if $p \in [\frac{4}{3}, 4]$. Nothing is known about the reverse inclusions and it might be that for every lattice in a higher rank group and ever $p \neq 2$, $L^p(\mathcal{L}\Gamma)$ lacks the CBAP. However each interval is optimal for the proofs (which are inspired by Lafforgue's work [5]) to work.

One sees from this list that for $\mathrm{SL}(n)$, although the proofs in the real and non-archimedean case are quite different, the numerology at the end is the same. By the forthcoming work [7], this is no longer the case for $\mathrm{Sp}(2)$. I wonder whether there is anything to say about this spontaneous symmetry breaking.

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Toposes in Noncommutative geometry

Simon Henry

Toposes are a generalization of ordinary topological spaces introduced by Grothendieck in order to define cohomology theories attached to algebraic varieties. One now knows how to attach interesting toposes to a large variety of geometric objects (foliation, dynamical system, topological groupoids, algebraic varieties, etc.). On the other hand, C*-algebras are also objects that one want to think of as “generalized” (locally compact) topological spaces, and it is very natural to wonder whether those two generalizations of topology are related or not. In this talk I tried to give an intuition of what is a topos, explain how C*-algebras and Von Neumann algebras can be attached to a topos and show that how to construct C*-algebras from geometric data (foliations, groupoids, graphs, semi-group actions, etc.) can be seen as a special case of the C*-algebra attached to a topos.

For a technical and precise introduction to topos theory we refer to ([3], [2]). Here we will try to give a more intuitive explanation. The starting point is the concept of a sheaf of sets over a topological space X, intuitively it is a continuous family $\mathcal{F}_x$ of sets indexed by $x \in X$. There are two equivalent definitions: a sheaf $\mathcal{F}$ can either be defined as a topological space $Y = \text{Et}\,\mathcal{F}$ together with a map $p : Y \to X$ which is locale homomorphism (the set $\mathcal{F}_x$ for $x \in X$ is then just $p^{-1}(\{x\})$), or it can be defined as the data of a set $\mathcal{F}(U)$ of so called “sections of $\mathcal{F}$ over U” for each open subset $U \subset X$, together with a compatible restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ when $V \subset U$ and such that when (U_i) is an open covering of U the restriction maps induce a bijection between $\mathcal{F}(U)$ and the subset of $\prod_i \mathcal{F}(U_i)$ of (s_i) such that for any i, j the restriction of s_i and s_j on $U_i \wedge U_j$ coincide. Those two definitions determine equivalent categories of sheaves over X: in the first case morphisms are continuous map over X and in the second case a family of maps $f_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ compatible with the restriction maps.

The first definition makes apparent that a sheaf is a sort of family of “generalized” open subsets: it is locally an open subset of X and globally it is some sort of gluing of open subsets along other open subsets, moreover, if $U \subset X$ is an open subset then the map $U \to X$ is a locale homeomorphism hence U can be seen as a special kind of sheaf.

One way to see topos theory is that it replaces open subsets by sheaves, which are more flexible: while a topological space is defined by giving the set of all its open subsets (or a basis of open subsets) a topos is defined instead by specifying the category of all sheaves over it, or a “basis” of sheaves. This gives two equivalent definitions of a topos : either a list of properties that the given category “of sheaves” should satisfy (the Giraud’s axioms) or as the category of sheaves over a Grothendieck site (which is the analogue of a basis). See [2] or [3] for precise definitions. So a topos is a “nice” category, but this not how we want to think about it: in the same way that we will never think of a topological space as the ordered set of its open subset, a topos is intuitively a geometric object and the corresponding category used to describe it is thought of as the category of sheaves over the topos.

A large part of what can be done with topological spaces can also be done with toposes: there is a good notion of “continuous map” between two toposes (called geometric morphism), of point of topos (it is a geometric morphism to the topos of sets, corresponding to the one point space) we know what a compact topos is, a proper map between toposes, an open map between toposes, we can attach homology and cohomology groups to topos or define the covering dimension or the homological dimension of a topos etc... Some of these notions are defined in [3] or in the encyclopedia [1].

Let us now give some examples of toposes:

- If X is a topological space, then it can be seen as a topos. Concretely, it is the category $\mathrm{Sh}(X)$ of sheaves over X.
- One can define a topos attached to a measured space such that continuous function over it are exactly measurable function up to equality almost everywhere (sheaves over it can be thought as measurable families of sets defined almost everywhere).
- If X is any topological space (or in fact any topos) and G is a group (preferably discrete) acting continuously on X then one can define a topos $X \rtimes G$ whose object are G-equivariant sheaves over X. If the action of G is free and proper then this topos is isomorphic to the topos of sheaves over the quotient. In general, its points are the orbits of the action of G on X (with the isotropy group as automorphism of the points) but it has a way finer structure than the mere topological quotient (see the next example). Applying this to the case where X is the topos described above on gets also a topos naturally attached to any measurable dynamical system.
- The previous construction generalizes to any topological groupoids (preferably etale). For etale groupoids the isomorphism of the topos is equivalent to the notion of equivalence of groupoids. One can reconstruct the groupoid C^*-algebras up to Morita equivalence from the topos. As a special case one can define toposes attached to a foliation whose objects are sheaves over the base manifold which are leaf-wise locally constant.
- One can also attach a topos to a graph and reconstruct the graph C*-algebras from this topos, and attach a topos to a semi-group or a semi-group action which is related to the corresponding algebras.
- There is a long list of toposes coming from algebraic geometry attached to a scheme: etale, Zariski, Nisnevich, crystalline...

In order to attach a C*-algebra to a topos, the first idea is the following: there is a good notion of continuous fields of Hilbert spaces over a topos (as well as continuous and semi-continuous fields of Banach spaces and C*-algebras). Moreover if $\mathcal{T}$ is any topos one can define a C*-category $\mathcal{H}(\mathcal{T})$ of continuous fields of Hilbert spaces over $\mathcal{T}$. This attaches to any topos a very natural C*-category hence a familly of C*-algebras all related to each other by Hilbert bi-modules. But those C*-algebras, although closely related, are not exactly the ones that we want to attach to the example mentioned above, we need to select nice sub-algebras of these algebras. One way to do that is by using the following theorem:

Theorem ([5]) **:** *Let $\mathcal{T}$ be a separated, locally compact, locally decidable topos then for any semi-continuous fields $\mathcal{C}$ of C^*-algebras over $\mathcal{T}$ there is a C^*-algebra $\mathcal{C} \rtimes \mathcal{T}$ such that the category of Hilbert $\mathcal{C} \rtimes \mathcal{T}$-modules is equivalent to the category of continuous fields of Hilbert $\mathcal{C}$-module over $\mathcal{T}$.*

The algebra $\mathcal{C} \rtimes \mathcal{T}$ is well-defined up to Morita equivalence, and in particular for such a topos one has an algebra $\mathrm{C}^*(\mathcal{T}) = \mathbb{C} \rtimes \mathcal{T}$ (well-defined up to Morita equivalence) whose category of Hilbert modules is equivalent to the category $\mathcal{H}(\mathcal{T})$ of continuous fields of Hilbert spaces over $\mathcal{T}$.

The hypothesis "separated" of the theorem is very restrictive: it corresponds to the idea of a proper groupoid or a proper group action in the case of the example above, and the theorem itself is hence just a topos theoretic form of the Green-Julg theorem. The trick is that all the example mentioned above satisfies the hypothesis of the theorem "locally" (in a topos theoretical sense). A topos which satisfies the theorem locally can always be written as an etale groupoid whose spaces of morphisms and of objects are toposes which satisfies the hypothesis of the theorem. One can then attach to all of these toposes a reduced and a maximal algebra by a modified version of the construction of the convolution algebra of an etale groupoid, using the algebra constructed by our theorem instead of the algebra of continuous functions. One can then prove that both the reduced and the maximal algebra attached to such a topos are well defined up to Morita equivalence, I am still looking for better universal characterizations of these algebras.

There is also an alternative approach, attaching Von Neuman algebras to a topos, which would behave like enveloping algebras and which play the role of the algebra of "measurable functions", see [4].

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Local spectral gap in simple Lie groups

Adrian Ioana

(joint work with Rémi Boutonnet and Alireza Salehi-Golsefidy)

Let G be a second countable locally compact group and $\Gamma < G$ be a countable dense subgroup. Then the *left translation* action $\Gamma \curvearrowright G$ given by $g \cdot x = gx$ preserves any left Haar measure m_G of G, and is free and ergodic.

Assume that G is compact. Then m_G can be taken to be a probability measure. A question that has received a lot of attention is whether the translation action $\Gamma \curvearrowright (G, m_G)$ has spectral gap? In other words, is there a finite symmetric set

$S \subseteq \Gamma$ such that the averaging operator $P : L^2(G) \to L^2(G)$ given by

$$P(\xi) = \frac{1}{|S|} \sum_{g \in S} g \cdot \xi$$

has a gap in its spectrum right below 1, i.e. $\sigma(P) \cap [1-\kappa, 1) = \emptyset$, for some $\kappa > 0$.

This question first attracted interest in the 1980's in connection with a classical problem of Ruziewicz. The latter asks whether the Lebesgue measure is the unique finitely additive measure on the n-dimensional sphere S^n which is invariant under rotations and is defined on all Lebesgue measurable sets. In 1923, Banach showed that the answer is negative if $n = 1$ [Ba23]. Strikingly, 50 years later, Margulis, Sullivan, and Drinfeld showed that the answer is positive for every $n \geq 2$ [Ma80, Su81, Dr84]. They achieved this by providing a countable dense subgroup Γ of $G := SO(n+1)$ such that the left translation action $\Gamma \curvearrowright G$ has spectral gap.

In recent years, this has been vastly generalized by Bourgain and Gamburd [BG06, BG10], and Benoist and de Saxcé [BdS14]. These works culminated in [BdS14, Theorem 1.2] which shows that $\Gamma \curvearrowright G$ has spectral gap, whenever G is a simple connected compact Lie group and Γ is a dense subgroup of G generated by matrices with algebraic entries.

One of the main motivations of our work [BISG15] is to formulate and prove an analogue of this result that applies to general (not necessarily compact) simple connected Lie groups G. The starting point is therefore to find an appropriate notion of spectral gap for infinite measure preserving actions.

If G is compact, then a left translation action $\Gamma \curvearrowright G$ has spectral gap iff there is no sequence of unit vectors $\xi_n \in L^2(G)$ which have mean zero and are almost invariant: $\|g \cdot \xi_n - \xi_n\|_2 \to 0$, for every $g \in G$. In [BISG15], we introduced a "local" version of spectral gap in the case G is locally compact, but not compact.

More precisely, let $B \subseteq G$ be a measurable set with non-empty interior and compact closure (e.g. let B be a ball in G). We say that a left translation action $\Gamma \curvearrowright G$ has *local spectral gap* if there is no sequence of vectors $\xi_n \in L^2(G)$ which have mean zero on B and are almost invariant on B: $\|g \cdot \xi_n - \xi_n\|_{2,B} \to 0$, for every $g \in G$. Here, we denote by $\|\xi\|_{2,B}$ the 2-norm of the restriction of ξ to B.

Theorem (see [BISG15, Theorem A]) *Let G be a connected simple Lie group. Denote by $\mathfrak{g}$ the Lie algebra of G and by Ad: $G \to GL(\mathfrak{g})$ its adjoint representation. Let $\Gamma < G$ be a dense subgroup. Assume that there is a basis $\mathfrak{B}$ of $\mathfrak{g}$ such that the matrix of $Ad(g)$ in the basis $\mathfrak{B}$ has algebraic entries, for any $g \in \Gamma$.*

Then the left translation action $\Gamma \curvearrowright (G, m_G)$ has local spectral gap.

This theorem generalizes all known results in the compact case, and is entirely new in the non-compact case. Moreover, this result and its proof lead to several novel applications, which we describe below.

Firstly, we deduce that the Haar measure m_G of G is, up to a multiplicative constant, the unique finitely additive measure which is Γ-invariant and is defined on all bounded measurable subsets of G. This provides a uniqueness characterization of the Haar measure of simple Lie groups as a finitely additive measure, in the spirit of the classical Banach-Ruziewicz problem.

Secondly, in combination with results from [Io14], we obtain new rigidity results for orbit equivalence of group actions. More precisely, assume that G has trivial center, let H be any connected Lie group with trivial center, and $\Lambda < H$ be any countable dense subgroup. We show that the left translation actions $\Gamma \curvearrowright G$ and $\Lambda \curvearrowright H$ are orbit equivalent iff there exists an isomorphism $\delta : G \to H$ such that $\delta(\Gamma) = \Lambda$.

Thirdly, the proof of the above theorem sheds some new light on the spectra of averaging operators on compact groups. Assume for simplicity that $G = SU(2)$. Let $P : L^2(G) \to L^2(G)$ be an averaging operator given by $P(\xi) = \frac{1}{|S|}\sum_{g\in S} g \cdot \xi$, for a finite symmetric set $S \subseteq \Gamma$. Then $\lambda_0 = 1$ is an eigenvalue of P. As explained above, in many cases it is now known that there is a gap in the spectrum of P right below 1. In other words, $\lambda_1 := \sup(\sigma(P) \setminus \{1\})$ satisfies $\lambda_1 < 1$. Moreover, as first noticed in [LPS86], "most" of the eigenvalues of P lie in the interval $[-\rho, \rho]$, where $\rho = \frac{\sqrt{2|S|-1}}{|S|}$.

However, it was an open problem whether λ_1 could ever be an eigenvalue of P. As a consequence of [BISG15, Theorem B], we were able to show that there are averaging operators P on G such that λ_1 *is* an eigenvalue, and moreover such that there is a gap in the spectrum of P right below λ_1.

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Quasidiagonality, unique ergodicty, and crossed products

David Kerr

In [2] Rosenberg showed that, for a countable discrete group G, if the reduced group C*-algebra $C^*_\lambda(G)$ is quasidiagonal then G is amenable. Whether the converse is true has remained an open problem ("Rosenberg's conjecture"). Recently Ozawa, Rørdam, and Sato succeeded in applying results from the classification program for simple separable nuclear C*-algebras to deduce that $C^*_\lambda(G)$ is quasidiagonal for every elementary amenable group G [3]. Their approach avoids a direct analysis of $C^*_\lambda(G)$ itself and proceeds by proving, using a bootstrap argument, that when G is elementary amenable the crossed product $(M_2^{\otimes\mathbb{N}})^{\otimes G} \rtimes G$ by the shift action on the CAR algebra is quasidiagonal, which then yields the quasidiagonlity of $C^*_\lambda(G)$ as an immediate corollary. The crossed product $(M_2^{\otimes\mathbb{N}})^{\otimes G} \rtimes G$ is simple and monotracial and hence is amenable to a combination of structure and classification results due to Matui-Sato, Winter, Lin-Niu, and Matui which allow one to conclude that quasidiagonality for such crossed products is preserved under extensions by $\mathbb{Z}$, which is the most difficult part of the proof.

The motivation for the present work is the question of whether one can replace the noncommutative shift $G \curvearrowright (M_2^{\otimes\mathbb{N}})^{\otimes G}$ with a free continuous action $G \curvearrowright X$ on a compact space. In order to carry out the Ozawa-Rørdam-Sato bootstrap argument, such an action must not only be strictly ergodic, but its restriction to each subgroup in a transfinite chain witnessing elementary amenability must also be strictly ergodic. This will ensure a unique tracial state at each stage of the transfinite recursion, which is automatic for the above noncommutative shift and is necessary in order to be able to apply classification results as in [3]. To show that such actions exists we apply a recent tiling theorem for amenable groups due to Downarowicz, Huczek, and Zhang [1] to establish the following version of the Jewett-Krieger theorem. In [5] Weiss introduced a method for extending the original Jewett-Krieger theorem for $\mathbb{Z}$-actions to actions of other amenable groups, and a treatment of the general countable amenable case using this approach is given in [4]. What is different here is the extra condition involving subgroups and the use of exact tilings.

Theorem 1. *Let $G \curvearrowright (X,\mu)$ be a free ergodic probability-measure-preserving action of a countable amenable discrete group, and let $\{H_i\}$ be a countable totally ordered collection of subgroups of G such that each restriction $H_i \curvearrowright (X,\mu)$ is ergodic. Then there is a minimal continuous action $G \curvearrowright Y$ on the Cantor set and a unique G-invariant Borel probability measure ν on Y such that $G \curvearrowright (X,\mu)$ and $G \curvearrowright (Y,\nu)$ are measure conjugate and each restriction $H_i \curvearrowright Y$ is strictly ergodic.*

This ends up giving us, for every countable elementary amenable group G, a wealth of continuous actions $G \curvearrowright X$ on the Cantor set whose crossed product is quasidiagonal and monotracial. We furthermore note that if such crossed product has finite nuclear dimension then it will fall within the scope of classification theorems based on the Elliott invariant.

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The resolvent expansion for second order elliptic differential multipliers

Matthias Lesch
(joint work with Henri Moscovici)

This is a report on the more technical aspects of the recent preprint [7]. We develop a complete asymptotic expansion of the heat resp. resolvent trace of Laplace type operators on vector bundles over the noncommutative torus (Heisenberg modules). Moreover we compute the second coefficient. The second coefficient contains significant geometric information, as in the case of classical Riemann surfaces. As discovered in [3] the noncommutativity of the symbol exhibits a completely new phenomenon: namely, the appearance of universal entire functions in the expression for the second heat coefficients. This has no counterpart in the commutative situation.

The main technical device which we are going to develop is a pseudodifferential calculus adapted to twisted C^*-dynamical systems, extending the well-known calculi due to Connes [2] and Baaj [1].

Heisenberg modules on the noncommutative torus. Connes and Rieffel [2], [4] gave a very beautiful description of the projective modules over the noncommutative torus. As usual, for a real number θ, we denote by A_θ the C^*–algebra generated by two unitaries $U_j, j = 1, 2$, subject to the commutation relation $U_2 U_1 = e^{2\pi i\theta} U_1 U_2$. The smooth structure is given by the subalgebra $\mathscr{A}_\theta \subset A_\theta$ consisting of the smooth elements w.r.t. to the natural $\mathbb{R}^2$–action on A_θ *i.e.*, of those $a = \sum_{k,l\in\mathbb{Z}} a_{k,l} U_1^k U_2^l \in A_\theta$ such that the sequence $\{a_{k,l}\} \subset \mathbb{C}$ is rapidly decreasing.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and let $\mathscr{E}(g,\theta) := \mathscr{S}(\mathbb{R})^{|c|} \equiv \mathscr{S}(\mathbb{R}\times\mathbb{Z}_c), \mathbb{Z}_c := \mathbb{Z}/c\mathbb{Z}$. Then $\mathscr{E}(g,\theta)$ has a natural $\mathscr{A}'_\theta - \mathscr{A}_\theta$ bimodule structure, $\theta' = g\theta := \frac{a\theta+b}{c\theta+d}$.

Furthermore, the three dimensional Heisenberg group acts on $\mathscr{E}(g,\theta)$ and hence $\mathbb{R}^2$ (its quotient modulo its center) acts projectively on $\mathscr{E}(g,\theta)$. This projective action is compatible with the natural $\mathbb{R}^2$–actions on $\mathscr{A}_\theta$ resp. $\mathscr{A}'_\theta$. Hence, there is a standard connection on $\mathscr{E}(g,\theta)$ compatible with the basic derivatives on $\mathscr{A}_\theta, \mathscr{A}_{\theta'}$.

Explicitly, $(\nabla_1 f)(t,\alpha) := \frac{\partial}{\partial t} f(t,\alpha)$, $(\nabla_2 f)(t,\alpha) := \frac{2\pi i}{\theta + d/c} \cdot t \cdot f(t,\alpha)$. This connection satisfies the *Heisenberg commutation relation* $[\nabla_1, \nabla_2] = \frac{2\pi i}{\theta+d/c} \cdot \mathrm{Id}$, hence it is a connection of constant curvature.

Twisted pseudodifferential multipliers. The action of the Heisenberg group on $\mathscr{E}(g,\theta)$ induces a C^*–dynamical system $(\mathscr{A}, \mathbb{R}^{n=2}, \alpha)$ ($\mathscr{A} = \mathscr{A}_\theta$ or $\mathscr{A} = \mathscr{A}_{\theta'}$). Equivalently, $\mathbb{R}^n$ acts by a *projective* representation with cocycle $e(x,y) := e^{i\langle Bx,y\rangle}$, with a *skew* symmetric matrix $B = (b_{kl})_{k,l=1}^n$.

$\mathscr{S}(\mathbb{R}^n, \mathscr{A}^\infty)$ is a pre-C^*–module with inner product $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)^* g(x) dx$. The natural covariant representation of the C^*–dynamical system on $\mathscr{S}(\mathbb{R}^n, \mathscr{A}^\infty)$ is implemented by the projective family of unitaries $\boldsymbol{U}_x^* = \boldsymbol{U}_{-x}$, $\boldsymbol{U}_x \boldsymbol{U}_y = e(x,y)\boldsymbol{U}_{x+y}$, $x, y \in \mathbb{R}^n$, $\boldsymbol{U}_x a \boldsymbol{U}_{-x} = \alpha_x(a)$, $a \in \mathscr{A}^\infty$.

By associating to $f \in \mathscr{S}(\mathbb{R}^n, \mathscr{A}^\infty)$ the multiplier $M_f = \int_{\mathbb{R}^n} f(x)\boldsymbol{U}_x dx$ the space $\mathscr{S}(\mathbb{R}^n, \mathscr{A}^\infty)$ becomes a $*$–algebra. Putting $P_f := M_{f^\vee}$ and allowing f to be a symbol of Hörmander class $\mathrm{S}^m(\mathbb{R}$ we obtain a class of multipliers extending the pseudodifferential multipliers à la Connes and Baaj. Essentially, the usual rules of calculus remain valid. However, the formula for the symbol of a composition is slightly more complicated due to the twisting.

As usual *differential* multipliers are defined as those pseudodifferential multiplierts having a symbol which is polynomial in the ξ–variable. The basic derivatives are defined by $\boldsymbol{\partial}^\gamma := P_{\xi^\gamma}$. It is important to note that due to the twisting in general $\boldsymbol{\partial}^\gamma \boldsymbol{\partial}^{\gamma'} \neq \boldsymbol{\partial}^{\gamma+\gamma'}$.

In dimension $n = 2$ the only invariant is the entry b_{12} of the twisting matrix B. Fixing $\tau \in \mathbb{C}$ with $\Im\tau > 0$ (a complex structure!) we have the following basic differential multipliers:

$$\begin{aligned} \boldsymbol{\partial}_{\boldsymbol{\tau}} &:= \boldsymbol{\partial}_1 + \overline{\tau}\boldsymbol{\partial}_2, \quad \boldsymbol{\partial}_{\boldsymbol{\tau}}^* = \boldsymbol{\partial}_1 + \tau\boldsymbol{\partial}_2, \quad \boldsymbol{\partial}_1 := \boldsymbol{\partial}^{1,0}, \boldsymbol{\partial}_2 := \boldsymbol{\partial}^{0,1} \\ [\boldsymbol{\partial}_{\boldsymbol{\tau}}, \boldsymbol{\partial}_{\boldsymbol{\tau}}^*] &= -4 \cdot \Im\tau \cdot b_{12} =: c_\tau, \\ \triangle_\tau &:= \frac{1}{2}(\boldsymbol{\partial}_{\boldsymbol{\tau}}^*\boldsymbol{\partial}_{\boldsymbol{\tau}} + \boldsymbol{\partial}_{\boldsymbol{\tau}}\boldsymbol{\partial}_{\boldsymbol{\tau}}^*) = \boldsymbol{\partial}_1^2 + |\tau|^2\boldsymbol{\partial}_2^2 + \Re\tau(\boldsymbol{\partial}_1\boldsymbol{\partial}_2 + \boldsymbol{\partial}_2\boldsymbol{\partial}_1). \end{aligned}$$

We will first analyze these operators acting as multipliers on the Hilbert module completion of $\mathscr{S}(\mathbb{R}^n, \mathscr{A}^\infty)$. Lateron we will have to pass to their concrete counterparts acting on the Heisenberg modules.

The resolvent expansion for second order elliptic differential multipliers. We consider the differential multiplier $P = P_{\varepsilon_1,\varepsilon_2} := k^2 \triangle_\tau + \varepsilon_1(\partial_\tau k^2)\boldsymbol{\partial}_{\boldsymbol{\tau}}^* + \varepsilon_2(\partial_\tau^* k^2)\boldsymbol{\partial}_{\boldsymbol{\tau}} + a_0$, where $a_0 \in \mathscr{A}$ and $\varepsilon_1, \varepsilon_2$ are real parameters. This multiplier contains all conformal Laplace type multipliers, which occur on Heisenberg modules over noncommutative tori, as special cases.

We want to compute the first three terms in the expansion of the resolvent $(P - \lambda)^{-1}$ in the parameter dependent pseudodifferential calculus.[1] The symbol

[1]Note that heat/resolvent invariants are enumerated from 0. We are after a_2 which is the second nontrivial heat invariant, as a_1 is always 0 for differential operators, but in the counting of the recursion system it is the third term.

of P takes the form $\sigma_P(\xi) := a_2(\xi) + a_1(\xi) + a_0$, where $a_0 \in \mathscr{A}^\infty$ is the same as above and

$$\begin{aligned} a_2(\xi) &= k^2|\xi_1 + \overline{\tau}\xi_2|^2 =: k^2|\eta|^2, \\ a_1(\xi) &= \varepsilon_1(\partial_\tau k^2)\overline{\eta} + \varepsilon_2(\partial_\tau^* k^2)\eta, && \eta := \xi_1 + \overline{\tau}\xi_2, \\ &=: \varrho_1\overline{\eta} + \varrho_2\eta, && \varrho_1 := \varepsilon_1\partial_\tau k^2, \varrho_2 := \varepsilon_2\partial_\tau^* k^2. \end{aligned}$$

Theorem 1. $(P-\lambda)^{-1}$ *is a* parameter dependent pseudodifferential *multiplier with polyhomogeneous symbol* $b_{-2} + b_{-3} + b_{-4} + \ldots$. *Up to a function of total* ξ*–integral* 0 *we have the following closed formulas for the first three terms in the symbol expansion of* $(P-\lambda)^{-1}$*:*

$$\begin{aligned} b_{-2} &= b = (k^2|\eta|^2 - \lambda)^{-1}, \quad b_{-3} = -bk^2\big(\eta\partial_\tau^* + \overline{\eta}\partial_\tau\big)b - ba_1 b, \\ b_{-4} &= \big(2bk^2|\eta|^2 - 1 - \varepsilon_1 - \varepsilon_2\big)bk^2\triangle_\tau b + \lambda bk^2\big((\partial_\tau^* b)(\partial_\tau b) + (\partial_\tau b)(\partial_\tau^* b)\big) \\ &\quad + \varepsilon_1 \cdot \lambda b(\partial_\tau k^2)b\partial_\tau^* b + \varepsilon_2 \cdot \lambda b(\partial_\tau^* k^2)b\partial_\tau b \\ &\quad + \varepsilon_1\varepsilon_2 \cdot |\eta|^2 b \cdot \big((\partial_\tau k^2)b(\partial_\tau^* k^2) + (\partial_\tau^* k^2)b\partial_\tau k^2\big) \cdot b - ba_0 b. \end{aligned}$$

These concise closed formulas should be compared to the somewhat lengthy earlier computer calculations, cf., e.g., [5].

Theorem 2 (Second heat coefficient in terms of $\log k^2$)**.** *There exist entire functions* $K(s), H^{\Re}(s,t), H^{\Im}(s,t)$*, such that with* $h := \log k^2$ *the second heat coefficient of* P *(w.r.t. the natural dual trace on the twisted crossed product) takes the form*

$$a_2(P,a) = \frac{1}{4\pi|\Im\tau|}\varphi_0\Bigg[a\Big)K(\nabla)(\triangle_\tau h) - k^{-2}a_0 + H^{\Re}(\nabla^{(1)},\nabla^{(2)})\big(\square^{\Re}(h)\big) + H^{\Im}(\nabla^{(1)},\nabla^{(2)})\big(\square^{\Im}(h)\big)\Big)\Bigg].$$

Here, $\square^{\Re/\Im}(h) := \frac{1}{2}\big(\partial_\tau h \cdot \partial_\tau^* \pm \partial_\tau^* h \cdot \partial_\tau h\big)$, $\nabla = -\operatorname{ad}(h)$*, and* $\nabla^{(i)}$ *signifies that it acts on the i-th factor (cf.* [3], [6]*).*

The functions $K, H^{\Re}, H^{\Im}$ *depend only on* P *but not on* τ*. They can naturally be expressed in terms of simple divided divided differences of* log.

Effective pseudodifferential operators and resolvent. The effective implementation of the pseudodifferential calculus amounts to passing from its realization on multipliers to a direct action on projective representation spaces (Heisenberg modules or on $L^2(\mathscr{A},\varphi_0)$ itself). More concretely, let $\boldsymbol{\pi} : G \to \mathcal{L}(\mathcal{H})$ be a projective unitary representation of $G = \mathbb{R}^n \times (\mathbb{R}^n)^\wedge$. For a symbol $f \in \mathrm{S}^m(\mathbb{R}$ the assignment $\mathrm{S}^m(\mathbb{R} \ni f \mapsto \mathrm{Op}(f) := \int_G f^\vee(y)\boldsymbol{\pi}(y)dy$ represents pseudodifferential multipliers as concrete operators in $\mathcal{H}$.

By exploiting the representation theory of the Heisenberg group we are able to relate the Hilbert space trace of parameter dependent pseudodifferential operators to the trace of the corresponding multiplier acting on $\mathscr{S}(\mathbb{R}^n,\mathscr{A}^\infty)$. For the class of operators in the Theorems 1 and 2 we prove full heat trace asymptotics and identify the second heat coefficient in terms of the expressions above and numerical invariants of the representation.

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Sofic mean length

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(joint work with Bingbing Liang)

Let R be a unital ring. By *a length function* L on (left) R-modules [13] we mean associating a value $\mathrm{L}(\mathscr{M}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for each R-module $\mathscr{M}$ such that the following conditions are satisfied: (1) $\mathrm{L}(0) = 0$; (2) (additivity) for any short exact sequence $0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$ of R-modules, one has $\mathrm{L}(\mathscr{M}_2) = \mathrm{L}(\mathscr{M}_1) + \mathrm{L}(\mathscr{M}_3)$; (3) (upper continuity) for any R-module $\mathscr{M}$, one has $\mathrm{L}(\mathscr{M}) = \sup_{\mathscr{N}} \mathrm{L}(\mathscr{N})$, for $\mathscr{N}$ ranging over all finitely generated R-submodules of $\mathscr{M}$.

A countable discrete group Γ is called *sofic* [4, 16] if there is a sequence of maps $\Sigma = \{\sigma_i : \Gamma \to S_{d_i}\}_{i \in \mathbb{N}}$, where $d_i \in \mathbb{N}$ and S_{d_i} denotes the permutation group of $[d_i] := \{1, \dots, d_i\}$, such that

(1) for any $s, t \in \Gamma$, $\lim_{i\to\infty} \frac{|\{v \in [d_i] : \sigma_{i,s}\sigma_{i,t}(v) = \sigma_{i,st}(v)\}|}{d_i} = 1$;

(2) for any $s \neq t \in \Gamma$, $\lim_{i\to\infty} \frac{|\{v \in [d_i] : \sigma_{i,s}(v) \neq \sigma_{i,t}(v)\}|}{d_i} = 1$.

When such a sequence exists, one can always find a sequence Σ satisfying the further requirement that $\lim_{i\to\infty} d_i = +\infty$. Then Σ is called a *sofic approximation sequence* for Γ. All amenable groups and residually finite groups are sofic, and it is an open question whether every group is sofic or not.

Fix a ring R with a length function L on (left) R-modules, and a countable sofic group Γ. A R-module $\mathscr{M}$ is called *locally* L*-finite* if $\mathrm{L}(Rx) < \infty$ for all $x \in \mathscr{M}$. When Γ is amenable, one can define a length function mL on locally L-finite $R\Gamma$-modules such that $\mathrm{mL}(R\Gamma \otimes_R \mathscr{M}) = \mathrm{L}(\mathscr{M})$ for all locally L-finite R-modules $\mathscr{M}$

[9, 15]. However, in general this is impossible for the free group $\mathbb{F}_2$ with two generators, since $R\mathbb{F}_2 \oplus R\mathbb{F}_2$ is isomorphic to a R-submodule of $R\mathbb{F}_2$ [14].

Fix a sofic approximation sequence Σ for Γ, and a free ultrafilter ω on $\mathbb{N}$. Let $\mathcal{M}$ be a (left) $R\Gamma$-module. Let $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\mathcal{M})$, $F \in \mathcal{F}(\Gamma)$, and σ be a map $\Gamma \to S_d$ for some $d \in \mathbb{N}$, where $\mathscr{F}(\mathcal{M})$ denotes the set of all finitely generated R-submodules of $\mathcal{M}$ and $\mathcal{F}(\Gamma)$ denotes the set of all finite subsets of Γ. For any $x \in \mathcal{M}$ and $v \in [d]$, denote by $\delta_v x$ the element of $\mathcal{M}^d$ taking value x at v and 0 everywhere else. Denote by $\mathscr{M}(\mathscr{B}, F, \sigma)$ the R-submodule of $\mathcal{M}^d$ generated by the elements $\delta_v b - \delta_{sv} sb$ for all $v \in [d], b \in \mathscr{B}$ and $s \in F$, and by $\mathscr{M}(\mathscr{A}, \mathscr{B}, F, \sigma)$ the image of $\mathscr{A}^d$ in $\mathcal{M}^d/\mathscr{M}(\mathscr{B}, F, \sigma)$ under the quotient map $\mathcal{M}^d \to \mathcal{M}^d/\mathscr{M}(\mathscr{B}, F, \sigma)$.

Definition 1. *For any locally* L*-finite* $R\Gamma$*-modules* $\mathcal{M}_1 \subseteq \mathcal{M}_2$*, we define the mean length of* $\mathcal{M}_1$ *relative to* $\mathcal{M}_2$ *as*

$$\mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_1|\mathcal{M}_2) = \sup_{\mathscr{A} \in \mathscr{F}(\mathcal{M}_1)} \inf_{\mathscr{B} \in \mathscr{F}(\mathcal{M}_2)} \inf_{F \in \mathcal{F}(\Gamma)} \lim_{i \to \omega} \frac{\mathrm{L}(\mathscr{M}(\mathscr{A}, \mathscr{B}, F, \sigma_i))}{d_i}.$$

The mean length of $\mathcal{M}_2$ *is defined as* $\mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_2) := \mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_2|\mathcal{M}_2)$.

Our main result is the following addition formula:

Theorem 1. *Let* $\mathcal{M}_1 \subseteq \mathcal{M}_2$ *be locally* L*-finite* $R\Gamma$*-modules. Then*

$$\mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_2) = \mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_1|\mathcal{M}_2) + \mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_2/\mathcal{M}_1).$$

For any $R\Gamma$-module $\mathcal{M}$, any $\mathscr{A} \in \mathscr{F}(\mathcal{M})$, and any $F \in \mathcal{F}(\Gamma)$, we set $\mathscr{A}^F = \sum_{t \in F^{-1}} t\mathscr{A}$. When Γ is amenable, by the Ornstein-Weiss lemma, the limit $\lim_F \frac{\mathrm{L}(\mathscr{A}^F)}{|F|}$ as $F \in \mathcal{F}(\Gamma)$ becomes more and more left invariant exists, which we denote by $\mathrm{mL}(\mathscr{A})$. Then we set $\mathrm{mL}(\mathcal{M}) := \sup_{\mathscr{A} \in \mathscr{F}(\mathcal{M})} \mathrm{mL}(\mathscr{A})$ as in [9, 15]. The following result says that Definition 1 extends with the definitions of mean length for $R\Gamma$-modules in [9, 15] for amenable groups:

Theorem 2. *Suppose that* Γ *is amenable. For any locally* L*-finite* $R\Gamma$*-modules* $\mathcal{M}_1 \subseteq \mathcal{M}_2$*, we have* $\mathrm{mL}_{\Sigma,\omega}(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{mL}(\mathcal{M}_1)$.

Kaplansky's direct finiteness conjecture says that for any field R and any group Γ, $R\Gamma$ is *directly finite* in the sense that for any $a, b \in R\Gamma$ with $ab = 1$, one has $ba = 1$. This is known to be true when R is a field with characteristic 0 by Kaplansky [7], when R is a skew-field and Γ is residually amenable by Ara et al. [1], when R is a skew-field and Γ is sofic by Elek and Szabó [3], when R is an Artinian ring and Γ is sofic by Ceccherini-Silberstein and Coornaert [2], and when R is a left Noetherian ring and Γ is amenable by Virili [15]. Using sofic mean length and the method of Virili, we get

Theorem 3. *For any left Noetherian ring* R *and any sofic group* Γ*, the group ring* $R\Gamma$ *is directly finite.*

For any countable group Γ, one has the left group von Neumann algebra $\mathcal{L}\Gamma$ and its canonical trace tr. One can extend the trace tr to $M_n(\mathcal{L}\Gamma)$ for any $n \in \mathbb{N}$ by $\mathrm{tr}((a_{jk})_{1 \le j,k \le n}) = \sum_{j=1}^n \mathrm{tr}(a_{jj})$. For any finitely generated projective (left)

$\mathcal{L}\Gamma$-module $\tilde{\mathcal{M}}$, its dimension $\dim(\tilde{\mathcal{M}})$ is defined as $\mathrm{tr}(P)$ for any $P \in M_n(\mathcal{L}\Gamma)$ for some $n \in \mathbb{N}$ with $P^2 = P$ and $\tilde{\mathcal{M}} \cong (\mathcal{L}\Gamma)^n P$. For an arbitrary (left) $\mathcal{L}\Gamma$-module $\tilde{\mathcal{M}}$, its *von Neumann-Lück dimension*, denoted by $\dim(\tilde{\mathcal{M}})$, is defined as $\sup_{\tilde{\mathcal{N}}} \dim(\tilde{\mathcal{N}})$ for $\tilde{\mathcal{N}}$ ranging over all finitely generated projective submodules of $\tilde{\mathcal{M}}$ [11, 12].

Mean topological dimension for continuous actions of countable amenable groups on compact metrizable spaces was introduced by Gromov [5, 10], as a dynamical analogue of the covering dimension of compact metrizable spaces. It was extended to actions of countable sofic groups [8]. For a compact space Y and two finite open covers $\mathcal{U}$ and $\mathcal{V}$ of Y, we say that $\mathcal{V}$ *refines* $\mathcal{U}$ if every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. For a finite open cover $\mathcal{U}$ of a compact space Y, we denote

$$\mathrm{ord}(\mathcal{U}) = \max_{y\in Y} \sum_{U\in\mathcal{U}} 1_U(y) - 1, \text{ and } \mathcal{D}(\mathcal{U}) = \min_{\mathcal{V}} \mathrm{ord}(\mathcal{V}),$$

where $\mathcal{V}$ ranges over finite open covers of Y refining $\mathcal{U}$. Let α be a continuous action of a countable sofic group Γ on a compact metrizable space X. Let ρ be a compatible metric on X. Let $F \in \mathcal{F}(\Gamma)$ and $\delta > 0$. Let σ be a map from Γ to S_d for some $d \in \mathbb{N}$. We define on the set of all maps from $[d]$ to X the metric $\rho_2(\varphi, \psi) = \left(\frac{1}{d}\sum_{v\in[d]}(\rho(\varphi(v), \psi(v)))^2\right)^{1/2}$, and then define $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be the set of all maps $\varphi : [d] \to X$ such that $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$ for all $s \in F$. We consider $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be a topological space with the topology inherited from X^d. For a finite open cover $\mathcal{U}$ of X, we denote by $\mathcal{U}^d$ the finite open cover of X^d consisting of $U_1 \times U_2 \times \cdots \times U_d$ for $U_1, \ldots, U_d \in \mathcal{U}$. Consider the restriction $\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)} = \mathcal{U}^d \cap \mathrm{Map}(\rho, F, \delta, \sigma)$ of $\mathcal{U}^d$ to $\mathrm{Map}(\rho, F, \delta, \sigma)$. Denote $\mathcal{D}(\mathcal{U}^d|_{\mathrm{Map}(\rho,F,\delta,\sigma)})$ by $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)$. Fix a sofic approximation sequence Σ for Γ, and a free ultrafilter ω on $\mathbb{N}$. The *sofic mean dimension* of the action $\Gamma \curvearrowright X$ is defined as

$$\mathrm{mdim}_{\Sigma,\omega}(\Gamma \curvearrowright X) = \sup_{\mathcal{U}} \inf_{F\in\mathcal{F}(\Gamma)} \inf_{\delta>0} \lim_{i\to\omega} \frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma_i)}{d_i}$$

for $\mathcal{U}$ ranging over finite open covers of X, and is independent of the choice of ρ.

Using the sofic mean length, we have

Theorem 4. *Let $\mathcal{M}$ be a countable (left) $\mathbb{Z}\Gamma$-module for a countable sofic group Γ. Consider the induced Γ-action on the Pontrjagin dual $\widehat{\mathcal{M}}$ of the discrete abelian group $\mathcal{M}$. Then* $\mathrm{mdim}_{\Sigma,\omega}(\Gamma \curvearrowright \widehat{\mathcal{M}}) = \dim(\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{M})$.

Theorem 4 was proved for the case Γ is amenable in [9], and for the case Γ is residually finite and $\mathcal{M}$ is a finitely presented $\mathbb{Z}\Gamma$-module in [6].

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Semigroup C*-algebras, Cartan subalgebras, and continuous orbit equivalence

Xin Li

This talk was mainly a report on [5].

1. Semigroup C*-algebras

As a motivation, let us start with semigroup C*-algebras: Let P be a left cancellative semigroup. For every $p \in P$, define the isometry $V_p : \ell^2 P \to \ell^2 P$, $\delta_x \mapsto \delta_{px}$. Set $C^*_\lambda(P) := C^*(\{V_p : p \in P\}) \subseteq \mathcal{L}(\ell^2 P)$. This is the (left reduced) semigroup C*-algebra attached to P.

From now on, let us assume that P is a subsemigroup of a group, say G, i.e., we have $P \subseteq G$. In order to analyze $C^*_\lambda(P)$, the following observation is crucial: Let $D_\lambda(P) = C^*_\lambda(P) \cap \ell^\infty(P)$. $\ell^\infty(P)$ acts on $\ell^2(P)$ by multiplication operators, so we can take the intersection in $\mathcal{L}(\ell^2 P)$. It turns out that G acts partially on $D_\lambda(P)$, or if we dualize, on $X = \mathrm{Spec}(D_\lambda(P))$. We obtain a canonical isomorphism $C^*_\lambda(P) \cong C(X) \rtimes_r G$, where $\rtimes$ stands for partial crossed product. This is explained in [7].

For instance, we obtain as a consequence

Theorem 1. *Let $P \subseteq G$ be a subsemigroup of a group G. If G is amenable, then $C^*_\lambda(P)$ is nuclear.*

It is interesting to compare this with the well-known fact that subsemigroups of amenable groups do not have to be amenable.

The motivation for the main part of the talk is the isomorphism problem: Given two semigroups P_1 and P_2, what does $C^*_\lambda(P_1) \cong C^*_\lambda(P_2)$ mean for P_1 and P_2? In [2], we were able to give a complete answer for right-angled Artin monoids. In [4, 6], partial answers were obtained for $ax+b$-semigroups over rings of algebraic integers in number fields. There are other classes of semigroups, for instance affine semigroups, where this isomorphism problem has not been studied.

2. Topological dynamics and C*-algebras

The isomorphism problem can be rephrased as follows: What does reduced crossed products tell us about the underlying partial dynamical systems? Already for classical dynamical systems, this is an interesting question. Let us discuss it now. More precisely, given two topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$, what does $C_0(X) \rtimes_r G \cong C_0(Y) \rtimes_r H$ mean for the underlying systems?

In the setting of measurable dynamics and von Neumann algebras, this question has been studied a lot. And there have been breakthrough results in recent years by Sorin Popa, Stefaan Vaes, and many others. In the topological setting, however, much less is known.

Let us first discuss the topological analogue of a classical result of Singer and Feldman-Moore in the measurable setting. In the following $G \curvearrowright X$ and $H \curvearrowright Y$ are topological dynamical systems. By this, we mean that G, H are discrete and countable groups, and X, Y are locally compact second countable Hausdorff spaces. In the following, we write $G \curvearrowright X \sim_{\text{conj}} H \curvearrowright Y$ if there exists a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ and a group isomorphism $\rho : G \xrightarrow{\cong} H$ such that $\varphi(g.x) = \rho(g).\varphi(x)$ for every $g \in G$ and $x \in X$. We write $G \curvearrowright X \sim_{\text{coe}} H \curvearrowright Y$ if there exists a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ and continuous maps $a : G \times X \to H$, $b : H \times Y \to G$ such that $\varphi(g.x) = a(g,x).\varphi(x)$ and $\varphi^{-1}(h.y) = b(h,y).\varphi^{-1}(y)$ for all $g \in G$, $x \in X$, $h \in H$, $y \in Y$.

Theorem 2. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free. Then $G \curvearrowright X \sim_{coe} H \curvearrowright Y$ if and only if there is a C*-isomorphism $\Phi : C_0(X) \rtimes_r G \xrightarrow{\cong} C_0(Y) \rtimes_r H$ with $\Phi(C_0(X)) = C_0(Y)$.*

Here, $G \curvearrowright X$ is topologically free if for every $e \neq g \in G$, $\{x \in X : g.x \neq x\}$ is dense in X. In that case, $C_0(X)$ is a Cartan subalgebra in $C_0(X) \rtimes_r G$. This definition goes back to Kumjian [3] and Renault [8].

To summarize, we have the following implications: Conjugacy $\Rightarrow$ COE $\Leftrightarrow$ Cartan-isomorphism $\Rightarrow$ C*-isomorphism.

Can we reverse one of these one-sided arrows? If yes, then we speak of rigidity. Let us discuss continuous orbit equivalence rigidity (COER), i.e., the question whether we can reverse the first on-sided arrow.

We always have COER for topologically free, topologically transitive actions of $G = H = \mathbb{Z}$ on compact spaces. This is a result by Boyle and Tomiyama [1]. But

COER does not hold for more complicated groups, for instance not for $\mathbb{Z}^n$ $(n \geq 2)$, and also not for $\mathbb{F}_n$ $(n \geq 2)$. Examples are given in [5].

Here are some positive results concerning COER (see [5] for details).

Theorem 3. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free, and let X and Y be compact spaces. Assume $G \curvearrowright X \sim_{coe} H \curvearrowright Y$.*

If G is finitely generated, then so is H, and G and H are quasi-isometric.

Theorem 4. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free.*

Assume that X is compact, that $C(X, \mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, and that $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$ $(*)$.

Further assume that G is a duality group, and that H is solvable.

Then $G \curvearrowright X \sim_{coe} H \curvearrowright Y \Rightarrow G \curvearrowright X \sim_{conj} H \curvearrowright Y$.

Here is an example: Let G be torsion-free. Let X_0 be a compact space. Then the Bernoulli action $G \curvearrowright X_0^G$ is $\mathbb{Z}G$-free, i.e., $C(X_0^G, \mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, where N is a free $\mathbb{Z}G$-module. Hence $\mathrm{pd}_{\mathbb{Z}G}(N) = 0$, and if $\mathrm{cd}(G) > 1$, then $G \curvearrowright X_0^G$ satisfies $(*)$.

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Hyperbolic groupoids and operator algebras

Volodymyr Nekrashevych

Hyperbolic groupoids (or pseudogroups) generalize two notions of hyperbolicity: Gromov hyperbolic groups (acting on their boundaries) and hyperbolic dynamical systems.

A *pseudogroup* of homeomorphisms of a topological space X is a collection G of homeomorphisms $F : U_1 \to U_2$ between open subsets of X closed under taking compositions, inverses, restrictions to open subsets, and unions of homeomorphisms that agree on intersections of their domains. Let G be a pseudogroup. A *G-germ* is an equivalence class of a pair (F, x), where $F \in G$ and x is a point of the domain of F. Two pairs (F_1, x_1) and (F_2, x_2) are equivalent (define the same germ) if $x_1 = x_2$ and there exists a neighborhood U of x_1 such that $F_1|_U = F_2|_U$. The set of all G-germs is a groupoid. A natural topology is given by the basis of open sets consisting of sets of the form $\{(F, x) : x \in \mathrm{Dom}(F)\}$. The groupoid of germs, as a topological groupoid, uniquely determines the associated pseudogroup. We will use, therefore, both terminologies interchangeably.

Let G be a pseudogroup of local homeomorphisms of a space X. A subset $X_1 \subset X$ is called a *topological transversal* if there exits an open subset $X_0 \subset X_1$ intersecting every G-orbit.

For a groupoid G of germs of a pseudogroup acting on X, and $A \subset X$, we denote by $G|_A$ the set of germs $(F, x) \in G$ such that $x \in A$ and $F(x) \in A$, seen as a topological groupoid (with topology induced from G).

A groupoid of germs G (and the associated pseudogroup) are said to be *compactly generated* if there exists a compact topological transversal X_1 and a compact subset S of the groupoid $G|_{X_1}$ such that for every $g \in G|_{X_1}$ there exists n such that $\bigcup_{1 \le k \le n} (S \cup S^{-1})^k$ is a neighborhood of g in $G|_{X_1}$. Then the associated *Cayley graph* $G(x, S)$, for $x \in X_1$, is the oriented graph with the set of vertices $\{(F, x) \in G|_{X_1} : x \in X_1\}$, where two germs g_1, g_2 are connected by an arrow from g_1 to g_2 if there exists $s \in S$ such that $g_2 = sg_1$.

A Hausdorff compactly generated pseudogroup G is said to be *hyperbolic*, if there exist X_1 and S as above, and a metric defined on a neighborhood of X_1 such that the following conditions hold.

(1) The Cayley graphs $G(x, S)$ are Gromov δ-hyperbolic (for a fixed δ, not depending on x).
(2) For every $x \in X_1$ there exists a point ω_x of the boundary $\partial G(x, S)$ such that every oriented path in the Cayley graph $G(x, S^{-1})$ is a quasigeodesic converging to ω_x (in a uniform way).
(3) The elements of S are germs of contractions.
(4) The elements of G are locally bi-Lipschitz (non-uniformly).
(5) The sets of sources and targets of the elements of S are equal to X_1.

For more details, see [Nek15].

Examples of hyperbolic pseudogroups are: pseudogroup generated by the action of a Gromov hyperbolic group on its boundary [Gro87], pseudogroup generated by a locally expanding self-covering of a compact metric space, pseudogroup generated by a shift of finite type, Ruelle pseudogroups associated with Smale spaces [Put14], pseudogroups associated with an Anosov flow.

There is an interesting duality theory for hyperbolic groupoids. Unlike hyperbolic groups, which act on their boundaries, for every hyperbolic groupoid G there is another groupoid $G^\top$ (defined up to a natural Morita equivalence of groupoids) acting on the boundary of the Cayley graph of G, and $(G^\top)^\top$ is equivalent to G. Groupoids of the actions of Gromov hyperbolic groups on their boundaries are self-dual, but in general G is different from $G^\top$. For example, if G is the groupoid of germs generated by the maps $x \mapsto 2x$ and $x \mapsto x + 1$ on the space of real numbers, then its dual is the groupoid of germs generated by the maps given by the same formulas on the space of dyadic integers.

J. Kaminker, I. F. Putnam, and M. F. Whittaker proved K-theoretic duality for algebras associated with stable and unstable foliations of a Smale space [KPW10].

These algebras are convolution algebras of mutually dual hyperbolic groupoids. Similarly, a Poincaré duality was proved for convolution algebras of the actions of hyperbolic groups on their boundaries, see [E03]. Both cases are examples of duality for hyperbolic groupoids, so it would be interesting to generalize their results for arbitrary hyperbolic groupoids.

We have the following properties of convolution algebras of hyperbolic groupoids.

Theorem. *Let G be a hyperbolic groupoid. Then it is amenable, hence the reduced and the full C^*-algebras of G coincide. The algebra $C^*(G)$ is purely infinite, simple, and satisfies UCT.*

K-theory of convolution algebras of groupoids associated with hyperbolic complex rational functions were studied in [Nek09].

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Categorical Poisson boundaries and applications

Sergey Neshveyev

Let $\mathcal{C}$ be a rigid C*-tensor category. Consider the set $I_\mathcal{C}$ of isomorphism classes of simple objects U in $\mathcal{C}$ and a probability measure μ on $I_\mathcal{C}$. Then we can define a random walk on $I_\mathcal{C}$ with transition probabilities

$$p_\mu(s,t) = \sum_r \mu(r) m^t_{rs} \frac{d(U_t)}{d(U_r)d(U_s)},$$

where $m^t_{rs} = \dim \operatorname{Hom}_\mathcal{C}(U_t, U_r \otimes U_s)$ is the multiplicity of U_t in $U_r \otimes U_s$ and $d = d^\mathcal{C}$ denotes the intrinsic dimension of an object in $\mathcal{C}$. The random walk defines a Markov operator P_μ on $\ell^\infty(I_\mathcal{C})$:

$$P_\mu(f)(s) = \sum_t p_\mu(s,t) f(t).$$

The space $H^\infty(I_\mathcal{C}; P_\mu)$ of bounded harmonic ($P_\mu(f) = f$) functions is an abelian von Neumann algebra with product

$$(f \cdot g)(s) = \lim_{n\to\infty} P^n_\mu(fg)(s).$$

This construction of the Poisson boundary of $I_{\mathcal{C}}$ can be lifted to the categorical level as follows. Take objects U and V and consider the functors $\iota \otimes U$ and $\iota \otimes V$ on $\mathcal{C}$. Consider bounded natural transformations between these functors: collections of natural in X morphisms

$$\eta_X : X \otimes U \to X \otimes V, \quad \sup_X \|\eta_X\| < \infty.$$

Define an operator P_μ on the space of such transformations by

$$P_\mu(\eta)_X = \sum_r \frac{\mu(r)}{d(U_r)} (\mathrm{Tr}_{U_r} \otimes \iota \otimes \iota)(\eta_{U_r \otimes X}),$$

where Tr denotes the categorical trace. Note that if $U = V$, then P_μ can be considered as a normal ucp map on

$$\ell^\infty\text{-}\bigoplus_s \mathrm{End}_{\mathcal{C}}(U_s \otimes U).$$

Then for $U = V = \mathbb{1}$ we get exactly the classical Markov operator introduced earlier. For arbitrary U and V, denote by $\mathcal{P}(U,V)$ the space of bounded harmonic ($P_\mu(\eta) = \eta$) natural transformations. Every morphism $T\colon U \to V$ in $\mathcal{C}$ defines such a transformation η: $\eta_X = \iota_X \otimes T$. We can then enlarge $\mathcal{C}$ to a C*-tensor category $\mathcal{P}$ such that every object in $\mathcal{P}$ is a subobject of an object in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{P}}(U,V) = \mathcal{P}(U,V),$$

and the composition of morphisms is defined by

$$(\eta \cdot \nu)_X = \lim_{n\to\infty} P_\mu^n(\eta\nu)_X.$$

We call $\mathcal{P}$ together with the embedding functor $\Pi\colon \mathcal{C} \to \mathcal{P}$ the *Poisson boundary* of $(\mathcal{C}, \mu)$.

This construction was introduced in a joint work with Makoto Yamashita [3], but its origin goes back to the notions of standard model in subfactor theory [6] and noncommutative Poisson boundary in quantum groups [2].

Our central result on the categorical Poisson boundaries is as follows. For every object U, let $\Gamma_U \in B(\ell^2(I_{\mathcal{C}}))$ be the operator such that its matrix coefficient corresponding to $s, t \in I_{\mathcal{C}}$ is the multiplicity of U_s in $U \otimes U_t$. Note that we always have $\|\Gamma_U\| \le d^{\mathcal{C}}(U)$.

Theorem 1.([3]) *Assume that μ is an ergodic measure, meaning that the classical random walk on $I_{\mathcal{C}}$ has trivial Poisson boundary ($H^\infty(I_{\mathcal{C}}; P_\mu) = \mathbb{C}$). Then the Poisson boundary $\Pi\colon \mathcal{C} \to \mathcal{P}$ of $(\mathcal{C}, \mu)$ is a universal unitary tensor functor such that*

$$d^{\mathcal{P}}(\Pi(U)) = \|\Gamma_U\|$$

for all objects U in $\mathcal{C}$.

This result unifies and generalizes various results on amenability for tensor categories, quantum groups and subfactors [4]. It also has applications to analysis of representations of Drinfeld doubles [5], non-existence of unitary fiber functors for certain categories [1], as well as to classification of some compact quantum groups [4].

The last two applications are based on the following description of the Poisson boundary of $\operatorname{Rep} G$, which is a categorical version of a result of Tomatsu [7]. Let G be a compact quantum group. Let K be the maximal quantum subgroup of G of Kac type, so that the algebra $\mathbb{C}[K]$ of regular functions on K is the quotient of $\mathbb{C}[G]$ by the ideal generated by the elements $a - S^2(a)$.

Theorem 2.([4]) *Assume G is coamenable and the set $\operatorname{Irr} G$ is at most countable. Then there exists an ergodic probability measure μ on $\operatorname{Irr} G$, and the Poisson boundary of $\operatorname{Rep} G$ with respect to any such measure can be identified with the forgetful functor $\operatorname{Rep} G \to \operatorname{Rep} K$.*

In particular, any unitary fiber functor $F\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ such that

$$\dim F(U) = \dim U \quad \text{for all} \quad U$$

factors, in an essentially unique way, through $\operatorname{Rep} K$.

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The Furstenberg boundary and C*-simplicity

Narutaka Ozawa

(joint work with Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy)

The reduced group C*-algebra $C^*_r G$ of a discrete group G is defined to be the norm closure of the complex group ring $\lambda(\mathbb{C}G)$ in $\mathbb{B}(\ell_2 G)$, where λ is the left regular representation of G on $\ell_2 G$. For example, in case G is abelian, the Fourier transform $\ell_2 G \cong L^2(\hat{G})$ gives rise to $C^*_r G \cong C(\hat{G})$. Here $C(\hat{G})$ is the abelian C*-algebra of the continuous functions on the Pontrjagin dual $\hat{G}$ of G and $C^*_r G$ is highly non-simple, but it is expected that $C^*_r G$ becomes simple when G is highly noncommutative. A discrete group G is said to be C*-*simple* if $C^*_r G$ is simple. This property is a priori not relevant to the simplicity. When N is a normal subgroup of G, the quotient map from $\mathbb{C}G$ onto $\mathbb{C}(G/N)$ extends to a continuous homomorphism between their reduced group C*-algebras if and only if N is amenable. Recall that every group G has the unique largest amenable normal subgroup $R(G)$, called the amenable radical. Hence, any C*-simple group G must have a trivial amenable radical. Thus the main problem on C*-simplicity is whether the converse holds true: Does $R(G) = \mathbf{1}$ imply C*-simplicity of G?

This problem is still open. The first result was due to R. Powers in 1975 who proved that the noncommutative free groups F_d are C*-simple. In fact, he found a combinatorial condition which implies C*-simplicity and checked it for the free groups. Since then, Powers's method has been streamlined and extended by many researchers, most notably by P. de la Harpe (see [dlH] for a survey). By now, the above problem is solved affirmatively for the following classes (see [BKKO] and references therein):

- Acylindrically hyperbolic groups (Dahmani–Guirardel–Osin 2011).
- Linear groups (Bekka–Cowling–de la Harpe 1994, Poznansky 2008).
- Groups with nontrivial ℓ_2-Betti numbers (Peterson–Thom 2011).
- Baumslag–Solitar groups and some other groups acting on a tree (de la Harpe–Préaux 2011).
- Free Burnside groups, etc. (Olshanskii–Osin 2014).

Recently, Haagerup and Olesen (2015) observed that if Thompson's group F is amenable (recall that whether F is amenable or not is a very famous open problem), then Thompson's group T is not C*-simple. Since T is simple and non-amenable and in particular $R(T) = \mathbf{1}$, this would provide the first counterexample if F is amenable. In fact, what Haagerup and Olesen found is an interesting condition on a pair $H \leq G$ of a group and a subgroup which implies that the quasi-regular representation $\lambda_{G/H}\colon \mathbb{C}G \to \mathbb{B}(\ell_2(G/H))$ is not faithful on the complex group ring $\mathbb{C}G$. Their condition applies to $F \leq T$. Note that the quasi-regular representation extends on $\mathrm{C}^*_{\mathrm{r}}G$ if and only if H is amenable.

Last year, M. Kalantar and M. Kennedy ([KK]) found a remarkable and totally new if-and-only-if characterization of C*-simplicity in terms of topological dynamical systems.

Theorem (M. Kalantar and M. Kennedy)**.** *A discrete group G is* C*-*simple if and only if its action on the Furstenberg boundary $\partial_{\mathrm{F}}G$ is (topologically) free.*

Recall that a compact topological space X on which G acts is called *minimal* if $\overline{Gx} = X$ for every $x \in X$. It is called *a G-boundary* in the sense of Furstenberg if for every $\mu \in \mathrm{Prob}(X)$ one has $\overline{G\mu} \supset \{\delta_x : x \in X\}$. Any G-equivariant quotient of a G-boundary is again a G-boundary, and the one-point space is a G-boundary for any group G—in fact this is the only G-boundary when G is amenable, because in which case there always exists a G-invariant probability measure $\mu \in \mathrm{Prob}(X)$. Furstenberg has observed that for any G there is a unique largest G-boundary $\partial_{\mathrm{F}}G$, which is called the (*maximal*) *Furstenberg boundary* of G. This should not be confused with the Furstenberg–Poisson boundary, which is a more famous measure-theoretic counterpart of $\partial_{\mathrm{F}}G$. It is proved in [KK] that the Furstenberg boundary $\partial_{\mathrm{F}}G$ coincides with the Hamana boundary and hence is an extremally disconnected space (a.k.a. a Stonean space), which is not second countable unless G is amenable. This means that there is no hope to describe $\partial_{\mathrm{F}}G$ concretely. The amenable radical $R(G)$ acts trivially on $\partial_{\mathrm{F}}G$ and the induced action of $G/R(G)$ on $\partial_{\mathrm{F}}G$ is faithful. The action G on X is said to be *topologically free* if for every non-neutral $g \in G$

the fixed point set $\{x \in X : gx = x\}$ has empty interior. Therefore, if G is not C*-simple, then the stabilizer groups $G_x = \{g \in G : gx = x\}$ are non-trivial for all $x \in \partial_{\mathrm{F}} G$, by the minimality of $\partial_{\mathrm{F}} G$. In that case, these subgroups G_x are all amenable and normalish. Here a subgroup $H \leq G$ is said to be *normalish* if $\bigcap_{t\in F} tHt^{-1}$ is non-trivial (infinite) for every finite subset $F \subset G$. Therefore, we arrive at the following.

Corollary ([BKKO]). *If $R(G) = \mathbf{1}$ but G is not* C*-*simple, then G must have uncountably many amenable normalish subgroups.*

This criterion gives a simple and unified approach to the C*-simplicity of all the above-mentioned classes and more. Moreover, we have used the boundary theory to prove the following very general result.

Theorem ([BKKO]). *For any group G, any tracial state ϕ on $\mathrm{C}^*_{\mathrm{r}}G$ is supported on $R(G)$, i.e., $\phi(\lambda(g)) = 0$ for every $g \in G \setminus R(G)$.*

I have talked about these results and outlined some of the proofs. The talk was based on [BKKO] and [Oz].

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A Dixmier-Douady Theory for strongly self-absorbing C^*-algebras

Ulrich Pennig

(joint work with Marius Dadarlat)

Continuous $C(X)$-algebras or continuous fields of C^*-algebras play the role of bundles in C^*-algebra theory. They are employed as versatile tools in several areas such as index and representation theory and in proofs of the Novikov and the Baum-Connes conjecture. The first classification result in this area is due to Dixmier and Douady: They proved that isomorphism classes of locally trivial continuous fields over a paracompact space X with the compact operators $\mathbb{K}$ as fibers form a group with respect to the natural tensor product. This group is isomorphic to the third Čech cohomology group $\check{H}^3(X, \mathbb{Z})$. Moreover, such a continuous field is locally trivial if and only if it satisfies Fell's condition. In joint work with Marius Dadarlat the speaker showed that the theorems of Dixmier and Douady can be generalized to continuous fields with fibers isomorphic to stabilized strongly self-absorbing C^*-algebras.

Deep theorems of Kirchberg and Phillips have shown the exceptional role played in the classification program by the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$. The class of strongly self-absorbing C^*-algebras defined by Toms and Winter [5] captures an essential property those two examples share: They tensorially absorb themselves in a very strong sense. A separable, unital C^*-algebra D is strongly self-absorbing if there exist an isomorphism $\varphi\colon D \to D \otimes D$ and a path $u\colon [0,1) \to U(D \otimes D)$, such that for all $d \in D$ we have $\lim_{t\to 1} \|\varphi(d) - u_t(d \otimes 1)u_t^*\| = 0$. Apart from the above two examples, the class includes the Jiang-Su algebra $\mathcal{Z}$, infinite UHF algebras and is closed under tensor products. Moreover, $K_0(D)$ is a ring.

The extension of the Fell condition to the case of stabilized strongly self-absorbing fibers takes the following form:

Theorem: [1, Thm. B] *A separable continuous field A with fibers isomorphic to $D \otimes \mathbb{K}$ over a locally compact metrizable space X of finite covering dimension is locally trivial if and only if for each point $x \in X$, there exist a closed neighborhood V of x and a projection $p \in A(V)$ such that $[p(v)] \in GL_1(K_0(A(v)))$ for all $v \in V$.*

The proof of the classification result of Dixmier and Douady rests on the fact that the classifying space $B\mathrm{Aut}(\mathbb{K})$ is an infinite loop space. Surprisingly, the same is true in the above case.

Theorem: [1, Thm. A] *The space $B\mathrm{Aut}(D \otimes \mathbb{K})$ is an infinite loop space. The set of isomorphism classes of locally trivial continuous fields over a compact metrizable space X with fiber $D \otimes \mathbb{K}$ is an abelian group with respect to the tensor product. It is isomorphic to the first group $E^1_D(X)$ of the generalized cohomology theory associated to $B\mathrm{Aut}(D \otimes \mathbb{K})$.*

The coefficients $E^*_D(pt)$ of the theory are completely determined by the K-theory of D. In fact, $B\mathrm{Aut}(D \otimes \mathbb{K})$ can be identified with the infinite loop space $BGL_1(KU^D)$ representing the unit spectrum of the multiplicative generalized cohomology theory $X \mapsto K_0(C(X) \otimes D)$ [2]. It is computable via the Atiyah-Hirzebruch spectral sequence. Via rationalization it is also possible to define (rational) higher analogues of the Dixmier-Douady class.

The above theorem has some nice applications in algebraic topology: The group $E^1_{\mathcal{O}_\infty}(X)$ classifies all homotopy theoretical twists of K-theory. It is bigger than the group $H^3(X,\mathbb{Z})$ usually studied in this context. A corresponding operator algebraic version of higher twisted K-theory has been worked out in [4].

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Noncommutative residues and higher indices

Denis Perrot

This report is based on the preprint [1]. Let $G \rightrightarrows B$ be a Lie groupoid acting smoothly on a submersion of smooth manifolds $\pi : M \to B$. The action is not supposed proper, nor isometric. By considering the induced action of G on the algebra $CL^0_\pi(M)$ of order zero pseudodifferential operators acting along the fibers of the submersion π, one gets an extension of crossed-product algebras

$$0 \to L^{-\infty}_\pi(M) \rtimes G \to CL^0_\pi(M) \rtimes G \to CS^0_\pi(M) \rtimes G \to 0$$

where $L^{-\infty}_\pi(M)$ is the subalgebra of smoothing operators, and $CS^0_\pi(M)$ denotes the quotient algebra of formal symbols. In general the elements of the crossed-product algebra $CL^0_\pi(M) \rtimes G$ are a mixture of pseudodifferential operators and diffeomorphisms: they do no longer belong to the class of pseudodifferential operators, but to the larger class of Fourier integral operators [2]. The quotient algebra $CS^0_\pi(M) \rtimes G$ should be viewed as the algebra of "G-equivariant symbols". An operator Q is called elliptic if its symbol is invertible. Such an operator has an index living in the K-theory of the ideal of smoothing operators:

$$Ind(Q) \in K_0(L^{-\infty}_\pi(M) \rtimes G)$$

Our main result is an index theorem which computes the evaluation of this index on cyclic cohomology classes contained in the range of the excision map associated to the extension above. More precisely, let $O \subset G$ be an isotropic Ad-invariant submanifold. We define the cyclic cohomology $HP^\bullet(C(G))_{[O]}$ of the smooth convolution algebra localized at O in terms of supports. Then if the action of O on M is non-degenerate in a certain sense, the excision map Exc fits in a commutative diagram

$$\begin{array}{ccc} HP^\bullet(L^{-\infty}_\pi(M) \rtimes G) & \xrightarrow{Exc} & HP^{\bullet+1}(CS^0_\pi(M) \rtimes G) \\ \uparrow & & \uparrow \sigma^* \\ HP^\bullet(C(G))_{[O]} & \xrightarrow{\pi^!_G} & HP^{\bullet+1}(C(S^*_\pi M \rtimes G))_{[\pi^* O]} \end{array}$$

where $S^*_\pi M$ is the fiberwise cosphere bundle of M, and the isotropic submanifold $\pi^* O \subset S^*_\pi M \rtimes G$ is the pullback of O under the submersion $S^*_\pi M \to B$. Moreover, the shriek map $\pi^!_G$ is given by an **explicit formula** involving a localized version of the Wodzicki residue. We give several examples, including the case of localization at units where explicit formulas are obtained in terms of the equivariant Todd class of the fiberwise tangent bundle of the submersion.

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Locally compact C*-simple groups

Sven Raum

In my talk I described recent work on locally compact C*-simple groups. Motivated by recent breakthrough results on discrete C*-simple groups of Kalantar-Kennedy [5] and Breuillard-Kalantar-Kennedy-Ozawa [2], I investigated possibilities to obtain non-discrete examples of C*-simple groups.

Our first result says that every C*-simple group is totally disconnected. I briefly outlined the proof, following a result of Bekka-Cowling-de la Harpe [1]. It is based on three ingredients. (1) An application of the structure theorem for locally compact groups, (2) representation theory of semisimple Lie groups and (3) results on the outer automorphism group of real semisimple Lie groups.

I then described the following example of a non-discrete C*-simple group. Let

$$\mathrm{BS}(m,n) = \langle a, t \mid t a^m t^{-1} = a^n \rangle$$

be the Baumslag-Solitar group with parameters m and n. If $2 \leq |m|, |n|$, then $\mathrm{BS}(m,n)$ is non-amenable. Further, if $|m| \neq |n|$, then $\langle a \rangle$ does not contain any non-trivial normal subgroup of $\mathrm{BS}(m,n)$. Fixing such parameters. The set of cosets $\mathrm{BS}(m,n)/\langle a \rangle$ becomes a tree T if we declare $g\langle a \rangle \sim gt\langle a \rangle$ for all $g \in \mathrm{BS}(m,n)$. (This is the Bass-Serre tree of $\mathrm{BS}(m,n)$). The action of $\mathrm{BS}(m,n)$ on T by left multiplication gives rise to an injective map $\mathrm{BS}(m,n) \to \mathrm{Aut}(T)$. The closure of $\mathrm{BS}(m,n)$ in $\mathrm{Aut}(T)$ is a non-discrete C*-simple group.

The previous example is covered by the following general result. We say that a locally compact group G satisfies condition $(*)$, if one of the following equivalent conditions.

- There is a locally finite tree T such that $G \leq \mathrm{Aut}(T)$ is closed. G is not amenable and does not contain any compact normal subgroup. Further there is a compact open subgroup $K \leq G$ such that $\mathcal{N}_G(K)/K$ contains an element of infinite order.
- There is a locally finite thick (i.e. all vertices have valency at least 3) tree T such that $G \leq \mathrm{Aut}(T)$ is closed. The action of G on the boundary ∂T is minimal and there is $x \in \partial T$ such that the stabiliser $G_x \leq G$ is open and $G_x \cap G_0$ contains a hyperbolic element, where G_0 denotes the kernel of the modular function of G.

My work says that if every group satisfying condition $(*)$ is C*-simple. The proof of this theorem is based on an adaption of Powers averaging for discrete group C*-algebras [6, 3]. Several obstacles have to be overcome to generalise it to totally disconnected groups. In particular, the work of Préeaux-de la Harpe on Powers groups acting on trees [4] has to be adapted.

I finished the talk by describing several von Neumann algebraic consequences that come forth from our work.

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Irreducible Representations of Bost-Connes systems

Takuya Takeishi

1. Classification Problem of Bost-Connes systems

For a number field K, we have a C^*-dynamical system $(A_K, \sigma_{t,K})$ which has a relation with class field theory. This C^*-dynamical system is called the *Bost-Connes system*, taken from the name of Bost and Connes [1] who created such system for $\mathbb{Q}$. The classification problem of the Bost-Connes system is that if two Bost-Connes systems $(A_K, \sigma_{t,K})$ and $(A_L, \sigma_{t,L})$ attached to number fields K and L are isomorphic, then K and L are isomorphic or not.

This problem was studied by Cornellissen and Marcolli [2] partially, but still remains unsolved. The best classification result is the classification theorem of KMS-states by Laca-Larsen-Neshveyev [4]. The KMS-classification theorem implies that the partition function of $(A_K, \sigma_{t,K})$ coincides with the Dedekind zeta function $\zeta_K(s)$. This means that $\zeta_K(s)$ is an invariant of Bost-Connes systems. When the condition $\zeta_K(s) = \zeta_L(s)$ implies $K \cong L$ was studied by R. Perlis [6]. As a part of his result, we know that for any number field K with $[K : \mathbb{Q}] \leq 6$, $\zeta_K(s)$ is a complete invariant.

In my recent work [7], we found that the narrow class number of K is also an invariant. More precisely, we have the following theorem:

Theorem 1. *Let (A_K, σ_t) be the Bost-Connes system for a number field K and let h_K^1 be the narrow class number of K. Then A_K has h_K^1-dimensional irreducible representations, and does not have n-dimensional irreducible representations for $n \neq h_K^1 < \infty$.*

It is known that the narrow class number is an invariant which is independent from zeta functions. For example, let

$$K = \mathbb{Q}(\sqrt[8]{-15}), L = \mathbb{Q}(\sqrt[8]{-240}).$$

Then they have the same zeta functions and $h_K^1/h_L^1 = 2$ (cf. [3]).

2. Dynamics of the primitive ideal space

Our strategy to prove Theorem 1 is to examine the primitive ideal space of A_K. There is a result of Laca and Raeburn [5] which concerns with the determination of the primitive ideal space of the original Bost-Connes C^*-algebra $A_{\mathbb{Q}}$. The key ingredient in that work was Williams' Theorem [8], and we will also use this theorem crucially. As a complementary result, we can also determine the primitive ideal space of A_K, which is a generalization of the work of Laca and Raeburn.

In the above strategy, we intentionally ignored the time evolution on A_K. Combining the above result and the information of $\sigma_{t,K}$, we get a similar result to KMS-classification theorem. Precisely, by looking at flows on the primitive ideal space, we know the dynamics $(\hat{P}^1_K, \sigma_{t,K})$ is embedded into $\mathrm{Prim}A_K$ and is preserved under $\mathbb{R}$-equivariant isomorphism. Here, P^1_K is the group of all principal ideals generated by totally positive elements, $\hat{P}^1_K$ is the Pontrjagin dual group of it, and $\mathbb{R}$ acts on $\hat{P}^1_K$ by the following formula:

$$\langle x, \sigma_t(\gamma)\rangle = N_K(x)^{it}\langle x, \gamma\rangle$$

for $x \in P^1_K, t \in \mathbb{R}, \gamma \in \hat{P}^1_K$. Here, N_K means the ideal norm map.

The dynamical system $(\hat{P}^1_K, \sigma_{t,K})$ is determined by the norm $N_K : P^1_K \to \mathbb{Q}$. Conversely, we can restore N_K from this dynamical system.

Theorem 2. *Let K, L be number fields. If their Bost-Connes systems $(A_K, \sigma_{t,K})$ and $(A_L, \sigma_{t,L})$ are $\mathbb{R}$-equivariantly isomorphic, then we have a group isomorphism $P^1_K \to P^1_L$ which preserves the norm map.*

Compared with KMS-classification theorem, we can restore the norm map $N_K : J_K \to \mathbb{Q}$ from the zeta function. In that sense, Theorem 2 is similar to KMS-classification theorem, and the difference amounts to the narrow class group.

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Topological K-theory for non-archimedean algebras and spaces

Georg Tamme
(joint work with Moritz Kerz, Shuji Saito)

This talk was a report on work in progress.

1. Motivation

Let X be an algebraic variety over $\mathbb{C}$. One way to study X is to look at its set of $\mathbb{C}$-valued points $X(\mathbb{C})$, which carries the structure of a complex analytic space. For these, usual topological K-theory is a well-behaved invariant.

If instead X is an algebraic variety over a complete non-archimedean field K, e.g. $K = \mathbb{Q}_p$ or $K = \mathbb{C}((t))$, then the canonical topology on $X(K)$ has unpleasant properties, for instance it is totally disconnected. To overcome this difficulty, Tate invented the notion of rigid analytic spaces. These are the analog of complex analytic spaces over non-archimedean fields and play an important role in arithmetic geometry.

The goal of this ongoing project is to develop a good notion of topological K-theory for rigid analytic spaces, which should be 'easier' than algebraic K-theory but still carry some interesting information. For example, to X as before one can associate a rigid analytic space X^{an}, and one hope is that one can get some control on the homotopy fibre of the comparison map

$$(\text{alg. } K\text{-theory of } X) \to (\text{top. } K\text{-theory of } X^{\mathrm{an}})$$

using cyclic homology. On the other hand, this new topological K-theory turns out to be related to so-called continuous K-theory of formal models (see below) which makes it interesting in the study of deformation problems of algebraic cycles [1, 2].

2. Definition

Locally, rigid analytic spaces are given by the max-spectra $\mathrm{Sp}(A)$ of so called affinoid K-algebras A, which by definition are quotients of a ring of convergent power series in finitely many variables with coefficients in K and radius of convergence 1. These are particular examples of Banach K-algebras. For the latter, Karoubi-Villamayor [6] and Calvo [3] proposed a definition of topological K-theory, which can be described as follows: For a Banach K-algebra A, one defines the algebra of power series with radius of convergence equal to 1 by

$$A\langle X_1, \dots, X_n\rangle := \{\sum_I a_I X^I \mid |a_I| \to 0 \text{ as } |I| \to \infty\}$$

and a simplicial ring

$$[n] \mapsto A\langle \Delta^n \rangle := A\langle X_0, \dots, X_n\rangle / (X_0 + \cdots + X_n - 1).$$

Then

$$KV_i(A) := \pi_{i-1}(\mathrm{GL}(A\langle \Delta^\bullet \rangle)), \quad i \geq 1.$$

Essentially by definition, KV_* is homotopy invariant with respect to the unit ball, i.e., the natural map

$$KV_i(A) \to KV_i(A\langle t\rangle)$$

is an isomorphism. However, an obstacle that prevents one from extending this definition to more general rigid spaces in a reasonable way is that there is no Mayer-Vietoris sequence.

Therefore, we modify the previous definition a little bit: Instead of fixing the radius of convergence 1, we consider power series $A\langle X_1,\dots,X_n\rangle_\rho$ with radius of convergence $\rho \geq 1$. As ρ goes to ∞, we get a pro-system of simplicial rings

$$\text{“}\lim_{\rho}\text{”}\, A\langle\Delta^\bullet\rangle_\rho.$$

As a substitute for the topological K-theory of A, we propose what we call the analytic KV-theory of A: It is given by the pro-abelian groups

$$KV_i^{\mathrm{an}}(A) := \text{“}\lim_{\rho}\text{”}\,\pi_{i-1}(\mathrm{GL}(A\langle\Delta^\bullet\rangle_\rho)), \quad i \geq 1.$$

These are no longer homotopy invariant with respect to the unit ball. Instead they satisfy the weaker property of being pro-homotopy invariant, i.e., the natural map of pro-abelian groups

$$KV_i^{\mathrm{an}}(A) \to \text{“}\lim_{\rho}\text{”}\, KV_i^{\mathrm{an}}(A\langle t\rangle_\rho)$$

is an isomorphism.

3. Results

From now on all rings are commutative and unital. The following results hold under more general assumptions. For simplicity, we only state them in the easiest cases.

Theorem 1. *Fix $\pi \in K^\times$ with $|\pi| < 1$ and assume that there is a regular, π-adically complete and separated ring A° such that $A = A^\circ[1/\pi]$. Then there is a long exact sequence*

$$\cdots \to G_i(A^\circ/(\pi)) \to \text{“}\lim_{n}\text{”}\, K_i(A^\circ/(\pi^n)) \to KV_i^{\mathrm{an}}(A) \to \dots$$

ending in

$$\cdots \to KV_1^{\mathrm{an}}(A) \to G_0(A^\circ/(\pi)) \to \text{“}\lim_{n}\text{”}\, K_0(A^\circ/(\pi^n)).$$

Here G_* is the algebraic K-theory of coherent modules, K_* is the usual algebraic K-theory. This result gives the relation with the continuous K-theory $K_*^{\mathrm{cont}}(A^\circ) := \text{“}\lim_{n}\text{”}\, K_*(A^\circ/(\pi^n))$ mentioned above.

Theorem 2. *Let A be an affinoid K-algebra. Let ℓ be an integer which is prime to the residue characteristic of K. Then the natural comparison map*

$$K_i(A;\mathbb{Z}/\ell) \to KV_i^{\mathrm{an}}(A;\mathbb{Z}/\ell)$$

from algebraic K-theory with $\mathbb{Z}/\ell$-coefficients to analytic KV-theory with $\mathbb{Z}/\ell$-coefficients is an isomorphism.

This is the analog of comparison results of Suslin [9], Fischer [4], and Prasolov [8] for commutative C^*-algebras. The proof uses Gabber's rigidity theorem [5] to reduce to a theorem of Weibel on homotopy invariance of algebraic K-theory with finite coefficients [11].

The final result is a version of the desired Mayer-Vietoris property.

Theorem 3. *Assume that K is discretely valued and of equal characteristic 0. Then KV_*^{an} satisfies Mayer-Vietoris for regular affinoids, i.e., if A is a regular affinoid K-algebra, $X := \mathrm{Sp}(A)$, and*

$$X = U \cup V$$

is an open covering by $U = \mathrm{Sp}(B)$ and $V = \mathrm{Sp}(C)$, then there is a long exact sequence of pro-abelian groups

$$\cdots \to KV_i^{\mathrm{an}}(A) \to KV_i^{\mathrm{an}}(B) \oplus KV_i^{\mathrm{an}}(C) \to KV_i^{\mathrm{an}}(B\widehat{\otimes}_A C) \to KV_{i-1}^{\mathrm{an}}(A) \to \dots$$

ending in $\cdots \to KV_1^{\mathrm{an}}(B) \oplus KV_1^{\mathrm{an}}(C) \to KV_1^{\mathrm{an}}(B\widehat{\otimes}_A C)$.

To prove this, one uses pro-cdh descent of algebraic K-theory [7] to reduce to Zariski descent for the algebraic K-theory of formal models [10].

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Isomorphism conjectures in algebraic and topological K-theory

Gisela Tartaglia
(joint work with Guillermo Cortiñas)

Let G be a discrete group, X a G-space, R a ring or a C^*-algebra equipped with an action of G by automorphisms and E a functor from the category of small $\mathbb{Z}$-linear categories to the category of spectra. We write $H^G(X, E(R))$ for the equivariant homology theory of X with coefficients in $E(R)$ (see [5]). If $H \subset G$ is a subgroup, then

$$H_*^G(G/H, E(R)) = E_*(R \rtimes H) \tag{1}$$

is just E_* evaluated at the crossed product ring. If R is a C^*-algebra and E is K^{top}, then $\rtimes$ stands for the reduced crossed product $C_r^*(G, R)$.

Let $\mathcal{F}$ be a family of subgroups of G and let $\mathcal{E}(G, \mathcal{F})$ be the classifying space associated to $\mathcal{F}$. The projection to the one point space $\mathcal{E}(G, \mathcal{F}) \to pt$ induces a morphism

$$H_*^G(\mathcal{E}(G, \mathcal{F}), E(R)) \to H_*^G(pt, E(R)) = H_*^G(G/G, E(R)) = E_*(R \rtimes G), \tag{2}$$

called *assembly map*. The *isomorphism conjecture* for $(G, \mathcal{F}, E, R)$ asserts that the assembly map is an isomorphism. The appropriate choice of $\mathcal{F}$ varies with E. For $E = K$, the nonconnective algebraic K-theory spectrum, one takes $\mathcal{F} = \mathcal{V}cyc$, the family of virtually cyclic subgroups; the isomorphism conjecture for $(G, \mathcal{V}cyc, K, R)$ is the K-theoretic *Farrell-Jones conjecture*. If $E = KH$ is homotopy K-theory, one can equivalently take $\mathcal{F}$ to be either $\mathcal{V}cyc$ or the family $\mathcal{F}in$ of finite subgroups. For $E = K^{top}$ one takes the family $\mathcal{F}in$ and in this case is called the *Baum-Connes conjecture* with coefficients in R.

Both the Farrell-Jones and the Baum-Connes conjectures have been proven for a large class of groups using a variety of different methods coming from operator theory, controlled topology and homotopy theory (see for example [1] for the case of Gromov hyperbolic groups and algebraic K-theory, and [7] for the case of a-T-menable groups and topological K-theory). It is worth mentioning that there are no counterexamples known to the Farrell-Jones conjecture and to the Baum-Connes conjecture with coefficients in $R = \mathbb{C}$.

We study different versions of the isomorphism conjectures with operator ideals as coefficient rings.

Let $\mathcal{B}$ be the ring of bounded operators in a complex, separable Hilbert space. Consider the following Farrell-Jones assembly map:

$$H_*^G(\mathcal{E}(G, \mathcal{V}cyc), K(\mathcal{S})) \to K_*(\mathcal{S}[G]). \tag{3}$$

Here $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$, and $\mathcal{L}^p \triangleleft \mathcal{B}$ is the Schatten ideal of those compact operators whose sequence of singular values is p-summable. Guoliang Yu proved that the assembly map (3) is rationally injective ([9]). His proof involves the construction of a certain Chern character tailored to work with coefficients $\mathcal{S}$ and the use of some results about algebraic K-theory of operator ideals and about controlled topology and coarse geometry. In [2] we give a different proof of Yu's result. Our proof uses the usual Chern character to cyclic homology. Like Yu's, it relies on results on algebraic K-theory of operator ideals, but no controlled topology or coarse geometry techniques are used. We formulate the result in terms of homotopy K-theory. We prove that the rational assembly map

$$H_*^G(\mathcal{E}(G,\mathcal{F}in), KH(\mathcal{L}^p)) \otimes \mathbb{Q} \to KH_*(\mathcal{L}^p[G]) \otimes \mathbb{Q} \tag{4}$$

is injective. Then we show that the latter map is equivalent to the assembly map considered by Yu, and thus obtain his result as corollary.

In [3] we prove that if (4) is surjective for $p = 1$ and F is a number field, then the following assembly maps are injective:

$$H_*^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} \to K_*(\mathbb{Z}[G]) \otimes \mathbb{Q} \tag{5}$$

$$H_*^G(\mathcal{E}(G,\mathcal{F}in), K(F)) \otimes \mathbb{Q} \to K_*(F[G]) \otimes \mathbb{Q}. \tag{6}$$

We remark that the K-theory Novikov conjecture asserts that the assembly (5) is injective for all G. Hence the validity of the rational isomorphism conjecture for KH with coefficients in $\mathcal{L}^1$ implies the validity of the Novikov conjecture for K-theory.

Finally, in [4] we study the techniques used by Higson, Kasparov and Trout in [6], [7] and [8] to prove the Baum-Connes conjecture for groups with the *Haagerup approximation property*, and we apply them to the algebraic case. More precisely, we prove the validity of the Farrell-Jones conjecture for such a group G with coefficients in a ring of the form $I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})$, where I is a K-excisive G-ring, $\mathfrak{A}$ is a G-C^*-algebra, $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ is the ideal of compact operators and $\underset{\sim}{\otimes}$ is the tensor product of C^*-algebras. Moreover, if we consider the following commutative diagram

$$\begin{array}{ccc} H_*^G(\mathcal{E}(G,\mathcal{F}in), K(\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})) & \longrightarrow & K_*((\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K}) \rtimes G) \\ \downarrow & & \downarrow \alpha \\ H_*^G(\mathcal{E}(G,\mathcal{F}in), K^{top}(\mathfrak{A})) & \longrightarrow & K_*^{top}(C^*_{red}(G,\mathfrak{A})) \end{array}$$

and we assume that $\mathfrak{A}$ is separable, we obtain as a corollary that α is an isomorphism.

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Non-commutative real algebraic geometry

Andreas Thom

In this talk I explained the use of sum-of-squares approaches in non-commutative geometry. A classical problem in real algebraic geometry is to find properties of real polynomials that certify that these polynomials are positive or non-negative on certain semi-algebraic subsets. A typical statement is that such a polynomial must be (plus some ε or not) be in the cone or the quadratic module formed by the defining inequalities and the sums of squares. Results of this sort are highly relevant in applications of semi-definite optimization.

The non-commutative framework has been studied by many authors, including William Helton and co-workers and Konrad Schmüdgen. One of the seminal results (by Helton) is that a polynomial in self-adjoint non-commuting variables is positive semi-definite in every (self-adjoint) matrix-evaluation if and only if it is a sum of hermitean squares.

In this talk, we present some results that are more particular for group rings and relate them to various fundamental problems in the theory of operator algebras. A first result (first proved by Schmüdgen) says that Helton's theorem also holds for the complex group ring of a free group. Using a model theoretic approach, we can give a new and conceptual proof of this fact – very much analogous to the modern approaches to Hilbert's 17th problem. This is one of the first applications of real closed fields in operator algebras.

Note also that any group that satisfies Helton's theorem must be residually finite dimensional. Result of Scheiderer say that Helton's theorem holds for the group ring of $\mathbb{Z}^2$, but not $\mathbb{Z}^3$. In general, we conjecture that it holds for group rings of groups with cohomological dimension at most two. In particular, this would cover surface groups (which are known to be residually finite dimensional by deep results of Lubotzky and Shalom) but also a product of two free groups – a group which is known to be residually finite dimensional if and only if Connes' Embedding Problem has a positive answer.

We end the talk be reviewing some work Ozawa on property (T) in the context of sums of squares methods. We present a new (elementary but computer-assisted) proof that $SL(3,\mathbb{Z})$ has property (T), a result that was obtained in joined work with Tim Netzer. Some more recent computations with Laurent Bartholdi indicate that the sum of squares approach might also be useful to prove property (T) for the integer points of Kac-Moody groups.

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A link between Krasner's valued hyperfields and Deligne's triples

Jeffrey Tolliver

An important problem in number theory is to link arithmetic in characteristic 0 with characteristic p. In the case of local fields two similar approaches to this problem have been developed by Marc Krasner[3] and Pierre Deligne[2]. Our goal is to understand how these approaches are related.

As motivation, it is useful to recall some results from local class field theory, which may be found in [4]. Let K denote a local field. The main result of local class field theory says that $\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}} \cong \widehat{K^\times}$. In particular, to classify the abelian extensions of K, we only need to understand the multiplicative structure. On the other hand, if we wish to understand the nonabelian extensions, we must take into account the additive structure on K as well.

Let $u > 0$. There is a variant of the above theorem which classifies abelian extensions satisfying the condition that $\mathrm{Gal}(L/K)^u = 1$. This condition should be interpreted as saying that the ramification of the extension is not too wild. The result states that for $u > 0$, $(\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u)^{\mathrm{ab}} \cong \widehat{K^\times/1+\mathfrak{m}_K^u}$. It is entirely possible for distinct local fields F, K to have $K^\times/1+\mathfrak{m}_K^u \cong F^\times/1+\mathfrak{m}_F^u$ even if $\mathrm{char}(F) \neq \mathrm{char}(K)$. In this case, the aforementioned result provides a link between characteristic p and characteristic 0. However this link is of limited value because it tells us nothing about the nonabelian extensions. Hence we would like a nonabelian generalization of this result.

Form the quotient $K/1+\mathfrak{m}_K^u$ of K by the action of the group $1+\mathfrak{m}_K^u$. Then we know that $(\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u)^{\mathrm{ab}}$ depends only on the multiplicative structure of $K/1+\mathfrak{m}_K^u$ in exactly the same ways as $\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}$ depends on the multiplicative structure of K. By analogy to the relation between $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and K, one might expect $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u$ to depend on both the additive structure and multiplicative structure of $K/1+\mathfrak{m}_K^u$. In particular, we should take the additive structure of $K/1+\mathfrak{m}_K^u$ seriously even though the addition is not well-defined. This motivates the study of hyperfields, which was initiated by Marc Krasner in [3]. Note that a map which is not well-defined may instead be regarded as a multivalued function.

Definition 1. *A hyperring consists of a multiplicative monoid H together with a multivalued operation $+ : H \times H \to 2^H$ (i.e an operation landing in the power set of H) such that the following properties hold.*

- $x + y = y + x$ *for all* $x, y \in H$.
- $(x+y)+z = x+(y+z)$ *for all* $x, y, z \in H$, *where we define the sum of an element and a subset by* $x + S = \bigcup_{t \in S} x + t$ *for any* $x \in H$ *and* $S \subseteq H$.
- *There exists an element* 0 *such that* $x + 0 = \{x\}$ *for all* $x \in H$.
- *For all* $x \in H$ *there is a unique element* $-x \in H$ *such that* $0 \in x + (-x)$.
- $x(y + z) = xy + xz$ *for all* $x, y, z \in H$.

A hyperfield is a hyperring in which every nonzero element has a multiplicative inverse

Before continuing the discussion of local fields, it is worth mentioning a few examples of hyperfields.

Given a field K and a multiplicative subset S, the quotient K/S is a hyperfield. A simple example is provided by the hyperfield of signs $\mathbb{S} = \mathbb{R}/\mathbb{R}_{>0}$. One has $\mathbb{S} = \{0, 1, -1\}$ with the obvious multiplication and with $1 + 1 = 1$, $-1 - 1 = -1$ and $1 - 1 = \{0, 1, -1\}$. This hyperfield encodes the arithmetic of zero, positive, and negative numbers in exactly the same way that $\mathbb{F}_2$ encodes the arithmetic of even and odd numbers.

For another example, take $\mathbb{Y} = \mathbb{R} \cup \{-\infty\}$. Define multiplication in $\mathbb{Y}$ to be addition of real numbers. Define addition in Y by $x + y = \max(x, y)$ if $x \neq y$ and $x + x = [-\infty, x]$. This addition operation bears an obvious relation to the ultrametric inequality. In fact if K is a non-archimedean field, then the logarithm of the absolute value is a homomorphism $K \to \mathbb{Y}$. This hyperfield was studied by Oleg Viro[5], who showed that tropical varieties are zero sets of polynomials over $\mathbb{Y}$.

Another example is the hyperfield $\mathcal{T}\mathbb{R}$. As a multiplicative monoid, $\mathcal{T}\mathbb{R} = \mathbb{R}$. The addition is defined by $x + y = x$ if $|x| > |y|$, $x + y = y$ if $|x| < |y|$, $x + x = x$ and $x - x = \{y \mid |y| \leq |x|\}$. In [5], Viro constructed $\mathcal{T}\mathbb{R}$ as a dequantization of $\mathbb{R}$ by analogy with the way that the max-plus algebra $\mathbb{R}_{\max}$ is a dequantization of the semiring $\mathbb{R}_{\geq 0}$. In the work of Alain Connes and Caterina Consani[1] it was necessary to replace the semiring $\mathbb{R}_{\max}$ with the hyperfield $\mathcal{T}\mathbb{R}$ in order to have a well behaved Witt construction in characteristic one.

The most important example for our purposes is the hyperfield $K/1+\mathfrak{m}_K^u$ where K is a local field. This example was studied by Marc Krasner who noted that it is possible to have $K/1 + \mathfrak{m}_K^u \cong F/1 + \mathfrak{m}_F^u$ even when K and F have different characteristics. In fact if K is a local field of characteristic p it is possible to have a sequence of fields K_n of characteristic zero with $K/1 + \mathfrak{m}_K^n \cong K_n/1 + \mathfrak{m}_{K_n}^n$ for all n. In this situation Krasner says that K is the limit of the local fields K_n.

Deligne suggested an alternate approach to limits of local fields. Given a local field K and an integer $n > 0$, one may define a ring $R_{K,n} = \mathcal{O}_K/\mathfrak{m}_K^n$, an $R_{K,n}$-module $M_{K,n} = \mathfrak{m}_K/\mathfrak{m}_K^{n+1}$, and a module homomorphism $\epsilon_{K,n} : M_{K,n} \to R_{K,n}$. Together these form what is called the triple $\mathrm{Tr}_n(K) = (R_{K,n}, M_{K,n}, \epsilon_{K,n})$ associated to K.

I have shown that the hyperfield $K/1+\mathfrak{m}_K^n$ determines the triple $Tr_n(K)$. In fact I have shown somewhat more:

Theorem 1. *There is a faithful essentially surjective functor from the category of discretely valued hyperfields which aren't fields (as defined by Krasner in* [3]*) to the category of triples (as defined by Deligne in* [2]*) which sends* $K/1+\mathfrak{m}_K^u$ *to* $\mathrm{Tr}_u(K)$ *for any local field* K *and any* $u > 1$*.*

The above functor fails to be full because morphisms of valued hyperfields are required to preserve the absolute value while morphisms of triples preserve it only up to equivalence. However one may show that there is a precise sense in which this is the only way it fails to be full, so that the functor is almost an equivalence of categories.

Deligne has shown that $\mathrm{Tr}_u(K)$ determines $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u$. Hence we have the following corollary.

Corollary 1. *The hyperfield* $K/1+\mathfrak{m}^u$ *determines* $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u$*.*

This generalizes the fact mentioned at the beginning of the report, which says that $K/1+\mathfrak{m}^u$ determines the abelianization of $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u$. A particular consequence of this result is that if $K/1+\mathfrak{m}_K^u \cong F/1+\mathfrak{m}_F^u$ then one has $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^u \cong \mathrm{Gal}(F^{\mathrm{sep}}/F)/\mathrm{Gal}(F^{\mathrm{sep}}/F)^u$, even if K and F have different characteristics.

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Representation theory and (co)homology for subfactors, λ-lattices and C*-tensor categories.

Stefaan Vaes

(joint work with Sorin Popa and Dimitri Shlyakhtenko)

Subfactors of finite Jones index $N \subset M$ give rise to several group like combinatorial structures, that can be axiomatized in different ways. In the joint work [6] with Sorin Popa, we introduced the unitary representation theory for these group like structures, which I presented in the first part of the talk.

This representation theory has several equivalent descriptions. The first makes use of Popa's symmetric enveloping (SE) inclusion $T \subset S$, associated in [5] with any extremal finite index subfactor $N \subset M$. Here $T = M \mathbin{\overline{\otimes}} M^{\mathrm{op}}$ and this SE-inclusion should be thought of as a crossed product type inclusion w.r.t. an outer action of the underlying group like structure: we have $T' \cap S = \mathbb{C}1$ and as a T-bimodule, $L^2(S)$ is a direct sum of irreducible finite index T-subbimodules, each appearing with multiplicity one. In [6], we called *SE-correspondence* of the subfactor $N \subset M$ any Hilbert S-bimodule $\mathcal{H}$ that is generated by T-central vectors. Every T-central vector $\xi \in \mathcal{H}$ gives rise to a coefficient of the representation, in the form of a T-bimodular completely positive map $\psi_\xi : S \to S$.

Denoting by $\mathcal{C}$ the tensor category generated by all M-bimodules that appear in the Jones tower of the subfactor $N \subset M$, it turns out that a normal T-bimodular map $\psi : S \to S$ is entirely determined by a function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$. In [6], we characterized, purely in terms of the rigid C*-tensor category $\mathcal{C}$, which functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ arise in this way and we called them *cp multipliers* of $\mathcal{C}$. These are the positive definite functions on $\mathcal{C}$, and we similarly defined completely bounded (cb) multipliers. This then leads to natural geometric group theory properties for C*-tensor categories, including weak amenability, the Haagerup property, property (T), etc.

In the case where $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$ is the representation category of a compact quantum group $\mathbb{G}$, we gave a description of cp/cb multipliers in terms of the quantum group $\mathbb{G}$. In combination with the work of [2], resp. [1], this allowed us to prove that the Temperley-Lieb-Jones category has both the Haagerup approximation property and the complete metric approximation property (CMAP), while the category $\mathrm{Rep}(\mathrm{SU}_q(3))$ has property (T).

Shortly after the publication of [6], two equivalent descriptions of the representation theory of a rigid C*-tensor category were given. In [4], Neshveyev and Yamashita consider the *Drinfeld center* of ind-$\mathcal{C}$, roughly speaking the category of all finite and infinite direct sums of objects in $\mathcal{C}$. When $\mathcal{C}$ is a category of finite index M-bimodules with associated SE-inclusion $T \subset S$, it is proved in [4] that there is a natural bijection between this Drinfeld center and *generalized SE-correspondences*, i.e. Hilbert S-bimodules that, as a T-bimodule, can be written as a direct sum of T-bimodules of the form $\mathcal{H}_\alpha \otimes \overline{\mathcal{H}_\beta}$, $\alpha, \beta \in \mathcal{C}$ (recall that $T = M \mathbin{\overline{\otimes}} M^{\mathrm{op}}$). Note that the SE-correspondences of [6] should be considered as the spherical part of the representation theory.

Given any rigid C*-tensor category $\mathcal{C}$, we denote by $\mathbb{C}[\mathcal{C}]$ its fusion $*$-algebra. By definition, $\mathrm{Irr}(\mathcal{C})$ is a vector space basis of $\mathbb{C}[\mathcal{C}]$, $\alpha^* = \overline{\alpha}$ and the product is given by the fusion rules. Whenever $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ is a cp multiplier in the sense of [6] and using the dimension function on $\mathcal{C}$, the map $\omega : \alpha \mapsto d(\alpha)\varphi(\alpha)$ extends to a positive functional on $\mathbb{C}[\mathcal{C}]$. If moreover $\varphi(\varepsilon) = 1$, we say that ω is an *admissible state* on $\mathbb{C}[\mathcal{C}]$. The completion w.r.t. all admissible states yields the universal C*-algebra $C_u(\mathcal{C})$ and we proved in [6] that there is a natural bijection between SE-correspondences and representations of the C*-algebra $C_u(\mathcal{C})$ (in the case where $\mathcal{C}$ is a category of M-bimodules).

Not all states on $\mathbb{C}[\mathcal{C}]$ are admissible. The fusion $*$-algebra $\mathbb{C}[\mathcal{C}]$ is a corner of Ocneanu's tube $*$-algebra $\mathcal{A}$. This $*$-algebra can be defined for any rigid C*-tensor category $\mathcal{C}$. It is typically non-unital, but comes with an orthogonal family of projections $(p_i)_{i\in \mathrm{Irr}(\mathcal{C})}$ whose sums serve as local units of $\mathcal{A}$. We canonically have $\mathbb{C}[\mathcal{C}] = p_\varepsilon \cdot \mathcal{A} \cdot p_\varepsilon$. In [3], Ghosh and C. Jones proved that a state $\omega : \mathbb{C}[\mathcal{C}] \to \mathbb{C}$ is admissible if and only if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A} \cdot p_\varepsilon$. This provides a new purely categorical description of cp multipliers on a rigid C*-tensor category. Pushing things a bit further, in [7], we provide a natural bijection between the non-degenerate $*$-representations of Ocneanu's tube $*$-algebra $\mathcal{A}$ and the Drinfeld center of ind$-\mathcal{C}$.

In the second part of the talk, I presented a work in progress with Sorin Popa and Dimitri Shlyakhtenko, [7], introducing homology and cohomology for subfactors and rigid C*-tensor categories with coefficients in a unitary representation as above. Using the regular representation, we can then define L^2-Betti numbers for subfactors and rigid C*-tensor categories.

We start off in the very general context of an arbitrary quasi-regular inclusion of tracial von Neumann algebras $T \subset S$. Here, quasi-regularity means that $L^2(S)$ can be decomposed as a direct sum of finite index T-subbimodules. Equivalently, the quasi-normalizer $\mathcal{S} = \mathrm{QN}_S(T)$ defined as the set of elements $x \in S$ such that

$$xT \subset \sum_{i=1}^{n} Tx_i \quad \text{and} \quad Tx \subset \sum_{j=1}^{m} y_j T$$

for some $x_i, y_j \in S$, is weakly dense in S.

In [7], we define homology and cohomology of $T \subset \mathcal{S}$ with coefficients in any Hilbert S-bimodule $\mathcal{H}$. When $\mathcal{H} = L^2(S) \,\overline{\otimes}_T\, L^2(S)$ is the regular representation and under the correct unimodularity assumption, this allows us to define the L^2-Betti numbers $\beta_n^{(2)}(T \subset \mathcal{S})$.

In the case where $T \subset S$ is a Cartan subalgebra with associated equivalence relation $\mathcal{R}$, we find Gaboriau's L^2-Betti numbers: $\beta_n^{(2)}(T \subset \mathcal{S}) = \beta_n^{(2)}(\mathcal{R})$.

In the case where $T \subset S$ is the SE-inclusion associated with a tensor category $\mathcal{C}$ of finite index M-bimodules, we prove that the homology of $T \subset \mathcal{S}$ is precisely given by the Hochschild homology of Ocneanu's tube $*$-algebra $\mathcal{A}$ with its canonical augmentation (co-unit) $\epsilon : \mathcal{A} \to \mathbb{C}$. Note that here, we use the natural bijection between generalized SE-correspondences (the coefficients for the homology of $T \subset \mathcal{S}$) and Hilbert $\mathcal{A}$-modules (the coefficients for the Hochschild homology of $\mathcal{A}$).

This last result provides us with the necessary computational tools. We prove that the L^2-Betti numbers of the Temperley-Lieb-Jones category all vanish and that the expected formulae for free products and direct products hold. In particular, the Fuss-Catalan category has a non-vanishing first L^2-Betti number.

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Dynamic Asymptotic dimension

Rufus Willett
(joint work with Erik Guentner, Guoliang Yu)

The aim of this work is to unify, generalise, and simplify some notions that have been important in controlled topology (e.g. [1]), operator K-theory (e.g. [6]), and C^*-classification theory (e.g. [5]). All these notions have in common that they allow one to 'decompose' an action of a discrete group G on a compact space X into simpler 'pieces', and then use information about these pieces to get information about the overall action. The main notions also work for étale groupoids, but for simplicity we will focus here on the case of free group actions.

The main definition is as follows. Let G be a group acting freely by homeomorphisms on a compact space X. For any open subset U of X and subset E of G, let $\sim$ be the equivalence relation on U generated by the relation

$$\{(x, xg) \mid x, xg \in U,\ g \in E\};$$

in other words, $x \sim y$ if y can be reached from x by applying a finite number of elements from $E \cup E^{-1}$, in such a way that each intermediate point remains in U. The subset U is called *small* for E if there is a uniform bound on the size of the equivalence classes for $\sim$. The *dynamic asymptotic dimension* of the action is then the smallest integer d such that for any finite subset E of G, there is an open cover

$$X = U_0 \cup \cdots \cup U_d$$

of X by sets that are small for E.

The definition is inspired by Gromov's asymptotic dimension [3][Section 1.E], which can be defined as follows. Say G is a group, and consider the right action of G on itself. The *asymptotic dimension* is the smallest integer d such that for any finite subset E of G there exists a finite cover

$$G = U_0 \cup \cdots \cup U_d$$

such that each U_i is small for E. It follows easily from this and the properties of the Stone-Čech compactification βG of G that the dynamic asymptotic dimension of the action of G on βG equals the asymptotic dimension of G.

There seems to be a large class of examples with this property, although much more remains to be understood here. Here are the examples that appear in our published work.

(1) Any minimal $\mathbb{Z}$ action has dynamic asymptotic dimension one.
(2) Conditions considered by Bartels, Lück, and Reich in their work on the Farrell-Jones conjecture [2] imply estimates on dynamic asymptotic dimension.
(3) Any group G admits a free and minimal action on the Cantor set with dynamic asymptotic dimension of the action equal to the asymptotic dimension of G.

Work of Szabó, Wu, and Zacharias [4] implies that there are also many examples coming from natural actions of $\mathbb{Z}^n$, and of more general nilpotent groups. On the other hand, the asymptotic dimension of G is an absolute upper bound on the dynamic asymptotic dimension of any action.

The main applications we get from dynamic asymptotic dimension are as follows.

(1) The nuclear dimension of a reduced crossed product of $C(X)$ by G is bounded by the product of the dimension of X, and the dynamic asymptotic dimension of the action (up to the usual additive constants '+1' that usually arise in dimension theory).
(2) If the dynamic asymptotic dimension of the action of G on X is finite, then the Baum-Connes conjecture is true for G with coefficients in $C(X)$.

We are currently writing up generalizations of the second theorem to the setting of algebraic K-theory and L-theory: combined with new descent techniques, this should lead to new results on the associated Novikov-type conjectures for the integral group ring $\mathbb{Z}[G]$.

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An Analytic Grothendieck Riemann Roch Theorem

Tang, Xiang
(joint work with Ronald G. Douglas and Guoliang Yu)

In my talk, I presented a generalized Toeplitz index theorem in [7], which can be viewed as an analytic version of the Grothendieck Riemann Roch theorem.

Let $\mathbb{D}$ be the unit disk inside $\mathbb{C}$. Consider the Hilbert space $L^2_a(\mathbb{D})$ of square integrable analytic functions on $\mathbb{D}$ with respect to the Lebesgue measure. Let z be the coordinate function on $\mathbb{C}$. We are interested in the Toeplitz operator $T_z \in B(L^2_a(\mathbb{D}))$ of the following form,

$$T_z : L^2_a(\mathbb{D}) \longrightarrow L^2_a(\mathbb{D}), \qquad T_z(f)(x) := xf(x),\ \forall x \in \mathbb{D}. \tag{1}$$

Let $T_z^* \in B(L^2_a(\mathbb{D}))$ be the adjoint of T_z. It is not hard to check the commutator

$$[T_z, T_z^*] := T_zT_z^* - T_z^*T_z \in B\big(L^2_a(\mathbb{D})\big)$$

is a compact operator. Furthermore, let $\mathfrak{K}\big(L^2_a(\mathbb{D})\big)$ be the C^*-algebra of compact operators on $L^2_a(\mathbb{D})$, and $\mathfrak{T}(\mathbb{S}^1)$ be the unital C^*-algebra generated by T_z and $\mathfrak{K}\big(L^2_a(\mathbb{D})\big)$. We have the following short exact sequence

$$0 \longrightarrow \mathfrak{K}\big(L^2_a(\mathbb{D})\big) \longrightarrow \mathfrak{T}(\mathbb{S}^1) \longrightarrow C(\mathbb{S}^1) \longrightarrow 0,$$

where $C(\mathbb{S}^1)$ is the C^*-algebra of continuous functions on $\mathbb{S}^1$. The above sequence gives an extension class $[\mathfrak{T}(\mathbb{S}^1)] \in \mathrm{Ext}(\mathbb{S}^1)$. The classical Toeplitz index theorem [2], [3] states that in the K-homology group of $\mathbb{S}^1$, the class $[\mathfrak{T}(\mathbb{S}^1)]$ can be identified with the one associated to the selfadjoint elliptic differential operator $[\frac{1}{\sqrt{-1}}\frac{d}{d\theta}]$ on $\mathbb{S}^1$.

In literature, there are many attempts to generalize this Toeplitz index theorem. The most notable is the Boutet de Monvel index theorem for manifolds with strongly pseudoconvex boundaries [15], [3], [2]. For example, let $\mathbb{B}^m$ be the open unit ball inside the complex plane $\mathbb{C}^m$. Consider the Hilbert space $L^2_a(\mathbb{B}^m)$ of square integrable analytic functions on $\mathbb{B}^m$ with respect to the Lebesgue measure. Let $z_1, \cdots, z_m$ be the coordinate functions on $\mathbb{C}^m$. Consider the Toeplitz operator $T_{z_i} \in B\big(L^2_a(\mathbb{D})\big)$, $i = 1, \cdots, m$ defined in the same way as Eq. (1). The above compact commutator property has a natural generalization, i.e.

$$[T_{z_i}, T^*_{z_j}] \in \mathfrak{K}\big(L^2_a(\mathbb{B}^m)\big),\ i, j = 1, \cdots, m.$$

In [7], we generalize the above index theorem on $\mathbb{B}^m$ by emphasizing the role of subvarieties. Let $A = \mathbb{C}[z_1, \cdots, z_m]$ be the polynomial ring of m variables. Let I be an ideal of A generated by $p_1, \cdots, p_M$. Define

$$Z_I = \{(z_1, \cdots, z_m) \in \mathbb{C}^m\ :\ a(z_1, \cdots, z_m) = 0, \forall a \in I\}.$$

We point out that the analytic space Z_I may have singularities. Let $\partial\mathbb{B}^m := \overline{\mathbb{B}}^m \backslash \mathbb{B}^m$ be the boundary of $\mathbb{B}^m$, the unit sphere $\mathbb{S}^{2m-1}$. Denote $Z_I \cap \mathbb{B}^m$ by Ω_I. The analytic space Ω_I is naturally a (singular) submanifold of $\mathbb{B}^m$ with the boundary $\partial\Omega_I := \overline{\Omega}_I \backslash \Omega_I = Z_I \cap \partial\mathbb{B}^m$.

Let $\overline{I}$ be the closure of I in $L^2_a(\mathbb{B}^m)$. As I is an ideal of A, $\overline{I}$ is closed under the Toeplitz operator T_{z_i}, $i = 1, \cdots, m$. Denote $T_{z_i}|_{\overline{I}}$ $(i = 1, \cdots, m)$ to be the restriction of T_{z_i} on $\overline{I}$. We proved the following theorem [7] for the operators $T_{z_i}|_{\overline{I}}$, $(i = 1, \cdots, m)$.

Theorem I. *Assume*

(1) The Jacobian matrix $(\partial p_i/\partial z_j)_{i,j}$ is of maximal rank on the boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$;
(2) $m - M \geq 2$;
(3) Z_I intersects $\partial\mathbb{B}^m$ transversely.

The commutator between $T_{z_i}|_{\overline{I}}$ and $T_{z_j}|^*_{\overline{I}}$ on $\overline{I}$ is compact, i.e.

$$[T_{z_i}|_{\overline{I}}, T_{z_j}|^*_{\overline{I}}] \in \mathfrak{K}(\overline{I}),\ i, j = 1, \cdots, m.$$

Theorem I confirms the conjecture by Arveson [1] and the first author [5] that the ideal $\bar{I}$ is an essentially normal A-module when I satisfies the assumption of Theorem I. We refer the reader to [8], [11]–[13], and [10] for related results.

Let $Q_I = L^2_a(\mathbb{B}^m)/\overline{I}$ be the quotient Hilbert space. The operator T_{z_i} $(i = 1, \cdots, m)$ naturally descends to a bounded operator on Q_I. We denote the associated operator on Q_I by $T_{z_i}|_{Q_I}$, $i = 1, \cdots, m$. As a corollary of Theorem I, we know that the commutator between $T_{z_i}|_{Q_I}$ and $T_{z_j}|^*_{Q_I}$ on Q_I is compact, i.e.

$$[T_{z_i}|_{Q_I}, T_{z_j}|^*_{Q_I}] \in \mathfrak{K}(Q_I),\ i, j = 1, \cdots, m.$$

Let $\mathfrak{T}(Q_I)$ be the unital C^*-algebra generated by $T_{z_i}|_{Q_I}$, $i = 1, \cdots, m$ and $\mathfrak{K}(Q_I)$. We studied in [7] the extension class associated to $\mathfrak{T}(Q_I)$. Under the assumption of Theorem I, Ω_I is an analytic space of complex dimension $k := m - M \geq 2$ and complex codimension M. Ω_I has a smooth strongly pseudoconvex boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$ and (possibly) a finite number of isolated singularities away from the boundary. As $\partial\Omega_I$ is smooth and strongly pseudoconvex, the restriction of the complex structure to the boundary defines a CR-structure and therefore a spin^c structure on $\partial\Omega_I$. Let $D_{\partial\Omega_I}$ be the Dirac operator associated to this spin^c structure, a fundamental class of $K_1(\partial\Omega_I)$.

Theorem II. *Under the assumption of Theorem I, the C^*-algebra $\mathfrak{T}(Q_I)$ defines an extension class on $\partial\Omega_I$. In $K_1(\partial\Omega_I)$, this class is equal to the one defined by the spin^c Dirac operator $D_{\partial\Omega_I}$ on $\partial\Omega_I$.*

As a special example of Theorem II, we consider the following polynomial

$$p_k(z_1, \cdots, z_5) = z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} \in \mathbb{C}[z_1, \cdots, z_5],\ k \geq 1.$$

The zero variety Z_{p_k} of p_k has an isolated singularity at the origin, and when $\epsilon > 0$ is sufficiently small, Z_{p_k} intersects with the sphere $\mathbb{S}^9_\epsilon = \partial\mathbb{B}^5_\epsilon = \overline{\mathbb{B}}^5_\epsilon \backslash \mathbb{B}^5_\epsilon$ transversely [4], [14], where $\mathbb{B}^5_\epsilon$ is the open ball of radius ϵ around the origin. Hence the conditions of Theorem II are satisfied on $\mathbb{B}^5_\epsilon$. We conclude that Q_{I_k} gives the fundamental class of the boundary $\partial\Omega^{\epsilon}_{I_k} = Z_{p_k} \cap \partial\mathbb{B}^5_{\epsilon}$. The boundary $\partial\Omega^{\epsilon}_{I_k}$ is a topological 7-sphere $\mathbb{S}^7$. When $k = 1, \cdots, 28$, the differentiable structures on $Z_{p_k} \cap \partial\mathbb{B}^5_{\epsilon}$ give all the different differentiable structures on $\mathbb{S}^7$. Theorem II offers a possibility to use operator algebra tools to study differentiable topology on $\mathbb{S}^7$.

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Higher Signatures of Witt spaces

Zhizhang Xie

(joint work with Nigel Higson)

The signature is a fundamental invariant for oriented manifolds. The Hirzebruch signature theorem expresses the signature of an oriented manifold M in terms of characteristic classes:

$$\mathrm{sig}(M) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z},$$

where $\mathcal{L}(M) \in H^*(M; \mathbb{Q})$ is the $\mathcal{L}$-class of M, a certain power series in the Pontrjagin classes. Since the definition of the signature only depends on the cohomology ring of the manifold, it is clearly a homotopy invariant. Now suppose M is not simply connected with $\pi_1(M) = \Gamma$. Let $B\Gamma$ be the classifying space for Γ and $f : M \to B\Gamma$ be a continuous map. For each cohomology class $[x] \in H^*(B\Gamma; \mathbb{Q})$, one has the following characteristic number, called a higher signature number:

$$\mathrm{sig}_{[x]}(M, f) = \langle \mathcal{L}(M) \cup f^*[x], [M] \rangle \in \mathbb{Q}.$$

The Novikov conjecture states that every higher signature number is homotopy invariant, that is, for all orientation preserving homotopy equivalences $g : N \to M$ of closed oriented manifolds and all continuous maps $f : M \to B\Gamma$,

$$\mathrm{sig}_{[x]}(M, f) = \mathrm{sig}_{[x]}(N, g \circ f).$$

This conjecture has been proved for a large class of groups [11, 6, 5, 16, 13, 14, 17, 18, 12, 4]. A common theme of the proofs for most of these cases is to first prove the *strong Novikov conjecture* by using mtheods from noncommutative geometry. The original Novikov conjecture follows as a consequence from the strong Novikov conjecture. Recall that the strong Novikov conjecture says the following map, called the Baum-Connes assembly map, $\mu : K_i^\Gamma(\underline{E}\Gamma) \to K_i(C_r^*(\Gamma))$ is injective, where $i = 0, 1$. Here $\underline{E}\Gamma$ is the universal space for proper Γ-actions, and $K_i^\Gamma(\underline{E}\Gamma)$ is the i-th Γ-equivariant K-homology of $\underline{E}\Gamma$. Roughly speaking, every K-homology class in $K_i^\Gamma(\underline{E}\Gamma)$ can be represented by a Dirac type operator on some closed manifold. What the assembly map μ does is to map each of these Dirac type operators to its corresponding K-theoretical higher index.

When the assembly map is applied to the signature operator of a manifold, we call the resulting K-theoretical higher index the higher signature index class of the manifold. The higher signature index class is one of the most fundamental invariants for studying manifolds. In this talk, I presented how to generalize the notion of higher signature index class from manifolds to a class of spaces with singularities, called Witt spaces. The signature for Witt spaces was first studied by Siegel [15], based on the work of Goresky and MacPherson [7]. This was also studied by Cheeger in the L^2-cohomology setting [2, 3]. More recently, by generalizing Cheeger's work, Albin, Leichtnam, Mazzeo and Piazza used an analytic approach to study the higher signature index class for Witt spaces [1]. In my joint work with Nigel Higson, we took a conceptual and combinatorial approach by using noncommutative geometric methods. Our approach is very much inspired by the work of Higson and Roe on mapping surgery exact sequence in topology to analytic exact sequence in K-theory [9, 10, 8]. Our main methods are a combination of the techniques from the original approach of Goresky, MacPherson and Siegel [7] [15], and techniques from noncommutative geometry.

Here is a brief summary of the main results. Suppose X is a pseudomanifold with a triangulation T. We denote the first barycentric subdivision of T by T'. Consider the stratification of X given by the skeleton of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$

Define $R_i^{\bar{p}}$ to be the subcomplex of T' consisting of all simplices which are $(\bar{p}, i)$-allowable with respect to this stratification, where $\bar{p}$ is a certain perversity. Let

$W_i^{\bar{p}}(X)$ be the vector space spanned by those simplicial i-chains with boundary supported on $R_{i-1}^{\bar{p}}$. We define $W_{\bar{p}}^i(X) = \mathrm{Hom}_{fin}(W_i^{\bar{p}}(X), \mathbb{C})$ the space of finitely supported $(\bar{p}, i)$-allowable simplicial i-cochains. We denote the corresponding chain complex by $(W_*^{\bar{p}}(X), b)$ and $(W_{\bar{p}}^*(X), b^*)$ respectively. We prove that if X is an oriented Witt space, then X naturally gives rise a geometrically controlled Poincaré complex.

Theorem 1. *Every n-dimensional oriented Witt space X is a geometrically controlled Poincaré pseudomanifold of dimension n, that is, the duality chain map $\mathbb{P} : (W_{\bar{m}}^*(X), b^*) \to (W_{n-*}^{\bar{m}}(X), b)$ associated to the fundamental class $[X]$ is a chain equivalence in the geometrically controlled category. Here $\bar{m}$ is the lower middle perversity.*

The theorem above allows us to define the higher signature index class for Witt spaces. More precisely, suppose X is a closed oriented Witt space of dimension n. Let $\widetilde{X}$ be a Γ-covering of X determined by a continuous map $f : X \to B\Gamma$. Here $B\Gamma$ is the classifying space of Γ. Consider the following analytically controlled Γ-equivariant Hilbert-Poincaré complex:

$$E_0^{\bar{m}}(\widetilde{X}) \xleftarrow{b} E_1^{\bar{m}}(\widetilde{X}) \xleftarrow{b} \cdots \xleftarrow{b} E_n^{\bar{m}}(\widetilde{X}),$$

where $E_i^{\bar{m}}(\widetilde{X})$ is the Hilbert space completion of $W_i^{\bar{m}}(\widetilde{X})$. We denote the associated higher signature index class in $K_n(C_r^*(\Gamma))$ by $\mathrm{sig}_\Gamma(X, f)$. Once casted in this framework, then the following invariance properties of the higher signature follow immediately from the general machinery for Hilbert-Poincaré complexes [8, Section 4 and Section 7].

Theorem 2. *(i) Higher signatures of Witt spaces are invariant under Witt cobordism. More precisely, suppose X_1 and X_2 are two closed oriented Witt spaces with continuous maps $f_1 : X_1 \to B\Gamma$ and $f_2 : X_2 \to B\Gamma$. If X_1 and X_2 are Γ-equivariantly cobordant, then*

$$\mathrm{sig}_\Gamma(X_1, f_1) = \mathrm{sig}_\Gamma(X_2, f_2)$$

in $K_n(C_r^*(\Gamma))$, where $n = \dim X_1 = \dim X_2$.

(ii) Higher signatures of Witt spaces are invariant under stratified homotopy equivalence. More precisely, X and Y are two closed oriented Witt spaces, and $f : Y \to B\Gamma$ is a continuous map. If $\varphi : X \to Y$ is a stratified homotopy equivalence, then $\mathrm{sig}_\Gamma(X, f \circ \varphi) = \mathrm{sig}_\Gamma(Y, f)$.

One naturally wonders whether the higher signature index class in our approach is equivalent to the higher signature index class defined by Albin, Leichtnam, Mazzeo and Piazza [1]. Indeed, by adapting some ideas of Cheeger from [2], we prove that the two approaches give rise to the same higher signature index class for Witt spaces.

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Non-rigidity of manifolds and K-theory of group C^*-algebras

Guoliang Yu

(joint work with Shmuel Weinberger)

In this talk, we use information about the K-theory of group C^*-algebras to estimate the degree of non-rigidity of manifolds.

Let G be a countable group. An element $g \in G$ is said to have order d if d is the smallest positive integer such that $g^d = e$, where e is the identity element of G. If no such positive integer exists, we say that the order of g is ∞.

If $g \in G$ is an element in G with finite order d, then we can define an idempotent in the group algebra $\mathbb{Q}G$ by:

$$p_g = \frac{1}{d}(\sum_{k=1}^{d} g^k).$$

For the rest of this paper, we denote the maximal group C^*-algebra of G by $C^*(G)$. We define $K_0^{fin}(C^*(G))$, the finite part of $K_0(C^*(G))$, to be the abelian subgroup of $K_0(C^*(G))$ generated by $[p_g]$ for all elements $g \neq e$ in G with finite order.

Conjecture 1. *If $\{g_1, \cdots, g_n\}$ is a collection of elements in G with distinct finite orders such that $g_i \neq e$ for all $1 \leq i \leq n$, then*
(1) $\{[p_{g_1}], \cdots, [p_{g_n}]\}$ generates an abelian subgroup of $K_0^{fin}(C^*(G))$ with rank n;
(2) any nonzero element in the abelian subgroup of $K_0^{fin}(C^*(G))$ generated by $\{[p_{g_1}], \cdots, [p_{g_n}]\}$ is not in the image of the assembly map $\mu : K_0(BG) \simeq K_0^G(EG) \to K_0(C^*(G))$, where EG is the universal space for proper free G-actions.

In fact, we can state a stronger conjecture in terms of K-theory elements coming from finite subgroups and the number of conjugacy classes of nontrivial finite order elements. Such a stronger conjecture follows from the strong Novikov conjecture but would not survive inclusion into large groups.

The following concept is due to Gromov.

Definition 2. *A countable discrete group G is said to be coarsely embeddable into Hilbert space H if there exists a map $f : G \to H$ satisfying*
(1) for any finite subset $F \subseteq G$, there exists $R > 0$ such that if $g^{-1}h \in F$, then $d(f(g), f(h)) \leq R$;
(2) for any $S > 0$, there exists a finite subset $E \subseteq G$ such that if $g^{-1}h \in G - E$, then $d(f(g), f(h)) \geq S$.

The class of groups coarsely embeddable into Hilbert space includes amenable groups, hyperbolic groups, and linear groups. However, Gromov's monster groups are not coarsely embeddable into Hilbert space. The importance of the concept of coarse embeddability is due to the theorem that the strong Novikov conjecture holds for groups coarsely embeddable into Hilbert space. Kasparov and Yu introduced a weaker condition, coarse embeddability into Banach spaces with Property H, and proved the strong Novikov conjecture for groups coarsely embeddable into Banach spaces with Property H.

The following concept is more flexible than coarse embeddability into Hilbert space.

Definition 3. *A countable discrete group G is said to be finitely embeddable into Hilbert space H if for any finite subset $F \subseteq G$, there exists a group G' coarsely embeddable into H such that there is a map $\phi : F \to G'$ satisfying*
(1) $\phi(gh) = \phi(g)\phi(h)$ if $g, h \in F$ and $gh \in F$;
(2) if g is a finite order element in F, then $order(\phi(g)) = order(g)$.

We mention that the class of groups finitely embeddable into Hilbert space include all residually finite groups, amenable groups, hyperbolic groups, Burnside groups, Gromov's monster groups, virtually torsion free groups (e.g. $Out(F_n)$), and any group of analytic diffeomorphisms of an analytic connected manifold fixing a given point. Narutaka Ozawa, Denise Osin and Thomas Delzant have independently constructed examples of groups which are not finitely embeddable into Hilbert space. We can similarly define a concept of finite embeddability into Banach spaces with Property H.

The general validity of the above conjecture is still open. The following result proves this conjecture for a large class of groups.

Theorem 4. *The above conjecture holds for groups finitely embeddable into Hilbert space.*

We define $N_{fin}(G)$ to be the cardinality of the following subset of positive integers:

$$\{d : \exists\ g \in G\ s.\ t.\ g \neq e,\ order(g) = d\}.$$

If M is a compact oriented manifold, the structure group $S(M)$ in the topological category is the abelian group of equivalence classes of all pairs (f, M') such that M' is a compact oriented manifold and $f : M' \to M$, is an orientation preserving homotopy equivalence. The rank of $S(M)$ measures the degree of non-rigidity for M.

The following result explains why it is interesting to study the finite part of $K_0(C^*(G))$.

Theorem 5. *Let M be a compact oriented manifold with dimension $4k-1$ $(k > 1)$ and $\pi_1(M) = G$. If Conjecture 1.1 holds for G, then the rank of the structure group $S(M)$ is greater than or equal to $N_{fin}(G)$.*

The following result is a consequence of the above theorems.

Corollary 6. *Let M be a compact oriented manifold with dimension $4k-1$ $(k > 1)$ and $\pi_1(M) = G$. If G is finitely embeddable into Hilbert space, then the rank of the structure group $S(M)$ is greater than or equal to $N_{fin}(G)$.*

We conjecture that elements of the structure group distinguished by the method of this paper are actually different manifolds. We shall make this precise in the following few paragraphs.

Let M be a compact oriented manifold. Let $S_0(M)$ be the abelian subgroup of $S(M)$ generated by elements $[(f, M')] - [(\psi \circ f, M')]$, where $f : M' \to M$, is an orientation preserving homotopy equivalence and $\psi : M \to M$, is an orientation preserving self homotopy equivalence. We define the reduced structure group $\tilde{S}(M)$ to be the quotient group $S(M)/S_0(M)$ (it is the coinvariants of the action of orientation preserving self homotopy equivalence of M on $S(M)$).

The following conjecture gives a lower bound on the "size" of the set of different manifolds in the structure group.

Conjecture 7. *If M is a compact oriented manifold with dimension $4k-1$ ($k > 1$) and $\pi_1(M) = G$, then the rank of the reduced structure group $\tilde{S}(M)$ is greater than or equal to $N_{fin}(G)$.*

In an interesting case, we can prove this conjecture.

Theorem 8. *If G has a homomorphism ϕ to a residually finite group such that kernel(ϕ) is torsion free, then the above conjecture holds.*

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