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Scaling Limits in Models of Statistical Mechanics

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Abstract. The emphasis of the workshop was on the deep relations between, on the one hand, recent advances in probabilistic investigation of statistical mechanical models and spatial stochastic processes and, on the other hand, rigorous field-theoretic and analytic methods of mathematical physics. There were 52 participants, including 6 postdocs and graduate students, working in diverse intertwining areas of probability, statistical mechanics and field theory. Specific topics addressed during the 24 talks include: Universality and critical phenomena, disordered models, Gaussian free field (GFF), stochastic representation of classical and quantum-mechanical models and related random interchange and permutation processes, random planar graphs and unimodular planar maps, random walks on critical graphs and the Alexander-Orbach conjecture, reinforced random walks and non-linear σ-models, metastability, aging, equilibrium and dynamics for continuum particles with hard core interactions, non-equilibrium dynamics and Toom's interfaces.

Mathematics Subject Classification (2010): 60, 82.

Introduction by the Organisers

The workshop was a sequel to three MFO conferences which took place in 2006, 2009 and 2012, and which were organized by Ken Alexander, Marek Biskup, Remco van der Hofstad and Vladas Sidoravicius. We tried to continue the tradition of organizing this series of workshops against the background of probabilistic and analytic methods of non-integrable statistical mechanics, this time with an emphasis on exchange of ideas between the experts in disjoint areas, specifically in rigorous Field Theory and in probabilistic Statistical Mechanics. The list of 52 invited participants reflects our attempts to maintain an optimal balance between diverse fields, leading experts and promising young researchers. Six participants were on postdoctoral and graduate level. One of the participants, Gordon Slade, was awarded Simons Visiting Professor fellowship.

In our choice of 24 talks we tried to illuminate major recent advances in the field and to expose and address at least some aspects of the works for each and every one of the participants. A more detailed account of the presentations is given below. Due to an intended intertwining of topics and themes it is hard to give an unambiguous classification.

Universality, renormalization and critical phenomena.
In a special session Roland Bauerschmidt explained profound ideas and techniques which were developed by Pierlugi Falco for an analysis of the Kosterlitz–Thouless transition line for the two-dimensional Coulomb gas.
Gordon Slade described recent results on criticality for four-dimensional weakly self-avoiding walks and ϕ^4 lattice fields via a rigorous renormalization group approach based on Berezin integration and the analysis of the flow of the effective coupling constants.
Alessandro Giuliani presented important results on: (1) universality of energy correlations and of free energy fluctuations for non-integrable two-dimensional Ising models and (2) universality of GFF-like height fluctuations of non-integrable two-dimensional dimer models.
Michael Aizenman explained how a random current representation leads to a proof of triviality of scaling theories in dimensions higher than four, and indicated how switching lemmas for the latter lead to fermionic Wick-type formulas in dimension two.
Francesco Caravenna presented results on universality of weak disorder limits, via chaos expansions, for a class of directed polymers, such as the $(2+1)$ directed polymer or the $(1+1)$ directed polymer with heavy tails.
A recent proof of the mean-field nature of critical percolation in dimensions larger than 10, via an enhanced method of lace expansions was outlined in the talk by Remco van der Hofstadt.
Asaf Nachmias presented results about critical branching random walk in 5 dimensions, showing that some spectral exponents deviate from their mean-field values.
A proof of full RSW bound for crossing probabilities in the critical model of two-dimensional Voronoi percolation was explained in the talk by Vincent Tassion.
Finally, Wendelin Werner gave an overview of Brownian loop soup and some new results connecting it to the gaussian free field.

Quantum models, reinforced walks and permutations.
A mixed random loop and random stirring representation for a class of quantum spin models was introduced by Daniel Ueltschi with a subsequent discussion of fascinating probabilistic interpretation of several open questions related to quantum phase transitions.

Shannon Starr explained his recent results on ordering of energy levels for quantum Heisenberg ferromagnet, and elucidated the relation with Aldous' spectral gap conjecture.
Emergence of microscopically large loops for random interchange process on the hyper-cube was discussed in the talk of Piotr Miłoś. Although the results fall short of proving Poisson-Dirichlet limiting statistics for macroscopic loop sizes, or even of proving existence of macroscopically large loops, an interesting fragmentation-coagulation structure was disclosed.
Ron Peled gave a different view on random permutations by discussing Mallows permutations model and band permutations.
Christophe Sabot presented results on the spectral properties of a random Schrödinger operator, which imply for instance recurrence of the Edge Reinforced Random Walk in dimension $d = 2$ and a functional central limit theorem at weak disorder in dimension $d \geq 3$.
Margherita Disertori surveyed relations between the vertex-reinforced jump process, the edge-reinforced random walk and the supersymmetric hyperbolic sigma model.

Discrete Gaussian free field (GFF), spin waves and related models.
Hubert Lacoin gave a lecture on the d-dimensional GFF interacting with a flat interface via a disordered pinning potential. For $d > 2$, the critical point of the pinning transition coincides with that of the annealed model, while the critical exponent is modified by quenched disorder. This contrasts the situation for $d = 1$.
Thomas Richthammer presented results in the challenging field of continuous particle systems. His results quantify the absence of breaking of translation invariance in two-dimensional hard disk models: the variance of a particle's position w.r.t. its ideal crystalline position is at least the logarithm of the system size.
Marek Biskup presented deep results which fully describe, in terms of a decorated Poisson-Dirichlet process, the extremal process of the two-dimensional lattice GFF and its conformal invariance properties.

Random planar graphs and unimodular planar maps.
Omer Angel showed a new characterization of hyperbolic planar triangulations using a notion of discrete curvature.
Nicolas Curien presented universality results for random planar maps, showing that, using various natural definitions of distance on these graphs, large balls grow in the same way, up to a constant multiplicative factor.

Random walks, long time behavior of equilibrium and non-equilibrium dynamics, metastability and aging.
Jiří Černý presented recent results on aging for the Metropolis dynamics of the Random Energy Model: aging is proved without the usual non-physical assumption that the process is a time change of the simple random walk on the hypercube.
Sabine Jansen gave a lecture on metastability phenomena for a Metropolis dynamics of continuous particles, and presented results on nucleation time and shape of critical droplets.

Nicholas Crawford discussed several results related to Toom's model, a nonequilibrium particle system.
Balint Toth presented a superdiffusive central limit theorem, valid in any dimension, for the displacement of a test particle in the periodic Lorentz gas in the limit of large times t and low scatterer densities (Boltzmann-Grad limit).

Summary. The workshop was an obvious success. In particular, it helped to update the participants on the state of the art and on the important pending open problems in the fields related to their domain of research, facilitated exchange of ideas between researchers in technically disconnected areas, and it gave rise to many interesting and informative discussions, which were conducted either during 10 minutes discussion time allocated after each and every 50 minute talk, or during afternoon breaks or during the evenings, all of which were kept free.

Acknowledgement: The organizers would like to thank the MFO personnel for the help and for the invaluable logistic support, as well as for creating a friendly and stimulating environment throughout the entire meeting.

The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Gordon Slade in the "Simons Visiting Professors" program at the MFO.

Workshop: Scaling Limits in Models of Statistical Mechanics

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Abstracts

Extreme points and shape of large peaks for 2D DGFF

Marek Biskup
(joint work with Oren Louidor)

In this report we will discuss detailed properties of extreme values of the Discrete Gaussian Free Field (DGFF) in finite subsets $V \subset \mathbb{Z}^2$. This is a mean-zero Gaussian process $\{h_x : x \in V\}$ whose covariance is the Green function G^V of the simple symmetric random walk killed upon exiting V. In $V_N := (0,N)^2 \cap \mathbb{Z}^2$ we have $G^{V_N}(x,y) = \log\bigl(\frac{N}{|x-y|}\bigr) + o(1)$ in the limit $N \gg |x-y| \gg 1$. Here $|x-y|$ denotes the Euclidean distance between x and y and so the resulting object is asymptotically scale and, in fact, conformally invariant.

Much recent effort went to the behavior of the absolute maximum $M_N := \max_{x\in V_N} h_x$ of the DGFF in V_N. This culminated in the work of Bramson, Ding and Zeitouni [5] who showed that the law of $M_N - m_N$, where

$$m_N := 2\sqrt{g}\,\log N - \frac{3}{4}\sqrt{g}\,\log\log N \tag{1}$$

with $g := 2/\pi$, converges to a non-degenerate limit as $N \to \infty$.

Our main focus here is the *extreme value process*. We wil use a description based on local maxima and keep record of the following triplet of "coordinates": (1) the scaled positions of (suitably defined) local maxima, (2) the reduced values of the field there and (3) the shape of the configuration in the vicinity of the local maxima. To address more general domains, we pick $D \subset \mathbb{C}$ bounded, open with "nice" boundary and let D_N be an appropriately defined scaled-up lattice version thereof. Then we define the *structured point-process measure* by

$$\eta^D_{N,r} := \sum_{x\in D_N} 1_{\{h_x = \max_{z\in\Lambda_r(x)} h_z\}}\, \delta_{x/N} \otimes \delta_{h_x - m_N} \otimes \delta_{\{h_x - h_{x+z} \colon z\in\mathbb{Z}^2\}}, \tag{2}$$

where $\Lambda_r(x) := \{z \in \mathbb{Z}^2 \colon |x-z| < r\}$ designates the neighborhood of x in which we regard (or require) h_x to be the local maximum of h — this is what is ensured by the indicator in the sum. Our main result, proved in [2, 4], is then:

Theorem 1. *There exist a random, a.s.-finite Borel measure Z^D on D and a probability measure ν on $[0,\infty)^{\mathbb{Z}^2}$ such that for any $r_N \to \infty$ with $r_N/N \to 0$,*

$$\eta^D_{N,r_N} \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathrm{PPP}\bigl(Z^D(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha h}\mathrm{d}h \otimes \nu(\mathrm{d}\phi)\bigr), \tag{3}$$

where $\alpha := 2/\sqrt{g} = \sqrt{2\pi}$ and where $\mathrm{PPP}(\lambda)$ denotes a Poisson point process with intensity measure λ.

This conclusion comes in two rather disjoint parts. This is because the theorem captures both the *global structure* of the extrema, governed by the mysterious random measure Z^D, as well as the *local structure* — the shape of the peaks —

which are trivial in terms of correlation but complicated within each peak. We proceed to discuss details of both of these aspects of the problem separately.

1. Global properties and conformal invariance

To begin with, we note that a couple of basic facts about the Z^D measure:

Proposition 2. *For the Z^D measure above, we have:*

(1) *If $A \subset D$ has $\mathrm{Leb}(A) = 0$, then $Z^D(A) = 0$ a.s.*
(2) *If $A \subset D$ is open and non-empty, then $Z^D(A) > 0$ a.s.*
(3) *Z^D is a.s. non-atomic.*

The last line shows that the measure cannot be supported on a set that is too small. Yet this is quite misleading as we also have:

Conjecture 3. *Z^D is a.s. supported on a set of zero Hausdorff dimension.*

Perhaps a more interesting question is that of dependence of Z^D on D. Here is a *transformation rule* under conformal bijections of the underlying domain:

Theorem 4. *Let D be an admissible domain and let f be a conformal bijection of D onto another admissible domain $f(D)$. Then*

$$Z^{f(D)} \circ f(\mathrm{d}x) \overset{\text{law}}{=} \left|f'(x)\right|^4 Z^D(\mathrm{d}x) \tag{4}$$

This should be contrasted with the Lebesgue measure, which transforms via the *second* power of $|f'(x)|$. Besides the scaling limit of the DGFF not being a function, the hard part of this conclusion is the fact that the extreme values of the DGFF do not transform canonically under conformal maps. The *Gibbs-Markov property*, satisfied both by the DGFF and its scaling limit, helps via a transformation rule of Z^D under partitions of D. See [2, 3] for proofs.

Naturally, one can ask whether the above measure could be constructed directly. Indeed, this is the case; the limit is conjectured to coincide (up to a constant) with the critical *Liouville Quantum Gravity* (LQG) constructed by Duplantier, Rhodes, Sheffield and Vargas [8]. The problem is that the LQG measure is not characterized uniquely and so its identification with Z^D is difficult. Notwithstanding, [3] contains a theorem giving conditions that do characterize Z^D and so all that remains to be done is to check that the critical LQG measure of [8] conforms to these as well.

2. Local properties and freezing phenomenon

Moving over to the local behavior at the peaks, we begin with an explicit description of measure ν. Let ϕ denote the DGFF in $\mathbb{Z}^2 \setminus \{0\}$ (a.k.a. pinned field). Let $\mathfrak{a}$ denote the so called *potential*; i.e., the unique solution of $\Delta\mathfrak{a} = \delta_0$ with $\mathfrak{a}(0) = 0$, where Δ is the discrete Laplacian (normalized by $\frac{1}{4}$).

Theorem 5. *For the measure ν above, we have:*

$$\nu = \lim_{r\to\infty} \text{Law of} \left(\phi + \frac{2}{\sqrt{g}}\,\mathfrak{a} \,\middle|\, \phi + \frac{2}{\sqrt{g}}\,\mathfrak{a} \geq 0 \textit{ for } |x| < r \right) \tag{5}$$

The limit exists by soft arguments (FKG inequality) but turns out to be singular: the probability of the conditional event in (5) decays as $\frac{c_\star}{\log r}$ as $r \to \infty$. The proof of all local results is by way of a novel concentric decomposition of the DGFF in the vicinity of large local maxima. See [4] for all details.

Our methods also settle some conjectures that have in the past been subject to debates in the spin-glass literature. These concern the Gibbs measure

$$\mu_N^D\bigl(\{x\}\bigr) := \frac{\mathrm{e}^{\beta h_x}}{\mathcal{Z}_{N,\beta}}, \qquad x \in D_N, \tag{6}$$

where $\beta \in (0,\infty)$ is the inverse temperature and $\mathcal{Z}_{N,\beta}$ is a normalizing constant. Carpentier and Le Doussal [6] pointed out that μ_N^D exhibits a phase transition akin to the Random Energy Model: Denoting $\beta_{\mathrm{c}} := \alpha$, the measure μ_N^D puts most of the mass at the level set $\bigl\{x \in D_N : h_x \approx \frac{\beta\wedge\beta_{\mathrm{c}}}{\beta_{\mathrm{c}}} 2\sqrt{g}\log N\bigr\}$. This can be proved rigorously by plugging in the results of Daviaud [7].

For $\beta > \beta_{\mathrm{c}}$, we have been able to describe the structure of μ_N^D far more precisely. Indeed, writing $\widehat{Z}^D$ to denote the probability measure $\widehat{Z}^D(A) := Z^D(A)/Z^D(D)$, where Z^D is above, we have:

Theorem 6. *Fix $\beta > \beta_{\mathrm{c}} := \alpha$. Then*

$$\sum_{z\in D_N} \mu_N^D\bigl(\{z\}\bigr)\delta_{z/N}(\mathrm{d}x) \;\underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \sum_{i\in\mathbb{N}} p_i\delta_{X_i}, \tag{7}$$

where $\{p_i\}$ has the Poisson-Dirichlet law with parameter β_{c}/β and $\{X_i\}$ are (conditionally on Z^D) independent samples from $\widehat{Z}^D$, independent of $\{p_i\}$.

A version of this result, formulated in the language of overlaps, has been derived by Arguin and Zindy [1]. The fact that the spatial distribution of the positions, expressed by $\{X_i : i \in \mathbb{N}\}$ above, does not depend on β is a manifestation of a *freezing phenomenon*. A subtle fact is that each limit point on the right of (7) is a conglomerate of the contributions of a whole cluster of points. This has to do with the behavior of the Gumbel law under independent shifts. See [4] for details.

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Universality in marginally relevant disordered systems

Francesco Caravenna
(joint work with Rongfeng Sun, Nikos Zygouras)

1. Abstract

We consider disordered systems of directed polymer type, for which disorder is so-called marginally relevant. This includes the disordered pinning model with tail exponent 1/2 and the usual (short-range) directed polymer model in dimension $(2+1)$, as well as the long-range directed polymer model with Cauchy tails in dimension $(1+1)$. We show that in a suitable weak disorder and infinite volume limit, the partition functions of these different models converge to a universal limit: a log-normal random field with a multi-scale correlation structure, which undergoes a phase transition as the disorder strength varies. As a by-product, we show that the solution of the two-dimensional Stochastic Heat Equation, suitably regularized, converges to the same limit.

2. Results

Many disordered systems arise as random perturbations of a pure (or homogenous) model. Examples include the random pinning model [G07], where the pure system is a renewal process, the directed polymer model [CSY04], where the pure system is a directed random walk, the random field Ising model and the stochastic heat equation [BC95]. A fundamental question for such systems is: *Does the addition of disorder alter the qualitative behavior of the pure model, such as its large-scale properties and/or critical exponents?*

If the answer is yes, regardless of how small is the disorder strength, then the model is called *disorder relevant*. If, on the other hand, disorder has to be strong enough to cause a qualitative change, then the model is called *disorder irrelevant*.

Inspired by the study of an intermediate disorder regime for directed polymers [AKQ14a], we gave in [CSZ13] a new perspective on disorder relevance: if a model is disorder relevant, then it should be possible to tune the strength of disorder down to zero (weak disorder limit) at the same time as one rescales space (continuum limit), so as to obtain a one-parameter family of *disordered continuum models*, indexed by a disorder strength parameter $\hat{\beta} \geq 0$.

The main step in the construction of such disordered continuum models is to identify their *partition functions*. In [CSZ13], we formulated general conditions on the pure model (that are consistent with the celebrated Harris criterion $d < 2/\nu$ from the physical literature [H74]) which allowed us to construct explicitly the continuum partition functions. However, the *marginally relevant* case ($d = 2/\nu$ in the Harris criterion) escapes the framework proposed in [CSZ13].

In the present work, we develop a novel approach to study marginally relevant systems of directed polymer type, which include pinning models (pin) and directed polymer models (dp), whose partition functions are defined as follows:

$$Z_{N,\beta}^{\omega,\mathrm{pin}} := \mathrm{E}\left[e^{\sum_{n=1}^N (\beta\omega_n - \lambda(\beta)) 1_{\{n\in\tau\}}}\right], \qquad Z_{N,\beta}^{\omega,\mathrm{dp}} := \mathrm{E}\left[e^{\sum_{n=1}^N \left(\beta\,\omega_{(n,S_n)} - \lambda(\beta)\right)}\right].$$

Here $\tau = (\tau_n)_{n\geq 0}$ is a renewal process on $\mathbb{N}_0$; $(S_n)_{n\geq 0}$ is a symmetric random walk on $\mathbb{Z}^d$; while ω (the *disorder*) is a family of i.i.d. random variables, independent of τ, S, with zero mean, unit variance and $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_1}] < \infty$ for small $\beta > 0$. These models are marginal in the following cases:

- for the pinning model, $\mathrm{P}(\tau_1 > n) = n^{-1/2+o(1)}$;
- for directed polymers, either $d = 2$ and $\mathrm{Var}(S_1) < \infty$ (short-range case), or $d = 1$ and $\mathrm{P}(|S_1| > x) \sim x^{-1+o(1)}$ (long-range case with Cauchy tails).

Surprisingly, there is a common structure among all these models. More precisely, one can defined a natural *replica overlap* $\mathbf{R}_N$ for each model, namely

$$\mathbf{R}_N := \sum_{1\leq n\leq N} \mathrm{P}(n\in\tau)^2, \qquad \text{resp.} \qquad \mathbf{R}_N := \sum_{1\leq n\leq N}\sum_{x\in\mathbb{Z}^d} \mathrm{P}(S_n = x)^2,$$

which is a *slowly varying function* of $N\to\infty$. Assuming that $\mathbf{R}_N$ *diverges as* $N\to\infty$ (usually $\mathbf{R}_N \sim C\log N$), one can prove the following results.

- If the disorder strength is sent to 0 as $\beta_N = \hat\beta/\sqrt{\mathbf{R}_N}$ for fixed $\hat\beta > 0$, then the partition function has a universal limit, *irrespective of the model*:

$$Z_{N,\beta_N}^{\omega} \xrightarrow[N\to\infty]{d} \boldsymbol{Z}_{\hat\beta} \overset{d}{=} \begin{cases} \text{log-normal} & \text{if } \hat\beta < 1 \\ 0 & \text{if } \hat\beta \geq 1 \end{cases}. \tag{1}$$

  More precisely, $\boldsymbol{Z}_{\hat\beta} \overset{d}{=} e^{\sigma_{\hat\beta}W_1 - \frac{1}{2}\sigma_{\hat\beta}^2}$ where $\sigma_{\hat\beta}^2 = \log\frac{1}{1-\hat\beta^2}$.
- A process-level version of (1) holds: for $\hat\beta < 1$, the family of log partition functions $\log Z_{N,\beta_N}(\mathsf{X})$, where the random walk starts at the space-time point $\mathsf{X}\in\mathbb{Z}^d\times\mathbb{N}_0$ (for directed polymers) or the renewal process starts at the time point $\mathsf{X}\in\mathbb{N}_0$ (for pinning models), converges to a limiting Gaussian random field with an explicit *multi-scale covariance structure*.

As a corollary, we gain new insights on how to define the solution of the *two-dimensional* Stochastic Heat Equation (2d SHE), which is formally written as

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \beta\,\dot W(t,x)\,u(t,x), \qquad u(0,\cdot)\equiv 1, \tag{2}$$

$(t,x)\in[0,\infty)\times\mathbb{R}^2$, $\beta > 0$ and $\dot W$ is (space-time) white noise on $[0,\infty)\times\mathbb{R}^2$.

One source of interest in the SHE is that it is connected via the Hopf-Cole transformation ($h := \log u$) to the KPZ equation [KPZ86]. Rigorously defining the solution of the SHE (or KPZ) is a serious challenge due to ill-defined terms such as $u\dot W$. The recent *Theory of Regularity Structures* bt Hairer [H13, H14] provides a robust and systematic way to make sense of singular SPDEs such as the SHE and KPZ; see also [K14], [GIP12] for alternative approaches. However, all these approaches fail at the *critical* dimension two for the SHE, and the SPDEs that can be treated so far are all known as *sub-critical* (or *super-renormalizable* in the physics literature [K14]). It turns out that the notion of *sub-criticality* for singular SPDEs correspond exactly to the notion of disorder relevance, while *criticality* corresponds to the case where the effect of disorder is marginal.

Since the solution of the SHE can be interpreted as the partition function of a continuum directed polymer via a generalized Feynman-Kac formula [BC95], our result for directed polymers implies a similar result for the 2d SHE. More precisely, consider the mollified 2d SHE

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \dot{W}^\varepsilon u^\varepsilon, \qquad u^\varepsilon(0,\cdot) \equiv 1, \tag{3}$$

where $\dot{W}^\varepsilon$ is the space-mollification of $\dot{W}$ via convolution with a smooth probability density $j_\varepsilon(x) := \varepsilon^{-2} j(x/\varepsilon)$ on $\mathbb{R}^2$. If the noise strength is scaled as $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}}$ for some $\hat{\beta} > 0$, then $u^\varepsilon(t,x)$ converges (as $\varepsilon \to 0$) in distribution to the same universal limit $\boldsymbol{Z}_{\hat{\beta}}$ in (1) as for the other marginally relevant models.

We hope that the universal structure we have uncovered among models of directed polymer type opens the door to further understanding of marginally relevant models in general, including both statistical mechanics models that are not of directed polymer type, as well as critical singular SPDEs with non-linearity. In particular, our results suggest that for marginally relevant models in general, there is a transition in the effect of disorder on an intermediate disorder scale. Establishing this transition in general, as well as understanding the behavior of the models at and above the transition point, will be the key future challenges.

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Disorder relevance for pinning of interfaces

Hubert Lacoin
(joint work with Giambattista Giacomin)

We present a rigorous study of the localization transition for a Gaussian free field on $\mathbb{Z}^d$, $d \geq 2$, interacting with a quenched disordered substrate that acts on the interface when the interface height is close to zero. If $\mathbf{P}_N$ denotes the measure of the centered lattice free-field in the box $[0, N]^d$, we are interested in the following exponential-tilt of $\mathbf{P}_N$

$$\mathbf{P}_N^{\beta,h,\omega}(\mathrm{d}\phi) := \frac{1}{Z_N^{\beta,h,\omega}} \exp\left(\sum_{x \in \Lambda_N} (\beta\omega_x + h)\mathbf{1}_{[-1,1]}(\phi_x) \right) \mathbf{P}_N(\mathrm{d}\phi)$$

where $(\omega_x)_{x\in\mathbb{Z}^d}$ is a field of IID centered random variables with finite exponential moments, and $\beta > 0$ and $h \in \mathbb{R}$ are two parameters, and

$$Z_N^{\beta,h,\omega} := \mathbf{E}_N \left[\exp\left(\sum_{x \in \Lambda_N} (\beta\omega_x + h)\mathbf{1}_{[-1,1]}(\phi_x) \right) \right],$$

is the partition function.

In the infinite volume $N \to \infty$, the system undergoes a wetting transition in h: there exists a critical value $h_c(\beta)$ which separates a localized phase where ϕ sticks to the interaction band, and a delocalized one where the contact fraction of ϕ with the band vanishes. More precisely if the asymptotic free-energy per unit of volume is given by [1]

$$\textsc{f}(\beta, h) := \lim_{N\to\infty} \frac{1}{N^d} \log Z_N^{\beta,h,\omega}, \tag{1}$$

This free-energy is non-negative. then the transition occurs when the free-energy

$$h_c(\beta) := \inf\{h \in \mathbb{R} : \textsc{f}(\beta, h) > 0\} = \sup\{h \in \mathbb{R} : \textsc{f}(\beta, h) = 0\}.$$

A simple upper bound for the free-energy is given by considering the annealed partition function. We have

$$\textsc{f}(\beta, h) \leq \lim_{N\to\infty} \frac{1}{N^d} \log Z_N^{\beta,h,\omega} = \textsc{f}(0, h + \lambda(\beta)). \tag{2}$$

where $\lambda(\beta) := \log \mathbb{E}[e^{\lambda(\beta)}]$. Note that it is known from [1] that the inequality (2) is always strict.

We want to study the influence of disorder for this system, or more precisely how this localization transition differs from that of the *pure* model (that for $\beta = 0$), which coincides with the annealed model. Note that for $\beta = 0$ we have

$$\textsc{f}(0, h) \sim \begin{cases} c_2 \frac{h}{\sqrt{|\log h|}} & \text{for } d = 2, \\ c_3 h & \text{for } d = 3. \end{cases}$$

We present a rather complete answer to the question in dimension 3 [3] and discuss on-going progress in dimension 2 [4]. We show that when there is disorder, the critical point is never shifted with respect to the annealed one but that the order of the transition is change: the free-energy grows quadratically for $d \geq 3$ (proved only for Gaussians) and the free-energy curve is believed to be even smoother in dimension $d = 2$.

Theorem 1. *(Critical point and critical behavior for the free energy)*

(i) *For $d \geq 2$, for all $\beta > 0$, we have $h_c(\beta) = \lambda(\beta)$.*
(ii) *For $d \geq 2$, for all $\beta > 0$, we have $\textsc{f}(0, u + \lambda(\beta)) \leq C_\beta u^2$ for $u \in [0, 1]$*
(iii) *For $d \geq 3$, for all $\beta > 0$, we have $\textsc{f}(0, u + \lambda(\beta)) \leq c_\beta u^2$ for $u \in [0, 1]$ if ω is Gaussian.*
For general ω we have $\textsc{f}(0, u + \lambda(\beta)) \leq c_\beta u^{66d}$.

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History dependent stochastic processes and non linear sigma models

Margherita Disertori
(joint work with T.Spencer, M.Zirnbauer, C. Sabot, P. Tarrès)

1. Two history dependent stochastic processes

Edge reinforced random walk (ERRW) and vertex reinforced jump processes (VRJP) are history dependent stochastic processes where the particle tends to come back more often on sites it has already visited in the past.

We will consider here a finite cube $\Lambda \subset \mathbb{Z}^d$ (though all definitions can be generalized to any finite graph). Let E be the set of non oriented edges (nearest neighbor pairs) in $\mathbb{Z}^d$. Each edge $e \in E$ is associated to a pair of vertices in $\mathbb{Z}^d$: $e = (i_e, j_e)$ (with some arbitrary order). Similarly we introduce E_Λ the set of edges inside Λ. A pair of nearest neighbor vertices i, j will be often denoted by $i \sim j$.

1.1. **Linearly Edge Reinforced Random Walk (ERRW)** [1, 2]**.** This is a dicrete time process $(X_n)_{n\in\mathbb{N}}$ on Λ with conditional transition probability at time n given by $\mathbb{P}\left(X_{n+1}=j|\, X_n=i, X_{n-1},\dots,X_0\right) = \mathbf{1}_{i\sim j}\frac{\omega_{ij}(n)}{\sum_{k,k\sim i}\omega_{ik}(n)}$. The local conductance $\omega_e(n)\geq 0$ across the edge $e=(i_e,j_e)$ at time n is given by $\omega_e(n) = \omega_{(i_e,j_e)}(n)=\omega_{(j_e,i_e)}(n) = a_e+T_e(n)$, where $a_e\geq 0$ is a fixed parameter and $T_e(n)$ is the numnber of crossings of the edge e up to time n. The conductance is linear in the local time $T_e(n)$.

1.2. **Vertex Reinforced Jump Process (VRJP)** [3, 4, 5]**.** This is a continuous time jump process $(Y_\tau)_{\tau\geq 0}$ on Λ. At each time τ, conditioned on $(Y_s)_{s\leq\tau}$, the process jumps from i to a neighbor j with rate $\omega_{ij}(\tau)=\beta_{ij}e^{T_i(\tau)+T_j(\tau)}$, where $\beta_e=\beta_{i_ej_e}=\beta_{j_ei_e}\geq 0$ is a fixed parameter and $T_i(\tau)$ is the total time spent by the process at the vertex i up to time τ. Note that in the original definition the rate was linear $\omega_{ij}(\tau)=\beta_{ij}[1+T_j(\tau)]$. Here we use a time changed version, introduced in [5]. A discrete time version is obtained by taking $Z_n=Y_{\tau_n}$, $n\in\mathbb{N}$, where τ_n is the instant of the $n-th$ jump.

1.3. **Special features.** In both processes the transition probability (resp. jump rate) depends the previous history and is larger for edges (resp. vertices) that have already been visited many times. The parameters a_e,β_e characterize the strenght of the reinforcement: large/small a_e,β_e correspond to weak/strong reinforcement. For the *particular scheme* of reinforcement defined above the two processes $(X_n)_n$ and $(Z_n)_n$ are **random walks in a random environment** (mixture of reversible Markov chains) . More precisely

$$\mathbb{P}^Z_{\Lambda,0}[\cdot]=\int d\mu_{\beta,\Lambda}(t)\mathbb{P}^{W(t,\beta)}_{\Lambda,0}[\cdot],\qquad \mathbb{P}^X_{\Lambda,0}[\cdot]=\int d\gamma_a(\beta)d\mu_{\beta,\Lambda}(t)\mathbb{P}^{W(t,\beta)}_{\Lambda,0}[\cdot]$$

where: *(a)* $\mathbb{P}^Z_{\Lambda,0}$ (resp.$\mathbb{P}^X_{\Lambda,0}$) is the probability associated to the process Z (resp. X) on Λ, starting at the vertex 0; *(b)* $\mathbb{P}^{W(t,\beta)}_{\Lambda,0}$ is the probability associated to a Markov chain with weights $W_e(t,\beta)=\beta_{i_ej_e}e^{t_{i_e}+t_{j_e}}$; *(c)* the field $t:\Lambda\to\mathbb{R}$ has probability measure $d\mu_{\beta,\Lambda}(t)$; *(d)* in the last equation the parameters β_e are independent gamma distributed random variables $\beta_e\sim\Gamma(a_e)$. The mixing measure $d\mu_{\beta,\Lambda}(t)$ is given by

$$d\mu_{\beta,\Lambda}(t)=e^{-\sum_{i\sim j}\beta_{ij}(\cosh(t_i-t_j)-1)}e^{-\beta_{0\delta}(\cosh t_0-1)}\sqrt{\det(D+\epsilon)}\,\prod_j\frac{dt_je^{-t_j}}{\sqrt{2\pi}}$$

where $D=-\Delta^{W(t,\beta)}$ is the discrete Laplacian with local conductance $W_{ij}(t,\beta)$ and $\epsilon_{ij}=\delta_{ij}\delta_{i0}\beta_{0\delta}e^{t_0}$. Finally δ is an auxiliary point (not in the lattice) that is connected only to the vertex 0.

1.4. **Some criterions for transience/recurrence.** Let e_0 be an arbitrary edge attached to the origin 0 and e any other edge.

Transience. Let $\mathbf{d}\geq\mathbf{3}$.

$$\mathbb{E}_\Lambda\left[\frac{W_{e_0}(t,\beta)}{W_e(t,\beta)}\right]\leq const\qquad \text{unif. in }\Lambda,e,e_0\quad\Rightarrow\ \text{transience.}$$

Positive recurrence. Let $\mathbf{d} \geq \mathbf{1}$. If $\exists 0 < \alpha \leq 1$ and $c > 0$ such that

$$\mathbb{E}_\Lambda \left[\left(\frac{W_e(t,\beta)}{W_{e_0}(t,\beta)} \right)^\alpha \right] \leq e^{-c|e|}$$

uniformly in Λ, then the process is positive recurrent.

1.5. **Transience for weak reinforcement.**

Theorem 1 (D., Spencer, Zirnbauer [6])**.** *If $d \geq 3$ and $\beta = \inf_e \beta_e \gg 1$ then*

$$\int d\mu_{\beta,\Lambda}(t)\ (\cosh(t_i - t_j))^m \leq 2$$

for all $i, j \in \Lambda$, $m \leq \beta^{1/8}$, uniformly in Λ.

Theorem 2 (D., Sabot, Tarrès [7])**.** *Let $d\tilde{\mu}_{a,\Lambda}(t)$ be the marginal t of the measure $d\gamma_a(\beta)d\mu_{\beta,\Lambda}(t)$. Then, if $d \geq 3$ and $a = \inf_e a_e \gg 1$ we have*

$$\int d\gamma_a(\beta)d\mu_{\beta,\Lambda}(t)\ (\cosh(t_i - t_j))^m = \int d\tilde{\mu}_{a,\Lambda}(t)\ (\cosh(t_i - t_j))^m \leq 2$$

for all $i, j \in \Lambda$, $m \leq a^{1/8}$, uniformly in Λ.

Using the criterions above, these results imply transience in $d \geq 3$ for ERRW and VRJP. The proof uses a relation between the mixing measures and a non-linear sigma model introduced in the context of random matrix models for quantum diffusion.

2. Relation with non linear sigma models

A non linear sigma model can be seen as a statistical mechanical model where the spin takes values on some non linear manifold and the measure is of gradient type. Here we consider the so called $H^{2|2}$ model [8] on Λ, where the spin $\phi : \Lambda \to \mathcal{M}$ takes values on a real Grassmann algebra. Precisely $\phi_j = (x, y, z, \xi, \eta)_j$, where x, y, z are even elements and ξ, η are odd elements in a real Grassmann algebra. We introduce the (non-positive definite) scalar product $\langle \phi, \phi' \rangle = xx' + yy' - zz' + \xi\eta' - \eta\xi'$. Adding the constraint $\langle \phi, \phi \rangle = -1$ our spin is restricted to the (non compact) non linear manifold parametrized by $z^2 = 1 + x^2 + y^2 + 2\xi\eta$. We consider the following two measures

$$d\rho_{\beta,\Lambda}(\phi) = e^{-\frac{1}{2}\beta_{0\delta}\langle(\phi_i - h),(\phi_i - h)\rangle} \prod_{i\sim j} e^{-\frac{1}{2}\beta_{ij}\langle(\phi_i - \phi_j),(\phi_i - \phi_j)\rangle} \prod_j d\phi_j \delta\left(\langle \phi_j, \phi_j \rangle + 1\right)$$

$$d\tilde{\rho}_{\beta,\Lambda}(\phi) = \frac{1}{(1+\langle(\phi_i - h),(\phi_i - h)\rangle)^{a_{0\delta}}} \prod_{i\sim j} \frac{1}{(1+\langle(\phi_i - \phi_j),(\phi_i - \phi_j)\rangle)^{a_{ij}}} \prod_j d\phi_j \delta\left(\langle \phi_j, \phi_j \rangle + 1\right)$$

where $h = (0, 0, 1, 0, 0)$ plays the role of a magnetic field and is necessary to ensure the integral is finite.

2.1. **VRJP, ERRW and $\mathbf{H}^{2|2}$.** After performing the change of coordinates $(x, y, \xi, \eta) \to (t, s, \bar{\psi}, \psi)$ defined by

$$x = \sinh t + e^t \left(\frac{s^2}{2} + \bar{\psi}\psi \right), \; y = e^t s, \; \xi = e^t \bar{\psi}, \; \eta = e^t \psi$$

the mixing measure $d\mu_{\beta,\Lambda}(t)$ is the t marginal of $d\rho_{\beta,\Lambda}(\phi)$, while $d\tilde{\mu}_{\beta,\Lambda}(t)$ is the t marginal of $d\tilde{\rho}_{\beta,\Lambda}(\phi)$.

2.2. **Proof of theorems 1 and 2.** Both measures $\rho_{\beta,\Lambda}(\phi)$ and $\tilde{\rho}_{\beta,\Lambda}(\phi)$ are invariant under global rotations $\phi_j \to \phi'_j = R\phi_j$, that leave the scalar product invariant. Some of these rotations mix even and odd elements in the Grassmann algebra. As a result we obtain a family of (highly non trivial) identities for the measures $\mu_{\beta,\Lambda}(t)$, $\tilde{\mu}_{\beta,\Lambda}(t)$.

The proof uses a special family of such identities plus an inductive argument on the distance $|i - j|$.

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A random Schrödinger operator associated with the Vertex Reinforced jump Process and the Edge Reinforced Random Walk

Christophe Sabot

(joint work with P. Tarrès, X. Zeng)

1. Abstract

The ERRW and the VRJP are self-interacting processes that have been shown to be related to a supersymetric sigma field investigated by Disertori, Spencer and Zirnbauer. In this talk we construct a random Schrödinger operator, with a 1-dependent potential, and show that some of its spectral properties at ground state are related to the behavior of the VRJP and ERRW. We deduce from this a functional central limit theorem for the ERRW and the VRJP in dimension $d \geq 3$ at weak disorder, and recurrence of the ERRW in dimension d=2 for any choice of initial constant weights, hence answering a longstanding open question of Diaconis.

2. Introduction

2.1. **The Edge Reinforced Random Walk (ERRW) and the Vertex Reinforced Jump Process (VRJP).** The ERRW is famous linearly reinforced process introduced by Diaconis and Coppersmith in the 80's [2]. Important progress have been done in the understanding of this process in the last years, in particular it has been proved in [7, 1], that ERRW is positive recurrent on graphs with bounded degree at strong reinforcement. A phase transition has been shown in [4], in dimension $d \geq 3$: more precisely, it has been shown that the ERRW is transient on $\mathbb{Z}^d$, $d \geq 3$, when the reinforcement is weak.

The VRJP has been introduced by Davis and Volkov and suggested by Werner. Let us recall its definition. Assume that $G = (V, E)$ is a non-directed connected graph with finite degree at each site. Consider $(W_e)_{e \in E}$ some positive conductances on the edges. The VRJP is the continuous time process (Y_s) that, conditionally on its past at time s, jumps from i to $j \sim i$ with rate $W_{i,j}\, L_j(s)$ where $L_j(s) = 1 + \int_0^s \mathbf{1}_{Y_u = j} du$. It has been investigated for some time independently of the ERRW, but in 2011, [7], it has been shown to be closely related to the ERRW. More precisely, the ERRW corresponds to a VRJP with random conductances, $(W_{i,j})$, independent at each edge with Gamma distribution of parameters $(a_{i,j})$, the initial weights of the ERRW. Hence, for a large part, the analysis of the ERRW and of the VRJP are equivalent.

The main consequence of the forthcoming theorem 4 is the following.

Theorem 1 ([8, 9]). *i)The ERRW is recurrent in dimension d=2, for all initial constant weights. (This part uses also crucial estimates from* [6]*.)*

ii) On $\mathbb{Z}^d$, $d \geq 3$, the ERRW (resp. the VRJP) at weak reinforcement, i.e. with constant weight $a_e = a$ for a large enough (resp. with constant conductances $W_e = W$ for W large enough), satisfies a functional central limit theorem with non-degenerate diffusion matrix.

2.2. **Relation with a supersymetric σ-model.** In [7], it was proved that, after some time change, the VRJP can be represented as a mixture of Markov Jump processes, with a mixing law which is a marginal of a supersymmetric σ-field investigated by Diserori, Spencer, Zirnbauer, [5]. Results of localization/delocalization which have been proved in [3, 5], can be translated in results on positive recurrence/transience of the VRJP, and with some extra work, [4], also of the ERRW. Nevertheless, the question of the behavior of these processes on $\mathbb{Z}^2$ at weak reinforcement was open. We give below a new representation of the VRJP in terms of the Green function of a random Schrödinger operator. This represention can be non-trivially extended to infinite graphs.

3. Main Results

3.1. **A (new) exponential family.** Let, as before, $G = (V, E)$ be a non-directed graph with conductances $(W_e)_{e\in E}$. Let $\Delta = (\Delta_{i,j})_{i,j\in V}$ be the discrete Laplacian

$$\Delta_{i,j} = \begin{cases} W_{i,j}, & \text{if } i \sim j,\ i \neq j \\ -W_i, & \text{if i=j,} \end{cases}$$

with $W_i = \sum_{j\sim i} W_{i,j}$. For $(\beta_j)_{j\in V} \in \mathbb{R}^V$, we set

$$H_\beta = -\Delta + 2\beta$$

where β is the operator of multiplication by $(\beta_j)_{j\in V}$. If V finite, we write $H_\beta > 0$ when it is positive definite. In this case $(H_\beta)^{-1}$ has positive coefficients (it is an M-Matrix).

Lemma 2 ([8]). *Assume $\mathcal{G}$ is finite. The following distribution on $\mathbb{R}^V$*

$$\nu^W(d\beta) = \mathbf{1}_{H_\beta>0}\frac{e^{-\sum_{i\in V}\beta_i}}{\sqrt{|H_\beta|}}\ \frac{\prod_{i\in V} d\beta_i}{\sqrt{2\pi}^{|V|}}$$

is a probability with Laplace transform

$$\int e^{-\lambda\cdot\beta}\nu^W(d\beta) = \frac{e^{\frac{1}{2}\sum_{i\sim j} W_{i,j}(\sqrt{1+\lambda_i}-\sqrt{1+\lambda_j})^2}}{\prod_{i\in V}\sqrt{1+\lambda_i}}$$

In particular $\beta_{|V_1}$ and $\beta_{|V_2}$ are independent if $dist_{\mathcal{G}}(V_1, V_2) \geq 2$.

Theorem 3 ([8]). *Let $\beta \sim \nu^W(du)$, and set $G = (H_\beta)^{-1}$. Define $u(i,j)$ by*

$$e^{u(i,j)} = \frac{G(i,j)}{G(i,i)}$$

Then the (time changed) VRJP starting from (i_0) is a mixture of Markov jump processes with jumping rate

$$\frac{1}{2}W_{i,j}e^{u(i_0,j)-u(i_0,i)} = \frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}$$

This representation has several advantages compared to the representation of section 2.2. Firstly, it gives a representation of the VRJP by the Green function of a random Schrödinger operator. Secondly, it couples the mixing laws of the VRJP starting from different points.

3.2. **Representation of the VRJP on infinite graphs.** The aim is now to extend the representation of theorem 3 to infinite graphs. Take for simplicity $V = \mathbb{Z}^d$ and $W_{i,j} = W$ constant. It is rather easy to prove that $(\beta_j)_{j\in V}$ can be defined on infinite graphs by Kolmogorov extension theorem, with the same properties of independence at distance larger or equal to 2. We can thus define $H_\beta = -\Delta + \beta$, the associated random Schrödinger operator. Then $H_\beta \geq 0$ and the limit

$$\hat{G} := \lim_{\epsilon\to 0, \epsilon>0} (H_\beta + \epsilon)^{-1}$$

exists a.s. and $0 < \hat{G}(i,j) < \infty$. We come now to the main theorem.

Theorem 4 ([9]). *i) We construct a β-measurable function $(\psi(i))_{i\in V}$, as a martingale limit, such that*

- *$\psi = 0$ when the VRJP is recurrent,*
- *$\psi(i) > 0$ for all $i \in V$ when the VRJP is transient, ψ is stationary ergodic and $H_\beta\psi = 0$*

ii) Let γ be an Gamma($\frac{1}{2}$) random variable indep. of β. Set

$$G(i,j) = \hat{G}(i,j) + \frac{1}{2\gamma}\psi(i)\psi(j).$$

Then VRJP starting at i_0 is a mixture of Markov jump processes with jump rates

$$\frac{1}{2} W_{i,j} \frac{G(i_0,j)}{G(i_0,i)}$$

The stationarity and ergodicity of ψ is the key ingredient in theorem 1.

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Graphical representations for quantum spin systems

Daniel Ueltschi

1. Introduction

Random loop approaches to quantum spin systems offer an elegant and different perspective to quantum correlations. They find their origin in Feynman-Kac representations of quantum lattice systems. In 1993, Tóth introduced a representation of the $S = \frac{1}{2}$ ferromagnetic Heisenberg model that is based on the random interchange model [2]. A similar representation was introduced shortly afterwards by Aizenman and Nachtergaele for the $S = \frac{1}{2}$ antiferromagnet model and certain models with higher spins [1]. It allowed them to relate the one-dimensional quantum chain to two-dimensional Potts and random cluster models, yielding new insights on the quantum spin chain.

A synthesis of these two representations was proposed in [3]. In the case $S = \frac{1}{2}$, it applies to models that interpolate between the Heisenberg ferromagnetic and antiferromagnetic models such as the quantum XY model. It also applies to certain SU(2)-invariant models of spin 1. Thanks to this representation, the existence of spin nematic long-range order was established in the model with $S = 1$ in dimension $d \geq 3$ [3]. In this report, we review the derivation of these graphical representations for the partition function and correlations of quantum spin systems. We restrict ourselves to the case $S = \frac{1}{2}$.

2. Quantum spin models

Let $(\Lambda, \mathcal{E})$ be a graph, with Λ the (finite) set of vertices and $\mathcal{E}$ the set of edges. Given $S \in \frac{1}{2}\mathbb{N}$, the Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^2$. The spin operators are denoted S^i_x, with $x \in \Lambda$ and $i = 1, 2, 3$. They satisfy the commutation relations $[S^1_x, S^2_y] = \mathrm{i}\delta_{x,y}S^3_x$, and further relations obtained by cyclic permutations of the indices $1, 2, 3$. Recall that "classical configurations" $\sigma \in \{-\frac{1}{2}, \frac{1}{2}\}^\Lambda$ form a basis of $\mathcal{H}_\Lambda$ where the operators S^3_x are diagonal: Using Dirac's notation, $S^3_x|\sigma\rangle = \sigma_x|\sigma\rangle$.

We consider the two operators T_{xy} and Q_{xy} on $\mathcal{H}_{\{x,y\}}$ (and their extensions on $\mathcal{H}_\Lambda$ by identifying T_{xy} with $T_{xy} \otimes \mathtt{Id}_{\Lambda\setminus\{x,y\}}$, etc...):

- T_{xy} is the transposition operator, $T_{xy}|a, b\rangle = |b, a\rangle$.
- Q_{xy} is the operator with matrix elements $\langle a, b|Q_{xy}|c, d\rangle = \delta_{a,b}\delta_{c,d}$.

We consider the following family of Hamiltonians indexed by the parameter $u \in [0, 1]$:

$$H^{(u)}_\Lambda = -\sum_{\{x,y\}\in\mathcal{E}} \Big(uT_{xy}+(1-u)Q_{xy}-1\Big) = -2\sum_{\{x,y\}\in\mathcal{E}} \big(S^1_xS^1_y+(2u-1)S^2_xS^2_y+S^3_xS^3_y-\tfrac{1}{4}\big).$$

The case $u = 1$ is the Heisenberg ferromagnet. The case $u = \frac{1}{2}$ is the quantum XY model. If the graph is bipartite, the case $u = 0$ is unitarily equivalent to the Heisenberg antiferromagnet. Let $Z^{(u)}(\beta, \Lambda) = \mathrm{Tr}\, \mathrm{e}^{-\beta H^{(u)}_\Lambda}$ denote the corresponding partition functions.

3. Random loop models

We now describe the models of random loops. At each edge $\{x, y\} \in \mathcal{E}$ is attached the interval $[0, \beta]$ and a Poisson point measure where "crosses" occur with intensity u and "double bars" occur with intensity $1-u$. Let ω denote a realisation and $\rho(\mathrm{d}\omega)$ denote independent Poisson point processes on $\mathcal{E} \times [0, \beta]$. To a given realisation ω of the Poisson point measure corresponds a set of loops, denoted $\mathcal{L}(\omega)$. The loops consist of vertical lines connected by crosses or bars. This is best understood by looking at pictures, see Fig. 1. A mathematically precise definition can be found in [3]. We define the partition function as $Y(\beta, \Lambda) = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega)$. The relevant measure for the model of random loops is given by $\frac{1}{Y(\beta,\Lambda)} \, 2^{|\mathcal{L}(\omega)|} \, \rho(\mathrm{d}\omega)$.

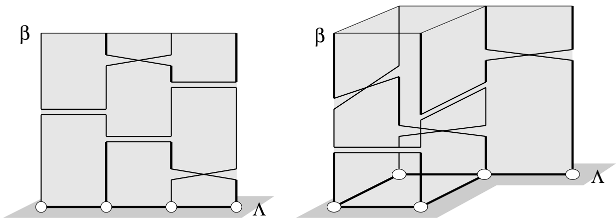

Figure 1. Graphs and realisations of Poisson point measures, and their loops. In both cases, the number of loops is $|\mathcal{L}(\omega)| = 2$.

4. Relations with quantum spin models

The first result is a formula for the Gibbs operator in terms of the Poisson point measure. To a realization ω corresponds a sequence $(A_1, t_1), \dots, (A_n, t_n)$ where $0 < t_1 < \cdots < t_n < \beta$ are the times for the occurrence of events in ω, and A_j is the operator T_{xy} if the event of time t_j is a cross at $\{x, y\} \in \mathcal{E}$, or the operator Q_{xy} if the event of time t_j is a double bar at $\{x, y\}$.

Lemma 1. *We have* $\mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\omega) A_n A_{n-1} \dots A_1.$

The proof proceeds by discretising the time interval $[0, \beta]$, linearising the Poisson point measure, grouping terms wisely and invoking the Trotter product formula.

Theorem 2. *For all* $u \in [0, 1]$, *we have* $Z^{(u)}(\beta, \Lambda) = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega).$

For the proof we need the concept of *space-time configurations.* These are functions $\sigma : \Lambda \times [0, \beta] \longrightarrow \{-\frac{1}{2}, \frac{1}{2}\}$ such that $\sigma_{x,t}$ is piecewise constant in t, for any x. Given a realisation ω of the Poisson point measure, let $\Sigma_{\mathrm{per}}(\omega)$ denote the set of space-time spin configurations that take constant values along each loop. See Fig. 2 for an illustration. Notice that $|\Sigma_{\mathrm{per}}(\omega)| = 2^{|\mathcal{L}(\omega)|}$. We use Lemma 1 and we insert the resolution of the identity $\mathtt{Id} = \sum_\sigma |\sigma\rangle\langle\sigma|$ on the left of each transition A_j. By the definitions of T_{xy} and Q_{xy}, we get Theorem 2 using

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\omega) \sum_{\sigma \in \Sigma_{\mathrm{per}}(\omega)} 1. \tag{1}$$

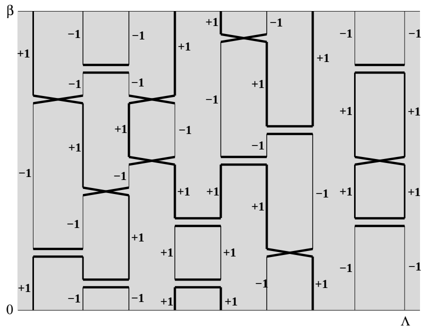

FIGURE 2. Illustration for a realisation of the measure ρ and a compatible space-time spin configuration.

The loop correlations are given by just three events: (i) $E^+_{x,y}$ is the set of all realizations ω such that x and y belong to the same loop, and with identical vertical direction at these points; (ii) $E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop, and with opposite vertical directions at these points; (iii) $E_{x,y} = E^+_{x,y} \cup E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop. These events are illustrated in Fig. 3. Let $\mathbb{P}(\cdot)$ denote the probability with respect to the random loop measure $2^{|\mathcal{L}(\omega)|}\rho(\mathrm{d}\omega)/Z^{(u)}(\beta,\Lambda)$.

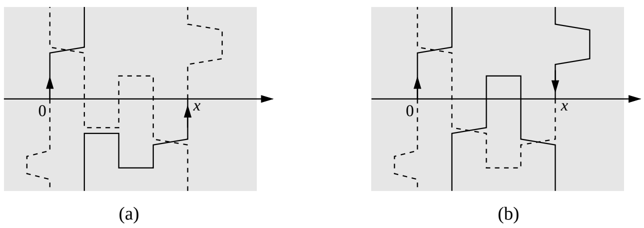

FIGURE 3. Illustration for (a) the event $E^+_{0,x}$; (b) the event $E^-_{0,x}$.

Theorem 3. *For $x \neq y$, the two-point correlation functions are given by*

$$\langle S^i_x S^i_y \rangle = \begin{cases} \frac{1}{4}\mathbb{P}(E_{x,y}) & \text{if } i = 1,3, \\ \frac{1}{4}[\mathbb{P}(E^+_{x,y}) - \mathbb{P}(E^-_{x,y})] & \text{if } i = 2. \end{cases}$$

This theorem can be proved as Theorem 2 with an expansion in space-time spin configurations. We refer to [3] for details.

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Deconstructing Brownian loop-soups

Wendelin Werner
(joint work with Wei Qian)

Background. Recall that when D is a simply connected domain in the plane, there exist two very natural measures on Brownian paths that do not require to choose any reference (or starting) point in the domain. These are on the one hand the Brownian loop-measure (introduced in [1] – this is a measure on unrooted Brownian loops) and on the other hand the Brownian excursion measure (this is a measure on Brownian excursions in D – these are paths that start and end on different non-prescribed points in ∂D and stay in D inbetween). Both these measures μ_D and ν_D are conformally invariant (under any conformal transformation from D onto itself, and the images of μ_D and ν_D under a conformal transformation from D onto D' are $\mu_{D'}$ and $\nu_{D'}$) when one views paths up to monotone time-reparametrization.

This makes it possible for each positive c and β to define a Poissonian cloud of Brownian loops with intensity $c\mu_D$ (this is the Brownian loop-soup with intensity c) and also a Poissonian cloud of Brownian excursions in D with parameter β. The laws of these random collections of paths are conformally invariant (because of the conformal invariance of their intensity measure).

It turns out that the structure of the Brownian loop-soups in two dimensions is particularly interesting. One can for instance study the loop-soup clusters (two loops b and b' will be in the same cluster if one can find a finite chain of other loops $b_1, \ldots, b_{n-1}$ in the soup so that if one puts $b_0 = b$ and $b_n = b'$, then $b_j \cap b_{j-1} \neq \emptyset$ for all $j \leq n$). It has been proved in [9] that all loops form a single dense cluster when $c > 1$, but that when $c \leq 1$, there are countably many disjoint clusters. Furthermore (see again [9]), the outer boundaries of these clusters are loop-variants of Schramm's SLE_κ curves where $\kappa \in (8/3, 4]$ and c are related by the formula $c = (3\kappa - 8)(6 - \kappa)/2\kappa$ (more precisely, the outer boundaries of the outermost loop-soup clusters form a CLE_κ — a conformal loop-ensemble with parameter κ).

Note that both the loop-soup with intensity $c = 1$ and CLE_4 can be coupled with a Gaussian Free Field: Miller-Sheffield [5] have shown that CLE_4 can be viewed as the set of level lines of the GFF with zero boundary conditions in D (building on earlier work by Schramm, Sheffield and Dubédat that relate the GFF to SLE_4 curves). In a different direction, Le Jan [2] has shown (and this is not specific to two dimensions) that one can interpret the (properly defined) square of the GFF as the (properly renormalized) total occupation time measure of a Brownian loop-soup with intensity $c = 1$. Note the previous result on loop-soups shows that there is also a coupling between the $c = 1$-loop-soup with CLE_4.

Statement and some remarks. In [7], we derive a decomposition of the Brownian loop-soups with $c = 1$, that can be described as follows. When one conditions a loop-soup cluster (say the outermost loop-soup cluster that surrounds the origin in a loop-soup in the unit disc) on its outer boundary γ (which is a CLE_4 loop),

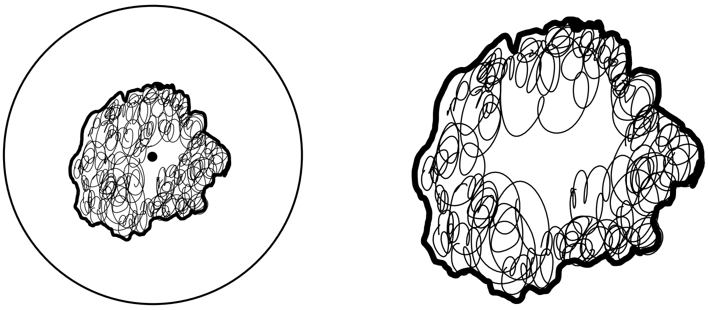

Figure 1. Sketch of the outermost loop-soup cluster K surrounding the origin in a loop-soup in the unit disc, and its outer boundary γ in bold. On the right, a magnification of γ and of the loops in K that touch γ.

one can consider the loops in the loop-soup that are inside of γ. There are two type of such Brownian loops – those that do not touch γ and stay in the open set $O(\gamma)$ encircled by γ, and on the other hand, those that do touch γ (but stay in $\overline{O}(\gamma)$). While (conditionally on γ) the former form a Brownian loop-soup in $O(\gamma)$ with $c = 1$ which is not surprising, the trace of the latter in $O(\gamma)$ coincides with that of a Poisson point process of Brownian excursions in $O(\gamma)$ with intensity $\beta = 1/4$.

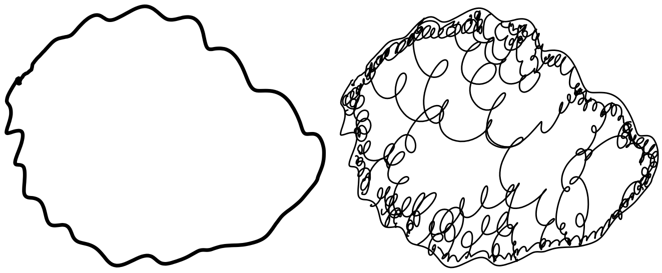

Figure 2. Sketch of a sample of a SLE_4 loop (which is distributed like the outer boundary γ of a loop-soup cluster K), and then of a Poisson point process of excursions inside γ. The obtained picture is distributed exactly like the picture on the right of the first Figure (i.e. like the union of the Brownian loops in K that touch the outer boundary of K.

This result can look at first fairly surprising, as one might have guessed that because these excursions away from γ have to hook up into loops, they should not be independent (as in a Poisson point process).

In the course of the proof, we use and relate the previously mentioned three couplings of the GFF, the loop-soup and CLE_4, and prove that they can be made to coincide (the outer boundary of the loop-soup clusters define the same CLE_4 as the GFF ϕ, while the occupation time of this loop-soup is exactly the square of ϕ). Other instrumental tools in the derivation of our decomposition are Dynkin's isomorphism theorem, Titus Lupu's recent construction of the GFF associated to a loop-soup, via the use of cable-systems [4, 3].

This result sheds some new light on the relation between the GFF, loop-soups and CLE_4, their Markovian properties, and it does also give an approach via the loop-soups to ideas developed by Schramm, Sheffield and Miller in the GFF context, such as the definition of local sets in [8, 6]. Details about motivations and implications of this result and about its proof can be found in [7].

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Hyperbolic planar maps

Omer Angel

(joint work with Tom Hutchcroft, Asaf Nachmias, Gourag Ray)

1. A dichotomy for unimodular planar maps

The classical Uniformization Theorem for Riemann surfaces (Koebe, Poincare, 1907), states that every simply connected, non-compact Riemann surface S is conformally equivalent to either the hyperbolic plane or the Euclidean plane. This dichotomy manifests itself in several different ways, relating to analytic, geometric and probabilistic properties of surfaces. In particular, either

> S is conformally equivalent to the plane, admits a Riemannian metric of constant curvature 0, does not admit non-constant bounded harmonic functions, and is recurrent for Brownian motion,

or else

> S is conformally equivalent to the hyperbolic plane, admits a Riemannian metric of constant curvature -1, admits non-constant bounded harmonic functions, and is transient for Brownian motion.

A discrete counterpart to this dichotomy appeared in the seminal work of He and Schramm [11, 10]. They proved that every plane triangulation can be circle packed in either in the unit disc or in the plane, but not both, calling the triangulation **CP hyperbolic** or **CP parabolic** accordingly.

In the case of triangulations with bounded degrees, they also connected the CP type to recurrence of the random walk. Benjamini and Schramm [6, 7] proved that every bounded degree, infinite planar map admits non-constant bounded harmonic functions if and only if it is transient for simple random walk, and in this case also admits non-constant bounded harmonic functions of finite Dirichlet energy.

Most of this theory fails without the assumption of bounded degrees. The goal of this work is to develop a similar theory for *unimodular random rooted maps*, without the assumption of bounded degree. A rooted map is a map together with a distinguished root vertex, and a random rooted map is said to be *unimodular* if it satisfies the *mass-transport principle*, which, in a precise sense, can be interpreted as meaning that 'every vertex of the map is equally likely to be the root', even when the maps in question are infinite. This theory applies to several maps that have been studied before, including the hyperbolic uniform triangulations [9] and the Poisson-Voronoi maps [5]. (A form of this dichotomy for triangulations appeared in earlier work [2].)

Theorem 1 (The Dichotomy Theorem)**.** *Every unimodular random rooted planar map is either hyperbolic or parabolic. The type of the map is determined by its average curvature, and determines many of its properties.*

The curvature is defined by constructing a manifold from a map. This is done by taking some manifold which is a topological disc for each face of the map, and gluing them together along edges. This can be done (not uniquely) is such a way that the manifold is flat except at the vertices. The average curvature is the expected curvature at the root ρ.

A more precise formulation of the theorem, is that for a unimodular rooted planar map (M, ρ), the following are almost surely equivalent:

(1) (M, ρ) has average curvature zero.
(2) (M, ρ) is invariantly amenable.
(3) (M, ρ) is a Benjamini-Schramm limit of finite planar maps.
(4) The Riemann surface associated to M is conformally equivalent to the plane $\mathbb{C}$ or the punctured plane $\mathbb{C} \setminus \{0\}$.
(5) M does not admit any non-constant bounded harmonic functions.
(6) M does not admit any non-constant harmonic functions of finite Dirichlet energy.
(7) The laws of the free and wired uniform spanning forests of M coincide.
(8) The wired uniform spanning forest of M is a.s. connected.
(9) Two independent random walks on M intersect infinitely often a.s.
(10) The wired and free minimal spanning forests of M have the same law.
(11) Bernoulli(p) bond percolation on M a.s. has at most one infinite connected component for every $p \in [0, 1]$.

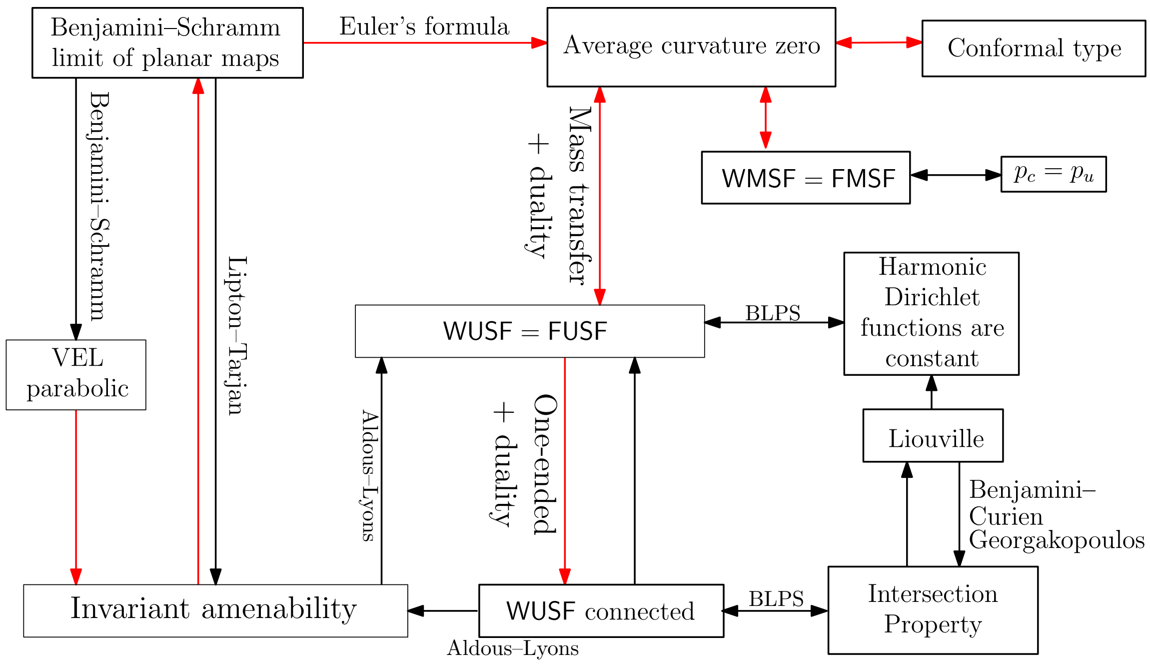

FIGURE 1. The logical structure, with new implications in red.

(12) M is vertex extremal length parabolic.
(13) Every bounded degree subgraph of M is recurrent.
(14) Every bounded degree subgraph of M is amenable.
(15) Every tree in M is recurrent almost surely.
(16) Every tree in M is amenable almost surely.

Maps where these properties hold are called parabolic, and maps where they do not are hyperbolic.

2. Soficity of unimodular planar maps

The **local topology** on rooted graphs is generated by the metric $d(G, G') = e^{-R}$ where R is maximal such that the balls $B_R(G,\rho) = B_R(G',\rho')$ are isomorphic. A random rooted graph (G,ρ) is the **Benjamini-Schramm limit** of a sequence of (possibly random) finite graphs G_n if, letting ρ_n be a uniform vertex of G_n for each $n \geq 1$, the random rooted graphs (G_n, ρ_n) converge weakly in the local topology to (G,ρ) as n tends to infinity. Benjamini-Schramm limits of finite maps and networks are defined similarly. A random rooted graph that is a limit of finite graphs in this way is called **sofic**.

Every finite graph with uniformly random root is unimodular, and unimodularity is preserved under weak limits, so every sofic graph is unimodular. It is a major open problem to determine whether the converse holds, that is, whether every unimodular random rooted graph is sofic (see [1]). We answer this question positively for simply connected unimodular random maps.

Theorem 2. *Every simply connected unimodular random rooted map is sofic.*

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The variance of particle positions in the hard disk model

Thomas Richthammer

A system of interacting particles may arrange itself into a regular pattern such as a lattice-like structure and thus may form a solid state. This phenomenon is usually referred to as crystallization. We restrict our attention to the case of two-dimensional systems. Here this phenomenon is expected to occur in a wide variety of models, provided that the temperature is sufficiently low or the particle density is sufficiently high. Indeed this behaviour can be observed in simulations, but so far there is no rigorous proof for crystallization in any realistic model. For two-dimensional systems the phase transition corresponding to crystallization is conjectured to be due to a breaking of rotational symmetry; i.e. two-dimensional solids exhibit directional order. In a simplified model this has been shown in [4]. In contrast, it is known that translational symmetry is not broken (see [6]); i.e. 2D solids can never exhibit positional order.

We would like to quantify the latter: If a 2D solid forms a lattice-like structure, then what can we say about the fluctuations of particle positions? For the 2D harmonic crystal the work of Peierls (see [5]) shows that the mean square displacement of a particle from its ideal lattice position is of order $\log n$ if n is the size of the system. We show a lower bound of the same size for more realistic models such as the hard disk model.

In the hard disk model the interaction between particles is a pure hard-core repulsion, i.e. any two point particles are forced to keep a distance of $> 2r$ but do not interact otherwise. Besides r the only parameter of the model is the activity z regulating the particle density. Such a model can be described in terms of a Gibbsian point process. In finite volume Λ it is obtained by prescribing a certain deterministic configuration of particles outside of Λ and generating a random configuration of particles inside of Λ by means of Poisson point process with intensity z, conditioned on any pair of particles to keep distance $> 2r$. Due to its simplicity the hard disk model is a good starting point for investigations. Simulations show (e.g. see [2]) that for an increasing value of z there is a phase transition from no order to orientational order. However, not much is known rigorously for the hard disk model apart from the result on the conservation of translational symmetry (see [6]) and a result on the percolation of disks with percolation distance $> 3r$ (see [1]).

Unlike in the harmonic crystal, in the hard disk model there is no a priori labelling of particles that would allow to pinpoint a specific particle and investigate the fluctuations of its position. Instead, we describe the fluctuations of positions in terms of a certain transformation of particle configurations with the following properties:

- The transformation shifts all particles of a given configuration in a predefined direction. The amount by which a particle is shifted may depend on its position and the positions of other particles of the configuration.
- Particles near the boundary of a box of size $2n \times 2n$ are kept fixed. Particles near the center are shifted by an amount of order $\sqrt{\log n}$.
- The translation amount for particles in between is chosen so that locally the transformation almost preserves the relative position of particles.
- After the transformation all particles still keep a distance $> 2r$.
- The transformation is bijective and only has a mild impact on the probability measure describing the hard disk model.

We note that the above conditions are in conflict for configurations of particles that are densely packed. Thus we define a set of good configurations (controlling the size of regions of the box where particle are densely packed) and proceed to give an explicit recursive construction of a transformation of the above type. We then are able to show that our construction has the following properties:

- Configurations are good with high probability if n is large.
- The transformation has all the above properties at least in case of good configurations.

If there is an a posteriori lattice structure by which we are able to identify a single particle, we show that the above construction implies that the fluctuations of the position of the particle are at least of order $\sqrt{\log n}$. For more details on our result we refer to [7].

A construction similar to the one described above was the main tool used in [6] for showing the absence of positional order. In [3] the method was adjusted to a lattice setting and used to show a delocalization result for the random Lipschitz surface model, including a lower bound on fluctuations of order $\sqrt{\log n}$. We use improvements and refinements of the construction from [3] and adjust them back to the continuous setting of the hard disk model to obtain our result on fluctuations of positions in the hard disk model. We note that our result can be extended to fairly arbitrary interactions.

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Superdiffusion in the periodic Lorentz gas

Bálint Tóth

(joint work with Jens Marklof)

1. Extended Abstract

We prove a superdiffusive central limit theorem for the displacement of a test particle in the periodic Lorentz gas in the limit of large times t and low scatterer densities (Boltzmann-Grad limit). The normalization factor is $\sqrt{t \log t}$, where t is measured in units of the mean collision time. This result holds in any dimension and for a general class of finite-range scattering potentials.

The periodic Lorentz gas is one of the iconic models of "chaotic" diffusion in deterministic systems. It describes the dynamics of a test-particle in an infinite periodic array of spherically symmetric scatterers. The main results characterizing the diffusive nature of the periodic Lorentz gas have to date been mainly restricted to the two-dimensional setting and hard-sphere scatterers. The first seminal result on this subject was the proof of a central limit theorem for the displacement of the test particle at large times t for the finite-horizon Lorentz gas by Bunimovich and Sinai [2]. In the case of the infinite-horizon Lorentz gas, Bleher [1] pointed out that the mean-square displacement grows like $t \log t$ when $t \to \infty$, as opposed to a linear growth in the finite-horizon case. The superdiffusive central limit theorem suggested in [1] was first proved by Szász and Varjú [7] for the discrete-time billiard map. Dolgopyat and Chernov [4] provided an alternative proof, and established the central limit theorem and invariance principle for the billiard flow. Both aforementioned results are valid in two dimensions only.

In the present work we prove an unconditional superdiffusive central limit theorem for the periodic Lorentz gas in any dimension $d \geq 2$, valid in the limit of low scatterer density (Boltzmann-Grad limit) and for a general class of finite-range scattering potentials.

In the present paper we prove unconditional superdiffusive central limit theorems and invariance principles for the periodic Lorentz gas in any dimension $d \geq 2$, valid in the limit of low scatterer density (Boltzmann-Grad limit) and for a general class of finite-range scattering potentials. That is, instead of fixing the radius r of each scatterer and considering the long time limit as in the above cited papers, we consider here the limit $r \to 0$ and then the limit of long times, where time is measured in units of the mean collision time. It remains a major interesting open problem to consider the two limits $r \to 0$, $t \to \infty$ jointly.

The precise setting is as follows. Let $\mathcal{L} \subset \mathbb{R}^d$ be a fixed Euclidean lattice of covolume one (such as the cubic lattice $\mathcal{L} = \mathbb{Z}^d$), and define the scaled lattice $\mathcal{L}_r := r^{(d-1)/d}\mathcal{L}$. At each point in $\mathcal{L}_r$ we center a sphere of radius r. We consider a test particle that moves along straight lines with unit speed until it hits a sphere, where it is scattered elastically. The above scaling of scattering radius vs. lattice spacing ensures that the mean free path length (i.e., the average distance between consecutive collisions) has the limit $\bar{\xi} = 1/v_{d-1}$ as $r \to 0$, where $v_{d-1} = \pi^{\frac{d-1}{2}}/\Gamma(\frac{d+1}{2})$ denotes the volume of the unit ball in $\mathcal{R}^{d-1}$.

In the case of the classic Lorentz gas the scattering mechanism is given by specular reflection, but as in [5] we will here also allow more general spherically symmetric scattering maps.

The position of our test particle at time t is denoted by

$$x(t) = x(t, x_0, v_0) \in \mathcal{K}_r := \mathbb{R}^d \setminus (\mathcal{L}_r + r\mathcal{B}_1^d),$$

where x_0 and v_0 are position and velocity at time $t = 0$, and $\mathcal{B}_1^d$ is the open unit ball in $\mathbb{R}^d$ centered at the origin.

We consider the time evolution of a test particle with random initial data (x_0, v_0), distributed according to a Borel probability measure Λ on $\mathbb{R}^d \times S_1^{d-1}$. The following superdiffusive central limit theorem and invariance principles, valid for small scattering radii and large times, are the main results of this work.

Theorem 1. *Let $d \geq 2$ and fix a Euclidean lattice $\mathcal{L} \subset \mathbb{R}^d$ of covolume one. Assume (x_0, v_0) is distributed according to an absolutely continuous Borel probability measure Λ on $\mathbb{R}^d \times S_1^{d-1}$. Then, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\lim_{t\to\infty} \lim_{r\to 0} \mathbf{E} f\left(\frac{x(t) - x(0)}{\Sigma_d \sqrt{t \log t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, e^{-\frac{1}{2}|x|^2} dx,$$

with

$$\Sigma_d^2 := \frac{2^{1-d} v_{d-1}}{d^2(d+1)\zeta(d)},$$

where $\zeta(d) := \sum_{n=1}^{\infty} n^{-d}$ denotes the Riemann zeta function.

The above result generalises to a functional central limit theorem, or, invariance principle. That is, for the same random initial data as in Theorem 1 the random curve

$$t \mapsto X_{T,r}(t) := \frac{x(Tt) = x(0)}{\Sigma_d \sqrt{t \log t}}$$

converges in distribution in $C_0([0,1])$ to the standard Brownian motion $t \mapsto W(t) \in \mathbb{R}^d$.

Theorem 2. *Under the conditions of Theorem 1, taking first $r \to 0$ and then $T \to \infty$, we have*

$$X_{T,r}(\cdot) \Rightarrow W(\cdot),$$

where $\Rightarrow$ denotes weak convergence of probability measures in $C_0([0,1])$.

The starting point of our analysis is the paper [5], which proves that, for every fixed $t > 0$, the first (inner) limit $r \to 0$ Theorem 1 exists and is given by a continuous-time Markov process. The main objective of this work is therefore to prove a superdiffusive central limit theorem and invariance principle for these Markov processes. This is realized by a sophisticated conditional Lindeberg argument.

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Aging of the Metropolis dynamics of the Random Energy Model

Jiří Černý

(joint work with Tobias Wassmer)

1. Introduction

In his influential paper [1], Bouchaud proposed a simple toy model suggesting a mechanism to understand the aging in low-temperature dynamics of spin glasses. This toy model, compared to dynamics of an actual spin glass model, introduces three main simplifications: (a) the state space of the spin glass is replaced by a complete graph, (b) the energies of states are taken to be a collection of i.i.d. random variables, and (c) the dynamics is very simple, the rate of jump from state x to state y does not depend on the energy of y, this dynamics is now called Random Hopping time (RHT) dynamics.

Starting with [2, 3], many authors tried to verify that Bouchaud's model prediction are valid for the dynamics of of mean-field spin glasses, removing some subsets of these simplification. In [4], it was observed that this essentially consists of proving that certain additive functional of an accelerated version of the dynamics, so called *clock process*, converges to a stable Lévy process; the aging as in Bouchaud's model can be then deduced from the classical arc-sine law. This observation allowed for a systematic treatment of the RHT dynamics of many spin-glass and related models in the last decade.

The RHT dynamics is however often attaced to be 'non-realistic'. It was thus a long-standing open problem in the field to show aging for if a 'realistic' dynamics of a spin glass model, like e.g. Metropolis dynamics.

Recently, some progress has been achieved in the context of the simplest mean-field spin glass model, the Random Energy Model (REM). First, in [5], the so called Bouchaud's asymmetric dynamics have been considered in the regime where the asymmetry parameter tends to zero with the size of the system. Second, the Metropolis dynamics have been studied in [6] for a truncated version of the REM. The purpose of the weak asymmetry assumption of [5] and the truncation of [6] is to to recover certain features of the RHT dynamics. In particular, they simplify considerably the 'local' estimates needed for proving the clock process convergence.

2. Results

The talk in Oberwolfach presents the recent work [7] with Tobias Wassmer, proving the clock process convergence of the Metropolis dynamics of the *non-modified* REM.

The REM is the simplest mean-field spin glass model whose state space is the N-dimensional hypercube $\Sigma_N = \{-1,1\}^N$, and its Hamiltonian is a collection $(E_x)_{x\in\Sigma_N}$ of i.i.d. standard Gaussian random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Its (non-normalized) Gibbs measure at inverse temperature $\beta > 0$ is given by $\tau_x = e^{\beta\sqrt{N}E_x}$. The Metropolis dynamics on the REM is the continuous-time Markov chain $X = (X_t)_{t\geq 0}$ on Σ_N with transition rates

$$r_{xy} = e^{-\beta\sqrt{N}(E_x-E_y)_+}1_{x\sim y} = \left(1 \wedge \frac{\tau_y}{\tau_x}\right)1_{x\sim y}, \qquad x, y \in \Sigma_N,$$

where $x \sim y$ means that x and y differ in exactly one coordinate. Obviously, the Gibbs measure τ is reversible for the Metropolis dynamics.

The Metropolis chain X is compared with its accelerated version $Y = (Y_t)_{t\geq 0}$ whose transition rates are

$$q_{xy} = \frac{\tau_x \wedge \tau_y}{1 \wedge \tau_x}1_{x\sim y}, \qquad x, y \in \Sigma_N.$$

Since $r_{xy} = (1 \vee \tau_x)^{-1}q_{xy}$, X can be written as a time change of Y,

$$X(t) = Y(S^{-1}(t))$$

with the *clock process* S being given by

$$S(t) = \int_0^t (1 \vee \tau_{Y_s})\mathrm{d}s.$$

Finally, for a fixed environment $\tau = (\tau_x)_{x\in\Sigma_N}$, let P^τ denote the law of the process Y started from its stationary distribution. The main result of [7] is the following

Theorem 1. *Let $\alpha \in (0, 1)$ and $\beta > 0$ be such that*

$$\frac{1}{2} < \frac{\alpha^2\beta^2}{2\ln 2} < 1,$$

and define

$$g_N = e^{\alpha\beta^2 N}(\alpha\beta\sqrt{2\pi N})^{-\frac{1}{\alpha}}.$$

Then there are random variables R_N which depend on the environment $(E_x)_{x\in\Sigma_N}$ only, such that the rescaled clock processes

$$S_N(t) = g_N^{-1}S(tR_N), \qquad t \geq 0,$$

converge in $\mathbb{P}$-probability as $N \to \infty$, in P_ν^τ-distribution on the Skorohod space equipped with the M_1-topology, to an α-stable Lévy process. The random variables R_N satisfy

$$\lim_{N\to\infty} \frac{\log R_N}{N} = \frac{\alpha^2\beta^2}{2}, \quad \mathbb{P}\text{-}a.s.$$

This theorem confirms Bouchaud's predictions, at least at the level of the clock process scalling. The deterministic scale g_N is the same as in [4], and the random scale R_N has the same exponential asyptototics as the corresponing scale r_N of [4]. That means that the scaling of the Metropolis dynamics is essentially the same as of the RHT dynamics, which can be interpreted as 'long-time' irrelevance of microscopic transition rules.

The techniques of [7] circumvent the necessity of 'local estimates' needed in previous papers, making, however, the verification of more precise aging statements (in terms of two-point functions or age process) difficult. This, as well as considering correlated mean-field spin glasses are interesting open questions.

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The Toom Interface Via Coupling

Nick Crawford

(joint work with Nick Crawford, Gady Kozma, Woijcech de Roeck)

We consider a one dimensional interacting particle system which describes the effective interface dynamics of the two dimensional Toom model at low temperature and noise. This model was first considered in [4]. The interest there (ultimately discarded) was due to the possibility of having a nontrivial, non-KPZ fluctuating hydrodynamics when the motion is symmetric with respect to the two particle types. More recently, in [1], the particle system was shown to be exactly solvable when only one of the two types of particles is allowed to actively move. The authors use the exact solvability to verify KPZ-type asymptotics, which are generally expected to hold for any bias between the rates of the two particle types.

In our work, we undertake a systematic study of the model for arbitrary choice of bias. First we consider the dynamics on a finite interval $[1, N)$, bounding the mixing time from above by $2N$. Then we consider the model defined on $\mathbb{Z}$. Due to infinite range interaction, this is a non-Feller process that we can define starting from product Bernoulli measures with density $p \in (0, 1)$ but not from arbitrary measures. We then give a number of regarding the constructed dynamics. First of all, under modest technical conditions which guarantee the existence of the dynamics started from a putative invariant measure, we show the measure must be a (mixture of) product Bernoulli measures.

It was known from the models definition in [4] that for each choice of bias, there is a unique stationary measure ν_∞ on $\mathbb{N}$. It is possible to couple the dynamics on $\mathbb{N}$ starting from ν_∞ with the dynamics on Z starting from a product Bernoulli measure. We use this coupling to further show that ν_∞ converges weakly product Bernoulli on $\mathbb{Z}$ as the boundary of $\mathbb{N}$ shifts to $-\infty$. Finally, on $\mathbb{Z}$ we consider functional CLTs of various observables including additive functionals of local functions, currents across a fixed vertex and the motion of a tagged push-particle.

All of our results are based on a coupling construction which is rather similar to what is known as the basic coupling for exclusion processes. The main additional tool we have here, which is not present for exclusion processes, is the observation that discrepancies (which are quasi-particles of the coupling) move monotonically and ballistically to ∞.

Unfortunately, we have been unable to settle the most interesting questions and conjectures raised in [4]. Of these, lets us mention two. First, one may ask what is the variance of the sum of spins S_N of the first N vertices under the stationary measure ν_N on $\{-1,1\}^{[1,N]}$. According to simulation, [4] predicts

$$\mathrm{Var}_{\nu_N}(S_N) \sim \begin{cases} N^{2/3} \text{ if there is a bias,} \\ N^{1/2}\log^{1/4} N \text{ for no bias.} \end{cases}$$

Currently our best estimate, obtained by combining the results of [2, 3], is of the order $N\mathrm{Poly}(\log N)$. The second interesting issue we wish to mention here (and suggested by the first conjecture) is as follows. A heuristic suggested in [4] is that the measure ν_∞ should be related to the fluctuating hydrodynamics of the process on $\mathbb{Z}$. Making this heuristic precise would represent considerable progress in our understanding of this model's behavior.

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Critical behaviour of spin systems and weakly self-avoiding walk in dimension 4

Gordon Slade

(joint work with Roland Bauerschmidt, David C. Brydges, Alexandre Tomberg)

1. Continuous-time weakly self-avoiding walk

For weakly self-avoiding walk, let $p \geq 1$ and consider p independent continuous-time simple random walks on $\mathbb{Z}^d$ started from the origin. We write this as a vector $X(T) = (X^1(T_1), \dots, X^p(T_p))$ with $T = (T_1, \dots, T_p) \in \mathbb{R}^p_+$. The intersection local time is

$$I_p(T) = \sum_{k,l=1}^{p} \int_0^{T_k} \int_0^{T_l} 1_{\{X^k(s)=X^l(t)\}} ds\, dt.$$

Let $g > 0$, $\nu \in \mathbb{R}$. The p-star network is defined by

$$S^{(p)}(g,\nu) = \int_{\mathbb{R}^p_+} E_0(e^{-gI_p(T)}) e^{-\nu \|T\|_1} dT,$$

and the p-watermelon network is defined by

$$W_{0x}^{(p)}(g,\nu) = \int_{\mathbb{R}_+^p} E_0(e^{-gI_p(T)} 1_{\{X^k(T_k)=x \,\forall k\}}) e^{-\nu\|T\|_1} dT.$$

These are networks of p weakly self- and mutually-avoiding walks. The two-point function is $W_{0x}^{(1)}(g,\nu)$. A simple subadditivity argument implies the existence of $\nu_c \le 0$ such that the susceptibility $\chi(g,\nu) := S^{(1)}(g,\nu) = \sum_x W_{0x}^{(1)}(g,\nu)$ obeys $\chi(g,\nu) < \infty$ if and only if $\nu > \nu_c$.

Let $\binom{p}{2}$ denote the binomial coefficient, with $\binom{1}{2} = 0$.

Theorem 1. *For $d = 4$, $g > 0$ small, $p \ge 1$, as $\varepsilon = \nu - \nu_c \downarrow 0$ or as $|x| \to \infty$,*

$$\begin{aligned}
\chi(g,\nu) &\sim A_g \varepsilon^{-1} (\log \varepsilon^{-1})^{1/4}, \\
\frac{S^{(p)}(g,\nu)}{\chi(g,\nu)^p} &\sim A_{g,p} \frac{1}{(\log \varepsilon^{-1})^{\frac{1}{4}\binom{p}{2}}}, \\
W_{0x}^{(p)}(g,\nu_c) &\sim C_{g,p} \frac{1}{|x|^{2p}} \frac{1}{(\log |x|)^{\frac{2}{4}\binom{p}{2}}}.
\end{aligned}$$

2. The $|\varphi|^4$ spin system

Related results are obtained for the n-component $|\varphi|^4$ spin system, for all $n \ge 1$. The model is first defined as the Gibbs measure on a finite torus $\Lambda = \Lambda_N = \mathbb{Z}^d/(L^N\mathbb{Z}^d)$ with $L > 1$ fixed (large):

$$\langle F(\varphi) \rangle_{g,\nu,N} = \frac{1}{Z_{g,\nu,N}} \int_{(\mathbb{R}^n)^\Lambda} F(\varphi) e^{-V_{g,\nu,N}(\varphi)} d\varphi.$$

Here $Z_{g,\nu,N}$ is a normalisation constant, $d\varphi$ is Lebesgue measure on $(\mathbb{R}^n)^\Lambda$, and

$$V_{g,\nu,N}(\varphi) = \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{4} \sum_{e : \|e\|_1 = 1} |\nabla^e \varphi_x|^2 \right),$$

with $g > 0$ and $\nu \in \mathbb{R}$. The susceptibility is defined by

$$\chi(g,\nu) = \lim_{N\to\infty} \sum_{x \in \Lambda_N} \langle \varphi_0^1 \varphi_x^1 \rangle_{g,\nu,N},$$

where φ_x^1 is the first component of the vector $\varphi_x \in \mathbb{R}^n$.

Theorem 2. *For $d = 4$, $g > 0$ small, $n \ge 1$, there exists $\nu_c < 0$ such that as $\varepsilon = \nu - \nu_c \downarrow 0$ or $|x| \to \infty$,*

$$\begin{aligned}
\chi(g,\nu) &\sim A_{g,n} \varepsilon^{-1} (\log \varepsilon^{-1})^{(n+2)/(n+8)}, \\
\langle \varphi_0^1 \varphi_x^1 \rangle_{g,\nu,N} &\sim C_{g,n} \frac{1}{|x|^2}.
\end{aligned}$$

Related results are also obtained for energy correlations.

3. Method of proof

The proof uses a rigorous renormalisation group argument. An interesting aspect of the proof is that the weakly self-avoiding walk can be treated as the $n = 0$ version of the spin model, via a supersymmetric integral representation.

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The 2D Coulomb gas at the Kosterlitz–Thouless transition: the work of Pierluigi Falco

Roland Bauerschmidt

1. Pierluigi Falco

Pierluigi Falco (born 1977) died in April 2014 from cancer. I present the result of his last work [5, 6], completed in November 2013.

2. 2D Coulomb gas

Let $\Lambda = \mathbb{Z}^d/R\mathbb{Z}^d$ denote the two-dimensional discrete torus of side length R. Let Ω_n^0 denote the space of neutral n particle configurations on Λ. Such a configuration $\omega \in \Omega_n^0$ is given by (x_i, σ_i), $i = 1, \dots, n$, where the $x_i \in \Lambda$ are the positions of the particles and the $\sigma_i \in \{\pm 1\}$ the signs of the charges; the neutrality condition is $\sum_i \sigma_i = 0$. The Coulomb energy for such a configuration is defined by

$$H_n(\omega) = \frac{1}{2}\sum_{i=1}^{n} \sigma_i\sigma_j(-\Delta)^{-1}(x_i - x_j) = \frac{1}{2}(f, (-\Delta)^{-1}f), \quad f = \sum_{i=1}^{n} \sigma_i \delta_{x_i}, \tag{1}$$

where Δ is the discrete Laplace operator whose inverse has kernel $(-\Delta)^{-1}(x-y) \sim \frac{1}{2\pi}\log\frac{1}{|x-y|}$ if applied to neutral change configurations.

The grand canonical partition function for activity z and inverse temperature β is defined by

$$Z_\Lambda(z, \beta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\omega\in\Omega_n^0} e^{-\beta H_n(\omega)}. \tag{2}$$

For $\eta \in (0,1)$, the fractional charge correlation function is defined by

$$\rho_\eta(x-y) = \lim_{\Lambda \uparrow \mathbb{Z}^2} \frac{Z^{p_1,p_2}}{Z}, \tag{3}$$

where Z^{p_1,p_2} denotes the partition function of a charge configuration in which two additional *fractional* charges $p_1 = (x, +\eta)$ and $p_2 = (y, -\eta)$ have been added.

3. Kosterlitz–Thouless transition

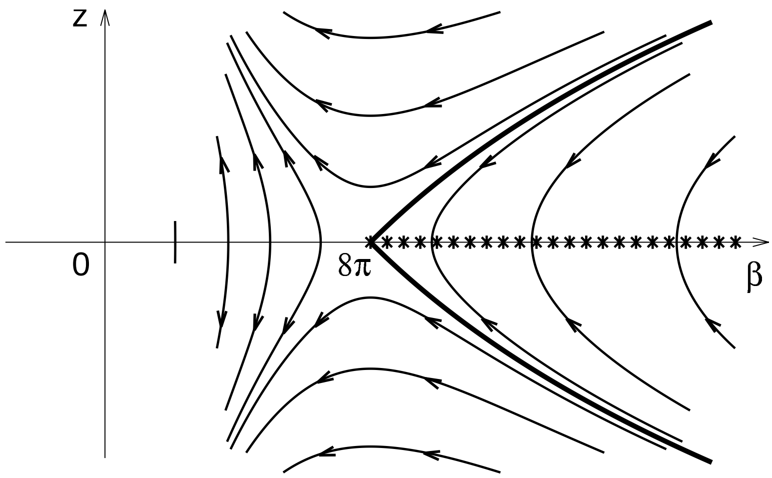

Figure 1. Kosterlitz–Thouless phase diagram

Berezinskii [1] and Kosterlitz–Thouless [9, 8] conjectured the phase diagram for the 2D Coulomb gas shown in Figure 1. In the diagram, the curves represent parameters (z, β) with the same long-distance behaviour in the sense of critical exponents. The curve through $(0, 8\pi)$ is called the Kosterlitz–Thouless transition line and separates a high temperature phase from a low temperature phase. The high temperature phase (or *screening phase*) is characterized by exponential decay of fractional charge correlation functions (see [2, 11]). In the low temperature phase (or *dipole phase* or *Kosterlitz–Thouless phase*), typical configurations consist of changes bound together into dipoles or multipoles, and fractional charge correlation functions decay algebraically. The existence of the low-temperature phase was proved in [7]. More precisely, in [7], it was proved that the fractional charge correlation functions are bounded from above and below by functions decaying algebraically in $|x-y|$ if β is sufficiently large. The parameter region for algebraic decay was extended in [10], and an alternative renormalisation group approach to the problem was initiated in [4, 3].

The main result of [5, 6] is the following theorem, which is the first result that applies directly on the Kosterlitz–Thouless transition line.

Theorem 1. *Fixed $\eta \in (0,1)$, there exist an $L_0 \equiv L_0(\eta) > 1$, a $z_0 \equiv z_0(\eta) > 0$ and an inverse temperature $\beta_{BKT}(z) \geq 8\pi$ such that if $L \geq L_0$, $0 < z \leq z_0$ and $\beta = \beta_{BKT}(z)$, the limit (3) exists with $R = L^N$, $N \to \infty$ and:*

(1) *If* $\eta \neq \frac{1}{2}$, *then*

$$\rho_\eta(x) = \rho_\eta^{(a)}(x) + \rho_\eta^{(b)}(x), \tag{4}$$

where, for x*-independent* f_a, f_b, f,

$$\begin{aligned} \rho_\eta^{(a)}(x) &= \frac{e^{8\pi\eta^2 c_E} + f_a}{|x|^{4\eta^2}\left(1 + f\ln|x|\right)^{2\eta^2}}\left(1 + o(1)\right), \\ \rho_\eta^{(b)}(x) &= \frac{f_b}{|x|^{4(1-\eta)^2}\left(1 + f\ln|x|\right)^{2(1-\eta)^2}}\left(1 + o(1)\right). \end{aligned} \tag{5}$$

(2) *If* $\eta = \frac{1}{2}$, *then, for* x*-independent* f_a, f,

$$\rho_{\frac{1}{2}}(x) = \frac{1}{2}\frac{e^{2\pi c_E} + f_a}{|x|}\left(1 + f\ln|x|\right)^{\frac{1}{2}}\left(1 + o(1)\right). \tag{6}$$

In the above formulas, $o(1)$ *are vanishing terms for* $|x| \to \infty$*;* $f = cz$ *for* $c > 0$*,* $f_b = c(\eta)^2 z^2(1 + \widetilde{f}_b)$ *for* $c(\eta) > 0$*;* f_a*,* $\widetilde{f}_b$ *are vanishing in the limit* $z \to 0$*. Besides* $z_0(\eta)$ *is such that, for every* $[a, b] \subset (0, 1)$*, one has* $\inf\{z_0(\eta) : \eta \in [a, b]\} > 0$*.*

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Universality in interacting Ising and dimer models

Alessandro Giuliani
(joint work with R. Greenblatt, V. Mastropietro, F. Toninelli)

In the last few years, the methods of constructive Fermionic Renormalization Group (RG) have successfully been applied to the study of the scaling limit of several two-dimensional (2D) statistical mechanics models at the critical point, including the 2D Ising with finite range interactions at the critical temperature and the close-packed interacting dimer model. Different instances of universality have been proved in these contexts, such as: universality of the energy-energy critical exponents and of the central charge in interacting Ising models; massless Gaussian free field (GFF) fluctuations of the height field in the interacting dimer model. Both results are briefly reviewed in the following.

1. The interacting Ising model

The class of Ising models we consider is defined by the Hamiltonian $H_\Lambda = -J\sum_{<\mathbf{x},\mathbf{y}>}\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - \lambda\sum_{\mathbf{x},\mathbf{y}}\sigma_{\mathbf{x}}v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{y}} \equiv H^{(0)}_\Lambda + \lambda W_\Lambda$, where J is a positive constant, $\Lambda \subset \mathbb{Z}^2$ is a finite rectangular box with periodic boundary conditions, $\sigma_{\mathbf{x}} = \pm 1$, the first sum runs over nearest neighbor pairs of sites in Λ, while in the second sum the interaction potential $v(\mathbf{x}-\mathbf{y})$ is rotation and reflection invariant and has finite range. As an illustrative example to keep in mind, $v(\mathbf{x}-\mathbf{y})$ could be a next-to-nearest-neighbor interaction (of either signs).

At $\lambda = 0$ (nearest neighbor case), the model is exactly solvable in a very strong sense: one can compute closed formulas for the free energy per site in the thermodynamic limit, as well as for the correlation functions. The truncated correlations decay exponentially to zero at large distance for all but one value of the inverse temperature, β_c, where correlations decay polynomially, with specific decay exponents. Correlations can be rescaled so that they admit a finite limit as the lattice mesh tends to zero. The collection of multipoint limiting correlations at the critical point are believed to define a Conformal Field Theory (CFT), corresponding to the so-called minimal model with central charge $c = 1/2$. The parameter c characterizes certain commutation relations among the fundamental fields of the theory, as well as the finite-size effects induced by the presence of a boundary or a finite box. Essentially all these properties have been rigorously proved, starting from the Onsager solution, dating back the mid 1940s, until very recent times, in particular with the ground-breaking works of Smirnov and collaborators, which finally substantiated the theoretical physics predictions about the existence, structure and conformal invariance of the critical scaling theory.

Both at a qualitative and quantitative level, the features of the model are believed to be robust under analytic changes of the Hamiltonian, in particular they should remain valid at $\lambda \neq 0$: the model should still admit a single critical temperature separating a high from a low temperature phase. Critical correlations at $\beta_c(\lambda)$ should decay polynomially, and their scaling limit should be the *same* as the $\lambda = 0$ case. This is a (strong) instance of the so-called universality hypothesis in statistical mechanics, which is still wide open from a rigorous, mathematical, point of view. In this talk, I present two results that rigorously prove two instances of this hypothesis at $\lambda \neq 0$ for the class of Ising models above.

The first concerns the scaling limit of the multipoint energy correlations: letting $e_j(\mathbf{x}) := a^{-1}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{e}_j}$ be the *energy* observable on a rescaled lattice with mesh a, we compute explicitly the scaling limit as $a \to 0$ of the multipoint energy correlations at the interacting critical point, for λ sufficiently small. The limiting m-point correlations are the same as the $\lambda = 0$ case, up to an overall factor $\big[Z(\lambda)\big]^m$, with $Z(\lambda)$ an analytic function of λ, which can be interpreted as a renormalization factor of the energy operator. Our result provides the first construction of the "energy-sector" of the critical scaling theory, and proves that it coincides as expected with the corresponding sector of the minimal model with $c = 1/2$.

Our second result concerns the universality of the sub-leading finite size corrections to the critical pressure. We prove that in a rectangle Λ of lattice mesh $a = 1$ and sides $L \gg \ell \gg 1$, the critical pressure at $\beta_c = \beta_c(\lambda)$ is $\frac{1}{L\ell}\log Z(\beta_c, \Lambda) = -\beta_c f(\beta_c) + \frac{1}{\ell^2}\frac{\pi}{12}$ up to lower order corrections in the limit $L \gg \ell \gg 1$. Remarkably, the dominant term $-\beta_c f(\beta_c)$ is a non-trivial analytic function of λ, while the first sub-leading correction is exactly independent of λ. From the factor $\frac{\pi}{12}$ one can read again the value of the central charge: in fact, CFT predicts that this factor should in general be equal to $c\pi/6$, with c the central charge.

Both results are proved by: (i) mapping exactly the model in an interacting 2D fermionic theory; (ii) computing the fermionic functional integral by rigorous renormalization group methods. An important fact is that the interaction of such an effective fermionic theory is *irrelevant* in the RG sense. For the precise statements of the results and their proofs, we refer to the original publications [1, 2].

2. The interacting dimer model

The second class of models we consider are interacting dimer models on $\mathbb{Z}^2$, defined as follows. Take a finite box $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions, and consider the configuration space $\mathcal{M}_\Lambda$ of dimer coverings of Λ. We shall assign every dimer covering $M \in \mathcal{M}_\Lambda$ a non-uniform statistical weight, proportional to $e^{\lambda W_\Lambda(M)}$, with $W_\Lambda(M)$ a translational invariant interaction, obtained by translating over Λ a finite range, rotational and reflection invariant, interaction among dimers. In particular, the partition function of the interacting dimer model is $Z_\Lambda(\lambda) = \sum_{M\in\mathcal{M}_\Lambda} e^{\lambda W_\Lambda(M)}$. As an illustrative example to keep in mind, $W_\Lambda(M)$ could be the number of plaquettes in Λ occupied by two parallel dimers.

$Z_\Lambda(\lambda)$ can be thought of either as a model of densely packed anisotropic molecules, or as model of random interfaces. The random interface interpretation is based on the following definition of height function: given a dimer covering M, two faces of Λ centered at $\mathbf{x}$ and $\mathbf{y}$ and a path $C_{\mathbf{x}\to\mathbf{y}}$ from $\mathbf{x}$ to $\mathbf{y}$ with trivial winding around the torus Λ, the height difference between $\mathbf{x}$ and $\mathbf{y}$ is defined as $h_{\mathbf{x}} - h_{\mathbf{y}} = \sum_{b\in\mathcal{C}_{\mathbf{x}\to\mathbf{y}}} \big(1_b(M) - \frac{1}{4}\big)\sigma_b$ where $\sigma_b = \pm 1$ depending on whether $C_{\mathbf{x}\to\mathbf{y}}$ crosses b with the white site on the right/left.

In complete analogy with the Ising model considered above, at $\lambda = 0$ the dimer model is exactly solvable: one can derive exact formulas for the free energy and for correlation functions, which decay polynomially at large distances. The height fluctuations are asymptotically Gaussian at large distances, with logarithmically divergent variance. If properly rescaled, all the correlations admit a limit as the lattice mesh is sent to zero. In particular, the height field correlations tend to those of the massless 2D GFF. The collection of limiting correlations defines a critical scaling theory, which is conformally invariant, with central charge $c = 1$. Essentially all these properties have been rigorously proved, starting from the Kasteleyn solution, dating back the early 1960s, until the recent works of Kenyon and collaborators, which finally substantiated the predictions about the existence, structure and conformal invariance of the critical scaling theory.

Our main result concerns the scaling limit of the height field at $\lambda \neq 0$. Remarkably, while the multipoint dimer correlations are *not* universal, and are characterized by critical exponents continuously changing with λ, the height fluctuations are. More precisely, if λ is sufficiently small, the infinite volume height correlation verifies $\langle (h_{\mathbf{x}} - h_{\mathbf{y}})^2 \rangle = \frac{K(\lambda)}{\pi^2} \log |\mathbf{x} - \mathbf{y}|$, up to lower order corrections at large distances. Here $\langle \cdot \rangle$ denotes the infinite volume interacting state, and $K(\lambda)$ is an analytic function such that $K(0) = 1$. The higher order truncated correlations are bounded uniformly in $|\mathbf{x} - \mathbf{y}|$. At large distances, the coarse graining of $h_{\mathbf{x}}$ converges to the GFF, in the sense that, if $\alpha \in \mathbb{R}$ and f is a smooth, compactly supported function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} f(u)du = 0$, one has $\langle \exp \left\{ i\alpha\epsilon^2 \sum_{\mathbf{x}\in\mathbb{Z}^2} h_{\mathbf{x}} f(\epsilon\mathbf{x}) \right\} \rangle \xrightarrow{\epsilon\to 0} \exp \left\{ \frac{K\alpha^2}{4\pi^2} \int f(u)f(v) \log |u - v| du dv \right\}$, where ϵ^{-1} represents the coarse-grain scale, to be sent to infinity after the thermodynamic limit.

Also in this case, the result is proved by first mapping the model in a lattice model of interacting fermions, and then by analyzing the latter by constructive RG methods. The analysis here is much more subtle, in that the interaction is marginal, rather than irrelevant, in a renormalization group sense. Universality of the height field emerges as a combined effect of hidden Ward Identities and lattice path invariance of the height difference. Once again, for the precise statement of the result and its proof, we refer to the original publications [3, 4].

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Ising model from the Random Current perspective

Michael Aizenman

(joint work with H. Duminil-Copin; related prior works also with D. Barsky, R. Fernandez, and V. Sidoravicious)

The random current (RC) representation [1] (which was presented in the talk) yields stochastic geometric insight on the phase transition in the Ising model, and the structure of its correlation functions at criticality. Mentioned here are:

A) General results of the Ising phase transition and its critical behavior on transitive graphs (seasoned works with D. Barsky, R. Fernandez), conditions for the continuity of the spontaneous magnetization in the amenable case (with H. Duminil-Copin, and V. Sidoravicious).

B) High dimensional results: mean-field critical exponents and asymptotically gaussian structure of the correlation functions in dimensions $d > 4$.

C) Results specific to the low dimension $d = 2$: new insights on the origins of the fermionic Wick rule for certain correlation functions, and the emergence of planarity at the critical points of two dimensional Ising spin systems with finite range interactions (work in progress with H. D-C).

The theorems are asserted here in natural generalizations of their original versions.

Theorem A.1 [3] *For any Ising model on a transitive graph with homogeneous ferromagnetic couplings for which $\sum_{x\in G} J_{0,x} e^{+\varepsilon \mathrm{dist}(0,x)} < \infty$ for some $\varepsilon > 0$ there exists a critical temperature $T_c \in [0, \|J\|]$ such that:*

i) *Along the line $h = 0$, the spontaneous magnetization satisfies*

$$M(T) \begin{cases} = 0 & \text{for all } T > T_c \\ > 0 & \text{for all } T < T_c \end{cases}$$

ii) *The high temperature exponential decay of corrections extends throughout the regime $T \in (T_c, \infty)$. In particular, for $h = 0$ and any $T \in (T_c, \infty)$ the two point correlation function decays exponentially*

$$\langle \sigma_x \sigma_y \rangle_{T,0+} \ \le \ A\, e^{-\mathrm{dist}(x,y)/\xi}$$

with some $A(T), \xi(T) < \infty$. [The summed version of the decay also holds].

iii) *In the vicinity of the critical point the model exhibits the following singular behavior (with C a generic symbol for J dependent constants):*

(a) *For $h = 0$, $T \searrow T_c$:* $\quad \chi(T) \ \ge \ \frac{C}{(T-T_c)^{\hat\gamma}} \quad$ *with* $\quad \hat\gamma = 1$

(b) *For $h = 0$, $T \nearrow T_c$:* $\quad M(T) \ \ge \ \frac{C}{(T-T_c)^{\hat\beta}} \quad$ *with* $\quad \hat\beta = 1/2$

(c) *For $T = T_c$, $h \searrow 0$:* $\quad M(T,h) \ \ge \ C\, h^{1/\hat\delta} \quad$ *with* $\quad \hat\delta = 3.$

Given the monotonicity properties of M which are implied by the GHS and FKG inequalities, these statements can be deduced from the following pair of non-linear partial-differential inequalities [2, 3]:

$$M \;\leq\; h\frac{\partial}{\partial h}M \;+\; \|J\|M^{n-1}\frac{\partial}{\partial \beta}M \;+\; M^n \tag{1}$$

$$\frac{\partial}{\partial \beta}M \;\leq\; \|J\|M\frac{\partial}{\partial h}M\,. \tag{2}$$

At different values of the power n these relations hold for both homogenous Ising models, at $n = 3$, and for independent percolation models (with suitable extension of its notions to $h \geq 0$) at $n = 2$ [2]. The proof relies on stochastic geometric representations of the quantities involved.

More recently, a natural extension of the RC representation to infinite graphs was instrumental in proving that in all dimensions for the n.n. models on $\mathbb{Z}^d$:

$$m^*(T_c) = 0\,. \tag{3}$$

The most elusive case has been $d = 3$, for which until recently the proof of (3) remained an open challenge. This was done though a clarification of the relation between the following two order parameters:

$$m^*(T_c) \;:=\; \langle\sigma_0\rangle_{T_c-0} \;=\; \lim_{T\nearrow T_c} M(T) \qquad \text{the residual magnetization}$$

$$M_{LRO} \;:=\; \lim_{\text{dist}(0,\text{x})\to\infty} \sqrt{\langle\sigma_0\sigma_x\rangle_{T_c+0}} \qquad \text{the long range order parameter}$$

The two provide information on the equilibrium states just below and just above T_C. For the latter, the infrared bound [9] allows to conclude that $M_{LRO}(T_c) = 0$ for the n.n. models on $\mathbb{Z}^d$ at $d > 2$. The open question was thus settled by:

Theorem A.2[5] *For any Ising model on a transitive amenable graph (in particular on $\mathbb{Z}^d$ for any $d \in (1,\infty)$) with homogeneous ferromagnetic couplings:*

$$m^*(T_c) = 0 \quad \Longleftrightarrow \quad M_{LRO}(T_c) = 0\,.$$

The graphs accompanying the derivations of (1) indicate relations of the Ising model to Φ^4 and of percolation to Φ^3 field theories. Consistently with that, through reversed versions of (a)-(c) it was possible to prove that for the n.n. Ising models the three critical exponents stabilize above $d_{uc} = 4$ (and $d_{uc} = 6$ for percolation models, with certain caveats [7, 11]), settling on their mean-field values:

Theorem B.1[1,3] *For the nearest neighbor Ising model on $\mathbb{Z}^d$ in $d > 4$ dimensions, the three bounds in Theorem A.1 part (iii) are valid also in the reversed direction (with different values of C) at the exponent values*

$$\gamma = 1, \qquad \beta = 1/2, \qquad \delta = 3\,. \tag{4}$$

Essential input for the reverse inequalities is provided by suitable 'infrared bounds' on the two-point function $\langle\sigma_x\sigma_y\rangle_{T,0+}$. Such bounds are provided by the reflection positivity method [9], when it is applicable, and alternatively, for spread enough interactions, the lace expansion [7]. Related to the above is:

Theorem B.2[1] *For the n.n. Ising models on $\mathbb{Z}^d$ in $d > 4$, if for some $\kappa(\delta) \to \infty$ the scaled correlation functions converge (pointwise for $x_1, ..., x_{2n} \in \mathbb{R}^d$)*

$$S_{2n}(x_1, ..., x_{2n}) \ = \ \lim_{\delta \to 0} \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c} \tag{5}$$

then the limiting functions satisfy the (Gaussian) Wick rule:

$$S_{2n}(x_1, ..., x_{2n}) \ = \ \sum_{\text{pairings}\pi} \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)}) \tag{6}$$

The RC representation played an essential role in both proving the above result, and in explaining why it does not hold in $d = 2$ dimensions. Recently the RC representation was recognized to offer a new perspective on the emergence of fermionic structures such as appear in the exact solution of the model on $\mathbb{Z}^d$. Following is an example of such (more will be presented in [4]).

Theorem C.1[4] *Let $\langle -- \rangle$ be an equilibrium state of a ferromagnetic Ising model on a planar graph with a connected boundary segment Γ. Then, for any collection of boundary sites $\{x_1, ..., x_{2n}\} \subset \Gamma$, ordered cyclicly along Γ:*

$$\langle \prod_{j=1}^{2n} \sigma_{x_j} \rangle \ = \ \sum_{\text{pairings}\pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \tag{7}$$

where $\varepsilon(\pi) = \pm 1$ is the pairing's parity.

The RC representation provides a stochastic geometric insight on the emergence of such relations in critical 2D models even in the non-planar case [10, 4].

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Crossing probabilities for Voronoi percolation

Vincent Tassion

One of the reasons that planar percolation has been studied so successfully is the fact that the large-scale connectivity properties of percolation clusters can be encoded through so-called "crossing probabilities". A fundamental tool in this approach is the box-crossing property at criticality. It was proved by Kesten for Bernoulli percolation on a lattice with a non-trivial rotation. But the original proof does not extend to models with dependencies.

We discuss a new proof and a generalization of the box-crossing property in the framework of Voronoi percolation [6]. Then, we present a joint work with Ahlberg, Griffith and Morris in which we prove a conjecture of Benjamini, Kalai and Schramm concerning the quenched crossing probabilities for Voronoi percolation [1].

1. Box-crossing property for planar Voronoi percolation

1.1. **Voronoi percolation.** First introduced in the context of first passage percolation, planar Voronoi percolation has been an active area of research, see [4] for an introduction. It is defined by the following two-step procedure. First, consider a Poisson Point Process in $\mathbb{R}^2$ with intensity 1, and form its associated Voronoi tiling. Then, color independently each cell of the tiling black with probability p, and white otherwise. Bollobás and Riordan [3] proved that the critical value for this model is $p_c = 1/2$.

1.2. **Box-crossing property.** Given a rectangle $R = [a, b] \times [c, d]$, write H_R for the event that there exists a path of black cells from left to right in R. The first result we discuss here is the box-crossing property for Voronoi percolation, proved in [6].

Theorem 1 (Box-crossing property)**.** *Consider critical Voronoi percolation in the plane (meaning for $p = 1/2$). For every $\rho > 0$, there exists a constant $c = c(\rho) > 0$ such that, for every n large enough,*

$$c < \mathrm{P}[H_R] < 1 - c, \text{ where } R = [0, \rho n] \times [0, n].$$

The central ingredient is a new Russo-Seymour-Welsh result, based on a renormalization procedure (similar to the one invented in [3]). The proof is very robust, and shows that the box-crossing property holds for a large class of planar percolation processes.

The box-crossing property has been instrumental in many works on Bernoulli percolation on a lattice, and has numerous applications. Therefore, we expect these applications to hold also for Voronoi percolation, as consequences of Theorem 1. These include Kesten's scaling relations, bounds on critical exponents (e.g. polynomial bounds on the one-arm event), the computation of the universal exponents and tightness arguments in the study of the scaling limit, to name a few.

2. Quenched Crossing probabilities

In this second part, we present a quenched version of the crossing probabilities for Voronoi percolation. In other words, we freeze a realization of the point process and then, we study the crossing probabilities conditioned on this realization.

Main Theorem. We focus here on a finite volume version of the Voronoi percolation process at criticality. Consider a set η of n points in the square $S = [0,1]^2$, each chosen independently and uniformly at random. Then form the Voronoi tiling associated to η inside the square S. Finally, color independently each tile black with probability $p = 1/2$, and white otherwise. As above, H_S denotes the event that S is crossed horizontally by a black horizontal crossing. Note that $\mathrm{P}(H_S) = 1/2$, by symmetry.

Theorem 2. *Let η be a set of n independent uniformly chosen points in S. As n tends to infinity, the quenched crossing probability $\mathrm{P}\big(H_S \,|\, \eta\big)$ converges to $1/2$.*

Sketch of proof. We identify the set $\{-1,1\}^\eta$ with the possible coloring of the tiling of η. Let $f_\eta \colon \{-1,1\}^\eta \to \{0,1\}$ be the function such that $f^\eta(\omega) = 1$ if and only if H_S holds. The proof is divided into three steps.

- We first prove the following Efron-Steif type bound on the variance of the probability of the crossing event in terms of the influences of f^η, which can be viewed as a random Boolean function:

  $$\mathrm{Var}\Big(\mathrm{P}\big(H_S \,|\, \eta\big)\Big) \;\leq\; \sum_{m=1}^{n} \mathrm{E}\big[\mathrm{Inf}_m(f^\eta)^2\big].$$

  Recall that the influence $\mathrm{Inf}_m(f_n)$ of the m^{th} variable of a Boolean function $f_n \colon \{-1,1\}^n \to \{0,1\}$ is defined to be the expected absolute change in f_n when the sign of the m^{th} variable is flipped, i.e.,

  $$\mathrm{Inf}_m(f_n) \;=\; \mathrm{P}\big(f_n(\omega) \neq f_n(\omega')\big),$$

  where ω is chosen uniformly, and ω' is obtained from ω by flipping the m^{th} variable.
- Then, we use a 'randomized algorithm method' introduced by Schramm and Steif [5] in order to bound $\sum_{m=1}^n \mathrm{Inf}_m(f^\eta)^2$ in term of a geometric estimate related to the one-arm event.
- Finally, we derive from Theorem 1 a quenched version of the box-crossing property in order to bound the "one-arm" estimate mentioned above.

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First passage percolation on random planar maps

Nicolas Curien
(joint work with Jean-François Le Gall)

In the recent years, there has been much effort to understand the large-scale geometry of random planar maps viewed as random metric spaces for the usual graph distance on their vertex set. A major achievement in the area is the construction and study of the so-called Brownian map, which has been proved to be the universal scaling limit of many different classes of planar maps equipped with the graph distance (see [6, 7] and more recently [1, 2, 4]). In this work, we replace the graph distance by other natural choices of distances on the vertex set or on the set of faces, and we show that in large scales these new distances behave like the original graph distance, up to a constant multiplicative factor.

Modified distances. If m is a rooted (finite or infinite) planar map, we let $\mathsf{V}(\mathrm{m}), \mathsf{E}(\mathrm{m})$ and $\mathsf{F}(\mathrm{m})$ denote respectively the set of vertices, edges and faces of m. The set $\mathsf{V}(\mathrm{m})$ is usually equipped with the graph distance, which is denoted by $\mathrm{d_{gr}}$. We will consider the following modifications of the graph distance.

Case 0: First-passage (bond) percolation. Assign i.i.d. positive random variables $w(e)$ to all edges $e \in \mathsf{E}(\mathrm{m})$. Assume that the common distribution of the "weights" $w(e)$ is supported on $[\kappa, 1]$ for some $\kappa \in (0, 1]$. The associated first-passage percolation distance is defined on $\mathsf{V}(\mathrm{m})$ by setting for any $x, y \in \mathsf{V}(\mathrm{m})$

$$\mathrm{d_{fpp}}(x, y) = \inf_{\gamma : x \to y} \sum_{e \in \gamma} w(e),$$

where the infimum runs over all paths going from x to y in the map m.

Case 1: Dual graph distance. Consider the dual map $\mathrm{m}^\dagger$, whose vertices are the faces of m, and each edge e of m corresponds to an edge of $\mathrm{m}^\dagger$ connecting the two (possibly equal) faces incident to e. We may then consider the graph distance on $\mathsf{V}(\mathrm{m}^\dagger) = \mathsf{F}(\mathrm{m})$, which we denote by $\mathrm{d}^\dagger_{\mathrm{gr}}$.

Case 2: Eden model. This is the first-passage percolation model on $\mathrm{m}^\dagger$ corresponding to exponential edge weights. More precisely, we assign independent exponential random variables with parameter 1 to the edges of $\mathrm{m}^\dagger$ (or equivalently to the edges of m) and the associated first-passage percolation distance on $\mathsf{F}(\mathrm{m}) = \mathsf{V}(\mathrm{m}^\dagger)$ is denoted by $\mathrm{d}^\dagger_{\mathrm{Eden}}$

We consider these new "modified distances" when $\mathrm{m} = \mathcal{T}_n$ is a random planar map chosen uniformly in the set of all rooted plane triangulations with $n + 1$ vertices. In each of the previous cases, we are able to prove that the modified distances behave in large scales like a deterministic constant times the graph distance on

$\mathsf{V}(\mathcal{T}_n)$. More precisely, there exist constants $\mathbf{c}_0, \mathbf{c}_1$ and $\mathbf{c}_2$ in $(0,\infty)$ such that we have the following three convergences in probability

$$n^{-1/4} \sup_{x,y\in\mathsf{V}(\mathcal{T}_n)} \left|\mathrm{d_{fpp}}(x,y) - \mathbf{c}_0 \cdot \mathrm{d_{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0, \tag{1}$$

$$n^{-1/4} \sup_{\substack{x,y\in\mathsf{V}(\mathcal{T}_n),\ f,g\in\mathsf{F}(\mathcal{T}_n)\\ x\lhd f \text{ and } y\lhd g}} \left|\mathrm{d}^{\dagger}_{\mathrm{gr}}(f,g) - \mathbf{c}_1 \cdot \mathrm{d_{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0, \tag{2}$$

$$n^{-1/4} \sup_{\substack{x,y\in\mathsf{V}(\mathcal{T}_n),\ f,g\in\mathsf{F}(\mathcal{T}_n)\\ x\lhd f \text{ and } y\lhd g}} \left|\mathrm{d}^{\dagger}_{\mathrm{Eden}}(f,g) - \mathbf{c}_2 \cdot \mathrm{d_{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0, \tag{3}$$

where we used the notation $x \lhd f$ to mean that the vertex x is incident to the face f. Since the convergence of rescaled triangulations to the Brownian map [6] implies that the typical graph distance between two vertices of $\mathcal{T}_n$ is of order $n^{1/4}$, convergence (1) shows that in large scales $\mathrm{d_{fpp}}(x,y)$ is proportional to $\mathrm{d_{gr}}(x,y)$. In fact (1) implies that the set $\mathsf{V}(\mathcal{T}_n)$ equipped with the metric $n^{-1/4}\mathrm{d_{fpp}}$ converges in distribution to (a scaled version of) the Brownian map, and that this convergence takes place jointly with that of $(\mathsf{V}(\mathcal{T}_n), n^{-1/4}\mathrm{d_{gr}})$ proved in [6].

In case 0., the constant $\mathbf{c}_0$ depends on the distribution of the weights and seems hopeless to explicitly compute. However in cases 1. and 2. (dual graph and Eden model) the constants can be computed exactly and we have

$$\mathbf{c}_1 = 1 + 2\sqrt{3} \quad \text{and} \quad \mathbf{c}_2 = 2\sqrt{3}.$$

The reason why these models are more tractable than the case 0. is that the metric exploration of the dual graph distance or the Eden model can be performed algorithmically using a peeling procedure of the underlying random map. These peeling explorations have been studied in details in the case of the Uniform Infinite Planar Triangulation (UIPT) in [5].

Results for the UIPT. We can also state versions of our results on the UIPT, the random infinite lattice first discussed by Angel & Schramm [3]. The UIPT, which will be denoted by $\mathcal{T}_\infty$, is the local limit of uniformly distributed triangulations with n faces when $n \to \infty$. We can equip the vertex set of the UIPT with the usual graph distance $\mathrm{d_{gr}}$ or with a modified distance as above. To simplify, let us only consider the first-passage percolation distance $\mathrm{d_{fpp}}$ defined as previously from i.i.d. edge weights (case 0.). For every $r > 0$, write $B_r(\mathcal{T}_\infty)$ for the planar map obtained by keeping only those faces of $\mathcal{T}_\infty$ that contain at least one vertex at graph distance strictly less than r from the root vertex, and define $B_r^{\mathrm{fpp}}(\mathcal{T}_\infty)$ analogously, replacing the graph distance by the first-passage percolation distance. Under the same assumptions on the weights, we prove that for the same $\mathbf{c}_0$ as above and for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \mathbb{P}\left(\sup_{x,y\in\mathsf{V}(B_r(\mathcal{T}_\infty))} \left|\mathrm{d_{fpp}}(x,y) - \mathbf{c}_0 \cdot \mathrm{d_{gr}}(x,y)\right| > \varepsilon r \right) = 0. \tag{4}$$

It follows that the inclusions

$$B_{(1-\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \subset B_r^{\mathrm{fpp}}(\mathcal{T}_\infty) \subset B_{(1+\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \tag{5}$$

hold with probability tending to 1 as $r \to \infty$. In other words, large balls for the first-passage percolation distance are close to balls for the graph distance. Similar results hold for the graph distance or the Eden distance on the dual of the UIPT. The particular case of the Eden model answers a question raised by Miller & Sheffield which served as a heuristic for the definition of the Quantum Loewner Evolution of parameter $(\frac{8}{3}, 0)$, [8, Question 9.14].

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Metastability for continuum interacting particle systems

Sabine Jansen

(joint work with Frank den Hollander)

1. Setting

We consider a system of point particles living on a finite torus Λ in $\mathbb{R}^2$, $d = 2$, and interacting with each other through a pair potential $v(r)$ that has a hard-core repulsion, a finite range, and a unique attractive minimum. The configuration space Ω consists of the finite subsets $\omega = \{x_1, \ldots, x_n\} \subset \Lambda$, with $N(\omega) = \#\omega$ the number of particles. The energy of such a configuration is

$$U(\omega) = \sum_{1 \leq i < j \leq n} v(|x_i - x_j|)$$

when there are at least two particles, and $U(\omega) = 0$ when $N(\omega) = 0$ or 1. Particles are randomly created and annihilated according to a *Metropolis dynamics*, with the outside of the torus acting as an infinite particle reservoir with a fixed chemical potential μ. Once on the torus, particles cannot move. Let

$$H(\omega) = U(\omega) - \mu N(\omega).$$

The infinitesimal generator is

$$\big(Lf\big)(\omega) = \int_\Lambda b(x,\omega)\,\big[f(\omega \cup x) - f(\omega)\big]\,\mathrm{d}x + \sum_{x\in\omega} d(x,\omega)\,\big[f(\omega \setminus x) - f(\omega)\big].$$

The unique reversible measure is the grand-canonical Gibbs measure $\mathbb{P} = \mathbb{P}_{\beta,\mu,\Lambda}$, which is absolutely continuous with respect to the Poisson point process $\mathbb{Q}$ with intensity 1 on Λ. The Radon-Nikodým derivative is

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(\omega) = \frac{1}{\Xi}\mathrm{e}^{-\beta H(\omega)},$$

with $\Xi = \Xi_{\beta,\mu,\Lambda} = \int_\Omega \mathrm{e}^{-\beta H(\omega)}\mathbb{Q}(\mathrm{d}\omega)$ a normalization.

The chemical potential is chosen such that the system is metastable: starting from the vacuum configuration where the torus is empty, the system wants to nucleate (i.e., fill up the torus with particles in the ground state lattice), but in order to do so it has to overcome an energetic threshold, namely, it has to create a *critical droplet* that is large enough to trigger the nucleation. We are interested in the nucleation time and in the size and shape of the critical droplets in the limit as the temperature tends to zero.

Subject to *four assumptions* on the energy landscape, we compute the average nucleation time, show that the nucleation time divided by its average is exponentially distributed, and identify the set of critical droplets. The average nucleation time follows the *Arrhenius law* with an activation energy given by the grand-canonical energy of the critical droplets and a *prefactor* that depends in a delicate way on the temperature, the chemical potential, and the shape of the pair potential near its minimum. Our proof of the Arrhenius law uses the *potential-theoretic approach to metastability* [2].

Our results extend earlier work for lattice system [1]. The problem with working in the continuum is that it is hard to control the energy landscape, especially in the vicinity of the set of critical droplets. We rely on properties derived in the literature for minimal energy configurations at fixed particle numbers [3, 4]. Our four assumptions on the energy landscape are expected to be true for a large class of pair potentials, but as yet can be proven only for a particular pair potential in $d = 2$, called the *soft disk potential* [3]. It is given by

$$v(r) = \begin{cases} \infty, & 0 \le r < 1, \\ -1 + 24(r-1), & 1 \le r \le \frac{25}{24}, \\ 0, & r > \frac{25}{24}. \end{cases}$$

For this potential every k-particle minimizer of the energy is a subset of a triangular lattice of spacing 1, with energy [3]

$$E_k := \min\{U(\omega) \mid N(\omega) = k\} = -3k + \lceil \sqrt{12k-3}\,\rceil.$$

2. Results

Let $\mu \in (-3,-2)$. Write $\mu = -3+h$ and assume that $h^{-1} \notin \frac{1}{2}\mathbb{N}$. Let $\ell_c = \lfloor h^{-1} \rfloor$,

$$k_c = \begin{cases} 3\ell_c^2 + 4\ell_c + 2, & h \in \big((\ell_c+1)^{-1}, (\ell_c+\frac{1}{2})^{-1}\big), \\ 3\ell_c^2 + 2\ell_c + 1, & h \in \big((\ell_c+\frac{1}{2})^{-1}, \ell_c^{-1}\big). \end{cases}$$

and $k_p = k_c - 1$. Then $k \mapsto E_k - k\mu$ has the unique maximizer $k = k_c$. Every k-particle ground state with protocritical number of particles $k = k_p$ is a hexagon of side-length ℓ_c (containing $3\ell_c^2+3\ell_c+1$ particles) for which a side bar has either been removed or added, i.e., a *quasi-hexagon*. We call these the *protocritical droplets*. Adding a particle to the longest side of the protocritical quasi-hexagons we obtain the set of *critical droplets* $\mathcal{C}$.

Let $\mathcal{N} \subset \{N(\omega) \geq 2k_c\}$. This target set satisfies the *no-deep-well property* of the potential-theoretic approach [2] and has communication height to the vacuum state $\mathsf{vac} \in \Omega$, $N(\mathsf{vac}) = 0$ given by

$$\Gamma = E_{k_c} - k_c\mu.$$

Write $\mathbf{P}_{\mathsf{vac}}$, $\mathbf{E}_{\mathsf{vac}}$ for the law and expectation of the process $(X_t)_{t\geq 0}$ with infinitesimal generator L started in the vacuum, and $\tau_{\mathcal{A}} = \inf\{t \geq 0:\ X_t \in \mathcal{A}, \exists s \in (0,t): X_s \notin \mathcal{A}\}$ to denote the first hitting time of $\mathcal{A}$ after the starting configuration has been left. Pick any $\beta \to \varepsilon(\beta)$ strictly positive such that $\lim_{\beta\to\infty} \varepsilon(\beta) = 0$ and $\lim_{\beta\to\infty} \beta\varepsilon(\beta) = \infty$, and define

$$\mathcal{C}(\beta) = \{\omega \in \Omega:\ d_H(\omega, \mathcal{C}) \leq \varepsilon(\beta)\}.$$

with d_H the Hausdorff distance. This set of *near-critical configurations* turns out to play an important role in our analysis of the nucleation time.

Theorem 1. [Scaling of the average nucleation time]
(i) *There exists a* $K_{\mathsf{stab}}(\mu) \in (0,\infty)$ *such that*

$$\lim_{\beta\to\infty} (24\beta)^{-(2k_c-3)}\, e^{-\beta\Gamma}\, \mathbf{E}_{\mathsf{vac}}(\tau_{\mathsf{stab}}) = \frac{K_{\mathsf{stab}}(\mu)}{2\pi|\Lambda|},$$

(ii) *There exists a* $\chi \in \mathbb{R}$ *such that*

$$\lim_{\mu\downarrow -3} (\mu+3)^2 \log K_{\mathsf{stab}}(\mu) = \chi.$$

Theorem 2. [Exponential limit law for the nucleation time]
$\lim_{\beta\to\infty} \mathbf{P}_{\mathsf{vac}}(\tau_{\mathsf{stab}}/\mathbf{E}_{\mathsf{vac}}(\tau_{\mathsf{stab}}) > t) = e^{-t}$ *for all* $t \geq 0$.

Theorem 3. [Gate for the nucleation]
$\lim_{\beta\to\infty} \mathbf{P}_{\mathsf{vac}}\big(\tau_{\mathcal{C}(\beta)} < \tau_{\mathsf{stab}}\big) = 1.$

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Asymptotics Ferromagnetic Ordering of Energy Levels for the Heisenberg Model on Boxes

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(joint work with Bruno Nachtergaele, Wolfgang Spitzer)

1. Description of the Result

1.1. **Definition of the Model.** Given a finite graph $G = (V, E)$, let $\Omega_V = \{+1, -1\}^V$ be the set of all $\sigma : V \to \{+1, -1\}$. Let $\mathcal{H}_V = \ell^2(\Omega_V)$, and, for each $x \in V$, define operators $S_x^{(1)}$, $S_x^{(2)}$ and $S_x^{(3)}$ on $\mathcal{H}_V$ as

$$S_x^{(3)} f(\sigma) = \frac{1}{2}\,\sigma(x) f(\sigma)\,, \quad S_x^{(1)} f(\sigma) = \frac{1}{2}\, f\left((-1)^{\delta_{x,\cdot}} \sigma(\cdot)\right)\,,$$
$$S_x^{(2)} = -i(S_x^{(3)} S_x^{(1)} - S_x^{(1)} S_x^{(3)})\,.$$

Two other important operators are S_x^+ and S_x^-, defined as $S_x^{\pm} = S_x^{(1)} \pm i S_x^{(2)}$:

$$S_x^{\pm} f(\sigma) = \delta_{\sigma(x),\mp 1} f\left((-1)^{\delta_{x,\cdot}} \sigma(\cdot)\right)\,.$$

Then the quantum Heisenberg ferromagnetic Hamiltonian is $H_G : \mathcal{H}_V \to \mathcal{H}_V$

$$H_G = \sum_{\{x,y\} \in E} \left(\frac{1}{4}\, I - S_x^{(1)} S_y^{(1)} - S_x^{(2)} S_y^{(2)} - S_x^{(3)} S_y^{(3)} \right)\,,$$

where I is the identity operator on $\mathcal{H}_V$. The interaction may also be written as $\frac{1}{4}\, I - S_x^{(3)} S_y^{(3)} - \frac{1}{2}\left(S_x^+ S_y^- + S_x^- S_y^+\right)$.

1.2. **Symmetries of the model.** The Hamiltonian commutes with each of the three components of the total spin operator, $S_V^{(1)}$, $S_V^{(2)}$ and $S_V^{(3)}$ where $S_V^{(a)}$ equals $\sum_{x \in V} S_x^{(a)}$, for $a \in \{1, 2, 3\}$. Hence it also commutes with S_V^+ and S_V^- with are $S_V^{\pm} = \sum_{x \in V} S_x^{\pm} = S_V^{(1)} \pm i S_V^{(2)}$. It also commutes with the total Casimir operator

$$\mathcal{C}_V = \left(S_V^{(1)}\right)^2 + \left(S_V^{(2)}\right)^2 + \left(S_V^{(3)}\right)^2 = \left(S_V^{(3)}\right)^2 + \frac{1}{2}\, S_V^+ S_V^- + \frac{1}{2}\, S_V^- S_V^+\,.$$

Moreover, the Casimir operator commutes with the three components of the total spin operator. Recall $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$, for each $x \in \mathbb{R}$. The spectrum of $\mathcal{C}_V$ is

$$\mathrm{spec}(\mathcal{C}_V) = \left\{ s(s+1) : s \in \left\{ \frac{1}{2}\,|V| - n : n \in \left\{ 0, \ldots, \left\lfloor \frac{1}{2}\,|V| \right\rfloor \right\} \right\} \right\}\,.$$

1.3. **The "Ferromagnetic Ordering of Energy Levels Property".** For each $n \in \{0, \dots, \lfloor \frac{1}{2}|V| \rfloor\}$, we may define the minimum energy of H_G among eigenvectors with total spin $s = \frac{1}{2}|V| - n$:

$$\lambda_{\min}(G,n) \,=\, \min\left\{\frac{\langle f, H_G f\rangle}{\|f\|^2} \,:\, f \in \ker(\mathcal{C}_V - s(s+1)I) \setminus \{0\}\right\} \text{ for } s \,=\, \frac{1}{2}\,|V|-n\,.$$

Then, it follows from a small part of a famous theorem of Lieb and Mattis [3] that

$$\lambda_{\min}(G,0) \,\leq\, \lambda_{\min}(G,n)\,, \tag{1}$$

for all $n > 0$. In [4], we defined the property "FOEL-n" for the graph G to be the property that

$$\lambda_{\min}(G,n) \,=\, \min\{\lambda_{\min}(G,m) \,:\, m \in \{n, \dots, \lfloor |V|/2 \rfloor\}\}\,.$$

The letters FOEL stand for "ferromagnetic ordering of energy levels." This property is supposed to be a ferromagnetic version of a property proved by Lieb and Mattis for antiferromagnets and ferrimagnets in the same reference [3].

It is known that the FOEL-n property need not hold for all graphs G and all $n \in \{0, \dots, \lfloor |V|/2 \rfloor\}$. However, from (1), it is known that FOEL-0 holds for all graphs. Moreover, in a remarkable result, Caputo, Liggett and Richthammer proved a result that does imply that FOEL-1 holds for all graphs G [1]. (Their result is a stronger result for the interchange process on arbitrary graphs, which verified a conjecture of Aldous.) The counterexamples reported on in [6] are that for an even cycle $G = C_{2n} = (\{1, \dots, n\}, \{\{1,2\}, \{2,3\}, \dots, \{2n-1, 2n\}, \{1, 2n\}\})$, it appears that FOEL-$(n-1)$ is violated when $n > 2$ because $\lambda_{\min}(C_{2n}, n) < \lambda_{\min}(C_{2n}, n-1)$. (This was numerically verified for $n = 3, \dots, 8$.) These counterexamples were anticipated in [1] because Caputo, Liggett and Richthammer's "octopus inequality" becomes an equality for C_4, and indeed $\lambda_{\min}(C_4, 2) = \lambda_{\min}(C_4, 1)$. Given all this, it is useful to know whether the FOEL-n property is generally true for some important family of graphs. Our main theorem is this:

Theorem 1. *For each $d, L \in \{1, 2, \dots\}$, let $B^d(L)$ denote the graph whose vertex set is $\{1, \dots, L\}^d \subset \mathbb{Z}^d$ and with the induced edge set, inherited as a subset of $\mathbb{Z}^d$. Then, for each $d, n \in \{1, 2, \dots\}$, there exists an $L_0(d,n) \in \{1, 2, \dots\}$ such that the graph $B^d(L)$ satisfies the FOEL-n property for all $L \geq L_0(d,n)$.*

The preprint containing the proof of this theorem has been posted to the arXiv. See [5].

2. Overview of the method

2.1. **Relation to a paper of Correggi, Giuliani and Seiringer.** For $d = 3$, this result may be deduced from two lemmas in [2]. Firstly, they proved a Poincaré type inequality, which implies that there exists some $c > 0$ such that

$$\lambda(B^d(L), n) \,\geq\, \frac{cn}{L^2}\,,$$

for all $d, n, L \in \{1, 2, \ldots\}$. This inequality follows from Proposition 5.2 in their paper. But, by some other results in their paper (Theorem 1.1, Lemma 5.4 and equations (5.32) and (5.33)), one may deduce that for $d = 3$ and each n, one has

$$\lambda_{\min}(B^3(L), n) - \frac{\pi^2 n}{2L^2} = O(L^{-3}).$$

More generally, this result holds, trivially, for each $d \geq 3$, and with a small effort, one may adapt their arguments to also conclude that $\lambda_{\min}(B^2(L), n) \sim \pi^2 n/(2L^2)$ as $L \to \infty$. For $d = 1$, one needs a different argument, which is actually the argument we included in our preprint.

2.2. **The Toth Graph.** For each $n \in \{0, \ldots, |V|\}$, let $\Theta_n(G) = (\Theta_n(V), \Theta_n(E))$ be the graph where $\Theta_n(V) = \{(x_1, \ldots, x_n) \in V^n : |\{x_1, \ldots, x_n\}| = n\}$ and where $\Theta_n(E)$ consists of all pairs $\{(x_1, \ldots, x_n), (y_1, \ldots, y_k)\} \subset \Theta_n(V)$ such that there is some $k \in \{1, \ldots, n\}$ such that $\{x_k, y_k\} \in E$, and $x_j = y_j$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$. Let $\Omega_{V,n} \subset \Omega_V$ denote the set of all σ's such that $|\sigma^{-1}(\{-1\})| = n$. Then $\ell^2(\Omega_{V,n})$ is an invariant subspace of H_G, and, when restricted to this invariant subspace, it is unitarily equivalent to $-\frac{1}{2}\Delta_{\Theta_n(G)}$, when restricted to symmetric functions $f \in \ell^2(\Theta_n(V))$, where $-\Delta_{\Theta_n(G)}$ is the graph Laplacian

$$-\Delta_{\Theta_n(G)} f(x_1, \ldots, x_n) = \sum_{\substack{(y_1,\ldots,y_n) \in \Theta_n(V) \\ \{(x_1,\ldots,x_n),(y_1,\ldots,y_n)\} \in \Theta_n(E)}} [f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)].$$

This is the graph such that symmetric exclusion process with n particles on G is the graph Laplacian of $\Theta_n(G)$. This equivalence was first found by Toth [7] who used it to obtain bounds on the pressure, which Correggi, Giuliani and Seiringer later improved.

By basic Sobolev inequality type techniques, one may prove that there is a constant c such that, for any function $f \in \ell^2(\Theta_n(B^d(L)))$ with average 0

$$\sum_{\substack{(x_1,\ldots,x_n) \in \Theta_n(\{1,\ldots,L\}) \\ \exists 1 \leq j < k \leq n\,,\ \{x_j, x_k\} \in E(B^d(L))}} |f(x_1, \ldots, x_n)|^2 \leq c \left(\langle f, -\Delta_{\Theta_n(G)} f \rangle\right)^{d/2}.$$

This shows that the effect of magnon collisions may be neglected in dimensions $d \geq 3$, as $L \to \infty$. In fact after a simple argument shows they may also be neglected in $d = 2$ if one only keeps track of the first K energy levels for any fixed K as $L \to \infty$. For $d = 1$ one needs to "fill-in" the function on the diagonal by taking an average of nearby points. For symmetric functions, then, the energy on the diagonal of the filled-in function may be neglected for the first K energy levels as $L \to \infty$. All this shows that $\lambda_{\min}(B^d(L), n) \sim n\pi^2/(2L^2)$ for all d, n as $L \to \infty$.

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Mean-field behavior for nearest-neighbor percolation in $d > 10$

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(joint work with Robert Fitzner)

We investigate nearest-neighbor percolation in $\mathbb{Z}^d$ with d large, where we set each bond $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$, with x and y nearest-neighbors, *occupied*, independently of all other bonds, with probability p and *vacant* otherwise. The corresponding product measure is denoted by $\mathbb{P}_p$ with corresponding expectation $\mathbb{E}_p$. We write $\{x \leftrightarrow y\}$ for the event that there exists a path of occupied bonds from x to y. For $x \in \mathbb{Z}^d$, the set $\mathscr{C}(x) := \{y \in \mathbb{Z}^d : y \leftrightarrow x\}$ is called the *cluster* of x. It is the size and geometry of these clusters close to criticality that we are interested in in high-dimensions. Our main result is the *infrared bound* for the percolation two-point function, as well as the existence of many critical exponents it implies.

We define p_c, the critical value of p, as

$$p_c(d) = \sup\{p : \mathbb{E}_p[|\mathscr{C}(0)|] < \infty\}. \tag{1}$$

We let the *two-point function* be given by

$$\tau_p(x) = \mathbb{P}_p(0 \leftrightarrow y), \tag{2}$$

and $\hat{\tau}_p(k)$ is its Fourier transform. Our main result is as follows:

Theorem 1 (Infrared bound)**.** *For nearest-neighbor percolation with $d \geq 11$, there exists a constant $A(d)$ such that, uniformly for $p \leq p_c(d)$,*

$$\hat{\tau}_p(k) \leq \frac{A(d)}{1 - \hat{D}(k)}. \tag{3}$$

Our proof also implies good bounds on $p_c(d)$ and $A(d)$ in Theorem 1:

Theorem 2 (Bounds on critical value and amplitude)**.** *For nearest-neighbor percolation with $d \geq 11$, the following upper bounds hold:*

d	11	12	13	14	15	20
$(2d-1)p_c(d) \leq$	1.0132	1.00861	1.006268	1.0048522	1.00391	1.00179
$A(d) \leq$	1.02476	0.995	0.986	0.98243	0.98088	0.98115

Since $(2d-1)p_c(d) \geq 1$ by a comparison to branching processes, the bound on $p_c(11)$ is provably at most 1.32% off the real value.

Our results further prove the existence of several percolation *critical exponents*. For example, it is predicted that

$$c_1(p-p_c)^\beta \leq \mathbb{P}_p(|\mathscr{C}(0)| = \infty) \leq c_2(p-p_c)^\beta \qquad \text{as} \qquad p \searrow p_c, \tag{4}$$

for some $\beta > 0$, which we will write as $\mathbb{P}_p(|\mathscr{C}(0)| = \infty) \sim (p-p_c)^\beta$. Then, we obtain the existence and values of the following critical exponents:

Theorem 3 (Critical exponents)**.** *For nearest-neighbor percolation with $d \geq 11$,*

$$\mathbb{P}_p(|\mathscr{C}(0)| = \infty) \sim (p-p_c)^1 \qquad \textit{as} \qquad p \searrow p_c; \tag{5}$$

$$\mathbb{E}_p[|\mathscr{C}(0)|] \sim (p-p_c)^{-1} \qquad \textit{as} \qquad p \nearrow p_c; \tag{6}$$

$$\mathbb{P}_{p_c}(|\mathscr{C}(0)| \geq n) \sim n^{-1/2} \qquad \textit{as} \qquad n \to \infty, \tag{7}$$

that is, $\beta = \gamma = 1, \delta = 2$. Further, $\eta = 0$ in x-space, i.e., there exists a constant $A(d)$ such that, as $|x| \to \infty$,

$$\tau_{p_c}(x) = \frac{a_d A(d)}{|x|^{d-2}}(1 + O(|x|^{-2/d})), \qquad \textit{with} \qquad a_d = \frac{d\Gamma(d/2-1)}{2\pi^{d/2}}. \tag{8}$$

The statements in Theorem 3 are direct consequences of Theorem 1, and several key results in the literature. Aizenman and Newman [1] proved that $\gamma = 1$ when the so-called *triangle condition* holds. The triangle condition states that

$$\triangle(p_c) = \sum_{x,y \in \mathbb{Z}^d} \tau_{p_c}(0,x)\tau_{p_c}(x,y)\tau_{p_c}(y,0) < \infty. \tag{9}$$

Barsky and Aizenman [2] in turn show that, under the same condition, $\beta = 1$ and $\delta = 2$. By the Fourier inversion theorem and the fact that $\triangle(p_c)$ is the three-fold convolution of τ_{p_c} with itself, the infrared bound in Theorem 1 immediately implies that the triangle condition holds. Hara [8] proves that (8) holds for $d \geq 19$, and his proof can be adapted to our setting. Finally, Kozma and Nachmias [12, 13] further identify two one-arm critical exponents.

It is now 25 years ago that Hara and Slade proved their seminal result [9] that percolation displays mean-field behavior for sufficiently high dimension for the nearest-neighbor model, and for $d > 6$ for so-called *spread-out* models, where all edges between x and y with $\|x-y\| \leq L$ are allowed for a sufficiently large L. By *universality*, it is believed that $d > 6$ is also enough for the nearest-neighbor model, but this is yet unproven. In 1994, Hara and Slade [11] reported that $d \geq 19$ is enough in the nearest-neighbor setting. This was achieved by adapting their seminal result for self-avoiding walk (SAW) proving that SAW is diffusive in $d \geq 5$, which is optimal, since in $d = 4$ there are logarithmic corrections, see the talk by Slade in this conference and the work by Bauerschmidt, Brydges and Slade [3]. The proof is *computer assisted*, since it relies on computing several random walk integrals. The Hara-Slade proof is a *perturbation analysis*, and thus needs a small parameter, which for the nearest-neighbor setting is $1/d$. Thus one needs to take d large to make the analysis work. For percolation, their methodology worked to $d \geq 19$ in 1994, and, Hara has been able to improve the analysis since to $d \geq 15$. These proofs have never been published. Also, the proof needs the finiteness of the *heptagram*, which holds when $d > 14$, so, in this sense, the proof is optimal.

The perturbation analysis makes use of the *lace expansion*, a perturbation of the two-point function $\tau_p(x)$ around the random walk's Green function. The lace-expansion analysis currently only works when the coefficients present in the analysis are sufficiently small, which in turn needs that the triangle diagram is small. Thus, while the Aizenman-Barsky-Newman result $\beta = \gamma = 1$, $\delta = 2$ require the triangle diagram to be *finite*, we can only prove that it is *small*, which is bound to only be true when the dimension is sufficiently higher than 7.

Our proof. Also our proof in [6] relies on the *lace expansion* and is *computer-assisted*. The main innovation resides in our expansion, which expands the two-point function $\tau_p(x)$ around *non-backtracking walk* (NBW) rather than simple random walk. A NBW is a simple random walk that is not allowed to traverse the edge it has last traversed. This expansion is called *non-backtracking lace expansion* or NoBLE. NoBLE removes the largest contribution in the perturbation analysis, which explains why it makes it possible to extend the analysis to lower dimensions compared to the Hara-Slade proof. The computer-assisted nature is again due to the need to compute various simple random walk integrals, which we perform in an identical was as Hara and Slade who prove rigorous bounds on such integrals using a representation in terms of integrals over powers of Bessel functions.

The necessary computations are performed in Mathematica notebooks that are available from Robert Fitzner's webpage. We then apply the general analysis in [5] to complete the proof. The fact that all the necessary Mathematica-notebooks are publicly available makes the proof as transparent as possible.

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The random interchange process on the hypercube

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(joint work with Roman Kotecký, Daniel Ueltschi)

1. Abstract

The interchange process is defined on a finite graph. With any edge is associated the transposition of its endvertices. The outcomes of the interchange process consist of sequences of random transpositions and the main questions of interest deal with the cycle structure of the random permutation that is obtained as the composition of these transpositions. As the number of random transpositions increases, a phase transition may occur that is indicated by the emergence of cycles of diverging lengths involving a positive density of vertices.

The most relevant graphs are regular graphs with an underlying "geometric structure" like a finite cubic box in $\mathbb{Z}^d$ with edges between nearest neighbours. But the problem of proving the emergence of long cycles is out of reach for now and recent studies have been devoted to simpler graphs such as trees [2, 5] and complete graphs [6, 3, 4]. (Note also the intriguing identities of Alon and Kozma based on the group structure of permutations [1].) The motivation for the present article is to move away from the complete graph towards $\mathbb{Z}^d$. We consider the hypercube $\{0,1\}^n$ in the large n limit and establish the occurrence of a phase transition demonstrated by the emergence of cycles larger than $2^{(\frac{1}{2}-\varepsilon)n}$. Our proof involves the recent method of Berestycki [3], which was used for the complete graph but is valid more generally, with an estimate of the rate of splits that invokes the isoperimetric inequality for hypercubes.

2. Main result

Let $G_n = (Q_n, E_n)$ be a graph whose $N = 2^n$ vertices form a *hypercube* $Q_n = \{0,1\}^n$ with edges joining nearest-neighbours—pairs of vertices that differ in exactly one coordinate, $E_n = \{\{x,y\} : x, y \in Q_n, |x-y|_1 = 1\}$, $|E_n| = \frac{Nn}{2}$.

Let Ω_n be the set of infinite sequences of edges in E_n. For $t \in \mathbb{N}$ by $\mathcal{F}_{n,t}$ we denote the σ-algebra generated by the first t coordinates. Further, for $t \in \mathbb{N}$ we use $\Omega_{n,t}$ to denote the set of sequences of t edges $\boldsymbol{e} = (e_1, \ldots, e_t)$, where $e_s \in E_n$

for all $s = 1, \dots, t$. The σ-algebra $\mathcal{F}_{n,t}$ will be identified with the total σ-algebra over $\Omega_{n,t}$. For an event $A \in \mathcal{F}_{n,t}$ we set

$$\mathbb{P}_n(A) = |A|\big(\tfrac{2}{Nn}\big)^t,$$

i.e. edges are chosen independently and uniformly from E_n. Here and after for any finite set A by $|A|$ we denote its cardinality.

Using τ_e to denote the transposition of the two endvertices of an edge $e \in E_n$, we can view the sequence $\boldsymbol{e} \in \Omega_{n,t}$ as a series of *random interchanges* generating a *random permutation* $\sigma_t = \tau_{e_t} \circ \tau_{e_{T-1}} \circ \cdots \circ \tau_{e_1}$ on Q_n. For any $\ell \in \mathbb{N}$, let $V_t(\ell)$ be the random set of vertices that belong to permutation cycles of lengths greater than ℓ in σ_t.

We start with the straightforward observation that only small cycles occur in σ_t when t is small. It is based on the fact that the random interchange model possesses a natural percolation structure when viewing any edge contained in $\boldsymbol{e}$ as opened.

Theorem 1. *Let $c < 1/2$ and $\epsilon > 0$. Then there exists n_0 such that*

$$\mathbb{P}_n(|V_t(\kappa n)| = 0) > 1 - \epsilon\kappa^{-3/2}$$

for all $t \leq cN$, all $\kappa \geq \frac{2\ln 2}{(1-2c)^2}$, and all $n > n_0$.

Our main result addresses the emergence of long cycles for sufficiently large t. We expect that cycles of order N occur for all large times; here we prove a weaker claim: cycles larger than $N^{\frac{1}{2}-\varepsilon}$ occur for a "majority of large times".

Theorem 2. *Let $c > 1$ and $a < \frac{1}{2}(1-\frac{1}{c})$. Further, consider a sequence of positive numbers (Δ_n) such that $\lim_{n\to\infty} \Delta_n n/\log(2n) = \infty$. Then there exists n_0 such that for all $n > n_0$ and all $T > cN$, we have*

$$\frac{1}{\Delta_n T} \sum_{t=T+1}^{\lfloor(1+\Delta_n)T\rfloor} \mathbb{E}_n\Big(\frac{|V_t(N^a)|}{N}\Big) \geq \tfrac{1}{2}(1-\tfrac{1}{c}) - a.$$

We can choose $\Delta_n \equiv \Delta > 0$, rather than a sequence that tends to 0. In this case, Theorem 2 takes a simpler form, which perhaps expresses the statement 'long cycles are likely' more directly.

Corollary 3. *Let $a \in (0, 1/2)$, $\Delta > 0$, and $\epsilon_1 \in (0, \frac{1}{2} - a)$. Then there exists $c > 1$ and $\epsilon_2 > 0$ such that for n large enough we have*

$$\frac{1}{\Delta_n T} \sum_{t=T+1}^{\lfloor(1+\Delta_n)T\rfloor} \mathbb{P}_n\Big(\frac{|V_t(N^a)|}{N} \geq \epsilon_1\Big) \geq \epsilon_2$$

for all $T > cN$.

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Electrical resistance of the critical branching random walk

Asaf Nachmias

(joint work with Antal A. Járai)

We study the electrical resistance of the trace of oriented critical branching random walk (BRW) in low dimensions. This trace is obtained by drawing a critical Galton-Watson tree $\mathcal{T}$ conditioned to survive forever and randomly mapping it into $\mathbb{Z}^d \times \mathbb{Z}^+$ in the following manner: we initialize by mapping the root of $\mathcal{T}$ to $(o, 0)$ and recursively, if $V \in \mathcal{T}$ was mapped to (x, n) and $U \in \mathcal{T}$ is a child of V, then we map U to $(y, n+1)$ where y is chosen according to a symmetric random walk distribution (we assume that this distribution has an exponential moment). Denote by $\Phi : \mathcal{T} \to \mathbb{Z}^d \times \mathbb{Z}^+$ this random mapping. The trace we consider in this paper is the graph induced by set of edges $\{\Phi(V), \Phi(U)\}$ for every edge $\{U, V\}$ of $\mathcal{T}$.

It follows from the work of Barlow, Járai, Kumagai and Slade [1, Example 1.8(iii)] (who studied the much more difficult model of critical oriented percolation (OP)) that when $d > 6$, the electrical resistance between the root and generation n in the BRW is linear in probability. This enabled them to calculate various exponents describing the behavior of the simple random walk on the trace. In particular, they show that the mean hitting time of graph distance n is $\Theta(n^3)$, that the spectral dimension equals 4/3 and more, see [1].

They asked [1, Section 1.4, Example 1.8 (iii)] whether the resistance of the critical BRW is still linear in n in dimensions $4 < d \leq 6$, that is, in any dimension above the critical dimension 4 of OP [2, 3, 4, 5]. Here we answer their question by showing that the resistance is $O(n^{1-\alpha})$ when $d \leq 5$.

Theorem 1. *Let $R(n)$ denote the expected effective resistance between the origin and generation n of a branching random walk in dimension $d < 6$ with progeny distribution that has mean 1, positive variance and finite third moment, conditioned to survive forever. Assume that the random walk steps are symmetric, non-degenerate and have exponential tails. There exists a universal constant $\alpha > 0$ such that*

$$R(n) = O(n^{1-\alpha}).$$

Unlike our firm understanding of anomalous diffusion in high dimensions, random fractals in low dimensions are not (stochastically) finitely ramified. That is, we do not see pivotal edges at every scale. This makes their analysis more challenging, even in the case of the critical BRW which is one of the simplest models of statistical physics. Our argument heavily relies on the built-in independence and self-similarity of the model to obtain recursive inequalities for the resistance. We first show that intersections within the trace occur at every scale; these intersections exist only when $d < 6$. Secondly, we show that the branches leading to each intersection are themselves distributed as BRW, allowing us to bound the electrical circuit using the parallel law and to form recursive estimates. There are additional technical difficulties to overcome. For instance, when intersections do not occur, the resistance is stochastically larger than it is unconditionally and one needs to get adequate bounds on it. Calculating the precise polynomial exponent which determines the growth of $R(n)$ when $d < 6$ remains a challenging open problem.

As mentioned before, it is believed that OP in $d = 5$ behaves similarly to BRW hence we expect an analogue of Theorem 1 to hold. Presumably, the general setup and proving existence of intersections can be done for OP (based on results of [2, 3]). However, due to the lack of distributional self-similarity in OP it seems difficult to obtain recursive bounds. Furthermore, we do not know whether the exponent determining the growth of the resistance in OP in $d = 5$ should be the same as the one for BRW (assuming they both exist).

It is easy to see (and stated in [1]) that the volume up to generation n of the BRW trace is of order $\Theta(n^2)$ in probability. Hence, Theorem 1 together with the commute time identity shows that the mean exit time of the simple random walk on the BRW trace from the ball of radius n in graph distance is at most $O(n^{3-\alpha})$, i.e., much faster than the $\Theta(n^3)$ in dimensions $d > 6$, see [1]. In fact, if one calculated the exponent determining the growth of the resistance, then many other random walk exponents (such as the spectral dimension, walk dimension etc.) could be determined. In particular, if the resistance exponent exists, it follows from our results that the spectral dimension is strictly larger than 4/3.

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Band Permutations

Ron Peled

(joint work with Nayantara Bhatnagar, Alexey Gladkich, Mathew Joseph and Partha Dey)

We consider two models of random permutations whose graph lies mostly within a diagonal band - the Mallows model and the band-Poisson model. For the Mallows model, we find the transition point for the emergence of macroscopic cycles and prove that the cycle structure converges to the Poisson-Dirichlet law above the transition point. Additionally, we provide a law of large numbers for the length of the longest increasing subsequence. For the band-Poisson model we prove a transition in the behavior of the longest increasing subsequence from a Tracy-Widom to a Gaussian regime.

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